Discretization schemes for diffusion operators on general meshes

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Aim and Scope

Design and analysis of schemes for the simulation of diffusive viscous flows

Free incompressible of compressible flows : *Navier Stokes equations*

Flow in porous media, *Darcy equation*.

$\rightsquigarrow$ discretization of $-\mathrm{div}(\Lambda \nabla u))$

$$A \text{ can be } \begin{cases} \text{heterogeneous } \Lambda = \lambda_i \mathrm{I}d \text{ on } \Omega_i \\ \text{anisotropic } \Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \end{cases}$$

on general **multidimensional** and possibly **non conforming** meshes

What kind of schemes ?

- Cheap, easy to implement...
- Local mass and flux conservation
- Preserves the physical bounds of the solution: *stability*
- Good mathematical properties: *existence, uniqueness, convergence*

Supplementary constraint from the oil reservoir simulation community: **cell centred schemes**

transport equations coupled to thermodynamics or chemistry: need of the definition of a volume for a "FLASH" to the thermodynamics or chemistry solver

Approximation of diffusion terms: the orthogonal case

1/ Discretization of the balance form

$$\int_K -\Delta u = -\int_{\partial K} \nabla u \cdot \mathbf{n}_K = \sum_{\sigma \subset \partial K} -\int_\sigma \nabla u \cdot \mathbf{n}_{K,\sigma}$$

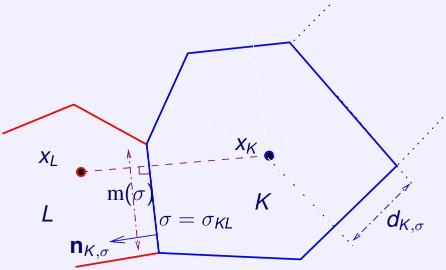
2/ Consistent approximation of the normal fluxes

$$-\int_\sigma \nabla u \cdot \mathbf{n}_{K,\sigma} \approx |\sigma| \frac{u_K - u_L}{d(x_K, x_L)}$$

$$(x_K, x_L) \perp \sigma$$

$$\Downarrow$$

consistency of the numerical fluxes



FV schemes for general grids or anisotropic problems

Numerous attempts to build approximate consistent approximations to the fluxes $A\nabla u \cdot \mathbf{n}$ on non-orthogonal grids

- full reconstruction of a discrete gradient
- computation of the (non orthogonal fluxes) with several points:

among which:

FV schemes for general grids or anisotropic problems

Cell centred schemes

- **Multi point** schemes (MPFA) (Aavatsmark et al. 98-08) (O scheme, U scheme, L scheme...): cell centered, in general non symmetric,
  Convergence proof of the O scheme under geometrical assumptions (Agelas Masson 08)
  Stability and coercivity problems for distorted meshes symmetric stabilized version (Le Potier 05)
- **Stabilized gradient** scheme "SUCCES" (Eymard, Gallouet H. 07) cell centred, symmetric, convergence proof with no regularity assumption.
  but: bad approximation of the fluxes at the heterogenities (partly cured by an adaptation of the "L" scheme (Agelas, Di Pietro Masson 2008)

FV schemes for general grids or anisotropic problems

Discrete duality finite volume (DDVF) schemes (Hermeline 2000, Domelevo Omnes 2005 Andreianov Boyer Hubert 2007)
full reconstruction of the gradient thanks to the introduction of a dual mesh (three meshes needed in 3D)
Unknowns at the centers and the vertices
Symmetric scheme, convergence is proven.
Unknowns at the centers and vertices (and edges in 3D)

Hybrid type schemes

- Mimetic finite difference (Brezzi, Lipnikov, Shashkov et al. 05),
- Mixed FV scheme (Droniou Eymard 06)
- Hybrid FV scheme (Eymard Gallouet H. 07),

Unknowns at the centres and the edges (or faces), may be hybridised: unknowns at the edges only.

SUSHI scheme (to cure SUCCES): hybrid at the heterogeneities, " (Eymard, Gallouet H. 08)

⋮

but unknowns at the interfaces

FVCA 5 benchark

Aim: to test existing discretizations for heterogeneous anisotropic diffusion problems

organized Florence Hubert and RH
2D linear diffusion
19 participants (finite volume, finite element, discontinuous Galerkin)
Some 10 test cases and meshes

`http://www.latp.univ-mrs.fr/fvca5/`
`http://www.cmi.univ-mrs.fr/~herbin/`

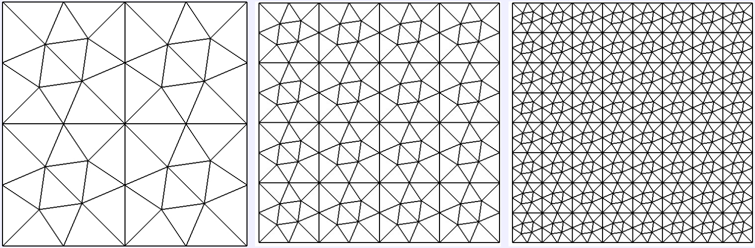
Principal properties under study:

- Rate of convergence w.r.t size of the mesh, number of unknowns.
- Positivity of the schemes: $f \geq 0 \Rightarrow u^{\tau} \geq 0$.
- Maximum principle: if $f = 0$, $u_e \in [a, b] \Rightarrow u^{\tau} \in [a, b]$.
- Approximation of physical quantities such as energy, boundary fluxes,...

FVCA5 benchmark Test 1: mild anisotropy

$$-\mathrm{div}(\Lambda\nabla u) = f \text{ in } \Omega, \qquad \Lambda = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$$

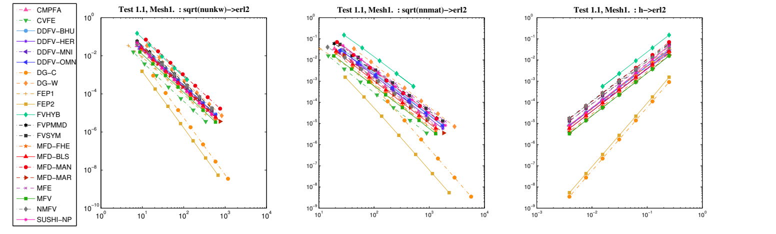
Non homogeneous Dirichlet boundary conditions Exact solution: $u(x, y) = 16x(1 - x)y(1 - y)$.



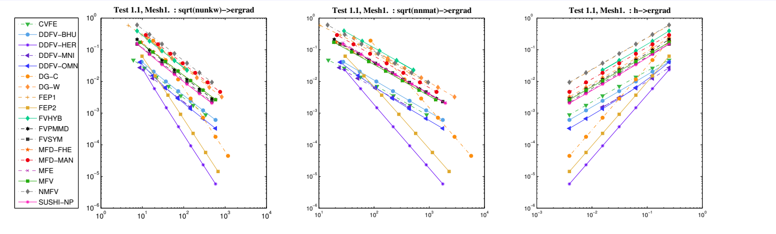
Mesh1

Test 1: mild anisotropy. Test 1.1 - mesh1_1

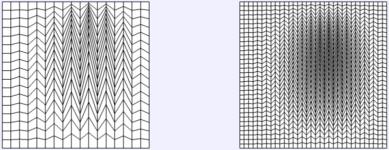
Comparison of L^2 norm of the solution (order in $\{2,3\}$)



and its gradient (order in $\{1,2\}$)



Test 1: mild anisotropy. Test 1.1 - mesh4_1, mesh4_2



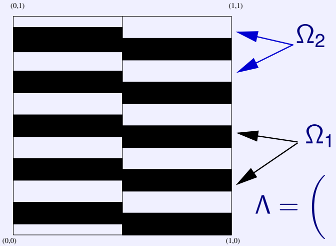
Minimum and maximum of the approximate solutions

	mesh 4_1		mesh 4_2	
	umin	umax	umin	umax
CMPFA	9.95E-03	1.00E+00	2.73E-03	9.99E-01
CVFE	0.00E+00	8.43E-01	0.00E+00	9.14E-01
DDFV-BHU	1.33E-02	9.96E-01	3.63E-03	9.99E-01
DDFV-HER	0.00E+00	**1.03E+00**	0.00E+00	**1.01E+00**
DDFV-MNI	**-3.09E-01**	**1.03E+00**	0.00E+00	1.00E+00
DDFV-OMN	1.34E-02	**1.03E+00**	3.65E-03	**1.01E+00**
DG-C	**-2.33E-03**	9.96E-01	**-3.24E-04**	9.99E-01
DG-W	**-7.90E-05**	9.22E-01	**-8.18E-06**	9.66E-01
FEQ1	0.00E+00	8.61E-01	0.00E+00	9.37E-01
FEQ2	0.00E+00	9.99E-01	0.00E+00	1.00E+00
FVHYB	2.14E-03	9.84E-01	7.16E-04	9.93E-01
FVSYM	7.34E-03	9.59E-01	2.33E-03	9.89E-01
MFD-BLS	8.54E-03	9.55E-01	2.44E-03	9.87E-01
MFD-FHE	9.73E-03	9.45E-01	2.90E-03	9.83E-01
MFD-MAN	6.64E-03	9.71E-01	1.50E-03	9.93E-01
MFD-MAR	8.82E-03	9.60E-01	2.47E-03	9.88E-01
MFV	1.08E-02	9.42E-01	3.34E-03	9.82E-01
NMFV	1.30E-02	**1.11E+00**	3.61E-03	**1.04E+00**
SUSHI-NP	7.64E-03	8.88E-01	2.33E-03	9.61E-01

Test 4: vertical fault

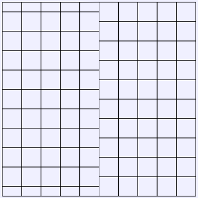
$$-\mathrm{div}(\Lambda \nabla u) = 0 \text{ in } \Omega$$

$$u = \bar{u} \text{ in } \partial\Omega$$

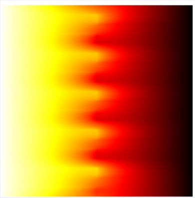


$$\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \text{ with } \begin{cases} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^2 \\ 10 \end{pmatrix} \text{ on } \Omega_1, \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10^{-2} \\ 10^{-3} \end{pmatrix} \text{ on } \Omega_2 \end{cases}$$

and $\bar{u}(x, y) = 1 - x$.



mesh5



Test 4: vertical fault

Maximum principle

- Problems only with the DG methods.

The values of the energies

	ener1 mesh5	eren mesh5	ener1 mesh5_ref	eren mesh5_ref
CVFE	45.9	1.04E-02	**43.3**	6.25E-04
DDFV-BHU	**42.1**	3.65E-02	**43.2**	1.27E-03
DDFV-HER	49.3	1.75E-01	43.8	1.64E-02
DDFV-MNI	/	/	43.8	6.23E-02
DDFV-OMN	**42.2**	3.65E-02	**43.2**	1.28E-03
DG-W	**43.5**	1.38E-02	**43.2**	7.63E-04
FEQ1	/	/	**43.3**	2.31E-03
FEQ2	/	/	**43.2**	0.00E+00
FVHYB	**41.4**	6.12E-02	/	/
MFD-BLS	33.9	7.93E-14	**43.2**	2.84E-12
MFD-FHE	/	/	**43.2**	3.53E-04
MFD-MAN	31.4	1.16E-12	**43.2**	4.71E-14
MFD-MAR	**41.1**	1.30E-13	**43.2**	2.69E-12
MFV	49.9	4.21E-05	**43.2**	1.88E-05
NMFV	/	/	**43.2**	5.92E-04
SUSHI-NP	39.1	6.67E-02	**43.1**	8.88E-04

- The methods that have trouble with PPMax are the most accurate for the energy on coarse meshes.

Test 4: vertical fault

The fluxes

	flux0 mesh5	flux0 mesh5_ref	flux1 mesh5	flux1 mesh5_ref	fluy0 mesh5	fluy0 mesh5_ref	fluy1 mesh5
CMPFA	-45.2	**-42.1**	46.1	**44.4**	-0.95	**-2.33**	4.84E-04
CVFE	-46.6	**-42.2**	48.5	**44.5**	0.87	-2.25	8.02E-04
DDFV-BHU	-40.0	**-42.1**	41.8	**44.4**	-1.81	**-2.33**	9.08E-04
DDFV-HER	-40.0	**-42.0**	41.8	**44.3**	-1.81	**-2.35**	9.08E-04
DDFV-MNI	-43.8	-39.9	45.5	42.6	-2.8	-2.68	1.18E+00
DDFV-OMN	-40.0	**-42.1**	41.8	**44.4**	-1.81	**-2.33**	9.08E-04
DG-W	**-43.1**	**-42.1**	**45.3**	**44.5**	-2.19	**-2.32**	1.50E-03
FEQ1	/	**-42.2**	/	**44.5**	/	-2.16	/
FEQ2	/	**-42.1**	/	**44.5**	/	**-2.32**	/
FVHYB	-44.3	/	46.3	/	0.49	/	1.55E-04
MFD-BLS	-32.3	**-42.1**	36.2	**44.4**	-3.94	**-2.33**	1.22E-03
MFD-FHE	/	**-42.1**	/	**44.5**	/	-2.47	/
MFD-MAN	-29.7	**-42.1**	34.1	**44.4**	-4.37	**-2.33**	1.01E-03
MFD-MAR	-39.8	**-42.1**	42.5	**44.4**	**-2.68**	**-2.33**	9.95E-04
MFE	/	**-42.1**	/	**44.4**	/	**-2.33**	/
MFV	-44.0	**-42.1**	50.3	**44.4**	-8.03	**-2.33**	1.72E+00
NMFV	**-43.2**	**-42.1**	**44.5**	**44.4**	-1.23	**-2.33**	2.32E-04
SUSHI-NP	**-40.9**	**-42.1**	**43.1**	**44.4**	**-2.21**	**-2.33**	6.94E-04

Stabilized gradient schemes SUCCES, Hybrid FV, SUSHI

Variational formulation: Find $u \in H_0^1(\Omega)$;
$$\int_\Omega \Lambda(x)\nabla u(x)) \cdot \nabla v(x)\, dx = \int_\Omega f(x)v(x)\, dx,\ \forall v \in H_0^1(\Omega).$$

First idea

$H_\mathcal{T}$ is the set of functions $u\ :\ \Omega \to \mathbb{R}$ such that u is constant on K, for all $K \in \mathcal{T}$; this constant is denoted by u_K.

For $u \in H_\mathcal{T}$, define $\nabla_\mathcal{T} u$ (convenient approximation of the gradient on u)

Approximate problem : Find $u \in H_\mathcal{T}$;

$$\int_\Omega \Lambda \nabla_\mathcal{T} u \cdot \nabla_\mathcal{T} v\, dx = \int f v\, dx,\ \forall v \in H_\mathcal{T}.$$

Ideas for convergence

Family of approximate problems : with
$v = P_{\mathcal{T}}(\varphi), \varphi \in C_c^\infty(\Omega)$

$$\int_\Omega \Lambda \nabla_{\mathcal{T}} u_{\mathcal{T}} \cdot \nabla_{\mathcal{T}} P_{\mathcal{T}}(\varphi)\, dx = \int f v\, dx, \ \forall v \in H_{\mathcal{T}}.$$

Imagine that, as $h_{\mathcal{T}} \to 0$:

(CompU) $u_{\mathcal{T}} \to \tilde{u}$ in $L^2(\Omega)$

(RegU) $\tilde{u} \in H_0^1(\Omega)$

(WCvG) $\nabla_{\mathcal{T}} u \rightharpoonup \nabla \tilde{u}$ in $L^2(\Omega)$ weakly

(SCvG) $\nabla_{\mathcal{T}} P_{\mathcal{T}}(\varphi) \to \nabla \varphi$ in $L^2(\Omega)$ (strongly)

Then: $\tilde{u} = u$ unique solution to

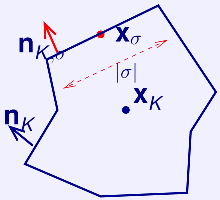
$$u \in H_0^1(\Omega)$$
$$\int_\Omega \Lambda(x) \nabla u(x)) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx, \ \forall v \in H_0^1(\Omega).$$

*

I - Definition of a consistent gradient

Question: for $v \in H_{\mathcal{T}}$, define $\nabla_{\mathcal{T}} v$;

(SCvG) $\quad \nabla_{\mathcal{T}} P_{\mathcal{T}}(\varphi) \to \nabla\varphi$ in $L^2(\Omega)$



$g(\mathbf{x}) = \mathbf{w}\cdot\mathbf{x}$, $\int_K \nabla g = \int_{\partial K} g\,\mathbf{n_K} \rightsquigarrow$

$$\forall \mathbf{w} \in \mathbb{R}^d,\ |K|\mathbf{w} = \int_{\partial K} \mathbf{w}\cdot\mathbf{x}\,\mathbf{n_K} \text{ (MF)}$$

If ψ is affine from $K \to \mathbb{R}$, then $\nabla\psi(\mathbf{x}_\sigma - \mathbf{x}_K) = \psi(\mathbf{x}_\sigma) - \psi(\mathbf{x}_K)$

$\mathbf{w} = (\nabla\psi)_K$ in (MF) and $\sum_{\sigma\subset\partial K} |\sigma|(\nabla\psi)_K \mathbf{n}_{K\sigma} = 0 \rightsquigarrow$

$$|K|(\nabla\psi)_K = \int_{\partial K} (\nabla\psi)_K\cdot\mathbf{x}\,\mathbf{n_K} = \sum_{\sigma\subset\partial K} |\sigma|(\psi(\mathbf{x}_\sigma) - \psi(\mathbf{x}_K))\,\mathbf{n}_{\mathbf{K},\sigma}$$

I - Definition of a consistent gradient

For ψ affine : $|K|(\nabla\psi)_K = \sum_{\sigma\subset\partial K} |\sigma|(\psi(\mathbf{x}_\sigma) - \psi(\mathbf{x}_K))\, \mathbf{n_{K,\sigma}}$

Discrete gradient on K : $|K|(\tilde{\nabla}_{\mathcal{T}} v)_K = \sum_{\sigma\subset K} |\sigma|(\Pi_\sigma v - v_K)\, \mathbf{n}_{K,\sigma}$

$$\tilde{\nabla}_{\mathcal{T}} P_{\mathcal{T}}(\varphi) \to \nabla\varphi in L^2(\Omega) \text{ as } h_{\mathcal{T}} \to 0?$$

YES: If $v_K = \varphi(\mathbf{x}_K))$ and $\Pi_\sigma v = \varphi(\mathbf{x}_\sigma)$ then
$\|\tilde{\nabla}_{\mathcal{T}} v - \nabla\varphi\|_{L^\infty} \leq C_\varphi h_{\mathcal{T}}$
Supplementary edge unknowns $\Pi_\sigma u$?

II - Choice of the values $\Pi_\sigma u$

For $K \in \mathcal{T}$, $x_K \in \mathcal{T}$ (K is "x_K star shaped").
For σ, interior edge (interface) of $\mathcal{T}$, x_σ is the center of σ.

(i) SUCCES Finite element approach: $\Pi_\sigma u_\sigma$ second order reconstruction from the $(u_K)_K$ "not too far":

$$x_\sigma = \sum_{M \in \mathcal{T}} \beta_{M,\sigma} x_M \; \Pi_\sigma u = \sum_{M \in \mathcal{T}} \beta_{M,\sigma} u_M.$$

(ii) HFV Finite volume approach: u_σ such that
$F_{K,L}(u) = |\sigma| \Lambda_K \tilde{\nabla}_{\mathcal{T}} u \cdot \mathbf{n}_{KL} = -|\sigma| \Lambda_L \tilde{\nabla}_{\mathcal{T}} u \cdot \mathbf{n}_{LK} = -F_{L,K}(u)$.

(iii) SUSHI approach: $\mathcal{E} = \mathcal{B} \cup \mathcal{H}$
   if $\sigma \in \mathcal{B}$, $\Pi_\sigma u$ by choice (i)
   if $\sigma \in \mathcal{H}$, $\Pi_\sigma u$ by choice (ii)

If σ is an edge on the boundary, one sets $\Pi_\sigma u = 0$.

Estimate

SUCCES choice (for simplicity)

$$\int_\Omega \Lambda \tilde{\nabla}_{\mathcal{T}} u_{\mathcal{T}} \cdot \tilde{\nabla}_{\mathcal{T}} v \, dx = \int f v \, dx, \ \forall v \in H_{\mathcal{T}}. \qquad (\tilde{S})$$

(CompU) $\quad u_{\mathcal{T}} \to \tilde{u}$ in $L^2(\Omega)$ as $h_{\mathcal{T}} \to 0$?

First estimate: (EST) $\quad \|u_{\mathcal{T}}\|_{L^2} \leq C$?

$v = u$ in $(\tilde{S})$:

$$\int_\Omega |\Lambda \tilde{\nabla}_{\mathcal{T}} u_{\mathcal{T}} \cdot \tilde{\nabla}_{\mathcal{T}} u_{\mathcal{T}}| \, dx = \int f u_{\mathcal{T}} \, dx \leq \|f\|_{L^2} \|u_{\mathcal{T}}\|_{L^2}$$

Assumptions on $\Lambda \Longrightarrow \underline{\lambda} \|\tilde{\nabla}_{\mathcal{T}} u_{\mathcal{T}}\|_{L^2}^2 \leq \|f\|_{L^2} \|u_{\mathcal{T}}\|_{L^2}$

$\|u_{\mathcal{T}}\|_{L^2} \leq \|\tilde{\nabla}_{\mathcal{T}} u_{\mathcal{T}}\|_{L^2}$? NO !!

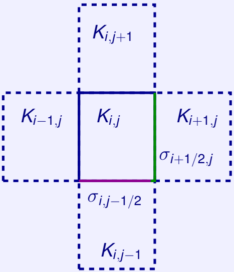
Unstability of $\tilde{\nabla}$

Uniform rectangular mesh,

$$(\tilde{\nabla}_{\mathcal{T}} u)_{K_{i,j}} = \begin{pmatrix} \frac{1}{h_x}\left(u_{i+1/2,j} - u_{i-1/2,j}\right) \\ \frac{1}{h_y}\left(u_{i,j+1/2} - u_{i,j-1/2}\right) \end{pmatrix}$$

If $u_{i+1/2,j} = \dfrac{u_{i+1,j} + u_{i,j}}{2}$ and $u_{i,j+1/2} = \dfrac{u_{i,j+1} + u_{i,j}}{2}$

$$(\tilde{\nabla}_{\mathcal{T}} u)_{K_{i,j}} = \begin{pmatrix} \frac{1}{2h_x}\left(u_{i+1,j} - u_{i-1,j}\right) \\ \frac{1}{2h_y}\left(u_{i,j+1} - u_{i,j-1}\right) \end{pmatrix}$$



Consistent, but unstable...

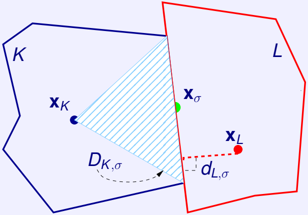
Modify $\tilde{\nabla}$ without destroying its consistency...

Idea: stabilize by a consistency error

Stabilized gradient

$$\tilde{\nabla}_K u_{\mathcal{T}} = \frac{1}{|K|} \sum_{\sigma\subset\partial K} |\sigma|(u_\sigma - u_K)\mathbf{n}_{K,\sigma}$$

$$R_{K,\sigma} u = \frac{\alpha_K}{d_{K,\sigma}}\left(u_\sigma - u_K - \tilde{\nabla}_K u \cdot (\mathbf{x}_\sigma - \mathbf{x}_K)\right)$$



$$\nabla_{\mathcal{T}} u \text{ constant in } D_{K,\sigma} \text{ and } \nabla_{K,\sigma} u = \tilde{\nabla}_K u + R_{K,\sigma} u\, \mathbf{n}_{K,\sigma}$$

The gradient ∇ still satisfies

(SCvG) $\nabla_{\mathcal{T}} P_{\mathcal{T}}(\varphi) \to \nabla\varphi$ in $L^2(\Omega)$ (strongly)

Moeover, thanks to the stabilization:

$$|u_{\mathcal{T}}|^2_{1,\mathcal{T}} = \sum_{\sigma=K|L} \frac{|\sigma|}{d_{K,\sigma}+d_{L,\sigma}}(u_K - u_L)^2 + \sum_{\sigma\subset\partial\Omega} \frac{|\sigma|}{d_{K,\sigma}}(u_K)^2 \le \|\nabla_{\mathcal{T}} u_{\mathcal{T}}\|^2_{L^2}$$

Estimates and weak discrete H^1 compactness

$$|u_{\mathcal{T}}|_{1,\mathcal{T}} \leq \|\nabla_{\mathcal{T}} u_{\mathcal{T}}\|_{L^2}$$
$$\text{and } \underline{\lambda}\|\tilde{\nabla}_{\mathcal{T}} u_{\mathcal{T}}\|_{L^2}^2 \leq \|f\|_{L^2}\|u_{\mathcal{T}}\|_{L^2}$$
$$\Downarrow$$
$$|u_{\mathcal{T}}|_{1,\mathcal{T}} \leq C_1\|_{L^2}\|u_{\mathcal{T}}\|_{L^2}$$

Discrete Poincaré inequality: $\|u_{\mathcal{T}}\|_{L^2} \leq C_2|u_{\mathcal{T}}|_{1,\mathcal{T}}$

$$\Downarrow$$
$$\|u_{\mathcal{T}}\|_{L^2} \leq C \text{ and } \|\nabla u_{\mathcal{T}}\|_{L^2} \leq C$$

Hence:

(CompU) $\quad u_{\mathcal{T}} \to \tilde{u}$ in $L^2(\Omega)$

and $\nabla_{\mathcal{T}} u \rightharpoonup G$ in $L^2(\Omega)$ weakly

+ a bit of work... $G = \nabla \tilde{u}$ and so:

(WCvG) $\nabla_{\mathcal{T}} u \rightharpoonup G$ in $L^2(\Omega)$ weakly

Prolonging $u_{\mathcal{T}}$ and $\nabla u_{\mathcal{T}}$ bu 0 outside of Ω and passing to the limit on the prolonged functions:

(RegU) $\quad \tilde{u} \in H_0^1(\Omega)$

Hence:

Strong convergence of $u_{\mathcal{T}}$

a/ Estimate on the translations in $L^1(\Omega)$:

$$\|u(\cdot + \xi) - u\|_{L^1(\mathbb{R}^d)} \leq \xi \|u\|_{BV}$$

b/ Discrete Hölder

$$\|u(\cdot + \xi) - u\|_{L^1(\mathbb{R}^d)} \leq C\xi \|u_{\mathcal{T}}\|_{1,\mathcal{T}}$$

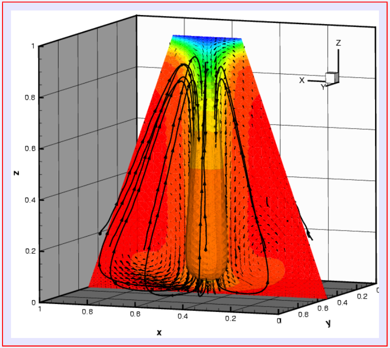
c/ Kolmogorov th. $\rightsquigarrow$ convergence of $u_{\mathcal{T}}$ in L^1
d/ Discrete Sobolev $\rightsquigarrow$ estimate in $L^{\frac{2d}{d-2}}$ (3D) or $L^q,\ q < +\infty$ (2D)
c+d/ $\Longrightarrow$ convergence of $u_{\mathcal{T}}$ in L^2 to $\bar{u}$
e/ $\bar{u} \in H_0^1$ (already known thanks to the weak cvce of the gradient)

Remarks on the scheme

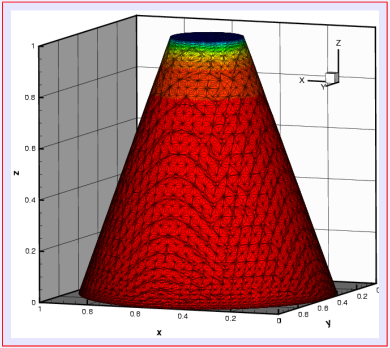
the best...

Application to Navier-Stokes: 3D code by E. Chénier

Natural convection in a cone heated by below
Adiabatic boundary condition on side walls.
Cartesian 3D mesh $40 \times 40 \times 40$, truncated by the cone.
Locally refined at the top of the cone.
$\mathrm{Ra} = 10^6$ and $\mathrm{Pr} = 0.71$



Temperature and velocity
in a cross section



Temperature at the wall

Some recent or on going work

- Equivalence between mixed FV, Hybrid FV and Mimetic (Droniou, Eymard, Gallouet, H.)
- Convergence of SUSHI for incompressible Navier Stokes (Eymard, H., Latché)
- 3D benchmark (H., Hubert)
- Monotone schemes (Droniou, Eymard, ... Le Potier)
- Cell centred scheme for heterogeneities (Agelas Eymard Gallouet H.)
- Convergence of numerical schemes for compressible flows (Eymard, Gallouet, H., Latché)