LOCALIZATION ON 5 SITES FOR VERTEX REINFORCED RANDOM WALKS: TOWARDS A CHARACTERIZATION

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Abstract. We continue the investigation of the localization phenomenon for a Vertex Reinforced Random Walk on the integer lattice. We provide some partial results towards a full characterization of the weights for which localization on 5 sites occurs with positive probability, and make some conjecture concerning the almost sure behavior.

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1. Introduction

Given a sequence $w = (w(n))_{n\geq 0}$ of positive real numbers, called the weight, one can define a process $(X_n)_{n\geq 0}$ on $\mathbb{Z}$, called Vertex Reinforced Random Walk (VRRW) as follows: first $X_0 = 0$, and then for any $n \geq 0$ and $x \in \mathbb{Z}$,

$$\mathbb{P}(X_{n+1} = x \pm 1 \mid \mathcal{F}_n) = \frac{w(Z_n(x \pm 1))}{w(Z_n(x+1)) + w(Z_n(x-1))}, \tag{1}$$

where $\mathcal{F}_n := \sigma(X_0, \ldots, X_n)$ and $Z_n(y)$ is the number of visits to site y by the process before time n (see below). This process was introduced by Pemantle [P] on the complete graph and for a linear weight, and then by Pemantle and Volkov on $\mathbb{Z}$, still for the linear weight, who showed that the process localizes on five sites with positive probability, that is with positive probability exactly five sites are visited infinitely often. This result was later improved by Tarrès who showed [T1, T2] that this behavior occurs in fact almost surely.

A few years later, Volkov [V] introduced the model with a general weight sequence, in the same fashion as Davis [Dav] did for Edge Reinforced Random Walks. He proved in particular that for weights of the form $w(n) = n^\alpha$, with $\alpha < 1$, localization on a finite subgraph is not possible. This was later improved in [CK, Sch, S] in the case $\alpha < 1/2$, where it was proved that the process visits almost surely all sites infinitely often.

In a previous work in collaboration with Basdevant and Singh [BSS], we managed to completely characterize the nondecreasing weights for which localization on 4 sites occurs with positive probability, or almost surely, in terms of some parameter $\alpha_c(w)$ (see below). Our aim here is to analyze the analogous question for the localization on 5 sites. For this we introduce some new parameter $\beta_c(w)$, which should play a similar role as $\alpha_c(w)$. To define it, we first extend w as a function on the positive reals by $w(t) := w(\lfloor t \rfloor)$, and then set

$$W(t) := \int_1^t \frac{1}{w(u)}\, du.$$

We will assume throughout the paper that

$$\sum_{n=0}^{\infty} \frac{1}{w(n)} = \infty, \tag{2}$$

which is equivalent to saying that W is a bijection from $\mathbb{R}_+$ to itself. Note however, that this is not a restrictive hypothesis, since when w is reciprocally summable, it is known [BSS, V] that the process localizes almost surely on two sites. Then we denote by W^{-1} its inverse, and define for $\alpha > 0$,

$$I_\alpha(w) := \int_0^\infty \frac{dx}{w(W^{-1}(W(x)+\alpha))}.$$

When w is nondecreasing, the map $\alpha \mapsto I_\alpha(w)$ is nonincreasing and one defines

$$\alpha_c(w) := \inf\{\alpha \geq 0 \ : \ I_\alpha(w) < \infty\} \in [0, \infty], \tag{3}$$

with the convention that $\inf \emptyset = \infty$. In [BSS] it was proved in particular that localization on 4 sites holds with nonzero probability if, and only if, $\alpha_c(w)$ is finite. We now define for $\beta \in \mathbb{R}$,

$$J_\beta(w) := \int_0^\infty \frac{dx}{w(W^{-1}(2W(x)+\beta))},$$

with the convention that $W^{-1}(u) = 0$, for $u < 0$, and set

$$\beta_c(w) := \inf\{\beta \in \mathbb{R} \ : \ J_\beta(w) < \infty\} \in [-\infty, +\infty].$$

We make the following conjecture (with R' standing for the set of sites which are visited infinitely often):

Conjecture 1.1. *Assume that w is nondecreasing and satisfies* (2). *Assume further that $\alpha_c(w) = \infty$. Then*

$$\mathbb{P}(|R'| = 5) > 0 \iff \mathbb{P}(|R'| = 5) = 1 \iff \beta_c(w) < \infty.$$

Remark 1.1. With the notation of [BSS2], we also conjecture that localization on 5 sites is equivalent to having $i_{1/2}(w) = 2$ (which by definition is itself equivalent to having $J_0(w) = \infty$). In fact as we will later explain, we conjecture that $\beta_c(w)$ always belongs to $\{\pm\infty\}$.

The hardest part here is the characterization of the almost sure localization, which is a notoriously difficult problem that we will not discuss in this paper; we simply recall that in the case of a linear weight, Tarrès proved that $|R'| = 5$ almost surely [T1, T2]. Proving that the same holds for some other weight function is possibly one of the most challenging problem on this model. Instead we will only be interested here on the easiest part of the conjecture, which is a characterization of the localization with positive probability. Our first result provides one direction of the conjecture:

Theorem 1.1. *Assume that w is nondecreasing. Then*

$$\mathbb{P}(|R'| = 5) > 0 \quad \Longrightarrow \quad \beta_c(w) < +\infty.$$

We note that this result was proved in [BSS2] (see the proof of Proposition 1.4 there) under some additional hypotheses on w, including the fact that w was a slowly varying function.

Our second result concerns the other direction. However, instead of $\beta_c(w)$ being finite, one needs to assume some slightly stronger condition (which we nevertheless

conjecture to be equivalent). Namely, we first define $H(x) := x + W^{-1}(W(x)+1)$, and note that H is increasing and continuous; thus it has an inverse which we denote by H^{-1}. Then set for $\beta \in \mathbb{R}$,

$$\widetilde{J}_\beta(w) := \int_0^\infty \frac{dx}{w(H^{-1}\left(W^{-1}(2W(x)+\beta)\right))},$$

and

$$\widetilde{\beta}_c(w) := \inf\{\beta \in \mathbb{R} \ : \ \widetilde{J}_\beta(w) < \infty\} \ \in [-\infty, +\infty].$$

Note that $H(x) \geq x$ and $H^{-1}(x) \leq x$, for all $x \geq 0$. Thus for any $\beta \in \mathbb{R}$, $\widetilde{J}_\beta(w) \geq J_\beta(w)$. In particular for any w,

$$\widetilde{\beta}_c(w) \ \geq \ \beta_c(w).$$

Our second result is the following:

Theorem 1.2. *Assume that w is nondecreasing and satisfies* (2)*. Assume further that $\alpha_c(w) = \infty$. Then*

$$\widetilde{\beta}_c(w) < \infty \implies \begin{cases} \mathbb{P}(5 \leq |R'| < \infty) = 1, \\ \mathbb{P}(|R'| \in \{5, 6\}) > 0. \end{cases}$$

As mentioned above we conjecture that in fact $\beta_c(w) = \widetilde{\beta}_c(w)$, for all weights w. We provide some evidence for this fact at the end of the paper, and show that it is true for a large class of weight functions (see Lemmas 5.1 and 5.2).

In particular Lemma 5.1 shows that for any surlinear weight function, such that $w(n) = o(n\sqrt{\log n})$, one has $\widetilde{\beta}_c(w) < \infty$. This is of course not surprising, regarding the known result for a linear weight, but let us observe that prior to this, nothing was known for weights with intermediate growth between linear and $n \log\log n$ (apart from the fact that the associated process could not localize on any finite subgraph with 4 or less sites). Note that when $\liminf w(n)/(n\log\log n) > 0$, then $\alpha_c(w)$ is finite, and localization was proved in [BSS] for such weights.

Now it is a bit disappointing that we cannot exclude the possibility of a localization on 6 sites in the conclusion of Theorem 1.2, especially since for a linear weight [PV, T2], as well as for weights satisfying the so-called condition $i_+(w) = 2$ from [BSS2], it has been proved that localization on 5 sites occurs with positive probability. Let us however stress that, concerning the weight $w(n) = n + 1$, both the original proof by Pemantle and Volkov [PV], as well as the simplified one by Tarrès [T2] heavily rely on the explicit form of the linear weight, and cannot be transposed (at least not directly) to the general setting we are considering here. On the other hand the hypotheses on w made in [BSS2] are stronger than ours (in particular in addition to the hypothesis $i_+(w) = 2$, which is already stronger than requiring $\widetilde{\beta}_c(w)$ being finite, some regularity condition was also imposed for w), and moreover, the proof in [BSS2] requires more sophisticated arguments than here.

Let us finally add that we also believe that localization on any even number of sites, larger than or equal to 6, is not possible for any weight function. In contrast it was proved in [BSS2] that localization on any odd number of sites – other than one and three – is possible.

The paper is organized as follows. In the next section, we recall some important and elementary facts about the VRRW, and some related martingales attached to each site. Then in Sections 3 and 4 we give the proofs of Theorems 1.1 and 1.2

respectively. The final section is concerned with the computation of the parameters $\beta_c(w)$ and $\widetilde{\beta}_c(w)$, and gives some cases where one can show equality between them.

2. Notation and background

2.1. **VRRW.** Given some initial distribution of local times $\mathcal{C} := (z_0(y))_{y\in\mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}$, we define the $\mathcal{C}$-VRRW as the process $(X_n)_{n\geq 0}$, whose transition probabilities are given by (1), with for any $y \in \mathbb{Z}$, $Z_0(y) = z_0(y)$, and for any $n \geq 1$,

$$Z_n(y) := z_0(y) + \sum_{k=1}^{n} \mathbf{1}\{X_k = y\}.$$

We call $\mathcal{C}_0$ the configuration with $z_0(y) = 0$, for all $y \neq 0$ and $z_0(0) = 1$. We then simply say that X is a VRRW when its initial local time distribution is given by $\mathcal{C}_0$. We also recall that a $\mathcal{C}$-VRRW can be defined as well on any subgraph of $\mathbb{Z}$, and we refer to [BSS] for details.

2.2. **The martingales $M_n(x)$.** For $x \in \mathbb{Z}$, define $Z_\infty(x) := \lim_{n\to\infty} Z_n(x)$. Recall that R' stands for the set of sites visited infinitely often by the walk:

$$R' := \{x \in \mathbb{Z} \ : \ Z_\infty(x) = \infty\}.$$

We define for any $n \geq 1$, and $x \in \mathbb{Z}$,

$$Y_n^{\pm}(x) := \sum_{k=0}^{n-1} \frac{\mathbf{1}\{X_k = x,\, X_{k+1} = x \pm 1\}}{w(Z_k(x \pm 1))}, \tag{4}$$

and

$$M_n(x) := Y_n^+(x) - Y_n^-(x).$$

We let also $Y_0^{\pm}(x) = 0$, and $M_0(x) = 0$, and consider the limits:

$$Y_\infty^{\pm}(x) := \lim_{n\to\infty} Y_n^{\pm}(x).$$

An important observation from Tarrès [T1, T2] is that $(M_n(x))_{n\geq 1}$ is a martingale for each $x \in \mathbb{Z}$. Moreover, if

$$\sum_{n=0}^{\infty} \frac{1}{w(n)^2} < \infty, \tag{5}$$

then these martingales are bounded in L^2, and thus converge almost surely and in L^2. Moreover, for any $\mathcal{C}$-VRRW, one has

$$Y_n^+(x-1) + Y_n^-(x+1) = W(Z_n(x)) - W(z_0(x)). \tag{6}$$

We will also use the following result due to Tarrès (see also [BSS, Lemma 3.3]):

Lemma 2.1 (Tarrès [T2])**.** *Assume that w is nondecreasing and that* (5) *holds. Then, for any $x \in \mathbb{Z}$, almost surely,*

$$\{Y_\infty^+(x) < \infty\} = \{Y_\infty^-(x) < \infty\} \ = \ \{Z_\infty(x-1) < \infty\} \cup \{Z_\infty(x+1) < \infty\}.$$

We further use the same notation as in [T2], and write $f(n) \equiv g(n)$, when the sequence $(f(n) - g(n))_n$ converges to some finite real. In particular, it follows from the above discussion that

$$\sum_{n\geq 0} \frac{1}{w(n)^2} < \infty \quad \Longrightarrow \quad Y_n^+(x) \equiv Y_n^-(x), \text{ for all } x \in \mathbb{Z}. \tag{7}$$

3. Proof of Theorem 1.1

We start the proof with the following lemma:

Lemma 3.1. *Assume that w is nondecreasing. Then*

$$\mathbb{P}(|R'| = 5) > 0 \quad \Longrightarrow \quad \sum_{n=0}^{\infty} \frac{1}{w(n)^2} < \infty.$$

Remark 3.1. This result has the same flavor than some others from [Sch, S, V], which all give different conditions on the weight w, ensuring that localization on any finite subgraph is not possible. In particular the proof in [S] shows that for any weight satisfying $\limsup n/w(n)^2 = \infty$, the walk cannot localize on any finite subgraph, which is close to imply our result (but not quite).

Proof of Lemma 3.1. We first note that if localization on five sites occurs with positive probability, then there exists some initial configuration $\mathcal{C}$, such that with positive probability the $\mathcal{C}$-VRRW spends all its time in the set $\{1, 2, 3, 4, 5\}$, and visits all sites from this set infinitely often. Call E this event. By the conditional Borel-Cantelli Lemma (see Theorem 4.3.2 in [Dur]), one can see that almost surely on the event E, one has $Y_\infty^+(1) < \infty$, since for some constant $c > 0$ (only depending on $\mathcal{C}$), one has

$$Y_\infty^+(1) \leq c \sum_{k \geq 0} \mathbb{P}\left[X_{k+1} = 0 \mid \mathcal{F}_k\right] \mathbf{1}\{X_k = 1\},$$

where we denote here by $\mathcal{F}_k$ the sigma field generated by the process X up to its k-th visit to site 1. Then we use a fact, already observed in [BSS] (see the end of Section 4 there), which is that the following process is a martingale:

$$Y_n^+(1) - \sum_{k=1}^{Z_n(2)-z_0(2)} \frac{p_k(2,1)}{w(k+z_0(2))},$$

where $p_k(2,1)$ denotes the probability to jump to site 1, at k-th visit to site 2. Since this martingale has bounded increments, we know that almost surely, either it converges, or its lim sup as well as its lim inf are both infinite (see Theorem 4.3.1 in [Dur]). However, we have just observed that on the event E, its lim sup is finite, which means that it must converge, and as a consequence on the event E, it holds almost surely

$$\sum_{k=0}^{\infty} \frac{p_k(2,1)}{w(k+z_0(2))} < \infty.$$

Now by definition of $p_k(2,1)$, one has for some constant $c > 0$ (depending only on $\mathcal{C}$), and on E,

$$\sum_{k=0}^{\infty} \frac{1}{w(k+z_0(2))w(Z_{\tau_k}(3))} \leq c \sum_{k=0}^{\infty} \frac{p_k(2,1)}{w(k+z_0(2))} < \infty,$$

where τ_k denotes the time of k-th visit to site 2. By symmetry one has as well

$$\sum_{k=0}^{\infty} \frac{1}{w(k+z_0(4))w(Z_{\widetilde{\tau}_k}(3))} < \infty,$$

with $\widetilde{\tau}_k$ the time of k-th visit to site 4. Finally observe that for any n, $Z_n(3) \leq Z_n(2) + Z_n(4) + C$, with C a constant depending only on $\mathcal{C}$. This implies that for

any k, either $Z_{\tau_k}(3) \le 2k + C + z_0(3)$ or $Z_{\widetilde{\tau}_k}(3) \le 2k + C + z_0(3)$. Using that w is nondecreasing, it follows that for some (possibly larger) constant $C > 0$,

$$\sum_{k\ge 0} \frac{1}{w(2k+C)^2} < \infty.$$

The lemma follows, using again that w is nondecreasing. □

We next prove the following result.

Lemma 3.2. *Let X be a $\mathcal{C}$-VRRW, for some initial local time configuration $\mathcal{C}$. Assume that w is nondecreasing, and satisfies (2) and (5). Then on the event $E = \{Z_\infty(0) = Z_\infty(4) = \infty\} \cap \{Y^+_\infty(0) < \infty\} \cap \{Y^-_\infty(4) < \infty\}$, it holds almost surely*

$$Z_n(2) - \max(Z_n(1), Z_n(3)) \to +\infty, \qquad \text{as } n \to \infty.$$

Proof. Let for $n \ge 1$,

$$N_n(y, y\pm 1) := \sum_{k=0}^{n-1} \mathbf{1}\{X_k = y,\, X_{k+1} = y \pm 1\},$$

denotes the number of jumps from y to $y \pm 1$ before time n, for any $y \in \mathbb{Z}$. Then $Z_n(1) \equiv N_n(0,1)+N_n(2,1)$ and similarly, $Z_n(2) \equiv N_n(1,2)+N_n(3,2)$. Now observe that $W(N_n(2,1)) - Y^-_n(2)$ is nondecreasing and that for all n,

$$0 \le W(N_n(2,1)) - Y^-_n(2) \le Y^+_n(0) + W(z_0(1)).$$

Since by definition $Y^+_\infty(0)$ is finite on the event E, we deduce that

$$W(N_n(2,1)) \equiv Y^-_n(2).$$

By symmetry, one has as well

$$W(N_n(2,3)) \equiv Y^+_n(2),$$

and since by (7), one also has $Y^-_n(2) \equiv Y^+_n(2)$, we get in fact

$$W(N_n(2,1)) \equiv W(N_n(2,3)). \tag{8}$$

Moreover, Lemma 2.1 implies that under the hypotheses of the lemma and on the event E, $Z_\infty(-1)$ is finite, and thus $Y^+_\infty(-1)$ also. Together with (6), it follows that

$$W(N_n(1,0)) \equiv W(Z_n(0)) \equiv Y^-_n(1) \equiv Y^+_n(1). \tag{9}$$

We claim now that

$$\delta_n := W(N_n(1,2)) - Y^+_n(1) \to +\infty. \tag{10}$$

To see this note first that δ_n is nondecreasing. Let then $(i_k)_{k\ge1}$ and $(j_k)_{k\ge1}$, be the increasing sequences of integers, such that for all $n \ge 0$,

$$Y^-_n(3) = \sum_{k=1}^{N_n(3,2)} \frac{1}{w(i_k)}, \quad \text{and} \quad Y^+_n(1) = \sum_{k=1}^{N_n(1,2)} \frac{1}{w(j_k)}.$$

Set also $n_0 = 0$, and for $k \ge 1$,

$$n_k := \inf\{n > n_{k-1} : w(j_n) \ge 2w(i_k)\}, \quad \text{and} \quad N_k := \inf\{n \ge 0 : N_n(1,2) \ge n_k\}.$$

Then for any $k \ge 1$,

$$\delta_{N_k} \ge \frac{1}{2} \sum_{k' \le k} \frac{1}{w(i_{k'})}.$$

Since $Z_\infty(4) = Z_\infty(2) = \infty$, Lemma 2.1 shows that $Y^-_\infty(3) = \infty$, which proves (10). By using next that $|N_n(1,2) - N_n(2,1)| \leq 1$, together with (8), (9) and (10), we obtain

$$W(N_n(2,3)) - W(N_n(1,0)) \to +\infty,$$

which implies that $N_n(2,3) - N_n(1,0) \to +\infty$. By symmetry one can prove as well that $N_n(1,2) - N_n(4,3) \to +\infty$, and the lemma follows. □

Let us resume now the proof of Theorem 1.1. Lemma 3.7 in [BSS] shows that there exists some local time configuration $\mathcal{C}$, such that for the $\mathcal{C}$-VRRW, the event

$$E := \{Z_\infty(0) = Z_\infty(4) = \infty\} \cap \{Y^+_\infty(0) < \infty\} \cap \{Y^-_\infty(4) < \infty\},$$

has some positive probability. Moreover, we know by (6) that on E,

$$W(Z_n(1)) - W(Z_n(3)) \equiv Y^-_n(2) - Y^+_n(2),$$

and using (7), we deduce that $W(Z_n(1)) - W(Z_n(3))$ converges as $n \to \infty$, towards some $\alpha \in \mathbb{R}$. Furthermore, Lemma 4.8 in [BSS] shows that almost surely $\alpha \neq 0$, and by symmetry we can assume without loss of generality that $\alpha > 0$. In particular, this gives $Z_n(1) \geq Z_n(3)$, for n large enough. Set now

$$h_n(2) := \sum_{k=1}^{Z_n(2)} \frac{2p_k - 1}{w(k)},$$

where p_k is the probability to jump to site 1 at k-th visit to site 2. As noticed already in the proof of Lemma 3.1, one has $h_n(2) \equiv Y^+_n(1) - Y^-_n(3)$. But since after some time the process has at least probability $1/2$ to jump to 1 when it is in 2, we see that for n large enough $h_n(2)$ is nondecreasing. In particular there exists some (random) constant $\gamma \in \mathbb{R}$, such that $h_n(2) \geq \gamma$, for all $n \geq 0$. By using also that

$$W(Z_n(0)) \equiv Y^-_n(1) \equiv Y^+_n(1) \equiv W(Z_n(2)) - Y^-_n(3),$$

we deduce that for some $\beta \in \mathbb{R}$,

$$W(Z_n(2)) \leq 2W(Z_n(0)) + \beta, \quad \text{for all } n \geq 0. \tag{11}$$

Together with Lemma 3.2, this yields for some constant $c > 0$,

$$Y^+_\infty(0) \equiv \sum_{n=0}^{\infty} \frac{\mathbf{1}\{X_n = 0\}}{w(Z_n(1))} \geq c \sum_{n=0}^{\infty} \frac{\mathbf{1}\{X_n = 0\}}{w(Z_n(2))} \geq c \sum_{n=0}^{\infty} \frac{1}{w(W^{-1}(2W(n) + \beta))},$$

which concludes the proof of the theorem, since $Y^+_\infty(0)$ is finite on E. □

4. Proof of Theorem 1.2

We start the proof with some elementary lemma.

Lemma 4.1. *Assume that w is nondecreasing. Then*

$$\beta_c(w) < \infty \quad \Longrightarrow \quad \sum_{n=0}^{\infty} \frac{1}{w(n)^2} < \infty.$$

Proof. Assume that $w(n) \leq \sqrt{n}$, for some $n \geq 1$. Since w is nondecreasing, this implies on one hand $W(n+1) \geq \sqrt{n}$, and also $w(k) \geq w(0)$, for all $k \geq 0$. The latter implies the existence of a constant $c > 0$, such that $W(k) \leq \sqrt{n}/3$, for all $k \leq c\sqrt{n}$ (namely one can take $c = w(0)/3$). Assume that n is large enough so that

$\sqrt{n}/3 < (\sqrt{n} - \beta_c(w) - 1)/2$. Then $2W(k) + \beta_c(w) + 1 < \sqrt{n}$, for all $k \leq c\sqrt{n}$. Therefore $W^{-1}(2W(k) + \beta_c(w) + 1) < n + 1$, for all such k, and it follows that

$$\sum_{k \in [c\sqrt{n}/2, c\sqrt{n}]} \frac{1}{w(W^{-1}(2W(k) + \beta_c(w) + 1))} \geq c/2.$$

In particular, by definition of $\beta_c(w)$, this can only happen for finitely many n, which proves that $\liminf w(n)/\sqrt{n} \geq 1$, and concludes the proof of the lemma. □

The next step is the following lemma.

Lemma 4.2. *Assume that w is nondecreasing and that $\widetilde{\beta}_c(w) < \infty$. Consider a $\mathcal{C}$-VRRW on a set of the form $\{-3, \dots, y\}$, with $0 \leq y \leq \infty$, and $\mathcal{C}$ satisfying $z_0(-1) \leq z_0(-2) + z_0(0)$. Let $\varepsilon_0 \in (0, 1]$, and $\beta > (\widetilde{\beta}_c(w) + 3\varepsilon_0)/2$ be given, and define for $N \geq 1$,*

$$T_{N,\beta,\varepsilon_0} := \inf \left\{ n \geq 0 \ : \ \begin{array}{l} Z_n(-1) \wedge Z_n(0) \geq N \\ W(Z_n(-2)) \leq W(Z_n(0)) - \varepsilon_0 \\ W(Z_n(-3)) \leq W(Z_n(-1))/2 - \beta \end{array} \right\}.$$

Then, as $N \to \infty$,

$$\mathbb{P}\left(Y^+_\infty(-3) < \infty \mid T_{N,\beta,\varepsilon_0} < \infty\right) \to 1.$$

Moreover, the convergence is uniform with respect to $\mathcal{C}$.

Proof. To simplify notation we denote by $\mathbb{P}_N$ the conditional probability:

$$\mathbb{P}_N(\cdot) := \mathbb{P}(\cdot \mid T_{N,\beta,\varepsilon_0} < \infty).$$

Consider $\overline{X}$ the process reflected in 0 after time $T_{N,\beta,\varepsilon_0}$, and denote by $\overline{Z}_n$, $\overline{Y}^{\pm}_n, \dots$ all the quantities associated to $\overline{X}$. First note that since $\overline{Y}^+_\infty(-3)$ stochastically dominates $Y^+_\infty(-3)$ (see Lemma 3.6 in [BSS]), it suffices to prove the result for the process $\overline{X}$ instead of X. Set now

$$T_0 := \inf\{n \geq T_{N,\beta,\varepsilon_0} \ : \ W(\overline{Z}_n(-2)) > W(N) - \varepsilon_0\},$$

and

$$T := \inf\{n \geq T_0 \ : \ W(\overline{Z}_n(-2)) \geq W(\overline{Z}_n(0)) - \varepsilon_0/2\}.$$

Since $W(\overline{Z}_n(-2)) - W(\overline{Z}_n(0)) \equiv \overline{Y}^+_n(-3)$, we have

$$\{T_{N,\beta,\varepsilon_0} < \infty\} \cap \{T = \infty\} \subset \{\overline{Y}^+_\infty(-3) < \infty\}, \tag{12}$$

and thus one can assume now that T is finite. Then note that $\overline{Z}_{T_0}(0) \geq N$ and $\overline{Z}_{T_0}(-2) \geq W^{-1}(W(N) - 1)$, since $\varepsilon_0 \leq 1$. Recall also that $\beta_c(w) \leq \widetilde{\beta}_c(w)$, and the latter is finite by hypothesis. Thus Lemma 4.1 shows that (5) holds. Let now $\varepsilon > 0$ be given. It follows from Doob's L^2-inequality for martingales, that for N large enough,

$$\mathbb{P}_N\left(\sup_{n \geq T_0} |\overline{M}_n(-1) - \overline{M}_{T_0}(-1)| \geq \frac{\varepsilon_0}{10}\right) \leq \frac{800}{\varepsilon_0^2} \sum_{i=N'}^{\infty} \frac{1}{w(i)^2} \leq \varepsilon, \tag{13}$$

with $N' := W^{-1}(W(N) - 1)$. Moreover, since the process $\overline{X}$ is reflected in 0 after time $T_{N,\beta,\varepsilon_0}$, one has for any $n \geq T_0$,

$$\overline{Y}^+_n(-1) - \overline{Y}^+_{T_0}(-1) = W(\overline{Z}_n(0)) - W(\overline{Z}_{T_0}(0)).$$

In addition, (6) gives

$$\left(\overline{Y}^+_n(-3) - \overline{Y}^+_{T_0}(-3)\right) + \left(\overline{Y}^-_n(-1) - \overline{Y}^-_{T_0}(-1)\right) = W(\overline{Z}_n(-2)) - W(\overline{Z}_{T_0}(-2)).$$

Assume that N' is large enough so that $w(N') \geq 6/\varepsilon_0$. By definition of T_0 and T, this implies that

$$W(\overline{Z}_T(-2)) - W(\overline{Z}_{T_0}(-2)) \geq W(\overline{Z}_T(0)) - W(\overline{Z}_{T_0}(0)) + \frac{\varepsilon_0}{3}.$$

Then it follows from the last displays and (13) that for N large enough,

$$\mathbb{P}_N\left(\overline{Y}^+_T(-3) - \overline{Y}^+_{T_0}(-3) \leq \frac{\varepsilon_0}{6}\right) \leq \mathbb{P}_N\left(|\overline{M}_T(-1) - \overline{M}_{T_0}(-1)| \geq \frac{\varepsilon_0}{6}\right) \leq \varepsilon. \tag{14}$$

Recall next the definition of N' above, and define

$$N'' := \inf\{n \geq N' \,:\, W(n) - W(N') \geq \frac{\varepsilon_0}{6}\}.$$

Since w is nondecreasing, and since we recall that by definition of T_0, one has $\overline{Z}_{T_0}(x+1) \geq N'$, it follows from (6) that

$$\left\{\overline{Y}^+_T(-3) - \overline{Y}^+_{T_0}(-3) \geq \frac{\varepsilon_0}{6}\right\} \subseteq \left\{\overline{Z}_T(-3) - \overline{Z}_{T_0}(-3) \geq N'' - N'\right\}. \tag{15}$$

However, $N'' - N' \to \infty$, as $N \to \infty$. Thus it follows again from Doob's L^2-inequality that for N large enough,

$$\mathbb{P}_N\left(\sup_{n \geq T} |\overline{M}_n(-2) - \overline{M}_T(-2)| \geq \frac{\varepsilon_0}{6}\right) \leq \varepsilon. \tag{16}$$

Now similarly as in the proof of Theorem 1.1, introduce

$$h(n) = \sum_{k=1}^{\overline{Z}_n(-1)} \frac{2p_k - 1}{w(k)},$$

where p_k is the probability to jump to 0 at k-th visit to -1 for the process $\overline{X}$. Recall that

$$\overline{Y}^-_n(0) - \overline{Y}^+_n(-2) \equiv h(n),$$

and on the other hand (6) and Lemma 2.1 yield

$$W(\overline{Z}_n(-1)) \equiv \overline{Y}^+_n(-2) + \overline{Y}^-_n(0), \quad \text{and} \quad \overline{Y}^+_n(-2) \equiv \overline{Y}^-_n(-2) \equiv W(\overline{Z}_n(-3)).$$

As a consequence,

$$U(n) := W(\overline{Z}_n(-3)) - \frac{W(\overline{Z}_n(-1))}{2} \equiv -\frac{h(n)}{2}.$$

Define next

$$T' := \inf\left\{ n \geq T \;:\; \begin{array}{l} W(\overline{Z}_n(-2)) \geq W(\overline{Z}_n(0)) - \varepsilon_0/4 \\ \text{or} \\ W(\overline{Z}_n(-2)) \leq W(\overline{Z}_n(0)) - 3\varepsilon_0/4 \end{array} \right\}.$$

Since $\overline{Z}_n(-2) \leq \overline{Z}_n(0)$, for all $n \in [T_{N,\beta,\varepsilon_0}, T']$, h is nondecreasing on this interval. Set now,

$$t_{N,\beta} := \inf\{n \geq T_{N,\beta,\varepsilon_0} \;:\; W(\overline{Z}_n(-3)) \geq W(N)/2 - \beta\},$$

and note that $U(T \wedge t_{N,\beta}) \leq -\beta + \varepsilon_0/2$, if N is taken large enough. Therefore by using (16), and again Doob's L^2-inequality, we get at least for N large enough,

$$\mathbb{P}_N \left(\sup_{T \wedge t_{N,\beta} \leq n \leq T'} U(n) \geq -\beta + \varepsilon_0 \right) \leq \varepsilon. \tag{17}$$

Remember then that H is defined by $H(x) = x + W^{-1}(W(x) + 1)$, and thus by using the hypothesis on $\mathcal{C}$, we get that for all $n \in [T, T']$,

$$\overline{Z}_n(-1) \leq \overline{Z}_n(-2) + \overline{Z}_n(0) \leq H(\overline{Z}_n(-2)).$$

It follows that on the event $\{\sup_{T \wedge t_{N,\beta} \leq n \leq T'} U(n) \leq -\beta + \varepsilon_0\}$, one has

$$\begin{aligned} \overline{Y}^+_{T'}(-3) - \overline{Y}^+_T(-3) &= \sum_{n=T}^{T'} \frac{\mathbf{1}\{\overline{X}_n = -3\}}{w(\overline{Z}_n(-2))} \leq \sum_{n=T}^{T'} \frac{\mathbf{1}\{\overline{X}_n = -3\}}{w(H^{-1}(\overline{Z}_n(-1)))} \\ &\leq \sum_{n=\overline{Z}_T(-3)}^{\infty} \frac{1}{w(H^{-1}(W^{-1}(2W(n) + 2\beta - 2\varepsilon_0)))}. \end{aligned} \tag{18}$$

Fix now $\beta \geq (\widetilde{\beta}_c(w) + 3\varepsilon_0)/2$, and let $K > 0$ be such that

$$\sum_{n=K}^{\infty} \frac{1}{w(H^{-1}(W^{-1}(2W(n) + \widetilde{\beta}_c(w) + \varepsilon_0)))} \leq \frac{\varepsilon_0}{10}.$$

Then by using (14), (15), (17) and (18), we get that if N is large enough,

$$\mathbb{P}_N \left(\overline{Y}^+_{T'}(-3) - \overline{Y}^+_T(-3) > \frac{\varepsilon_0}{10} \right) \leq 2\varepsilon. \tag{19}$$

But by definition of T and T', on the event $\{T' < \infty\}$, we have for N large enough,

$$\begin{aligned} \varepsilon_0/5 &< \{W(\overline{Z}_{T'}(-2) - W(\overline{Z}_T(-2))\} - \{W(\overline{Z}_{T'}(0) - W(\overline{Z}_T(0))\} \\ &= \{\overline{Y}^+_{T'}(-3) - \overline{Y}^+_T(-3)\} - \{\overline{M}_{T'}(-1) - \overline{M}_T(-1)\}. \end{aligned}$$

Therefore (13) and (19) imply

$$\mathbb{P}_N(T' < \infty) \leq 3\varepsilon, \quad \text{and} \quad \mathbb{P}_N \left(\overline{Y}^+_{\infty}(-3) - \overline{Y}^+_T(-3) \geq 1 \right) \leq 5\varepsilon.$$

With (12) we deduce that

$$\mathbb{P}(\overline{Y}^+_{\infty}(-3) = \infty \mid T_{N,\beta,\varepsilon_0} < \infty) \leq 5\varepsilon,$$

for N large enough. Since $\varepsilon > 0$ can be chosen arbitrarily small, this proves the lemma. □

We can now finish the proof of Theorem 1.2. Fix some $\varepsilon_0 \in (0,1)$ and $\beta > (\widetilde{\beta}_c(w) + 3\varepsilon_0)/2$, and consider some initial local time configuration $\mathcal{C}$, such that $z_0(-1) \geq N$, $z_0(0) \geq N$, and $z_0(x) = 0$, for $x \notin \{-1, 0\}$, with $N \geq 1$. Note that in this case, for the $\mathcal{C}$-VRRW, one has $T_{N,\beta,\varepsilon_0}(-3) = 0$, for any N large enough, by definition of this stopping time. Thus Lemma 4.2 shows that if N is large enough, then $\mathbb{P}(Y^+_{\infty}(-3) < \infty) \geq 1/2$, for any $\mathcal{C}$ as above, when the process is reflected in -3. By using a coupling and Lemmas 3.6 and 3.7 in [BSS], we deduce that the $\mathcal{C}$-VRRW never visits site -4 with probability at least some constant $c_0 > 0$, independently of its trajectory on the set of positive integers, and for any N larger than some constant N_0.

By symmetry, we deduce that with probability at least c_0^2, it will never visit neither -4, nor 3. But then the same holds for the $\mathcal{C}_0$-VRRW, since for any $N \geq 1$, with positive probability at time $2N$, we have $Z_N(-1) = N-1$, $Z_N(0) = N$, and $X_{2N} = 0$, and one can then apply the previous result at time $2N$. This proves in particular that $|R'| \in \{5, 6\}$, with positive probability.

Now it just remains to show that almost surely the walk visits only a finite number of sites. However, each time the VRRW on $\mathbb{Z}$ visits a new site $x < 0$, two cases may appear. If at this time the local time in $x+1$ is not larger than N_0, then the process has some positive probability (depending only on N_0) to jump immediately to $x-1$, and then to localize on the set $\{x-7, \ldots, x-1\}$ and never come back to x, by the above argument. If instead at this hitting time of x, the local time in $x+1$ is larger than N_0, then necessarily the local time in $x+2$ has to be also not smaller than N_0, and we deduce by using again the above argument, that the process has some (constant) positive probability to never visit $x-2$. Then the conditional Borel-Cantelli lemma (see Theorem 4.3.2 in [Dur]) shows that almost surely $\inf_n X_n > -\infty$. By symmetry we also get that almost surely $\sup_n X_n < \infty$, and this concludes the proof of Theorem 1.2. □

5. On the values of the parameters $\beta_c(w)$ and $\widetilde{\beta}_c(w)$.

Let us first observe that for any nondecreasing w, and any $\lambda > 0$, one has $\beta_c(\lambda w) = \beta_c(w)/\lambda$, and $\widetilde{\beta}_c(\lambda w) = \widetilde{\beta}_c(w)/\lambda$ (which follows from the facts that $J_\beta(\lambda w) = J_{\lambda\beta}(w)/\lambda$, and $\widetilde{J}_\beta(\lambda w) = \widetilde{J}_{\lambda\beta}(w)/\lambda$).

Now our aim here is to convince the reader that in most cases (and we believe this is true in fact for any nondecreasing weight function), one has:

$$\beta_c(w) = \widetilde{\beta}_c(w) \in \{-\infty, \infty\}. \tag{20}$$

On one hand we prove in Lemma 5.1 that this is true for any weight function growing at least linearly and not faster than $n\sqrt{\log n}$. On the other hand, we show in Lemma 5.2 that it holds as well for a large class of sublinear weights.

Now recall that one can restrict our attention to weights satisfying (2) and such that $\alpha_c(w) = \infty$, since otherwise we already know the behavior of the process by the results of [BSS]. But it is also proved there that if $\liminf w(n)/(n \log\log n) > 0$, then $\alpha_c(w)$ is finite; thus the upper bound on w, which is imposed in the hypotheses of Lemma 5.1 below is not a strong restriction.

Lemma 5.1. *Let w be some nondecreasing weight function, such that*

$$\liminf_{n\to\infty} \frac{w(n)}{n} > 0.$$

- *If $w(n) = o(n \log n)$, then $\beta_c(w) = -\infty$.*
- *If $w(n) = o(n\sqrt{\log n})$, then $\beta_c(w) = \widetilde{\beta}_c(w) = -\infty$.*

Proof. Assume first that $w(n) = o(n \log n)$. Let $\varepsilon > 0$, be such that $w(n) \leq \varepsilon^2 n \log n$, and $w(n) \geq \varepsilon n$, for all n large enough. Then at least for t large enough,

$$W(t) \geq \frac{1}{2\varepsilon^2} \log\log t. \tag{21}$$

Now by definition, for any $\beta \in \mathbb{R}$, and t large enough,

$$W(t) + \beta = \int_t^{W^{-1}(2W(t)+\beta)} \frac{du}{w(u)}.$$

Thus using that $w(n) \geq \varepsilon n$, we get that for t large enough,

$$W(t) + \beta \leq \frac{1}{2\varepsilon} \log \left(\frac{W^{-1}(2W(t)+\beta)}{t} \right).$$

Combining this with (21), we get,

$$W^{-1}(2W(t)+\beta) \geq t(\log t)^{\frac{1}{2\varepsilon}},$$

for all t large enough. Then by choosing $\varepsilon < 1/2$, and using again that $\liminf w(n)/n > 0$, the first assertion of the lemma follows.

Assume now that $w(n) = o(n\sqrt{\log n})$, so that for n large enough $w(n) \leq \varepsilon^2 n\sqrt{\log n}$. Then for x large enough,

$$\begin{aligned} 1 &= \int_x^{W^{-1}(W(x)+1)} \frac{du}{w(u)} \geq \frac{1}{\varepsilon^2} \int_x^{W^{-1}(W(x)+1)} \frac{du}{u\sqrt{\log u}} \\ &\geq \frac{2}{\varepsilon^2} \left(\sqrt{\log W^{-1}(W(x)+1)} - \sqrt{\log x} \right). \end{aligned}$$

Thus for x large enough (and ε small enough),

$$H(x) = x + W^{-1}(W(x)+1) \leq 2x \cdot e^{\varepsilon^2 \sqrt{\log x}}.$$

On the other hand, a similar argument as above shows that for any $\beta \in \mathbb{R}$, for x large enough,

$$W^{-1}(2W(x)+\beta) \geq x \exp(\sqrt{\log x}).$$

In particular, by taking ε small enough, we get that for x large enough,

$$H^{-1}(W^{-1}(2W(x)+\beta)) \geq x(\log x)^2,$$

and the second assertion of the lemma follows, using again that $\liminf w(n)/n$ is positive. □

Our second result is concerned with sublinear weights.

Lemma 5.2. *Let w be some nondecreasing weight function satisfying* (2)*. If the two following conditions hold:*

$$\limsup \; W^{-1}(W(n)+\alpha)/n < \infty, \quad \text{for any } \alpha > 0, \tag{22}$$

and

$$\limsup \; w(cn)/w(n) < \infty, \quad \text{for any } c > 1, \tag{23}$$

then $\beta_c(w) = \widetilde{\beta}_c(w) \in \{-\infty, \infty\}$. In particular (22) *holds when $w(n) = \mathcal{O}(n)$.*

Proof. By using a change of variables, we can write for any $\beta > \beta'$, for some constant $c > 0$,

$$\begin{aligned} J_\beta(w) &= \int_0^\infty \frac{dt}{w(W^{-1}(2W(t)+\beta))} = \int_0^\infty \frac{w(W^{-1}(u))}{w(W^{-1}(2u+\beta))}\, du \\ &\geq c \int_0^\infty \frac{w(W^{-1}(u))}{w(W^{-1}(2u+\beta'))}\, du = cJ_{\beta'}(w), \end{aligned}$$

using the two hypotheses of the lemma, and the first assertion follows.
Now if there exists $C > 0$, such that $w(n) \leq Cn$, for all $n \geq 1$, then for any $\alpha > 0$,

$$\alpha = \int_n^{W^{-1}(W(n)+\alpha)} \frac{dt}{w(t)} \geq \frac{1}{C} \int_n^{W^{-1}(W(n)+\alpha)} \frac{dt}{t} = \frac{1}{C} \log \left(\frac{W^{-1}(W(n)+\alpha)}{n} \right),$$

which proves the second assertion of the lemma. □

Let us conclude this section by mentioning that by combining the results of [BSS2] with our Theorem 1.1, and the previous lemma, we obtain that any non-decreasing weight function w, such that $w(n) \sim n\exp(-(\log n)^\alpha)$, for some $\alpha \in (0, 1/2)$ satisfies $\widetilde{\beta}_c(w) = -\infty$. Indeed, we know from [BSS2], that for such weight localization on 5 sites occurs with positive probability. Then Theorem 1.1 shows that $\beta_c(w)$ is finite, and finally Lemma 5.2 gives that in fact $\beta_c(w) = \widetilde{\beta}_c(w) = -\infty$. On the other hand, when $w(n) \sim n\exp(-(\log n)^\alpha)$, with $\alpha > 1/2$, the results of [BSS2] show that $J_0(w) = \infty$, which imply $\beta_c(w) \geq 0$. Applying then Lemma 5.2 gives $\beta_c(w) = \widetilde{\beta}_c(w) = \infty$.

References

[BSS] **Basdevant A.L., Schapira Br., Singh A.** Localization on 4 sites for Vertex Reinforced Random Walk on $\mathbb{Z}$, *Ann. Probab.* 42, (2014), 527–558.

[BSS2] **Basdevant A.-L., Schapira Br., Singh A.** Localization of a vertex reinforced random walk on $\mathbb{Z}$ with sub-linear weight, *Probab. Theory Related Fields* 159, (2014), 75–115.

[CK] **Chen J., Kozma G.** Vertex-reinforced random walk on $\mathbb{Z}$ with sub-square-root weights is recurrent. *C. R. Math. Acad. Sci. Paris* 352, (2014), 521–524.

[Dav] **Davis B.** Reinforced random walk, *Probab. Theory Related Fields* 84, (1990), 203–229.

[Dur] **Durrett R.** Probability: theory and examples, Cambridge University Press, Cambridge, Fourth edition, (2010).

[P] **Pemantle R.** Vertex-reinforced random walk, *Probab. Theory Related Fields* 92 (1992), 117–136.

[PV] **Pemantle R., Volkov S.** Vertex-reinforced random walk on $\mathbb{Z}$ has finite range, *Ann. Probab.* 27, (1999), 1368–1388.

[Sch] **Schapira Br.** A 0-1 law for Vertex Reinforced Random Walk on $\mathbb{Z}$ with weight of order k^α, $\alpha < 1/2$, *Electron. Commun. Probab.* 17, (2012), no. 22, 8 pp.

[S] **Singh A.** Recurrence for vertex-reinforced random walks on $\mathbb{Z}$ with weak reinforcements. *Electron. Commun. Probab.* 19, (2014), 6 pp.

[T1] **Tarrès P.** Vertex-reinforced random walk on $\mathbb{Z}$ eventually gets stuck on five points, *Ann. Probab.* 32, (2004), 2650–2701.

[T2] **Tarrès P.** Localization of reinforced random walks, arXiv:1103.5536.

[V] **Volkov S.** Phase transition in vertex-reinforced random walks on $\mathbb{Z}$ with non-linear reinforcement, *J. Theoret. Probab.* 19, (2006), 691–700.