# Natural coding of minimal rotations of the torus, induction and exduction

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**I - Motivations**

1. The remarkable case of dimension 1
2. One remarkable example in dimension 2: the Tribonacci word
3. Towards a generalization?
4. Unfortunately, the naive definition is trivial.

**II - A purely topological definition**

**III - The miracle: borders can be wisely assigned**

1. This is *natural* in dimension 1.
2. But not in higher dimension.

**IV - Expected properties are satisfied**

1. Stability through induction
2. Stability through exduction

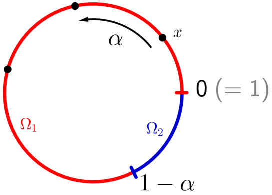
**V - Applications to Arnoux-Rauzy and Cassaigne-Selmer words**

# I. Motivations

## 1. The remarkable case of dimension 1

Consider:

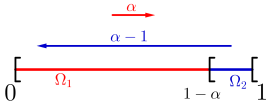
- $\star$ $\alpha \in \mathbb{R}\backslash\mathbb{Q}$
- $\star$ $R_\alpha : \begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \to & \mathbb{R}/\mathbb{Z} \\ x & \mapsto & x+\alpha \end{array}$
- $\star$ the partition $(\Omega_1, \Omega_2)$




The partition $(\Omega_1, \Omega_2)$ is remarkable:

1. The symbolic trajectory of any $x$ under the iterations of $R_\alpha$ is a Sturmian word with frequency vector $(\frac{1}{1+\alpha}, \frac{\alpha}{1+\alpha})$.

2. Once lifted to $[0, 1)$, $\Omega_1$,$\Omega_2$ are two intervals and $R_\alpha$ is the exchange of these two intervals.



## In higher dimension?

Roughly speaking

A word $w \in \{1, \ldots, d+1\}^{\mathbb{N}}$ is a natural coding of rotation with angle $\alpha \in \mathbb{R}^d$ on the d-dimensional torus if there exists a partition $\Omega_1, \ldots, \Omega_{d+1}$ of the fundamental domain $[0,1)^d$ such that:

- there exists a point whose symbolic trajectory is $w$
- the map induced by the rotation on the fondamental domain coincides with a **piecewise translation** (with pieces $\Omega_1, \ldots, \Omega_{d+1}$).

$\nwarrow$ this partition is **special!**

Old questions:

*1. Do such objects exist in dimension $d \geq 2$?*
*2. If so, describe a class of words $\mathcal{C}(d)$ such that for all $\alpha \in \mathbb{R}^d$, there exists $w \in \mathcal{C}(d)$ which is a natural coding of the rotation with angle $\alpha$.*

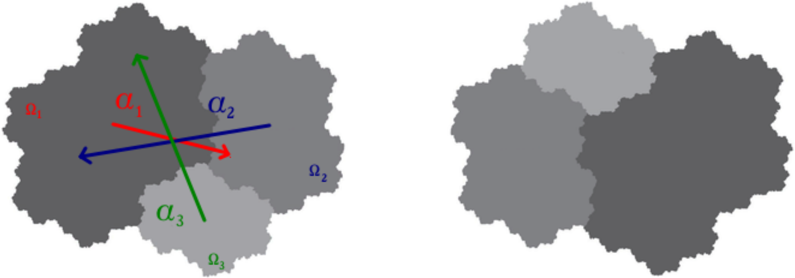
## One encouraging example in dimension 2: the Tribonacci word (1982)

The *Tribonacci word*:

$$w_{trib} := \lim_{n\to\infty} \sigma^n(1) = 1213121121312121\ldots$$

where $\sigma: \begin{array}{ccc} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{array}$

encodes an **exchange of 3 pieces**.




[Immediate consequence] It encodes the rotation by $\alpha_1$ on the torus $\mathbb{R}^2/L$, with $L := (\alpha_2 - \alpha_1)\mathbb{Z} + (\alpha_3 - \alpha_1)\mathbb{Z}$.

## One encouraging example in dimension 2: the Tribonacci word (1982)

$\longrightarrow$ The **fundamental domain** is not $[0,1)^d$, but a **fractal**!

the "Rauzy fractal"

**Some Properties:**
- connected
- regular (its closure = the closure of its interior)
- borders have measure 0.

## Towards a generalization: the class of Arnoux-Rauzy words (1991)

Consider $S_{AR} = \{\sigma_1, \sigma_2, \sigma_3\}$ where:

$\sigma_1 : \begin{array}{ccc} 1 & \to & 1 \\ 2 & \to & 12 \\ 3 & \to & 13 \end{array}$ ; $\sigma_2 : \begin{array}{ccc} 1 & \to & 21 \\ 2 & \to & 2 \\ 3 & \to & 23 \end{array}$ and $\sigma_3 : \begin{array}{ccc} 1 & \to & 31 \\ 2 & \to & 32 \\ 3 & \to & 3. \end{array}$

**def:** $w$ is an **Arnoux-Rauzy word** if there exists $(s_n)_n \in {S_{AR}}^{\mathbb{N}}$ (the "directive sequence") such that:

(i) $\sigma_1, \sigma_2$ and $\sigma_3$ appear infinitely often in $(s_n)$;

$\longrightarrow$ In this case, the sequence $(s_0 \circ \ldots \circ s_n(1))_n$ converges to an infinite word $w_0$.

(ii) $w$ has the same set of factors than $w_0$.

$\nwarrow$ *"factor"* = subword read with consecutive letters

example:

$w_{trib}$ is the limit of $(\sigma_1 \circ \sigma_2 \circ \sigma_3)^n(1))_n$ and thus, is an AR word.

Naive questions:

- *are all AR words natural codings of minimal rotations of a 2 torus?*
- *for all minimal rotation of the 2 torus, does there exist an AR word which is a natural coding?*

## The naive definition

**definition 1:** A word $w \in \{1, ..., d+1\}^{\mathbb{N}}$ is a **natural coding** of a minimal rotation $R_\alpha$ if there exist a fundamental domain $\Omega$ of $\mathbb{R}^d/\mathbb{Z}^d$, together with a partition $\Omega = \Omega_1 \sqcup ... \sqcup \Omega_{d+1}$, such that:

i. on each piece $\Omega_i$, the covered rotation coincides with a translation by a vector $\alpha_i$;

ii. the sequence $w$ is the symbolic coding of the orbit of a point $x \in \Omega$ with respect to the partition.

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*In fact...*

**Proposition 2**

*Under the axiom of choice, any word in $\{1, \dots, d+1\}^{\mathbb{N}}$ is a natural coding [under def 1] of any minimal rotation $R_\alpha$.*

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**Proposition 2**

*Under the axiom of choice, any word in $\{1, ..., d+1\}^{\mathbb{N}}$ is a natural coding [under def 1] of any minimal rotation $R_\alpha$.*

$\longrightarrow$ *What did we miss? Which* **structure** *should we require from the pieces $\Omega_i$?*

## And in the litterature?

- Natural codings of rotations of the torus are often referred to, rarely defined
- The naive definition is sometimes used...
- Some authors add working hypotheses (e.g. boundedness of the FD)

$\longrightarrow$ But we have examples of **unbounded** Rauzy fractals...

# II. A topological definition

# 1. A topological definition

Let $d \geq 1$.
Let $L \subset \mathbb{R}^d$ a lattice and $\alpha \in \mathbb{R}^d$ such that $R_{\alpha,L}$ (the rotation with angle $\alpha$ on the torus $\mathbb{R}^d/L$) is minimal.

**Definition 3**

The word $w_0 \in \{1, ..., d\}^{\mathbb{N}}$ is a natural coding of $R_{\alpha,L}$ if:

- [partition of a pseudo-fundamental domain] There exist $\Omega_1,...,\Omega_{d+1}$ **nonempty**, **open** sets of $\mathbb{R}^d$ such that:
  - the sets $\Omega_1,...,\Omega_{d+1}$ are pairwise disjoint;
  - the projection $p_L : \Omega \to \mathbb{R}^d/L$, with $\Omega := \cup\Omega_i$, is one-to-one;
  - the image set $p_L(\Omega)$ is dense in the torus $\mathbb{R}^d/L$.
- [exchange of pieces] There exist $\alpha_1, ..., \alpha_{d+1} \in \mathbb{R}^d$ such that for all indices $i \in \{1, ..., d+1\}$ and for all point $\tilde{x} \in p_L(\Omega_i) \cap R_\alpha^{-1}(p_L(\Omega))$, $r_{\Omega,L}(R_\alpha(\tilde{x})) = r_{\Omega,L}(\tilde{x}) + \alpha_i$, with $r_{\Omega,L} : p_L(\Omega) \mapsto \Omega$ the lift map.
- [a coding trajectory] There exists $\tilde{x}_0$ in $p_L(\Omega)$ such that, for all $n \in \mathbb{N}$, $R_\alpha^n(\tilde{x}_0) \in p_L(\Omega_{w_0[n]})$, where $w_0[n]$ denotes the (n+1)-th letter of $w_0$.

**def:** $((L, \alpha); (\Omega : \Omega_1, ..., \Omega_{d+1}), x_0, (\alpha_1, ..., \alpha_{d+1}))$ are the *elements* of the natural coding.

## Examples in dimension 1

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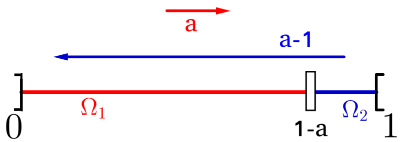
Let $a \in [0,1] \backslash \mathbb{Q}$.
The standard sturmian word with slope $a$, denoted $w_0$, is a natural coding with elements:

$$\begin{cases} L = \mathbb{Z} \\ \Omega = (0,1); \quad \Omega_1 = (0, 1-a); \quad \Omega_2 = (1-a, 1) \\ x_0 = \alpha \\ \alpha = a; \quad \alpha_1 = a; \quad \alpha_2 = a - 1. \end{cases}$$



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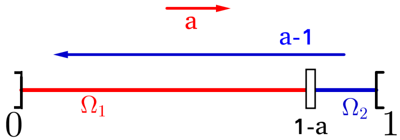



(!) In fact, **all sturmian words** with slope $a$ are natural codings... **except** those such that $S^n(w) = w_0$ for some $n > 0$.

**def:** a minimal subshift is a *natural coding* of rotation if there exists $w \in X$ which is a natural coding...

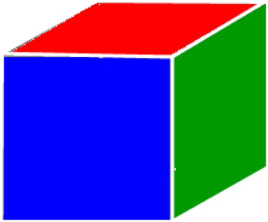
## One example in dimension 2

The Tribonacci subshift $X := \overline{\{S^n(w_{trib}) | n \in \mathbb{N})\}}$ is a natural coding of rotation...

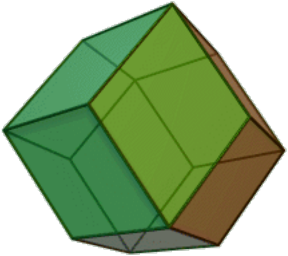
Since borders have measure 0, almost all points remain in the interior throughout time.

## Another family of examples

Minimal (d+1)-letter cubic billiard subshifts are natural codings of minimal rotation of the d-torus.



$d = 2$ (convex hexagonal parallelogon)

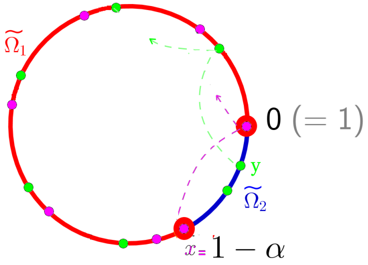


$d = 3$ (rhombic dodecahedron)

# III - The miracle: borders can be wisely assigned

## Borders assignment is natural in dimension 1

What happens if we choose the partition $\Omega_1 = [0, 1-\alpha]$, $\Omega_2 = (1-\alpha, 1)$?




**Problems:**

1. The lifted rotation does not coincide with a translation on $\Omega_1$.
2. The words 1111111 and 2111112 are factors of the coding word $w$... which cannot be sturmian (not 1-balanced). We lost the minimality of the coding subshift.

$\longrightarrow$ How to prevent this bad bahavior in higher dimension? We have an uncountable decisions to make...

## 2. Borders assignment in dimension $d \geq 2$

**Proposition 4:** Under the axiom of choice, we can **wisely assign borders**, i.e., complete each piece $\Omega_i$ so as to obtain a true partition of a true fundamental domain $\Omega' = \Omega'_1 \sqcup ... \sqcup \Omega'_d$, while preserving:

- the exchange of pieces property
- the "continuity" of the coding function $f : \Omega' \to \{1, ..., d\}^{\mathbb{N}}$:

**Lemma 5** [weak sequencial continuity]:
For all $x \in \Omega'$, there exists a sequence $(y_n)_n \in \Omega^{\mathbb{N}}$ such that $y_n \to x$ and $f(y_n) \to f(x)$.

$\longrightarrow$ In particular, we don't add new factors!

**Strengh of our definition is to fully know what happens on borders.**

# IV - Expected properties are satisfied

**Theorem A (stability by induction)**

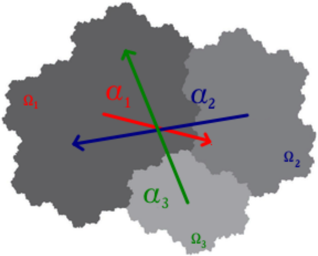
Let $w_0$ be a natural coding of a minimal rotation of a $d$-torus, $((\alpha, L); (\Omega : \Omega_1, ..., \Omega_{d+1}); x_0; (\alpha_1, ..., \alpha_{d+1}))$ its elements, and $(\Omega' : \Omega'_1, ..., \Omega'_{d+1})$ a borders assignment. Denote by $T := r_{\Omega', L} \circ R_{\alpha, L} \circ p_L$ the covered rotation.
Assume that $a$ is a letter which admits $d+1$ return words $u_1, ..., u_{d+1}$.
Then there exist a second lattice $M$ together with an angle $\beta \in \mathbb{R}^d$ such that:

1. the rotation $R_{\beta, M}$ is minimal;
2. the set $\Omega'_a$ is a fundamental domain of $M$;
3. for all $x$ in $\Omega'_a$, $T_{ind,a}(x) = x + \beta \mod M$, with $T_{ind,a}$ the first return map to the set $\Omega'_a$ of the covered rotation $T$;
4. $D_a(w_0)$, is a natural coding of the rotation of $\mathbb{R}^d / M$ with angle $\beta$, whose elements and borders assignment are explicit.

⋆ There is no need to resort to the axiom of choice a second time.
⋆ Theorem A is still true for factors instead of letters.

## On the example of Tribonacci: first return map to $\Omega_1$

$w_{trib} = 1213121121312121\ldots$



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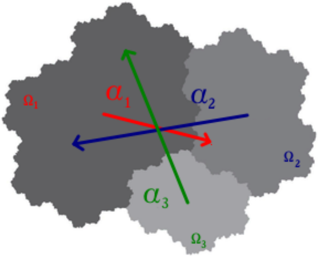
Return words to 1: $\begin{cases} u_{\mathbf{1}} &= 1 \\ u_{\mathbf{2}} &= 12 \\ u_{\mathbf{3}} &= 13 \end{cases}$



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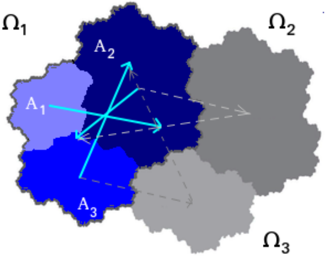


⋆ The translation vectors: $\begin{cases} \beta_1 = \alpha_1 \\ \beta_2 = \alpha_1 + \alpha_2 \\ \beta_3 = \alpha_1 + \alpha_3 \end{cases}$

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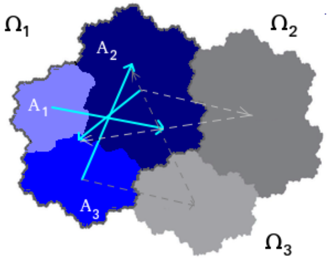


⋆ The induced pseudo-partition:
$$\begin{cases} A_1 = \Omega_1 \cap T^{-1}(\Omega_1) \\ A_2 = \Omega_1 \cap T^{-1}(\Omega_2) \cap T^{-2}(\Omega_1) \\ A_3 = \Omega_1 \cap T^{-1}(\Omega_3) \cap T^{-2}(\Omega_1) \end{cases}$$

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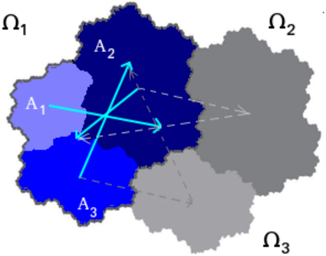


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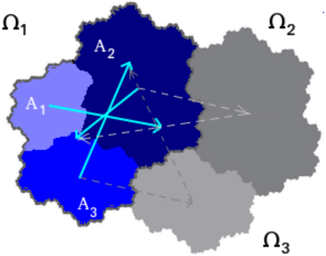


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⋆ The lattice: $M = (\beta_2 - \beta_1)\mathbb{Z} + (\beta_3 - \beta_1)\mathbb{Z}$.

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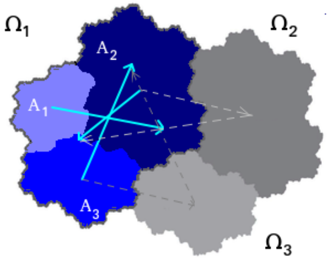


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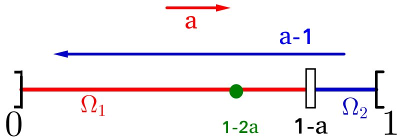
⋆ The translation vectors: $\begin{cases} \beta_1 = \alpha_1 \\ \beta_2 = \alpha_1 + \alpha_2 \\ \beta_3 = \alpha_1 + \alpha_3 \end{cases}$ ⋆ The lattice: $M = (\beta_2 - \beta_1)\mathbb{Z} + (\beta_3 - \beta_1)\mathbb{Z}$.

Finally, $D_1(w_{trib}) = 23212322\ldots$ is a natural coding of rotation with elements $((M, \beta_1), (\Omega_1 : A_1, A_2, A_3); x_0; (\beta_1, \beta_2, \beta_3))$ and borders assignment $(\Omega'_1 : A'_1, A'_2, A'_3)$.

## Aside question: what would have happened without the borders assignment?




Let $T_{ind,\Omega_\mathbf{1}}$ be the first return map to $\Omega_\mathbf{1}$.

For all $x \in \Omega_\mathbf{1}$, we have $T_{ind,\Omega_\mathbf{1}} = x + a \mod (1-a)$...

**except** for $x = 1 - 2a$ !

Contrary to the initial map, the induced map does not coincide with a rotation of the torus everywhere it could.

## A reverse construction: *the exduction*

(X,S) a dynamical system

⋆ **Induce** = study the dynamical system $(A, S_{ind,A})$ with $A \subset X$ and $S_{ind,A}$ the first return map to A.

$\sim$ zoom in + contract time

⋆ **Exduce** = study a [simple] dynamical system $(Y, T)$ such that $X \subset Y$ and $S = T_{ind,X}$.

$\sim$ zoom out + dilate time

## Natural codings of rotation are stable by exduction

**Theorem B** stability by *exduction*

Let $w$ be a natural coding of a minimal rotation of a $d$-torus and $i$ a letter. If $\sigma : \{1, ..., d+1\}^* \to \{1, ..., d+1\}^*$ is a substitution such that:
- all images of letters start with $i$ and contain no other occurrences of $i$
- the incidence [integer] matrix of $\sigma$ is invertible,

then $\sigma(w)$ is a natural coding of a minimal rotation of a $d$-torus.

Again:
- the lattice, the angle, the fundamental domain and its partition are explicitely given;
- borders assignment are inherited.

# V - Consequences

## Consequences of stability through induction

Old questions (of slide 5):

1. *Do such objects [natural codings] exist in dimension $d \geq 2$? YES*
2. *If so, describe a class of words $\mathcal{C}(d)$ such that for all $\alpha \in \mathbb{R}^d$, there exists $w \in \mathcal{C}(d)$ which is a natural coding of the rotaton with angle $\alpha$.*

**Good candidates should be classes of words stable under derivation.**

$\longrightarrow$ This is not the case of d-letter cubic billiard words for $d \geq 3$ !

$\longrightarrow$ This is the case for Arnoux-Rauzy and Cassaigne-Selmer words!

## Definition of Cassaigne-Selmer words

Consider the set of substitutions $C = \{c_1, c_2\}$ with:

$$c_1 : \begin{array}{lcl} 1 & \mapsto & 1 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 2 \end{array} \qquad \text{and} \qquad c_2 : \begin{array}{lcl} 1 & \mapsto & 2 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 3. \end{array}$$

**def:** $w \in \{1,2,3\}^{\mathbb{N}}$ is a **Cassaigne-Selmer word** if there exists $(s_n)_n \in C^{\mathbb{N}}$ (the "directive sequence") such that $((s_0 \circ \ldots \circ s_n(1))_n$ converges to $w$.

**Remark:** As Arnoux-Rauzy words, the class of Cassaigne-Selmer words is associated with a multidimensional continued fraction algorithm, and thus, can be seen as a generalization of sturmian words.

## Consequences of stability through induction

A theorem of Rauzy for bounded remainder sets gives:

**Corollary 6**

No Arnoux-Rauzy / Cassaigne-Selmer word with infinite imbalance is a natural coding of a minimal rotation of the 2-torus with a bounded pseudo-fundamental domain.

$\star$ Remaider: infinite imbalance $\iff$ unbounded Rauzy fractal

$\longrightarrow$ **True question**: does this still hold **without the assumption of boundedness??**

## Consequences of stability through induction and exduction

By studying the S-adic expression of their return words, we obtain:

**Corollary 7**

For Arnoux-Rauzy and Cassaigne-Selmer words, the property of being a natural coding of a minimal rotation of the 2-torus does not depend on any prefix of the directive sequence.

$\longrightarrow$ neither does the infinite imbalance...

Thank you!

## Summary

### I - Motivations

1. The remarkable case of dimension 1
2. One remarkable example in dimension 2: the Tribonacci word
3. Towards a generalization?
4. Unfortunately, the naive definition is trivial.

### II - A purely topological definition

### III - The miracle: borders can be wisely assigned

1. This is *natural* in dimension 1.
2. But not in higher dimension.

### IV - Expected properties are satisfied

1. Stability through induction
2. Stability through exduction

### V - Applications to Arnoux-Rauzy and Cassaigne-Selmer words