Convergence of approximate solutions for Stationary compressible Stokes equations

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First step for proving the convergence of approximate solutions for the evolution compressible Navier-Stokes equations (which gives, in particular, the existence of solutions, $d = 3$, $p = \rho^{\gamma}$, $\gamma > \frac{3}{2}$).

Stationary compressible Stokes equations

Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma > 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho\, \mathbf{u}) = 0 \text{ in } \Omega,\ \rho \geq 0 \text{ in } \Omega,\ \int_\Omega \rho(x)dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

Functional spaces : $\mathbf{u} \in (H_0^1(\Omega))^d$, $p \in L^2(\Omega)$, $\rho \in L^{2\gamma}(\Omega)$

($p \in L^q, 1 \leq q < 2$ in the case of Navier-Stokes if $d = 3$ and $\gamma < 3$)

Aim

Prove the existence of a weak solution to the compressible Stokes equations

by the convergence of a sequence of approximate solutions

given by a numerical scheme as the mesh size goes to 0

(up to a subsequence, since, up to now, no uniqueness result is available for this problem)

Discretization spaces

- Mesh: partition of Ω in simplices, regular in the usual finite element sense. Additional assumption:

$$\inf\{\frac{h_L}{h_K}, \frac{h_K}{h_L}, \ \sigma = K|L\} \geq \theta_0$$

- Approximation spaces: $(\mathbf{u}, p, \rho) \in \mathbf{W}_h \times L_h \times L_h$ Crouzeix Raviart spaces

  $\mathbf{W}_h$: piecewise linear functions discontinuous through the edges, with equal mean value on both sides of an edge

  L_h: piecewise constant functions

  Unknowns: $(\mathbf{u}_\sigma)_{\sigma \in \mathcal{E}_{\mathrm{int}}}$, $(p_K)_{K \in \mathcal{T}}$, $(\rho_K)_{K \in \mathcal{T}}$.

$$\mathbf{u} = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \mathbf{u}_\sigma \varphi_\sigma(\mathbf{x}) \qquad p = \sum_{K \in \mathcal{T}} p_K 1_K \qquad \rho = \sum_{K \in \mathcal{T}} \rho_K 1_K$$

Discretization of the momentum equation

- Weak form of the momentum equation $-\Delta \mathbf{u} + \nabla p = \mathbf{f}$.

$$\forall \mathbf{v} \in (H_0^1(\Omega))^d, \qquad \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \, \mathrm{div} \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}$$

- Discrete operators for the velocity field:

$$\nabla_h \mathbf{u} \in \mathrm{L}^2(\Omega)^{d \times d} \quad \text{with} \quad \nabla_h \mathbf{u} = \nabla \mathbf{u} \quad \text{inside the cells,}$$

$$\mathrm{div}_h \mathbf{u} \in \mathrm{L}^2(\Omega) \quad \text{with} \quad \mathrm{div}_h \mathbf{u} = \mathrm{div} \mathbf{u} \quad \text{inside the cells.}$$

- Discrete equation for $\mathbf{u} \in \mathbf{W}_h$:

$$\forall \mathbf{v} \in \mathbf{W}_h, \qquad \int_\Omega \nabla_h \mathbf{u} : \nabla_h \mathbf{v} - \int_\Omega p \, \mathrm{div}_h \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}$$

Properties of the discrete operators

- Broken Sobolev H^1 semi-norm:

$$\|v\|_{1,b}^2 = \sum_{K \in \mathcal{T}} \int_K |\boldsymbol{\nabla} v|^2 \,\mathrm{d}\mathbf{x} = \int_\Omega |\boldsymbol{\nabla}_h v|^2 \,\mathrm{d}\mathbf{x}$$

- Approximation operator $u \in \mathrm{H}^1_0(\Omega) \mapsto r_h u = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \left(\frac{1}{|\sigma|} \int_\sigma u \right) \varphi_\sigma \in W_h$
- Stability and approximation properties: $v \in \mathrm{H}^1_0(\Omega)$

$$\|r_h v\|_{1,b} \le c\, |v|_{\mathrm{H}^1(\Omega)}$$
$$\|v - r_h v\|_{\mathrm{L}^2(K)} + h_K \, \|\boldsymbol{\nabla}_h (v - r_h v)\|_{\mathrm{L}^2(K)} \le c\, h_K^2 \, |v|_{\mathrm{H}^2(K)}$$

- *inf-sup* condition $p \in L_h$

$$\sup_{\mathbf{v} \in \mathbf{W}_h} \frac{\displaystyle\int_\Omega p \, \mathrm{div}_h \mathbf{v} \,\mathrm{d}\mathbf{x}}{\|\mathbf{v}\|_{1,b}} \ge c\, \|p - \pi\|_{\mathrm{L}^2(\Omega)}, \text{ with } \pi = \frac{1}{m(\Omega)} \int_\Omega p \,\mathrm{d}\mathbf{x}$$

- Compactness: If $\lim_{n\to\infty} h^n = 0$ and $u^n \in W_h^n$; $\|u^n\|_{1,b} \le C$ then $\exists\ \bar{u} \in \mathrm{H}^1_0(\Omega)$; $u^n \to \bar{u}$ in $\mathrm{L}^2(\Omega)$ (up to a subsequence).

The scheme

The scheme:

$$\left|\begin{array}{ll} \forall \mathbf{v} \in \mathbf{W}_h, & \displaystyle\int_\Omega \nabla_h \mathbf{u} : \nabla_h \mathbf{v} - \int_\Omega p \, \mathrm{div}_h \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \\ \forall K \in \mathcal{T}, & \displaystyle\sum_{\sigma=K|L} \rho_\sigma \, \mathrm{v}_{\sigma,K} + (T_M)_K + (T_{\mathrm{stab}})_K = 0 \\ \forall K \in \mathcal{T}, & p_K = (\rho_K)^\gamma \end{array}\right.$$

where:

- $\mathrm{v}_{\sigma,K} = |\sigma| \, \mathbf{u}_\sigma \cdot \mathbf{n}_{KL}$, upwind value for ρ: $\rho_\sigma = \begin{cases} \rho_K \text{ if } \mathrm{v}_{\sigma,K} \geq 0 \\ \rho_L \text{ otherwise} \end{cases}$
- $(T_M)_K = h^\alpha \, |K| \, (\rho_K - \rho^*)\,, \quad \rho^* = M/|\Omega|, \ \alpha > 0$
- $(T_{\mathrm{stab}})_K = \displaystyle\sum_{\sigma=K|L} (h_K + h_L)^\xi \, \frac{|\sigma|}{h_\sigma} \left(|\rho_K|^\zeta + |\rho_L|^\zeta \right) (\rho_K - \rho_L)$,

  $\zeta = \max(0, 2 - \gamma)$, $\xi \in (0, 2)$

$T_M \rightsquigarrow$ regularity of the system, $\rho_K > 0$ and $\sum_K |K| \rho_K = M$.

$T_{\mathrm{stab}} \rightsquigarrow$ control of $|\rho|^2_{\mathcal{T}} = \displaystyle\sum_{\sigma=K|L} \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2$

Existence of a solution to the scheme

Theorem

There exists a solution to the scheme.

Proof

Consider the function $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\rho}) \rightarrow (\mathbf{u}, p, \rho)$ defined for positive $\tilde{p}$ and $\tilde{\rho}$ as follows:

1− compute ρ from the mass balance with $\tilde{\mathbf{u}}$.
2− compute p from ρ by the equation of state.
3− compute $\mathbf{u}$ by the momentum balance.

Then:

1. $\rho \geq 0$, $\displaystyle\int_\Omega \rho \,\mathrm{d}\mathbf{x} = M$, so $\|\rho\|_{\mathrm{L}^1(\Omega)} \leq C_1$.
2. ... so $\|p\|_{\mathrm{L}^2(\Omega)} \leq C_2$
3. ... so $\|\mathbf{u}\|_{1,b} \leq C_3$

and, by Brouwer's fixed point theorem, this function admits a fixed point in:

$$\mathcal{C} = \{(\mathbf{u}, p, \rho) \in \mathbf{W}_h \times L_h \times L_h \text{ s.t.}$$
$$\|\mathbf{u}\|_{1,b} \leq C_3,\ \|p\|_{\mathrm{L}^2(\Omega)} \leq C_2,\ \|\rho\|_{\mathrm{L}^1(\Omega)} \leq C_1,\ \rho \geq 0, p \geq 0\}$$

Convergence result

Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of meshes with $\lim_{n\to\infty} h_n = 0$.

Let $(\mathbf{u}_n, p_n, \rho_n)_{n\in\mathbb{N}}$ be the corresponding sequence of solutions to the scheme.

Then, when $n \to \infty$, up to a subsequence:

$$\mathbf{u}_n \to \bar{\mathbf{u}} \in \mathrm{H}^1_0(\Omega)^d \text{ strongly in } \mathrm{L}^2(\Omega)^d$$

$$p_n \to \bar{p} \text{ weakly in } \mathrm{L}^2(\Omega), \text{ strongly in } \mathrm{L}^q(\Omega), q < 2,$$

$$\rho_n \to \bar{\rho} \text{ weakly in } \mathrm{L}^{2\gamma}(\Omega), \text{ strongly in } \mathrm{L}^q(\Omega), q < 2\gamma.$$

with $(\bar{\mathbf{u}}, \bar{p}, \bar{\rho})$ solution to the continuous problem

Technique of proof:

1. Estimates
2. Passing to the limit on the equation

Simpler result: "continuity" with respect to the data

$$-\Delta \mathbf{u}_n + \nabla p_n = \mathbf{f}_n \text{ in } \Omega, \ \ \mathbf{u}_n = 0 \text{ on } \partial\Omega,$$

$$\operatorname{div}(\rho_n \mathbf{u}_n) = 0 \ \text{ in } \Omega, \ \rho_n \geq 0 \ \text{ in } \Omega, \ \int_\Omega \rho_n(x)dx = M_n,$$

$$p_n = \rho_n^\gamma \text{ in } \Omega$$

$\mathbf{f}_n \to \mathbf{f}$ in $(L^2(\Omega))^d$ and $M_n \to M$. Then, up to a subsequence,

- $\mathbf{u}_n \to \overline{\mathbf{u}}$ in $L^2(\Omega)^d$ and weakly in $H_0^1(\Omega)^d$,
- $p_n \to \overline{p}$ in $L^q(\Omega)$ for any $1 \leq q < 2$ and weakly in $L^2(\Omega)$,
- $\rho_n \to \overline{\rho}$ in $L^q(\Omega)$ for any $1 \leq q < 2\gamma$ and weakly in $L^{2\gamma}(\Omega)$,

where $(\overline{\mathbf{u}}, \overline{p}, \overline{\rho})$ is a weak solution of the compressible Stokes equations (with $\mathbf{f}$ and M as data)

The case $\gamma = 1$ is also possible, but we obtain only weak convergence of p_n and ρ_n in $L^2(\Omega)$ (strong conv. are not needed).

Preliminary lemma

$\rho \in L^{2\gamma}(\Omega)$, $\rho \geq 0$ a.e. in Ω, $\mathbf{u} \in \left(H_0^1(\Omega)\right)^d$, $\mathrm{div}(\rho\mathbf{u}) = 0$, then:

$$\int_\Omega \rho \mathrm{div}(\mathbf{u}) = 0$$

$$\int_\Omega \rho^\gamma \mathrm{div}(\mathbf{u}) = 0$$

Proof of the preliminary lemma

For simplicity : $\rho \in C^1(\bar{\Omega})$, $\rho \geq \alpha$ a.e. in Ω, $\alpha > 0$,
$1 < \beta \leq \gamma$. Take $\varphi = \rho^{\beta-1}$ as test function in $\mathrm{div}(\rho\,\mathbf{u}) = 0$:

$$\int_\Omega \rho\,\mathbf{u}\cdot\boldsymbol{\nabla}\rho^{\beta-1} = (\beta-1)\int_\Omega \rho^{\beta-1}\mathbf{u}\cdot\boldsymbol{\nabla}\rho = 0.$$

Then

$$\frac{\beta-1}{\beta}\int_\Omega \mathbf{u}\cdot\boldsymbol{\nabla}\rho^\beta = 0,$$

and finally

$$\int_\Omega \rho^\beta \mathrm{div}(\mathbf{u}) = 0.$$

Two cases :
$\beta = \gamma$
$\beta = 1 + \frac{1}{k}$ and $k \to \infty$ (or $\varphi = \ln(\rho)$)

Estimate on $\mathbf{u}_n$

Taking $\mathbf{u}_n$ as test function in $-\Delta \mathbf{u}_n + \nabla p_n = \mathbf{f}_n$:

$$\int_\Omega \nabla \mathbf{u}_n : \nabla \mathbf{u}_n - \int_\Omega p_n \mathrm{div}(\mathbf{u}_n) = \int_\Omega \mathbf{f}_n \cdot \mathbf{u}_n \, dx.$$

But $p_n = \rho_n^\gamma$ a.e. and $\mathrm{div}(\rho_n \, \mathbf{u}_n) = 0$, then $\int_\Omega p_n \mathrm{div} \mathbf{u}_n = 0$. This gives an estimate on $\mathbf{u}_n$:

$$\|\mathbf{u}_n\|_{(H_0^1(\Omega))^d} \leq C_1.$$

Estimate on p_n

Let $q \in L^2(\Omega)$ s.t. $\int_\Omega q dx = 0$.
Then, there exists $\mathbf{v} \in (H_0^1(\Omega))^d$ s.t.

$$\mathrm{div}(\mathbf{v}) = q \text{ a.e. in } \Omega,$$

$$\|\mathbf{v}\|_{(H_0^1(\Omega))^d} \leq C_2 \|q\|_{L^2(\Omega)},$$

where C_2 only depends on Ω.

Estimate on p_n

$$\pi_n = \frac{1}{|\Omega|} \int_\Omega p_n dx$$

, $\mathbf{v}_n \in H_0^1(\Omega)^d$, $\mathrm{div}(\mathbf{v}_n) = p_n - \pi_n$

Taking $\mathbf{v}_n$ as test function in $-\Delta \mathbf{u}_n + \nabla p_n = \mathbf{f}_n$:

$$\int_\Omega \nabla \mathbf{u}_n : \nabla \mathbf{v}_n \, dx - \int_\Omega p_n \mathrm{div}(\mathbf{v}_n) = \int_\Omega \mathbf{f}_n \cdot \mathbf{v}_n.$$

Using $\int_\Omega \mathrm{div}(\mathbf{v}_n) dx = 0$:

$$\int_\Omega (p_n - \pi_n)^2 dx = \int_\Omega (\mathbf{f}_n \cdot \mathbf{v}_n - \nabla \mathbf{u}_n : \nabla \mathbf{v}_n) dx.$$

Since $\|\mathbf{v}_n\|_{(H_0^1(\Omega))^d} \leq C_2 \|p_n - \pi_n\|_{L^2(\Omega)}$ and $\|\mathbf{u}_n\|_{(H_0^1(\Omega))^d} \leq C_1$, the preceding inequality leads to:

$$\|p_n - \pi_n\|_{L^2(\Omega)} \leq C_3.$$

where C_3 only depends on the L^2-bound of $(f_n)_{n \in \mathbb{N}}$ and on Ω.

Estimates on p_n and ρ_n

$$\|p_n - \pi_n\|_{L^2(\Omega)} \leq C_3.$$

$$\int_\Omega p_n^{\frac{1}{\gamma}}\, dx = \int_\Omega \rho_n dx \leq \sup\{M_p, p \in \mathbb{N}\}.$$

Then:

$$\|p_n\|_{L^2(\Omega)} \leq C_4;$$

where C_4 only depends on the L^2-bound of $(f_n)_{n\in\mathbb{N}}$, the bound of $(M_n)_{n\in\mathbb{N}}$, γ and Ω.

$p_n = \rho_n^\gamma$ a.e. in Ω, then:

$$\|\rho_n\|_{L^{2\gamma}(\Omega)} \leq C_5 = C_4^{\frac{1}{\gamma}}.$$

Weak-convergence on $\mathbf{u}_n$, p_n, ρ_n

Thanks to the estimates on $\mathbf{u}_n$, p_n, ρ_n, it is possible to assume (up to a subsequence) that, as $n \to \infty$:

$$\mathbf{u}_n \to \overline{\mathbf{u}} \text{ in } L^2(\Omega)^d \text{ and weakly in } H_0^1(\Omega)^d,$$

$$p_n \to \overline{p} \text{ weakly in } L^2(\Omega),$$

$$\rho_n \to \overline{\rho} \text{ weakly in } L^{2\gamma}(\Omega).$$

Is $(\overline{\mathbf{u}}, \overline{p}, \overline{\rho})$ solution to the problem with data $\mathbf{f}$ and M?

Passing to the limit on the equations, except EOS

Linear equation :

$$-\Delta \overline{\mathbf{u}} + \nabla \overline{p} = \mathbf{f} \text{ in } \Omega, \quad \overline{u} = 0 \text{ on } \partial\Omega,$$

Strong times weak convergence

$$\mathrm{div}(\overline{\rho}\, \overline{\mathbf{u}}) = 0 \ \text{ in } \Omega,$$

L^1-weak convergence of ρ_n gives positivity of ρ and convergence of mass:

$$\overline{\rho} \geq 0 \ \text{ in } \Omega, \ \int_\Omega \overline{\rho} = M.$$

Question (if $\gamma > 1$):

$$\overline{p} = \overline{\rho}^\gamma \text{ in } \Omega \text{ ?}$$

Idea : prove $\int_\Omega p_n \rho_n dx \to \int_\Omega \overline{p}\, \overline{\rho}$ and deduce a.e. convergence (of p_n and ρ_n) and $\overline{p} = \overline{\rho}^\gamma$.

$\nabla : \nabla = \mathrm{div}\ \mathrm{div} + \mathrm{curl} \cdot \mathrm{curl}$

For all $\mathbf{u}, \mathbf{v}$ in $H_0^1(\Omega)^d$,

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \int_\Omega \mathrm{div}\mathbf{u}\ \mathrm{div}\mathbf{v} + \int_\Omega \mathrm{curl}\mathbf{u} \cdot \mathrm{curl}\mathbf{u}.$$

Then, the weak form of $-\Delta \mathbf{u}_n + \nabla p_n = \mathbf{f}_n$ gives for all $\mathbf{v}$ in $H_0^1(\Omega)^d$

$$\int_\Omega \mathrm{div}\mathbf{u}_n\ \mathrm{div}\mathbf{v} dx + \int_\Omega \mathrm{curl}\mathbf{u}_n \cdot \mathrm{curl}\mathbf{v} - \int_\Omega p_n\ \mathrm{div}\mathbf{v} = \int_\Omega \mathbf{f}_n \cdot \mathbf{v}\, dx.$$

Choice of $\mathbf{v}$? $\mathbf{v} = \mathbf{v}_n$;

- $\mathbf{v}_n \in (H_0^1(\Omega))^d$, (unfortunately, 0 is impossible).
- $\mathrm{div}\mathbf{v}_n = \rho_n$ a.e. in Ω,
- $\mathrm{curl}\mathbf{v}_n = 0$ a.e. in Ω,
- $\|\mathbf{v}_n\|_{(H^1(\Omega))^d} \leq C_6 \|\rho_n\|_{L^2(\Omega)}$, where C_6 only depends on Ω.

Then, up to a subsequence,

$\mathbf{v}_n \to \mathbf{v}$ in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$, $\mathrm{curl}\mathbf{v} = 0$, $\mathrm{div}\mathbf{v} = \rho$.

Proof using an ideal $\mathbf{v}_n$ (1)

$$\int_\Omega \mathrm{div}\mathbf{u}_n \mathrm{div}\mathbf{v}_n + \int_\Omega \mathrm{curl}\mathbf{u}_n \cdot \mathrm{curl}\mathbf{v}_n - \int_\Omega p_n \mathrm{div}\mathbf{v}_n = \int_\Omega \mathbf{f}_n \cdot \mathbf{v}_n.$$

But, $\mathrm{div}\mathbf{v}_n = \rho_n$ and $\mathrm{curl}\mathbf{v}_n = 0$. Then:

$$\int_\Omega (\mathrm{div}\mathbf{u}_n - p_n)\ \rho_n = \int_\Omega \mathbf{f}_n \cdot \mathbf{v}_n.$$

Weak convergence of $\mathbf{f}_n$ in $L^2(\Omega)^d$ to $\mathbf{f}$ and convergence of $\mathbf{v}_n$ in $L^2(\Omega)^d$ to $\mathbf{v}$:

$$\lim_{n\to\infty} \int_\Omega (\mathrm{div}\mathbf{u}_n - p_n)\ \rho_n = \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

Proof using an ideal $\mathbf{v}_n$ (2)

But, since $-\Delta\overline{\mathbf{u}} + \nabla\overline{p} = \mathbf{f}$:

$$\int_\Omega \mathrm{div}\overline{\mathbf{u}}\ \mathrm{div}\mathbf{v} + \int_\Omega \mathrm{curl}\overline{\mathbf{u}} \cdot \mathrm{curl}\mathbf{v} - \int_\Omega \overline{p}\mathrm{div}\mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

which gives (using $\mathrm{div}\mathbf{v} = \overline{\rho}$ and $\mathrm{curl}\mathbf{v} = 0$):

$$\int_\Omega (\mathrm{div}\overline{\mathbf{u}} - \overline{p})\ \rho = \int_\Omega \mathbf{f} \cdot \mathbf{v}.$$

Then:

$$\lim_{n\to\infty} \int_\Omega (p_n - \mathrm{div}\mathbf{u}_n)\ \rho_n = \int_\Omega (\overline{p} - \mathrm{div}\overline{\mathbf{u}})\ \overline{\rho}.$$

Finally, the preliminary lemma gives $\int_\Omega \rho_n \mathrm{div}\mathbf{u}_n = \int_\Omega \rho\mathrm{div}\overline{\mathbf{u}} = 0$ (since $\mathrm{div}(\rho_n\mathbf{u}_n) = \mathrm{div}(\overline{\rho}\overline{\mathbf{u}}) = 0$)

$$\lim_{n\to\infty} \int_\Omega p_n\rho_n = \int_\Omega \overline{p}\ \overline{\rho}.$$

Unfortunately, it is impossible to have $\mathbf{v}_n \in H^1_0(\Omega)$.

Curl-free test function

Let B be a ball containing Ω and $w_n \in H_0^1(B)$, $-\Delta w_n = \rho_n$,

$$\mathbf{v}_n = \nabla w_n$$

- $\mathbf{v}_n \in (H^1(\Omega))^d$,
- $\mathrm{div}\mathbf{v}_n = \rho_n$ a.e. in Ω,
- $\mathrm{curl}\mathbf{v}_n = 0$ a.e. in Ω,
- $\|\mathbf{v}_n\|_{(H^1(\Omega))^d} \leq C_6 \|\rho_n\|_{L2(\Omega)}$, where C_6 only depends on Ω.

Then, up to a subsequence,

$\mathbf{v}_n \to v$ in $L^2(\Omega)$ and weakly in $H^1(\Omega)$,

$\mathrm{curl}\mathbf{v} = 0$, $\mathrm{div}\mathbf{v} = \rho$.

(Remark : $\|\mathbf{v}_n\|_{(H^2(\Omega))^d} \leq C_6 \|\rho_n\|_{H^1(\Omega)}$)

Proving $\int_\Omega (p_n - \mathrm{div}\mathbf{u}_n)\rho_n\varphi dx \to \int_\Omega (\overline{p} - \mathrm{div}\overline{\mathbf{u}})\ \overline{\rho}\varphi dx$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $\mathbf{v}_n\varphi \in H_0^1(\Omega)^d$)). Taking $\mathbf{v} = \mathbf{v}_n\varphi$:

$$\int_\Omega \mathrm{div}\mathbf{u}_n \mathrm{div}(\mathbf{v}_n\varphi) + \int_\Omega \mathrm{curl}\mathbf{u}_n \cdot \mathrm{curl}(\mathbf{v}_n\varphi) - \int_\Omega p_n \mathrm{div}(\mathbf{v}_n\varphi) = \int_\Omega \mathbf{f}_n \cdot (\mathbf{v}_n\varphi)$$

Using a proof similar to that given if $\varphi = 1$ (with additional terms involving φ), we obtain :

$$\lim_{n\to\infty} \int_\Omega (p_n - \mathrm{div}\mathbf{u}_n)\rho_n\varphi = \int_\Omega (\overline{p} - \mathrm{div}\overline{\mathbf{u}})\ \overline{\rho}\ \varphi$$

Proving $\int_\Omega (p_n - \mathrm{div}\mathbf{u}_n)\rho_n dx \to \int_\Omega (\overline{p} - \mathrm{div}\overline{\mathbf{u}})\rho$

$(p_n - \mathrm{div}\mathbf{u}_n)\rho_n \to (\overline{p} - \mathrm{div}\overline{\mathbf{u}})\overline{\rho}$ in $D'(\Omega)$

$p_n - \mathrm{div}\mathbf{u}_n$ bounded in $L^2(\Omega)$, ρ_n bounded in $L^{2\gamma}(\Omega)$

Lemma : $F_n \to F$ in $D'(\Omega)$, $(F_n)_{n\in\mathbb{N}}$ bounded in L^q for some $q > 1$. Then $F_n \to F$ weakly in L^1.

With $F_n = (p_n - \mathrm{div}\mathbf{u}_n)\rho_n$, $F = (\overline{p} - \mathrm{div}\overline{\mathbf{u}})\overline{\rho}$ and since $\gamma > 1$, the lemma gives

$$\int_\Omega (p_n - \mathrm{div}\mathbf{u}_n)\rho_n dx \to \int_\Omega (\overline{p} - \mathrm{div}\overline{\mathbf{u}})\overline{\rho} dx.$$

Proving $\int_\Omega p_n \rho_n dx \to \int_\Omega \overline{p\rho} dx$

$$\int_\Omega (p_n - \mathrm{div}\mathbf{u}_n)\rho_n dx \to \int_\Omega (\overline{p} - \mathrm{div}\mathbf{u})\overline{\rho} dx.$$

But since $\mathrm{div}(\rho_n\ \mathbf{u}_n) = 0$, $\mathrm{div}(\rho\ \mathbf{u}) = 0$, the preliminary lemma gives:

$$\int_\Omega \mathrm{div}(\mathbf{u}_n)\rho_n dx = 0, \ \int_\Omega \mathrm{div}\overline{\mathbf{u}}\ \overline{\rho} dx = 0;$$

Then:

$$\int_\Omega p_n \rho_n \to \int_\Omega \overline{p\rho}$$

a.e. convergence of ρ_n and p_n

Let $G_n = (\rho_n^\gamma - \overline{\rho}^\gamma)(\rho_n - \overline{\rho}) \in L^1(\Omega)$ and $G_n \geq 0$ a.e. in Ω. Futhermore $G_n = (p_n - \overline{\rho}^\gamma)(\rho_n - \overline{\rho}) = p_n\rho_n - p_n\overline{\rho} - \rho^\gamma\rho_n + \overline{\rho}^\gamma\overline{\rho}$ and:

$$\int_\Omega G_n = \int_\Omega p_n\rho_n - \int_\Omega p_n\overline{\rho} - \int_\Omega \overline{\rho}^\gamma\rho_n + \int_\Omega \overline{\rho}^\gamma\overline{\rho}.$$

Using the weak convergence in $L^2(\Omega)$ of p_n and ρ_n and $\int_\Omega p_n\rho_n \to \int_\Omega \overline{p}\,\overline{\rho}$:

$$\lim_{n\to\infty} \int_\Omega G_n dx = 0,$$

Then (up to a subsequence), $G_n \to 0$ a.e. and then $\rho_n \to \overline{\rho}$ a.e. (since $y \mapsto y^\gamma$ is an increasing function on $\mathbb{R}_+$). Finally:

$\rho_n \to \overline{\rho}$ in $L^q(\Omega)$ for all $1 \leq q < 2\gamma$,

$p_n = \rho_n^\gamma \to \overline{\rho}^\gamma$ in $L^q(\Omega)$ for all $1 \leq q < 2$,

and $\overline{p} = \overline{\rho}^\gamma$.

Back to the scheme

The scheme:

$$\left|\begin{array}{ll} \forall \mathbf{v} \in \mathbf{W}_h, & \displaystyle\int_\Omega \boldsymbol{\nabla}_h \mathbf{u} : \boldsymbol{\nabla}_h \mathbf{v} - \int_\Omega p \,\mathrm{div}_h \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \\ \forall K \in \mathcal{T}, & \displaystyle\sum_{\sigma=K|L} \rho_\sigma \, \mathrm{v}_{\sigma,K} + (T_M)_K + (T_{\mathrm{stab}})_K = 0 \\ \forall K \in \mathcal{T}, & p_K = (\rho_K)^\gamma \end{array}\right.$$

where:

- $\mathrm{v}_{\sigma,K} = |\sigma| \, \mathbf{u}_\sigma \cdot \mathbf{n}_{KL}$, upwind value for ρ: $\rho_\sigma = \begin{cases} \rho_K \text{ if } \mathrm{v}_{\sigma,K} \geq 0 \\ \rho_L \text{ otherwise} \end{cases}$
- $(T_M)_K = h^\alpha \, |K| \, (\rho_K - \rho^*)\,, \quad \rho^* = M/|\Omega|,\ \alpha > 0$
- $(T_{\mathrm{stab}})_K = \displaystyle\sum_{\sigma=K|L} (h_K + h_L)^\xi \, \frac{|\sigma|}{h_\sigma} \left(|\rho_K|^\zeta + |\rho_L|^\zeta\right) (\rho_K - \rho_L),$

  $\zeta = \max(0, 2 - \gamma),\ \xi \in (0, 2)$

Estimates for the discrete solutions

Lemma ("Preliminary lemma", continuous case)
if $\rho \in L^{2\gamma}(\Omega)$, $\rho > 0$ *and* $\mathbf{u} \in (H_0^1(\Omega))^d$ *satisfy* $\mathrm{div}(\rho\, \mathbf{u}) = 0$ *then*

$$\int_\Omega \rho^\beta \mathrm{div}\mathbf{u}\, \mathrm{d}\mathbf{x} = 0,\ 1 \le \beta \le \gamma$$

Lemma ("Preliminary lemma", discrete case)
if $\rho, \mathbf{u}$ *satisfy* $\sum_{\sigma=K|L} \rho_\sigma\, \mathrm{v}_{\sigma,K} + (T_M)_K + (T_{\mathrm{stab}})_K = 0$ *then*

$$\int_\Omega \rho^\beta \mathrm{div}_h \mathbf{u}\, \mathrm{d}\mathbf{x} \le C(\beta, \Omega, M) h^\alpha,\ \forall \beta \ge 1$$

upwind choice for ρ_σ and T_{stab} $\rightsquigarrow$ "$\le$"
(T_M) $\rightsquigarrow$ "$\le$" and h^α

Estimates

Theorem

Any solution to the scheme satisfies:

$$\|\mathbf{u}\|_{1,b} + \|p\|_{\mathrm{L}^2(\Omega)} + \|\rho\|_{\mathrm{L}^{2\gamma}(\Omega)} + h^{\xi/2}\,|\rho|_{\mathcal{T}} \leq C$$

Proof:

1. $\rho > 0$
2. Take $\mathbf{v} = \mathbf{u}$ in the momentum balance:
   $\|\mathbf{u}\|_{1,b}^2 - \int_\Omega p \,\mathrm{div}_h \mathbf{u}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{u}\,\mathrm{d}\mathbf{x}$
   $\int_\Omega p \,\mathrm{div}_h \mathbf{u}\,\mathrm{d}\mathbf{x} = \int_\Omega \rho^\gamma \,\mathrm{div}_h \mathbf{u}\,\mathrm{d}\mathbf{x} \leq C(\gamma, \Omega, M) h^\alpha \rightsquigarrow \|\mathbf{u}\|_{1,b} \leq c$
3. Stability of the gradient (test function $r_h \mathbf{v}$ where $\mathrm{div}\mathbf{v} = p - \pi$ in momentum eq.) $\rightsquigarrow \|p - \pi\|_{\mathrm{L}^2(\Omega)} \leq C$,
   $\int_\Omega \rho\,\mathrm{d}\mathbf{x} = M \rightsquigarrow \|p\|_{\mathrm{L}^2(\Omega)} \leq C$.
4. T_{stab} in mass balance $\rightsquigarrow h^{\frac{\xi}{2}}\,|\rho|_{\mathcal{T}} \leq C$

Convergence: the momentum balance equation

- **Momentum balance equation**

Let $\varphi \in \mathrm{C}_c^\infty(\Omega)^d$, and φ_n its Crouzeix-Raviart interpolate. We have:

$$\underbrace{\int_\Omega \boldsymbol{\nabla}_h \mathbf{u}_n : \boldsymbol{\nabla}_h \varphi_n \,\mathrm{d}\mathbf{x}}_{T_1} - \underbrace{\int_\Omega p_n \,\mathrm{div}_h \varphi_n \,\mathrm{d}\mathbf{x}}_{T_2} = \underbrace{\int_\Omega \mathbf{f} \cdot \varphi_n \,\mathrm{d}\mathbf{x}}_{T_3}$$

And:

$$\begin{aligned}
T_3 &= \int_\Omega \mathbf{f} \cdot \varphi \,\mathrm{d}\mathbf{x} + \underbrace{\int_\Omega \mathbf{f} \cdot (\varphi_n - \varphi) \,\mathrm{d}\mathbf{x}}_{\leq c\,(h_n)^2} \\
T_2 &= \int_\Omega p_n \,\mathrm{div}\varphi \,\mathrm{d}\mathbf{x} \\
T_1 &= \int_\Omega \boldsymbol{\nabla}_h \mathbf{u}_n : \boldsymbol{\nabla} \varphi \,\mathrm{d}\mathbf{x} + \underbrace{\int_\Omega \boldsymbol{\nabla}_h \mathbf{u}_n : (\boldsymbol{\nabla}_h \varphi_n - \boldsymbol{\nabla} \varphi) \,\mathrm{d}\mathbf{x}}_{\leq c\,h_n} \\
&= -\int_\Omega \mathbf{u}_n \cdot \Delta \varphi \,\mathrm{d}\mathbf{x} + \underbrace{\textit{jump terms}}_{\leq c\,h_n} + \underbrace{\int_\Omega \boldsymbol{\nabla}_h \mathbf{u}_n : (\boldsymbol{\nabla}_h \varphi_n - \boldsymbol{\nabla} \varphi) \,\mathrm{d}\mathbf{x}}_{\leq c\,h_n}
\end{aligned}$$

So, passing to the limit:

$$\int_\Omega \boldsymbol{\nabla} \bar{\mathbf{u}} : \boldsymbol{\nabla} \varphi \,\mathrm{d}\mathbf{x} - \int_\Omega \bar{p} \,\mathrm{div}\varphi \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \varphi \,\mathrm{d}\mathbf{x}$$

Convergence: the mass balance equation

- **Mass balance equation**
  Let $\varphi \in \mathrm{C}^\infty(\Omega)$. We have:

$$\sum_{K\in\mathcal{T}} \Big[\sum_{\sigma=K|L} (\rho_\sigma)_n \, (\mathrm{v}_{\sigma,K})_n + (T_M)_K^n + (T_{\mathrm{stab}})_K^n \Big] \frac{1}{|K|} \int_K \varphi \, \mathrm{d}\mathbf{x} = 0$$

  When $n \to \infty$:

  - $$\sum_{K\in\mathcal{T}} \Big[\sum_{\sigma=K|L} (\rho_\sigma)_n \, (\mathrm{v}_{\sigma,K})_n \Big] \frac{1}{|K|} \int_K \varphi \, \mathrm{d}\mathbf{x} \to - \int_\Omega \bar\rho \, \bar{\mathbf{u}} \cdot \boldsymbol{\nabla}\varphi \, \mathrm{d}\mathbf{x}$$
    thanks to $|\rho_n|_\mathcal{T} \leq c \; h_n^{-\frac{\xi}{2}}$ with $\xi < 2$.
  - $$\sum_{K\in\mathcal{T}} (T_M)_K^n \frac{1}{|K|} \int_K \varphi \, \mathrm{d}\mathbf{x} = h^\alpha \int_\Omega (\rho_n - \rho^*) \, \varphi \to 0$$ thanks to $\alpha > 0$.
  - $$\sum_{K\in\mathcal{T}} (T_{\mathrm{stab}})_K^n \frac{1}{|K|} \int_K \varphi \, \mathrm{d}\mathbf{x} \simeq h^\xi \int_\Omega (\rho_n)^\zeta \, \boldsymbol{\nabla}\rho_n \cdot \boldsymbol{\nabla}\varphi \to 0$$ thanks to $\xi > 0$.

$$(0 \leq \zeta = \max(2-\gamma, 0) \leq 1)$$

Therefore: $\mathrm{div}\,\overline{\rho\mathbf{u}} = 0$

Convergence: the equation of state in the case $\gamma > 1$

Lemma

$$\forall \varphi \in \mathrm{C}_c^\infty(\Omega), \quad \lim_{n\to\infty} \int_\Omega (\mathrm{div}_h \mathbf{u}_n - p_n)\, \rho_n \varphi = \int_\Omega (\mathrm{div} \bar{\mathbf{u}} - \bar{p})\, \bar{\rho} \varphi$$

Idea of the proof

1. Regularization of the sequence ρ_n: let $\tilde{\rho}_n$ be defined as the P_1-interpolate of ρ_n. Then:

$$\|\tilde{\rho}_n\|_{\mathrm{H}^1(\Omega)} \leq c\, |\rho_n|_{\mathcal{T}}\ , \quad \|\tilde{\rho}_n - \rho_n\|_{\mathrm{L}^2(\Omega)} \leq c\ h_n\, |\rho_n|_{\mathcal{T}}\ \leq c\ h_n^{1-\frac{\xi}{2}}$$

2. Let $\mathbf{v}_n$ be such that:

$$\mathrm{div} \mathbf{v}_n = \tilde{\rho}_n, \quad \mathrm{rot} \mathbf{v}_n = 0, \quad \|\mathbf{v}_n\|_{\mathrm{H}^2(\Omega)^d} \leq c \|\tilde{\rho}_n\|_{\mathrm{H}^1(\Omega)} \leq c\ h_n^{-\frac{\xi}{2}}$$

3. Take $r_h \varphi \mathbf{v}_n$ (the Crouzeix-Raviart interpolate of $\varphi \mathbf{v}_n$) as test function in the momentum balance equation, proceed as in the continuous case and use the "regularity" of the sequences ($\mathbf{v}_n \in (H^2)^d$) to control the error terms...

   Example: $\|r_h \varphi \mathbf{v}_n - \varphi \mathbf{v}_n\|_{1,b} \leq c\ h_n \|\varphi \mathbf{v}_n\|_{\mathrm{H}^2(\Omega)^d} \leq c\ h_n^{1-\frac{\xi}{2}}$

Convergence: the equation of state in the case $\gamma > 1$

Lemma (a.e. convergence)

Up to a subsequence, $\rho_n \to \bar\rho$, $p_n \to \bar p$ *a.e..*

Idea of the proof

As in the continuous case,

$$\lim_{n\to\infty} \int_\Omega (\mathrm{div}_h \mathbf{u}_n - p_n)\, \rho_n = \int_\Omega (\mathrm{div}\bar{\mathbf{u}} - \bar p)\, \bar\rho$$

Continuous preliminary lemma $\rightsquigarrow \displaystyle\int_\Omega \bar\rho \mathrm{div}\bar{\mathbf{u}} = 0.$

Discrete preliminary lemma $\rightsquigarrow \displaystyle\int_\Omega \rho_n \mathrm{div}_h \mathbf{u}_n \le c\ h_n^\alpha$

$$\int_\Omega p_n \rho_n \le \int_\Omega (p_n - \mathrm{div}_h \mathbf{u}_n)\rho_n + c\ h_n^\alpha$$

Therefore:

$$\limsup \int_\Omega p_n \rho_n \le \int_\Omega \bar p\, \bar\rho$$

As in the continuous case, up to a subsequence, $\rho_n \to \bar\rho$, $p_n \to \bar p$ a.e..

Convergence of the scheme

Theorem

If $0 < \alpha$ *and* $0 < \xi < 2$,

1. *the sequence* $(\mathbf{u}_n)_{n\in\mathbb{N}}$ *converges in* $\mathrm{L}^2(\Omega)^d$ *to a limit* $\bar{\mathbf{u}} \in \mathrm{H}^1_0(\Omega)^d$,
2. *the sequence* $(p_n)_{n\in\mathbb{N}}$ *converges weakly in* $\mathrm{L}^2(\Omega)$ *and strongly in* $\mathrm{L}^p(\Omega),\ 1 \leq p < 2$ *to* $\bar{p} \in \mathrm{L}^2(\Omega)$,
3. *the sequence* $(\rho_n)_{n\in\mathbb{N}}$ *converges weakly in* $\mathrm{L}^{2\gamma}(\Omega)$ *and strongly in* $\mathrm{L}^p(\Omega),\ 1 \leq p < 2\gamma$ *to* $\bar{\rho} \in \mathrm{L}^{2\gamma}(\Omega)$,
4. $(\bar{\mathbf{u}}, \bar{p}, \bar{\rho})$ *are solution to the continuous problem.*

Conclusion

- Replacing $-\Delta\mathbf{u}$ by $-\mu\Delta\mathbf{u} - \mu/3\nabla\operatorname{div}\mathbf{u}$ (with a constant viscosity μ) brings no additional difficulty.
- The term T_{stab} never appears in practice. Probably only a technical tool for the proof of convergence. Seems useless in the case of the MAC scheme (ongoing work)
- The convergence (but not the stability, if one restricts to the L^1 norm of p) relies on the stability of the gradient. Should *inf-sup stable* discretizations be used for the compressible Navier-Stokes equations?
- Higher order in pressure does not seem easy to achieve.
- Stability has been proven for coupled or pressure correction schemes for the barotropic transient Navier-Stokes equations (GGHL, M2AN 08) and for a drift-flux model (GHL, submitted).
- Proof is generalized to steady state compressible Navier-Stokes ($\gamma \geq 3/2$ in 3d for Crouzeix Raviart)
- On going work : MAC scheme, time dependent NS...

Additional difficulty for stat. comp. NS equations

Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma > 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \mathrm{div}(\rho u \otimes u) + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega, \; \rho \geq 0 \text{ in } \Omega, \; \int_\Omega \rho(x) dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

$d = 2$: no aditional difficulty

$d = 3$: no additional difficulty if $\gamma \geq 3$. But for $\gamma < 3$, no estimate on p in $L^2(\Omega)$.

Estimates in the case of NS equations, $\frac{3}{2} < \gamma < 3$

Estimate on u : Taking u as test function in the momentum leads to an estimate on u in $(H_0^1(\Omega)^d$ since

$$\int_\Omega \rho u \otimes u : \nabla u dx = 0.$$

Then, we have also an estimate on u in $L^6(\Omega)^d$ (using Sobolev embedding).

Estimate on p in $L^q(\Omega)$, with some $1 < q < 2$ and $q = 1$ when $\gamma = \frac{3}{2}$ (using Nečas Lemma in some L^r instead of L^2).

Estimate on ρ in $L^q(\Omega)$, with some $\frac{3}{2} < q < 6$ and $q = \frac{3}{2}$ when $\gamma = \frac{3}{2}$ (since $p = \rho^\gamma$).

Remark : $\rho u \otimes u \in L^1(\Omega)$, since $u \in L^6(\Omega)^d$ and $\rho \in L^{\frac{3}{2}}(\Omega)$ (and $\frac{1}{6} + \frac{1}{6} + \frac{2}{3} = 1$).

NS equations, $\gamma < 3$, how to pass to the limit in the EOS

We prove

$$\lim_{n\to\infty} \int_\Omega p_n \rho_n^\theta dx = \int_\Omega p \rho^\theta dx,$$

with some convenient choice of $\theta > 0$ instead of $\theta = 1$.

This gives, as for $\theta = 1$, the a.e. convergence (up to a subsequence) of p_n and ρ_n.