Numerical experiments with the DDFV method

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ABSTRACT. *We present the numerical results we have obtained with the finite volume method introduced in [HER 98],[HER 00],[HER 03] for approximating diffusion operators with variable (continuous or discontinuous, linear or non-linear), full tensor coefficients on arbitrary meshes. In [DEL 05] this type of method has been called Discrete-Duality Finite Volume (DDFV) in order to emphasize that it satisfies a discrete integration by parts.*

KEYWORDS: *Finite volume method, anisotropic diffusion.*

1. Presentation of the scheme

The main idea lies in using two different meshes, namely an arbitrary (given) *primal* mesh and a *dual* mesh that is made up from the primal mesh. Here we have chosen the *median* dual mesh whose vertices are the centers of gravity of the primal cells and the middle of the primal sides. The diffusion equation to be dealt with is integrated both on the cells of the primal mesh and those of the dual mesh while the degrees of freedom are the values of the unknown function at the centers of gravity of the primal cells, the middle of the boundary sides and the vertices of the primal mesh. The method provides symmetric positive definite matrices. Moreover it provides natural, accurate, a posteriori approximations of both the gradient and the Hessian of the solution. The approximated gradient is calculated at the middle of the cell sides while the approximated Hessian is calculated both at the centers of gravity and vertices of the primal cells. If needed, the approximated gradient can be calculated at the centers of gravity and vertices of the primal cells (by using a least square method for example).

Convergence analysis have been carried out in [DOM 05],[YUA 07] for the linear case. Generalizations to non-linear diffusion operators of Leray-Lions type and convergence analysis in this framework have been carried out in [AND 05],[AND 07] and [BOY 06]. A generalization to Laplace-Beltrami diffusion operators over surfaces has been carried out in [CAL 08]. 3-D generalizations are proposed in [PIE 05],[COU 06] and [HER 07].

Besides the requested numerical results, we have provided (if possible) the relative error in the L^2 norm on the Hessian (erhess and ratiohess). Furthermore we have tested the heterogeneous rotating anisotropy (test 5) with $\alpha = 1$ and $\beta = 1.e - 12$. All the linear systems are diagonally preconditioned before being solved by the conjugate gradient method. As a reference mesh we have chosen a 160×160 square mesh.

Let denote by S_s a primal side, $\boldsymbol{n}_s$ its unit normal vector and D_s, D_{sp}, D_{sq} its associated *diamond* and *half diamond* cells. As detailed in [HER 03] we obtain approximations $(\boldsymbol{\nabla} u)_{sp}$, $(\boldsymbol{\nabla} u)_{sq}$ of $\boldsymbol{\nabla} u$ in D_{sp}, D_{sq} by using:

1) the values u_d, u_e of u at the vertices of S_s,

2) the value u_p (u_q) of u at the center of gravity of the first (second) primal cell whose S_s is a side,

3) the value u_s at the middle of S_s.

For *boundary* primal sides S_s, u_s is a degree of freedom of the method. For *interior* primal sides S_s, u_s is an *auxiliary* degree of freedom calculated so that:

$$(\boldsymbol{K}_p(\boldsymbol{\nabla} u)_{sp}).\boldsymbol{n}_s = (\boldsymbol{K}_q(\boldsymbol{\nabla} u)_{sq}).\boldsymbol{n}_s \stackrel{\text{def}}{=} (\boldsymbol{K}\boldsymbol{\nabla} u)_s.\boldsymbol{n}_s,$$

where $\boldsymbol{K}_p$, $\boldsymbol{K}_q$ are the values of $\boldsymbol{K}$ at the centers of gravity of the cells whose S_s is a side. If $\boldsymbol{K}$ is continuous we replace $\boldsymbol{K}_p$ and $\boldsymbol{K}_q$ by the value $\boldsymbol{K}_s$ at the middle of S_s, so we obtain an unique approximation $(\boldsymbol{\nabla} u)_s$ of $\boldsymbol{\nabla} u$ in the *whole* diamond cell $D_s = D_{sp} \cup D_{sq}$. Let $(\boldsymbol{\nabla} u)_s^{ex}$, $(\boldsymbol{K}\boldsymbol{\nabla} u)_s^{ex}$ be the exact value of $\boldsymbol{\nabla} u$, $\boldsymbol{K}\boldsymbol{\nabla} u$ at the middle of primal sides S_s, we have set:

$$ergrad = \frac{(\sum_{D_s} |D_s||(\boldsymbol{\nabla} u)_s - (\boldsymbol{\nabla} u)_s^{ex}|^2)^{\frac{1}{2}}}{(\sum_{D_s} |D_s||(\boldsymbol{\nabla} u)_s^{ex}|^2)^{\frac{1}{2}}},$$

$$erflm = \max_{S_s} |(\boldsymbol{K}\boldsymbol{\nabla} u)_s.\boldsymbol{n}_s - (\boldsymbol{K}\boldsymbol{\nabla} u)_s^{ex}.\boldsymbol{n}_s|,$$

$$ener1 = \sum_{D_s} |D_s|(\boldsymbol{K}_s(\boldsymbol{\nabla} u)_s).(\boldsymbol{\nabla} u)_s,$$

$$ener2 = \sum_{S_s \in \partial\Omega} |S_s|(\boldsymbol{K}\boldsymbol{\nabla} u)_s.\boldsymbol{n}_s u_s.$$

Since the previous definition of $(\boldsymbol{\nabla} u)_s$ in the whole D_s presupposes that $\boldsymbol{\nabla} u$ is continuous in D_s, we have not given the value of $ergrad$ for Test 7.

2. Numerical results

• **Test 1.1 Mild anisotropy,** $u(x, y) = 16x(1 - x)y(1 - y)$**,** $min = 0$, $max = 1$**, regular triangular meshes**

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	109	864	-1.79e-10	2.13e-2	2.33e-2		
2	385	3252	-1.89e-11	5.77e-3	5.92e-3	2.07	2.17
3	1441	12564	2.60e-11	1.47e-3	1.48e-3	2.07	2.09
4	5569	49332	7.57e-11	3.68e-4	3.72e-4	2.04	2.05
5	21889	195444	9.47e-11	9.21e-5	9.32e-5	2.02	2.02
6	86785	777972	6.36e-11	2.30e-5	2.33e-5	2.01	2.01
7	345601	3104244	-3.93e-11	5.76e-6	5.82e-6	2.01	2.00

i	erhess	ratiohess
1	1.56e-1	
2	5.42e-2	1.67
3	1.98e-2	1.52
4	7.76e-3	1.39
5	3.25e-3	1.27
6	1.45e-3	1.27
7	6.77e-4	1.10

ocvl2=2.00, **ocvgradl2**=2.00, **ocvhessl2**=1.10.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	1.06e-2	1.06e-2	1.06e-2	1.06e-2	1.37e-1	0.0	0.99101
2	2.65e-3	2.65e-3	2.65e-3	2.65e-3	3.83e-2	0.0	0.99785
3	6.63e-4	6.63e-4	6.63e-4	6.63e-4	9.99e-3	0.0	0.99946
4	1.65e-4	1.65e-4	1.65e-4	1.65e-4	2.54e-3	0.0	0.99986
5	4.14e-5	4.14e-5	4.14e-5	4.14e-5	6.42e-4	0.0	0.99997
6	1.03e-5	1.03e-5	1.03e-5	1.03e-5	1.62e-4	0.0	0.99999
7	2.60e-6	2.60e-6	2.60e-6	2.60e-6	4.07e-5	0.0	0.99999

• **Test 1.1 Mild anisotropy,** $u(x, y) = 16x(1 - x)y(1 - y)$**,** $min = 0$, $max = 1$**, distorted quadrangular meshes (1)**

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	681	5707	-5.85e-11	2.04e-2	1.95e-2		
2	2517	21820	-2.73e-11	5.16e-3	5.08e-3	2.10	2.06
3	5509	48340	3.90e-11	2.29e-3	2.27e-3	2.07	2.05
4	9657	85264	1.89e-12	1.29e-3	1.28e-3	2.05	2.04
5	14961	132592	-3.38e-12	8.25e-4	8.21e-4	2.04	2.03
6	21421	190324	1.31e-11	5.73e-4	5.71e-4	2.03	2.02

i	erhess	ratiohess
1	9.03e-2	
2	3.24e-2	1.56
3	1.77e-2	1.54
4	1.15e-2	1.53
5	8.27e-3	1.53
6	6.28e-3	1.52

ocvl2=2.00, **ocvgradl2=**1.99, **ocvhessl2=**1.52.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	1.53e-3	1.10e-2	3.87e-3	6.90e-3	1.47e-1	0.0	1.026
2	3.01e-4	2.86e-3	9.34e-4	1.61e-3	6.28e-2	0.0	1.005
3	1.36e-4	1.27e-3	4.13e-4	7.18e-4	3.77e-2	0.0	1.002
4	7.77e-5	7.19e-4	2.32e-4	4.06e-4	2.50e-2	0.0	1.001
5	5.02e-5	4.60e-4	1.48e-4	2.61e-4	1.77e-2	0.0	1.001
6	3.50e-5	3.20e-4	1.03e-4	1.81e-4	1.32e-2	0.0	1.000

• **Test 1.1 Mild anisotropy,** $u(x, y) = 16x(1-x)y(1-y)$**,** $min = 0$, $max = 1$**, distorted quadrangular meshes (2)**

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	2377	20587	1.79e-11	5.54e-3	5.29e-3		
2	9109	80380	-1.46e-11	1.38e-3	1.34e-3	2.06	2.04
3	20197	179380	-1.65e-11	6.16e-4	5.96e-4	2.03	2.03
4	35641	317584	-1.36e-11	3.46e-4	3.36e-4	2.03	2.02
5	55441	494992	-1.62e-12	2.21e-4	2.15e-4	2.02	2.01
6	79597	711604	-9.66e-12	1.54e-4	1.49e-4	2.02	2.01

i	erhess	ratiohess
1	4.27e-2	
2	1.53e-2	1.52
3	8.37e-3	1.52
4	5.44e-3	1.52
5	3.89e-3	1.52
6	2.95e-3	1.52

ocvl2=1.96, **ocvgradl2=**1.99, **ocvhessl2=**1.50.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	4.61e-4	3.33e-3	1.00e-3	1.47e-3	4.42e-2	0.0	1.008
2	1.12e-4	8.39e-4	2.45e-4	3.68e-4	1.17e-2	0.0	1.002
3	4.99e-5	3.73e-4	1.08e-4	1.64e-4	5.25e-3	0.0	1.001
4	2.81e-5	2.10e-4	6.11e-5	9.21e-5	3.03e-3	0.0	1.000
5	1.79e-5	1.34e-4	3.91e-5	5.89e-5	2.03e-3	0.0	1.000
6	1.24e-5	9.34e-5	2.71e-5	4.09e-5	1.46e-3	0.0	1.000

• **Test 1.2 Mild anisotropy,** $u(x, y) = \sin((1-x)(1-y)) + (1-x)^3(1-y)^2$**,** $min = 0$, $max = 1 + \sin 1 = 1.84147$**, regular triangular meshes**

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	109	864	-2.20e-10	2.65e-3	7.93e-3		
2	385	3252	-8.63e-10	6.55e-4	2.22e-3	2.21	2.01
3	1441	12564	-6.95e-10	1.65e-4	6.11e-4	2.09	1.95
4	5569	49332	-8.17e-9	4.14e-5	1.66e-4	2.04	1.93
5	21889	195444	-9.93e-9	1.04e-5	4.44e-5	2.02	1.92
6	86785	777972	3.09e-8	2.59e-6	1.18e-5	2.01	1.92
7	345601	3104244	3.76e-8	6.47e-7	3.12e-6	2.00	1.92

i	erhess	ratiohess
1	8.13e-2	
2	3.19e-2	1.48
3	1.39e-2	1.26
4	6.61e-3	1.10
5	3.34e-3	0.99
6	1.73e-3	0.95
7	9.11e-4	0.93

ocvl2=2.00, **ocvgradl2**=1.92, **ocvhessl2**=0.92.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	4.08e-3	4.86e-3	5.36e-3	1.60e-2	4.83e-2	0.0	1.84147
2	1.20e-3	1.32e-3	2.03e-3	5.13e-3	2.47e-2	0.0	1.84147
3	3.51e-4	3.37e-4	6.69e-4	1.57e-3	1.30e-2	0.0	1.84147
4	1.08e-4	8.48e-5	2.11e-4	4.74e-4	6.75e-3	0.0	1.84147
5	2.82e-5	2.14e-5	6.35e-5	1.37e-4	3.44e-3	0.0	1.84147
6	7.79e-6	5.33e-6	1.85e-5	3.87e-5	1.73e-3	0.0	1.84147
7	2.07e-6	1.34e-6	5.42e-6	1.09e-5	8.73e-4	0.0	1.84147

• **Test 1.2 Mild anisotropy,** $u(x, y) = \sin((1-x)(1-y)) + (1-x)^3(1-y)^2$**,** $min = 0$, $max = 1 + \sin 1 = 1.84147$**, nonconforming refined rectangular meshes**

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	121	929	-1.67e-10	7.83e-3	2.57e-2		
2	401	3304	-8.58e-10	1.94e-3	8.96e-3	2.33	1.76
3	1441	12376	3.86e-10	4.84e-4	3.10e-3	2.16	1.66
4	5441	47800	8.43e-9	1.21e-4	1.08e-3	2.08	1.59
5	21121	187768	9.12e-9	3.04e-5	3.77e-4	2.04	1.55

i	erhess	ratiohess
1	1.49e-1	
2	1.08e-1	0.54
3	8.00e-2	0.47
4	5.86e-2	0.47
5	4.24e-2	0.47

ocvl2=1.99, **ocvgradl2=**1.52, **ocvhessl2=**0.47.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	7.58e-3	1.26e-3	9.86e-4	1.44e-2	1.27e-1	0.0	1.84147
2	1.23e-3	1.01e-4	9.57e-5	6.70e-3	6.90e-2	0.0	1.84147
3	1.30e-4	1.41e-5	4.51e-5	2.50e-3	3.59e-2	0.0	1.84147
4	1.29e-5	3.30e-6	2.07e-5	8.34e-4	1.84e-2	0.0	1.84147
5	1.48e-5	6.49e-7	7.54e-6	2.61e-4	9.37e-3	0.0	1.84147

• **Test 2 Numerical locking,** $u(x,y) = \sin(2\pi x)e^{-2\pi\sqrt{\frac{1}{\delta}}y}$, $\delta = 10^5$**,** $min = -1$, $max = 1$**, regular triangular meshes**

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	109	864	-7.09e+1	7.19e-1	5.27e-1		
2	385	3252	-3.25e+0	5.92e-1	5.34e-1	0.31	-0.02
3	1441	12564	-1.53e+1	5.12e-1	3.93e-1	0.22	0.44
4	5569	49332	2.88e+0	4.36e-1	2.69e-1	0.23	0.54
5	21889	195444	1.18e+0	2.85e-1	1.07e-1	0.62	1.33
6	86785	777972	-7.92e-1	8.10e-2	3.60e-2	1.81	1.57
7	345601	3104244	-6.50e-7	1.19e-4	3.14e-4	9.44	6.84

i	erflx0	erflx1	fluy0	fluy1	erflm	umin	umax
1	5.34e-3	1.89e-3	-1.07e+2	-5.35e+1	6.80e-1	-0.47	0.77
2	6.92e-4	1.12e-3	1.00e+0	-2.14e+0	1.24e+0	-0.42	1.05
3	2.16e-5	2.57e-5	3.42e+0	-1.58e+1	8.31e-1	-0.64	0.89
4	2.84e-5	1.25e-4	5.02e+0	7.68e-1	6.34e-1	-0.75	9.12
5	9.03e-7	1.29e-6	-1.85e-1	-1.11e+0	2.50e-1	-0.74	1.12
6	7.49e-7	2.65e-7	6.47e-3	9.74e-1	1.72e-1	-0.93	1.03
7	2.50e-10	2.52e-10	1.16e-7	-1.59e-7	7.16e-3	-1.00	1.00

• **Test 2 Numerical locking,** $u(x, y) = \sin(2\pi x)e^{-2\pi\sqrt{\frac{1}{\delta}}y}$, $\delta = 10^6$, $min = -1$, $max = 1$**, regular triangular meshes**

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	109	864	4.67e+1	9.24e-1	9.20e-1		
2	385	3252	-1.17e+1	7.99e-1	9.41e-1	0.23	-0.02
3	1441	12564	-3.32e+1	7.06e-1	7.13e-1	0.18	0.39
4	5569	49332	-6.21e-1	5.52e-1	4.61e-1	0.36	0.62
5	21889	195444	1.40e+0	4.56e-1	3.07e-1	0.28	0.59
6	86785	777972	3.31e-1	3.79e-1	2.18e-1	0.27	0.49
7	345601	3104244	7.72e-1	2.64e-1	7.60e-2	0.52	1.52

i	erflx0	erflx1	fluy0	fluy1	erflm	umin	umax
1	9.74e-3	4.19e-3	4.23e+1	1.18e+2	1.20e+0	-0.33	0.71
2	9.54e-4	1.06e-3	-3.58e-1	-1.06e+2	1.47e+0	-0.21	0.91
3	1.09e-4	1.86e-4	9.75e-1	-2.27e+1	1.77e+0	-0.35	0.67
4	6.59e-9	4.61e-4	5.72e+1	2.19e+0	8.18e-1	-0.60	0.87
5	4.87e-6	7.01e-6	-1.36e+0	-6.76e+0	8.13e-1	-0.74	0.86
6	2.63e-6	3.98e-6	1.15e+0	-1.25e+0	5.36e-1	-0.70	0.97
7	7.33e-8	1.72e-7	-8.00e-2	7.83e-1	1.64e-1	-0.78	1.11

• **Test 3 Oblique flow,** $min = 0$, $max = 1$**, uniform rectangular meshes**

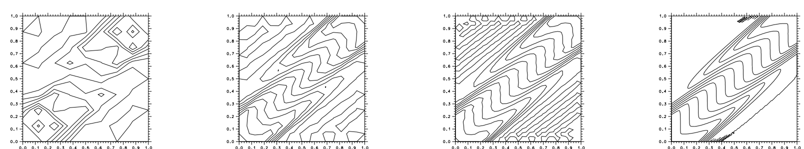

Figure 1. *Solutions for the oblique flow: mesh2_i for i=2,3,4 and reference mesh (from left to right).*

i	nunkw	nnmat	sumflux	umin	umax
1	57	403	4.99e-16	-4.72e-3	1.00472
2	177	1387	2.38e-10	-5.63e-2	1.05634
3	609	5083	2.41e-10	-1.52e-2	1.01528
4	2241	19387	1.95e-10	-1.82e-2	1.01824
5	8577	75643	2.76e-10	-1.07e-3	1.00106
ref	52161	465595	8.74e-10	-5.96e-8	1.00000

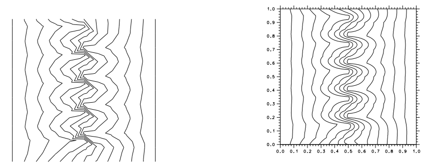

Figure 2. *Solutions for the vertical fault: coarse non conforming rectangular mesh (left) and reference mesh (right).*

i	flux0	flux1	fluy0	fluy1
1	-1.82e-1	1.82e-1	-1.21e-1	1.21e-1
2	-1.91e-1	1.91e-1	-1.03e-1	1.03e-1
3	-1.94e-1	1.94e-1	-9.83e-2	9.83e-2
4	-1.94e-1	1.94e-1	-9.79e-2	9.79e-2
5	-1.93e-1	1.93e-1	-9.82e-2	9.82e-2
ref	-1.93e-1	1.93e-1	-9.85e-2	9.85e-2

i	ener1	ener2	eren
1	2.14e-1	2.36e-1	9.46e-2
2	2.46e-1	2.68e-1	8.03e-2
3	2.40e-1	2.43e-1	1.01e-2
4	2.42e-1	2.44e-1	7.52e-3
5	2.42e-1	2.42e-1	9.52e-4
ref	2.42e-1	2.42e-1	1.91e-5

• **Test 4 Vertical fault,** $min = 0.$, $max = 1.$**, non conforming rectangular mesh**

i	nunkw	nnmat	sumflux	umin	umax
1	282	2278	7.60e-9	0.0	1.0
ref	52161	465595	3.15e-8	0.0	1.0

i	flux0	flux1	fluy0	fluy1
1	-4.00e+1	4.18e+1	-1.81e+0	9.08e-4
ref	-4.20e+1	4.43e+1	-2.35e+0	7.97e-4

i	ener1	ener2	eren
1	4.93e+1	4.06e+1	1.75e-1
ref	4.38e+1	4.31e+1	1.64e-2

• **Test 5 Heterogeneous rotating anisotropy,**
$u(x, y) = \sin(\pi x)\sin(\pi y),\ \alpha = 1,\ \beta = 10^{-3}$**,** $min = 0,\ max = 1$ **, uniform rectangular meshes**

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	57	403	3.21e-10	4.83e-2	5.86e-2		
2	177	1387	3.65e-11	1.11e-2	1.83e-2	2.59	2.05
3	609	5083	2.02e-12	2.68e-3	5.14e-3	2.30	2.05
4	2241	19387	7.62e-12	6.62e-4	1.39e-3	2.14	2.00
5	8577	75643	-8.70e-13	1.65e-4	3.68e-4	2.07	1.98
6	33537	298747	1.56e-13	4.12e-5	9.64e-5	2.03	1.97
7	132609	1187323	3.37e-14	1.03e-5	2.50e-5	2.02	1.96
8	527361	4733947	-3.62e-13	2.58e-6	6.47e-6	2.00	1.96
9	2103297	18905083	-3.74e-13	6.44e-7	1.67e-6	2.00	1.96

i	erhess	ratiohess
1	2.29e-1	
2	9.33e-2	1.58
3	3.88e-2	1.42
4	1.68e-2	1.28
5	7.65e-3	1.17
6	3.61e-3	1.10
7	1.75e-3	1.05
8	8.63e-4	1.03
9	4.28e-4	1.01

ocvl2=2.00, **ocvgradl2**=1.95, **ocvhessl2**=1.01.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	1.25e-1	4.48e-2	1.25e-1	4.48e-2	4.73e-1	0.0	1.01089
2	4.07e-2	1.04e-2	4.07e-2	1.04e-2	2.37e-1	0.0	1.00473
3	1.24e-2	2.56e-3	1.24e-2	2.56e-3	1.19e-1	0.0	1.00158
4	3.67e-3	6.35e-4	3.67e-3	6.35e-4	5.95e-2	0.0	1.00045
5	1.06e-3	1.58e-4	1.06e-3	1.58e-4	2.97e-2	0.0	1.00012
6	2.98e-4	3.90e-5	2.98e-4	3.95e-5	1.49e-2	0.0	1.00003
7	8.33e-5	9.40e-6	8.33e-5	9.88e-6	7.43e-3	0.0	1.00000
8	2.30e-5	2.02e-6	2.30e-5	2.50e-6	3.71e-3	0.0	1.00000
9	6.30e-6	1.75e-7	6.30e-6	6.55e-7	1.36e-3	0.0	1.00000

• **Test 5 Heterogeneous rotating anisotropy,** $u(x,y) = \sin(\pi x)\sin(\pi y),\ \alpha = 1,\ \beta = 10^{-12}$**,** $min = 0,\ max = 1$ **, uniform rectangular meshes**

i	nunkw	nnmat	sumflux	erl2	ergrad	ratiol2	ratiograd
1	57	403	2.10e-11	4.84e-2	5.88e-2		
2	177	1387	5.80e-11	1.11e-2	1.85e-2	2.59	2.03
3	609	5083	2.66e-11	2.69e-3	5.33e-3	2.30	2.02
4	2241	19387	-3.34e-12	6.64e-4	1.47e-3	2.14	1.97
5	8577	75643	3.11e-12	1.66e-4	4.00e-4	2.07	1.94
6	33537	298747	7.85e-13	4.14e-5	1.07e-4	2.03	1.93
7	132609	1187323	-4.91e-12	1.03e-5	2.87e-5	2.02	1.92
8	527361	4733947	-1.85e-11	2.59e-6	7.60e-6	2.01	1.92
9	2103297	18905083	-3.75e-11	6.47e-7	2.00e-6	2.00	1.93

i	erhess	ratiohess
1	2.29e-1	
2	9.42e-2	1.57
3	4.02e-2	1.37
4	1.80e-2	1.23
5	8.48e-3	1.12
6	4.10e-3	1.06
7	2.01e-3	1.03
8	1.00e-3	1.01
9	4.97e-4	1.01

ocvl2=2.00, **ocvgradl2=**1.92, **ocvhessl2=**1.00.

i	erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1	1.25e-1	4.10e-2	1.25e-1	4.10e-2	4.74e-1	0.0	1.01062
2	4.07e-2	6.63e-3	4.07e-2	6.63e-2	2.38e-1	0.0	1.00450
3	1.24e-2	1.22e-3	1.24e-2	1.22e-3	1.19e-1	0.0	1.00149
4	3.67e-3	3.15e-3	3.67e-3	3.15e-3	5.96e-2	0.0	1.00011
5	1.06e-3	3.64e-3	1.06e-3	3.64e-3	2.98e-2	0.0	1.00003
6	2.98e-4	3.76e-3	2.98e-4	3.76e-3	1.49e-2	0.0	1.00000
7	8.33e-5	3.80e-3	8.33e-5	3.80e-3	7.45e-3	0.0	1.00000
8	2.30e-5	3.80e-3	2.30e-5	3.80e-3	3.72e-3	0.0	1.00000
9	6.30e-6	3.80e-3	6.30e-6	3.80e-3	1.35e-3	0.0	1.00000

• **Test 6 Oblique drain,** $min = -1.2,\ max = 0$**, conforming and non-conforming coarse oblique mesh**

nunkw	nnmat	sumflux	erl2	ergrad
514	4237	2.26e-9	1.21e-8	1.61e-7
574	4777	-9.02e-9	9.25e-9	1.72e-7

erflx0	erflx1	erfly0	rfly1	erflm	umin	umax
1.82e-8	1.80e-8	1.70e-8	1.07e-8	9.52e-7	-1.20	0
6.40e-8	6.27e-8	1.30e-8	2.01e-8	1.24e-6	-1.20	0

• **Test 7 Oblique barrier,** $min = -5.575$, $max = 0.575$**, conforming and non-conforming coarse oblique mesh**

nunkw	nnmat	sumflux	erl2	ergrad
514	4237	-6.01e-9	6.53e-8	
574	4777	- 3.08e-9	6.00e-8	

erflx0	erflx1	erfly0	erfly1	erflm	umin	umax
1.02e-7	6.68e-8	4.31e-8	3.00e-8	1.82e-6	-5.575	0.57499
7.42e-8	3.13e-8	1.09e-9	1.42e-8	2.02e-6	-5.575	0.57499

• **Test 8 Perturbed parallelograms,** $min = 0.$**, perturbed parallelogram mesh**

nunkw	nnmat	sumflux	umin	umax
309	2503	-1.32e-10	-1.61e-3	8.99e-2

flux0	flux1	fluy0	fluy1
2.45e-10	-1.83e-10	4.80e-1	5.11e-1

The discrete (cell-centered) solution is :

```
0.199E-10 0.171E-11 -.721E-10 -.445E-09 -.940E-10
-.479E-09 -.117E-09 -.615E-09 -.281E-09 -.201E-09
-.888E-10 0.734E-09 0.169E-08 0.183E-08 0.115E-07
-.247E-07 -.382E-08 -.218E-07 0.110E-08 -.125E-07
-.132E-07 -.439E-08 0.175E-07 0.433E-07 0.736E-07
-.574E-06 0.728E-07 -.107E-05 -.413E-06 -.393E-06
-.380E-06 -.185E-06 -.141E-06 0.108E-04 0.107E-04
-.188E-04 0.407E-04 -.487E-04 0.376E-04 0.763E-05
-.120E-04 -.262E-04 -.775E-05 -.342E-05 0.716E-03
0.243E-02 0.465E-03 0.277E-02 0.484E-02 0.103E-02
0.151E-02 -.851E-03 -.137E-03 -.116E-02 -.159E-03
0.922E-02 0.266E-01 0.398E-01 0.590E-01 0.764E-01
0.899E-01 0.712E-01 0.530E-01 0.395E-01 0.246E-01
0.716E-02 -.240E-03 -.104E-02 -.565E-03 0.168E-02
-.161E-02 0.244E-03 0.108E-02 0.435E-03 -.150E-02
0.255E-02 0.589E-03 -.680E-05 -.279E-04 -.926E-05
-.700E-04 -.217E-04 0.530E-05 -.866E-05 -.170E-04
0.406E-04 -.793E-06 0.651E-05 -.224E-06 -.482E-06
-.407E-06 -.113E-05 -.591E-06 -.111E-05 -.798E-06
0.376E-06 -.656E-06 -.177E-06 0.556E-07 -.464E-08
-.492E-08 -.184E-07 -.808E-08 -.177E-07 -.129E-07
```

```
-.141E-07 -.197E-07 -.978E-08 0.302E-09 0.598E-10
-.853E-10 -.288E-09 -.220E-09 -.346E-09 -.335E-09
-.278E-09 -.163E-09 -.155E-09 -.846E-10 -.590E-10
-.532E-11
```

- **Test 9 Anisotropy with wells,** $min = 0.$, $max = 1.$**, square uniform grid**

i	nunkw	nnmat	sumflux	umin	umax
1	309	2503	2.38e-5	-1.03e-1	1.100e+0
ref	52161	465595	-1.82e-11	-1.47e-3	1.001e+0

The discrete (cell-centered) solution is :

```
-.229E-02 -.193E-02 -.124E-02 -.458E-03 0.143E-03
0.521E-03 0.797E-03 0.854E-03 0.714E-03 0.607E-03
0.414E-03 0.733E-02 0.594E-03 -.893E-06 0.879E-03
0.156E-02 0.176E-02 0.149E-02 0.892E-03 0.182E-03
-.742E-03 -.267E-02 0.106E+00 0.384E-01 0.915E-02
0.113E-02 0.341E-03 0.933E-04 0.138E-03 0.280E-03
-.150E-02 -.775E-02 -.185E-01 0.498E+00 0.299E+00
0.139E+00 0.401E-01 0.884E-03 0.000E+00 -.226E-03
-.106E-01 -.289E-01 -.415E-01 -.352E-01 0.914E+00
0.797E+00 0.630E+00 0.426E+00 0.215E+00 0.363E-01
-.574E-01 -.712E-01 -.272E-01 0.350E-01 0.762E-01
0.106E+01 0.101E+01 0.907E+00 0.780E+00 0.641E+00
0.500E+00 0.359E+00 0.220E+00 0.926E-01 -.694E-02
-.631E-01 0.923E+00 0.964E+00 0.103E+01 0.107E+01
0.106E+01 0.964E+00 0.786E+00 0.574E+00 0.371E+00
0.205E+00 0.879E-01 0.103E+01 0.104E+01 0.103E+01
0.101E+01 0.100E+01 0.100E+01 0.999E+00 0.961E+00
0.864E+00 0.707E+00 0.511E+00 0.102E+01 0.101E+01
0.999E+00 0.997E+00 0.998E+00 0.100E+01 0.100E+01
0.101E+01 0.100E+01 0.975E+00 0.910E+00 0.999E+00
0.998E+00 0.999E+00 0.100E+01 0.100E+01 0.101E+01
0.101E+01 0.101E+01 0.102E+01 0.102E+01 0.101E+01
0.999E+00 0.100E+01 0.100E+01 0.101E+01 0.101E+01
0.101E+01 0.102E+01 0.102E+01 0.102E+01 0.102E+01
0.102E+01
```

3. Comments on the scheme

The proposed method provides satisfactory results except for test 2 for which the conjugate gradient method does not converge (no attempt has been made to optimize the preconditioner). Note that it generally does not satisfy the discrete maximum principle for anisotropic diffusion coefficients and/or distorted meshes (like numerous other methods).

Thanks to the previous numerical experiments, our method seems to be second-order accurate in the L^2 norm, almost second-order accurate in the H^1 norm and first-order accurate in the H^2 norm, except for the non-conforming meshes for which it remains second-order accurate in the L^2 norm but is only 1.5-order accurate in the H^1 norm and 0.5-order accurate in the H^2 norm. Other numerical experiments with various type of distorted meshes confirm these observations. However using *highly* distorted meshes for solving *highly* anisotropic diffusion equations can deteriorate the convergence properties.

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