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Mathematical and Computer Modelling 43 (2006) 838–853

MATHEMATICAL AND COMPUTER MODELLING

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On the Cauchy problem for a quantum kinetic equation linked to the Compton effect

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Received 6 September 2005; accepted 20 September 2005

Abstract

The Cauchy problem is studied for an homogeneous quantum kinetic equation describing the Compton effect. Since the collision kernel commonly used in physics is highly singular, numerical simulations are performed for related collision kernels to get a preliminary insight into the behavior of the solutions. Some of the numerical results are then given a theoretical explanation. Global existence of a solution to the Cauchy problem is proven when the L^1 initial data are a.e. smaller than the Planck distribution function, and non-existence of solutions to the Cauchy problem is proven when the L^1 initial data are a.e. bigger than the Planck distribution function.
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Keywords: Compton effect; Quantum kinetic equation

1. Introduction

The Compton effect describes the change in wavelength of X-rays and gamma rays due to scattering by electrons. Its discovery by Arthur H. Compton in 1922 strengthened the case for quantum mechanics in an epoch when it was still much debated. Indeed it supported the notion that not only radiation, but matter as well, had both wave and corpuscular properties. From a modelization point of view, it represents one of the few known cases in quantum kinetic theory where the collision kernel can be derived precisely.

Among the physics papers in the area we mention the following pioneering ones. G. Cooper developed the Fokker–Planck equation for the Compton scattering in a plasma without recourse to a non-relativistic approximation [1]. Dreicer presented a simple kinetic theory which includes the interactions between electrons and photons and may thus describe relaxation phenomena [2]. Kompaneets studied the thermal equilibrium between quanta and electrons [3]. Zel'dovich and Levich studied the process of equilibrium in a system consisting of radiation and totally ionized plasma [4].

The mathematical perspective was developed in a.o. the following articles. Entropy increase and comparison principles for the Fokker–Planck equation describing the radiation distribution in a homogeneous plasma were derived

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doi:10.1016/j.mcm.2005.09.034

by Caflish and Levermore in [5]. There the entropy function was used to find the equilibrium distributions for the scattering alone and for scattering with emission and absorption. In [6] solutions to the Fokker–Planck equation developing singularities in finite time, independently of the total initial number of photons, were analyzed. Escobedo and Mischler obtained existence results for a quantum kinetic equation with a simplified regular and bounded kernel [7]. They studied the asymptotic behavior of the solutions, and showed that the photon distribution function may condensate at energy zero asymptotically in time. Numerical methods were developed in [8] for the same quantum kinetic equation, as well as for the Kompaneets equation. Chane-Yook and Nouri kept the physical kernel in the same quantum kinetic equation and derived a local existence in time theorem for the Cauchy problem in the case of small initial data [9]. In the context of parabolic problems, Brandle, Groisman and Rossi proposed numerical schemes for the approximation of solutions with possible blow-ups [10]. Adaptive methods were performed in [11] in order to reproduce the asymptotic behaviour of the solutions.

In this paper, global existence in time for the Cauchy problem is considered for the model with physical kernel. Section 2 recalls the model and some of its properties. Section 3 describes earlier results for the model. Section 4 presents numerical simulations for various collision kernels, from the constant one considered in [7] to the physical one. This is done with a scheme that preserves the main physical properties of the model. Different initial data are considered, essentially distinguishing the case where the initial datum corresponds to a total number of photons bigger than the Planck distribution one. Section 5 gives a theoretical explanation of some numerical simulations, in the particular cases where the initial datum is an L^1 function almost everywhere smaller (resp. bigger) than the Planck distribution function. Global existence of a solution to the Cauchy problem is derived when the initial datum is almost everywhere smaller than the Planck distribution function. For an initial datum a.e. bigger than the Planck distribution function, non-existence of any solution to the Cauchy problem is proven. This is a limitation of the possible initial data for the Cauchy problem.

2. The model

As considered in [7], the following quantum relativistic homogeneous equation describes the interaction via Compton scattering between a gas of low energy electrons of mass m and weakly dense photons at low temperature,

$$\frac{\partial f}{\partial t}(t, P) = Q(f, g)(P), \quad t > 0, P \in \mathbb{R}^4, \tag{2.1}$$

with

$$Q(f, g)(P) = \frac{8c}{p^0} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} s\, \sigma(s, \theta) q(f, g) \delta_{\{P+P_*-P'-P'_*=0\}} \chi_2(P^0_*) \chi_1(P'^0) \chi_2(P'^0_*) \mathrm{d}P' \mathrm{d}P'_* \mathrm{d}P_*. \tag{2.2}$$

The non-negative scalar function $f(t, P)$ (resp. $g(t, P)$) is the distribution function of photons (resp. electrons). c denotes the speed of the light. P and P' (resp. P_* and P'_*) are the momenta of the photons (resp. electrons) before and after a collision. A particle is determined by the pair $(X, P) \in \mathbb{R}^4 \times \mathbb{R}^4$ of position $X = (t, x)$ and momentum $P = (P^0, p)$. Let

$$p^0 = |p|, \quad p'^0 = |p'|, \quad p^0_* = \sqrt{|p_*|^2 + m^2c^2}, \quad p'^0_* = \sqrt{|p'_*|^2 + m^2c^2}.$$

Let $s = (P + P_*)^2 := (P^0 + P^0_*)^2 - |p + p_*|^2$, and by θ the scattering angle given by

$$\cos\theta = \frac{(P_* - P).(P'_* - P')}{(P_* - P)^2}.$$

The differential cross section $\sigma(s, \theta)$ depends on energy and scattering angle and is given by the Klein Nishina formula [12]. It behaves like

$$\frac{1}{2} r_0^2 (1 + \cos^2\theta), \tag{2.3}$$

with $r_0 = \frac{e^2}{4\pi mc^2}$ when $c \to \infty$. Here, e is the charge of the electron.

The functions $\chi_1(P'^0)$, $\chi_2(P_*^0)$ and $\chi_2(P_*'^0)$ are defined by

$$\chi_1(P'^0) = \frac{1}{2p'^0}\delta_{\{P'^0=p'^0\}}, \quad \chi_2(P_*^0) = \frac{1}{2p_*^0}\delta_{\{P_*^0=p_*^0\}}, \quad \chi_2(P_*'^0) = \frac{1}{2p_*'^0}\delta_{\{P_*'^0=p_*'^0\}},$$

and

$$q(f,g) = g(p'_*)f(p')(1+\hbar f(p))(1+\tau g(p_*)) - f(p)g(p_*)(1+\hbar f(p'))(1+\tau g(p'_*)), \tag{2.4}$$

with $\tau \in \{-\hbar, 0, \hbar\}$ and $\hbar$ the Planck constant.

Here and below, the following notations are used for any function f,

$$f' = f(t,p'), \quad f_* = f(t,p_*), \quad f'_* = f(t,p'_*).$$

In Eq. (2.1), emission and absorption of photons have not been taken into account, so that the transitions are produced exclusively by the Compton scattering. In order to simplify the formulas, m and $\hbar$ are taken equal to 1.

By integrating (2.2) with respect to P_*^0, P'^0 and $P_*'^0$, $Q(f,g)$ becomes

$$Q(f,g)(p) = c\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{s}{p^0p'^0p_*^0p_*'^0}\sigma(s,\theta)q(f,g)\delta_\Sigma \mathrm{d}p'\mathrm{d}p'_*\mathrm{d}p_*,$$

where Σ is the manifold of 4-uplets (p, p_*, p', p'_*) such that

$$p + p_* = p' + p'_*,$$
$$c|p| + \frac{|p_*|^2}{2} = c|p'| + \frac{|p'_*|^2}{2}.$$

To simplify the model, only the highest-order terms with respect to c are kept in $Q(f,g)(p)$. The term $\frac{s}{p^0p'^0p_*^0p_*'^0}$ is equivalent to $\frac{1}{|p||p'|}$, when $c \to \infty$. Together with (2.3), this implies that the collision operator can be approximated by

$$Q(f,g)(p) = \frac{cr_0^2}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(1+\cos^2\theta)}{|p||p'|}q(f,g)\delta_\Sigma \mathrm{d}p'\mathrm{d}p'_*\mathrm{d}p_*.$$

The electrons are assumed to be at non-relativistic equilibrium, i.e.,

$$\tau = 0 \quad \text{and} \quad g(p) = e^{-\frac{|p|^2}{2c}}.$$

Then,

$$q(f,g) = g(p'_*)f(p')(1+f(p)) - f(p)g(p_*)(1+f(p')).$$

The collision integral becomes

$$Q(f,g)(p) = \frac{cr_0^2}{2}\int_{\mathbb{R}^3} \frac{(1+\cos^2\theta)}{|p||p'|}e^{|p'|}q(f)\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\delta_\Sigma e^{-\frac{|p_*|^2}{2c}}\mathrm{d}p_*\mathrm{d}p'_*\right)\mathrm{d}p',$$

with

$$q(f) = e^{-|p|}f(p')(1+f(p)) - e^{-|p'|}f(p)(1+f(p')).$$

It can be simplified in the following way.

Lemma 2.1. *Let*

$$S(p,p') = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\delta_\Sigma e^{-\frac{|p_*|^2}{2c}}\mathrm{d}p_*\mathrm{d}p'_*, \quad A = |p'| - |p| + \frac{|p-p'|^2}{2c}, \quad w = p' - p.$$

Then,

$$S(p,p') = \frac{2\pi c^2}{|w|}e^{-\frac{A^2c}{2|w|^2}}.$$

It is then assumed that the photons distribution function is radial. Denote by $k = |p|$, $k' = |p'|$, $F(t,k) = k^2 f(t,k)$. The quantum kinetic homogeneous equation describing the interaction between photons and electrons is then

$$\frac{\partial F}{\partial t}(t,k) = Q(F,F)(t,k), \tag{2.5}$$

where

$$\begin{aligned} Q(F,F)(t,k) &= \int_0^\infty b(k,k')[F'(k^2+F)e^{-k} - F(k'^2+F')e^{-k'}]\mathrm{d}k', \\ b(k,k') &= \frac{2c^3 r_0^2 \pi^2}{k\,k'} \int_0^\pi (1+\cos^2\theta) \frac{\sin\theta}{|w|} e^{-\frac{A^2 c}{2|w|^2}+k'} \mathrm{d}\theta, \\ A &= k' - k + \frac{|w|^2}{2c}, \quad |w|^2 = k^2 + k'^2 - 2kk'\cos\theta. \end{aligned} \tag{2.6}$$

The kernel set of the collision operator Q consists of the distribution functions

$$B_\mu(k) = \frac{k^2}{e^{k+\mu} - 1}, \quad \mu \geq 0, \quad \text{and} \quad B_0(k) + \alpha\,\delta_{k=0}, \quad \alpha \geq 0.$$

The specific distribution function B_0 is called the Planck distribution function.

Let us recall [7] the a priori estimates for the Cauchy problem

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,k) = Q(F,F)(t,k), & t > 0,\ k \geq 0, \\ F(0,k) = F_i(k), & k \geq 0. \end{cases} \tag{2.7}$$

Proposition 2.1. *Let $M(F)(t) = \int_0^\infty F(t,k)\,\mathrm{d}k$ be the total number of photons at time t. Then,*

$$M(F)(t) = M(F_i), \quad t > 0. \tag{2.8}$$

Proposition 2.2. *The entropy defined by*

$$H(F)(t) = \int_0^\infty [(k^2+F)\ln(k^2+F) - F\ln F - k^2\ln k^2 - kF]\mathrm{d}k$$

is a non-decreasing function of time.

Proposition 2.3. *The energy $M(kF)$ and the entropy $H(F)$ are bounded. More precisely,*

$$M(kF) \leq C(1 + M(F) - H(F)), \tag{2.9}$$
$$|H(F)| \leq M((1+k)F), \tag{2.10}$$

for some constant C.

3. Description of the previous results

Contrary to the common $\int f\ln f$-type entropy, the present one of Proposition 2.2 is of no use for equi-integrability questions. Here, instead, a suitable mathematical frame for solving the Cauchy problem (2.7) is $L^\infty(\mathbb{R}_+, \mathcal{M}^1(\mathbb{R}_+))$, where $\mathcal{M}^1(\mathbb{R}_+)$ denotes the space of bounded measures in the variable k. In [7], Escobedo and Mischler proved the global existence and uniqueness of a measure solution of the Cauchy problem (2.7) for the following three particular types of cross section $b(k,k')$.

(i) $b \geq 0$ and bounded;

(ii) $b(k, k') = e^{\eta k} e^{\eta k'} \sigma(k' - k)$, with $\eta \in (0, 1)$ and the function σ satisfying

$$0 < \sigma_\star e^{-\nu |z|^\gamma} \leq \sigma(z) \leq \sigma^\star, \quad \text{for all } z \in \mathbb{R},$$

for some $\sigma_\star, \sigma^\star, \nu > 0, \gamma \in [0, 1)$;

(iii) $0 \leq b(k, k')\, e^{-\eta k} e^{-\eta k'}$, bounded for some $\eta \in [0, 1]$.

Moreover, the time asymptotics of a solution to the Cauchy problem (2.7) is derived in [7,13]. When $b > 0$, let $m = M(F_i)$ be the total number of photons of the initial distribution. Denote by $M_0 = M(B_0)$ the Planck distribution one. Let $B_m = B_\mu + \alpha\, \delta_{k=0}$ be the Bose distribution of the total number of photons m, with $\alpha = 0$ and $\mu \geq 0$ such that $M(B_\mu) = m$ if $m \leq M_0$, and $\mu = 0$ and $\alpha = m - M_0$ if $m > M_0$. In this setting, the following result holds.

Theorem 3.1. *Let $F \in C([0, T], \mathcal{M}^1(\mathbb{R}_+))$ be the solution to the Cauchy problem (2.7). Then*

$$\begin{aligned} &F(t, \cdot) \underset{t\to\infty}{\overset{\star}{\rightharpoonup}} B_m \quad \textit{weakly } \star \textit{ in } (C_c(\mathbb{R}_+))', \\ &\lim_{t\to\infty} \left\| g(t, \cdot) - B_\mu \right\|_{L^1((k_0,\infty))} = 0 \quad \textit{for every } k_0 > 0. \end{aligned} \tag{3.1}$$

Here $F = g + G$, with $g \in L^1(\mathbb{R}_+)$ and G the singular part of F with respect to the Lebesgue measure in $\mathbb{R}_+$.

Moreover, if $m \leq M_0$ or $0 \leq g_i \leq B_0$, k_0 can be taken as 0.

The assumption (i), (ii), or (iii) made by Escobedo and Mischler when deriving global existence in time of a solution to the Cauchy problem (2.7) does not allow b to be singular in a neighborhood of zero, which is the case for the physical kernel (2.6). On the other hand, for the physical kernel, Chane-Yook and Nouri have proved a local existence in time theorem for small initial data [9]. Moreover, the following proposition is proven there, in order to give a sense to the collision operator $Q(F, F)$ for distribution functions F in $C\left([0, T], \mathcal{M}^1(\mathbb{R}_+)\right)$.

Proposition 3.1. *Let $F \in C([0, T], \mathcal{M}^1(\mathbb{R}_+))$ be such that*

$$F(\tau, \cdot) \neq B_0 + \alpha\, \delta_{k=0}, \quad \alpha \in \mathbb{R}_+$$

for all $\tau \in [0, T]$. If for any continuous and bounded function ϕ of second order with respect to k in the neighborhood of 0 and for any interval $J \subset [0, T]$,

$$\left| \int_J \int_0^\infty \phi(\tau, k) Q(F, F)(\tau, k) \mathrm{d}k\, \mathrm{d}\tau \right| < +\infty,$$

then

$$\int_0^\infty \frac{F}{k}(\tau, k) \mathrm{d}k < +\infty, \quad a.a.\ \tau \in [0, T].$$

In order to approach the possible global existence of solutions to the Cauchy problem (2.7), numerical simulations are performed in Section 4. Some of the results obtained there are then theoretically explained in Section 5.

4. Numerical simulations

4.1. Discretizations

Approximations of the solution to the Cauchy problem (2.7) are performed in the following way. We restrict to a bounded domain $[0, R]$, $R > 0$, in energy. The collision operator in (2.5) becomes

$$\int_0^R b(k, k') \left(F'(k^2 + F) e^{-k} - F(k'^2 + F') e^{-k'} \right) \mathrm{d}k'. \tag{4.1}$$

We consider a uniform grid in energy, given by the points

$$k_1 < k_2 < \cdots < k_N \in [0, R],$$

with $k_1 > 0$ in order to avoid the singularity of b at 0, and $\Delta k = R/N$. Hence the collision operator $Q(F, F)(k_l)$ can be discretized as

$$\Delta k \sum_{j=1}^{N} b(k_l, k_j) \left(F_j (k_l^2 + F_l) e^{-k_l} - F_l (k_j^2 + F_j) e^{-k_j} \right), \quad 1 \le l, j \le N. \tag{4.2}$$

By choosing $k_j = \left(j - \frac{1}{2}\right) \Delta k,\ \ j = 1, \dots, N$, the discretization (4.2) is an approximation of (4.1) of second order in energy. The time integration is performed by using the standard second-order explicit Runge–Kutta scheme given by the following scheme,

$$\begin{cases} p_1 = Q(F^n, F^n), \\ p_2 = Q(F^n + \Delta t\, p_1, F^n + \Delta t\, p_1), \\ F^{n+1} = F^n + \dfrac{\Delta t}{2}(p_1 + p_2). \end{cases} \tag{4.3}$$

Here F^n denotes the approximate solution computed at t^n.

Proposition 4.1. *The scheme preserves the total number of photons. Moreover, it is entropy non-decreasing.*

Sketch of the proof. As shown in [8], the discrete weak form for the collision operator $Q(F, F)(k_l)$ is

$$\sum_{l=1}^{N} \Psi_l \frac{\partial F_l}{\partial t} = \frac{(\Delta k)^2}{2} \sum_{l=1}^{N} \sum_{j=1}^{N} b(k_l, k_j) F_l\, F_j \left(\frac{(k_l^2 + F_l) e^{-k_l}}{F_l} - \frac{(k_j^2 + F_j) e^{-k_j}}{F_j} \right) (\Psi_l - \Psi_j). \tag{4.4}$$

Choosing $\Psi \equiv 1$ leads to the conservation of the total number of photons. Then choosing $\Psi_l := \ln \left(\frac{k_l^2 + F_l}{F_l}\, e^{-k_l} \right)$ gives a discrete version of the proof of Proposition 2.2.

Proposition 4.2. *Let $\beta \in]0, 1[$, $k_1 = \lambda \Delta k$ and*

$$M_i = \Delta k \sum_{j=1}^{N} F^0(k_j)$$

be the discrete number of photons of the distribution function F at time $t = 0$.

Assume that

$$\frac{\Delta t}{\Delta k} \le \frac{\lambda \beta}{4 R} e^{-\frac{c}{2}}, \tag{4.5}$$

and

$$\frac{\Delta t}{(\Delta k)^2} \le \frac{3 \lambda^2 (1 - \beta)}{8 M_i} e^{-\frac{c}{2}}. \tag{4.6}$$

Then the scheme preserves the positivity. More precisely,
if $F^0(k_j) \ge 0, 1 \le j \le N$, then $F^n(k_j) \ge 0,\ 0 \le n, 1 \le j \le N$.

Proof of Proposition 4.2. First prove the result by induction for the explicit in time Euler scheme. Start from $F_l^0 \ge 0,\ 1 \le l \le N$, and assume $F_l^n \ge 0,\ 1 \le l \le N$ for some $n > 0$. For simplicity we split the physical kernel (2.6) in two parts as $b = b1\, b2$, where

$$b1(k, k') = \frac{1}{k\, k'},$$

$$b2(k, k') = \int_0^{\pi} \left(1 + \cos^2 \theta\right) \frac{\sin \theta}{|w|} e^{-\frac{A^2 c}{2|w|^2} + k'} \mathrm{d}\theta.$$

Then, in order to prove that $F_l^{n+1} \geq 0,\ 1 \leq l \leq N$, it is sufficient to prove that

$$F_l^n + \Delta t\,\Delta k \sum_{j=1}^{N} b2(k_l, k_j) \left(\frac{F_j^n}{k_j} \left(k_l + \frac{F_l^n}{k_l} \right) e^{-k_l} - \frac{F_l^n}{k_l} \left(k_j + \frac{F_j^n}{k_j} \right) e^{-k_j} \right) \geq 0, \tag{4.7}$$

for $1 \leq l \leq N$. Splitting the former inequality into its linear and bilinear terms, (4.7) is satisfied as soon as the following inequalities hold,

$$\beta F_l + \Delta t\,\Delta k \left(b2(k_l, k_l)\, F_l\, e^{-k_l} - \frac{F_l}{k_l} \sum_{j=1}^{N} b2(k_l, k_j) k_j\, e^{-k_j} \right) \geq 0, \quad 1 \leq l \leq N, \tag{4.8}$$

and

$$\Delta t\,\Delta k \sum_{\substack{j=1 \\ j \neq l}}^{N} b2(k_l, k_j)\, \frac{F_j}{k_j} \left(e^{-k_j} - e^{-k_l} \right) \leq (1 - \beta) k_1, \quad 1 \leq l \leq N. \tag{4.9}$$

Then

$$\int_0^{\pi} \frac{\sin\theta}{|w|} \mathrm{d}\theta = \frac{2}{\max\{k, k'\}}, \quad k, k' > 0,$$

and

$$e^{-\frac{1}{8c} w^2 - \frac{c(k-k')^2}{2w^2} + \frac{(k+k')}{2}}\, e^{-k'} \leq e^{-\frac{1}{8c}(k-k')^2 + \frac{(k-k')}{2}} \leq e^{\frac{c}{2}}, \quad k, k' \geq 0.$$

Consequently,

$$b2(k_l, k_j)\, k_j\, e^{-k_j} \leq 4\, e^{\frac{c}{2}}, \quad 1 \leq l, j \leq N.$$

And so, (4.8) follows from (4.5).

Moreover,

$$\begin{aligned} \frac{1}{|w|} e^{-\frac{1}{8c} w^2 - \frac{c(k-k')^2}{2w^2} + \frac{(k+k')}{2}} \left(e^{-k} - e^{-k'} \right) &\leq e^{-\frac{1}{8c}(k'-k)^2} \frac{\sinh \frac{(k'-k)}{2}}{\frac{(k'-k)}{2}} \\ &\leq e^{-\frac{1}{8c}(k'-k)^2 + \frac{(k'-k)}{2}} \\ &\leq e^{\frac{c}{2}}, \quad 0 < k < k', \end{aligned}$$

so that

$$b2(k, k') \left(e^{-k} - e^{-k'} \right) \leq \frac{8}{3}\, e^{\frac{c}{2}}, \quad k, k' > 0. \tag{4.10}$$

And so,

$$\begin{aligned} \sum_{j \neq l} b2(k_l, k_j)\, \frac{F_j}{k_j} \left(e^{-k_j} - e^{-k_l} \right) &\leq \frac{8}{3}\, e^{\frac{c}{2}} \sum_{j \neq l} \frac{F_j}{k_j} \\ &\leq \frac{8}{3}\, e^{\frac{c}{2}} \frac{M_i}{k_1}, \quad 1 \leq l \leq N. \end{aligned}$$

Hence, (4.9) follows from (4.6). Therefore, if $F^n \geq 0$, $F^n + \Delta t\, Q(F^n, F^n) \geq 0$.

Finally, the result also holds for the second-order explicit Runge–Kutta scheme (4.3), since

$$F^{n+1} = \frac{1}{2} \left(F^n + \Delta t\, Q(F^n, F^n) \right) + \frac{1}{2} (F^n) + \Delta t\, Q(F^n + \Delta t\, Q(F^n, F^n), F^n + \Delta t\, Q(F^n, F^n)).$$

If $F^n \geq 0$, both terms of the right-hand side of the previous inequality are non-negative. □

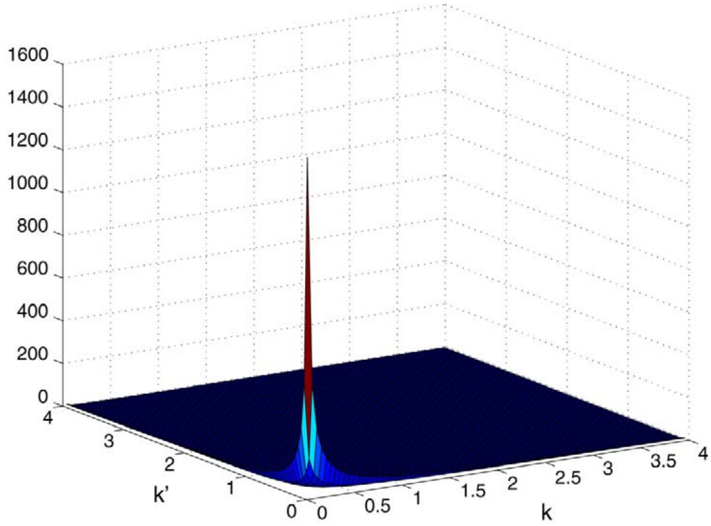

Fig. 1. Plot of $b1(k, k')$.

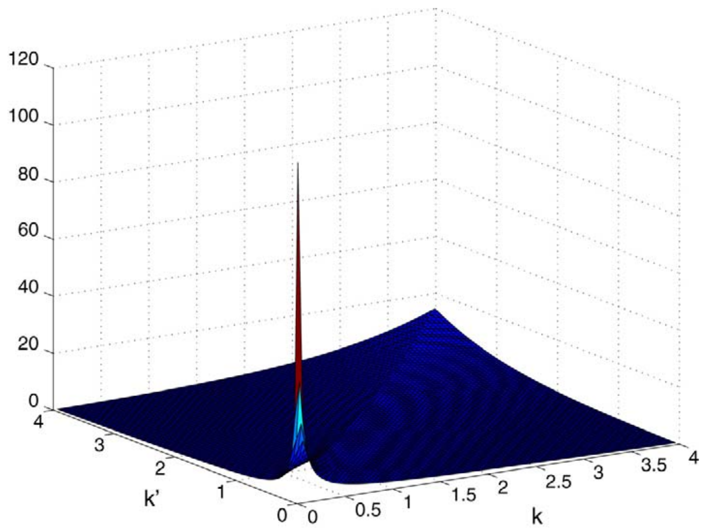

Fig. 2. Plot of $b2(k, k')$.

4.2. Plots of the physical kernel

The graphics of $b1, b2$ and b are respectively shown in Figs. 1–3. The critical case $k = k'$ is approximated by choosing $|k - k'| = 10^{-12}$.

As expected, the critical zone is located where k and k' are small, but the singularity is stronger in $b1$ than in $b2$.

4.3. Numerical tests

Let

$$\tilde{M}_0 = \Delta k \sum_{j=1}^{N} B_0(k_j)$$

be the discrete number of photons of the Planck distribution function. The initial number of photons being fundamental to define the time asymptotics of the solution to the Cauchy problem, the following initial data are chosen.

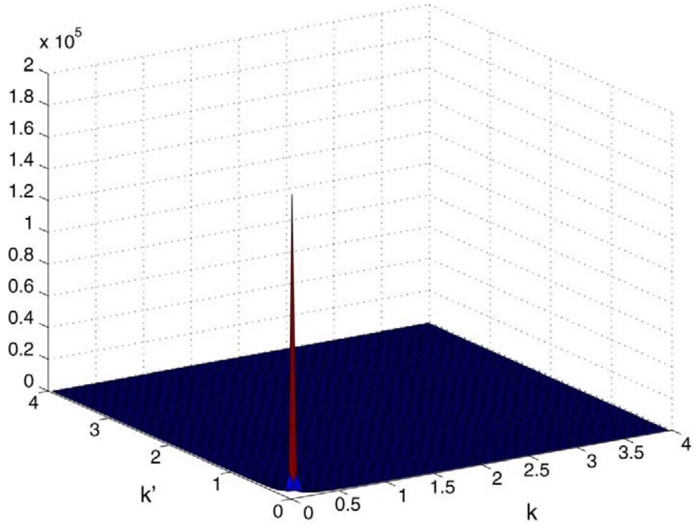

Fig. 3. Plot of the physical kernel $b(k, k')$.

i1. The Planck distribution $F_i = B_0$.

i2. The Gaussian profile $F_i(k) = C_1\, k^2\, e^{-4\left(k-\frac{R}{2}\right)^2}$, with C_1 such that $M_i = \frac{1}{2}\tilde{M}_0$.

i3. $F_i = \frac{1}{2}\, B_0$.

i4. The Gaussian profile $F_i(k) = C_2\, k^2\, e^{-4\left(k-\frac{R}{2}\right)^2}$, with C_2 such that $M_i = \frac{3}{2}\tilde{M}_0$.

i5. $F_i = \frac{3}{2}\, B_0$.

In order to deal with the strong singularity of the physical kernel in the neighborhood of zero, numerical simulations with the simplest collision kernel, i.e., the constant one used by Escobedo and Mischler, are first performed, then with an intermediate singular kernel, and finally with the physical one, as follows.

k1. $b(k, k') = 1$.

k2. $b(k, k') = \frac{1}{k\, k'}$.

k3. The physical kernel $b(k, k') = \frac{1}{k\, k'} \int_0^{\pi} \left(1 + \cos^2\theta\right) \frac{\sin\theta}{|w|} e^{-\frac{A^2 c}{2|w|^2} + k'} \mathrm{d}\theta$.

The simulation is stopped at time step t^n, when

$$\left|F^n(k_l, k_j) - F^{n-1}(k_l, k_j)\right| \leq 10^{-5}, \quad l, j \in 1, \ldots, N.$$

The graphics in this section show the initial function F_i, the final distribution F_f and the Bose distribution B_μ with the same total number of photons as F_i.

They are obtained by using the space discretization given in Section 4.1, with $N = 40$. As long as the distribution function remains non-negative (see Section 4.3.3), the time step used is $\Delta t = \frac{\Delta k}{4}$.

4.3.1. Initial datum: The Planck distribution

The Planck distribution function B_0 is an equilibrium state for the solution of the Cauchy problem associated to (2.5) and the numerical kernels. Hence it is a good test for the validation of the schemes. As expected, for all the kernels **ki**, **i** $= 1, 2, 3$, the Planck distribution is numerically preserved with time.

4.3.2. Initial data with total number of photons smaller than the one of B_0

The evolution of the distribution function F in the cases **i2** and **i3**, i.e., F_i given by a Gaussian profile or a Bose distribution with $M_i = \frac{1}{2}\tilde{M}_0$, are quite similar, so only the results of the case **i2** are shown here. The result for the initial datum **i2** and the kernel **k1**, as well as the comparison with the Bose distribution B_μ having the same number of photons of F_i, is shown in Fig. 4.

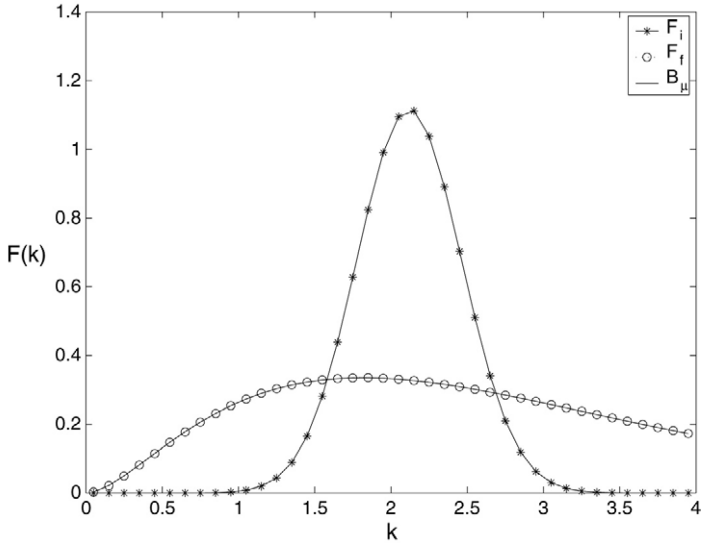

Fig. 4. Initial and final distribution for **i2** and **k1**.

The computations for **i2**, as well as the ones for **i3**, with the other kernels **ki**, **i** $= 2, 3$, lead to the same asymptotic behavior. However, the cases **i2** and **i3** differ in their time of convergence. The computations for **i2**, with the kernels **ki**, **i** $= 1, 2, 3$, lead to the following times of convergence.

k1. $T_f = 5.300$;
k2. $T_f = 16.825$;
k3. $T_f = 9.525$.

Instead, the times of convergence for **i3** and **ki**, **i** $= 1, 2, 3$, are the following.

k1. $T_f = 5.325$;
k2. $T_f = 7.875$;
k3. $T_f = 6.100$.

4.3.3. Initial data with total number of photons bigger than the B_0 one

In this case, the total number of photons of the initial datum F_i is equal to $\frac{3}{2}\tilde{M}_0$, so the equilibrium state is expected to be

$$B_0 + \frac{1}{2}\tilde{M}_0 \ \delta_{k=0}. \tag{4.11}$$

Both cases **i4** and **i5** have the same asymptotic behavior. In this section we show the graphics obtained starting from the datum **i5**, corresponding to $\frac{3}{2}B_0$. The results for the initial datum **i5** and the kernel **k1**, as well as the comparison with the Planck distribution B_0, are shown in Fig. 5.

Notice that a Dirac part appears in F at $k = 0$ for large times, and that it only concentrates at the first energy point k_1. In particular, the value taken by F at k_1, when the number of points in the energy discretization is $N = 40$, is $F(k_1) = 8.2265$. By increasing N up to 80, this value becomes $F(k_1) = 17.8337$, as shown in Fig. 6. One expects the value $F(k_1)\,\Delta k_N$, where $\Delta k_N = \frac{R}{N}$, to approach the coefficient $\frac{1}{2}\tilde{M}_0 = 0.9622$ of the Dirac measure $\delta_{k=0}$ in (4.11), for large values of N. The result of this computation for increasing values of N is shown in Fig. 7.

The times of convergence obtained for **i5** (with $N = 40$) and **ki**, **i** $= 1, 2$ are the following.

k1. $T_f = 41.850$;
k2. $T_f = 2.800$.

In an analogous way, for **i4** (with $N = 40$) and **ki**, **i** $= 1, 2$ the following times of convergence are obtained.

k1. $T_f = 46.375$;
k2. $T_f = 3.250$.

The physical case **i5** and **k3** is quite different. Indeed, in order to preserve the positivity of the scheme (see Proposition 4.2), it is necessary to keep a smaller time step. In particular, the following simulations are performed

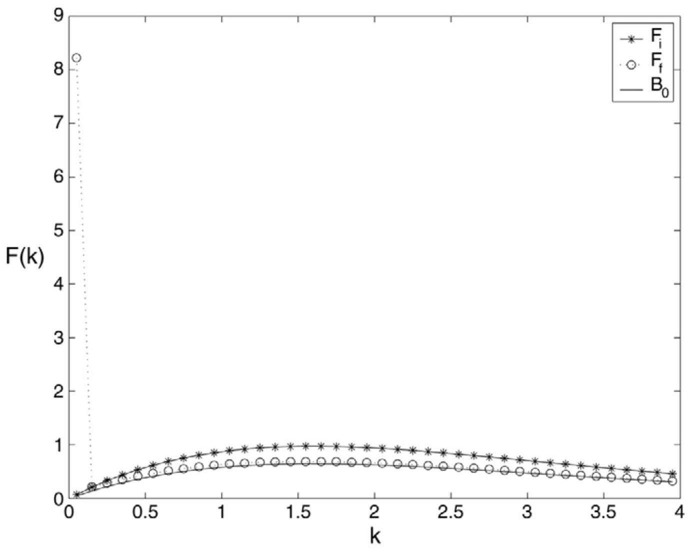

Fig. 5. Initial and final distribution for **i5** and **k1** with $N = 40$.

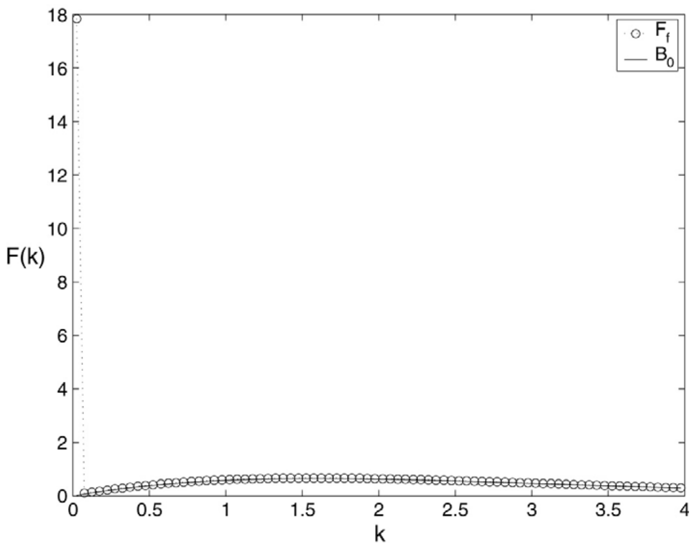

Fig. 6. Final distribution for **i5** and **k1** with $N = 80$.

by using $\Delta t = 10^{-4}$. The time of convergence obtained for **i5** and **k3** with $N = 40$ is $T_f = 1.6330$, and with $N = 80$ it is $T_f = 1.1193$. The value of T_f seems to decrease when k_1 is closer to 0. In order to confirm this behavior, the following simulations are performed. Let $N = 80$, $\Delta t = 10^{-4}$ and $k_1 = \lambda \, \Delta k$ for decreasing values of $\lambda \in (0, \frac{1}{2}]$. The relation between λ and the time of convergence T_f is shown in Fig. 8. It shows that T_f tends to 0, when $k_1 \to 0$. A concentration of photons occurs at $k = 0$ at time 0.

So far, L^1 initial data have been considered. In order to study what happens for data that are bounded measures, similar simulations are done for the following initial data.

i6. $F_i = C_3\,(\delta_{k=0} + \delta_{k=2})$ or $F_i = C_3\,(\delta_{k=1} + \delta_{k=2})$, with C_3 such that $M_i \leq \tilde{M}_0$.

i7. $F_i = C_4\,(\delta_{k=0} + \delta_{k=2})$ or $F_i = C_4\,(\delta_{k=1} + \delta_{k=2})$, with C_4 such that $M_i > \tilde{M}_0$.

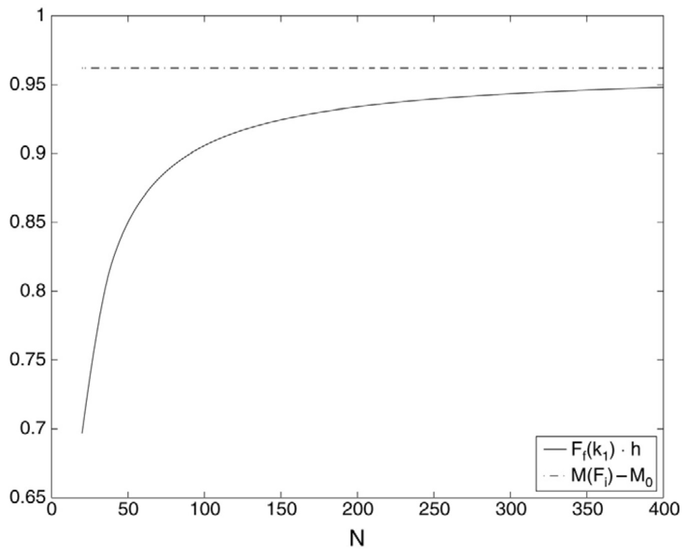

Fig. 7. Computation of $F(k_1)\,\Delta k_N$ for different values of N.

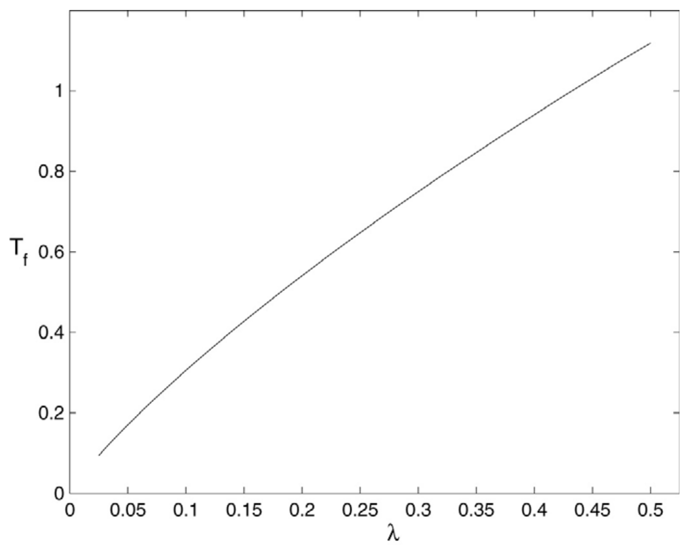

Fig. 8. Relation between λ and T_f.

The evolution obtained for **i6** and **ki**, **i** $= 1, 2, 3$ leads to the distribution B_μ having the same total number of photons as F_i. If the initial datum is **i7**, it leads to $B_0 + \left(M_i - \tilde{M}_0\right)\,\delta_{k=0}$ for **ki**, **i** $= 1, 2, 3$, by choosing $\Delta t = 10^{-4}$ in the simulation for **k3**.

5. Global existence or non-existence, depending on the initial datum

Denote by $G = F - B_0$. Solving the Cauchy problem (2.7) is equivalent to solving

$$\begin{aligned}&\frac{\partial G}{\partial t} = B_0 \int b(k,k')G'(1-e^{-k'})\mathrm{d}k' + G(t,k)\int b(k,k')(G'(e^{-k}-e^{-k'}) - B_0'(1-e^{-k}))\mathrm{d}k',\\ &G(0,k) = F_i(k) - B_0(k).\end{aligned} \tag{5.1}$$

Notice that, since for a non-negative G, $B_0 \int b(k,k')G'(1-e^{-k'})\mathrm{d}k'$ is also non-negative, an initial datum F_i smaller (resp. bigger) than the Planck distribution function B_0 will provide a $F(t,\cdot)$ remaining so for a.e. positive time t.

Theorem 5.1. *Assume that* $0 \le F_i(k) \le B_0(k)$, *a.a.* $k > 0$. *Then there is a solution* $F \in C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ *to the Cauchy problem* (2.7), *such that*

$$0 \le F(t,k) \le B_0(k), \quad t \ge 0, \quad a.a.\ k > 0.$$

Proof of Theorem 5.1. Let $b_j \in C^1(\mathbb{R}_+^2)$ be such that

$$b_j(k,k') = b(k,k'), \quad k > \frac{1}{j}, \quad k' > \frac{1}{j},$$
$$b_j(k,k') = 0, \quad k < \frac{1}{2j} \text{ or } k' < \frac{1}{2j}.$$

Let K be the closed subset of $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ defined by

$$K = \{f;\ 0 \le f(t,k) \le B_0(k), t \ge 0, \text{a.a. } k > 0\}.$$

Define the map $\mathcal{T}$ on K by $\mathcal{T}(f) = F$ solution to

$$\frac{\partial F}{\partial t} = \int b_j (f'(k^2+F)e^{-k} - F(k'^2+f')e^{-k'})\mathrm{d}k',$$
$$F(0,k) = F_i(k), \quad \text{a.a. } k > 0.$$

By the exponential form of F, the non-negativity of $f \in K$ and F_i implies that $F = \mathcal{T}(f)$ is non-negative. Then, if $g := f - B_0$, $G := \mathcal{T}(f) - B_0$,

$$\frac{\partial G}{\partial t} = B_0(1-e^{-k}) \int b_j g' \mathrm{d}k' + G \int b_j (g'(e^{-k} - e^{-k'}) - B_0'(1-e^{-k}))\mathrm{d}k',$$

so that the non-positivity of g and $G(0,\cdot)$ implies that G is non-positive. Consequently $\mathcal{T}(K) \subset K$. Then $\mathcal{T}$ is continuous for the $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ topology. Indeed, if $F_1 = \mathcal{T}(f_1)$ and $F_2 = \mathcal{T}(f_2)$,

$$\frac{\partial}{\partial t}(F_1 - F_2) = \int b_j ((f_1' - f_2')((k^2+F_1)e^{-k} + F_2 e^{-k'}) + (F_1 - F_2)(f_2' e^{-k} - (k'^2 + f_1')e^{-k'}))\mathrm{d}k',$$

so that

$$\int |F_1 - F_2|(t,k)\mathrm{d}k \le \tilde{c} \int_0^t e^{t-s} \int |(f_1 - f_2)(s,k)| \mathrm{d}k\, \mathrm{d}s,$$

for some constant $\tilde{c}$. It finally follows from the exponential form of $F = \mathcal{T}(f)$ that $\mathcal{T}$ is compact for the $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ topology.

By a Schauder fixed point theorem, there is a solution $F \in K \cap C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ to

$$\frac{\partial F_j}{\partial t} = \int b_j \left(F_j'(k^2 + F_j)e^{-k} - (k'^2 + F_j')e^{-k'} \right) \mathrm{d}k', \quad F_j(0,k) = F_i(k), \text{a.a. } k > 0.$$

Denote by $h_j(k,k') = kk' b_j(k,k')$. The sequence $(G_j) := (F_j - B_0)$ satisfies $-B_0 \le G_j \le 0$ and

$$\frac{\partial G_j}{\partial t} = \frac{G_j}{k} \int h_j(k,k')(e^{-k} - e^{-k'}) \frac{G_j'}{k'} \mathrm{d}k' + \frac{B_0}{k} \int h_j G_j' \frac{1-e^{-k'}}{k'} \mathrm{d}k' - G_j \frac{1-e^{-k}}{k} \int h_j \frac{B_0'}{k'} \mathrm{d}k'.$$

Then, by (4.10), $h_j(k,k')(e^{-k} - e^{-k'})$ is continuous and uniformly bounded w.r.t. j, so that one can pass to the limit in the weak formulation of the last equation when $j \to +\infty$. And so, there is a solution $G = \lim_{j\to+\infty} G_j$ to the Cauchy problem (2.7). □

The situation is completely different when the initial datum is almost everywhere bigger than the Planck distribution function.

Theorem 5.2. *Let the initial datum F_i be an L^1 function a.e. bigger than the Planck distribution function and have bounded number of photons and energy. Then, there is no time $T > 0$ such that the Cauchy problem (2.7) has a solution in $F \in C([0,T], \mathcal{M}_1(0,+\infty))$.*

Proof of Theorem 5.2. In order to simplify the notation, let $h(k,k') = b2(k,k')$ denote the second part of the physical kernel introduced in Section 4. Then the Cauchy problem (5.1) writes

$$\frac{\partial G}{\partial t} = \frac{G}{k}\int e^{-k}\frac{G'}{k'}h\mathrm{d}k' - \frac{G}{k}\int e^{-k'}\frac{G'}{k'}h\mathrm{d}k' + \frac{B_0}{k}\int G'\frac{1-e^{-k'}}{k'}h\mathrm{d}k' - G\frac{1-e^{-k}}{k}\int \frac{B_0'}{k'}h\mathrm{d}k', \tag{5.2}$$

$$G(0,k) = (F_i - B_0)(k), \quad \text{a.a. } k > 0. \tag{5.3}$$

Assume there exists a solution G to the Cauchy problem (5.2) and (5.3) on a time interval $[0,T]$ with $T > 0$. Since the initial datum $F_i - B_0$ is positive, it will be the same for the absolutely Lebesgue continuous part of F on an interval of time $[0,T_1]$, $0 < T_1 \le T$. And so, Proposition 3.1 applies on $[0,T_1]$. Integrate (5.2) w.r.t. $k \in (0,\epsilon)$, for some $\epsilon > 0$. Performing the change of variables $(k,k') \to (k',k)$ in the second and fourth terms of the collision operator, the integration of the right-hand side gives

$$\int_0^\epsilon \left(\int_\epsilon^{+\infty}\frac{GG'}{kk'}h(e^{-k}-e^{-k'})\mathrm{d}k'\right)\mathrm{d}k + \int_\epsilon^{+\infty} G\frac{1-e^{-k}}{k}\left(\int_0^\epsilon \frac{B_0'}{k'}h\mathrm{d}k'\right)\mathrm{d}k$$
$$- \int_0^\epsilon G\frac{1-e^{-k}}{k}\left(\int_\epsilon^{+\infty}\frac{B_0'}{k'}h\mathrm{d}k'\right)\mathrm{d}k.$$

Taking the constant $2c^3r_0^2\pi^2$ as 1 for the sake of simplicity, the function h is given by

$$h(k,k') = \int_0^\pi (1+\cos^2\theta)\frac{\sin\theta}{|w|}e^{-\frac{1}{8c}w^2 - \frac{c(k-k')^2}{2w^2} + \frac{1}{2}(k+k')},$$

where

$$|w| = \sqrt{k^2 + k'^2 - 2kk'\cos\theta}.$$

Moreover,

$$\int_0^\pi \frac{\sin\theta}{|w|}\mathrm{d}\theta = \int_{-1}^1 \frac{\mathrm{d}y}{\sqrt{k^2+k'^2-2kk'y}} = \frac{2}{\max\{k,k'\}}.$$

Then, for a.a. $0 < k < \epsilon < k'$,

$$h(k,k')(e^{-k}-e^{-k'}) \ge \frac{2}{k'}e^{-\frac{1}{8c}(k+k')^2 - \frac{c}{2} + \frac{1}{2}(k+k')}(e^{-k}-e^{-k'})$$
$$\ge 2e^{-\frac{1}{8c}(\epsilon+k')^2 - \frac{c}{2}}\left(1 - \frac{k}{k'}\right).$$

Hence, for $\epsilon < \frac{1}{4}$ and for a.a. $0 < k < \epsilon < \sqrt{\epsilon} < k'$,

$$h(k,k')(e^{-k}-e^{-k'}) \ge e^{-\frac{1}{8c}(\frac{1}{4}+k')^2 - \frac{c}{2}}.$$

And so,

$$\int_0^\epsilon \left(\int_\epsilon^{+\infty}\frac{GG'}{kk'}h(e^{-k}-e^{-k'})\mathrm{d}k'\right)\mathrm{d}k \ge \frac{1}{\epsilon}e^{-\frac{c}{2}}\left(\int_{\sqrt{\epsilon}}^{+\infty}\frac{G'}{k'}e^{-\frac{1}{8c}(\frac{1}{4}+k')^2}\mathrm{d}k'\right)\left(\int_0^\epsilon G\mathrm{d}k\right).$$

Then, for $\epsilon < \frac{1}{4}$ and a.a. $0 < k' < \epsilon < k$,

$$\frac{B_0'}{k'} \ge \frac{1}{2} \quad \text{and} \quad h(k,k') \ge \frac{2}{k}e^{-\frac{c}{2} - \frac{1}{8c}(k+\frac{1}{4})^2},$$

so that

$$\int_{\epsilon}^{+\infty} G\frac{1-e^{-k}}{k}\left(\int_0^{\epsilon}\frac{B_0'}{k'}h\mathrm{d}k'\right)\mathrm{d}k \geq \epsilon\, e^{-\frac{c}{2}}\int_{\epsilon}^{+\infty} G\frac{1-e^{-k}}{k^2}e^{-\frac{1}{8c}(k+\frac{1}{4})^2}\mathrm{d}k.$$

Moreover, for a.a. $0 < k < \epsilon < k'$,

$$\frac{1-e^{-k}}{k}\int_{\epsilon}^{+\infty}\frac{B_0'}{k'}h(k,k')\mathrm{d}k' \leq 8\frac{s\,h\frac{k}{2}}{k}\int_{\epsilon}^{+\infty}\frac{e^{\frac{1}{2}k'}}{e^{k'}-1}\mathrm{d}k' \leq 8\ln\left(1+\frac{4}{\epsilon}\right).$$

Let $\epsilon_2 := \frac{1}{4}\wedge\epsilon_1$, where $8\ln(1+\frac{4}{\epsilon}) \leq \frac{1}{\sqrt{\epsilon}}$, $\epsilon < \epsilon_1$. For $\epsilon < \epsilon_2$ and a.a. $0 < k < \epsilon < k'$,

$$\frac{1-e^{-k}}{k}\int_{\epsilon}^{+\infty}\frac{B_0'}{k'}h(k,k')\mathrm{d}k' \leq \frac{1}{\sqrt{\epsilon}}.$$

And so, the integration of (5.2) w.r.t. $k \in (0,\epsilon)$, $\epsilon < \epsilon_2$, implies that

$$e^{\frac{c}{2}}\frac{\partial}{\partial t}\int_0^{\epsilon} G(t,k)\mathrm{d}k \geq \frac{1}{\epsilon}\left(\int_{\sqrt{\epsilon}}^{+\infty}\frac{G(t,k')}{k'}e^{-\frac{1}{8c}(\frac{1}{4}+k')^2}\mathrm{d}k' - e^{\frac{c}{2}}\sqrt{\epsilon}\right)\int_0^{\epsilon} G(t,k)\mathrm{d}k + \epsilon\int_{\epsilon}^{+\infty} G(t,k)\frac{1-e^{-k}}{k^2}e^{-\frac{1}{8c}(k+\frac{1}{4})^2}\mathrm{d}k,$$

so that

$$\int_0^{\epsilon} G(t,k)\mathrm{d}k \geq \left(\int_{\frac{t}{4}}^{\frac{t}{2}}\int_{\epsilon}^{+\infty} G(s,k)\frac{1-e^{-k}}{k^2}e^{-\frac{1}{8c}(k+\frac{1}{4})^2}\mathrm{d}k\mathrm{d}s\right)\left(\epsilon\, e^{\frac{1}{\epsilon}(\int_{\frac{t}{2}}^{t}\int_{\sqrt{\epsilon}}^{+\infty}\frac{G(\tau,k)}{k}e^{-\frac{1}{8c}(\frac{1}{4}+k)^2}\mathrm{d}k\mathrm{d}\tau - \frac{1}{2}e^{\frac{c}{2}}\sqrt{\epsilon}t)}\right).$$

By Propositions 2.1 and 2.3 the integral $\int G(t,k)\mathrm{d}k$ is conserved with time. If there were a time $t \in (0,T_1]$ such that for any $\epsilon < \epsilon_2$,

$$\int_{\frac{t}{2}}^{t}\int_{\sqrt{\epsilon}}^{+\infty}\frac{G(\tau,k)}{k}e^{-\frac{1}{8c}(\frac{1}{4}+k)^2}\mathrm{d}k\mathrm{d}\tau \leq e^{\frac{c}{2}}\sqrt{\epsilon}t,$$

then making ϵ tend to zero in this last inequality would imply $G(\tau,\cdot) = 0$, $\tau \in (\frac{t}{2}, t)$, which would contradict the conservation of $\int G(t,k)\mathrm{d}k$ with time. Consequently, for any $t \in (0,T_1]$, there is an $\epsilon_t < \epsilon_2$ such that

$$\int_{\frac{t}{2}}^{t}\int_{\sqrt{\epsilon_t}}^{+\infty}\frac{G(\tau,k)}{k}e^{-\frac{1}{8c}(\frac{1}{4}+k)^2}\mathrm{d}k\mathrm{d}\tau > e^{\frac{c}{2}}\sqrt{\epsilon_t}t.$$

Then, for any $\epsilon < \epsilon_t$,

$$\int_{\frac{t}{2}}^{t}\int_{\sqrt{\epsilon}}^{+\infty}\frac{G(\tau,k)}{k}e^{-\frac{1}{8c}(\frac{1}{4}+k)^2}\mathrm{d}k\mathrm{d}\tau > e^{\frac{c}{2}}\sqrt{\epsilon}t.$$

Consider any $t \in (0,T_1]$. For any $\epsilon < \epsilon_t$,

$$\int_0^{+\infty} G_i(k)\mathrm{d}k \geq \int_0^{\epsilon} G(t,k)\mathrm{d}k \geq \left(\int_{\frac{t}{4}}^{\frac{t}{2}}\int_{\epsilon}^{+\infty} G(s,k)\frac{1-e^{-k}}{k^2}e^{-\frac{1}{8c}(k+\frac{1}{4})^2}\mathrm{d}k\mathrm{d}s\right)\epsilon e^{\frac{t}{2\sqrt{\epsilon}}e^{\frac{c}{2}}}.$$

Since $\lim_{\epsilon\to 0}\epsilon e^{\frac{t}{2\sqrt{\epsilon}}e^{\frac{c}{2}}} = +\infty$,

$$\int_{\frac{t}{4}}^{\frac{t}{2}}\int_{\epsilon}^{+\infty} G(s,k)\frac{1-e^{-k}}{k^2}e^{-\frac{1}{8}(k+\frac{1}{4})^2}\mathrm{d}k\mathrm{d}s$$

should tend to zero when ϵ tends to zero. This would imply that $\frac{G(s,k)}{k}$, $s \in (\frac{t}{4}, \frac{t}{2})$ be a Dirac measure at $k = 0$, which would contradict the continuity of F at time zero. □

6. Conclusion

The quantum kinetic equation linked to the Compton effect studied in this paper presents a strong singularity at energy zero. Numerical simulations, with more and more singular kernels up to the physical one, point out the real singular part of the collision kernel. Moreover, they show that the problem behaves quite differently depending on the value of the initial total number of photons $M(F_i)$ compared to the Planck one M_0. In particular, blow-up immediately occurs with the physical kernel if $M(F_i) > M_0$. Theoretical results stress these features for L^1 initial data that are almost everywhere smaller (resp. bigger) than the Planck distribution function.

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