On a Quantum Kinetic Equation Linked to the Compton Effect.

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Abstract. The Compton effect, that describes the interaction via scattering between photons and electrons, is modelled by a homogeneous quantum kinetic equation. The electrons are assumed to be at nonrelativistic equilibrium, and the scattering of photons by electrons is studied. The kernel in the collision operator presents a strong singularity. The local existence in time of an entropy solution to the Cauchy problem is proven for small initial data.

Keywords. Boltzmann equation, relativistic particles, Bose distribution, COmpton scattering

1 Introduction.

The Compton effect was discovered in 1922. It takes place when high X-ray energy photons collide with electrons. This results in deflections of the particles trajectories. The incident photon emerges with longer wavelength due to some loss of energy during the interaction. These deflections, together with a change of wavelength, are known as the Compton effect. A.H. Compton found that, due to the scattering of X-rays from free electrons, the wavelength of the scattered rays is measurably longer than that of the incident light. His discovery was of special importance in 1922, when quantum mechanics was debated.
From the physical point of view, G. Cooper ([**?**]) developed the Fokker-Planck equation for the Compton scattering in a plasma without having recourse to a nonrelativistic approximation. H. Dreicer ([**?**]) presented a simple kinetic theory including the interactions between electrons and photons, and describing relaxation phenomena. A.S. Kompaneets ([**?**]) studied the thermal equilibrium of quanta and electrons. Ya. B. Zel'Dovich and

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V. Levich ([?]) studied the process of equilibrium of radiation in a totally ionized plasma.
From the mathematical point of view, R.E. Caflish and C.D. Levermore ([?]) studied the Fokker-Planck equation for the Compton scattering in a homogeneous plasma. The entropy function was used to find the equilibrium distributions. More recently, M. Escobedo and S. Mischler ([?]) stated existence results for a quantum kinetic equation with a simplified regular and bounded kernel. They studied the asymptotic behaviour of the solutions, and showed that the photon distribution function may condensate at energy zero, asymptotically in time.
This paper is devoted to prove an existence result for a quantum kinetic equation describing the Compton effect. Its kernel is kept singular as it is derived when keeping the higher order term with respect to the speed of light in the relativistic model. Like in [?] already, the boundedness of the photons entropy is not sufficient to stay in an L^1 frame. Measure solutions for the photon distribution function are expected. Moreover, the singularity in the collision kernel brings severe restrictions. Existence results to the Cauchy problem are obtained for initial data small enough, and locally in time. The entropy of the solution is controlled.

2 The model.

As considered in [?], the following quantum relativistic homogeneous equation describes the interaction via Compton scattering between a gas of low energy electrons of mass m and weakly dense photons at low temperature,

$$\frac{\partial f}{\partial t}(t,P) = Q(f,g)(P), \quad t>0,\ P \in I\!R^4, \tag{2.1}$$

with

$$\begin{aligned} Q(f,g)(P) = \frac{8c}{p^0} \int_{I\!R^4}\int_{I\!R^4}\int_{I\!R^4} & s\sigma(s,\theta)\ q(f,g)\ \delta_{\{P+P_*-P'-P'_*=0\}} \\ & \chi_2(P^0_*)\ \chi_1(P'^0)\ \chi_2(P'^0_*)\ dP'\ dP'_*\ dP_*. \end{aligned} \tag{2.2}$$

The nonnegative scalar function $f(t,P)$ (resp. $g(t,P)$) is the distribution function of photons (resp. electrons). c denotes the speed of the light.
P and P' (resp. P_* and P'_*) are the momentum of the photons (resp. electrons) before and after a collision.
A particle is determined by the pair $(X,P) \in I\!R^4 \times I\!R^4$ of position $X=(t,x)$ and momentum $P=(P^0,p)$. Let

$$p^0 = |p|,\ p'^0 = |p'|,\ p^0_* = \sqrt{|p_*|^2+m^2c^2},\ {p'_*}^0 = \sqrt{|p'_*|^2+m^2c^2}.$$

Denote by $s=(P+P_*)^2 := (P^0+P^0_*)^2 - |p+p_*|^2$, and by θ the scattering angle, given by

$$\cos\theta \ = \ \frac{(P_*-P).(P'_*-P')}{(P_*-P)^2}.$$

The differential cross section $\sigma(s,\theta)$ is a function of the energy and the scattering angle, and is given by the Klein Nishina formula ([?]). It behaves like

$$\frac{1}{2} r_0^2 (1+\cos^2\theta), \tag{2.3}$$

as $c \to \infty$, with $r_0 = \dfrac{e^2}{4\pi m c^2}$. Here, e is the charge of the electron.
The functions $\chi_1(P'^0)$, $\chi_2(P^0_*)$ and $\chi_2(P'^0_*)$ are defined by

$$\chi_1(P'^0) = \frac{1}{2p'^0}\delta_{\{P'^0=p'^0\}},\ \ \chi_2(P^0_*) = \frac{1}{2p^0_*}\delta_{\{P^0_*=p^0_*\}},\ \ \chi_2(P'^0_*) = \frac{1}{2p'^0_*}\delta_{\{P'^0_*=p'^0_*\}},$$

and

$$q(f,g) = g(p'_*)f(p')(1+\hbar f(p))(1+\tau g(p_*)) - f(p)g(p_*)(1+\hbar f(p'))(1+\tau g(p'_*)), \tag{2.4}$$

with $\tau \in \{-\hbar, 0, \hbar\}$ and $\hbar$ the Planck constant.
Here and below, the following notations are used for any function f,

$$f' = f(t,p'), \quad f_* = f(t,p_*), \quad f'_* = f(t,p'_*).$$

In equation (**??**), emission and absorption of photons have not been taken into account, so that the transitions are produced exclusively by the Compton scattering.
In order to simplify the formulas, m and $\hbar$ are taken equal to 1.
By integrating (**??**) with respect to P^0_*, P'^0 and P'^0_*, $Q(f,g)$ becomes

$$Q(f,g)(p) = c\int_{I\!R^3}\int_{I\!R^3}\int_{I\!R^3} \frac{s}{p^0 p'^0 p^0_* p'^0_*}\ \sigma(s,\theta)\ q(f,g)\ \delta_\Sigma\ dp' dp'_* dp_*,$$

where Σ is the manifold of 4-uplets (p, p_*, p', p'_*) such that,

$$\begin{aligned} p + p_* &= p' + p'_*, \\ c|p| + \frac{|p_*|^2}{2} &= c|p'| + \frac{|p'_*|^2}{2}. \end{aligned}$$

To simplify the model, only the highest-order terms with respect to c are kept in $Q(f,g)(p)$. The term $\dfrac{s}{p^0 p'^0 p^0_* p'^0_*}$ is equivalent to $\dfrac{1}{|p||p'|}$, when $c \to \infty$.
Together with (**??**), this implies that the collision operator can be approximated by

$$Q(f,g)(p) = \frac{c\ r_0^2}{2}\int_{I\!R^3}\int_{I\!R^3}\int_{I\!R^3} \frac{(1+\cos^2\theta)}{|p||p'|}\ q(f,g)\ \delta_\Sigma\ dp' dp'_* dp_*.$$

The electrons are assumed to be at nonrelativistic equilibrium, i.e.

$$\tau = 0 \ \text{ and } \ g(p) = e^{-\frac{|p|^2}{2c}}.$$

Then,

$$q(f,g) = g(p'_*)f(p')(1+f(p)) - f(p)g(p_*)(1+f(p')).$$

The collision integral becomes

$$Q(f,g)(p) = \frac{c\, r_0^2}{2} \int_{I\!R^3} \frac{(1+\cos^2\theta)}{|p||p'|}\, e^{|p'|}\, q(f) \left(\int_{I\!R^3} \int_{I\!R^3} \delta_\Sigma\, e^{-\frac{|p_*|^2}{2c}}\, dp_*\, dp'_* \right) dp',$$

with

$$q(f) = e^{-|p|} f(p')(1+f(p)) - e^{-|p'|} f(p)(1+f(p')).$$

It can be simplified in the following way.

Lemma 2.1
Denote by

$$S(p,p') = \int_{I\!R^3} \int_{I\!R^3} \delta_\Sigma\, e^{-\frac{|p_*|^2}{2c}}\, dp_*\, dp'_*, \quad A = |p'| - |p| + \frac{|p-p'|^2}{2c}, \quad w = p' - p.$$

Then

$$S(p,p') = \frac{2\pi c^2}{|w|} e^{-\frac{A^2 c}{2|w|^2}}.$$

Lemma **??** is proven in [**?**].
It is then assumed that the photon distribution function is radial. Denote by $k = |p|$, $k' = |p'|$, $F(t,k) = k^2 f(t,k)$. The quantum kinetic homogeneous equation describing the interaction between photons and electrons is then

$$\frac{\partial F}{\partial t}(t,k) = Q(F)(t,k),$$

where

$$Q(F)(t,k) = \int_0^\infty b(k,k')[F'(k^2+F)e^{-k} - F(k'^2+F')e^{-k'}]\, dk',$$

$$b(k,k') = \frac{2c^3 r_0^2 \pi^2}{k\ k'} \int_0^\pi (1+\cos^2\theta)\, \frac{\sin\theta}{|w|}\, e^{-\frac{A^2 c}{2|w|^2} + k'}\, d\theta, \qquad (2.5)$$

$$A = k' - k + \frac{|w|^2}{2c}, \quad |w|^2 = k^2 + k'^2 - 2kk'\cos\theta.$$

Sections 3 and 4 prove the existence of a solution F to the Cauchy problem,

$$\frac{\partial F}{\partial t}(t,k) = Q(F)(t,k),\ \ t \in [0,T],\ k \geq 0, \qquad F(0,k) = F_0(k),\ k \geq 0, \quad (2.6)$$

where the initial datum F_0 is given. The following a priori estimates for (**??**) hold.

Proposition 2.1
Let $M(F)(t) = \int_0^\infty F(t,k)\ dk$ *be the total number of photons at time* t.
Then,

$$\frac{d}{dt} M(F)(t) = 0.$$

Proof.

Proposition **??** follows from an integration of (**??**) with respect to k and the change of the variable k by k'.

Proposition 2.2
The entropy, defined by

$$H(F)(t) = \int_0^\infty [(k^2+F(t,k))\ln(k^2+F(t,k)) - F(t,k)\ln F(t,k) - k^2\ln k^2 - kF(t,k)]\, dk,$$

is a non-decreasing function of time.

Proof.

Multiply equation (**??**) by $ln\Big(\frac{(k^2+F)e^{-k}}{F}\Big)$,
so that

$$2\,\frac{d}{dt}H(F) = \int_0^\infty \int_0^\infty b(k,k')\; j(F(k'^2+F')e^{-k'}, F'(k^2+F)e^{-k})\; dk'\; dk.$$

Here,

$$j(u,v) = (v-u)(\ln v - \ln u) \quad \text{if} \quad u>0,\; v>0,$$
$$j(u,v) = 0 \quad \text{if} \quad u=v=0,$$
$$j(u,v) = +\infty \quad \text{elsewhere.}$$

The nonnegativity of b and j implies the result. □

Proposition 2.3
*There exists a constant $C>0$ such that for any solution F to (**??**), the following inequalities hold,*

$$M(kF) \le C(1 + M(F) - H(F)), \tag{2.7}$$

$$|H(F)| \le M((1+k)F). \tag{2.8}$$

A proof of Proposition **??** is given in [**?**].

3 Main results.

As recalled in the introduction, M. Escobedo and S. Mischler ([**?**]) proved the existence and uniqueness of a measure solution of the problem (**??**) for three different types of the cross section b.
(i) $b \ge 0$ and bounded,
(ii) $b(k,k') = e^{\eta k}e^{\eta k'}\sigma(k'-k)$, for some $\eta \in (0,1)$, with the function σ satisfying

$$\sigma_* e^{-\nu|z|^\gamma} \le \sigma(z) \le \sigma_*, \quad z \in I\!R,$$

for some $\sigma_* > 0$, $\nu > 0, \gamma \in [0,1]$.
(iii) $0 \le b(k,k')e^{-\eta k}e^{-\eta k'}$, bounded for some $\eta \in [0,1]$.

In this paper, the cross-section b defined in (**??**) does not satisfy any of these three conditions. $b(k,k')$ is singular at $k = k' = 0$. Still, we have remarked that measure solutions are expected for the Cauchy problem. We need to give a sense to

$$\int_s^t \int_0^\infty \phi(t,k)\ Q(F)(\tau,k)\ dkd\tau,$$

for any continuous and bounded test function ϕ. The a priori estimates of Propositions **??**-**??** are not sufficient to obtain finite

$$\Big| \int_s^t \int_0^\infty \phi(t,k)\ Q(F)(\tau,k)\ dkd\tau \Big|,$$

for any test function ϕ. Denote by $M^1(I\!\!R_+)$ the space of bounded measures on $I\!\!R_+$, and by $m(k) := \frac{k^2}{e^k - 1}$. In order to deal with solutions F to (**??**) in $C([0,T], M^1(I\!\!R_+))$, the following bound on F is required.

Proposition 3.1
Let $F \in C([0,T], M^1(I\!\!R_+))$ be such that $F(\tau,.) \neq m + \alpha\delta_{k=0},\ \alpha \in I\!\!R_+$ for all τ in $[0,T]$. If for any continous and bounded function ϕ with second order with respect to k in the neighborhood of 0 and for any interval $J \subset [0,T]$,

$$\Big| \int_J \int_0^\infty \phi(\tau,k)\ Q(F)(\tau,k)\ dk\ d\tau \Big| \ <\ +\infty,$$

then

$$\int_0^\infty \frac{F}{k}(\tau,k)\ dk\ <\ +\infty, \quad a.a.\ \tau \in [0,T].$$

Remark 3.1
The condition "$F(\tau,.) \neq m + \alpha\ \delta_{k=0},\ \alpha \in I\!\!R_+$ for all τ in $[0,T]$" is not restrictive in the frame of the existence (and not for the uniqueness) of a solution to the Cauchy problem. At a first possible time t_* such that $F(t_*,.) = m + \alpha\delta_{k=0}$, we extend the solution obtained on $[0,t_*]$ by

$$F(\tau,.) = F(t_*,.), \quad \tau \in [t_*,T].$$

Remark 3.2
This boundedness of $\frac{F}{k}$ is important to establish the local existence result developped in the following theorem.

Let $c_1,\ c_2,\ c_3$ be the constants independant of F_0 defined further on in Lemmas **??**, **??**, **??**.

Theorem 3.1
Let $T > 0$ and the initial datum F_0 satisfy

$$U := \Big(\int_0^{\frac{c_1}{8\pi}} (1+k) \frac{F_0(k)}{k} dk + \int_{\frac{c_1}{8\pi}}^{\infty} F_0(k) dk \Big) \ \exp(T \max\{c_1, \frac{32\pi^{3/2}}{c_1} + 8\pi \frac{c_3}{c_2}\}) \\ \leq \frac{c_1}{c_2} \frac{c_1}{c_1 + 8\pi}. \tag{3.1}$$

*Then, there exists a nonnegative solution $F \in C([0,T], M^1(0,+\infty))$ to the problem (**??**), such that*

$$\frac{F(t,k)}{k} \in L^\infty_+(0,T; M^1(0,+\infty))$$

and

$$\int_0^\infty kF(t,k)dk < d, \ \text{a.a. } t > 0,$$

for some constant $d > 0$.
Moreover, if the initial datum F_0 has a finite entropy, then F is an entropy solution in the sense that

$$H(F)(t) \geq \alpha, \quad \text{a.a. } t > 0, \tag{3.2}$$

for some constant α.

Remark 3.3 *The solution F to the problem (**??**) in theorem **??** is meant in a weak sense, for continuous and bounded test functions, with second order with respect to k in the neighborhood of 0.*

Proof of Proposition **??**.

Let

$$\mathcal{I}(\phi) = \int_J \int_0^\infty \phi(\tau,k) \ Q(F)(\tau,k) \ dkd\tau.$$

It can be written as

$$\begin{aligned} \mathcal{I}(\phi) &= \int_J \int_0^\infty \int_0^\infty \frac{\phi(\tau,k)}{kk'} h(k,k')[F'(k^2+F)e^{-k} - F(k'^2+F')e^{-k'}] \ dk'dkd\tau \\ &= \mathcal{I}_1(\phi) + \mathcal{I}_2(\phi) + \mathcal{I}_3(\phi) + \mathcal{I}_4(\phi), \end{aligned}$$

with

$$h(k,k') = \int_0^\pi \frac{(1+\cos^2\theta)\sin\theta}{|w|} \ \exp(-\frac{A^2c}{2|w|^2} + k') \ d\theta,$$

$$A = k' - k + \frac{|w|^2}{2c}, \quad |w|^2 = k^2 + k'^2 - 2kk'\cos\theta,$$

and,

$$\mathcal{I}_1(\phi) = -\int_J\int_0^\infty\int_0^\infty \frac{\phi(\tau,k)}{k}\ h(k,k')\ F\ k'e^{-k'}\ dk'\ dk\ d\tau,$$

$$\mathcal{I}_2(\phi) = \int_J\int_0^\infty\int_0^\infty \frac{\phi(\tau,k)}{kk'}\ h(k,k')[FF'(1-e^{-k'})]\ dk'\ dk\ d\tau,$$

$$\mathcal{I}_3(\phi) = \int_J\int_0^\infty\int_0^\infty \frac{\phi(\tau,k)}{kk'}\ (h(k,k')-h(k,0))\ [F'((k^2+F)e^{-k}-F)]\ dk'\ dk\ d\tau,$$

$$\mathcal{I}_4(\phi) = \int_J\int_0^\infty \frac{\phi(\tau,k)}{k}\ h(k,0)\ [(k^2+F)e^{-k}-F]\Big(\int_0^\infty \frac{F'}{k'}\ dk'\Big)\ dk\ d\tau.$$

Lemma 3.1
Let $F \in C([0,T],M^1(\mathbb{R}_+))$ be such that the mass $M(F)(t)$ is uniformly bounded from above.
For any continuous and bounded function ϕ of second order with respect to k in the neighborhood of 0,

$$|\mathcal{I}_j(\phi)| < \infty, \quad 1 \le j \le 3.$$

Proof of lemma ??.

$|\mathcal{I}_1(\phi)|$ and $|\mathcal{I}_2(\phi)|$ are finite because the second order with respect k of ϕ in the neighborhood of 0 deals with the singularity of $h(k,k')$ at $k=k'=0$.

$\mathcal{I}_3(\phi)$ can be written as

$$\mathcal{I}_3(\phi) = \int_J\int_0^\infty\int_0^\infty\int_0^1 F'[ke^{-k}+F\ \frac{e^{-k}-1}{k}]\phi(\tau,k)\ \frac{\partial h}{\partial k'}(k,\gamma k')\ d\gamma\ dk\ dk'\ d\tau.$$

Here again, the second order with respect to k of ϕ in the neighborhood of 0 deals with the singularity of $\frac{\partial h}{\partial k'}$.
Thus, $|\mathcal{I}_3(\phi)|$ is finite.

So, proving proposition **??** comes back to prove the following lemma.

Lemma 3.2

For any continuous and bounded function ϕ vanishing in a neighborhood of 0 with respect to k,

$$|\mathcal{I}_4(\phi)| < \infty \Longrightarrow \int_0^\infty \frac{F}{k}(\tau,k)\ dk\ <\ +\infty, \quad a.a.\ \tau \in [0,T].$$

Proof.

Consider

$$\mathcal{I}_4(\phi) = \int_J \int_0^\infty \psi(\tau,k)\ (F(\tau,k) - m(k))\ \Big(\int_0^\infty \frac{F'}{k'}\ dk'\Big)\ dk\ d\tau,$$

where

$$\psi(\tau,k) = \frac{8}{3}\frac{\phi(\tau,k)}{k^2}\ \exp(-\frac{1}{2}(-1+\frac{1}{2}k)^2)\ (e^{-k}-1).$$

It is sufficient to prove that for all $t \in [0,T]$, there exists a neighborhood V_t of t such that for almost all $s \in V_t$,

$$\int_0^\infty \frac{F(s,k')}{k'}\ dk'$$

is finite. We prove it by contradiction. Let

$$S := \Big\{ t \in [0,T];\quad \int_0^\infty \frac{F(t,k')}{k'}\ dk' = +\infty \Big\},$$

and

$$F(t,.) = L(t,.) + H_c(t,.) + H_d(t,.)$$

be the decomposition of the bounded measure $F(t,.)$. $L(t,.)$ is the continuous absolute Lebesgue part of $F(t,.)$. $H_c(t,.)$ and $H_d(t,.)$ are respectively the continuous singular part and the discrete singular part of $F(t,.)$. $H_d(t,.)$ can be written as

$$H_d(t,.) = \sum_{j\geq 1} a_j(t)\delta_{k_j},$$

with a decreasing sequence of positive coefficients $a_j(t)$. We assume that there exists $t \in [0,T]$ such that for every neigborhood V_t of t, $|V_t \cap S| > 0$. The three following cases are considered. Either $\int_0^\infty H_c(t,k)\ dk > 0$, or $H_c(t,\cdot) = 0$ and $\int |\ L(t,k) - m(k)\ |\ dk > 0$, or $H_c(t,\cdot) = 0$, $L(t,\cdot) = m$.
In the first case, $\int_\alpha^\infty H_c(t,k)\ dk > 0$ for some $\alpha > 0$. The support of H_c restricted to $]\alpha, +\infty[$ is included in a denumerable union of open intervals with small arbitrarily measure. In particular, it is included in $\cup I_n$, with

$$\int_{\cup I_n} m\ dk < \frac{1}{2}\int_\alpha^\infty H_c(t,k)\ dk.$$

If for every integer n,

$$\int_{I_n} H_c(t,k)\ dk < \int_{I_n} m(k)\ dk,$$

then

$$\begin{aligned}\int_\alpha^\infty H_c(t,k)\ dk &< \sum_n \int_{I_n} H_c(t,k)\ dk \\ &\leq \int_{\cup I_n} m\ dk < \frac{1}{2}\int_\alpha^\infty H_c(t,k)\ dk.\end{aligned}$$

Thus, there exists an interval $I \subset]\alpha, +\infty[$ such that

$$\int_I (H_c(t,k) + L - m)\ dk > 0.$$

By continuity in time of F, this is also true for a neighborhood V_t of t. Restricting eventually V_t, there exists an integer n such that

$$\sum_{j>n} a_j(s) < \frac{1}{4} \int_I (H_c + L - m)(s,k)\ dk, \quad s \in V_t.$$

We construct a continuous function ψ, which is equal to 1 on I, vanishes quickly on the boundary of I and on some small neigborhoods of eventual k_i, $1 \leq i \leq n$, being in I. For this function ψ, $\mathcal{I}_4 = +\infty$.

In the second case where $H_c(t,.) = 0$ and $\int_0^\infty |L(t,k) - m(k)|\ dk > 0$, the inequality $\int_\alpha^{+\infty} \mid L(t,k) - m(k) \mid dk > 0$ holds for some $\alpha > 0$. By the continuity in time of F, V_t can be restricted so that

$$\int_0^\infty |L(s,k) - m(k)|\ dk > 0, \quad a.a.\ s \in V_t.$$

For almost all $s \in V_t$, there exists a set I_s of positive measure in $[\alpha, \infty[$, such that

$$a(s,k) := L(s,k) - m(k) \neq 0, \quad k \in I_s.$$

Let $n(t)$ be such that

$$\sum_{j>n(t)} a_j(t) < \frac{1}{4} \int \mid L(t,k) - m(k) \mid dk.$$

Let $\mathcal{X}$ be the function defined by

$$\begin{aligned} \mathcal{X}(s,k) &= \operatorname{sgn}(a(s,k)), \quad s \in V_t,\ k \in I_s, \\ \mathcal{X}(s,k) &= 0 \quad otherwise. \end{aligned}$$

Let $\bar{\mathcal{X}}(s,k)$ be the characteristic function of the complementary of the support of $H_c(t,.) + H_d(t,.)$. Let

$$G_n(s) = \min\Big\{ \int_0^\infty \frac{F(s,k')}{k'}\ dk', n \Big\}.$$

Then $G_n = n$ on V_t. The function a belonging to $L^1((0,T) \times I\!R_+)$, let $\varepsilon > 0$ be such that

$$\int_\Omega |a| < \frac{1}{4} \int_{V_t \times I\!R_+} |a|, \quad |\Omega| < \varepsilon.$$

Take ψ continuous such that $|\psi| \leq 1$, $\psi = \bar{\mathcal{X}}\mathcal{X}$ outside of a set Ω of measure smaller that ε, and which vanishes at k_1, ..., k_n. For such a function ψ,

$$\Big| \int_{(0,T)\times I\!R_+} (\psi(F-m)G_n - \bar{\mathcal{X}}\mathcal{X}(F-m)G_n)\ d\tau dk \Big|$$

$$\begin{aligned} &\leq (2n+1) \int_\Omega |a|\ d\tau dk \\ &\leq \frac{2n+1}{4} \int_{V_t \times I\!R_+} |a|\ d\tau dk \\ &\leq \frac{2n+1}{4n} \int_{(0,T)\times I\!R_+} \bar{\mathcal{X}}\mathcal{X}(F-m)\ G_n\ d\tau dk. \end{aligned}$$

Thus,

$$\int_{(0,T)\times I\!R_+} \bar{\mathcal{X}}\mathcal{X}\ (F-m)\ G_n\ d\tau\ dk \leq \frac{4n}{2n-1} \int_{(0,T)\times I\!R_+} \psi\ (F-m)\ G_n\ d\tau\ dk.$$

Passing to the limit in the previous inegality when $n \to +\infty$ leads to $\mathcal{I}_4 = +\infty$.

In the third case where $H_c(t,.) = 0$, $\displaystyle\int_0^\infty |L(t,k) - m(k)|\ dk = 0$ and $H_d(t,.) \neq 0$, let n be such that

$$\sum_{j>n} a_j < \frac{1}{4}\ a_1.$$

Let I be a neighborhood of k_1 such that $k_2, \ldots, k_n \notin I$, so that

$$\int_I (H_d(t,.) - a_1\ \delta_{k_1}) < \frac{1}{4}\ a_1.$$

Restricting I if necessary, and by continuity in time of $F(t,.)$, there exists a neighborhood W_t of t such that

$$\int_I (F(s,k) - m(k))\ dk > \frac{1}{2}\ a_1, \quad s \in W_t.$$

Choose a continuous function ψ which approaches the characteristic function of I and is equal to 0 outside of I. Then,

$$\int_{W_t} \Big(\int_0^\infty \psi(s,k)\ (F(s,k) - m(k))\ dk \Big) \Big(\int_0^\infty \frac{F(s,k')}{k'}\ dk' \Big)\ ds = +\infty.$$

□

In this paper, solutions $F \in C(0,T; M^1(I\!R_+))$ to (**??**), such that $\dfrac{F(t,k)}{k} \in L^\infty_+(0,T; M^1(I\!R_+))$ are considered .

4 Proof of Theorem ??.

Compared to the existence results in [?], the main problem here is to reach the frame $\frac{F(t,k)}{k} \in L^\infty(0,T;M^1(I\!R_+))$. Hence, the function $G(t,k) = \frac{F(t,k)}{k}$ is introduced. The problem to be solved is

$$\frac{\partial G}{\partial t} = 2c^3 r_0^2 \pi^2 \int_0^\infty h(k,k')[G'(1+\frac{G}{k})e^{-k} - \frac{G}{k}(k'+G')e^{-k'}]\, dk', \quad t \in [0,T],\ k>0,$$
$$G(0,k) = \frac{F_0(k)}{k}, \tag{4.1}$$

with $G \in L^\infty(0,T;M^1(I\!R_+))$. Here,

$$h(k,k') = \int_0^\pi \frac{(1+\cos^2\theta)\sin\theta}{|w|}\ e^{-\frac{A^2 c}{2|w|^2}+k'}\ d\theta.$$

For the sake of simplicity, $2c^3 r_0^2 \pi^2$ is taken equal to 1.
The proof of the theorem splits into three parts. The first part provides bounds on h, that will be useful in dealing with its singularity. The second part proves the existence of a nonnegative solution $F \in C([0,T], M^1(0,\infty))$ to (**??**), such that $\frac{F(t,k)}{k} \in L^\infty(0,T;M^1(I\!R_+))$. The third part states the entropy feature of F.

4.1 Technical bounds on the cross-section h.

Lemma 4.1
There exists a constant $c_1 > 0$ such that,

$$4c_1 < \int_0^\infty h(k,k')k'e^{-k'}dk', \quad 0<k<c_1.$$

Proof.

Let l be the positive limit of $\int_0^\infty h(k,k')k'e^{-k'}dk'$ when $k \to 0$. Then,

$$\int_0^\infty h(k,k')k'e^{-k'}dk' > \frac{l}{2}, \quad k<\eta, \text{ for some } \eta>0.$$

Choose $c_1 = \min\{\frac{l}{8}, \eta\}$. □

Lemma 4.2
There exists a constant $c_2 > 0$ such that

$$h(k,k')(e^{-k} - e^{-k'}) \le c_2, \quad 0<k<c_1.$$

Proof.

For $0 < k' < 2c_1$,

$$h(k,k')|e^{-k} - e^{-k'}| \leq d \, \frac{|e^{-k} - e^{-k'}|}{|k' - k|} \leq d, \quad d > 0.$$

For $k' > 2c_1$,

$$\begin{aligned} h(k,k')(e^{-k} - e^{-k'}) &\leq \int_0^\pi \frac{(1 + \cos^2\theta)\sin\theta}{|w|} \, e^{-\frac{A^2 c}{2|w|^2} + k' - k} \, d\theta \\ &\leq \frac{d}{|k' - k|}, \quad d > 0, \end{aligned}$$

since $-\dfrac{A^2 c}{2|w|^2} + k' - k \leq 0$. □

Lemma 4.3
There exists a constant $c_3 > 0$ such that

$$h(k,k')e^{-k} \leq c_3, \quad k' \in (0,\infty), \quad k > c_1.$$

Proof.

First, $-\dfrac{A^2 c}{2|w|^2} + k' - k \leq 0$, so that $e^{-\frac{A^2 c}{2|w|^2} + k' - k} \leq 1$. Then,

$$h(k,k')e^{-k} \leq \frac{2\pi}{|k - k'|}, \quad k > c_1, \quad k' \leq \frac{c_1}{2}.$$

Moreover,

$$h(k,k')e^{-k} \leq \frac{2\pi}{\sqrt{k}\sqrt{k'}}, \quad k > c_1, \quad k' \geq \frac{c_1}{2}.$$

Choose $c_3 = \dfrac{4\pi}{c_1}$. □

Truncated cross-section h_n will be used in the existence proof in order to avoid the singularity of h at $k = k' = 0$ in the approximation procedure. Let $(h_n)_{n \in I\!N^*}$ be defined such that

$$h_n(k,k') = h(k,k') \; {}_{\{k \in [\frac{1}{n}, n]\}}.$$

Remark 4.1 *The constants c_1, c_2 and c_3 are linked to the function $h(k,k')$.*

4.2 Existence of a solution to the problem.

Throughout the proof, fixed point arguments will be used in the convex set K of nonnegative measures G, such that

$$\int_0^\infty G(t,k)dk \leq \frac{c_1}{c_2}, \quad a.a. \; t \in [0,T].$$

First step : proof of the existence and unicity of the truncated equation (**??**).

Let $g(t,k) : [0,T] \times [0,\infty[\to I\!R^+$ such that $\displaystyle\int_0^\infty g(t,k)dk \leq \frac{c_1}{c_2}$ and $n \geq 1$ be given. In this first step, the problem

$$\begin{aligned} \frac{\partial G_n}{\partial t} &= e^{-k}\int_0^\infty h_n(k,k')G_n' dk' + \frac{G_n}{k}\int_0^\infty h_n(k,k')(e^{-k}-e^{-k'})g' dk' \\ &\quad - \frac{G_n}{k}\int_0^\infty h_n(k,k')k' e^{-k'} dk', \\ G_n(0,k) &= \frac{F_0(k)}{k}, \quad k \geq 0, \end{aligned} \tag{4.2}$$

with unknown G_n will be solved in K.
For $u \in L^\infty_+(0,T;L^1(I\!R^+))$, define $\mathcal{F}(u) = U$ as the solution to

$$\begin{aligned} \frac{\partial U}{\partial t} &= e^{-k}\int_0^\infty h_n(k,k')u(t,k')dk' + \frac{U}{k}\int_0^\infty h_n(k,k')(e^{-k}-e^{-k'})g(t,k')dk' \\ &\quad - \frac{U}{k}\int_0^\infty h_n(k,k')k' e^{-k'} dk', \\ U(0,k) &= \frac{F_0(k)}{k}, \quad k \geq 0. \end{aligned} \tag{4.3}$$

It follows from the exponential form of U, that $U(t,k) \geq 0 \;\; a.a.\ t \geq 0, \;\; k \geq 0$. Integrating (**??**) with respect to the variable k implies that

$$\frac{\partial}{\partial t}\int_0^\infty U(t,k)dk \leq \lambda_n \|u\|_{L^\infty(0,T;L^1(R_+))} + \tilde{\lambda}_n \; c_1 \int_0^\infty U(t,k)dk.$$

The constants λ_n and $\tilde{\lambda}_n$ take into account the compact support $[\frac{1}{n}, n]$ with respect to k of h_n. And so, using Gronwall's argument, the function U belongs to $L^\infty(0,T;L^1(I\!R_+))$.

Analogously, for any u, $\tilde{u} \in L^1_k(0,+\infty)$, the corresponding solutions U, $\tilde{U}$ to (**??**) satisfy

$$\frac{\partial}{\partial t}|U-\tilde{U}| \leq c_3 \int_0^\infty |u-\tilde{u}| \; dk - 4\pi \; |U-\tilde{U}|.$$

Hence,

$$\int_0^\infty |U-\tilde{U}|(t,k) \; dk \leq \frac{c_3}{4\pi}(1-e^{-4\pi t})\int_0^\infty |u-\tilde{u}|(t,k) \; dk, \quad a.a.\ t \in [0,\tilde{T}].$$

This implies that for $\tilde{T}$ small enough only depending on n, the map $\mathcal{F}$: $u \to U$ is a contraction from $L^\infty(0,\tilde{T};L^1(I\!R^+))$ into itself. Then, $\mathcal{F}$ admits a unique fixed point on $L^\infty(0,\tilde{T};L^1(I\!R^+))$, denoted by U. The argument can be iterated to obtain a unique solution $U = G_n$ in $L^\infty(0,T;L^1(I\!R^+))$ to (**??**). □

Furthermore, G_n belongs to the convex set K. Indeed, it follows from (**??**) and Lemma **??**, that, for $0 < k < \frac{c_1}{8\pi}$,

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^{\frac{c_1}{8\pi}} G_n(t,k)dk &\leq \int_0^{\frac{c_1}{8\pi}} e^{-k}\int_0^\infty h(k,k')G_n'dk'dk \\ &+ \int_0^{\frac{c_1}{8\pi}} \frac{G_n}{k}\int_0^\infty h_n(e^{-k}-e^{-k'})g'dk'dk \\ &- 2c_1\int_0^{\frac{c_1}{8\pi}} \frac{G_n}{k}dk.\end{aligned}$$

Then, by Lemma **??**,

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^{\frac{c_1}{8\pi}} G_n(t,k)dk &\leq \int_0^{\frac{c_1}{8\pi}} e^{-k}\int_0^\infty h(k,k')G_n'dk'dk \;-\; c_1\int_0^{\frac{c_1}{8\pi}} \frac{G_n}{k}dk \\ &+ c_2\int_0^{\frac{c_1}{8\pi}} \frac{G_n}{k}\Big(\int_0^\infty g'dk' - \frac{c_1}{c_2}\Big)dk \\ &\leq \int_0^{\frac{c_1}{8\pi}} e^{-k}\int_0^\infty h(k,k')G_n'dk'dk \;-\; c_1\int_0^{\frac{c_1}{8\pi}} \frac{G_n}{k}dk,\end{aligned}$$

since $g \in K$.
And so,

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^{\frac{c_1}{8\pi}} G_n(t,k)dk &\leq 4\pi\int_0^{\frac{c_1}{8\pi}} \frac{1}{\sqrt{k}}dk\int_0^\infty \frac{G_n'}{\sqrt{k'}}dk' - c_1\int_0^{\frac{c_1}{8\pi}} \frac{G_n}{k}dk \\ &\leq \sqrt{2c_1\pi}\int_0^\infty \frac{G_n}{\sqrt{k}}dk - c_1\int_0^{\frac{c_1}{8\pi}} \frac{G_n}{k}dk \\ &\leq \int_0^{\frac{c_1}{8\pi}} \frac{G_n}{\sqrt{k}}(\sqrt{2c_1\pi} - \frac{c_1}{\sqrt{k}})dk + \sqrt{2c_1\pi}\int_{\frac{c_1}{8\pi}}^\infty \frac{G_n}{\sqrt{k}}dk \\ &\leq \sqrt{2c_1\pi}\int_{\frac{c_1}{8\pi}}^\infty \frac{G_n}{\sqrt{k}}dk.\end{aligned}$$

Hence,

$$\frac{\partial}{\partial t}\int_0^{\frac{c_1}{8\pi}} G_n(t,k)dk \leq \frac{32\pi^{3/2}}{c_1}\int_{\frac{c_1}{8\pi}}^\infty kG_ndk. \tag{4.4}$$

Using (**??**) and Lemmas **??** and **??** implies that

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^\infty kG_n(t,k)dk &= \int_0^\infty G_n \int_0^\infty h_n(k,k')(e^{-k}-e^{-k'})g'dk'dk \\
&= \int_0^{\frac{c_1}{8\pi}} G_n \int_0^\infty h_n(k,k')(e^{-k}-e^{-k'})g'dk'dk \\
&+ \int_{\frac{c_1}{8\pi}}^\infty G_n \int_0^\infty h_n(k,k')(e^{-k}-e^{-k'})g'dk'dk \\
&\le c_2\int_0^{\frac{c_1}{8\pi}} G_n \int_0^\infty g'dk'dk + \int_{\frac{c_1}{8\pi}}^\infty G_n \int_0^\infty h_n(k,k')e^{-k}g'dk'dk \\
&\le c_1\int_0^{\frac{c_1}{8\pi}} G_n dk + c_3\frac{c_1}{c_2}\int_{\frac{c_1}{8\pi}}^\infty G_n dk.
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t}\int_0^\infty kG_n(t,k)dk \le c_1\Big(\int_0^{\frac{c_1}{4\pi}} G_n dk + \frac{8\pi c_3}{c_1\ c_2}\int_{\frac{c_1}{8\pi}}^\infty kG_n dk\Big). \tag{4.5}$$

It follows from (**??**) and (**??**) that

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\int_0^{\frac{c_1}{8\pi}}(1+k)G_n\ dk + \int_{\frac{c_1}{8\pi}}^\infty kG_n dk\Big) &\le c_1\int_0^{\frac{c_1}{8\pi}} G_n dk + \Big(\frac{32\pi^{3/2}}{c_1}+8\pi\frac{c_3}{c_2}\Big)\int_{\frac{c_1}{8\pi}}^\infty kG_n dk \\
&\le \max\{c_1, \frac{32\pi^{3/2}}{c_1}+8\pi\frac{c_3}{c_2}\}\Big(\int_0^{\frac{c_1}{8\pi}}(1+k)G_n dk + \int_{\frac{c_1}{8\pi}}^\infty kG_n dk\Big).
\end{aligned}$$

And so, by Gronwall's argument,

$$\begin{aligned}
&\int_0^{\frac{c_1}{8\pi}}(1+k)G_n(t,k)dk + \int_{\frac{c_1}{8\pi}}^\infty kG_n dk \\
&\quad\le \Big(\int_0^{\frac{c_1}{8\pi}}(1+k)G(0,k)dk + \int_{\frac{c_1}{8\pi}}^\infty kG(0,k)dk\Big)\ \exp(T\max\{c_1, \frac{32\pi^{3/2}}{c_1}+8\pi\frac{c_3}{c_2}\}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^\infty G_n(t,k)dk &\le \int_0^{\frac{c_1}{8\pi}}(1+k)G_n dk + \int_{\frac{c_1}{8\pi}}^\infty G_n dk \\
&\le \int_0^{\frac{c_1}{8\pi}}(1+k)G_n dk + \frac{8\pi}{c_1}\int_{\frac{c_1}{8\pi}}^\infty kG_n dk.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_0^\infty G_n(t,k)\ dk \\
&\quad\le (1+\frac{8\pi}{c_1})\Big(\int_0^{\frac{c_1}{8\pi}}(1+k)\ G(0,k)\ dk + \int_{\frac{c_1}{8\pi}}^\infty k\ G(0,k)\ dk\Big)e^{T\max\{c_1,\frac{32\pi^{3/2}}{c_1}+8\pi\frac{c_3}{c_2}\}} \\
&\quad:= (1+\frac{8\pi}{c_1})\ U,
\end{aligned}$$

which implies that $G_n \in K$ by assumption (**??**) of Theorem **??**. $\square$

Second step : proof of the existence of a solution G_n to (**??**).

In this second step, a Schauder fixed point theorem is used to prove the existence of a solution $G_n \in K$ to

$$
\begin{aligned}
\frac{\partial G_n}{\partial t} &= e^{-k}\int_0^\infty h_n(k,k')G_n' dk' + \frac{G_n}{k}\int_0^\infty h_n(k,k')(e^{-k}-e^{-k'})G_n' dk' \\
&\quad - \frac{G_n}{k}\int_0^\infty h_n(k,k')k'e^{-k'}dk', \\
G_n(0,k) &= \frac{F_0(k)}{k}, \quad k \geq 0,
\end{aligned}
\tag{4.6}
$$

such that $\displaystyle\int_0^\infty G_n(t,k)dk \leq \frac{c_1}{c_2}, \quad a.a\ t \in (0,T).$

Let $\mathcal{H}$ be the map defined on K by $\mathcal{H}(g) = G_n$ where $G_n \in K$ is the solution of (**??**).
The map $\mathcal{H}$, taking its values in the convex set K, is compact for the weak * topology of $L^\infty(0,T;M^1(I\!R_+))$. It is moreover continuous. Indeed, let $g_j \rightharpoonup g$ for the weak * topology of $L^\infty(0,T;M^1(I\!R^+))$. Denote by $(G_j) = (\mathcal{H}(g_j))_{j\in I\!N}$. By the compactness of $\mathcal{H}$, there is a subsequence (G_{jl}) of (G_j) and a function G in K such that $G_{jl} \rightharpoonup G$. Moreover G is the unique solution to (**??**). Hence, the whole sequence (G_j) converges to G for the weak * topology of $L^\infty(0,T;M^1(I\!R_+))$. By the Schauder fixed point theorem, $\mathcal{H}$ admits a fixed point, denoted by G_n, solution in K to (**??**).
The nonnegative function $F_n = kG_n$ is such that

$$
\int_0^\infty F_n(t,k)dk \leq \int_0^\infty F(0,k)dk, \qquad \int_0^\infty \frac{F_n}{k}(t,k)dk \leq \frac{c_1}{c_2}, \quad a.a.\ t \in (0,T),
\tag{4.7}
$$

and is solution to

$$
\begin{aligned}
\frac{\partial F_n}{\partial t} &= ke^{-k}\int_0^\infty h_n(k,k')\frac{F_n'}{k'}dk' + \frac{F_n}{k}\int_0^\infty h_n(k,k')(e^{-k}-e^{-k'})\frac{F_n'}{k'}dk' \\
&\quad - \frac{F_n}{k}\int_0^\infty h_n(k,k')k'e^{-k'}dk', \\
F_n(0,k) &= F_0(k).
\end{aligned}
\tag{4.8}
$$

Third step : passage to the limit in (4.8) when n tends to infinity.

In this third step, the passage to the limit when $n \to +\infty$ in (**??**) is performed, which leads to a solution F to the genuine problem (**??**).
By (**??**), there is a measure $G \in L^\infty(0,T;M^1(I\!R_+))$ such that $F_n \rightharpoonup kG$ and $\frac{F_n}{k} \rightharpoonup G$ in $L^\infty(0,T;M^1(I\!R_+))$ for the weak $*$ topology.
Let $(t,k) \to \phi(t,k) \in C^1([0,T] \times [0,\infty[)$ be a bounded test function with

second order with respect to k in the neigborhood of 0.
Multiplying (**??**) by ϕ and integrating on $[0,t]\times\mathbb{R}_+$ leads to

$$\int_0^\infty F(t,k)\ \phi(t,k)\ dk - \int_0^\infty F_0(k)\ \phi(0,k)\ dk - \int_0^\infty \int_0^t F(s,k)\frac{\partial\phi}{\partial s}(s,k)\ ds\ dk$$
$$= A_n + B_n + C_n,$$

with

$$\begin{aligned}
A_n &= \int_0^t \int_0^\infty \frac{F_n}{k}\ \phi(s,k) \int_0^\infty h_n(k,k')\ (e^{-k}-e^{-k'})\ \frac{F_n'}{k'}\ dk'\ dk\ ds,\\
B_n &= \int_0^t \int_0^\infty k\ e^{-k}\ \phi(s,k) \int_0^\infty h_n(k,k')\ \frac{F_n'}{k'}\ dk'\ dk\ ds,\\
C_n &= -\int_0^T \int_0^\infty \frac{F_n}{k}\ \phi(s,k) \int_0^\infty h_n(k,k')\ k'\ e^{-k'}\ dk'\ dk\ ds.
\end{aligned}$$

Let $U(k,k') = h(k,k')(e^{-k}-e^{-k'})$. For all $K>0$, A_n can be written as

$$A_n = X_{n,K} + \bar{X}_n + A_{n,K} + \bar{A}_{n,K},$$

where

$$\begin{aligned}
X_{n,K} &= \int_0^t \int_0^K \frac{F_n}{k}\ \phi(s,k) \int_0^K U(k,k')\ \frac{F_n'}{k'}\ dk'\ dk\ ds\\
\bar{X}_n &= -\int_0^t \int_0^{\frac{1}{n}} \frac{F_n}{k}\ \phi(s,k) \int_0^\infty U(k,k')\ \frac{F_n'}{k'}\ dk'\ dk\ ds\\
A_{n,K} &= \int_0^t \int_0^K \frac{F_n}{k}\ \phi(s,k) \int_K^\infty U(k,k')\ \frac{F_n'}{k'}\ dk'\ dk\ ds\\
\bar{A}_{n,K} &= \int_0^t \int_K^n \frac{F_n}{k}\ \phi(s,k) \int_0^\infty U(k,k')\ \frac{F_n'}{k'}\ dk'\ dk\ ds.
\end{aligned}$$

First, $\bar{X}_n \underset{n\to\infty}{\longrightarrow} 0$ thanks to the second order with respect to k of ϕ in the neigborhood of 0.
Then $A_{n,K}$ and $\bar{A}_{n,K}$ tend to 0 when $K\to\infty$, uniformly with respect to n.
Finally, by the Stone-Weierstrass theorem, for $K>0$ large enough and every $\varepsilon \in I\!R_+^*$, there exist $J\in I\!N^*$ and continuous functions

$$\beta_1,\cdots,\beta_J,\gamma_1,\cdots,\gamma_J : I\!R_+ \to I\!R,$$

such that

$$\text{for all }\ 0<k<K,\ 0<k'<K,\quad \Big|U(k,k') - \sum_{j=1}^J \beta_j(k)\gamma_j(k')\Big| \le \varepsilon.$$

Let
$$U_J(k,k') = \sum_{j=1}^J \beta_j(k)\gamma_j(k').$$

Then,

$$|X_{n,K}-X| \leq \Big|\Big\langle \frac{F_n}{k}\otimes\frac{F_n'}{k'}-\frac{F}{k}\otimes\frac{F'}{k'}, \phi\;(U-U_J)\Big\rangle\Big|+\Big|\Big\langle \frac{F_n}{k}\otimes\frac{F_n'}{k'}-\frac{F}{k}\otimes\frac{F'}{k'}, \phi\;U_J\Big\rangle\Big|.$$

The first term tends to 0 when $J \to \infty$ uniformly with respect to n because

$$\Big|\Big\langle \frac{F_n}{k}\otimes\frac{F_n'}{k'}-\frac{F}{k}\otimes\frac{F'}{k'}, \phi\;(U-U_J)\Big\rangle\Big| \quad \leq \quad 2|\phi|_\infty\Big(\int\frac{F_n}{k}dk\Big)^2\sup_{k,k'}|(U-U_J)(k,k')|.$$

The second term tends to 0 when $n \to \infty$ for all J.

Therefore,

$$\int_0^t\int_0^\infty\frac{F_n}{k}\;\phi(s,k)\int_0^\infty h_n(k,k')\;(e^{-k}-e^{-k'})\;\frac{F_n'}{k'}\;dk'\;dk\;ds$$

tends to

$$\int_0^t\int_0^\infty\frac{F}{k}\;\phi(s,k)\int_0^\infty h(k,k')\;(e^{-k}-e^{-k'})\;\frac{F'}{k'}\;dk'\;dk\;ds,$$

when n tends to infinity.
The passage to the limit in B_n and C_n when $n \to \infty$ can be done analogously. So, performing the passage to the limit when $n \to +\infty$ in (**??**) implies that F is a solution to

$$\begin{aligned}\frac{\partial F}{\partial t} &= ke^{-k}\int_0^\infty h(k,k')\frac{F'}{k'}dk' + \frac{F}{k}\int_0^\infty h(k,k')(e^{-k}-e^{-k'})\frac{F'}{k'}dk' \\ &- \frac{F}{k}\int_0^\infty h(k,k')k'e^{-k'}dk', \\ F(0,k) &= F_0(k),\end{aligned}$$

which also means that F is a solution of the problem (**??**). The continuity of F with respect to time follows from the boundedness of $Q(F)$ in $L^\infty((0,T);M^1(I\!R_+))$.

4.3 Study of the entropy.

In order to prove the entropy feature of F stated in Theorem **??**, the following Lemma is established.

Lemma 4.4
If $F_n(t,\cdot) \rightharpoonup F(t,\cdot) = \bar{F}(t,\cdot)dk + \mu_s$, then

$$\liminf_{n\to\infty} -H(F_n)(t) \geq -H(F)(t) - <\mu_s, k>, \tag{4.9}$$

$\bar{F}(t,\cdot)$ and μ_s being respectively the absolute Lebesgue part and the singular part of $F(t,\cdot)$.

Proof of lemma **??**.

Recall that

$$H(\bar F)(t) = \int_0^\infty [(k^2+\bar F(t,k))\ln(k^2+\bar F(t,k)) - \bar F(t,k)\ln \bar F(t,k) - k^2\ln k^2 - k\bar F(t,k)]\, dk.$$

Let

$$h(y,k) = -(k^2+y)\ln(k^2+y) + y\ln y + k^2\ln k^2 + ky.$$

It is a convex function with respect to the variable y.
Prove that

$$\int_\gamma^\delta h(k,\bar F(t,k))dk \le \liminf_{n\to\infty} \int_\gamma^\delta h(k,F_n(t,k))dk, \quad \delta \ge \gamma \ge 0,$$

in the following way.
Let $j \in I\!N^*$ and O be an open neighborhood of support μ_s such that $|O| < (\frac{\delta-\gamma}{j})^2$.
O is the denumerable union of open intervals. Denote by O_1 one of the intervals where μ_s has its bigger mass, ..., O_{l+1} one of the intervals where μ_s has its bigger mass after O_l, $l \ge 1$. μ_s being of finite mass, for any $\alpha > 0$, there is an integer l_α such that

$$\mu_s(\bigcup_{l \ge l_\alpha} O^l) < \alpha,$$

and for l_α large enough,

$$\int_{\bigcup_{l\ge l_\alpha} O^l} F(t,k)dk < \alpha.$$

Hence,

$$\lim_{n\to\infty} \int_{\bigcup_{l\ge l_\alpha} O^l} F_n(t,k)dk \quad < \quad 2\alpha. \tag{4.10}$$

Let α be such that $\alpha << \frac{\delta-\gamma}{j}$ and $\tilde I_i = I_i \setminus (O \cap I_i)$, with

$$I_i =]\gamma + i(\frac{\delta-\gamma}{j}), \gamma + (i+1)(\frac{\delta-\gamma}{j})[.$$

Then,

$$\int_{\tilde I_i} F_n(t,k)dk = \int_{I_i \setminus \bigcup_{l=1}^{l_\alpha -1} O^l} F_n dk - \int_{\bigcup_{l\ge l_\alpha} O^l} F_n dk \;\; \to_{n\to\infty} \int_{I_i \setminus \bigcup_{l=1}^{l_\alpha -1} O^l} (\bar F dk + d\mu_S) - A' := U,$$

where $A' < 2\alpha$ by (**??**).
Thus,

$$U = \int_{\tilde I_i} \bar F dk + B,$$

with

$$B := \int_{\bigcup\limits_{l\geq l_\alpha} O^l} \bar{F} dk + \int_{I_i \setminus \bigcup\limits_{l=1}^{l_\alpha - 1} O^l} d\mu_s - A' < 4\alpha.$$

Hence,

$$\liminf_{n\to\infty} h(\gamma + i(\frac{\delta-\gamma}{j}), \frac{1}{|\tilde{I}_i|}\int_{\tilde{I}_i} F_n dk) = h(\gamma + i(\frac{\delta-\gamma}{j}), \frac{1}{|\tilde{I}_i|}\int_{\tilde{I}_i} \bar{F} dk + \frac{B}{|\tilde{I}_i|}). \tag{4.11}$$

Let $\varepsilon > 0$ be given. First, it holds that for some $\lambda_\varepsilon > 0$,

$$h(k,\lambda) < \varepsilon, \quad \lambda > \lambda_\varepsilon, \quad k \in [\gamma, \delta].$$

Then, by the uniform continuity of $h(k,\lambda)$ on $[\gamma,\delta]\times[0,\lambda_\varepsilon]$, it holds that

$$\int_\gamma^\delta h(k,F_n) dk = \lim_{j\to\infty} \sum_{i=0}^{j-1} \int_{\tilde{I}_i} h(\gamma + i(\frac{\delta-\gamma}{j}), F_n) dk.$$

Thanks to Jensen's inequality,

$$\frac{1}{|\tilde{I}_i|}\int_{\tilde{I}_i} h(\gamma + i(\frac{\delta-\gamma}{j}), F_n(t,k)) dk \geq h(\gamma + i(\frac{\delta-\gamma}{j}), \frac{1}{|\tilde{I}_i|}\int_{\tilde{I}_i} F_n(t,k) dk).$$

It follows from the constant sign of $h(k,F_n) - kF_n$ that

$$\begin{aligned}
\liminf_{n\to\infty} \int_\gamma^\delta h(k,F_n) dk &\geq \liminf_{n\to\infty} \lim_{j\to\infty} \sum_{i=0}^{j-1} |\tilde{I}_i| \; h(\gamma + i(\frac{\delta-\gamma}{j}), \frac{1}{|\tilde{I}_i|}\int_{\tilde{I}_i} F_n dk) \\
&= \lim_{j\to\infty} \liminf_{n\to\infty} \sum_{i=0}^{j-1} |\tilde{I}_i| \; h(\gamma + i(\frac{\delta-\gamma}{j}), \frac{1}{|\tilde{I}_i|}\int_{\tilde{I}_i} F_n dk).
\end{aligned}$$

And so, by (**??**),

$$\liminf_{n\to\infty} \int_\gamma^\delta h(k,F_n) dk \geq \lim_{j\to\infty} \sum_{i=0}^{j-1} |\tilde{I}_i| \; h(\gamma + i(\frac{\delta-\gamma}{j}), \frac{1}{|\tilde{I}_i|}\int_{\tilde{I}_i} \bar{F} dk + \frac{B}{|\tilde{I}_i|}).$$

Hence, for $j \to \infty$ and $\alpha \to 0$,

$$\liminf_{n\to\infty} \int_\gamma^\delta h(k,F_n) dk \geq \int_\gamma^\delta h(k,\bar{F}) dk.$$

For $\delta \to \infty$ and $\gamma \to 0$,

$$\liminf_{n\to\infty} \int_0^\infty h(k,F_n)(t,k) dk \geq \int_0^\infty h(k,\bar{F})(t,k) dk,$$

$$\text{i.e} \quad \liminf_{n\to\infty} -H(F_n)(t) \geq -H(\bar{F})(t), \quad a.a.\ t > 0.$$

By definition ([**?**]), $H(F) = H(\bar{F}) - M(k\mu_s)$, so that,

$$\liminf_{n\to\infty} -H(F_n)(t) \geq -H(F)(t) - <\mu_s, k>, \quad a.a.\ t > 0.$$

□

Proof of (**??**).

The proof of Proposition **??** implies that

$$\frac{d}{dt}H(F_n) = \frac{1}{2}\int_0^\infty \int_0^\infty b(k,k')\ j(F_n(k'^2+F_n')e^{-k'}, F_n'(k^2+F_n)e^{-k})\ dk'\ dk, \tag{4.12}$$

with j defined in the proof of Proposition **??**. This implies that

$$H(F_n)(t) \geq H(F_n(0)) = H(F_0)\ \ a.a.\ t > 0, \tag{4.13}$$

so that

$$H(F)(t) \geq H(F_0) - < \mu_s, k > .$$

Moreover, $< \mu_s, k >$ is bounded from above. Indeed, by (**??**), (**??**) and (**??**),

$$\int_0^\infty kF_n(t,k)dk < c,$$

uniformly with respect to n. □

5 Conclusion.

In this paper, we have proven the existence of a solution to an homogeneous quantum kinetic evolutionary problem describing the Compton effect. Due to a strong singularity in the collision operator, the mathematical framework is the set of photon distribution functions F such that F and $\frac{F(t,k)}{k}$ are bounded measures. A local in time existence theorem is proven for small initial data. The mathematical entropy of the solutions is bounded from below.

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