Deviations for the Capacity of the Range of a Random Walk

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Abstract

We obtain estimates for large and moderate deviations for the capacity of the range of a random walk on $\mathbb{Z}^d$, in dimension $d \geq 5$, both in the upward and downward directions. The results are analogous to those we obtained for the volume of the range in two companion papers [AS17a, AS19]. Interestingly, the main steps of the strategy we developed for the latter apply in this seemingly different setting, yet the details of the analysis are different.

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1 Introduction

We consider a simple random walk $(S_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}^d$ starting from the origin. The range of the walk between two times k, n with $k \leq n$, is denoted as $\mathcal{R}[k,n] := \{S_k, \dots, S_n\}$ with the shortcut $\mathcal{R}_n = \mathcal{R}[0,n]$. Its Newtonian capacity, denoted $\mathrm{Cap}\,(\mathcal{R}_n)$, can be seen as the hitting probability of $\mathcal{R}_n$ by an independent random walk starting from *far away* and properly normalized. Equivalently, using reversibility, it can be expressed as the sum of escape probabilities from $\mathcal{R}_n$ by an independent random walk starting along the range. In other words, $\mathrm{Cap}\,(\mathcal{R}_n)$ is random and has the following representations:

$$\mathrm{Cap}\,(\mathcal{R}_n) \;=\; \lim_{z\to\infty} \frac{\mathbb{P}_{0,z}(\widetilde{H}_{\mathcal{R}_n} < \infty \mid S)}{G(z)} = \sum_{x\in\mathcal{R}_n} \mathbb{P}_{0,x}\big(\widetilde{H}^+_{\mathcal{R}_n} = \infty \mid S\big), \tag{1.1}$$

where $\mathbb{P}_{0,z}$ is the law of two independent walks S and $\widetilde{S}$ starting at 0 and z respectively, $G(\cdot)$ is Green's function, and $\widetilde{H}_\Lambda$ (resp. $\widetilde{H}^+_\Lambda$) stands for the hitting (resp. return) time of Λ by the walk $\widetilde{S}$.

In view of (1.1), the study of the capacity of the range is intimately related to the question of estimating probabilities of intersection of random walks. This chapter has grown quite large, with several motivations from statistical mechanics keeping the interest alive (see Lawler's celebrated monograph [Law91]). The last decade has witnessed revival interests both after a link between uniform spanning trees and loop erased random walks was discovered (see [LawSW18], [Hut18] for recent results) and after the introduction of random interlacements by Sznitman in [S10] which mimic a random walk confined in a region of volume comparable to its time span.

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The study of the capacity of the range of a random walk has a long history. Jain and Orey [JO69] show that in any dimension $d \geq 3$, there exists a constant $\gamma_d \in [0, \infty)$, such that almost surely

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Cap}(\mathcal{R}_n) = \gamma_d, \quad \text{and} \quad \gamma_d > 0, \text{ if and only if, } d \geq 5. \tag{1.2}$$

The first order asymptotics is obtained in dimension 3 in [C17], where $\mathrm{Cap}(\mathcal{R}_n)$ scales like $\sqrt{n}$. Dimension 4 is *the critical dimension*, and a central limit theorem with a non-Gaussian limit is established in [ASS19b]. In higher dimensions, a central limit theorem is proved in [Sch19] for $d = 5$, and in [ASS18] for $d \geq 6$.

Here, we mainly study the downward deviations for the capacity of the range in dimension $d \geq 5$, in the moderate and large deviations regimes. We also establish a large deviations principle in the upward direction. Our analysis is, as in our previous works [AS17a, AS19], related to the celebrated large deviation analysis of the volume of the Wiener sausage by van den Berg, Bolthausen and den Hollander [BBH01]. The folding of the Wiener sausage, under squeezing its volume, became a paradigm of *folding*, with localization in a domain with holes of order one (the picture of a Swiss Cheese popularized in [BBH01]). The variational formula for the rate function was shown to have minimizers of different nature in $d = 3$ and in $d \geq 5$ suggesting dimension-dependent optimal scenarii to achieve the deviation. For the discrete analogue of the Wiener sausage, we established in [AS17a, AS20a] some path properties confirming some observations of [BBH01].

Main results Our first result concerns the large and moderate deviations in dimension 7 and higher. In this case, we obtain upper and lower bounds which are of the same order (on a logarithmic scale), and we cover (almost) the whole set of possible moderate deviations in the non-Gaussian regime.

Theorem 1.1. *Assume $d \geq 7$. There exist positive constants ε, $\underline{\kappa}$ and $\overline{\kappa}$ (only depending on the dimension), such that for any $n^{\frac{d-2}{d}} \cdot \log n \leq \zeta \leq \varepsilon n$, and for n large enough,*

$$\exp\left(-\underline{\kappa} \cdot \zeta^{1-\frac{2}{d-2}}\right) \leq \mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_n\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)] \leq -\zeta\right) \leq \exp\left(-\overline{\kappa} \cdot \zeta^{1-\frac{2}{d-2}}\right). \tag{1.3}$$

Recall that a central limit theorem is proved in [ASS18], where we show in particular that $\mathrm{var}(\mathrm{Cap}\left(\mathcal{R}_n\right)) \sim \sigma^2 n$, for some constant $\sigma > 0$. Our next result proves now a Moderate Deviation Principle in the Gaussian regime.

Theorem 1.2. *Assume $d \geq 7$. For any sequence $\{\zeta_n\}_{n\geq 0}$, satisfying $\lim_{n\to\infty} \zeta_n/\sqrt{n} = \infty$, and $\lim_{n\to\infty} \zeta_n(\log n)/n^{\frac{d-2}{d}} = 0$, we have*

$$\lim_{n\to\infty} \frac{n}{\zeta_n^2} \cdot \log \mathbb{P}\left(\pm(\mathrm{Cap}\left(\mathcal{R}_n\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)]) > \zeta_n\right) = -\frac{1}{2\sigma^2}. \tag{1.4}$$

In dimension 5, we obtain estimates similar to Theorem 1.1, but we do not reach the Gaussian regime:

Theorem 1.3. *Assume $d = 5$. There exist positive constants ε, $\underline{\kappa}$ and $\overline{\kappa}$, such that for any $n^{5/7} \cdot \log n \leq \zeta \leq \varepsilon n$, and n large enough,*

$$\exp\left(-\underline{\kappa} \cdot (\frac{\zeta^2}{n})^{1/3}\right) \leq \mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_n\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)] \leq -\zeta\right) \leq \exp\left(-\overline{\kappa} \cdot (\frac{\zeta^2}{n})^{1/3}\right).$$

Remark 1.4. In $d=5$, the variance of $\mathrm{Cap}(\mathcal{R}_n)$ is of order $n\log n$, [Sch19]. Thus, the moderate deviations should go from a Gaussian regime with a speed of order $\zeta^2/(n\log n)$, to a large deviation regime with a speed of order $(\zeta^2/n)^{1/3}$, and with a transition occurring for ζ of order $\sqrt{n}(\log n)^{3/4}$. For an explanation of the exponent $5/7$ which limits us here, see Remark 3.3. Note that in the case of the volume of the range, a similar transition has been established by Chen [Chen10] in dimension 3, and by the authors in $d\ge 5$ in the companion paper [AS19].

Remark 1.5. In dimension 6 our result is less precise. One can only show that

$$\exp\left(-\underline{\kappa}\cdot\zeta^{1/2}\right)\ \le\ \mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_n\right)-\mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)]\le -\zeta\right)\ \le\ \exp\left(-\frac{\overline{\kappa}}{\log(n/\zeta)}\cdot\zeta^{1/2}\right).$$

Our next result provide path properties of the trajectory under the constraint of moderate deviations. To state it, one needs more notation. For $r>0$, and $x\in\mathbb{Z}^d$, set

$$Q(x,r):=[x-\frac{r}{2},x+\frac{r}{2})^d\cap\mathbb{Z}^d.$$

Given $\Lambda\subseteq\mathbb{Z}^d$, and $n\ge 0$, let $\ell_n(\Lambda)$ be the time spent by random walk in Λ before time n. For $\rho\in(0,1]$, and r,n positive integers, we let

$$\mathcal{C}_n(r,\rho):=\{x\in r\mathbb{Z}^d\ :\ \ell_n(Q(x,r))\ge\rho r^d\},\qquad\text{and}\qquad \mathcal{V}_n(r,\rho):=\bigcup_{x\in\mathcal{C}_n(r,\rho)}Q(x,r). \tag{1.5}$$

Define also for a sequence of values of deviation $(\zeta_n)_{n\ge1}$,

$$\rho_{\mathrm{typ}}:=\begin{cases}\zeta_n^{5/3}/n^{7/3} & \text{if } d=5\\ \zeta_n^{-2/(d-2)} & \text{if } d\ge 7,\end{cases}\qquad \tau_{\mathrm{typ}}:=\begin{cases} n & \text{if } d=5\\ \zeta_n & \text{if } d\ge 7,\end{cases}\qquad\text{and}\quad \chi_d:=\begin{cases} 5/7 & \text{if } d=5\\ \frac{d-2}{d} & \text{if } d\ge 7.\end{cases}$$

Theorem 1.6. *Assume $d=5$, or $d\ge 7$. There are positive constants α, β, ε and C_0, such that for any sequence $(\zeta_n)_{n\ge1}$, satisfying*

$$n^{\chi_d}\cdot\log n\le\zeta_n\le\varepsilon n,$$

defining $(r_n)_{n\ge1}$ by

$$r_n^{d-2}\rho_{typ}=C_0\log n,$$

one has

$$\lim_{n\to\infty}\mathbb{P}\left(\ell_n(\mathcal{V}_n(r_n,\beta\rho_{typ}))\ge\alpha\,\tau_{typ}\mid \mathrm{Cap}\left(\mathcal{R}_n\right)-\mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)]\le-\zeta_n\right)=1. \tag{1.6}$$

Moreover, there exists $A>0$, such that

$$\lim_{n\to\infty}\mathbb{P}\left(\mathrm{Cap}(\mathcal{V}_n(r_n,\beta\rho_{typ}))\le A|\mathcal{V}_n(r_n,\beta\rho_{typ})|^{1-2/d}\mid\mathrm{Cap}\left(\mathcal{R}_n\right)-\mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)]\le-\zeta_n\right)=1. \tag{1.7}$$

Theorem 1.6 provides some information on the density the random walk has to realize in order to achieve the deviation. We obtain that $\mathcal{V}_n(r_n,\beta\rho_{\mathrm{typ}})$ is typically ball-like, in the sense that its capacity is of the order of its volume to the power $1-2/d$, as it is the case for Euclidean balls.

The final result concerns the upward deviations. Our decomposition (1.8) allows us to adapt the argument of Hamana and Kesten, [HK], written for the volume of the range of a random walk.

Theorem 1.7. *Assume $d \geq 5$. The following limit exists for all $x > 0$:*

$$\psi_d(x) := -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\text{Cap}\,(\mathcal{R}_n) \geq n \cdot x\big).$$

Furthermore, there exists a constant $\gamma_d^* > \gamma_d$ (defined in (1.2)), such that the function ψ_d is continuous and convex on $[0, \gamma_d^*]$, increasing on $[\gamma_d, \gamma_d^*]$, and satisfies

$$\psi_d(x) \begin{cases} = 0 & \text{if } x \leq \gamma_d \\ \in (0, \infty) & \text{if } x \in (\gamma_d, \gamma_d^*] \\ = \infty & \text{if } x > \gamma_d^*. \end{cases}$$

We also obtain Gaussian upper bounds (up to a logarithmic factor) in the regime of moderate deviations, see Proposition 2.4.

Our approach to downward deviations. The cornerstone of our approach is a decomposition formula obtained in [ASS19a]:

$$\forall A, B \quad \text{finite sets of } \mathbb{Z}^d, \qquad \text{Cap}\,(A \cup B) = \text{Cap}\,(A) + \text{Cap}\,(B) - \chi_{\mathcal{C}}(A, B), \tag{1.8}$$

where $\chi_{\mathcal{C}}(A, B)$ called *the cross-term* has a nice expression. In this work, the decomposition (1.8) allows us to follow a simple approach devised in [AS17a], and later improved in [AS19], to study downward deviations for the volume of the range in dimensions $d \geq 3$. We partition the time-period of length n into intervals of length $T \leq n$, and by iterating (1.8) appropriately one can write our functional of the range, $\text{Cap}\,(\mathcal{R}_n)$, as a sum of i.i.d. terms minus a certain sum of cross-terms of the form $\chi_{\mathcal{C}}(\mathcal{R}_{iT}, \mathcal{R}[iT, (i+1)T])$, with i going from 1 to $\lfloor n/T \rfloor$. The so-called corrector, is the sum of these cross-terms that we integrate over $\mathcal{R}[iT, (i+1)T]$. We then show that for some appropriate time-scale T it is this corrector which is responsible for (most of) the deviations. The final step is to estimate the cost for such deviations. This analysis is similar to the corresponding one for the volume of the range that we performed in [AS19], but it also requires some new ingredients, in particular Lemmas 4.1, 4.2 and 4.3.

On the other hand the proof of Theorem 1.2 relies on the following estimate, similar to the result for the intersection of two ranges that was obtained in [AS20c]: first we observe that $\chi_{\mathcal{C}}(A, B)$ is bounded above by (twice) another functional $\widetilde{\chi}(A, B)$, defined for any $A, B \subseteq \mathbb{Z}^d$ by

$$\widetilde{\chi}(A, B) := \sum_{x\in A} \sum_{y\in B} \mathbb{P}_x\big(H_A^+ = \infty\big) \cdot G(x-y) \cdot \mathbb{P}_y\big(H_B^+ = \infty\big).$$

We then show that for some $\kappa > 0$, if $\mathcal{R}_\infty$ and $\widetilde{\mathcal{R}}_\infty$ are the ranges of two independent walks,

$$\mathbb{E}\left[\exp\left(\kappa \cdot \widetilde{\chi}(\mathcal{R}_\infty, \widetilde{\mathcal{R}}_\infty)^{1-\frac{2}{d-2}}\right)\right] < \infty. \tag{1.9}$$

Heuristics. We use the sign $\approx$ to express that two quantities are *of the comparable order* (which here will have a deliberately vague meaning, and precise statements come later). As already mentioned, the first step in this work is a simple decomposition for the capacity of a union of sets in term of a cross-term

$$\chi_{\mathcal{C}}(A, B) \;\approx\; 2\sum_{x\in A} \sum_{y\in B} \mathbb{P}_x\big(H_A^+ = \infty\big) \cdot G(x-y) \cdot \mathbb{P}_y\big(H_B^+ = \infty\big), \tag{1.10}$$

see (2.9) and (2.12) for a precise expression. The key phenomenon responsible for producing a small capacity for the range of a random walk is *an increase of the cross-term on an appropriate scale.* In other words, the walk *folds* into a ball-like domain in order to increase some *self-interaction* captured by the cross-term. Now to be more concrete, let us divide the range $\mathcal{R}[0,2n]$ into two subsets $\mathcal{R}[0,n]$ and $\mathcal{R}[n,2n]$. Let us call, for simplicity $\mathcal{R}_n^1=\mathcal{R}[0,n]-S_n$, and $\mathcal{R}_n^2=\mathcal{R}[n,2n]-S_n$ the two subranges translated by S_n so that they become independent. By (1.10) and translation invariance of the capacity we see that

$$\mathrm{Cap}\,(\mathcal{R}[0,2n]) = \mathrm{Cap}\,\big(\mathcal{R}_n^1\big) + \mathrm{Cap}\,\big(\mathcal{R}_n^2\big) - \chi_{\mathcal{C}}(\mathcal{R}_n^1,\mathcal{R}_n^2).$$

Now, assume that both walks stay inside a ball of radius R a time of order $\tau \le n$, and are unconstrained afterward. Thus, under the strategy we mentioned, and writing $G(R)$ for the Green's function taken at some point z with Euclidean norm R,

$$\begin{aligned}\chi_{\mathcal{C}}(\mathcal{R}_n^1,\mathcal{R}_n^2) &\approx G(R)\times \mathrm{Cap}\,\big(\mathcal{R}_\tau^1\big)\times \mathrm{Cap}\,\big(\mathcal{R}_\tau^2\big) + \mathcal{O}\big(G(\sqrt{n})n^2\big)\\ &\approx G(R)\big(\min(\tau,R^{d-2})\big)^2 + \mathcal{O}\big(n^{\frac{6-d}{2}}\big).\end{aligned} \tag{1.11}$$

The term $\mathcal{O}\big(G(\sqrt{n})n^2\big)$ appears if τ is smaller than n, and accounts for the unconstrained contribution to the cross-term. In obtaining (1.11), we have used that if $\mathcal{R}_\tau^1$ and $\mathcal{R}_\tau^2$ are inside a ball of radius R, then their capacity is bounded by the capacity of the ball, which is of order R^{d-2}, as well as by their volume bounded by τ. Thus, it is useless to consider τ larger than R^{d-2}, since then τ no more affects the cross-term and increasing τ (or decreasing R below τ) only makes the strategy more costly. Now a deviation of order ζ is reached if

$$\frac{1}{R^{d-2}}\tau^2 \approx \zeta. \tag{1.12}$$

Recall that the cost of being localized a time τ in a ball of radius R is of order $\exp(-\tau/R^2)$ (up to a constant in the exponential). So we need to find a choice of (τ,R) which minimizes this cost under the constraint (1.12). In other words one needs to maximize $\sqrt{\zeta}\cdot R^{(d-6)/2}$. This leads to two regimes.

- When $d=5$, R (and then τ) is as large as possible. So, $\tau=n$ and $R^{d-2}=n^2/\zeta$ by (1.12). The strategy is time homogeneous for any ζ!
- When $d\ge 7$, then τ is as small as possible, that is $\tau=R^{d-2}=\zeta$. The strategy is time-inhomogeneous.

When $d=6$, the strategy remains unknown, but the cost should be of order $\exp(-\sqrt{\zeta})$.

Application to a polymer melt. The model of random interlacements, introduced by Sznitman [S10], is roughly speaking the union of the ranges of trajectories obtained by a Poisson point process on the space of doubly infinite trajectories, and is such that the probability of avoiding a set K is $\exp(-u\cdot \mathrm{Cap}\,(K))$, where $u>0$ is a fixed parameter. With this in mind, let us consider the following model of polymer among a polymer melt interacting by exclusion. We distinguish one polymer, a simple random walk, interacting with a cloud of other random walk trajectories modeled by random interlacements which we call for short *the melt.* The interaction is through exclusion: the walk and the melt do not intersect. When integrating over the interlacements law, the measure on the walk with the effective interaction has a density proportional to $\exp(-u\cdot \mathrm{Cap}\,(\mathcal{R}_n))$, with respect to the law of a simple random walk.

As a corollary of our deviation estimates, one can address some issues on this polymer. Since this follows in the same way as the study of the Gibbs measure tilted by the volume of the range was a corollary of [AS17a], we repeat neither the statements corresponding to Theorem 1.8 of [AS17a], nor the proofs here. The simplest and most notable difference with the latter theorem is that the proper scaling of the intensity parameter u which provides a phase transition is when it is of order $n^{-2/(d-2)}$ in dimension $d \geq 5$. Thus, one would consider the polymer partition function as a function of $u \in \mathbb{R}^+$

$$Z_n(u) = \mathbb{E}\Big[\exp\big(-\frac{u}{n^{2/(d-2)}}(\mathrm{Cap}\,(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}\,(\mathcal{R}_n)])\,\big)\Big].$$

Theorem 1.8 of [AS17a] is true, here also after the drop in dimension is performed, and establishes the existence of a phase transition as one tunes u. On the other hand, considering the quenched model, where the random interlacements is given a typical realization, is an interesting open problem, beyond the reach of the present techniques.

Organization. The paper is organized as follows. In the next section, we recall some basic estimates on the random walk, the capacity, and the range that we will need. Section 3, and more precisely Proposition 3.2 makes the link between downward deviations for the capacity and upward deviations of a corrector. The corrector itself is studied in Section 4, where we prove the upper bounds in Theorems 1.1 and 1.3, as well as Theorem 1.6. In Section 5, we prove the lower bounds in Theorems 1.1 and 1.3. The proof of Theorem 1.2 is done in Section 6. Finally, we prove Theorem 1.7 concerning the upward deviations in Section 7.

2 Preliminaries

2.1 Further notation

For $z \in \mathbb{Z}^d$, $d \geq 5$, we denote by $\mathbb{P}_z$ the law of the simple random walk starting from z, and simply write it $\mathbb{P}$ when $z = 0$. We let

$$G(z) := \mathbb{E}\left[\sum_{n=0}^{\infty} \mathbf{1}\{S_n = z\}\right],$$

be the Green's function. It is known (see [Law91]) that for some positive constants c and C,

$$\frac{c}{\|z\|^{d-2}+1} \;\leq\; G(z) \;\leq\; \frac{C}{\|z\|^{d-2}+1}, \quad \text{for all } z \in \mathbb{Z}^d, \tag{2.1}$$

with $\|\cdot\|$ the Euclidean norm. We also consider for $T > 0$, and $z \in \mathbb{Z}^d$,

$$G_T(z) := \mathbb{E}\left[\sum_{n=0}^{T} \mathbf{1}\{S_n = z\}\right].$$

In particular for any $z \in \mathbb{Z}^d$, and $T \geq 1$,

$$\mathbb{P}(z \in \mathcal{R}_T) \leq G_T(z). \tag{2.2}$$

For $A \subset \mathbb{Z}^d$, we denote by $|A|$ the cardinality of A, and by

$$H_A := \inf\{n \geq 0 \,:\, S_n \in A\}, \quad \text{and} \quad H_A^+ := \inf\{n \geq 1 \,:\, S_n \in A\},$$

respectively the hitting time of A and the first return time to A.

We also need the following well known fact, see [Law91]. There exists a constant $C > 0$, such that for any $R > 0$ and $z \in \mathbb{Z}^d$,

$$\mathbb{P}_z\left(\inf_{k\geq 0} \|S_k\| \leq R\right) \leq C \cdot \left(\frac{R}{\|z\|}\right)^{d-2}. \tag{2.3}$$

2.2 On the capacity

The capacity of a finite subset $A \subset \mathbb{Z}^d$, with $d \geq 3$, is defined by

$$\mathrm{Cap}(A) := \lim_{\|z\|\to\infty} \frac{1}{G(z)} \mathbb{P}_z(H_A < \infty). \tag{2.4}$$

It is well known, see Proposition 2.2.1 of [Law91], that the capacity is monotone for inclusion:

$$\mathrm{Cap}(A) \leq \mathrm{Cap}(B), \quad \text{for any } A \subset B, \tag{2.5}$$

and satisfies the sub-additivity relation

$$\mathrm{Cap}(A \cup B) \leq \mathrm{Cap}(A) + \mathrm{Cap}(B) - \mathrm{Cap}(A \cap B), \quad \text{for all } A, B \subset \mathbb{Z}^d. \tag{2.6}$$

Another equivalent definition of the capacity is the following (see (2.12) of [Law91]).

$$\mathrm{Cap}(A) = \sum_{x\in A} \mathbb{P}_x(H_{A^+} = \infty). \tag{2.7}$$

In particular it implies that

$$\mathrm{Cap}(A) \leq |A|, \quad \text{for all } A \subset \mathbb{Z}^d. \tag{2.8}$$

The starting point for our decomposition is the definition (2.4) of the capacity in terms of a hitting time. It implies that for any two finite subsets $A, B \subset \mathbb{Z}^d$,

$$\mathrm{Cap}\,(A \cup B) = \mathrm{Cap}\,(A) + \mathrm{Cap}\,(B) - \chi_{\mathcal{C}}(A, B), \tag{2.9}$$

with

$$\chi_{\mathcal{C}}(A, B) := \lim_{z\to\infty} \frac{1}{G(z)} \mathbb{P}_z\big(\{H_A < \infty\} \cap \{H_B < \infty\}\big).$$

In particular by (2.4) and the latter formula, one has

$$0 \leq \chi_{\mathcal{C}}(A, B) \leq \min(\mathrm{Cap}(A), \mathrm{Cap}(B)). \tag{2.10}$$

Now, we have shown in [ASS19b] that

$$\chi_{\mathcal{C}}(A, B) = \chi(A, B) + \chi(B, A) - \varepsilon(A, B), \tag{2.11}$$

with

$$\chi(A, B) = \sum_{x\in A}\sum_{y\in B} \mathbb{P}_x\big(H^+_{A\cup B} = \infty\big) \cdot G(x - y) \cdot \mathbb{P}_y\big(H^+_B = \infty\big), \tag{2.12}$$

and,

$$0 \leq \varepsilon(A, B) \leq \mathrm{Cap}\,(A \cap B) \leq |A \cap B|, \tag{2.13}$$

where the last inequality follows from (2.8).

We will need some control on the speed of convergence in (1.2).

Lemma 2.1. *Assume $d \geq 5$. One has*

$$|\mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] - \gamma_d n| = \mathcal{O}(\psi_d(n)),$$

with

$$\psi_d(n) = \begin{cases} \sqrt{n} & \text{if } d = 5 \\ \log n & \text{if } d = 6 \\ 1 & \text{if } d \geq 7. \end{cases}$$

Proof. By (2.9), (2.11), (2.12), and (2.13) one has the rough lower bound:

$$\mathrm{Cap}(\mathcal{R}_{n+m}) \geq \mathrm{Cap}(\mathcal{R}_n) + \mathrm{Cap}(\mathcal{R}[n, n+m]) - 2\sum_{k=0}^{n}\sum_{\ell=n}^{n+m} G(S_k - S_\ell), \tag{2.14}$$

for any integers $n, m \geq 1$ (a better inequality will be used later, but this one is enough here). Then one concludes exactly as in [AS17b], using (2.6), Hammersley's lemma and Lemma 3.2 in [ASS18], which controls the moments of the error term in the right-hand side of (2.14). For the details, we refer to the proof of (1.13) in [AS17b]. □

The next result provides some useful bounds on the variance of the capacity of the range, which were obtained in [Sch19] in case of dimension 5, and in [ASS18] in higher dimension.

Proposition 2.2. *One has,*

$$\mathrm{var}(\mathrm{Cap}(\mathcal{R}_n)) = \begin{cases} \mathcal{O}(n \log n) & \text{if } d = 5 \\ \mathcal{O}(n) & \text{if } d \geq 6. \end{cases}$$

Remark 2.3. Actually sharp asymptotics are known: in dimension 5, one has $\mathrm{var}(\mathrm{Cap}(\mathcal{R}_n)) \sim \sigma_5 n \log n$, and in higher dimension $\mathrm{var}(\mathrm{Cap}(\mathcal{R}_n)) \sim \sigma_d n$, for some positive constant $(\sigma_d)_{d \geq 5}$, see respectively [Sch19] and [ASS18].

As a consequence of the previous results one can obtain Gaussian type upper bounds for the moderate deviations in the upward deviations.

Proposition 2.4. *There exist positive constants $(c_d)_{d\geq 5}$, such that for any $n \geq 2$, and $\zeta > 0$,*

$$\mathbb{P}\big(\mathrm{Cap}\left(\mathcal{R}_n\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)] \geq \zeta\big) \leq \begin{cases} \exp\left(-c_5 \cdot \frac{\zeta^2}{n(\log n)^3}\right) & \text{if } d = 5 \\ \exp\left(-c_6 \cdot \frac{\zeta^2}{n(\log \log n)^2}\right) & \text{if } d = 6 \\ \exp\left(-c_d \cdot \frac{\zeta^2}{n}\right) & \text{if } d \geq 7. \end{cases}$$

Proof. For simplicity let us concentrate on the proof when $d = 5$. We will explain at the end the necessary modifications to the proof when $d \geq 6$. Note first that one can always assume that ζ is smaller than $n/2$. We use now (1.8) repeatedly along a dyadic decomposition of $\{0, \dots, n\}$. This gives for $L \geq 1$, with $m_L := \lfloor n/2^L \rfloor$,

$$\mathrm{Cap}\left(\mathcal{R}_n\right) = \sum_{i=1}^{2^L} \mathrm{Cap}\left(\mathcal{R}_{m_L}^{(i)}\right) - \sum_{\ell=1}^{L} \Sigma_\ell,$$

where the $\mathcal{R}_{m_L}^{(i)}$ are consecutive pieces of the range of length either m_L or m_L+1, and

$$\Sigma_\ell := \sum_{j=1}^{2^{\ell-1}} \chi_C(\mathcal{R}_{m_\ell}^{(2j-1)}, \mathcal{R}_{m_\ell}^{(2j)}),$$

with similar notation as above, in particular $m_\ell = \lfloor n/2^\ell \rfloor$. Thus,

$$\mathbb{P}\big(\mathrm{Cap}\,(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}\,(\mathcal{R}_n)] > \zeta\big) \le \mathbb{P}\left(\sum_{i=1}^{2^L} \mathrm{Cap}\left(\mathcal{R}_{m_L}^{(i)}\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_{m_L}^{(i)}\right)] > \frac{\zeta}{2}\right) + \sum_{\ell=1}^{L} \mathbb{P}\left(\mathbb{E}[\Sigma_\ell] - \Sigma_\ell > \frac{\zeta}{2L}\right). \tag{2.15}$$

We fix now L, such that $n/\zeta \le m_L \le 2n/\zeta$. The first term in (2.15) is ruled out using Bernstein's inequality and Proposition 2.2, which give for some constant $c>0$.

$$\mathbb{P}\left(\sum_{i=1}^{2^L} \mathrm{Cap}\left(\mathcal{R}_{m_L}^{(i)}\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_{m_L}^{(i)}\right)] > \zeta/2\right) \le \exp\left(-c\frac{\zeta^2}{n\log m_L}\right). \tag{2.16}$$

Concerning the sum in (2.15), note first that by Lemma 2.1, one has

$$\mathbb{E}[\Sigma_\ell] = \mathcal{O}(2^{\ell/2}\sqrt{n}),$$

for any $\ell \ge 1$. Therefore, one can assume that ℓ is such that $2^{\ell/2}\sqrt{n} > c\zeta/L$, for some constant $c>0$, for otherwise the corresponding probability is zero. For such ℓ one has by using standard concentration results (see Theorem 4.4. in [CL06]):

$$\mathbb{P}\left(\mathbb{E}[\Sigma_\ell] - \Sigma_\ell > \frac{\zeta}{2L}\right) \le \exp\left(-c\frac{(\zeta/L)}{\sqrt{m_\ell} + Ln(\log m_\ell)/\zeta}\right) \le \exp\left(-\frac{c\zeta^2}{n(\log n)^3}\right),$$

which completes the proof in case $d=5$. In case $d \ge 6$, the variance is linear. So first, the term $\log m_L$ can be removed in (2.16). Moreover, in case $d=6$, one has $\mathbb{E}[\Sigma_\ell] = \mathcal{O}(2^\ell \log n)$, and thus only the ℓ's such that $\zeta \ge 2^\ell \ge c\zeta/\log n$ need to be considered. There are order $\log\log n$ such integers, and for each of them one has by the same argument as above,

$$\mathbb{P}\left(\mathbb{E}[\Sigma_\ell] - \Sigma_\ell > \frac{\zeta}{C\log\log n}\right) \le \exp(-c\zeta^2/(\log\log n)^2),$$

which concludes the proof in case $d=6$. The case $d \ge 7$ is similar, since this time $\mathbb{E}[\Sigma_\ell] = \mathcal{O}(2^\ell)$, and thus there are only a bounded number of integers ℓ's that need to be considered. □

3 Transfer of downward deviations to the corrector

The possibility of establishing the heuristic picture described in the introduction stems from writing the capacity of a union of sets as a sum of capacities and a cross-term. The latter though typically small is nonetheless responsible for the fluctuations. Iterating this decomposition leads to an expression of the capacity of the range as a sum of i.i.d. terms minus a sum of cross-terms. The so-called corrector is obtained by summing appropriate conditional expectations of these cross-terms.

Our first result in this section, Lemma 3.1, provides an explicit expression for (what turns out to be an upper bound for) this corrector in terms of a sum of convoluted Green's functions taken along the trajectory and weighted by escape probability terms. We then recall a result from [AS19], which relates the deviations of the capacity to those of the corrector, which we state here as Proposition 3.2.

Thus, the strategy is similar to the one used to treat downward deviations for the range developed in [AS19]. However the form of the corrector is slightly different. Roughly it involves a convolution of Green's function with itself together multiplied by escape probability terms, where in [AS17a] only Green's function appeared.

A detailed analysis of this corrector is carried out in Sections 4.3 and 4.4. Before we can state precisely the result, some preliminary notation is required.

For $I \subset \mathbb{N}$, we write $\mathcal{R}(I) := \{S_k,\ k \in I\}$, for the set of visited sites during times $k \in I$. Since for any two intervals $I, J \subset \mathbb{N}$, one has $\mathcal{R}(I \cup J) = \mathcal{R}(I) \cup \mathcal{R}(J)$, (2.9) gives

$$\mathrm{Cap}\,(\mathcal{R}(I \cup J)) = \mathrm{Cap}\,(\mathcal{R}(I)) + \mathrm{Cap}\,(\mathcal{R}(J)) - \chi_{\mathcal{C}}\big(\mathcal{R}(I), \mathcal{R}(J)\big). \tag{3.1}$$

Next, given two sets A and B, their symmetric difference is defined as $A\Delta B := (A \cap B^c) \cup (B \cap A^c)$. Note in particular that for any $I, J \subset \mathbb{N}$, one has $\mathcal{R}(I)\Delta\mathcal{R}(J) \subset \mathcal{R}(I\Delta J)$. Moreover, it follows from (2.5), (2.6) and (2.8) that for any $A, B \subset \mathbb{Z}^d$,

$$|\mathrm{Cap}(A) - \mathrm{Cap}(B)| \le \mathrm{Cap}(A\Delta B) \le |A\Delta B|.$$

Applying this inequality to ranges on some intervals I and J, we get

$$|\mathrm{Cap}(\mathcal{R}(I)) - \mathrm{Cap}(\mathcal{R}(J))| \le |I\Delta J|. \tag{3.2}$$

Now given some integer $T \le n$, we define for $j \ge 0$ and $\ell \ge 1$,

$$I_{j,\ell} := [j + (\ell-1)T, j + \ell T], \quad \text{and} \quad \widetilde{I}_{j,\ell} := I_{j,1} \cup \dots \cup I_{j,\ell}.$$

It follows from (3.2) that almost surely

$$|\mathrm{Cap}(\mathcal{R}_n) - \frac{1}{T}\sum_{j=0}^{T-1} \mathrm{Cap}(\mathcal{R}(\widetilde{I}_{j,\lfloor n/T \rfloor}))| \le T. \tag{3.3}$$

On the other hand, applying (3.1) recursively we obtain for any $j = 0, \dots, T-1$,

$$\mathrm{Cap}(\mathcal{R}(\widetilde{I}_{j,\lfloor n/T \rfloor})) = \sum_{\ell=1}^{\lfloor n/T \rfloor} \mathrm{Cap}(\mathcal{R}(I_{j,\ell})) - \sum_{\ell=1}^{\lfloor n/T \rfloor - 1} \chi_{\mathcal{C}}\big(\mathcal{R}(\widetilde{I}_{j,\ell}), \mathcal{R}(I_{j,\ell+1})\big). \tag{3.4}$$

Define now

$$\chi_n(T) := \frac{1}{T}\sum_{j=0}^{T-1} \sum_{\ell=1}^{\lfloor n/T \rfloor - 1} \chi_{\mathcal{C}}\big(\mathcal{R}(\widetilde{I}_{j,\ell}), \mathcal{R}(I_{j,\ell+1})\big),$$

and note that (3.3) and (3.4) give for any $T \le \zeta/2$,

$$\mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \le -\zeta\right) \le \mathbb{P}\left(\frac{1}{T}\sum_{j=0}^{T-1} \mathrm{Cap}(\mathcal{R}(\widetilde{I}_{j,\lfloor n/T \rfloor})) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}(\widetilde{I}_{j,\lfloor n/T \rfloor}))] \le -\frac{\zeta}{2}\right)$$

$$\leq \mathbb{P}\left(\frac{1}{T}\sum_{j=0}^{T-1}\sum_{\ell=1}^{\lfloor n/T\rfloor} \mathrm{Cap}(\mathcal{R}(I_{j,\ell})) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}(I_{j,\ell}))] \leq -\frac{\zeta}{4}\right) + \mathbb{P}\left(\chi_n(T) \geq \frac{\zeta}{4}\right). \tag{3.5}$$

The first term on the right-hand side of (3.5) is dealt with Bernstein's inequality and Proposition 2.2, which show that for any $\zeta > \frac{n\log n}{T}$, for some constant $c>0$.

$$\mathbb{P}\left(\frac{1}{T}\sum_{j=0}^{T-1}\sum_{\ell=1}^{\lfloor n/T\rfloor} \mathrm{Cap}(\mathcal{R}(I_{j,\ell})) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}(I_{j,\ell}))] \leq -\frac{\zeta}{4}\right)$$
$$\leq T \max_{0\leq j\leq T-1} \mathbb{P}\left(\sum_{\ell=1}^{\lfloor n/T\rfloor} \mathrm{Cap}(\mathcal{R}(I_{j,\ell})) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}(I_{j,\ell}))] \leq -\frac{\zeta}{4}\right) \leq T\exp(-c\frac{\zeta}{T}). \tag{3.6}$$

For the second term in the right-hand side of (3.5), we will use a general result of [AS19], which allows to compare the (moderate) deviations of $\chi_n(T)$ to those of its compensator, defined by

$$\xi_n^*(T) := \frac{1}{T}\sum_{j=0}^{T-1}\sum_{\ell=1}^{\lfloor n/T\rfloor -1} \mathbb{E}\left[\chi_{\mathcal{C}}\big(\mathcal{R}(\widetilde{I}_{j,\ell}), \mathcal{R}(I_{j,\ell+1})\big) \mid \mathcal{F}_{j+\ell T}\right]. \tag{3.7}$$

More specifically, the proof of Proposition 4.1 in [AS19] (see also the proof of Corollary 4.2 there) shows that for some constant $c>0$, for any $\zeta>0$,

$$\mathbb{P}(\chi_n(T) \geq \frac{\zeta}{4},\ \xi_n^*(T) \leq c\zeta) \leq \exp(-c\frac{\zeta}{T}), \tag{3.8}$$

(where here we use also that by (2.8) and (2.10), each term of the sum in the definition of $\chi_n(T)$ is bounded by T). We next define

$$\xi_n(T) := \sum_{k=0}^{n}\sum_{x\in\mathcal{R}_k} \mathbb{P}_x\big(H^+_{\mathcal{R}_k} = \infty\big)\cdot \frac{G\star G_T(x-S_k)}{T}. \tag{3.9}$$

Lemma 3.1. *One has, for any $n\geq 1$ and $1\leq T\leq n$,*

$$\xi_n^*(T) \leq 2\xi_n(T).$$

Proof. By (2.12), for any sets A and B,

$$\chi(A,B) \leq \widetilde{\chi}(A,B) := \sum_{x\in A}\sum_{y\in B} \mathbb{P}_x\big(H_A^+ = \infty\big)\cdot G(x-y)\cdot \mathbb{P}_y\big(H_B^+=\infty\big).$$

Note that $\widetilde{\chi}$ is symmetric in the sense that $\widetilde{\chi}(A,B) = \widetilde{\chi}(B,A)$, for any A,B. Bounding the last probability term by one, we get

$$\chi_{\mathcal{C}}(A,B) \overset{(2.11)}{\leq} \chi(A,B)+\chi(B,A) \leq 2\overline{\chi}(A,B), \quad \text{with} \quad \overline{\chi}(A,B) := \sum_{x\in A}\sum_{y\in B}\mathbb{P}_x\big(H_A^+=\infty\big)\cdot G(x-y).$$

Now for any j,ℓ, the Markov property and translation invariance of the simple random walk give

$$\mathbb{E}\left[\overline{\chi}\big(\mathcal{R}(\widetilde{I}_{j,\ell}), \mathcal{R}(I_{j,\ell+1})\big) \mid \mathcal{F}_{j+\ell T}\right] = \sum_{x\in\mathcal{R}(\widetilde{I}_{j,\ell})} \mathbb{P}_x(H^+_{\mathcal{R}(\widetilde{I}_{j,\ell})} = \infty) \sum_{y\in\mathbb{Z}^d} G(x-y)\cdot\mathbb{P}\big(y\in\mathcal{R}(I_{j,\ell+1}) \mid \mathcal{F}_{j+\ell T}\big)$$
$$\overset{(2.2)}{\leq} \sum_{x\in\mathcal{R}(\widetilde{I}_{j,\ell})} \mathbb{P}_x(H^+_{\mathcal{R}(\widetilde{I}_{j,\ell})} = \infty)\cdot G\star G_T(x-S_{j+\ell T}),$$

and the lemma follows from the definition (3.9) and (3.7) of $\xi_n(T)$ and $\xi_n^*(T)$ respectively. □

Combining (3.5), (3.6), (3.8), and Lemma 3.1 we obtain the main result of this section.

Proposition 3.2. *There exists a positive constant c, such that for any $n \geq 2$, $\zeta > 0$, and $T \geq 1$ satisfying $T \leq \zeta/2$, and $\zeta \geq \frac{n \log n}{T}$,*

$$\mathbb{P}\Big(\{\xi_n(T) \leq c\zeta\} \cap \{\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \leq -\zeta\}\Big) \leq 2T \exp(-c\frac{\zeta}{T}),$$

and as a consequence,

$$\mathbb{P}\Big(\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \leq -\zeta\Big) \leq 2T \exp(-c\frac{\zeta}{T}) + \mathbb{P}(\xi_n(T) \geq c\zeta).$$

Remark 3.3. In dimension 5, the mean of $\xi_n(T)$ is of order $n/\sqrt{T}$. So the upper deviations for $\xi_n(T)$ start to decay only for $\zeta > n/\sqrt{T}$, and since on the other hand one needs to take T at most of order $(\zeta n)^{1/3}$, to ensure the term $\exp(-c\zeta/T)$ to have the right decay, this imposes the condition $\zeta > n^{5/7}$. In particular the approach we have here has no chance to work up to the Gaussian regime. On the other hand in dimension 7 and higher, the mean of $\xi_n(T)$ is of order n/T, and T can be chosen of order $\zeta^{2/(d-2)}$, which only imposes the a priori condition $\zeta > n^{(d-2)/d}$, leaving a chance to cover entirely the non-Gaussian regime.

4 Upper Bounds

We prove here the upper bounds in (1.3) and in Theorems 1.3 and 1.6. We start by some preliminaries, which shall be used as well in Section 6, concerning the Gaussian regime.

4.1 Basic estimates

For $r > 0$, and $x \in \mathbb{R}^d$, we recall that $Q(x,r) := [x - r/2, x + r/2)^d \cap \mathbb{Z}^d$, and for simplicity $Q(r) := Q(0,r)$.

Lemma 4.1. *Assume that $d \geq 5$. There exists a constant $C_1 > 0$, such that for any $r \geq 1$, and any $\Lambda \subset Q(r)$,*

$$\sum_{x \in \Lambda} \frac{1}{\|x\|^{d-4} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) \;\leq\; C_1\, r^2. \tag{4.1}$$

Proof. Without loss of generality, one can assume $r \geq 2$. For $i \geq 0$, write

$$\Lambda_i := \Lambda \cap \big(Q(r2^{-i}) \backslash Q(r2^{-i-1})\big)\,,$$

and define $L := \lfloor \log_2(r) \rfloor$. Then, for some positive constants C_0 and C_1,

$$\begin{aligned}\sum_{x \in \Lambda} \frac{1}{\|x\|^{d-4} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) \;&\leq\; \sum_{i=0}^{L} \sum_{x \in \Lambda_i} \frac{1}{\|x\|^{d-4} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) \\ &\leq \sum_{i=0}^{L} \big(\frac{2^{i+1}}{r}\big)^{d-4} \mathrm{Cap}\,(\Lambda_i) \leq \sum_{i=0}^{L} \big(\frac{2^{i+1}}{r}\big)^{d-4} \mathrm{Cap}\,\Big(Q(\frac{r}{2^i})\Big) \\ &\leq C_0 \sum_{i=0}^{L} \big(\frac{2^{i+1}}{r}\big)^{d-4} \cdot \big(\frac{r}{2^i}\big)^{d-2} \leq\; C_1\, r^2.\end{aligned}$$

□

The second result we need is the following.

Lemma 4.2. *Assume $d \geq 5$. There exists a constant $C_2 > 0$, such that for any $x \in \mathbb{Z}^d$, and any $T \geq 1$,*

$$\varphi_T(x) := \frac{G \star G_T(x)}{T} \ \leq \ C_2 \cdot \min\left(\frac{1}{1+\|x\|^{d-2}}, \frac{1}{T(1+\|x\|^{d-4})}\right).$$

Proof. First $G_T \leq G$, so that $G \star G_T \leq G \star G$, and an elementary computation gives that $G \star G(x) \leq C_2/(1+\|x\|^{d-4})$, for all $x \in \mathbb{Z}^d$, and some $C_2 > 0$. This already proves one of the two desired bounds.

For the other one write, by definition of G_T,

$$G \star G_T(x) = \sum_{y \in \mathbb{Z}^d} G(x-y) G_T(y) = \sum_{k=1}^{T} \mathbb{E}[G(x - S_k)]. \tag{4.2}$$

Let τ be the hitting time of the cube $Q(x,2)$ for the walk starting at 0, and note that one can assume $\|x\| \geq 4$. Since G is harmonic on $\mathbb{Z}^d \backslash \{0\}$, we have for any $k \geq 0$, $\mathbb{E}[G(x - S_{k\wedge\tau})] = G(x)$. This entails

$$G(x) = \mathbb{E}[\mathbf{1}\{\tau \geq k\} G(x - S_k)] + \mathbb{E}[\mathbf{1}\{\tau < k\} G(x - S_\tau)] \geq \mathbb{E}[G(x - S_k)] - \mathbb{E}[\mathbf{1}\{\tau < k\} G(x - S_k)].$$

Now, we use that $G(x)$ is bounded by $G(0)$, so that the previous inequality gives

$$\mathbb{E}[G(x - S_k)] \leq G(x) + G(0)\mathbb{P}(\tau < \infty) \overset{(2.3)}{\leq} (1 + CG(0)) \cdot G(x),$$

for some constant $C > 0$. Injecting this in (4.2) and using (2.1), proves the second inequality. □

Our last estimate requires some new notation. For a (deterministic) function $S : \mathbb{N} \to \mathbb{Z}^d$ (not necessarily to the nearest neighbor), and for any $\mathcal{K} \subset \mathbb{N}$, we define for any $\Lambda \subset \mathbb{Z}^d$,

$$\ell_{\mathcal{K}}(\Lambda) := \sum_{k \in \mathcal{K}} \mathbf{1}\{S(k) \in \Lambda\}.$$

Lemma 4.3. *Assume $d \geq 3$. Let $S : \mathbb{N} \to \mathbb{Z}^d$, and $\mathcal{K} \subset \mathbb{N}$, be such that for some $\rho \in (0,1)$ and $r \geq 1$,*

$$\ell_{\mathcal{K}}(Q(x,r)) \leq \rho r^d, \qquad \text{for all } x \in r\mathbb{Z}^d.$$

There exists a constant $C_3 > 0$ (independent of ρ, r, S, and $\mathcal{K}$), such that for any $z \in \mathbb{Z}^d$,

$$\sum_{k \in \mathcal{K}} \frac{\mathbf{1}(\|S(k) - z\| \geq 2r)}{\|S(k) - z\|^{d-2}} \ \leq \ C_3\, \rho^{1-\frac{2}{d}}\, |\mathcal{K}|^{2/d}. \tag{4.3}$$

Proof. We start by proving that for any $R \geq 2r$, and any $z \in \mathbb{Z}^d$,

$$\sum_{k \in \mathcal{K}} \frac{\mathbf{1}(2r \leq \|S(k) - z\| \leq R)}{\|S(k) - z\|^{d-2}} \ \leq \ C_3\, \rho\, R^2. \tag{4.4}$$

Consider a covering of the cube $Q(z,R)$ by a partition made of smaller cubes which are translates of $Q(r)$, with centers in the set $z + r\mathbb{Z}^d$. For each $x \in z + r\mathbb{Z}^d$, with $x \neq z$, the contribution of the points $S(k)$ lying in $Q(x,r)$ to the sum we need to bound, is upper bounded (up to some constant) by $\rho r^d \cdot \|x-z\|^{2-d}$, and (4.4) follows as we observe that, for some constant $C>0$,

$$\sum_{x\in z+r\mathbb{Z}^d} \frac{\mathbf{1}\{r\le \|z-x\|\le R\}}{\|z-x\|^{d-2}} \le C\frac{R^2}{r^d}.$$

We then deduce (4.3), by observing that by rearranging the points $(S(k))_{k\in\mathcal{K}}$, one can only increase the sum (at least up to a multiplicative constant) by assuming they are all in $Q(z,2(\frac{|\mathcal{K}|}{\rho})^{1/d})$, and still satisfy the hypothesis of the lemma. □

4.2 The sets $\mathcal{K}_n$

We recall here our main tools from [AS19], which require some new notation. For $n\ge 0$, and $\Lambda\subseteq\mathbb{Z}^d$, define the time spent in Λ by the walk up to time n as

$$\ell_n(\Lambda) := \sum_{k=0}^n \mathbf{1}\{S_k\in\Lambda\}.$$

Then given $\rho>0$, $r\ge 1$, and $n\ge 1$, set

$$\mathcal{K}_n(r,\rho) := \{k\le n:\ \ell_n(Q(S_k,r))\ge \rho r^d\}. \tag{4.5}$$

The following result is Theorem 1.5 and Proposition 3.1 from [AS19].

Theorem 4.4 ([AS19])**.** *There exist positive constants C_0 and κ, such that for any $\rho>0$, $r\ge 1$, and $n\ge 1$, satisfying*

$$\rho\, r^{d-2}\ge C_0\log n, \tag{4.6}$$

one has for any $L\ge 1$,

$$\mathbb{P}\big(|\mathcal{K}_n(r,\rho)|\ge L\big) \ \le\ C_0\exp\Big(-\kappa\,\rho^{\frac{2}{d}}\,L^{1-\frac{2}{d}}\Big).$$

Furthermore, for any $A>0$, there exists $\alpha>0$, such that

$$\mathbb{P}\big(|\mathcal{K}_n(r,\rho)|\ge L,\, \ell_n(\mathcal{V}_n(r,2^{-d}\rho))\le \alpha L\big) \ \le\ C_0\exp\Big(-A\rho^{\frac{2}{d}}\,L^{1-\frac{2}{d}}\Big).$$

4.3 Dimension seven and larger

We assume here that $d\ge 7$, and fix the value of T as

$$T := \lceil \gamma\cdot\zeta^{\frac{2}{d-2}}\rceil, \tag{4.7}$$

for some constant $\gamma\in(0,1)$ (depending on dimension d) that will be fixed later (in the proof of Theorem 1.6 below). Under the event of moderate deviations considered here (when the capacity of the range up to time n is reduced by an amount ζ from its mean value), the walk typically folds its trajectory a time of order ζ, in a region of volume $\zeta^{d/(d-2)}$. Thus the typical density of the range in the folding region is

$$\overline{\rho} := \zeta^{-\frac{2}{d-2}}.$$

Define ρ_i, r_i, and L_i, for $i \in \mathbb{Z}$, by

$$\rho_i := 2^{-i} \cdot \overline{\rho}, \qquad r_i^{d-2} \cdot \rho_i = C_0 \log n, \qquad \text{and} \qquad L_i := \zeta \cdot 2^{\frac{2i}{d-2}},$$

with C_0 as in Theorem 4.4. Define

$$N := \lceil \frac{d-2}{2} \cdot \log_2(n/\zeta) \rceil, \quad \text{and} \quad M := \lceil \log_2(1/\overline{\rho}) \rceil,$$

so that $n \le L_N \le 2n$, and $1 \le \rho_{-M} \le 2$. For $-M \le i \le N$, set

$$\widehat{\mathcal{K}}_i := \mathcal{K}_n(r_i, \rho_i) \setminus \bigcup_{-M \le j < i} \mathcal{K}_n(r_j, \rho_j),$$

with the convention that $\widehat{\mathcal{K}}_{-M} = \mathcal{K}_n(r_{-M}, \rho_{-M})$. Finally for $A > 0$, $\delta > 0$, and $I < \min(M, N)$, define

$$\mathcal{E}(A, \delta, I) := \left(\bigcap_{-I \le i \le I} \left\{ |\widehat{\mathcal{K}}_i| \le \delta L_i \right\} \right) \cap \left(\bigcap_{I < i \le N} \left\{ |\widehat{\mathcal{K}}_i| \le A L_i \right\} \right) \cap \left(\bigcap_{-M \le i < -I} \left\{ |\widehat{\mathcal{K}}_i| \le A L_i \right\} \right).$$

Our main result here is the following proposition.

Proposition 4.5. *For any $A > 0$, there exist $\delta > 0$ and $I \ge 0$, such that for any $n \ge 2$, and $n^{\frac{d-2}{d}} \cdot \log n \le \zeta \le n$,*

$$\mathcal{E}(A, \delta, I) \ \subseteq \ \{\xi_n(T) \le \zeta\}.$$

Before we give the proof of this proposition, let us show how it implies the upper bound in Theorem 1.1, as well as Theorem 1.6 for dimension 7 and higher, assuming for a moment the lower bound in Theorem 1.1 (which will be proved later and independently in Section 5).

Proof of Theorem 1.1: the upper bound. Note first that Proposition 4.5 and Theorem 4.4 give

$$\mathbb{P}(\xi_n(T) > \zeta) \le \mathbb{P}(\mathcal{E}(1, \delta, I)^c) \le C \exp(-c \zeta^{1 - \frac{2}{d-2}}),$$

for some constant $c, C > 0$, where δ and I are those given by Proposition 4.5, associated to $A = 1$. Note also that by definition T is of order $\zeta^{2/(d-2)}$, see (4.7), and thus the above estimate together with Proposition 3.2 prove the upper bound in Theorem 1.1. □

Proof of Theorem 1.6. Assume the lower bound in Theorem 1.1, and let us start with the proof of (1.6). First choose γ small enough in the definition (4.7) of T, so that conditionally on the event of moderate deviations $MD(n, \zeta) := \{\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \le -\zeta\}$, the probability of the event $\{\xi_n(T) \le c\zeta\}$ goes to zero, with c some appropriately chosen constant. Note that this is possible thanks to Proposition 3.2 and the assumed lower bound in Theorem 1.1. Then choose A large enough, so that conditionally on $MD(n, \zeta)$, the probability of any of the events $\{|\widehat{\mathcal{K}}_i| > A L_i\}$, for $i \in \mathbb{Z}$, goes to zero (which is always possible thanks to Theorem 4.4), where implicitly ζ is replaced by $c\zeta$ in the definition of these events. Then Propositions 3.2 and 4.5 show that conditionally on $MD(n, \zeta)$, one of the events $\{|\widehat{\mathcal{K}}_i| > \delta L_i\}$, with $-I \le i \le I$, holds with probability going to 1 (where δ and I are given by Proposition 4.5), and (1.6) follows from the second part of Theorem 4.4.

Finally, the characterization of the capacity in (1.7), is a simple consequence of a general result of [AS20b], namely (1.15) of Theorem 1.5, once we know (1.6). □

Proof of Proposition 4.5. Let $\widehat{\mathcal{K}}_{N+1}$ be such that

$$\widehat{\mathcal{K}}_{N+1} := \{0,\dots,n\} \setminus \bigcup_{-M\le i\le N} \widehat{\mathcal{K}}_i. \tag{4.8}$$

Now, we decompose $\xi_n(T)$ over the various $\widehat{\mathcal{K}}_i$. By (3.9), for any $I \le \min(-M,N)$,

$$\xi_n(T) \le \Sigma_1 + \Sigma_2 + 2\Sigma_3 + 2\Sigma_4 + 2\Sigma_5,$$

where (note that $r_i \le \sqrt{T}$ when $i \le 0$, and $\varphi_T(z) = \frac{1}{T} G \star G_T(z)$ is defined in Lemma 4.2)

$$\Sigma_1 := \sum_{i=-I}^{N+1} \sum_{k\in\widehat{\mathcal{K}}_i} \sum_{x\in\mathcal{R}_k} \varphi_T(x-S_k)\mathbb{P}_x(H^+_{\mathcal{R}_k}=\infty)\cdot \mathbf{1}\{x\in Q(S_k,r_{i-1})\},$$

$$\Sigma_2 := \sum_{i=-M}^{-I} \sum_{k\in\widehat{\mathcal{K}}_i} \sum_{x\in\mathcal{R}_k} \varphi_T(x-S_k)\mathbb{P}_x(H^+_{\mathcal{R}_k}=\infty)\cdot \mathbf{1}\{x\in Q(S_k,\sqrt{T})\},$$

$$\Sigma_3 := \sum_{i=-M}^{N+1}\sum_{j=i}^{N+1} \sum_{k\in\widehat{\mathcal{K}}_i} \sum_{k'\in\widehat{\mathcal{K}}_j} \varphi_T(S_{k'}-S_k)\cdot \mathbf{1}\{S_{k'}\in Q(S_k,r_{j-1})\setminus Q(S_k,r_{i-1})\},$$

$$\Sigma_4 := \sum_{i=-M}^{0}\sum_{j=i}^{0} \sum_{k\in\widehat{\mathcal{K}}_i} \sum_{k'\in\widehat{\mathcal{K}}_j} \varphi_T(S_{k'}-S_k)\cdot \mathbf{1}\{S_{k'}\notin Q(S_k,\sqrt{T})\},$$

$$\Sigma_5 := \sum_{i=-M}^{N+1}\sum_{j=\max(i,0)}^{N+1} \sum_{k\in\widehat{\mathcal{K}}_i} \sum_{k'\in\widehat{\mathcal{K}}_j} \varphi_T(S_{k'}-S_k)\cdot \mathbf{1}\{S_{k'}\notin Q(S_k,r_{j-1})\},$$

Note that the third term Σ_3 is not included in Σ_1 and Σ_2, since in these last two terms we sum over points of the space, not over time indices. This is important since one important tool used to control them is Lemma 4.1.

Now assume that $\mathcal{E}(A,\delta,I)$ holds, and let us bound Σ_1 first. For $-I\le i\le N+1$, define

$$J(i) = -i + \lfloor \frac{d-4}{2}\log_2(T) - \frac{d}{2}\log_2(\log n) - \frac{d-2}{2}h\rfloor,$$

with h some positive constant to be chosen later, so that for any $-I\le i\le N+1$,

$$\frac{L_i\cdot r_{J(i)}^2}{T} \le C2^{-h}\cdot \frac{\zeta}{\log n},$$

for some constant $C>0$ (that might change from line to line). Note here that since $-M\le \log_2(\gamma)-\log_2(T)$, by choosing γ small enough (once h is fixed), one can always assume that $J(N+1)\ge -M$, which we will do now.

Then Lemmas 4.1 and 4.2 show that for any $-I\le i\le N+1$, on $\mathcal{E}(A,\delta,I)$,

$$\sum_{k\in\widehat{\mathcal{K}}_i}\sum_{x\in\mathcal{R}_k} \varphi_T(x-S_k)\mathbb{P}_x(H^+_{\mathcal{R}_k}=\infty)\cdot\mathbf{1}\{x\in Q(S_k,r_{J(i)})\} \le C|\widehat{\mathcal{K}}_i|\frac{r_{J(i)}^2}{T} \le CL_i\frac{r_{J(i)}^2}{T} \le C2^{-h}\frac{\zeta}{\log n}.$$

On the other hand, for i such that $r_{J(i)} < r_{i-1}$, and $k \in \widehat{\mathcal{K}}_i$, we use that by definition the time spent on concentric shells around S_k is bounded, up to distance r_{i-1}. This gives for such i, using again Lemma 4.2,

$$\sum_{k\in\widehat{\mathcal{K}}_i}\sum_{x\in\mathcal{R}_k}\varphi_T(x-S_k)\cdot\mathbf{1}\{x\in Q(S_k,r_{i-1})\setminus Q(S_k,r_{J(i)})\}$$
$$\leq C|\widehat{\mathcal{K}}_i|\sum_{J(i)\leq j\leq i-1}\frac{\rho_j r_j^d}{Tr_j^{d-4}}\leq C|\widehat{\mathcal{K}}_i|\sum_{J(i)\leq j\leq i-1}\frac{\log n}{Tr_j^{d-6}}\leq C|\widehat{\mathcal{K}}_i|\frac{\log n}{T}.$$

Moreover, by hypothesis on ζ, one has $\frac{n\log n}{T}\leq C\zeta\cdot(\log n)^{-\frac{2}{d-2}}$, and by (4.8) it also holds $\sum_i|\widehat{\mathcal{K}}_i| = n$. Therefore, by fixing now the constant h large enough, we get for all n large enough,

$$\Sigma_1\leq C\left\{(N-M)2^{-h}\frac{\zeta}{\log n}+\frac{n\log n}{T}\right\}\leq\frac{\zeta}{8}.$$

Similarly, using Lemmas 4.1 and 4.2, we get by choosing I large enough,

$$\Sigma_2\leq C\sum_{-M\leq i\leq -I}|\widehat{\mathcal{K}}_i|\leq C2^{-\frac{2I}{d-2}}\cdot\zeta\leq\frac{\zeta}{8}.$$

We consider the term Σ_3. Note that for any k, by definition of $\widehat{\mathcal{K}}_j$, there are at most $C\rho_j r_j^d$ indices $k'\in\widehat{\mathcal{K}}_j$, such that $S_{k'}\in Q(S_k,r_{j-1})$. Therefore, using Lemma 4.2, we get for n large enough,

$$\Sigma_3\leq C\sum_{-M\leq i\leq N+1}|\widehat{\mathcal{K}}_i|\sum_{i\leq j\leq N+1}\frac{\rho_j r_j^d}{Tr_{i-1}^{d-4}}\leq C\sum_{-M\leq i\leq N+1}\zeta\cdot\frac{r_{i-1}^2}{T^{\frac{2}{d-2}}}\sum_{i\leq j\leq N+1}\frac{r_j^2\log n}{Tr_{i-1}^{d-4}}$$
$$\leq C\,\frac{\zeta\cdot\log n}{T}\sum_{M\leq i\leq N+1}\frac{r_{N+1}^2}{r_{i-1}^{d-6}T^{\frac{2}{d-2}}}\leq C\frac{n\log n}{T}\leq\frac{\zeta}{8}.$$

Next, using simply Lemma 4.2, we obtain (choosing first I large enough, and then δ small enough)

$$\Sigma_4\leq C\sum_{-M\leq i\leq 0}\sum_{-M\leq j\leq 0}\frac{|\widehat{\mathcal{K}}_i|\cdot|\widehat{\mathcal{K}}_j|}{T^{\frac{d-2}{2}}}\leq C\zeta\sum_{-M\leq i\leq -I}A2^{\frac{2i}{d-2}}+C\delta\zeta\sum_{-I\leq i\leq 0}2^{\frac{2i}{d-2}}\leq\frac{\zeta}{8}.$$

By Lemma 4.3, one has for some $C>0$, (choosing first I large enough, and then δ small enough)

$$\Sigma_5\leq\sum_{-M\leq i\leq N+1}|\widehat{\mathcal{K}}_i|\sum_{j\geq\max(i,0)}|\widehat{\mathcal{K}}_j|^{2/d}\rho_j^{1-\frac{2}{d}}$$
$$\leq C\sum_{-M\leq i\leq N+1}|\widehat{\mathcal{K}}_i|\sum_{j\geq\max(i,0)}2^{\frac{4j}{d(d-2)}-j(1-\frac{2}{d})}$$
$$\leq C\sum_{-M\leq i\leq N+1}|\widehat{\mathcal{K}}_i|2^{-\max(i,0)(1-\frac{2}{d-2})}$$
$$\leq C\zeta\left\{\delta\sum_{-I\leq i\leq I}2^{-i\frac{d-6}{d-2}}+A\sum_{i\geq I}2^{-i\frac{d-6}{d-2}}+A\sum_{M\leq i\leq -I}2^{\frac{2i}{d-2}}\right\}\leq\frac{\zeta}{8}.$$

This concludes the proof of the proposition. □

4.4 Dimension five

We assume here that $d = 5$ and let

$$T := \lceil \gamma \cdot (\zeta n)^{1/3} \rceil,$$

with γ some constant (chosen similarly as in the previous subsection), and

$$\overline{\rho} := \zeta^{5/3}\, n^{-7/3}.$$

Define next ρ_i, r_i, and L_i, for $i \in \mathbb{Z}$, by

$$\rho_i := 2^i \cdot \overline{\rho}, \qquad r_i^3 \cdot \rho_i = C_0 \log n, \qquad \text{and} \qquad L_i := n \cdot 2^{-\frac{2i}{3}},$$

with C_0 as in Theorem 4.4. Let for $i \in \mathbb{Z}$, $\widehat{\mathcal{K}}_i := \mathcal{K}_n(r_i, \rho_i) \setminus \bigcup_{j>i} \mathcal{K}_n(r_j, \rho_j)$. Then, let N be the smallest integer, such that $1 \le r_N \le 2$, and for $A > 0$, $\delta > 0$, and $0 \le I \le N$, let

$$\mathcal{E}(A, \delta, I) := \left(\bigcap_{-I \le i \le I} \left\{ |\widehat{\mathcal{K}}_i| \le \delta L_i \right\} \right) \cap \left(\bigcap_{I < i \le N} \left\{ |\widehat{\mathcal{K}}_i| \le A L_i \right\} \right).$$

Our main result here is the following proposition, which implies both the upper bound in Theorem 1.3, as well as Theorem 1.6 for $d = 5$. Since this can be done in exactly the same way as in dimension 7 and higher, we will not repeat the arguments here.

Proposition 4.6. *For any $A > 0$, there exist $\delta > 0$ and $I \ge 0$, such that for any $n \ge 2$, and $n^{5/7} \cdot \log n \le \zeta \le n$,*

$$\mathcal{E}(A, \delta, I) \;\subseteq\; \{\xi_n(T) \le \zeta\}.$$

Proof. Given some $I \ge 0$, let

$$\widetilde{\mathcal{K}}_0 := \{0, \dots, n\} \setminus \bigcup_{-I \le i \le N} \mathcal{K}_n(r_i, \rho_i).$$

Note that for any I,

$$\xi_n(T) \le 2\Sigma_1 + 2\Sigma_2 + \Sigma_3 + \Sigma_4,$$

where,

$$\Sigma_1 := \sum_{-I \le i \le N} \sum_{k \in \widehat{\mathcal{K}}_i} \sum_{k'=0}^{n} \varphi_T(S_{k'} - S_k) \cdot \mathbf{1}\{S_{k'} \in Q(S_k, r_{i+1})\},$$

$$\Sigma_2 := \sum_{-I \le i \le N} \sum_{k \in \widehat{\mathcal{K}}_i} \sum_{-I \le j \le i} \sum_{k' \in \widehat{\mathcal{K}}_j} \varphi_T(S_{k'} - S_k) \cdot \mathbf{1}\{S_{k'} \notin Q(S_k, r_{j+1})\},$$

$$\Sigma_3 := \sum_{k \in \widetilde{\mathcal{K}}_0} \sum_{x \in \mathcal{R}_k} \varphi_T(x - S_k) \cdot \mathbf{1}\{x \in Q(S_k, r_{-I})\},$$

$$\Sigma_4 := \sum_{k \in \widetilde{\mathcal{K}}_0} \sum_{x \in \mathcal{R}_k} \varphi_T(x - S_k) \cdot \mathbf{1}\{x \notin Q(S_k, r_{-I})\},$$

Assume now that $\mathcal{E}(A, \delta, I)$ holds. Let J be the smallest integer, such that $r_J \le \sqrt{T}$. Using Lemma 4.2, and the bound $\sum_i |\widehat{\mathcal{K}}_i| \le n$, we get for some $C > 0$, and n large enough,

$$\Sigma_1 \le C \sum_{-I \le i \le N} |\widehat{\mathcal{K}}_i| \left(\sum_{i \le j \le J} \frac{\rho_j r_j^5}{r_j^3} + \sum_{j > \max(i,J)} \frac{\rho_j r_j^5}{T r_j} \right)$$

$$\leq C\log n \sum_{-I\leq i\leq N} |\widehat{\mathcal{K}}_i| \left(\sum_{i\leq j\leq J} \frac{1}{r_j} + \sum_{j>\max(i,J)} \frac{r_j}{T}\right) \leq C\frac{n\log n}{\sqrt{T}} \leq \frac{\zeta}{8},$$

using also the hypothesis on ζ for the last inequality. The same argument gives as well $\Sigma_3 \leq \zeta/4$. Using in addition Lemma 4.3, we get taking first I large enough, and then δ small enough.

$$\begin{aligned}\Sigma_2 &\leq C \sum_{-I\leq i\leq N} |\widehat{\mathcal{K}}_i| \sum_{-I\leq j\leq i} |\widehat{\mathcal{K}}_j|^{2/5}\rho_j^{3/5} \leq C\frac{\zeta}{n}\cdot \sum_{-I\leq i\leq N} |\widehat{\mathcal{K}}_i| \sum_{-I\leq j\leq i} 2^{j(-\frac{4}{15}+\frac{3}{5})} \\ &\leq C\delta\zeta \sum_{-I\leq i\leq I} 2^{-i/3} + C\cdot\zeta\cdot A\sum_{i\geq I} 2^{-i/3} \leq \frac{\zeta}{8},\end{aligned}$$

The same argument gives as well $\Sigma_4 \leq \zeta/4$, concluding the proof. □

5 Lower Bounds

We prove here the lower bounds in Theorems 1.1 and 1.3. In fact in dimension 5 the result covers a larger range of possible values for ζ.

Proposition 5.1. *Assume $d = 5$. There exist positive constants ε_0 and $\underline{\kappa}$, such that for any $n \geq 2$, and any $\sqrt{n}(\log n)^3 \leq \zeta \leq \varepsilon_0 n$, one has*

$$\mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_n\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)] \leq -\zeta\right) \geq \exp(-\underline{\kappa}\cdot(\frac{\zeta^2}{n})^{1/3}).$$

Proof. The proof of (2.12) in [ASS19b] reveals that for any finite $A, B \subset \mathbb{Z}^d$, one has also

$$\mathrm{Cap}(A\cup B) \leq \mathrm{Cap}(A) + \mathrm{Cap}(B) - \chi_0(A,B), \tag{5.1}$$

with

$$\chi_0(A,B) := \sum_{x\in A\setminus B}\sum_{y\in B} \mathbb{P}_x(H^+_{A\cup B} = \infty)G(y-x)\mathbb{P}_y(H^+_B = \infty).$$

Now given $n \geq 1$, set $\ell = \lfloor \frac{n}{10}\rfloor$, and $m = n - \ell$. We apply (5.1) with $A = \mathcal{R}_m$ and $B = \mathcal{R}[m,n]$. Fix $\varepsilon_0 > 0$ (later chosen small enough), and define

$$E := \left\{\overline{\mathrm{Cap}}\left(\mathcal{R}_n\right) \geq -\varepsilon_0 n\right\},$$

where we use the notation $\overline{\mathrm{Cap}}\left(\mathcal{R}_n\right)$ for the centered capacity. Using (5.1), Lemma 2.1, and Proposition 2.4, we deduce that for some constant $c > 0$,

$$\begin{aligned}\mathbb{P}\left(-\varepsilon_0 n \leq \overline{\mathrm{Cap}}\left(\mathcal{R}_n\right) \leq -\zeta\right) \geq\ & \mathbb{P}(E,\, \chi_0(\mathcal{R}_m, \mathcal{R}[m,n]) \geq 4\zeta) - \mathbb{P}(\overline{\mathrm{Cap}}\left(\mathcal{R}_m\right) \geq \zeta) \\ & - \mathbb{P}(\overline{\mathrm{Cap}}\left(\mathcal{R}[m,n]\right) \geq \zeta) \\ \geq\ & \mathbb{P}(E,\, \chi_0(\mathcal{R}_m, \mathcal{R}[m,n]) \geq 4\zeta) - 2\exp(-c\frac{\zeta^2}{n(\log n)^3}).\end{aligned} \tag{5.2}$$

Note that when $\zeta \geq \sqrt{n}(\log n)^3$, then $\zeta^2/(n(\log n)^3) \geq (\log n)(\zeta^2/n)^{1/3}$, and therefore the last term above is negligible. Now, let $\rho > 0$ be some small constant (to be fixed later) and consider the event

$$F := \{\|S_k\| \leq \rho\cdot n^{2/3}\,\zeta^{-1/3}, \quad \text{for all } k \leq n\}.$$

Note that by (2.1) and (2.7), on the event F,

$$\chi_0(\mathcal{R}_m, \mathcal{R}[m,n]) \ \geq \ c_\rho \cdot \frac{\zeta}{n^2} \cdot \mathrm{Cap}(\mathcal{R}[m,n]) \cdot (\mathrm{Cap}(\mathcal{R}_n) - \mathrm{Cap}(\mathcal{R}[m,n])) \,, \tag{5.3}$$

for some constant $c_\rho > 0$, going to infinity as ρ goes to zero. Furthermore, by (2.6), one has

$$\mathrm{Cap}(\mathcal{R}_n) \leq \mathrm{Cap}(\mathcal{R}_m) + \mathrm{Cap}(\mathcal{R}[m,n]),$$

and thus by Lemma 2.1 and Proposition 2.4, by taking ε_0 small enough, we get for n large enough,

$$\begin{aligned}\mathbb{P}\left(\mathrm{Cap}(\mathcal{R}[m,n]) \leq \gamma_5 \frac{\ell}{2},\, E\right) \ &\leq \ \mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_m) \geq \gamma_5(m+\ell/3)\right) \\ &\leq \ \mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_m) \geq \gamma_5 \frac{\ell}{10}\right) \leq \ \exp\left(-c' \frac{n}{(\log n)^3}\right),\end{aligned}$$

for some constant $c' > 0$, and with γ_5 as in (1.2). Similarly one has for some possibly smaller constant $c' > 0$,

$$\mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_n) - \mathrm{Cap}(\mathcal{R}[m,n]) \leq \gamma_5 \frac{n}{4},\, E\right) \ \leq \ \mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}[m,n]) \geq \gamma_5 \ell\right) \leq \ \exp\left(-c' \frac{n}{(\log n)^3}\right).$$

Then (5.3) gives

$$\begin{aligned}\mathbb{P}(\chi_0(\mathcal{R}_m, \mathcal{R}[m,n]) \geq \frac{c_\rho \gamma_5^2}{100} \cdot \zeta,\, E) \ &\geq \ \mathbb{P}(\chi_0(\mathcal{R}_m, \mathcal{R}[m,n]) \geq \frac{c_\rho \gamma_5^2}{100} \cdot \zeta,\, E \cap F) \\ &\geq \ \mathbb{P}(E \cap F) - 2\exp\left(-c' \frac{n}{(\log n)^3}\right) \\ &\geq \ \mathbb{P}(F) - \mathbb{P}(E^c) - 2\exp\left(-c' \frac{n}{(\log n)^3}\right).\end{aligned}$$

Coming back to (5.2), and choosing ρ, such that $c_\rho \geq 300/\gamma_5^2$, we deduce that

$$\begin{aligned}\mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_n) \leq -\zeta\right) \ &= \ \mathbb{P}\left(-\varepsilon_0 n \leq \overline{\mathrm{Cap}}\,(\mathcal{R}_n) \leq -\zeta\right) + \mathbb{P}(E^c) \\ &\geq \ \mathbb{P}(F) - 2\exp\left(-c' \frac{n}{(\log n)^3}\right) - 2\exp(-c\,(\log n) \cdot \zeta^{2/3} n^{-1/3}).\end{aligned}$$

Moreover, it is well known that for any $\rho > 0$, there exists $\kappa > 0$, such that

$$\mathbb{P}(F) \ \geq \ \exp(-\kappa \cdot \zeta^{2/3} n^{-1/3}),$$

and this concludes the proof. □

In dimension 6 and more the result reads as follows.

Proposition 5.2. *Assume $d \geq 7$. There exist positive constants ε_0, K and $\underline{\kappa}$, such that for any $n \geq 2$ and any $K n^{\frac{d-2}{d}} \leq \zeta \leq \varepsilon_0\, n$, one has*

$$\mathbb{P}\left(\mathrm{Cap}\,(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}\,(\mathcal{R}_n)] \leq -\zeta\right) \ \geq \ \exp\left(-\underline{\kappa} \cdot \zeta^{1-\frac{2}{d-2}}\right).$$

In dimension $d = 6$, the same result holds for $n^{\frac{d-2}{d}} (\log\log n)^2 \leq \zeta \leq \varepsilon_0\, n$.

Proof. We prove the result for $d \geq 7$ to keep notation simple, but the same argument works as well for $d = 6$. Set $\ell := \lfloor 5\zeta/\gamma_d \rfloor$. Using (2.6), Lemma 2.1, and Proposition 2.4, we obtain that for some constant $c > 0$,

$$\begin{aligned}\mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_n) \leq -\zeta\right) &\geq \mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_\ell) \leq -3\zeta\right) - \mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}[\ell, n]) \geq \zeta\right)\\ &\geq \mathbb{P}\left(\mathrm{Cap}\,(\mathcal{R}_\ell) \leq \zeta\right) - \exp(-c \cdot \frac{\zeta^2}{n}),\end{aligned}$$

at least provided ζ is large enough, which one can always assume. Now the hypothesis on ζ implies that the last term is negligible, provided K is chosen large enough, and by the same argument as in the proof of Proposition 5.1, one can see that the first term on the right-hand side is of the right order (which is of the order of the event F where the walk stays confined in a ball of radius $c'\zeta^{1/(d-2)}$, with $c' > 0$ small enough, during the whole time ℓ). This concludes the proof of the proposition. □

6 The Gaussian regime

The starting point to proving Theorem 1.2 is a standard dyadic decomposition which follows from using (2.9) repeatedly along a dyadic scheme. For any $L \geq 1$, and $n \geq 2^L$,

$$\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] = \sum_{i=1}^{2^L} \left(\mathrm{Cap}(\mathcal{R}_i^L) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_i^L)]\right) - \sum_{\ell=1}^{L} \sum_{i=1}^{2^{\ell-1}} Y_i^\ell, \tag{6.1}$$

where $Y_i^\ell := \chi_{\mathcal{C}}(\mathcal{R}_{2i-1}^\ell, \mathcal{R}_{2i}^\ell) - \mathbb{E}[\chi_{\mathcal{C}}(\mathcal{R}_{2i-1}^\ell, \mathcal{R}_{2i}^\ell)]$, and the $\{\mathcal{R}_i^\ell\}_{i=1,\dots,2^\ell}$, are independent ranges of length $n2^{-\ell}$ (the time-length is not exactly equal for each of them since we do not suppose that n is of the form $n = 2^K$, for some $K \geq 1$, but they differ by at most one unit).

A Gaussian-type fluctuation is due to the sum of the 2^L self-similar terms in (6.1), after L is chosen appropriatly. It is classical (see [Chen10]) to use Gärtner-Ellis' Theorem after we show that the contribution of the Y_i^ℓ is negligible. Thus, the main technical novelty of this section is the stretched exponential moment bound (1.9), which is performed in Section 6.2.

After recalling some well-known results in Section 6.1, we conclude the proof of Theorem 1.2 in Section 6.3.

6.1 Preliminary results

We first state an instance of Gärtner-Ellis' Theorem (see Theorem 2.3.6 in [DZ98]).

Theorem 6.1 (Gärtner-Ellis)**.** *Let $\{X_n\}_{n\geq 0}$ be a sequence of real random variables, and let $\{b_n\}_{n\geq 0}$ going to infinity. If for any $\theta \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}[\exp(\theta b_n \cdot X_n)] = \frac{\sigma^2}{2} \cdot \theta^2,$$

then, for all $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}(X_n > \lambda) = -\frac{\lambda^2}{2\sigma^2}.$$

We recall now a large deviation estimates for variables with stretched exponential moment.

Theorem 6.2 (A. Nagaev [Na69]). *Let $\{Y_n\}_{n\ge 0}$ be a sequence of centered random variables, such that $\mathbb{E}[\exp(\kappa|Y_1|^\alpha)] < \infty$, for some constants $\kappa > 0$, and $\alpha \in (0,1]$. Then there are positive constants c and C, such that for any $n \ge 1$ and any $t > n^{\frac{1}{2-\alpha}}$,*

$$\mathbb{P}\big(Y_1 + \cdots + Y_n > t\big) \le C\exp(-ct^\alpha).$$

6.2 Stretched exponential moment of the cross term

The heart of the proof of Theorem 1.2 use Theorem 6.3 below which is more general than (1.9), and has interest of its own. It is analogous to the arguments of [AS20c].

Define for any subsets $A, B \subseteq \mathbb{Z}^d$,

$$\Gamma(A,B) = \sum_{x\in A}\sum_{y\in B} G(y-x)\mathbb{P}_y(H_B^+ = \infty).$$

Recall that $0 \le \chi_{\mathcal{C}}(A,B) \le 2\Gamma(A,B)$, for any $A, B \subseteq \mathbb{Z}^d$.

Theorem 6.3. *Let $\mathcal{R}_\infty$ and $\widetilde{\mathcal{R}}_\infty$ be the ranges of two independent random walks on $\mathbb{Z}^d$, with $d \ge 7$. There exist positive constants c_1, c_2, such that for all t large enough,*

$$\exp(-c_1 t^{1-\frac{2}{d-2}}) \le \mathbb{P}(\Gamma(\widetilde{\mathcal{R}}_\infty, \mathcal{R}_\infty) > t) \le \exp(-c_2 t^{1-\frac{2}{d-2}}).$$

Let us emphasize that in the definition of Γ it is fundamental to keep the escape probabilities, in other words one cannot simply bound them by one. Indeed one could show that the tail distribution of $\Gamma'(\mathcal{R}_\infty, \widetilde{\mathcal{R}}_\infty) := \sum_{x\in\mathcal{R}_\infty}\sum_{y\in\widetilde{\mathcal{R}}_\infty} G(x-y)$ obeys a different decay at infinity.

Proof of Theorem 6.3. We start with the lower bound. Observe that $\Gamma(\cdot,\cdot)$ is increasing in both arguments for the inclusion of sets, thus for any $n \ge 1$,

$$\Gamma(\widetilde{\mathcal{R}}_\infty, \mathcal{R}_\infty) \ge \Gamma(\widetilde{\mathcal{R}}_n, \mathcal{R}_n).$$

Therefore the lower bound is obtained by forcing the two walks to stay confined in a ball of radius $t^{\frac{1}{d-2}}$ for a time Ct, with $C > 0$ large enough, exactly as in the proof of Proposition 5.1.

We now move to the upper bound. The proof is obtained in three steps. In the first step, we reduce the time window of one walk to a finite interval, as follows. Observe that for any integer $n \ge 1$,

$$\begin{aligned}\mathbb{E}[\Gamma(\widetilde{\mathcal{R}}_\infty, \mathcal{R}[n,\infty))] &\le \mathbb{E}\left[\sum_{k=0}^\infty\sum_{\ell=n}^\infty G(S_k - \widetilde{S}_\ell)\right] = \sum_{k=0}^\infty\sum_{\ell=n}^\infty \mathbb{E}[G(S_{k+\ell})] \\ &= \sum_{k=n}^\infty (k+1-n)\mathbb{E}[G(S_k)] \le C\sum_{k=n}^\infty \frac{k+1-n}{k^{\frac{d-2}{2}}} \le \frac{C}{n^{\frac{d-6}{2}}},\end{aligned}$$

for some constant $C > 0$. Therefore if we let $n := \exp(t^{1-\frac{2}{d-2}})$, then by Markov's inequality,

$$\mathbb{P}(\Gamma(\widetilde{\mathcal{R}}_\infty, \mathcal{R}[n,\infty)) \ge 1) \le \mathbb{E}[\Gamma(\widetilde{\mathcal{R}}_\infty, \mathcal{R}[n,\infty))] \le C\exp(-\frac{d-6}{2}\cdot t^{1-\frac{2}{d-2}}),$$

and thus, due to the inequality

$$\Gamma(\widetilde{\mathcal{R}}_\infty, \mathcal{R}_\infty) \le \Gamma(\widetilde{\mathcal{R}}_\infty, \mathcal{R}_n) + \Gamma(\widetilde{\mathcal{R}}_\infty, \mathcal{R}[n,\infty)),$$

it just remains to bound the first term on the right-hand side.

In a second step we claim that for any subset $\Lambda \subseteq \mathbb{Z}^d$, and any $t \geq 1$,

$$\mathbb{P}(\Gamma(\widetilde{\mathcal{R}}_\infty, \Lambda) > t) \leq \exp\left(-\frac{t \cdot \log 2}{2 \sup_{x \in \mathbb{Z}^d} \mathbb{E}_x[\Gamma(\widetilde{\mathcal{R}}_\infty, \Lambda)]}\right). \tag{6.2}$$

To see this, we use again that for any $A, B \subseteq \mathbb{Z}^d$, one has $\Gamma(A \cup B, \Lambda) \leq \Gamma(A, \Lambda) + \Gamma(B, \Lambda)$. Thus the Markov property and Markov's inequality show that the random variable $\frac{\Gamma(\widetilde{\mathcal{R}}_\infty, \Lambda)}{2 \sup_{x \in \mathbb{Z}^d} \mathbb{E}_x[\Gamma(\widetilde{\mathcal{R}}_\infty, \Lambda)]}$ is stochastically bounded by a Geometric random variable with mean 2, from which (6.2) follows immediately. Note also that for any x,

$$\mathbb{E}_x[\Gamma(\widetilde{\mathcal{R}}_\infty, \Lambda)] \leq \sum_{z \in \Lambda} G \star G(z - x) \cdot \mathbb{P}_z(H_\Lambda^+ = \infty) =: \mathcal{F}(\Lambda - x),$$

where we recall that $G \star G$ is the convolution of G with itself, and

$$\mathcal{F}(\Lambda) := \sum_{z \in \Lambda} G \star G(z) \cdot \mathbb{P}_z(H_\Lambda^+ = \infty).$$

Thus it amounts to show that for some positive constants c and C, one has

$$\mathbb{P}\left(\sup_{x \in \mathbb{Z}^d} \mathcal{F}(\mathcal{R}_n - x) > C t^{\frac{2}{d-2}}\right) \leq C \exp(-c t^{1 - \frac{2}{d-2}}), \quad \text{with } n = \exp(t^{1 - \frac{2}{d-2}}), \tag{6.3}$$

which is our third and last step. Note that $\mathcal{F}$ is also subadditive in the sense that for any $A, B \subseteq \mathbb{Z}^d$, $\mathcal{F}(A \cup B) \leq \mathcal{F}(A) + \mathcal{F}(B)$. This allows to partition the range into different pieces, according to the occupation density in a certain neighborhood, and then bound $\mathcal{F}$ on each of them. To be more precise, set $\rho_0 := t^{-\frac{2}{d-2}}$, and then for $i \geq 0$, define ρ_i and r_i by

$$\rho_i := 2^{-i} \rho_0, \quad \text{and} \quad \rho_i r_i^{d-2} = C_0 \log n,$$

with C_0 as in (4.6). Then let $\mathcal{R}_n(r_i, \rho_i) := \{S_k,\ k \in \mathcal{K}_n(r_i, \rho_i)\}$, and

$$\Lambda_i := \mathcal{R}_n(r_i, \rho_i) \setminus \bigcup_{0 \leq j < i} \mathcal{R}_n(r_j, \rho_j), \qquad \Lambda_i^* := \mathcal{R}_n \setminus \bigcup_{0 \leq j < i} \Lambda_i.$$

By Theorem 4.4, one has for any $i \geq 0$,

$$\mathbb{P}(|\Lambda_i| \geq 2^{\frac{2i}{d-2}} t) \leq C \exp(-\kappa t^{1 - \frac{2}{d-2}}),$$

for some positive constants C and κ, and in fact for $i > \frac{d-2}{2} \log_2(n+1)$, the above probability is zero, since by definition $|\Lambda_i| \leq n + 1$. Therefore, if we let

$$\mathcal{E} := \left\{|\Lambda_i| \leq 2^{\frac{2i}{d-2}} t, \text{ for all } i \geq 0\right\},$$

then the above discussion shows that

$$\mathbb{P}(\mathcal{E}^c) \leq C \exp(-(\kappa/2) \cdot t^{1 - \frac{2}{d-2}}),$$

at least for t large enough. We now show that for some constant $C > 0$,

$$\mathcal{E} \subseteq \left\{\sup_{x \in \mathbb{Z}^d} \mathcal{F}(\mathcal{R}_n - x) \leq C t^{\frac{2}{d-2}}\right\}, \tag{6.4}$$

which will conclude the proof of the theorem. To simplify notation we only bound $\mathcal{F}(\mathcal{R}_n - x)$ for $x = 0$, but it should be clear from the proof that all our estimates are uniform with respect to x. We partition space into shells $(\mathcal{S}_k)_{k\geq 0}$, defined by $\mathcal{S}_0 := Q(0, r_0)$, and $\mathcal{S}_k := Q(0, r_k)\backslash Q(0, r_{k-1})$ for $k \geq 1$. By subadditivity, one has

$$\mathcal{F}(\mathcal{R}_n) \leq \sum_{k\geq 0} \mathcal{F}(\mathcal{S}_k \cap \mathcal{R}_n).$$

The proof of Lemma 4.2 shows that $G \star G(z) \leq C\|z\|^{4-d}$, and thus Lemma 4.1 gives

$$\mathcal{F}(\mathcal{S}_0 \cap \mathcal{R}_n) \leq \mathcal{F}(\mathcal{S}_0) \leq Cr_0^2 \leq Ct^{\frac{2}{d-2}}.$$

Then for $k \geq 1$, we write

$$\mathcal{F}(\mathcal{S}_k \cap \mathcal{R}_n) \leq \sum_{i=0}^{k} \mathcal{F}(\mathcal{S}_k \cap \Lambda_i) + \mathcal{F}(\mathcal{S}_k \cap \Lambda_{k+1}^*).$$

On one hand one has on the event $\mathcal{E}$,

$$\mathcal{F}(\Lambda_0 \cap \mathcal{S}_0^c) \leq C\frac{|\Lambda_0|}{r_0^{d-4}} \leq Ct^{\frac{2}{d-2}}.$$

On the other hand, for any $i \geq 1$,

$$\sum_{k\geq i} \mathcal{F}(\mathcal{S}_k \cap \Lambda_i) \leq \sum_{z\in\Lambda_i\cap Q(0,r_{i-1})^c} G \star G(z) \leq \sum_{z\in\Lambda_i\cap Q(0,r_{i-1})^c} \frac{C}{1+\|z\|^{d-4}} \leq C\rho_i^{1-\frac{4}{d}}|\Lambda_i|^{4/d},$$

using the same argument as in the proof of Lemma 4.3 for the last inequality. Thus on the event $\mathcal{E}$, we get

$$\sum_{k\geq i} \mathcal{F}(\mathcal{S}_k \cap \Lambda_i) \leq C2^{-i\frac{d-6}{d-2}}t^{\frac{2}{d-2}}.$$

It follows that on $\mathcal{E}$,

$$\sum_{i\geq 1}\sum_{k\geq i} \mathcal{F}(\mathcal{S}_k \cap \Lambda_i) \leq C2^{-i\frac{d-6}{d-2}}t^{\frac{2}{d-2}} \leq Ct^{\frac{2}{d-2}}.$$

Similarly, one has

$$\sum_{k\geq 1} \mathcal{F}(\mathcal{S}_k \cap \Lambda_{k+1}^*) \leq C\sum_{k\geq 1} \frac{\rho_k r_k^d}{r_{k-1}^{d-4}} \leq C\frac{\log n}{r_0^{d-6}} \leq Ct^{\frac{2}{d-2}}.$$

Altogether this proves (6.4), and concludes the proof of the theorem. □

6.3 Proof of Theorem 1.2

Let $\{\zeta_n\}_{n\geq 0}$ be a sequence as in the statement of Theorem 1.2, and let L be the integer such that $2^{L-1} \leq \zeta_n < 2^L$.

We first show that the cross terms appearing in (6.1) are negligible. Applying Theorems 6.2 and 6.3, we get that for any $\delta > 0$, and any $\ell \leq L$,

$$\limsup_{n\to\infty} \frac{n}{\zeta_n^2} \cdot \log \mathbb{P}\left(\pm\sum_{i=1}^{2^\ell} Y_i^\ell \geq \frac{\delta\zeta_n}{L}\right) = -\infty.$$

By using a union bound we also deduce

$$\limsup_{n\to\infty} \frac{n}{\zeta_n^2}\cdot \log \mathbb{P}\left(\pm \sum_{\ell=1}^{L}\sum_{i=1}^{2^\ell} Y_i^\ell \geq \delta\zeta_n\right) = -\infty.$$

Thus indeed the cross terms in (6.1) can be ignored, and we focus now on proving the Moderate Deviation Principle for the first sum.

For simplicity, let $Z_i := |\mathcal{R}_i^L| - \mathbb{E}[|\mathcal{R}_i^L|]$. We apply Theorem 6.1 with $X_n := \frac{\pm 1}{\zeta_n}\sum_{i=1}^{2^L} Z_i$, and $b_n := \zeta_n^2/n$. One has using independence, and the fact that $\frac{\zeta_n}{n}\cdot |Z_1|$ is bounded,

$$\mathbb{E}[\exp(\theta b_n X_n] = \Big(\mathbb{E}[\exp(\theta \frac{\zeta_n}{n} Z_1]\Big)^{2^L} = \Big(1 + \frac{\theta^2}{2}\big(\frac{\zeta_n}{n}\big)^2\cdot \mathbb{E}[Z_1^2] + \mathcal{O}\big(\big(\frac{\zeta_n}{n}\big)^3\cdot \mathbb{E}[|Z_1|^3]\big)\Big)^{2^L}.$$

Note that $2^L\cdot \mathbb{E}[Z_1^2]/n$ converges to $\sigma^2>0$, and that the fourth centered moment of $\mathrm{Cap}(\mathcal{R}_n)$ is $\mathcal{O}(n^2(\log n)^2)$. This can be seen as for the volume of the range, following the same proof as in [LG86]. Thus, using that $\mathbb{E}[|Z_1|^3]\leq \mathbb{E}[Z_1^4]^{3/4}$, we have

$$\big(\frac{\zeta_n}{n}\big)^3\cdot \mathbb{E}[|Z_1|^3] \leq C\big(\frac{\zeta_n\log n}{n}\big)^{3/2}.$$

It follows that for any $\theta\in\mathbb{R}$,

$$\lim_{n\to\infty}\frac{n}{\zeta_n^2}\log \mathbb{E}[\exp\left(\theta\frac{\zeta_n}{n}X_n\right) = \frac{\sigma^2}{2}\theta^2,$$

and one can then apply Gärtner–Ellis' Theorem, which concludes the proof of Theorem 1.2.

7 Upward Deviations

We prove here Theorem 1.7. Thanks to our decomposition (2.9), we can adapt the approach of Hamana and Kesten [HK], who proved a similar result for the size of the range.

The approach of Hamana and Kesten is based on first proving an approximate subadditivity relation for the probability of upward deviations, that is the existence of some constants $\chi\in(0,1)$, $c>0$, and $C>0$, such that for any $m,n\geq 1$ integers, and y,z positive reals,

$$\mathbb{P}\big(|\mathcal{R}_{m+n}|\geq y+z-Ca(m,n)\big) \;\geq\; c\,\chi^{a(m,n)}\,\mathbb{P}\big(|\mathcal{R}_n|\geq y\big)\mathbb{P}\big(|\mathcal{R}_m|\geq z\big), \tag{7.1}$$

with

$$a(m,n) := (n\cdot m)^{\frac{1}{d+1}}.$$

The second step, which is general and only based on (7.1) and the fact that (when $d\geq 2$) one has $\lim_{m,n\to\infty}\frac{a(m,n)}{n\vee m}=0$, shows that the following limit exists,

$$\psi(x) := -\lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}\big(|\mathcal{R}_n|\geq x\cdot n\big), \quad \text{for all } x>0,$$

and that ψ is continuous and convex on $[0,1]$. Here we prove an analogous result as (7.1), and use their general argument to conclude.

Proof of Theorem 1.7. We first prove an analogous result as (7.1), but with $a(m,n)$ replaced by the function:

$$\widetilde{a}(m,n) = (n \cdot m)^{\frac{1}{d-1}}.$$

In other words we establish the following inequality. There exists $\chi \in (0,1)$, and $C > 0$, such that for any m, n integers and y, z positive reals,

$$\mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_{m+n}\right) \geq y + z - C\,\widetilde{a}(m,n)\right) \geq \frac{1}{2}\chi^{\widetilde{a}(m,n)}\mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_n\right) \geq y\right)\mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_m\right) \geq z\right). \tag{7.2}$$

The first step is to obtain the analogue of the following simple deterministic bound used in [HK]: if $\mathcal{R}_n$ and $\widetilde{\mathcal{R}}_m$ are two independent copies of the range, there is a positive constant C, such that for any $r \geq 1$

$$\frac{1}{|Q(r)|}\sum_{z\in Q(r)} |(z + \mathcal{R}_n) \cap \widetilde{\mathcal{R}}_m| \leq C\,\frac{n \cdot m}{r^d}.$$

The corresponding bound in our context reads as follows:

$$\frac{1}{|Q(r)|}\sum_{z\in Q(r)}\sum_{x\in\mathcal{R}_n}\sum_{y\in\widetilde{\mathcal{R}}_m} G(x-y+z) \leq C\,\frac{n\cdot m}{r^{d-2}}, \tag{7.3}$$

and is a direct consequence of (2.1) and the fact that for any $x \in \mathbb{Z}^d$, and for some constant $C > 0$,

$$\sum_{z\in Q(r)} \frac{1}{1 + \|z-x\|^{d-2}} \leq C\,r^2.$$

Now to lighten notation, we simply write $a = \widetilde{a}(m,n) = \lfloor (mn)^{\frac{1}{d-1}} \rfloor$. Using that the capacity is translation-invariant, we deduce

$$\begin{aligned} \mathrm{Cap}\left(\mathcal{R}_{m+n+a}\right) &\overset{(2.5)}{\geq} \mathrm{Cap}\left(\mathcal{R}_n \cup \mathcal{R}[n+a, n+m+a]\right) \\ &\overset{(2.9)}{=} \mathrm{Cap}\left(\overline{\mathcal{R}}_n\right) + \mathrm{Cap}\left(\widetilde{\mathcal{R}}_m\right) - \chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + S_a'), \end{aligned} \tag{7.4}$$

with $\overline{\mathcal{R}}_n := \mathcal{R}_n - S_n$, $S_a' := S_{n+a} - S_n$, and $\widetilde{\mathcal{R}}_m := \mathcal{R}[n+a, n+m+a] - S_{n+a}$. The Markov property implies that $\overline{\mathcal{R}}_n$ and $\widetilde{R}_m$ are independent, and distributed as $\mathcal{R}_n$ and $\mathcal{R}_m$ respectively. Furthermore,

$$\chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + S_a') \overset{(2.12)}{\leq} \sum_{x\in\overline{\mathcal{R}}_n}\sum_{y\in\widetilde{\mathcal{R}}_m} G(x - y - S_a'). \tag{7.5}$$

Now, one idea of Hamana and Kesten [HK] is to bound the law of S_a' by a uniform law on the cube $Q(a/d)$. Indeed for any $x \in Q(a/d)$, for which $\mathbb{P}(S_a = x) \neq 0$, one has

$$\mathbb{P}(S_a' = x) \geq \frac{1}{(2d)^a}, \tag{7.6}$$

since there is at least one path of length a going from 0 to x. Write $\overline{Q}(a/d)$ for the set of sites $x \in Q(a/d)$, for which $\mathbb{P}(S_a = x) \neq 0$. Then for any $x \in \overline{Q}(a/d)$, and any $\alpha > 0$,

$$\mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_{m+n+a}) \geq z + y - \frac{\alpha}{2}\right) \overset{(7.4)}{\geq} \mathbb{P}(S_a' = x)\cdot\mathbb{P}\left(\mathrm{Cap}(\overline{\mathcal{R}}_n) \geq z, \mathrm{Cap}(\widetilde{\mathcal{R}}_m) \geq y, \chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + x) \leq \frac{\alpha}{2}\right).$$

Integrating with respect to the uniform measure on $\overline{Q}(a/d)$, we get

$$\mathbb{P}\big(\mathrm{Cap}(\mathcal{R}_{m+n+a}) \geq z+y-\frac{\alpha}{2}\big) \overset{(7.6)}{\geq} \frac{1}{(2d)^a} \times \frac{1}{|\overline{Q}(a/d)|} \sum_{x\in\overline{Q}(a/d)} \mathbb{P}\left(\mathrm{Cap}(\overline{\mathcal{R}}_n)\geq z, \mathrm{Cap}(\widetilde{\mathcal{R}}_m)\geq y, \chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m+x)\leq \frac{\alpha}{2}\right). \tag{7.7}$$

We need now to estimate the mean of $\chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m+\cdot)$ with respect to the uniform measure. According to (7.3), there is a positive constant C, such that

$$\frac{1}{|\overline{Q}(a/d)|}\sum_{x\in\overline{Q}(a/d)} \chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m+x) \leq C\frac{m\cdot n}{a^{d-2}} \leq Ca, \tag{7.8}$$

where the last inequality follows from the definition of a. Then by Chebychev's inequality, we obtain

$$\frac{1}{|\overline{Q}(a/d)|}\sum_{x\in\overline{Q}(a/d)} \mathbf{1}(\chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m+x)\leq 2Ca) \geq \frac{1}{2}. \tag{7.9}$$

As a consequence,

$$\begin{aligned}
\mathbb{P}\big(\mathrm{Cap}(\mathcal{R}_{m+n})\geq z+y-a-4Ca\big) &\overset{(2.6),(2.8)}{\geq} \mathbb{P}\big(\mathrm{Cap}(\mathcal{R}_{m+n+a})\geq z+y-4Ca\big)\\
&\overset{(7.7)}{\geq} \frac{1}{(2d)^a}\cdot\mathbb{E}\Big[\mathbf{1}(\mathrm{Cap}(\overline{\mathcal{R}}_n)\geq z)\cdot\mathbf{1}(\mathrm{Cap}(\widetilde{\mathcal{R}}_m)\geq y)\times\frac{1}{|\overline{Q}(a/d)|}\sum_{x\in\overline{Q}(a/d)}\mathbf{1}(\chi_C(\overline{\mathcal{R}}_n,\widetilde{\mathcal{R}}_m+x)\leq 2Ca)\Big]\\
&\overset{(7.9)}{\geq} \frac{1}{2(2d)^a}\cdot\mathbb{P}\big(\mathrm{Cap}\,(\mathcal{R}_n)\geq z\big)\mathbb{P}\big(\mathrm{Cap}\,(\mathcal{R}_m)\geq y\big),
\end{aligned}$$

proving (7.2), with $\chi = 1/(2d)$.

It then follows from the general arguments of Hamana and Kesten, see Lemma 3 in [HK], that the following limit exists for all $x>0$:

$$\psi_d(x) \;:=\; -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_n)\geq nx\right).$$

We now prove that the range for which $\psi_d(x)$ is finite is not empty. Define for $n\geq 0$,

$$c_n := \max_{\gamma:\{0,\dots,n\}\to\mathbb{Z}^d}\mathrm{Cap}(\{\gamma(0),\dots,\gamma(n)\}), \tag{7.10}$$

where the max is taken over all nearest neighbor paths of length $n+1$. By (2.6), it follows that $c_{n+m}\leq c_n+c_m$, for all $n,m\geq 0$, so that by Fekete's lemma, the limit $\lim_{n\to\infty}c_n/n$ exists. Call γ_d^* this limit. Note that $\psi_d(x)$ is finite on $[\gamma_d,\gamma_d^*]$, since the probability that the simple random walk follows the path realizing the maximum in (7.10) is larger than or equal to $1/(2d)^{n+1}$, so that $\psi_d(x)\leq\log(2d)$, for all $x\leq\gamma_d^*$. Conversely, by definition of c_n, one has $\psi_d(x)=\infty$ for all $x>\gamma_d^*$. Furthermore, it follows from Lemma 3 and Proposition 4 in [HK], that ψ_d is continuous, and convex on $(0,\gamma_d^*]$. Now Proposition 2.4 and Lemma 2.1 show that when $d=5$, $\psi_d(x)\geq c(x-\gamma_5)^3$, for all $x\geq\gamma_d$. Likewise, when $d\geq 6$, we get $\psi_d(x)\geq c(x-\gamma_d)^3$, for $\gamma_d\leq x\leq 1$. Using convexity, this also shows that ψ_d is increasing on $[\gamma_d,\gamma_d^*]$. In addition one has $\psi_d(x)=0$ for all $x<\gamma_d$, by definition of γ_d as the limit of the (normalized) expected capacity, and using that by (2.8), $\mathrm{Cap}(\mathcal{R}_n)\leq n$.

Finally we show that $\gamma_d^* > \gamma_d$.

Consider $\mathcal{D}_n$ the set of *no double backtrack at even times* paths of length $n+1$ that we introduced in [AS17b]. By definition, this is simply the set of paths $\gamma : \{0, \dots, n\} \to \mathbb{Z}^d$, such that for any even $k \le n$, one has $\gamma(k+2) \neq \gamma(k)$. The only important property we need is that from a no-backtrack walk $\widetilde{S}$, and a sum of independent geometric variables $\{\xi_i,\ i \in \mathbb{N}\}$, with parameter $1/(2d)^2$, we can build a simple random walk S such that

$$\mathcal{R}[0, n + 2\sum_{i \le n/2} \xi_i] = \widetilde{\mathcal{R}}_n.$$

Thus, for any $\alpha > 0$, we have by (2.6) and (2.8),

$$\mathrm{Cap}(\widetilde{\mathcal{R}}_n) \ge \mathrm{Cap}(\mathcal{R}_{(1+\alpha)n}) - \mathbf{1}\left(\sum_{i \le n/2} \xi_i < \frac{\alpha n}{2}\right) \cdot (1+\alpha)(n+1).$$

By taking the maximum over $\mathcal{D}_n$ on the left hand side, and then the expectation on the right hand side, we obtain

$$c_n \ge \max_{\pi \in \mathcal{D}_n} \mathrm{Cap}(\pi) \ge \mathbb{E}[\mathrm{Cap}(\mathcal{R}_{(1+\alpha)n})] - (1+\alpha)(n+1) \cdot \mathbb{P}\left(\sum_{i \le n/2} \xi_i < \frac{\alpha n}{2}\right). \tag{7.11}$$

Now take $\alpha < 1/(2d)^2$, and use Chebyshev's inequality, to see that the last term of (7.11) is $\mathcal{O}(1)$. Together with Lemma 2.1 it implies that

$$c_n \ge \gamma_d(1+\alpha)n - \mathcal{O}(\sqrt{n}),$$

which indeed proves that $\gamma_d < \gamma_d^*$. □

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