The gradient discretization method

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Framework and purpose

▶ **Framework:** approximation of elliptic and parabolic problems, possibly for anistropic, heterogeneous and on general polytopal meshes.

▶ **Purpose:** identification of the key properties satisfied by all methods and sufficient for convergence analysis.

▶ **Model problem:**

$$\begin{cases} -\mathrm{div}(\Lambda\nabla\overline{u}) + \overline{u} = f & \text{in } \Omega \\ \overline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

- Ω open bounded subset of $\mathbb{R}^d$,
- $f \in L^2(\Omega)$,
- $\Lambda \in \mathbb{R}^{d\times d}$, symmetric, s.t. $0 < \underline{\lambda}\xi\cdot\xi \leq \Lambda\xi\cdot\xi \leq \overline{\lambda}\xi\cdot\xi$.

Weak formulation and Galerkin approximations

Weak formulation:
Find $\overline{u} \in H_0^1(\Omega) : \forall \overline{v} \in H_0^1(\Omega), \int_\Omega \Lambda \nabla \overline{u} \cdot \nabla \overline{v} + \int_\Omega \overline{u}\ \overline{v} = \int_\Omega f \overline{v}.$

Conforming Galerkin approximation:
Find $u_h \in V_h \subset H_0^1(\Omega) : \forall v \in V_h, \int_\Omega \Lambda \nabla u_h \cdot \nabla v + \int_\Omega u_h\ v = \int_\Omega f v.$

Non-conforming Galerkin approximation:
Find
$u_h \in V_h \not\subset H_0^1(\Omega)\ :\ \forall v \in V_h,\ \int_\Omega \Lambda \nabla_h u_h \cdot \nabla_h v + \int_\Omega u_h\ v = \int_\Omega f v.$

From Galerkin approximations to the GDM

Galerkin approximation:

Find $u_h \in V_h \;:\; \forall v \in V_h,\ \int_\Omega \Lambda \nabla_h u_h \cdot \nabla_h v + \int u_h v = \int_\Omega f v.$

Gradient discretization method:

Find $u \in X_{\mathcal{D}} \subset \mathbb{R}^{N_{\mathcal{D}}} \;:\; \forall v \in X_{\mathcal{D}},$

$$\int_\Omega \Lambda \nabla_{\mathcal{D}} u \cdot \nabla_{\mathcal{D}} v + \int_\Omega \Pi_{\mathcal{D}} u \; \Pi_{\mathcal{D}} v = \int_\Omega f \Pi_{\mathcal{D}} v.$$

- u: the family of discrete unknowns
- $\nabla_{\mathcal{D}} u$: reconstructed gradient
- $\Pi_{\mathcal{D}} u$: reconstructed function

GDM defined by the triplet $(X_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$
GDM includes :

- (non) conforming Galerkin methods $\Pi_{\mathcal{D}} u = u_h \in V_h$
- mass lumping (with a suitable operator $\Pi_{\mathcal{D}}$)
- "finite volume style" methods (MPFA, SUSHI)

Conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh, $\mathcal{N}$ = set of nodes of the mesh.

Gradient discretisation:

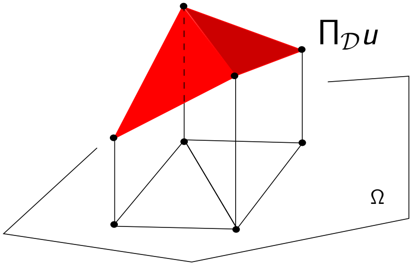
- $X_{\mathcal{D},0} = \{u = (u_N)_{N\in\mathcal{N}} \,:\, u_N = 0 \text{ if } N \in \partial\Omega\}$,
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to C(\Omega)$

  $$u \mapsto u_h = \sum_{N\in\mathcal{N}} u_N \phi_N,$$

  with ϕ_N P1 f.e. shape function associated to node N.
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)$

  $u \mapsto \nabla_{\mathcal{D}} u = \nabla u_h$ (piecewise constant function)

  - $(\nabla_{\mathcal{D}} u)_{|K} = \nabla(\Pi_{\mathcal{D}} u)_{|K}$.

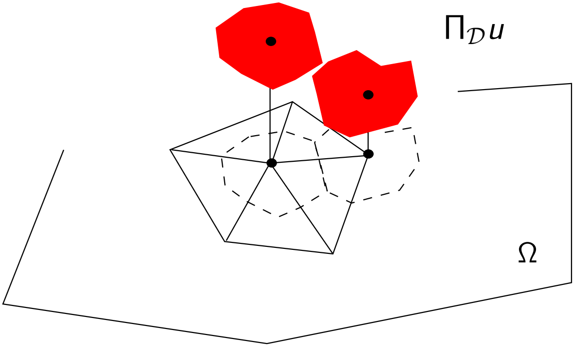


Mass-lumped conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh.

Gradient discretisation:

- $X_{\mathcal{D},0}$,
- $\Pi_{\mathcal{D}} u = u_N$ on the Donald cell associated to N.
- $\nabla_{\mathcal{D}} u = \nabla u_h$
  - $(\nabla_{\mathcal{D}} u)_{|K} \neq \nabla(\Pi_{\mathcal{D}} u)_{|K}$.

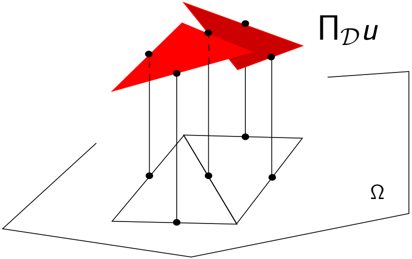


Non-conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh; $\mathcal{F}$ is the set of faces (or edges)

Gradient discretisation:

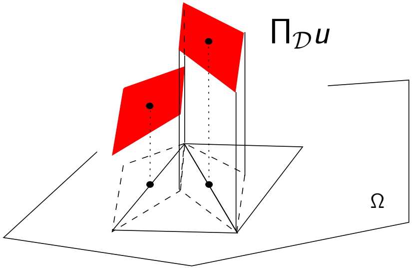
- $X_{\mathcal{D},0} = \{u = (u_\sigma)_{\sigma\in\mathcal{F}} \,:\, u_\sigma = 0 \text{ if } \sigma \in \mathcal{F}_{\rm ext}\}$,
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^p(\Omega)$
  $$u \mapsto u_h = \sum_{\sigma\in\mathcal{F}} u_\sigma \phi_\sigma,$$
  with ϕ_σ non conforming P1 f.e. shape function associated to face (or edge) σ (piecewise continuous and affine function).
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to W_0^{1,p}(\Omega)$ such that $(\nabla_{\mathcal{D}} u)_{|K} = \nabla(\Pi_{\mathcal{D}} u)_{|K}$ (broken gradient).



Mass-lumped non-conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh.
Gradient discretisation:

- $X_{\mathcal{D},0}$ and $\nabla_{\mathcal{D}}$ as for non-conforming finite elements,
- $\Pi_{\mathcal{D}} u = u_\sigma$ on a "dual" cell around σ.



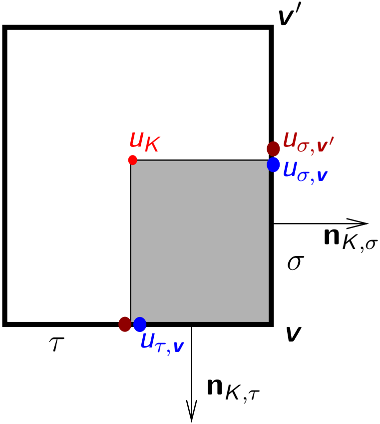
- $(\nabla_{\mathcal{D}} u)_{|K} \neq \nabla(\Pi_{\mathcal{D}} u)_{|K}$.

MPFA-O finite volume scheme

$\mathcal{M}$=Cartesian mesh (also possible with triangular/tetrahedral).

Gradient discretisation:

- $u \in X_{\mathcal{D},0}$ if $u = ((u_K)_K, (u_{\sigma,\boldsymbol{v}})_{\sigma,\boldsymbol{v}})$ with K cells and $(\sigma, \boldsymbol{v})$ pairs edge-vertex s.t. $\boldsymbol{v} \in \sigma$, and $u_{\sigma,\boldsymbol{s}} = 0$ if $\sigma \in \mathcal{F}_{\rm ext}$.
- $\Pi_{\mathcal{D}} u = u_K$ in K,
- $\nabla_{\mathcal{D}} u = \dfrac{u_{\sigma,\boldsymbol{v}} - u_K}{\mathrm{d}(x_K,\sigma)}\mathbf{n}_{K,\sigma} + \dfrac{u_{\tau,\boldsymbol{v}} - u_K}{\mathrm{d}(x_K,\tau)}\mathbf{n}_{K,\tau}$ in the cube defined by K and $\boldsymbol{v}$.



The fully hybrid SUSHI scheme

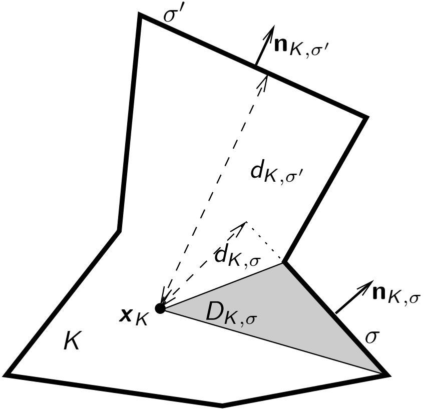
$\mathcal{M}$=polytopal mesh.

Gradient discretisation:

- $u \in X_{\mathcal{D},0}$ if
  $u = ((u_K)_K, (u_\sigma))_{K\in\mathcal{M},\sigma\in\mathcal{F}}$
- $\Pi_{\mathcal{D}} u = u_K$ in K,
- $\nabla_{\mathcal{D}} u = \overline{\nabla}_{\mathcal{D}} u + S_{\mathcal{D}} u$

with:

$$\overline{\nabla}_{\mathcal{D}} u|_K = \frac{1}{|K|} \sum_{\sigma\in\mathcal{F}_K} |\sigma| v_\sigma \mathbf{n}_{K,\sigma}$$



$$S_{\mathcal{D}} u|_{K,\sigma} = \frac{\sqrt{d}}{d_{K,\sigma}} \sum_{\sigma\in\mathcal{F}_K} \left[(v_\sigma - v_K - \overline{\nabla}_K v \cdot (\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K))\mathbf{n}_{K,\sigma}\right]$$

Other examples of GDMs

- Galerkin methods: (non) conforming $\mathbb{P}_k$ FE method, mixed finite elements *($H_{\rm div}$-conforming gradient discretisations: $\nabla_{\mathcal{D}} u \in H_{\rm div}$)*, SIPG.
- Hybrid Mimetic Mixed methods: SUSHI, mixed finite volumes, mixed-hybrid mimetic finite Differences.
- CeVeFE-DDFV, Nodal mimetic finite differences.
- Hybrid high-order methods, non-conforming Virtual Element Methods, non-conforming Mimetic Finite Difference.

Coercivity, GD-consistency, limit-conformity

- $\mathcal{D} = (X_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ GD, $(\mathcal{D}_m)_{m\in\mathbb{N}}$ sequence of GDs

(P1) Coercivity:

$$C_{\mathcal{D}} = \max_{v_{\mathcal{D}}\in X_{\mathcal{D},0}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} v_{\mathcal{D}}\|_{L^p}}{\|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^p}}.$$

$(C_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded (discrete Poincaré inequality).

(P2) GD-consistency: ("interpolation error" in FE)

$$S_{\mathcal{D}}(\varphi) = \min_{v_{\mathcal{D}}\in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v_{\mathcal{D}} - \varphi\|_{L^p} + \|\nabla_{\mathcal{D}} v_{\mathcal{D}} - \nabla\varphi\|_{L^p}\right).$$

For all $\varphi \in W_0^{1,p}(\Omega)$, $S_{\mathcal{D}_m}(\varphi) \to 0$ as $m \to \infty$.

(P3) Limit-conformity:

$$W_{\mathcal{D}}(\boldsymbol{\psi}) = \max_{v_{\mathcal{D}}\in X_{\mathcal{D},0}\setminus\{0\}} \frac{1}{\|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^p}} \left|\int_\Omega \nabla_{\mathcal{D}} v_{\mathcal{D}} \cdot \boldsymbol{\psi} + \Pi_{\mathcal{D}} v_{\mathcal{D}} \mathrm{div}\boldsymbol{\psi}\right|.$$

$\forall \boldsymbol{\psi} \in W_{\mathrm{div}}^{p'}(\Omega)$, $W_{\mathcal{D}_m}(\boldsymbol{\psi}) \to 0$ as $m \to \infty$.

- *Actually, (P3)$\Rightarrow$(P1).*

Error estimate

- Weak formulation:
  Find $\overline{u} \in H_0^1(\Omega)$:
  $\int_\Omega \Lambda \nabla \overline{u} \cdot \nabla \overline{v} = \int_\Omega f \overline{v}$,
  $\forall \overline{v} \in H_0^1(\Omega)$

- Gradient scheme:
  $u \in X_{\mathcal{D},0}$:
  $\int_\Omega \Lambda \nabla_{\mathcal{D}} u \cdot \nabla_{\mathcal{D}} v = \int_\Omega f \Pi_{\mathcal{D}} v$,
  $\forall v \in X_{\mathcal{D},0}$.

- **Error estimate**:

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|_{L^2} + \|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla \overline{u}\|_{L^2} \le C(1 + C_{\mathcal{D}}) \left[S_{\mathcal{D}}(\overline{u}) + W_{\mathcal{D}}(\nabla \overline{u}) \right].$$

$C_{\mathcal{D}}$ Coercivity, $S_{\mathcal{D}}(\overline{u})$ Consistency, $W_{\mathcal{D}}$ Limit conformity

- Error estimate also obtained for the p-Laplace equation.

Additional properties for non-linear problems

(P4) Compactness: $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is compact if for all $u_m \in X_{\mathcal{D}_m,0}$ such that $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^p})_{m\in\mathbb{N}}$ is bounded, $(\Pi_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ is relatively compact in L^p.
(Discrete Rellich theorem).

- Useful for $-\mathrm{div}(a(\overline{u})\nabla\overline{u}) = f$ for example.

(P5) Piecewise constant reconstruction:
$\exists$ $(e_i)_{i\in I}$ basis of $X_{\mathcal{D},0}$ $\exists(\Omega_i)_{i\in I}$ partition of Ω (some of them can be empty) such that, for all $u = \sum_i u_i e_i \in X_{\mathcal{D},0}$,

$$\Pi_{\mathcal{D}} u = \sum_i u_i \mathbf{1}_{\Omega_i}.$$

- Mass lumping is a way to obtain (P5)
- Essential for degenerate evolution problems : permutation of nonlinearity and discrete unknown.

Polytopal toolbox to prove (P1), (P3), (P4)

Polytopal tools:

polytopal mesh
+ discrete unknowns on cells and faces : $X_{\mathcal{M}}$
+ reconstruction operator
+ "natural" non-stabilized discrete gradient
+ norm

Polytopal toolbox:

set of tools adapted to the considered BCs

Control by a polytopal toolbox:

- mapping the discrete unknowns of a GD onto cell- and face-unknowns on a polytopal mesh;
- three estimates on this mapping $\rightsquigarrow$ *coercivity*, *compactness* and *limit-conformity* of GDs, thanks to Discrete Funtional Analysis.

Local linearly exact (LLE) gradients to prove (P2)

- Rigorous writing of "$\nabla_{\mathcal{D}}$ exactly reconstructs linear functions"

nearly all numerical methods try to satisfy this property.

- Provides easy proof of (P2) consistency for all methods.

Main results with GDM

- Error estimates and convergence for linear elliptic problems and the p-Laplacian
- Error estimate for linear parabolic problems
- Convergence analysis for non linear (degenerate) parabolic problems (Richards, Stefan)

Conclusion, perspectives and ongoing work

- Framework for the convergence analysis
  - of a number of methods: FE, MFE, MPFA, MFV, CeVeDDFV, MFD, dG
  - for a number of problems: heat equation, miscible flow, multi-phase flow, stefan problem, image processing, Richards equation, Navier-Stokes...
- Abstract setting for the analysis of all boundary conditions at once.
- Take into account all the referees remarks, finish the revised version and publish the book ! revised version online soon!