On a class of numerical schemes for compressible flows

R. Herbin[⋆],

with

T. Gallouët[⋆], J.-C. Latché[†]
L. Gastaldo[†], D. Grapsas[⋆], W. Kheriji[⋆†],
T.T. N'Guyen[⋆†], N. Therme [⋆†], C. Zaza[⋆♯].

⋆ Aix-Marseille Université
† I.R.S.N. Cadarache
♯ C.E.A Cadarache

Some key words:

- Compressible Euler and Navier Stokes
- Implicit or semi-implicit (or explicit) schemes
- Internal energy balance formulation (rather than total energy formulation) even in the presence of shocks
- “All Mach” scheme
- Finite volume / non conforming finite element
- Upwinding with respect to the material velocity (no Riemann solver)
- Colocated or staggered grids
- Stability via a kinetic energy inequality
- Consistency in the Lax-Wendroff sense “*if a conservative numerical scheme for a hyperbolic system of conservation laws converges, then it converges towards a weak solution.*”

The Euler (NS) equations: total energy vs. internal energy

- Euler (Navier-Stokes) equations:

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \qquad \text{(mass)}$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{\tau} + \boldsymbol{\nabla} p = 0, \qquad \text{(mom)}$$

$$\partial_t(\varrho E) + \operatorname{div}\big[(\varrho E + p)\boldsymbol{u}\big] = \operatorname{div}(\boldsymbol{\tau} \boldsymbol{u}), \qquad \text{(tot.en)}$$

$$p = (\gamma - 1)\, \varrho e, \quad E = \frac{1}{2}\,|\boldsymbol{u}|^2 + e.$$

- For regular functions, (mom) $\cdot \boldsymbol{u}$ & (mass) $\rightsquigarrow$ (kin.en):

$$\frac{1}{2}\partial_t(\varrho|\boldsymbol{u}|^2) + \frac{1}{2}\operatorname{div}(\varrho|\boldsymbol{u}|^2\boldsymbol{u}) + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = \operatorname{div}(\boldsymbol{\tau}) \cdot \boldsymbol{u}. \qquad \text{(kin.en)}$$

Subtracting from (tot.en) yields the internal energy balance:

$$\partial_t(\varrho e) + \operatorname{div}(\varrho e \boldsymbol{u}) + p \operatorname{div} \boldsymbol{u} = \boldsymbol{\tau} : \boldsymbol{\nabla} \boldsymbol{u}, \qquad \text{(int.en)}$$

which implies $e \geq 0$.

“Incompressible” schemes use the internal energy (or temperature) equation.

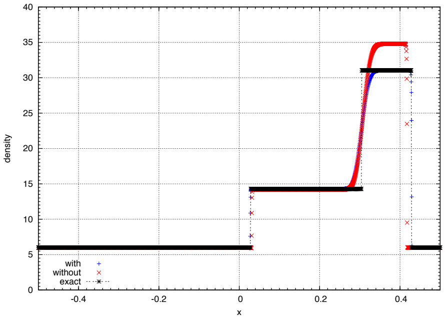
Internal energy

Dealing with the internal energy:

- ♯ positive internal energy
- ♯ convenient for incompressible problems

- ♭ ρe is not a conserved variable – conserved variables : ρ, ρu, ρE

"Formulations based on variables other than the conserved variables (non-conservative variables) fail at shock waves. They give the wrong jump conditions; consequently they give the wrong shock strength, the wrong shock speed and thus the wrong shock position." (Toro, 1999)



Test 5 of [Toro chapter 4] - Density at $t = 0.035$, $n = 2000$ cells, with and without corrective source terms, and analytical solution.

Right and wrong shock speed for Bürgers

Bürgers equation: for regular positive solutions

$$(\mathrm{B}) : \partial_t u + \partial_x (u^2) = 0 \Longleftrightarrow (\mathrm{BS}) : \partial_t u^2 + \frac{4}{3}\, \partial_x u^3 = 0.$$

No longer true with irregular solutions:

Rankine Hugoniot gives

$$\sigma = u_\ell + u_r \text{ and } \sigma = \tfrac{4}{3}(u_\ell + u_r).$$

Weak solutions of (B) $\neq$ weak solutions of (BS).

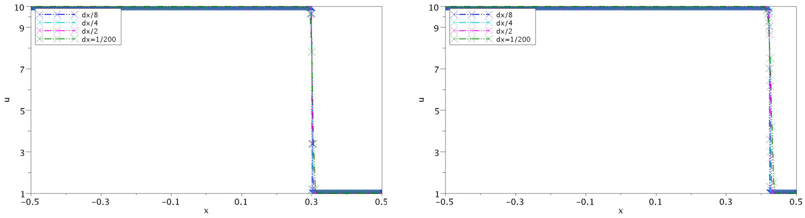


Figure: Explicit upwind Scheme for (B) (left) and (BS) (right) with different mesh sizes, $CFL = 1$.

Bürgers, numerical diffusion

Scalar conservation law: $\partial_t u + \partial_x(f(u)) = 0, f' \geq 0$

Upwind scheme: $\frac{h}{\delta t}(u_i^{n+1} - u_i^n) + f(u_i^n) - f(u_{i-1}^n) = 0.$

Upwinding is " formarly similar" to add a numerical diffusion.

$$\partial_t u + \partial_x(f(u)) - \partial_x\left(\frac{hf'(u) - \delta t\, f'^2(u)}{2}\partial_x u\right) = 0$$

The CFL condition states that $hf'(u) - \delta t\, f'^2(u) \geq 0$ (i.e. $\delta t\, f'(u) \leq h$)

In the case of the Bürgers equation it gives

$$\partial_t u + \partial_x(u^2) - \partial_x((hu - 2\delta t\, u^2)\partial_x u) = 0,\ x \in \mathbb{R},\ t \in \mathbb{R}_+$$

Bürgers, non conservative numerical diffusion

In the case of the "equivalent" equation

$$\partial_t(u^2) + (4/3)\partial_x(u^3) = 0,$$

$u > 0 \Rightarrow$ upwinding is formally consistent with

$$\partial_t(u^2) + \frac{4}{3}\partial_x(u^3) - \partial_x((2hu^2 - 4\delta t\ u^3)\partial_x u) = 0,$$

Dividing by $2u$, this leads to

$$\partial_t u + \partial_x(u^2) - \frac{1}{u}\partial_x((hu^2 - 2\delta t\ u^3)\partial_x u) = 0,\ x \in \mathbb{R},\ t \in \mathbb{R}_+$$

This is a numerical diffusion (thanks to the CFL condition) but not in conservative form.

A non conservative diffusion may lead to wrong discontinuities

Benefit from a non conservative numerical diffusion ?

$$\partial_t u + \partial_x(u^2) = 0 + \text{initial condition} \quad (B) \qquad \partial_t(u^2) + \frac{4}{3}\partial_x(u^3) = 0 + \text{initial condition} \quad (BS)$$

Explicit upwind scheme on (BS) formally equivalent to:

$$\partial_t u + \partial_x(u^2) \underbrace{- \frac{1}{u}\partial_x((hu^2 - 2\delta t \ u^3)\partial_x u)}_{\text{non conservative numerical diffusion.}} = 0.$$

- Negative result for a non conservative diffusion

  :-(Non conservative numerical diffusion on (B) yields
  - wrong shock velocity for (B)
  - correct shock velocity for (BS)

- Positive result for a non conservative diffusion ?

  :-) Non conservative numerical diffusion on (BS) yields
  - wrong shock velocity for (BS)
  - correct shock velocity for (B)

  ?

  How do we choose the non conservative numerical diffusion ?

Non conservative numerical diffusion on (BS)

Viscous Bürgers:

$$\partial_t u + \partial_x(u^2) - \varepsilon\partial_{xx} u = 0. \qquad (\mathrm{B})_\varepsilon$$

Multiplying by $2u$:

$$\partial_t(u^2) + \frac{4}{3}\partial_x(u^3) - 2u\varepsilon\partial_{xx} u = 0. \qquad (\mathrm{BS})_\varepsilon$$

Discretize $(\mathrm{BS})_\varepsilon$ instead of (BS):

$$\partial_t(u^2) + \frac{4}{3}\partial_x(u^3) - u\varepsilon_0 h\partial_{xx} u = 0, \quad (\mathrm{BS})_\varepsilon \text{ with } 2\varepsilon = \varepsilon_0 h.$$

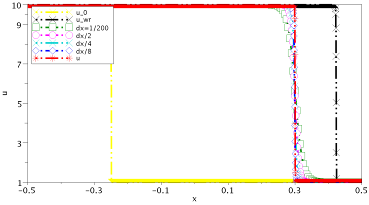
Non conservative numerical diffusion on (BS)

$$\partial_t(u^2) + \frac{4}{3}\partial_x(u^3) - u\varepsilon_0 h\partial_{xx}u = 0, \quad (\mathrm{BS})_\varepsilon \text{ with } 2\varepsilon = \varepsilon_0 h.$$

Centered finite volume with non conservative diffusion

$$(u_i^{(n)})^2 = (u_i^{(n-1)})^2 + \frac{4\delta t}{3h}\left[\left(\frac{u_{i-1}^{(n-1)} + u_i^{(n-1)}}{2}\right)^3 - \left(\frac{u_i^{(n-1)} + u_{i+1}^{(n-1)}}{2}\right)^3\right]$$

$$+ -\frac{\delta t}{h^2}\,\varepsilon_0 h\, u_i^{(n-1)}\left[u_{i-1}^{(n-1)} - 2u_i^{(n-1)} + u_{i+1}^{(n-1)}\right].$$



Centered Scheme for $(\mathrm{BS})_{\varepsilon_0 h}$,

$$u\varepsilon_0 h\partial_{xx}u = \varepsilon_0 h\left[\partial_x(u\partial_x u) + \partial_x(\frac{u^2}{2})\right]$$

From Bürgers to Euler

For regular solutions,

Bürgers:

$$\partial_t u + \partial_x (u^2) = 0 \iff \partial_t (u^2) + \frac{4}{3}\,\partial_x (u^3) = 0.$$

Euler:

$$\left\{\begin{array}{l} \partial_t \varrho + \mathrm{div}(\varrho \boldsymbol{u}) = 0, \\ \partial_t (\varrho \boldsymbol{u}) + \mathrm{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0, \\ \partial_t (\varrho E) + \mathrm{div}\big[(\varrho E + p)\boldsymbol{u}\big] = 0. \end{array}\right. \iff \left\{\begin{array}{l} \partial_t \varrho + \mathrm{div}(\varrho \boldsymbol{u}) = 0, \\ \partial_t (\varrho \boldsymbol{u}) + \mathrm{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0, \\ \partial_t (\varrho e) + \mathrm{div}(\varrho e \boldsymbol{u}) + p\,\mathrm{div}\boldsymbol{u} = 0. \end{array}\right.$$

- we had an equation, we now have a system...
- Idea: add a non conservative corrective term to the internal energy equation.
- Which term ? Inspiration comes from copying the formal derivation of the internal energy equation at the discrete level.

Total energy= kinetic energy + internal energy

Total energy (Euler):

$$\partial_t(\varrho E) + \operatorname{div}\big[(\varrho E + p)\boldsymbol{u}\big] = 0, \quad E = e + \frac{1}{2}|u|^2$$

$$\Longrightarrow \partial_t(\rho e) + \operatorname{div}(\rho e \boldsymbol{u}) + p \operatorname{div} \boldsymbol{u} + \frac{1}{2}\partial_t(\rho\,|\boldsymbol{u}|^2) + \frac{1}{2}\operatorname{div}(\rho\,|\boldsymbol{u}|^2\,\boldsymbol{u}) + \boldsymbol{u}\cdot\nabla p = 0.$$

From mass balance, for “regular” z:

$$\partial_t(\rho z) + \operatorname{div}(\rho z \boldsymbol{u}) = \underbrace{(\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}))}_{=0.} z + \rho\,\partial_t z + \rho \boldsymbol{u}\cdot\nabla z,$$

$$\Longrightarrow \frac{1}{2}\partial_t(\rho u_i^2) + \frac{1}{2}\operatorname{div}(\rho u_i^2 \boldsymbol{u}) = \rho u_i \partial_t u_i + \rho u_i \boldsymbol{u}\cdot\nabla u_i = u_i\,\big[\rho\partial_t u_i + \rho\boldsymbol{u}\cdot\nabla u_i\big] =$$

$$u_i\,\big[\partial_t(\rho u_i) + \operatorname{div}(\rho u_i \boldsymbol{u})\big] = -u_i\,\partial_i p,\ 1 \le i \le 3.$$

From momentum balance:

$$\Longrightarrow \frac{1}{2}\partial_t(\rho\,|\boldsymbol{u}|^2) + \frac{1}{2}\operatorname{div}(\rho\,|\boldsymbol{u}|^2\,\boldsymbol{u}) = \boldsymbol{u}\cdot\big[\partial_t(\rho\boldsymbol{u}) + \operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u})\big] = -\boldsymbol{u}\cdot\nabla p.$$

and $\partial_t(\rho e) + \operatorname{div}(\rho e \boldsymbol{u}) + p \operatorname{div}(\boldsymbol{u}) = 0.$

Strategy for a numerical scheme for Euler

How to get correct weak solutions of Euler equations while solving the internal energy balance ?

General idea: $(\text{int.en})_d + (\text{kin.en})_d \rightsquigarrow$ "$(\text{tot.en})_d$"

More precisely:

1- $(\text{mom})_d$ and $(\text{mass})_d \rightsquigarrow$ discrete kinetic energy (E_k) balance

$$\eth_t E_k + \operatorname{div}_d(E_k \boldsymbol{u}) + \boldsymbol{u} \cdot \nabla_d p + R = 0$$

R : non conservative residual term

2- $(\text{int.en})_d$:

$$\eth_t(\rho e) + \operatorname{div}_d(e\boldsymbol{u}) + p \operatorname{div}_d \boldsymbol{u} = R$$

$$R \geq 0$$

3- Perform Lax-Wendroff consistency analysis of the scheme:

(a) Suppose bounds and convergence for a sequence of discrete solutions
- control in BV and L^∞,
- convergence in L^p, for $p \geq 1$.

(b) Let φ be a regular function,
- interpolate,
- test the kinetic energy balance,
- test the internal energy balance,
- and pass to the limit in the scheme.

The corrective term in the internal energy balance is such that, at the limit, the weak form of the total energy equation is recovered.

Required discrete properties

- Discrete transport property, i.e. discrete equivalent of
  $\partial_t(\rho z) + \mathrm{div}(\rho z \boldsymbol{u}) = \rho\, \partial_t z + \rho \boldsymbol{u} \cdot \nabla z, \qquad z = u_i.$

  $\Longrightarrow$ Compatible discretization of mass and momentum balance equation

- Discrete duality
  i.e. discrete equivalent of $\mathrm{div}(p\boldsymbol{u}) = p\,\mathrm{div}\boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p$.

- Positivity of the residual $R \geq 0$ in the discrete kinetic energy balance equation (to ensure the positivity of the internal energy).

$\rightsquigarrow$ Points to be taken care of when designing the scheme(s).

Time discretization : I - Decoupled (explicit) choice

- Decoupled 1: natural ordering, bad idea..., $\rho \to \boldsymbol{u} \to e \to p$

$$\frac{1}{\delta t}(\varrho^{n+1} - \varrho^n) + \mathrm{div}(\varrho^n \boldsymbol{u}^{n)}) = 0, \qquad (\text{mass})_d \quad \rightsquigarrow \quad \varrho^{n+1}$$

$$\frac{1}{\delta t}(\varrho^{n+1}\boldsymbol{u}^{n+1} - \varrho^n\boldsymbol{u}^n) + \mathrm{div}(\varrho^n\boldsymbol{u}^n \otimes \boldsymbol{u}^n) + \boldsymbol{\nabla} p^n = 0, \qquad (\text{mom})_d \quad \rightsquigarrow \quad \boldsymbol{u}^{n+1}.$$

$$\frac{1}{\delta t}(\varrho^{n+1}e^{n+1} - \varrho^n e^n) + \mathrm{div}(\varrho^n e^n \boldsymbol{u}^n) + p^n\,\mathrm{div}\boldsymbol{u}^{n+1} = R^n, \qquad (\text{int.en})_d \quad \rightsquigarrow \quad e^{n+1}$$

$$p^{n+1} = \wp(\varrho^{n+1}, e^{n+1}) = (\gamma - 1)\,\varrho^{n+1}e^{n+1}, \qquad (\text{eos})_d \quad \rightsquigarrow \quad p^{n+1}$$

- Decoupled 2: better idea..., $\rho \to e \to p \to \boldsymbol{u}$

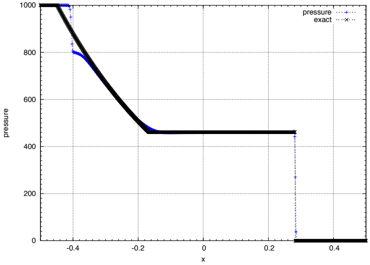
$$\frac{1}{\delta t}(\varrho^{n+1} - \varrho^n) + \mathrm{div}(\varrho^n \boldsymbol{u}^{n)}) = 0, \qquad (\text{mass})_d \quad \rightsquigarrow \quad \varrho^{n+1}$$

$$\frac{1}{\delta t}(\varrho^{n+1}e^{n+1} - \varrho^n e^n) + \mathrm{div}(\varrho^n e^n \boldsymbol{u}^n) + p^n\,\mathrm{div}\boldsymbol{u}^n = R^n, \qquad (\text{int.en})_d \quad \rightsquigarrow \quad e^{n+1}$$

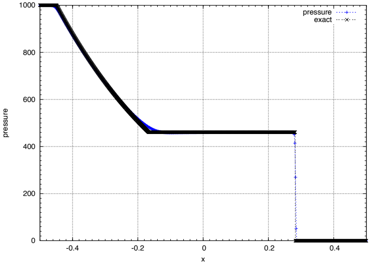
$$p^{n+1} = \wp(\varrho^{n+1}, e^{n+1}) = (\gamma - 1)\,\varrho^{n+1}e^{n+1}, \qquad (\text{eos})_d \quad \rightsquigarrow \quad p^{n+1}$$

$$\frac{1}{\delta t}(\varrho^{n+1}\boldsymbol{u}^{n+1} - \varrho^n\boldsymbol{u}^n) + \mathrm{div}(\varrho^n\boldsymbol{u}^n \otimes \boldsymbol{u}^n) + \boldsymbol{\nabla} p^{n+1} = 0, \qquad (\text{mom})_d \quad \rightsquigarrow \quad \boldsymbol{u}^{n+1}.$$

Time discretization : I -Decoupled choice



$\rho \to \boldsymbol{u} \to e \to p$



$\rho \to e \to p \to \boldsymbol{u}$

[E. Toro, *Riemann solvers and numerical methods for fluid dynamics*, third edition, test 3 of chapter 4].

Decoupled scheme : Why choose $\rho \to e \to p \to \boldsymbol{u}$?

- Derivation of the entropy balance at the continuous level. For regular functions, an entropy s satisfies

$$\partial_t s + \boldsymbol{u} \cdot \boldsymbol{\nabla} s = 0 \text{ or equivalently } \partial_t(\rho s) + \mathrm{div}(\rho \boldsymbol{u} s) = 0;$$

$$\text{for Euler perfect gas } s = \psi(\rho, e) = \ln(\rho) - \frac{1}{\gamma - 1} \ln e.$$

$$\left| \begin{array}{ll} \partial_t \varrho + \boldsymbol{u} \cdot \boldsymbol{\nabla} \varrho + \varrho\, \mathrm{div} \boldsymbol{u} = 0 & \times \partial_\varrho \psi(\rho, e) \\ \partial_t e + \boldsymbol{u} \cdot \boldsymbol{\nabla} e + \frac{p}{\varrho}\, \mathrm{div} \boldsymbol{u} = 0 & \times \partial_e \psi(\rho, e) \end{array} \right.$$

$$\rightsquigarrow \partial_t s + \boldsymbol{u} \cdot \boldsymbol{\nabla} s + \underbrace{\left[\varrho\, \partial_\varrho s + \frac{p}{\varrho} \partial_e s \right]}_{=0} \mathrm{div} \boldsymbol{u} = 0.$$

- We need $\mathrm{div}(\varrho \boldsymbol{u})$ and $p\, \mathrm{div} \boldsymbol{u}$ at the same time level to mimick this computation at the discrete level.
  - Choice 1 $\rho \to \boldsymbol{u} \to e \to p \rightsquigarrow \mathrm{div}(\varrho^n \boldsymbol{u}^n)$ and $p^n\, \mathrm{div} \boldsymbol{u}^{n+1}$
  - Choice 2 $\rho \to e \to p \to \boldsymbol{u} \rightsquigarrow \mathrm{div}(\varrho^n \boldsymbol{u}^n)$ and $p^n\, \mathrm{div} \boldsymbol{u}^n$

Time discretization : II - Implicit or semi-implicit choice

Implicit

$$\frac{\rho^{n+1} - \rho^n}{\delta t} + \mathrm{div}(\rho^{n+1}\, \boldsymbol{u}^{n+1}) = 0, \qquad (\text{mass})_d$$

$$\frac{1}{\delta t}(\varrho^{n+1} \boldsymbol{u}^{n+1} - \varrho^n \boldsymbol{u}^n) + \mathrm{div}(\varrho^{n+1} \boldsymbol{u}^{n+1} \otimes \boldsymbol{u}^{n+1}) + \boldsymbol{\nabla} p^{n+1} = 0, \qquad (\text{mom})_d$$

$$\frac{1}{\delta t}(\varrho^{n+1} e^{n+1} - \varrho^n e^n) + \mathrm{div}(\varrho^{n+1} e^{n+1} \boldsymbol{u}^{n+1}) + p^{n+1}\, \mathrm{div} \boldsymbol{u}^{n+1} = R^{n+1}, \qquad (\text{int.en})_d$$

$$p^{n+1} = \wp(\varrho^{n+1}, e^{n+1}) = (\gamma - 1)\, \varrho^{n+1} e^{n+1}, \qquad (\text{eos})_d$$

Semi-implicit

Pressure gradient scaling step: $(\overline{\boldsymbol{\nabla} p})^{n+1} = \left(\dfrac{\rho^n}{\rho^{n-1}}\right)^{1/2} (\boldsymbol{\nabla} p^n)$

Prediction step – Solve for $\tilde{\boldsymbol{u}}^{n+1}$:

$$\frac{1}{\delta t}\, (\rho^n \tilde{\boldsymbol{u}}^{n+1} - \rho^{n-1} \boldsymbol{u}^n) + \mathrm{div}(\rho^n \tilde{\boldsymbol{u}}^{n+1} \otimes \boldsymbol{u}^n) + (\overline{\boldsymbol{\nabla} p})^{n+1} = 0, \qquad (\text{mom})_d$$

Correction step – Solve for p^{n+1}, e^{n+1}, ρ^{n+1} and $\boldsymbol{u}^{n+1}$:

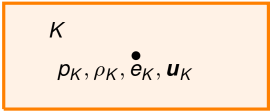
$$\frac{1}{\delta t}\, \rho^n\, (\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}) + (\boldsymbol{\nabla} p^{n+1}) - (\overline{\boldsymbol{\nabla} p})^{n+1} = 0,$$

$$\frac{\rho^{n+1} - \rho^n}{\delta t} + \mathrm{div}(\rho^{n+1}\, \boldsymbol{u}^{n+1}) = 0, \qquad (\text{mass})_d$$

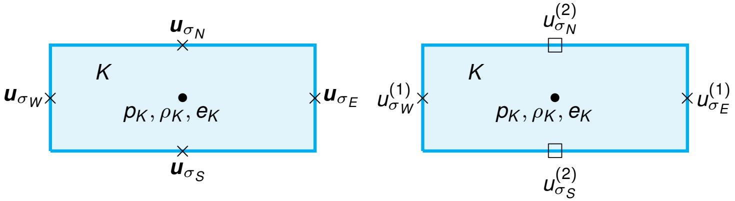
$$\frac{1}{\delta t}(\rho^{n+1} e^{n+1} - \rho^n e^n) + \mathrm{div}(\rho^{n+1} e^{n+1} \boldsymbol{u}^{n+1}) + p^{n+1}(\mathrm{div}(\boldsymbol{u}^{n+1})) = R^{n+1}, \qquad (\text{int.en})_d$$

$$\rho^{n+1} = \varrho(e^{n+1}, p^{n+1}), \qquad (\text{eos})_d$$

Meshes



Colocated
Advantage: Easier Data structure
Easily refined
Pressure correction scheme studied for the Euler equations (C. Zaza's thesis).



Staggered:

Crouzeix-Raviart (on simplices)
Rannacher-Turek (on quadrangles) | $\rightsquigarrow$ full velocities on the edges (faces)

MAC: $\rightsquigarrow$ normal velocities on the edges (faces)

Advantage: Stable in the incompressible limit

Space discretization: Finite volume discretization of the mass equation

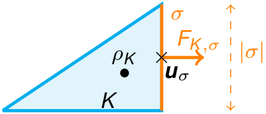
$$\frac{\rho^{n+1}-\rho^{n}}{\delta t} + \mathrm{div}(\rho^{n+1}\,\boldsymbol{u}^{n+1}) = 0, \qquad \text{(mass)}$$

- $\displaystyle\int_K$ (mass) $\rightsquigarrow$

$$\int_K \frac{\rho^{n+1}-\rho^{n}}{\delta t} + \int_{\partial K} (\rho^{n+1}\,\boldsymbol{u}^{n+1}\cdot \mathbf{n}_K) = 0.$$

- discretization of the fluxes:

$$\frac{|K|}{\delta t}\,(\rho_K^{n+1}-\rho_K^{n}) + \sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma}^{n+1} = 0,$$



$F_{K,\sigma}^{n+1} = |\sigma|\,\rho_\sigma^{n+1}\boldsymbol{u}_\sigma^{n+1}\cdot\mathbf{n}_{K,\sigma}$,
numerical flux through σ.

- ρ_σ^{n+1} upwind approximation of ρ^{n+1} at the face σ with respect to $\boldsymbol{u}_\sigma^{n+1}\cdot\boldsymbol{n}_{K,\sigma}$.
- $\rightsquigarrow$ Positive density: $\rho^{n+1} > 0$ if ($\rho^n > 0$ and $\rho > 0$ at inflow boundary)

Discrete internal energy equation and E.O.S.

$$\frac{1}{\delta t}(\rho^{n+1} e^{n+1} - \rho^n e^n) + \mathrm{div}(\rho^{n+1} e^{n+1} \boldsymbol{u}^{n+1}) + p^{n+1}(\mathrm{div}(\boldsymbol{u}^{n+1})) = R^{n+1}$$

- Discretization by upwind finite volume of the discrete internal energy

$$\frac{|K|}{\delta t}(\rho_K^{n+1} e_K^{n+1} - \rho_K^n e_K^n) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{n+1} e_\sigma^{n+1} + |K|\, p_K^{n+1}\, (\mathrm{div}\boldsymbol{u}^{n+1})_K = R_K^{n+1},$$

  - e_σ^{n+1} upwind choice $\rightsquigarrow$ positivity of e (if $R_K^{n+1} \geq 0$)
  - $|K|\, (\mathrm{div}\boldsymbol{u})_K = \sum_{\sigma \in \mathcal{E}(K)} |\sigma|\, u_\sigma \cdot \mathbf{n}_{K,\sigma}$.

- discrete E.O.S. $\quad p_K^{n+1} = (\gamma - 1)\rho_K^{n+1}\, e_K^{n+1}. \qquad (\mathrm{eos})_d$

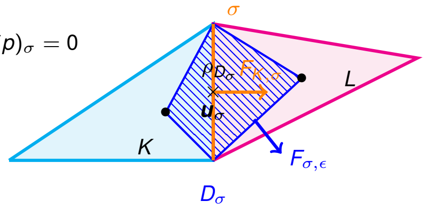
Discretization of the momentum equation

$$\frac{1}{\delta t}\left(\rho^n \tilde{\boldsymbol{u}}^{n+1} - \rho^{n-1}\boldsymbol{u}^n\right) + \mathrm{div}(\rho^n \tilde{\boldsymbol{u}}^{n+1} \otimes \boldsymbol{u}^n) + (\overline{\nabla p})^{n+1} \qquad (\mathrm{mom})^n$$

- $\displaystyle\int_{D_\sigma} (\mathrm{mom})^n \rightsquigarrow \underbrace{\int_{D_\sigma} \frac{\rho^n \tilde{\boldsymbol{u}}^{n+1} - \rho^{n-1}\boldsymbol{u}^n}{\delta t} + \int_{\partial D_\sigma} \rho^n \tilde{\boldsymbol{u}}^{n+1} \otimes \boldsymbol{u}^n \cdot \boldsymbol{n}_K}_{\mathcal{C}(\rho^n, \boldsymbol{u}^n)} + \int_{D_\sigma} (\overline{\nabla p})^{n+1} = 0.$
- Space discretization

$$\underbrace{\frac{|D_\sigma|}{\delta t}\left(\rho^n_{D_\sigma}\, \tilde{\boldsymbol{u}}^{n+1}_\sigma - \rho^{n-1}_{D_\sigma}\, \boldsymbol{u}^n_\sigma\right) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F^n_{\sigma,\epsilon}\, \tilde{\boldsymbol{u}}^{n+1}_\epsilon}_{\mathcal{C}_d(\rho^n, \boldsymbol{u}^n)} + |D_\sigma| \sqrt{\frac{\rho^n_{D_\sigma}}{\rho^{n-1}_{D_\sigma}}} (\nabla p^n)_\sigma = 0,$$

- Grad-div duality :
  $$\sum_{K \in \mathcal{T}} |K|\, p_K\, (\mathrm{div}\boldsymbol{u})_K + \sum_{\sigma \in \mathcal{E}} |D_\sigma|\, \boldsymbol{u}_\sigma \cdot (\nabla p)_\sigma = 0$$
  $\rightsquigarrow |D_\sigma| (\nabla p^n)_\sigma = |\sigma|\, (p^n_L - p^n_K)\, \boldsymbol{n}_{K,\sigma}$
  for $\sigma = K|L$.



- $\rho^n_{D_\sigma}$? $F^n_{\sigma,\epsilon}$?

Discretization of the convection operator

- Choice of $\rho^n_{D_\sigma}$, $\rho^{n-1}_{D_\sigma}$ and $F^n_{\sigma,\epsilon}$ in $\mathcal{C}_d(\rho^n, \boldsymbol{u}^n, \tilde{\boldsymbol{u}}^{n+1})$?
- Discretize $\mathcal{C}_d(\rho^n, \boldsymbol{u}^n, \tilde{\boldsymbol{u}}^{n+1})$ so as to obtain a discrete kinetic energy balance.
- Copy the continuous kinetic energy balance:

$$\text{(mom)} \cdot \boldsymbol{u} \;\&\; \text{(mass)} \rightsquigarrow \text{(kin.en)}$$

with some formal algebra... using $\partial_t \rho + \mathrm{div}(\rho\, \boldsymbol{u}) = 0$.

- Do the same at the discrete level ?

♭ Momentum on dual cells, mass on primal cells...

♯ Idea: ***reconstruct a mass balance on the the dual cells***

Choose

- $\rho_{D_\sigma} = \frac{1}{|D_\sigma|} (|D_{K,\sigma}|\ \rho_K + |D_{L,\sigma}|\ \rho_L)$
- $F_{\sigma,\epsilon}$: linear combination of the primal fluxes $(F_{K,\sigma})_{\sigma \in \mathcal{E}(K)}$.

so that a discrete mass balance holds on the dual cells D_σ:

$$\forall \sigma \in \mathcal{E}_{\text{int}}, \qquad \frac{|D_\sigma|}{\delta t} (\rho^{n+1}_{D_\sigma} - \rho^n_{D_\sigma}) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F^{n+1}_{\sigma,\epsilon} = 0.$$

Then take $\mathcal{C}_d(\rho^n, \boldsymbol{u}^n, \tilde{\boldsymbol{u}}^{n+1}) = \dfrac{|D_\sigma|}{\delta t} (\rho^n_{D_\sigma}\ \tilde{\boldsymbol{u}}^{n+1}_\sigma - \rho^{n-1}_{D_\sigma}\ \boldsymbol{u}^n_\sigma) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F^n_{\sigma,\epsilon}\ \tilde{\boldsymbol{u}}^{n+1}_\epsilon$

with $\boldsymbol{u}^{n+1}_\epsilon = \frac{1}{2} (\boldsymbol{u}^{n+1}_\sigma + \boldsymbol{u}^{n+1}_{\sigma'})$

Discrete kinetic energy balance: computation of R_σ

- Continuous setting: Multiply continuous momentum by $\boldsymbol{u}$:

$$\Big(\partial_t(\rho\,\boldsymbol{u}) + \operatorname{div}(\rho\,\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = 0 \quad\Big)\cdot\boldsymbol{u}$$

... with some formal algebra... using $\partial_t\rho + \operatorname{div}(\rho\,\boldsymbol{u}) = 0$,
$\rightsquigarrow$ continuous kinetic energy balance:

$$\partial_t\big(\frac{1}{2}\,\rho\,|\boldsymbol{u}|^2\big) + \operatorname{div}\big(\big(\frac{1}{2}\,\rho\,|\boldsymbol{u}|^2\big)\,\boldsymbol{u}\big) + \boldsymbol{\nabla} p\cdot\boldsymbol{u} = 0 \qquad \text{(kin.en)}$$

- Discrete setting: Similarly, multiply discrete momentum by $\tilde{\boldsymbol{u}}_\sigma^{n+1}$:

$$\Big(\frac{|D_\sigma|}{\delta t}\,(\rho_{D_\sigma}^n\,\tilde{\boldsymbol{u}}_\sigma^{n+1} - \rho_\sigma^{n-1}\,\boldsymbol{u}_\sigma^n) + \sum_{\epsilon\in\mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n\,\tilde{\boldsymbol{u}}_\epsilon^{n+1} + |D_\sigma|(\overline{\boldsymbol{\nabla} p}^{n+1})_\sigma = 0\Big)\cdot\tilde{\boldsymbol{u}}_\sigma^{n+1}$$

... with some algebra... using

  - Mass on dual cell: $\dfrac{|D_\sigma|}{\delta t}\,(\rho_{D_\sigma}^n - \rho_\sigma^{n-1}) + \displaystyle\sum_{\epsilon\in\mathcal{E}(D_\sigma)} F_{\sigma,\epsilon}^n = 0.$
  - Scaling of the pressure $\rho^{n-1}|(\overline{\boldsymbol{\nabla} p})^{n+1}|^2 = \rho^n|\boldsymbol{\nabla} p^n|^2$.
  - Correction equation $\dfrac{1}{\delta t}\,\rho^n\,\boldsymbol{u}^{n+1} + \boldsymbol{\nabla} p^{n+1} = \dfrac{1}{\delta t}\,\rho^n\tilde{\boldsymbol{u}}^{n+1} + (\overline{\boldsymbol{\nabla} p})^{n+1}$

$\rightsquigarrow$ discrete kinetic energy balance:

$$\frac{1}{2}\,\frac{|D_\sigma|}{\delta t}\Big[\rho_\sigma^n|\boldsymbol{u}_\sigma^{n+1}|^2 - \rho_{D_\sigma}^{n_1}\,|\boldsymbol{u}_\sigma^n|^2\Big] + \frac{1}{2}\sum_{\epsilon=D_\sigma|D_{\sigma'}} F_{\sigma,\epsilon}^n\;\tilde{\boldsymbol{u}}_\sigma^{n+1}\cdot\tilde{\boldsymbol{u}}_{\sigma'}^{n+1}$$

$$+|D_\sigma|\,(\boldsymbol{\nabla} p^{n+1})_\sigma\cdot\boldsymbol{u}_\sigma^{n+1} + R_\sigma^{n+1} + \mathcal{P}_\sigma^{n+1} = 0 \text{ with } R_\sigma^{n+1}\geq 0, \qquad \text{(kin.en)}_\sigma$$

Choice of R_K

$R_\sigma = \dfrac{|D_\sigma|}{\delta t} \rho_{D_\sigma}^{n-1} |\widetilde{\boldsymbol{u}}_\sigma^{n+1} - \boldsymbol{u}_\sigma^n|^2 \to 0$ for regular functions, but NOT for discontinuous functions.

By definition of ρ_{D_σ}, for $\sigma = K|L$,

$$R_\sigma = \frac{|D_{K,\sigma}|}{\delta t} \rho_K^{n-1} \, |\widetilde{\boldsymbol{u}}_\sigma^{n+1} - \boldsymbol{u}_\sigma^n|^2 + \frac{|D_{L,\sigma}|}{\delta t} \rho_L^{n-1} \, |\widetilde{\boldsymbol{u}}_\sigma^{n+1} - \boldsymbol{u}_\sigma^n|^2$$

$$\rightsquigarrow R_K = \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K)} \frac{|D_{K,\sigma}|}{\delta t} \rho_K^{n-1} \, |\widetilde{\boldsymbol{u}}_\sigma^{n+1} - \boldsymbol{u}_\sigma^n|^2$$

$$\Rightarrow \sum_{K \in \mathcal{T}} R_K - \sum_{\sigma \in \mathcal{E}} R_\sigma = 0$$

Passage to the limit: total energy recovered

▷ Kinetic energy

$$(\mathrm{kin})^n_\sigma = \frac{|D_\sigma|}{2\delta t}(\varrho^{n+1}_\sigma |\boldsymbol{u}^{n+1}_\sigma|^2 - \varrho^n_\sigma |\boldsymbol{u}^n_\sigma|^2) + \frac{1}{2}\sum_{\epsilon = D_\sigma | D_{\sigma'}} F^n_{\sigma,\epsilon}\, \tilde{\boldsymbol{u}}^{n+1}_\sigma \cdot \tilde{\boldsymbol{u}}^{n+1}_{\sigma'} + (\boldsymbol{\nabla} p)^{n+1}_\sigma \cdot \boldsymbol{u}^{n+1}_\sigma = -R^{n+1}_\sigma + \mathcal{P}^{n+1}_\sigma,$$

▷ Internal energy

$$(\mathrm{int})^n_K = \frac{|K|}{\delta t}(\rho^{n+1}_K e^{n+1}_K - \rho^n_K e^n_K) + \sum_{\sigma \in \mathcal{E}(K)} F^n_{K,\sigma} e^n_\sigma + |K|\, p^{n+1}_K\, (\mathrm{div}\, \boldsymbol{u}^{n+1})_K = R^{n+1}_K,$$

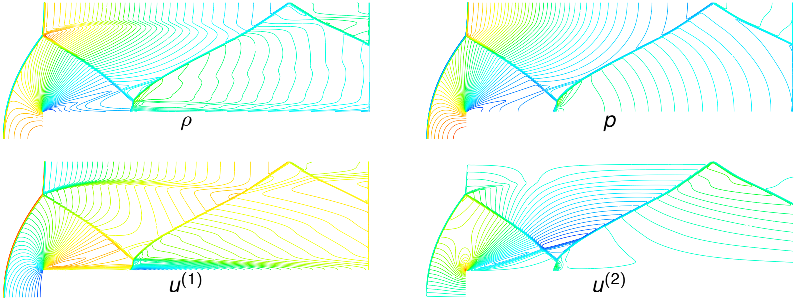
▷ φ : test function
Multiply $(\mathrm{kin})_\sigma$ by interpolate φ^n_σ and $(\mathrm{int})_K$ by interpolate φ^n_K

$$\underbrace{\sum_n \sum_{\sigma\in\mathcal{E}} (\mathrm{kin})^n_\sigma \varphi^n_\sigma + \sum_n \sum_{K\in\mathcal{T}} (\mathrm{int})^n_K \varphi^n_K}_{\downarrow} = \underbrace{\sum_n \sum_{K\in\mathcal{T}} \delta t R_K \varphi^n_K - \sum_n \sum_{\sigma\in\mathcal{E}} \delta t R_\sigma \varphi^n_\sigma}_{\downarrow} + \underbrace{\sum_n \sum_{\sigma\in\mathcal{E}} \delta t \mathcal{P}_\sigma \varphi^n_\sigma}_{\downarrow}.$$

$$-\int_0^T\int_\Omega \Big[\rho\, E\, \partial_t\varphi + (\rho\, E + p)\, u\, \nabla\varphi\Big] - \int_\Omega \rho_0(x)\, E_0(x)\, \varphi(x,0) \qquad 0 \qquad 0$$

In particular, the pressure terms combine themselves to converge to $-p\boldsymbol{u}\cdot\boldsymbol{\nabla}\varphi$.

Numerical tests - I Euler, high Mach

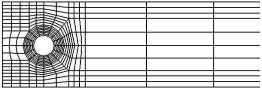


Mach 3 facing step (Woodward Collela)
MAC space discretization, 1200×400 uniform grid, $\delta t = h/4 = 0.001$, ($u_1 + c = 4$ at the inlet boundary).

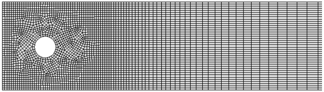
Artificial viscosity $\mu = 0.001$, which roughly corresponds to the numerical viscosity associated to an upwinding of the convection term $\mu_{\text{upw}} \simeq \rho\,|\boldsymbol{u}|\,h/2$ divided by 5.

Numerical tests - II Flow past cylinder, low Mach

Flow past a cylinder, benchmark Schäfer and S. Turek, Mach $\simeq 0.003$, $Re \simeq 100$.
Pressure correction scheme, Rannacher-Turek FE.



coarse mesh



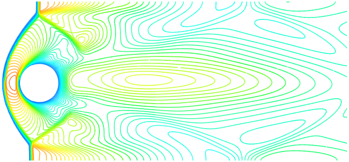
fine mesh

Mesh	Space unks	$c_{d,max}$	$c_{l,max}$	St
m2	64840	3.4937	0.9141	0.2850
m3	215545	3.2887	0.9891	**0.2955**
m4	381119	3.2614	**1.0062**	**0.2972**
m5	531301	**3.2365**	**1.0148**	**0.2976**
Reference range		3.22 − 3.24	0.99 − 1.01	0.295 − 0.305

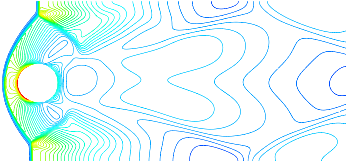
Table: Drag and lift coefficients and Strouhal number.

Numerical tests - II Flow past cylinder, high Mach

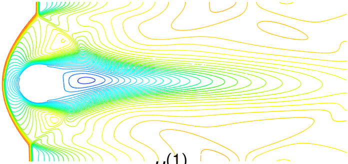
Flow past a cylinder, Mach $\simeq 3$, $Re \simeq 100$.
$p_{\text{ext}} = \gamma / 10\, \rho$, $c = 0.1$.
$\text{mes} = 10^{-3}$.
impermeability and and perfect slip condition at the upper and lower boundaries
$\boldsymbol{u} = (1, 0)^t$. at inlet.
$\delta t = 10^{-4}$. Rannacher-Turek FE.



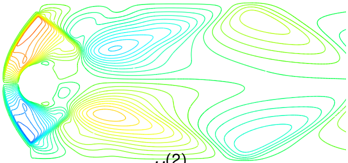
ρ



p



$u^{(1)}$



$u^{(2)}$

$t = 5$, mesh of 10^6

Summary: main features of the class of schemes

- Colocated or staggered discretization
- Upwind choice for ρ, $\rightsquigarrow$ positivity of the density,
- Compatible discretization of (mass) and (mom)
  & careful choice of ρ_{D_σ} and fluxes in (mom) to recover a mass conservation on the dual cells
  $\rightsquigarrow$ discrete kinetic energy inequality
- Compatible discretization of (mass) and (int.en) & upwind choice for e_K under CFL
  $\rightsquigarrow$ positivity of e
- Conservation of total mass
- Existence of a solution to the scheme (topological degree argument)
- Velocity and pressure are constant through the contact discontinuity: p^n and u^n constant $\Rightarrow$ p^{n+1} and u^{n+1} constant.
- Consistency : under compactness assumptions, the discrete solution tends to a weak solution of the Euler systems.
- Numerical discontinuous solutions have correct shocks.

Recent and on going work

- Higher order schemes for the explicit MUSCL and additional viscosity.
- Convergence of the scheme to an entropy weak solution
- Low Mach limit for barotropic NS (with K. Saleh, M.-H. Vignal).
- Convergence for barotropic NS ($\gamma \geq 3$) and strong-weak error estimates for barotropic NS (with D. Maltese, and A. Novotny).
- Reactive flows.
- Analysis of the MAC scheme in primitive variables for incompressible flows (K. Mallem's thesis)