The Lane-Emden equation for the p-Laplacian

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The Lane-Emden equation

We are interested in the following equation:

$$(1) \qquad \begin{cases} -\Delta_p u &= \lambda |u|^{q-2}u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here Ω is an open, connected subset of $\mathbb{R}^n$ with Lipschitz boundary, $1 < p < q < \infty$, and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the *p-Laplacian*. Solutions of these equation are found as critical points of the energy functional

$$\varphi_q(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q.$$

Minimization of the energy functional on the *Nehari manifold*

$$\mathcal{N}_q := \{u \in W_0^{1,p}(\Omega) \setminus \{0\} \mid d\varphi_q(u)(u) = 0\},$$

where

$$d\varphi_q(u)(v) := \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \lambda \int_\Omega |u|^{q-2}uv$$

is the first variation of φ_q at the point u in the direction v, provides a *ground-state solution*. Since every critical point of φ_q must belong to the Nehari manifold, it is clear that a ground-state solution has minimal energy among all possible solutions of (1). Moreover, such solutions are of one sign. Therefore one may wonder if there exist sign-changing solutions whose energy is minimal among all sign-changing solutions of (1). The answer is affirmative; they can be found by minimizing the energy functional on the set

$$\mathcal{M}_q := \{u \in W_0^{1,p}(\Omega) \mid u^+, u^- \in \mathcal{N}_q\},$$

which necessarily contains all sign-changing critical points of φ_q. The minimizers of the energy functional on $\mathcal{M}_q$ are called *least energy nodal solutions*.

Behaviour as $q \to p$

We intend to investigate the behaviour of sequences of solutions to (1) as q goes to p. One expects convergence to eigenfunctions of the p-Laplacian, and this is what actually happens; of course the value of λ must be taken into account, due to the scaling properties of the equation. Let us denote by $\lambda_1(p)$ and $\lambda_2(p)$ the first and the second eigenvalue of $-\Delta_p$ under Dirichlet boundary conditions.

Theorem 1. [1, Theorem 3.1] *As $q \to p$, the ground state solutions of Problem (1):*

(i) *diverge to infinity if $\lambda < \lambda_1(p)$;*
(ii) *converge to a first eigenfunction of the p-Laplacian if $\lambda = \lambda_1(p)$;*
(iii) *converge to zero if $\lambda > \lambda_1(p)$.*

Theorem 2. [1, Theorem 3.2] *As $q \to p$, the least energy nodal solutions of Problem (1):*

(i) *diverge to infinity if $\lambda < \lambda_2(p)$;*
(ii) *converge to a second eigenfunction of the p-Laplacian if $\lambda = \lambda_2(p)$;*
(iii) *converge to zero if $\lambda > \lambda_2(p)$.*

The proof relies on uniform bounds for the $W^{1,p}$-norm of the solutions in order to insure convergence (upper bounds) and non-degeneracy of the limit (lower bounds).

Behaviour as $p \to \infty$

The investigation of the case $p \to \infty$ is more involved. More precisely, we consider a family of problems of the type

$$(2) \qquad \begin{cases} -\Delta_p u &= \lambda_p |u|^{q(p)-2}u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

The difference with (1) is that the exponent q and the value λ now depend on p. We define

$$Q := \lim_{p\to\infty} \frac{q(p)}{p}, \qquad \Lambda = \lim_{p\to\infty} (\lambda_p)^{\frac{1}{p}},$$

assuming that such limits exist. It is clear that $Q \geq 1$. Moreover, it is known that

$$\lim_{p\to\infty} \lambda_1(p)^{\frac{1}{p}} = \Lambda_1, \qquad \lim_{p\to\infty} \lambda_2(p)^{\frac{1}{p}} = \Lambda_2,$$

where Λ_1 and Λ_2 are the first and the second eigenvalue of the infinity Laplacian.
The first step is to identify the limit problem. It can be shown that, if the solutions of (2) converge uniformly as $p \to \infty$ to a function u, then u is a viscosity solution of

$$\begin{cases} F_\Lambda(u, \nabla u, D^2 u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where

$$F_\Lambda(s,\xi,X) = \begin{cases} \min\{|\xi| - \Lambda s^Q, -X\xi\cdot\xi\} & \text{if } s > 0 \\ -X\xi\cdot\xi & \text{if } s = 0 \\ \max\{-\Lambda|s|^{Q-1}s - |\xi|, -X\xi\cdot\xi\} & \text{if } s < 0. \end{cases}$$

The following convergence results hold.

Theorem 3. [2] *Let $\{u_{\lambda_p,p}\}_p$ be a sequence of ground state solutions of (2). Suppose that $Q > 1$. Then $u_{\lambda_p,p}$ converge uniformly to a positive viscosity solution u of (2) with $\|u\|_\infty = 1$.*

Theorem 4. [2] *Let $\{v_{\lambda_p,p}\}_p$ be a sequence of least energy nodal solutions of (2). Suppose that $Q > 1$. Then $v_{\lambda_p,p}$ converge to a sign-changing viscosity solution v of (2) with $1 \leq \|v\|_\infty \leq \Lambda_2\Lambda_1^{-1}$.*

The case $Q = 1$ is more delicate and needs to be treated separately.

Theorem 5. [2, Theorem 6.1] *Let $\{u_{\lambda_p,p}\}_p$ be a sequence of ground state solutions of (2). Suppose that $Q = 1$. Then:*

(i) *If $\Lambda > \Lambda_1$, then $\lim_{p\to\infty} ||u_{\lambda_p,p}||_\infty = 0$.*
(ii) *If $\Lambda < \Lambda_1$, then $\lim_{p\to\infty} ||u_{\lambda_p,p}||_\infty = \infty$.*

A similar result holds for least energy nodal solutions, with Λ_1 replaced by Λ_2.

Bibliography

References

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