Probab. Th. Rel. Fields 82, 489–526 (1989)

Probability Theory and Related Fields

Linear Stochastic Differential Equations with Boundary Conditions

Daniel Ocone [1] * and Etienne Pardoux [2] **

[1] Mathematics Department, Rutgers University, New Brunswick, NJ 08903, USA

[2] Mathématiques, UA 225, Université de Provence, F-13331 Marseille Cedex 3, France

Summary. We study linear stochastic differential equations with affine boundary conditions. The equation is linear in the sense that both the drift and the diffusion coefficient are affine functions of the solution. The solution is not adapted to the driving Brownian motion, and we use the extended stochastic calculus of Nualart and Pardoux [16] to analyse them. We give analytical necessary and sufficient conditions for existence and uniqueness of a solution, we establish sufficient conditions for the existence of probability densities using both the Malliavin calculus and the co-aera formula, and give sufficient conditions that the solution be either a Markov process or a Markov field.

§ 1. Introduction

Let $\{W_t\}$ denote a Brownian motion. Recently, progress has been made in developing a useful theory of stochastic integrals $\int_0^t \varphi(s, w)\, dW_s$ in which the integrand $\{\varphi(s, w)\}$ anticipates $\{W_t\}$. In particular, Nualart and Pardoux [16] derive an extended stochastic calculus both for the Skorohod integral and for a generalized Stratonovich integral $\int_0^t \varphi(s, w) \circ dW_s$. This allows one to formulate stochastic differential equations containing parameters that anticipate the driving noise. One natural way to do this is to impose two-point, or even distributed, boundary conditions on the solution of a stochastic d.e., and in this paper we study the following particular case:

$$dX_t = [AX_t + a(t)]\, dt + \sum_{i=1}^{k} [B_i X_t + b_i(t)] \circ dW_t^i, \quad 0 \leqq t \leqq 1 \tag{1.1}$$

$$F_0 X_0 + F_1 X_1 = f. \tag{1.2}$$

* Supported in part by NSF Grant No. MCS-8301880

** The research was carried out while this author was visiting the Institute for Advanced Study, Princeton NJ, and was supported by a grant from the RCA corporation

Our purpose is to show that a fairly complete analysis of this problem can be made.

The boundary condition (1.2) includes the usual adapted, initial value problem ($F_1=0$), periodic boundary condition ($X_0=X_1$), and two-point boundary value problems. The case of linear-gaussian dynamics, in which $B_i=0$, $1\leqq i\leqq k$, $a(t)=0$, and $b_i(t)$, $1\leqq i\leqq k$, are deterministic constants, has been studied by Krener [13], Adams, Willsky and Levy [1] and Kwakernaak [14]. Cinlar and Wang [5] treat one-dimensional random processes on circles which essentially have the form (1.1)–(1.2) with linear-gaussian dynamics and periodic boundary condition and they study infinite-dimensional generalizations. For the linear-gaussian case, the extended stochastic calculus is not needed because the stochastic integrals do not contain an anticipating term.

In Sect. 2 of this paper we define the problem (1.1)–(1.2) more precisely and discuss the natural definition of its solution. In Sect. 4, we establish analytical necessary and sufficient conditions for the existence of solutions and show that these are unique in a certain class of processes. We rely here on the results of Nualart and Pardoux [16], which are reviewed in Sect. 3. In Sect. 5, we discuss the issue of existence of a density for the probability distribution induced by X_t. First we do this by calculating and analyzing the Malliavin covariance matrix of X_t. Then, in more specialized circumstances, we employ the co-area formula of geometric measure theory to represent densities and to derive a necessary and sufficient condition for existence of densities.

In Sect. 6, we consider the Markov property of solutions. Two types of Markov property are relevant here; the usual Markov property requiring conditional independence of past and future given the present, and the Markov field property requiring conditional independence between $\{X_t|t\in[a,b]\}$ and $\{X_t|t\in[a,b]^c\}$ given (X_a, X_b) for any interval $[a,b]$. The Markov field property always holds in the linear-gaussian case; indeed, one focus of previous research has been the realization of Gaussian Markov fields by linear stochastic differential equations (Krener [13]; see also Chay [4] and Jamison [9, 10] for related references). We do not know whether the Markov field property holds in general. In Sect. 6, we develop sufficient conditions to determine when $\{X_t\}$ is a Markov process or a Markov field. In 6.1 we give probabilistic criteria for either type of Markov property and show that $\{X_t\}$ always has a weak Markov field property with respect to enlarged filtrations (Theorem 6.4 and Proposition 6.8). In 6.2, we use the co-area formula and explicit representation of densities to give analytical conditions for the Markov or the Markov field property in the two-point boundary value problem, and we give examples to illustrate the use of our criteria. In particular, it is not necessary for the filtration of $\{X_t\}$ to be adapted to either the forward or backward filtrations of $\{W_t\}$ in order that $\{X_t\}$ be Markov.

There are two styles of argument in this paper; the probabilistic style using stochastic calculus which is not so heavily dependent on the particular structure of (1.1)–(1.2); and the analytical style which takes strong advantage of the linearity in (1.1)–(1.2). In the former class fall the uniqueness theory, the calculation of the Malliavin calculus and the probabilistic criteria for Markovianity. These aspects of the theory may perhaps be generalized; see the discussion in Sect. 2.

In the analytical class falls our use of the co-area formula, which we have found to be a powerful tool for questions concerning densities and conditional densities.

§ 2. Presentation of the Problem

Let $\Omega = C(\mathbb{R}_+; \mathbb{R}^k)$ be equipped with the topology generated by the sup norm on compact intervals, let $\mathscr{F}$ denote the Borel σ-field of subsets of Ω, and let P denote Wiener measure on $(\Omega, \mathscr{F})$. We define

$$\begin{aligned} W_t(\omega) &= (W_t^1(\omega), \ldots, W_t^k(\omega)) \\ &= \omega(t). \end{aligned}$$

The aim of this paper is to study the following stochastic differential equation, whose solution will be a d-dimensional process defined on the time-interval $[0, 1]$:

$$d X_t = (A X_t + a(t))\, dt + (B_i X_t + b_i(t)) \circ d W_t^i \tag{2.1}$$

where we use the convention of summation from 1 to k of the repeated index i, together with the boundary condition:

$$F_0 X_0 + F_1 X_1 = f \tag{2.2}$$

where $A, B_1, \ldots, B_k, F_0, F_1$ are $d \times d$ matrices, such that

$$\operatorname{rank}(F_0 : F_1) = d \tag{2.3}$$

$\{a(t), b_1(t), \ldots, b_k(t); t \in [0, 1]\}$ are d-dimensional processes satisfying assumptions to be specified later, and f is a d-dimensional (possibly) random vector defined on $(\Omega, \mathscr{F})$. From (2.2), we do not expect in general that the solution $\{X_t, t \in [0, 1]\}$ be adapted to any filtration with respect to which $\{W_t\}$ might be a Wiener process. Since we want to keep our problem symmetric with respect to time reversal, we will not try to take advantage of the filtration enlargement technique (see Jeulin [11], Jeulin-Yor [12]). The stochastic integrals in (2.1) will be understood in the sense of generalized Stratonovich integrals (see Nualart-Pardoux [16]). The reason for choosing this type of integral, rather than the Itô-Skorohod integral, will be given below. We will present in the next section the results we need on the generalized Stratonovich integral and its associated calculus.

Let us now explain what we mean by a solution to (2.1)–(2.2). We can associate to Eq. (2.1) a fundamental solution Φ_t, which is a $d \times d$ matrix valued process, solution of:

$$\begin{aligned} d\Phi_t &= A\, \Phi_t\, dt + B_i\, \Phi_t \circ W_t^i \\ \Phi_0 &= I. \end{aligned} \tag{2.4}$$

Note that $\{\Phi_t, t\in[0,1]\}$ is adapted to the natural filtration of $\{W_t\}$, and the stochastic integrals in (2.4) are standard Stratonovich-type stochastic integrals. We define further:

$$\Phi(t,s)=\Phi_t\Phi_s^{-1}; \qquad s, t\in[0,1]$$

$$V_t=\int_0^t a(s)\,ds+\int_0^t b_i(s)\circ dW_s^i, \qquad t\in[0,1].$$

We then have the following variation of constants formula:

$$X_t=\Phi(t,0)\,X_0+\int_0^t \Phi(t,s)\circ dV_s. \tag{2.5}$$

Let us admit for a moment that (2.1) is equivalent to (2.4)–(2.5). Then from (2.2):

$$[F_0+F_1\,\Phi(1,0)]\,X_0=f-F_1\int_0^1 \Phi(1,s)\circ dV_s. \tag{2.6}$$

If the matrix $F_0+F_1\,\Phi(1,0)$ is invertible a.s., then X_0 is uniquely determined by (2.6) a.s., and $\{X_t, t\in[0,1]\}$, defined by (2.5)–(2.6) will be the solution to (2.1)–(2.2). We see that (except possibly for the definition of $\{V_t\}$) $\{X_t\}$ is constructed with the standard tools of stochastic calculus, and the generalized stochastic integral and calculus with non-adapted integrands will be necessary only to give sense to (2.1), and establish the equivalence between (2.1)–(2.2) and (2.4)–(2.6).

Let us remark that, using the flow associated to (2.1) instead of the fundamental matrix, and the generalized Itô-Ventzell formula (see Ocone-Pardoux [17]), we could replace (2.1) by an arbitrary nonlinear stochastic differential equation, and (2.2) by a nonlinear relation between X_0 and X_1. But the situation which we consider here is the only general framework in which we are able to give conditions on the data which insure the a.s. existence and uniqueness of a solution X_0 to (2.2).

Let us now discuss the nature of the boundary condition (2.2). If $F_1=0$ (resp. $F_0=0$), then (2.1)–(2.2) becomes an initial value (resp. final value) problem, which is of course well understood, except that we allow the initial (or final) condition to depend on the driving Wiener process. We now describe two particular cases of the boundary condition (2.2):

Two-point Boundary Value Problem

Let $l\in\mathbb{N}$, $0<l<d$, and suppose that $F_0=\begin{pmatrix}F_0'\\0\end{pmatrix}$, $F_1=\begin{pmatrix}0\\F_1''\end{pmatrix}$ where F_0' is a $l\times d$ matrix, F_1'' is a $(d-l)\times d$ matrix. Condition (2.3) requires that F_0' has rank l and F_1'

has rank $d-l$. If we write $f=\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$, where f_0 is l dimensional and f_1 is $d-l$ dimensional, then (2.2) becomes:

$$F_0' X_0 = f_0 \qquad F_1' X_1 = f_1. \tag{2.2'}$$

Note that $\operatorname{Image}\begin{bmatrix} F_0' \\ 0 \end{bmatrix} \cap \operatorname{Image}\begin{bmatrix} 0 \\ F_1' \end{bmatrix} = \{0\}$. Conversely, if $\operatorname{Image} F_0 \cap \operatorname{Image} F_1 = \{0\}$, one can always find by row reduction an invertible G such that:

$$G[F_0 \vdots F_1] = \left[\begin{array}{c|c} F_0' & 0 \\ 0 & F_1' \end{array}\right].$$

Thus, (2.1)–(2.2) can be expressed as a two-point boundary value problem if and only if $\operatorname{Image} F_0 \cap \operatorname{Image} F_1 = \{0\}$.

Periodic Solution of a S.D.E.

Suppose $F_0 = -F_1 = I$, the $d \times d$ identity matrix; and $f=0$. Then (2.2) becomes:

$$X_0 = X_1. \tag{2.2''}$$

Clearly, (2.2) fixes exactly d degrees of freedom, exactly like an initial condition would do. In other words, (2.2) is exactly the kind of condition required in order for (2.1)–(2.2) to have a unique solution. Therefore, there is no analogy between the solution to our equation and a Brownian bridge, which is a process whose values at both endpoints $t=0$ and $t=1$ are completely prescribed. A Brownian bridge is a conditioned Brownian motion, whereas no conditioning enters in our construction.

Let us indicate finally that the boundary condition (2.2) could be replaced by a more general condition of the type:

$$\int_0^1 F(t)\, X_t\, d\gamma(t) = f \tag{2.2'}$$

where $\{F(t), t \in [0, 1]\}$ is a measurable collection of $d \times d$ matrices, and γ is a finite measure on $[0, 1]$. (2.6) would then have to be replaced by:

$$\left(\int_0^1 F(t)\, \Phi_t\, d\gamma(t)\right) X_0 = f - \int_0^1 \int_0^t F(t)\, \Phi(t, s) \circ d V_s\, d\gamma(t) \tag{2.6'}$$

and we would need to ensure that the random matrix $\int_0^1 F(t)\,\Phi_t\,d\gamma(t)$ is a.s. invertible. Part of the analysis below can be generalized to this case, but we will restrict outself to consider the boundary condition (2.2), which will lead to a tractable necessary and sufficient condition for existence and uniqueness.

§ 3. Generalized Stratonovich Stochastic Integral and Calculus

All processes will be defined on the probability space $(\Omega, \mathscr{F}, P)$ introduced in the previous section.

The results below which are not proved are taken from Nualart-Pardoux [16].

In the next definition, $t_n^l = l\,2^{-n}$; $k, n \in \mathbb{N}$.

Definition 3.1. A real valued process $\{u_t, t \in [0, 1]\}$ is said to be *Stratonovich integrable with respect to* dW_t^i if for any $t \in [0, 1]$, the sequence $\{\xi_n(t), n \in \mathbb{N}\}$ defined by:

$$\xi_n(t) = \sum_{l=0}^{n-1} \frac{W^i_{t_n^{l+1} \wedge t} - W^i_{t_n^l \wedge t}}{t_n^{l+1} - t_n^l} \int_{t_n^l}^{t_n^{l+1}} u_s\,ds$$

converges in probability as $n \to \infty$. In that case, the limit will be denoted:

$$\int_0^t u_s \circ dW_s^i.$$

A real valued process which is Stratonovich integrable with respect to dW_t^i, $i = 1, \ldots, k$, will be said to be *Stratonovich integrable.* □

This definition differs slightly from that in Nualart-Pardoux [16], where convergence to the same limit along any refining sequence of partitions of $[0, 1]$ is required. The present definition will be sufficient for our purpose.

Let us consider the forward filtration $\mathscr{F}_t = \sigma\{W_s^i,\ 0 \leqq s \leqq t;\ i = 1, \ldots, k\}$ and the backward filtration $\mathscr{F}^t = \sigma\{W_s^i - W_t^i;\ t \leqq s \leqq 1;\ i = 1, \ldots, k\}$. We will say that a process $\{v_t; t \in [0, 1]\}$ is a "forward semi-martingale" if it is a $\mathscr{F}_t$ semi-martingale (see e.g. Meyer [15]). We will say that a process $\{v_t; t \in [0, 1]\}$ is a "backward semi-martingale" if v_{1-t} is a $\mathscr{F}^{1-t}$ semi-martingale. The following result is well-known (see e.g. Meyer [15]):

Proposition 3.2. *Let $u_t = g(v_t)$, where $g \in C^1(\mathbb{R})$, and $\{v_t\}$ is a continuous process, which is either a forward or a backward semi-martingale. Then u_t is Stratonovich integrable.* □

The following is an immediate consequence of the Definition:

Proposition 3.3. *Let $\{v_t\}$ be a Stratonovich integrable process, θ a random variable, and $u_t = \theta v_t$, $t \in [0, 1]$.*

Then $\{u_t\}$ is Stratonovich integrable, and:

$$\int_0^t u_s \circ d W_s^i = \theta \int_0^1 v_s \circ d W_s^i, \qquad u \geqq 0,\ i = 1, \ldots, k. \quad \square \tag{3.1}$$

In order to describe other types of nonadapted Stratonovich integrable processes, let us recall the notion of derivation of random variables defined on Wiener space $(\Omega, \mathscr{F}, P)$.

Let $H = L^2(0, 1)$. If $h \in H$, we denote by $\delta_i(h)$ the Wiener integral

$$\int_0^1 h(s)\, d W_s^i.$$

We denote by $\mathbb{S}$ the set of random variables of the type:

$$F = f(\delta_{i_1}(h_1), \ldots, \delta_{i_n}(h_n)) \tag{3.2}$$

where $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, \ldots, h_n \in H$, $i_1, \ldots, i_n \in \{1, \ldots, k\}$. Note that $\mathbb{S}$ is dense in $L_2(\Omega)$. If $F \in \mathbb{S}$ is of the form (3.2), we define its "derivative in the i-th direction", for $1 \leqq i \leqq k$, as the process $\{D_t^i F, t \in [0, 1]\}$ given by:

$$D_t^i F = \sum_{\{l;\, i_l = i\}} \frac{\partial f}{\partial x_l} (\delta_{i_1}(h_1), \ldots, \delta_{i_n}(h_n))\, h_l(t).$$

More generally, we define the p-th order derivative of F, $D_{t_1 \ldots t_p}^{i_1 \ldots i_p} F$, as given by:

$$D_{t_p}^{i_p} \ldots D_{t_1}^{i_1} F.$$

If $F \in \mathbb{S}$, $h \in H$, $1 \leqq i \leqq k$, we define the random variable:

$$D_h^i F = \int_0^1 D_t^i F\, h(t)\, dt.$$

We will denote by $D_t F$ (resp. $D_h F$) the k dimensional vector whose i-th coordinate is $D_t^i F$ (resp. $D_h^i F$). DF stands for the process $\{D_t F, t \in [0, 1]\}$.

Proposition 3.4. *As an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega \times (0, 1); \mathbb{R}^k)$ (resp. $L^2(\Omega; \mathbb{R}^k)$), D (resp. D_h) is closable. We denote by $\mathbb{D}_{2,1}$ (resp. $\mathbb{D}_{2,h}$) the domain of D (resp. D_h), identified with its closed extension.*

Moreover, D and D_h are local operators, in the sense that:

(i) *If $F \in \mathbb{D}_{2,1}$, $D_t F = 0$ $P \times \lambda$ a.e. on $\{F = 0\} \times [0, 1]$.*

(ii) *If $F \in \mathbb{D}_{2,h}$, $D_h F = 0$ a.s. on the set $\{F = 0\}$.* $\square$

More generally, $\mathbb{D}_{p,l}$ ($p \geqq 1$, $l \in \mathbb{N}$) denotes the completion of $\mathbb{S}$ with respect to the norm:

$$\|F\|_{p,l} = \|F\|_p + \| \, \| D^{(l)} F \|_{HS} \|_p$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$, and

$$\|D^{(l)} F\|_{HS}^2 = \sum_{j_1,\ldots,j_l=1}^{k} \int_{(0,1)^l} (D_{t_1}^{j_1} \ldots D_{t_l}^{j_l} F)^2 \, dt_1 \ldots dt_l.$$

$\mathbb{D}_{p,l,\mathrm{loc}}$ will denote the set of r.v. F which are such that there exists a sequence $\{(\Omega_n, F_n);\, n\in\mathbb{N}\} \subset \mathscr{F} \times \mathbb{D}_{p,l}$ with the two following properties:

(i) $\Omega_n \uparrow \Omega$ a.s.

(ii) $F = F_n$ a.s. on Ω_n.

We then say that $\{\Omega_n, F_n\}$ localizes F in $\mathbb{D}_{p,l}$, and $D_t F$ is defined without ambiguity by: $D_t F = D_t F_n$ on Ω_n, $n\in\mathbb{N}$.

Lemma 3.5. *Let $d\in\mathbb{N}$, G be an open subset of $\mathbb{R}^d$, and $\varphi\in C^1(G)$. If F is a d-dimensional random vector s.t. $F\in G$ a.s. and $F^i\in\mathbb{D}_{p,l,\mathrm{loc}}$, $1\leqq i\leqq d$, then:*

$$\varphi(F)\in\mathbb{D}_{p,l,\mathrm{loc}}$$

and

$$D_t^j\, \varphi(F) = \sum_{i=1}^{d} \frac{\partial \varphi}{\partial x^i}(F)\, D_t^j F^i.$$

Proof. Let $\{\varphi_n\}$ be a sequence in $C^1(G)$, such that each φ_n has a compact support in G, and

$$G_n = \{x;\, \varphi_n(x) = \varphi(x)\} \uparrow G.$$

Let $\{\Omega_n, F_n\}$ localize F in $(\mathbb{D}_{p,1})^d$. Then $\{\bar{\Omega}_n, \varphi_n(F_n)\}$, where $\bar{\Omega}_n = \Omega_n \cap \{F\in G_n\}$, localizes $\varphi(F)$ in $\mathbb{D}_{p,1}$, and:

$$D_t^j\, \varphi_n(F_n) = \sum_{i-1}^{d} \frac{\partial \varphi_n}{\partial x^i}(F_n)\, D_t^j F_n^i. \quad \square$$

Let us now introduce some classes of processes: $\mathbb{L}^{p,l}$ will denote the set of processes $\{u_t,\, t\in[0,1]\}$ which are such that $u_t\in\mathbb{D}_{p,l}$ t a.e., and:

$$\int_0^1 \|u_t\|_{p,l}^p \, dt < \infty$$

$\mathbb{L}_C^{p,l}$ will denote the set of processes $u\in\mathbb{L}^{p,l}$ which satisfy:

(i) $s\to D_t u_s$ is continuous with values in $L^p(\Omega)$, both on $(0,t)$ and on $(t,1)$, uniformly with respect to t.

(ii) $\operatorname*{ess\,sup}_{(s,t)\in(0,1)^2} E(|D_s u_t|^p) < \infty$.

$\mathbb{L}_{\mathrm{loc}}^{p,l}$ and $\mathbb{L}_{C,\mathrm{loc}}^{p,l}$ are defined in an obvious way. If $u\in L_{C,\mathrm{loc}}^{p,l}$, we define:

$$D_t^+ u_t = \lim_{\substack{s\to t\\ s>t}} D_t u_s, \qquad D_t^- u_t = \lim_{\substack{s\to t\\ s<t}} D_t u_s, \qquad (\nabla u)_t = D_t^+ u_t + D_t^- u_t.$$

The i-th coordinate of $(\nabla u)_t$ will be denoted $(\nabla^i u)_t$.

Proposition 3.6. *If $u \in \mathbb{L}^{2;1}_{C,\mathrm{loc}}$, then u is Stratonovich integrable.*
The process:

$$\delta^i_t(u) = \int_0^t u_s \circ d W^i_s - \tfrac{1}{2} \int_0^t (\nabla^i u)_s \, ds$$

is called the Skorohod integral of u with respect to $d W^i$. If, moreover, $u \in \mathbb{L}^{2;1}_C$, then:

$$E[(\delta^i_t(u))^2] = E \int_0^t u_s^2 \, ds + E \int_0^t \int_0^t D^i_s u_r \, D^i_r u_s \, ds \, dr$$

$$E(\delta^i_t(u)) = 0. \quad \square$$

Let $\mathbb{L}_S \triangleq \{u \in \mathbb{L}^{4;2}_C; \nabla u \in \mathbb{L}^{4,1}\}$.
We can now state the extended Stratonovich stochastic calculus rule.

Theorem 3.7. *Let $\varphi \in C^2(\mathbb{R}^d)$, and $\{Z_t, v(t), u_1(t), \ldots, u_k(t);\ t \in [0, 1]\}$ be d-dimensional processes such that: $u^j_i \in \mathbb{L}_{S,\mathrm{loc}}$, $v^j \in \mathbb{L}^{4,1}_{\mathrm{loc}}$, $Z^j_0 \in \mathbb{D}_{4,1,\mathrm{loc}}$; $1 \leqq i \leqq k$, $1 \leqq j \leqq d$; and moreover:*

$$Z_t = Z_0 + \int_0^t v(s) \, ds + \int_0^t u_i(s) \circ d W^i_s.$$

Then

$$\varphi(Z_t) = \varphi(Z_0) + \int_0^t \varphi'(Z_s) \, v(s) \, ds + \int_0^t \varphi'(Z_s) \, u_i(s) \circ d W^i_s. \quad \square$$

We will also need the:

Theorem 3.8. *Let $u \in \mathbb{L}^{2;2}_{C,\mathrm{loc}}$, and be such that there exists a localizing sequence $\{u^n\}$ of u in $\mathbb{L}^{2;2}_C$ with the property that for every $n \in \mathbb{N}$,*

$$t \to D_t u^n$$

belongs to $L^2(0, 1; (\mathbb{L}^{2;1}_C)^k)$. Then, for any $t \in [0, 1]$,

$$\int_0^t u_r \circ d W^i_r \in \mathbb{D}_{2,1,\mathrm{loc}}$$

and

$$D^j_s \left(\int_0^t u_r \circ d W^i_r \right) = \int_0^t D^i_s u_r \circ d W^i_r + \delta_{ij} \, 1_{\{s \leq t\}} \, u_s .$$

Proof. It suffices to consider the case where $u \in \mathbb{L}^{4;2}_C$, with $t \to D_t u$ belonging to $L^2(0, 1; (\mathbb{L}^{2;1}_C)^k)$.

We approximate $\int_0^t u_r \circ d W_r^i$ by the sequence $\{\xi_n, n\in\mathbb{N}\}$ given in Definition 3.1. Clearly, $\xi_n \in \mathbb{D}_{2,1}$, and:

$$D_s^j \xi_n = \sum_{l=0}^{n-1} \left\{ \frac{W^i_{t_n^{l+1}\wedge t} - W^i_{t_n^l \wedge t}}{t_n^{l+1} - t_n^l} \int_{t_n^l}^{t_n^{l+1}} D_s^j u_r \, dr \right.$$
$$\left. + \delta_{ij} 1_{\{t_n^l \wedge t < s \leq t_n^{l+1} \wedge t\}} (t_n^{l+1} - t_n^l)^{-1} \int_{t_n^l}^{t_n^{l+1}} u_r \, dr \right\}.$$

We claim that:

$$\xi_n \to \int_0^t u_r \circ d W_r^i \quad \text{in } L^2(\Omega)$$

$$D_s^j \xi_n \to \int_0^t (D_s^j u_r) \circ d W_r^i + \delta_{ij} 1_{\{s \leq t\}} u_s \quad \text{in } L^2(\Omega \times (0, 1))$$

which proves the theorem, since D is a closed operator. We prove the first claim; the proof of the second one is analogous.

From Nualart-Pardoux [16] Proposition 4.3 and Theorem 7.3, it suffices that the following convergence holds in $L^2(\Omega)$ (and not just in probability):

$$\sum_{l=0}^{n-1} (t_n^{l+1} - t_n^l)^{-1} \int_{t_n^l}^{t_n^{l+1}} \int_{t_n^l \wedge t}^{t_n^{l+1} \wedge t} D_s^j u_r \, ds \, dr \to \tfrac{1}{2} \int_0^t (\nabla^j u)_r \, dr.$$

For that sake, we need only to show that the sequence is dominated in absolute value by a sequence which converges in $L^2(\Omega)$. But

$$\left| \sum_l 2^n \int_{t^l \wedge t}^{t^{l+1} \wedge t} \int_{t^l}^{t^{l+1}} D_s^i u_r \, ds \, dr \right| \leq c \left(\sum_l 2^n \int_{t^l \wedge t}^{t^{l+1} \wedge t} \int_{t^l}^{t^{l+1}} (D_s^i u_r)^2 \, ds \, dr \right)^{1/2},$$

and the right hand side is a sequence of positive random variables which converges in probability towards:

$$c \left(\int_0^t (\nabla^j u)_r^2 \, dr \right)^{1/2}$$

and the $L^2(\Omega)$ norms converge to the $L^2(\Omega)$ norm of the limit. Therefore the convergence holds in $L^2(\Omega)$. □

Remark 3.9. As in the standard theory of stochastic calculus, most of the results on Stratonovich integrals are obtained via their translation into Itô language. The generalized Itô integral is called the Skorohod integral. The reason why the extended Stratonovich integral is the one to be used in our problem is the identity (3.1), as we will explain below. The Skorohod integral does not possess the same property. Indeed, let $v \in \mathbb{L}_C^2$, and $\theta \in \mathbb{S}$, $u_t = \theta v_t$. Then $u \in \mathbb{L}_C^2$, both u and v are Stratonovich integrable, and from Proposition 3.5,

$$\theta \int_0^t v_s \circ d W_s^i = \theta\, \delta_t^i(v) + \tfrac{1}{2} \int_0^t \theta (\nabla^i v)_s\, ds$$

$$\int_0^t u_s \circ d W_s^i = \delta_t^i(u) + \tfrac{1}{2} \int_0^t \theta (\nabla^i v)_s\, ds + \int_0^t v_s\, D_s^i \theta\, ds.$$

Then, from (3.1),

$$\delta_t^i(u) = \theta\, \delta_t^i(v) - \int_0^t v_s\, D_s^i \theta\, ds. \tag{3.3}$$

§ 4. Existence and Uniqueness of a Solution to the Two-sided Stochastic Differential System

Let us rewrite our system:

$$d X_t = (A X_t + a(t))\, dt + (B_i X_t + b_i(t)) \circ d W_t^i \tag{4.1}$$

$$F_0 X_0 + F_1 X_1 = f. \tag{4.2}$$

We define:

$$V_t = \int_0^t a(s)\, ds + \int_0^t b_i(s) \circ d W_s^i.$$

$\{\Phi_t, t \in [0, 1]\}$ is the $d \times d$ matrix valued process, solution of the equation:

$$\Phi_t = I + \int_0^t A \Phi_s\, ds + \int_0^t B_i \Phi_s \circ d W_s^i$$

$$\Phi(t, s) = \Phi_t \Phi_s^{-1}; \qquad s, t \in [0, 1].$$

We then consider the system:

$$X_t = \Phi(t, 0) X_0 + \int_0^t \Phi(t, s) \circ d V_s \tag{4.3}$$

$$[F_0 + F_1 \Phi(1, 0)] X_0 = f - F_1 \int_0^1 \Phi(1, s) \circ d V_s. \tag{4.4}$$

Our study of existence and uniqueness for (4.1)–(4.2) will be made in two steps: first we will show the equivalence of (4.1)–(4.2) with (4.3)–(4.4); second we will study existence and uniqueness for (4.3)–(4.4), which is equivalent to the a.s. invertibility of the $d \times d$ random matrix $F_0 + F_1 \Phi(1, 0)$.

Before proceeding to the proof, let us state the hypotheses on the random data $\{a(t), b_1(t), \ldots, b_k(t); t\in[0,1]\}$ and f, which we suppose to hold throughout this section.

$$a^j \in \mathbb{L}^{4,2}_{\text{loc}}; \qquad 1 \leqq j \leqq d \tag{H.1}$$

$$b^j_i \in \mathbb{L}_{S,\text{loc}}; \qquad 1 \leqq i \leqq k,\ 1 \leqq j \leqq d \tag{H.2}$$

$$f^j \in \mathbb{D}_{4,2,\text{loc}}; \qquad 1 \leqq j \leqq d. \tag{H.3}$$

Remark 4.1. i) The only reason for not allowing $A, B_1, \ldots, B_k$ to depend on t is to obtain a necessary and sufficient condition for existence and uniqueness. In case $B_1 = \ldots = B_k = 0$, that restriction is clearly irrelevant.

ii) In case $a, b_1, \ldots, b_k, f$ are deterministic, it suffices to assume that $a\in L^2(0,1;\mathbb{R}^d)$; $b_1, \ldots, b_k \in L^2(0,1;\mathbb{R}^d)$; $f\in\mathbb{R}^d$. □

First remark that $\Phi.$ and $\Phi(1,.)$ belong to $\mathbb{L}^{p,l}$ for any $p\geqq 1$, $l\in\mathbb{N}$. We have in particular:

$$D_s\Phi_t = \Phi_s + \int_s^t A\, D_s\Phi_r\, dr + \int_s^t B_i D_s \Phi_r \circ dW^i_r, \qquad t\in[s,1].$$

Theorem 4.2. *Suppose that the random matrix $F_0 + F_1\Phi(1,0)$ is a.s. invertible. Then the two-sided stochastic differential system* (4.1)–(4.2) *has a unique solution among those continuous processes whose components belong to $\mathbb{L}_{S,\text{loc}}$.*

Proof. Existence. Under our standing hypotheses, (4.3)–(4.4) determine a unique process $\{X_t, t\in[0,1]\}$. From (3.1), we can rewrite (4.3) as:

$$X_t = \Phi(t,0)X_0 + \Phi_t \int_0^t \Phi_s^{-1} \circ dV_s.$$

It then follows from (3.1) and Theorem 3.7 that $\{X_t\}$ satisfies (4.1). (4.2) follows from (4.4). Clearly, $\{X_t\}$ is a continuous process. It remains to show that $X\in\mathbb{L}_{S,\text{loc}}$. From (4.3)–(4.4) we infer that:

$$X_t = [F_0\Phi(0,t) + F_1\Phi(1,t)]^{-1}\left(f + F_0\int_0^t \Phi(0,s)\circ dV_s - F_1\int_t^1 \Phi(1,s)\circ dV_s\right).$$

It then follows from standard estimates and Lemma 3.5 that $X_t\in\mathbb{D}_{4,2,\text{loc}}$, and each X_t can be localized in $\mathbb{D}_{4,2}$ by a sequence $\{X^n_t, n\in\mathbb{N}\}$, which can be chosen such that $\forall n\in\mathbb{N}$; $X^n\in(\mathbb{L}_S)^d$.

Uniqueness. Let $Y\in(\mathbb{L}_{S,\text{loc}})^d$ be a solution to (4.1)–(4.2). Consider the process $\{\Phi_t^{-1}, t\in[0,1]\}$. We have:

$$\Phi_t^{-1} = I - \int_0^t \Phi_s^{-1} A\, ds - \int_0^t \Phi_s^{-1} B_i \circ dW^i_s.$$

Again, $\Phi^{-1} \in \mathbb{L}^{p,l}$, $\forall p \geqq 1$, $l \leqq \mathbb{N}$. It follows from Theorem 3.7 that:

$$\Phi_t^{-1} Y_t = Y_0 + \int_0^t \Phi_s^{-1} \circ d V_s$$

i.e.:

$$Y_t = \Phi(t, 0) Y_0 + \int_0^t \Phi(t, s) \circ d V_s.$$

Then $\{Y_t\}$ satisfies (4.3). But (4.4) follows from (4.2)+(4.3), and $\{Y_t\}$ satisfies (4.3)+(4.4). It follows that Y and X are indistinguishable. □

Remark 4.3. Let us see by a simple example that we cannot expect in general that the coordinates of X_0 (as well as those of X_t, $t \in [0, 1]$) have any moment. Choose $d=2$, $k=1$, $a=b=0$, $A=0$, $B=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $G=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $f=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; i.e., we consider the system:

$$d X_t^1 = X_t^2 \circ d W_t$$

$$d X_t^2 = 0$$

$$X_0^1 = 0, \qquad X_1^1 = 1.$$

This system has a unique solution: $X_t^1 = (W_1)^{-1} W_t$, $X_1^2 = (W_1)^{-1}$. Clearly, $E\,|X_t^2| = +\infty$, $\forall t \in [0, 1]$. □

One can easily check the:

Proposition 4.4. *If we assume, in addition to* (H.1), (H.2), (H.3) *and the hypothesis of Theorem* 4.2, *that* a^j, $b_i^j \in \mathbb{L}_{\mathrm{loc}}^{p,l}$ *and* $f^j \in \mathbb{D}_{p,l,\mathrm{loc}}$; $\forall p \geqq 1$, $l \in \mathbb{N}$, $1 \leqq j \leqq d$, $1 \leqq i \leqq k$, *the solution* $\{X_t\}$ *to* (4.1), (4.2) *satisfies:*

$$X \in \mathbb{L}_{\mathrm{loc}}^{p,l} \;\; \forall p \geqq 1, \quad l \in \mathbb{N}. \;\; \square$$

Remark 4.5. Let us explain why our approach is not applicable to the Itô-Skorohod version of (4.1). For simplicity, we consider here the case $a=b_1=\ldots=b_k=0$. Let $\{\Psi_t, t \in [0, 1]\}$ be the $d \times d$ matrix valued process, solution of:

$$\Psi_t = I + \int_0^t A \Psi_s \, ds + \int_0^t B_i \Psi_s \, d W_s^i \tag{4.5}$$

and, X_0 being a somewhat regular d-dimensional random vector, define $X_t = \Psi_t X_0$. We then deduce from (3.3) (the stochastic integrals below are Skorohod integrals):

$$X_t = X_0 + \int_0^t A X_s \, ds + \int_0^t B_i \Psi_s (D_s^i X_0) \, ds + \int_0^t B_i X_s \, d W_s^i$$

which can be rewritten as:

$$X_t = X_0 + \int_0^t (A - \tfrac{1}{2}\sum_i B_i^2) X_s\, ds + \tfrac{1}{2}\int_0^t B_i (V^i X)_s\, ds + \int_0^t B_i X_s\, dW_s^i.$$

It does not seem possible to modify (4.5), such that the last equation coincides with the Itô-Skorohod version of (4.1). □

We want now to give a necessary and sufficient condition, in terms of the matrices $A, B_1, \ldots, B_k, F_0, F_1$, for $F_0 + F_1\, \Phi(1,0)$ to be a.s. invertible. Let us first, as a preparation, consider two extreme cases.

Suppose first that the ideal generated by $B_1, B_2, \ldots, B_k$, in the Lie algebra of matrices generated by $A_1\, B_1, \ldots, B_k$, has rank d^2. Then the law of $\Phi(1,0)$ possesses a density with respect to Legesgue measure on $Gl(d, \mathbb{R})$, which can be identified with an open subset of $\mathbb{R}^{d^2}$. On the other hand, the mapping:

$$M \to \Pi(M) = \det(F_0 + F_1 M)$$

from $Gl(d, \mathbb{R})$ into $\mathbb{R}$ is analytic. Therefore, provided Π is nonzero at some point $M \in Gl(d; \mathbb{R})$, the set of zeros of Π has Lebesgue measure zero, and:

$$P(F_0 + F_1\, \Phi(1,0) \text{ is invertible}) = 1.$$

It is not hard to see that the condition (2.3) is equivalent to the existence of an $M \in Gl(d; \mathbb{R})$ s.t. $F_0 + F_1 M$ is invertible. Therefore, in the hypoelliptic case, (2.3) is a necessary and sufficient condition for existence and uniqueness to (4.1)–(4.2).

Let us now consider the case where $B_1 = \ldots = B_k = 0$. Then existence and uniqueness is equivalent to the fact that $F_0 + F_1\, e^A$ be invertible. The condition is stronger than (2.3).

We want now to treat the general case. To this end, it is useful to study first the manifold on which $\Phi_t = \Phi(t, 0)$ evolves. Let $\mathcal{G}$ denote the Lie algebra of matrices generated by $A, B_1, \ldots, B_k$, $\mathcal{T}$ the ideal in $\mathcal{G}$ generated by $B_1, \ldots, B_k$. Let G (resp. G_0) denote the connected component containing the identity of the matrix Lie group generated by $\mathcal{G}$ (resp. $\mathcal{T}$).

Since for any $t > 0$, $e^{-tA} B_j\, e^{tA} \in \mathcal{T}$, the equation

$$d\psi_t = e^{-tA} B_i\, e^{tA}\, \psi_t \circ dW_t^i$$
$$\psi_0 = I$$

may be considered as a stochastic differential equation on G_0. Since $e^{tA}\, \psi_t$ solves (2.4), $\Phi_t = e^{tA}\, \psi_t$, and hence we can assume that:

$$\Phi_t \in e^{tA}\, G_0, \qquad t \geqq 0; \quad \text{a.s.}$$

For $t \geqq 0$, let v_t denote the induced (from $\mathbb{R}^{d \times d}$) volume measure on $e^{tA}\, G_0$.

Proposition 4.6. *For every $t > 0$, the law of Φ_t on $e^{tA}\, G_0$ admits a C^∞ density with respect to v_t.*

Proof. Consider:

$$\tilde{G}_0 = \{(e^{tA} M, t);\ t \in \mathbb{R},\ M \in G_0\} \subset G \times \mathbb{R}.$$

The following is a composition law in $\tilde{G}_0$:

$$(e^{tA} M, t) \square (e^{sA} N, s) = (e^{tA} M e^{sA} N, t+s).$$

Indeed,

$$e^{tA} M e^{sA} N = e^{(t+s)A} e^{-sA} M e^{sA} N,$$

and $e^{-sA} M e^{sA} N \in G_0$, whenever $M, N \in G_0$, since, $\mathcal{T}$ being an ideal of $\mathcal{G}$, G_0 is a normal subgroup of G, see Helgason [7, p. 128 Chap. II, § 5].

If we define the vector fields on $G \times \mathbb{R}$:

$$Z_A(N, t) = (AN, 0)$$

$$Z_{B_i}(N, t) = (B_i N, 0) \qquad 1 \leqq i \leqq k$$

$$Z_0(N, t) = (0, 1),$$

we have that $Z_A + Z_0$, $Z_{B_1}, \ldots, Z_{B_k}$ can be restricted to vector fields on $\tilde{G}_0$. Let us define the operator:

$$\frac{\partial}{\partial t} + L = \tfrac{1}{2} \sum_1^k Z_{B_i}^2 + Z_A + Z_0.$$

For $t \geqq 0$, let μ_t denote the law of Φ_t. $\{\mu_t\}$ solves the Fokker-Planck equation:

$$\left(-\frac{\partial}{\partial t} + L^*\right) \mu_t = 0, \qquad t > 0$$

$$\mu_t \to \delta_I, \quad \text{as} \quad t \downarrow 0.$$

Let Λ denote the Lie algebra of vector fields over $\tilde{G}_0$ generated by $Z_A + Z_0$, $Z_{B_1}, \ldots, Z_{B_k}$. It is easily seen that

$$\operatorname{rank} \Lambda(e^{tA} M, t) = \dim \tilde{G}_0$$

$\forall t > 0$, $M \in G_0$. The result then follows from Hörmander's hypo-ellipticity theorem. □

Let us define:

$$\mathcal{C} = \{M \in Gl(d; \mathbb{R});\ \Pi(M) = 0\}.$$

We have the following dichotomy:

Corollary 4.7. *Either $e^A G_0 \subset \mathcal{C}$, and*

$$P(\Pi(\Phi_1) = 0) = 1$$

or else

$$P(\Pi(\Phi_1) \neq 0) = 0.$$

Proof. $\Pi(M)$ is an analytic function on the analytic manifold $e^A G_0$. Therefore, either $e^A G_0 \subset \mathscr{C}$, or else $\mathscr{C} \cap e^A G_0$ is a subvariety of lower dimension in $e^A G_0$. Since $P(\Pi(\Phi_1)=0)=\mu_1(\mathscr{C} \cap e^A G_0)$, the result follows from Proposition 4.6. □

From Corollary 4.7, $\Pi(\Phi_1) \neq 0$ a.s. if and only if the following holds:

$$\text{there exists } M \in G_0 \text{ s.t. } \Pi(e^A M) \neq 0. \tag{4.6}$$

Let $E_1, \ldots, E_m$ be a basis of $\mathscr{T}$. From the analyticity of Π, the following is equivalent to (4.6):

There exists a multi-index $\alpha=(\alpha_1, \ldots, \alpha_m)$ s.t.:

$$\frac{\partial^{|\alpha|}}{\partial \varepsilon_1^{\alpha_1} \ldots \partial \varepsilon_m^{\alpha_m}} \Pi(e^A e^{\varepsilon_1 E_1 + \ldots + \varepsilon_m E_m})\Big|_{\varepsilon_1 = \ldots = \varepsilon_m = 0} \neq 0. \tag{4.7}$$

It remains to make the computation of the derivatives explicit, and show that one needs to check only a finite number of derivatives. It is convenient to use the wedge product notation. Define $\bar{F}_1 = F_1 e^A$.

Let V denote the space of $d \times 2d$ real matrices. A typical element of V will be denoted $[M, N]$, where M and N denote $d \times d$ matrices. We introduce the following d-linear form on V:

$$v([N_1, M_1], \ldots, [N_d, M_d]) = (F_0 N_1 + \bar{F}_1 M_1)^1 \wedge \ldots \wedge (F_0 N_d + \bar{F}_1 M_d)^d$$

where Q^i denotes the i-th column of the $d \times d$ matrix Q. In other words, $v \in V^{\otimes d}$. Let E be a $d \times d$ matrix. We define $\hat{E} v \in V^{\otimes d}$ as:

$$\begin{aligned} &\hat{E} v([N_1, M_1], \ldots, [N_d, M_d]) \\ &\quad = \sum_{i=1}^{d} (F_0 N_1 + \bar{F}_1 M_1)^1 \wedge \ldots \wedge (F_0 N_{i-1} + \bar{F}_1 M_{i-1})^{i-1} \wedge (\bar{F}_1 M_i E)^i \\ &\quad\quad \wedge (F_0 N_{i+1} + \bar{F}_1 M_{i+1})^{i+1} \wedge \ldots \wedge (F_0 N_d + \bar{F}_1 M_d)^d \end{aligned}$$

$\hat{E}$ can be extended as a linear operator on $V^{\otimes d}$.

We can now prove the:

Theorem 4.8. *The three following conditions are equivalent:*

(i) $\Pi(\Phi_1) \neq 0$ *a.s.*

(ii) $\exists M \in e^A G_0$ *s.t.* $\Pi(M) \neq 0$

(iii) $\exists$ *a multiindex* $\alpha=(\alpha_1, \ldots, \alpha_m)$, *with* $\alpha_i \leqq 2^d d^{2d} - 1$, $1 \leqq i \leqq m$, *such that*:

$$\hat{E}_m^{\alpha_m} \ldots \hat{E}_1^{\alpha_1} v([I, I], \ldots, [I, I]) \neq 0.$$

Proof. From the formula:

$$\begin{aligned} &\frac{\partial^{(\alpha)}}{\partial \varepsilon_1^{\alpha_1} \ldots \partial \varepsilon_m^{\alpha_m}} \Pi(e^{\varepsilon_1 E_1} \ldots e^{\varepsilon_m E_m})\Big|_{\varepsilon_1 = \ldots = \varepsilon_m = 0} \\ &\quad = \hat{E}_m^{\alpha_m} \ldots \hat{E}_1^{\alpha_1} v([I, I], \ldots, [I, I]), \end{aligned}$$

we see that the theorem is the consequence of the above discussion, except that it suffices to check condition (iii) for α's with $\alpha_i \leqq \dim(V^{\otimes d})-1$. But this is a consequence of the Cayley-Hamilton theorem, which tells us that, for $\alpha_i \geqq \dim V^{\otimes d}$, $\hat{E}_i^{\alpha_i}$ is a linear combination of lower powers of $\hat{E}_i$. □

Note that checking condition (iii) amounts to computing a finite number of determinants which are expressible in terms of $A, B_1, \ldots, B_k, F_0$ and F_1. Nevertheless, the condition can be used in practice only with small d's.

§ 5. Existence of Probability Densities

In this section we consider the question of when solutions X_t to (2.1)–(2.2) admit probability densities. We first calculate the Malliavin covariance matrix of X_t and use this to give a criterion for existence of densities. Secondly, we show that under more restrictive assumptions, it is possible to compute the distribution of X_t fairly explicitly using the co-area formula. This is then used to considerably sharpen the criterion for existence of densities.

Throughout this section, X_t solves

$$dX_t = [a + A X_t]\,dt + [b_i + B_i X_t] \circ dW_t^i \tag{5.1}$$

$$F_0 X_0 + F_1 X_1 = f \tag{5.2}$$

for constant $a, b_1, \ldots, b_k$, and f. We assume always that $F_0 + F_1\,\Phi(1,0)$ is a.s. invertible, so that existence and uniqueness of solutions is guaranteed. We shall use the following notations:

i) $M(t) = F_0\,\Phi(0,t) + F_1\,\Phi(1,t)$,

ii) $Q(x) = \sum_{j=1}^{k} (B_j x + b_j)(B_j x + b_j)^T$, $x \in \mathbb{R}^d$,

and

iii) $\tilde{A}(x) = a + Ax$, $\tilde{B}_i(x) = b_i + B_i x$; $x \in \mathbb{R}^d$.

Notice that $M(t)$ is a.s. invertible if $F_0 + F_1\,\Phi(1,0)$ is. We shall think of $\tilde{A}(x)$ and $\tilde{B}_i(x)$ as defining vector fields on $\mathbb{R}^d$. Thus, we associate to $\tilde{A}(x)$ the vector field $\sum_1^d (a+Ax)_i \frac{\partial}{\partial x_i}$, and, if $[\tilde{A}, \tilde{B}_i]$ denotes the Lie bracket of the corresponding vector fields, $[\tilde{A}, \tilde{B}_i](x) = [B_i A - A B_i]\,x + B_i\,a - A\,b_i$. Finally, we let $\tilde{\mathscr{G}}$ = Lie algebra generated by the vector fields $\tilde{A}, \tilde{B}_1, \ldots, \tilde{B}_k$, and we let $\tilde{\mathscr{I}}$ denote the ideal in $\tilde{\mathscr{G}}$ generated by $\tilde{B}_1, \ldots, \tilde{B}_k$.

5.1 The Malliavin Covariance Matrix and Densities

If $F = (F_1, \ldots, F_n)$ and $F_i \in \mathbb{D}_{2,1,\mathrm{loc}}$ for $1 \leqq i \leqq n$, we let

$$\langle\!\langle DF, DF \rangle\!\rangle = \left[\int_0^1 \sum_{l=1}^{k} D_s^l F_i\, D_s^l F_j\, ds\right]_{1 \leqq i,j \leqq n} \tag{5.3}$$

$\langle\!\langle DF, DF\rangle\!\rangle$ is called the Malliavin covariance matrix of F. A result of Bouleau and Hirsch [3] implies that if $F\in\mathbb{D}_{2,1,\text{loc}}$ and $\langle\!\langle DF, DF\rangle\!\rangle>0$ a.s., then F admits a probability density w.r.t. Lebesgue measure. That is, if P_F denotes the probability distribution of F, $P_F\ll m$ where m denotes Lebesgue measure on $\mathbb{R}^d$. For solutions $\{X_t\}$ to (5.1)–(5.2) we shall compute $\langle\!\langle DX_t, DX_t\rangle\!\rangle$ and give a sufficient condition that $\langle\!\langle DX_t, DX_t\rangle\!\rangle>0$ a.s. We do not consider the question of whether the density of X_t is a C^∞-function. The standard criterion of the Malliavin calculus for C^∞-regularity requires that $X_t\in\mathbb{D}_\infty=\bigcap_{\substack{l\in\mathbb{N}^k\\ p\geqq 2}} D_{l,p}$ and $\det\langle\!\langle DF, DF\rangle\!\rangle^{-1}\in L^p(P)\ \forall p\geqq 1$. But we have seen that X_t is not integrable in general, i.e., $X_t\in\mathbb{D}_{\infty,\text{loc}}$ only, and hence the standard theory will not directly apply.

Proposition 5.1. *Given the above assumptions,*

$$\langle\!\langle DX_t, DX_t\rangle\!\rangle=M^{-1}(t)\left\{\int_0^t F_0\,\Phi(0,s)\,Q(X_s)\,\Phi^T(0,s)\,F_0^T\,ds\right.$$
$$\left.+\int_t^1 F_1\,\Phi(1,s)\,Q(X_s)\,\Phi^T(1,s)\,F_1^T\,ds\right\}(M^{-1}(t))^T.$$

Proof. Using the properties of the derivation operator D, including Theorem 3.8, we obtain:

$$F_0\,D_s^i X_0+F_1\,D_s^i X_1=0, \tag{5.4}$$

and,

$$D_s^i X_t=D_s^i X_0+\int_0^t A\,D_s^i X_u\,du+\int_0^t B_j(D_s^i X_u)\circ dW_u^j+\tilde{B}_i(X_s)\,1_{\{s<t\}}. \tag{5.5}$$

Now (5.4)–(5.5) is a two-sided system precisely of the form of (4.1)–(4.2) but with the additional term $\tilde{B}_i(X_s)\,1_{[s<t]}$. It is evident from the form (4.3)–(4.4) of the solution X_t, that $D_s^i X_t\in\mathbb{L}_{S,\text{loc}}$ for fixed s. By a repeat of the existence and uniqueness argument of Theorem 4.2 separately on the time intervals $[0,s]$ and $[s,t]$, we can conclude that:

$$D_s^i X_t=\Phi(t,0)[D_s^i X_0+\Phi(0,s)\,\tilde{B}_i(X_s)\,1_{\{s<t\}}].$$

In particular

$$D_s^i X_0=\Phi(0,t)\,D_s^i X_t-\Phi(0,s)\,\tilde{B}_i(X_s)\,1_{\{s<t\}}$$

and similarly

$$D_s^i X_1=\Phi(1,t)\,D_s^i+\Phi(1,s)\,\tilde{B}_i(X_s)\,1_{\{s\geqq t\}}.$$

Then, substituting these into (5.4) and solving for $D_s^i X_t$ gives:

$$D_s^i X_t=M^{-1}(t)[F_0\,\Phi(0,s)\,1_{\{0\leqq s\leqq t\}}-F_1\,\Phi(1,s)\,1_{\{t\leqq s\leqq 1\}}]\,\tilde{B}_i(X_s).$$

The proposition follows immediately from this and the definition (5.3) of $\langle\!\langle DX_t, DX_t\rangle\!\rangle$. $\square$

In the spirit of the Malliavin-Stroock-Bismut approach to existence of densities for solutions to stochastic d.e.'s, we shall establish a Lie algebraic sufficient condition that $\langle\!\langle DX_t, DX_t\rangle\!\rangle > 0$ a.s. Let G_0 be defined as in Sect. 4 and define the subset $BC \subset \mathbb{R}^d \times \mathbb{R}^d$

$$BC = \{(x_0, x_1) \mid F_0 x_0 + F_1 x_1 = f\}$$

BC contains the set of all possible initial and final values of the X_t process.

Theorem 5.2. *If*

$$\{F_0 C_0(X_0(\omega)) \mid C_0 \in \tilde{\mathscr{Z}}\} + \{F_1 C_1(X_1(\omega)) \mid C_1 \in \tilde{\mathscr{Z}}\} = \mathbb{R}^d \quad a.s., \tag{5.9}$$

then $\langle\!\langle DX_t, DX_t\rangle\!\rangle > 0$ *a.s. for any* $0<t<1$ *and (see Bouleau-Hirsch [3])* X_t *admits a density for* $0<t<1$. *In particular,* (5.9) *is satisfied if*

$$\{F_0 C_0(x_0) \mid C \in \tilde{\mathscr{Z}}\} + \{F_1 C_1(x_1) \mid C \in \tilde{\mathscr{Z}}\} = \mathbb{R}^d \quad \textit{for every } (x_0, x_1) \in BC. \quad \square \tag{5.10}$$

Condition (5.10) is really much too strong. It is obvious from (5.9) that we could replace BC in (5.10) by

$$BC' = \bigcup_{\omega} (X_0(\omega), X_1(\omega))$$

which, in general, is smaller than BC. However BC' has no simple analytic characterization. In the case that $a = b_1 = \ldots = b_k = 0$ a useful reduction of BC is possible, because $X_0(\omega)$ and $X_1(\omega)$ are constrained by the conditions that $X_1 = \Phi(1, 0) X_0$ and that $F_0 + \Phi(1, 0) F_1$ be invertible.

Corollary 5.3. *Let* $a = b_1 = \ldots = b_k = 0$. *Let* BC'' *be the subset of* $(x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d$ *such that* $F_0 x_0 + F_1 x_1 = f$ *and there exists a* $T \in e^A G_0$ *such that* $F_0 + F_1 T$ *is invertible and* $x_1 = T x_0$ *(see Sect. 4 for the definition of* G_0.*) Then*

$$\{F_0 C_0(x_0) \mid C_0 \in \tilde{\mathscr{Z}}\} + \{F_1 C_1(x_1) \mid C_1 \in \tilde{\mathscr{Z}}\} = \mathbb{R}^d \quad \textit{for every } (x_0, x_1) \in BC'' \tag{5.11}$$

implies that $\langle\!\langle DX_t, DX_t\rangle\!\rangle > 0$ *a.s.*

To illustrate, we specialize to some particular cases, the first of which is well-known.

Corollary 5.4 (Adapted case). *If* $F_1 = 0$, *then*

$$\Lambda(F_0^{-1} f) := \mathrm{Span}\{C(F_0^{-1} f) \mid C \in \tilde{\mathscr{Z}}\} = \mathbb{R}^d$$

implies that $\langle\!\langle DX_t, DX_t\rangle\!\rangle > 0$ *a.s.*

Proof. Use 5.10, noting that F_0 is invertible, and $BC = \{F_0^{-1} f\} \times \mathbb{R}^d$. $\square$

As a second example, consider the two point boundary value problem defined by the boundary conditions $F_0' X_0 = f_0$ and $F_1' X_1 = f_1$, where, as usual, rank $F_0' = l$ and rank $F_1' = d - l$. In this case

$$\langle\!\langle D X_t, D X_t \rangle\!\rangle = M^{-1}(t) \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} [M^{-1}(t)]^T \tag{5.12}$$

where

$$L_1 = \int_0^t F_0' \,\Phi(0, s)\, Q(X_s)\, \Phi^T(0, s)(F_0')^T \, ds$$

and

$$L_2 = \int_0^t F_0' \,\Phi(1, s)\, Q(X_s)\, \Phi^T(1, s)(F_0')^T \, ds.$$

We obtain

Corollary 5.5. *The following is a sufficient condition that* $\langle\!\langle D X_1, D X_1 \rangle\!\rangle > 0$ *a.s.* $\forall\, t \in (0, 1)$ *for the two-point boundary value problem: for every* $(x_0, x_1) \in BC$

$$\{F_0' \,C(x_0) \mid C \in \tilde{\mathscr{Q}}\} = \mathbb{R}^l \quad \textit{and} \quad \{F_1' \,C(x_1) \mid C \in \tilde{\mathscr{Q}}\} = \mathbb{R}^{d-l}. \quad \square$$

Proof of Theorem 5.2. We adapt an argument due to Bismut [2]. From Proposition 5.1, it suffices to show that for $0 < t < 1$ the following random matrix is a.s. positive definite:

$$R(t) = \int_0^t F_0 \,\Phi(0, s)\, Q(X_s)\, \Phi^T(0, s)\, F_0^T \, ds + \int_t^1 F_1 \,\Phi(1, s)\, Q(X_s)\, \Phi^T(1, s)\, F_1^T \, ds.$$

We define some random vector subspaces of $\mathbb{R}^d$:

$$\mathscr{U}(t) = \operatorname{Span}\{F_0 \,\Phi(0, s)\, \tilde{B}_i(X_s);\ 0 \leqq s \leqq t,\ 1 \leqq i \leqq k\}$$

$$\mathscr{U}(0^+) = \bigcap_{t > 0} \mathscr{U}(t)$$

$$\mathscr{V}(t) = \operatorname{Span}\{F_1 \,\Phi(1, s)\, \tilde{B}_i(X_s);\ t \leqq s \leqq 1,\ 1 \leqq i \leqq k\}$$

$$\mathscr{V}(1^-) = \bigcap_{t < 1} \mathscr{V}(t).$$

Note that $\mathscr{U}(t)$ [resp. $\mathscr{V}(t)$] is an increasing (resp. decreasing] function of t, so that $\mathscr{U}(0^+)$ and $\mathscr{V}(1^-)$ are well defined. It clearly suffices to show that for $0 < t < 1$, $\mathscr{V}(t) + \mathscr{U}(t) = \mathbb{R}^d$ a.s., which is implied by the stronger condition:

$$\mathscr{U}(0^+) + \mathscr{V}(1^-) = \mathbb{R}^d \quad \text{a.s.}$$

Let us admit for a moment the:

Lemma 5.6. *There exists a measurable random vector* $q(\omega)$ *s.t. a.s.:*

i) $q(\omega) \in (\mathscr{U}(0^+)(\omega) + \mathscr{V}(1^-)(\omega))^\perp$

ii) $|q(\omega)| = 1$ *if* $\mathscr{U}(0^+)(\omega) + \mathscr{V}(1^-)(\omega) \neq \mathbb{R}^d$. $\square$

It clearly suffices to show that for any such $q(\omega)$,

$$P(q(\omega)=0)=1.$$

To this end, we define $\mathscr{H}$ to be the set of $C\in\tilde{\mathscr{G}}$ such that, a.s.:
i) $\exists\, t_0(\omega)>0$ such that $F_0\,\Phi(0,s)\,C(X_s)(\omega)\perp q(\omega)$ for $s\leqq t_0(\omega)$ and
ii) $\exists\, t_1(\omega)<1$ such that $F_1\,\Phi(1,s)\,C(X_s)(\omega)\perp q(\omega)$ for $t_1(\omega)\leqq s\leqq 1$.
For every ω there is a $t_0(\omega)>0$ such that $\mathscr{U}(0+)(\omega)=\mathscr{U}(t_0(\omega))(\omega)$. Similarly, for every ω there is a $t_1(\omega)<1$ such that $\mathscr{V}(1^-)(\omega)=\mathscr{V}(t_1(\omega))(\omega)$. Therefore, since $q\in[\mathscr{U}(0^+)+\mathscr{V}(1^-)]^\perp$ a.s., $\tilde{B}_1,\ldots,\tilde{B}_k\in\mathscr{H}$. To complete the proof, it then remains to establish that $\mathscr{H}$ is an ideal in $\tilde{\mathscr{G}}$, and for this it is enough to show that if $\tilde{C}\in\mathscr{H}$, then $[\tilde{A},\tilde{C}]\in\mathscr{H}$ and $[\tilde{B}_i,\tilde{C}]\in\mathscr{H}$ for $1\leqq i\leqq k$. To this end the following two lemmas are crucial.

Lemma 5.7. *Let $\tilde{C}(x)$ be a vector field represented by $\tilde{C}(x)=Cx+c$. Note that all vector fields in $\tilde{\mathscr{G}}$ are of this type. Then*

$$\Phi(0,s)\,\tilde{C}(X_s)=\tilde{C}(X_0)+\int_0^s\Phi(0,r)[\tilde{A},\tilde{C}](X_r)\,dr+\int_0^s\Phi(0,r)[\tilde{B}_i,\tilde{C}](X_r)\circ dW_r^i \tag{5.13}$$

and

$$\Phi(1,s)\,\tilde{C}(X_s)=\tilde{C}(X_1)-\int_s^1\Phi(1,r)[\tilde{A},\tilde{C}](X_r)\,dr-\int_s^1\Phi(1,r)[\tilde{B}_i,\tilde{C}](X_r)\circ dW_r^i. \tag{5.14}$$

Proof. The second formula follows from the first using the identity $\Phi(1,s)=\Phi(1,0)\,\Phi(0,s)$. The first formula is a consequence of the chain rule for Stratonovich integrals and $d\Phi(0,s)=-\Phi(0,s)A\,ds-\Phi(0,s)\,B_i\circ dW_s^i$. □

For a stochastic process $\{Z_t\}$, let

$$QV_t(Z)=\lim_{n\to\infty}\text{in prob.}\sum_{j=0}^{n-1}(Z_{t_{j+1}^n}-Z_{t_j^n})^2$$

[where $\{t_j^n\}$ is a sequence of partitions of $[0,t]$ with $\max_{j\leqq n}(t_{j+1}^n-t_j^n)\to 0$] if this limit exists and does not depend on the particular sequence of partitions.

Lemma 5.8. *Let q be a random vector in $\mathbb{R}^d$. Then*

$$QV_t\langle F_0\,\Phi(0,t)\,\tilde{C}(X_t),q\rangle=\int_0^t\sum_{i=1}^k\langle F_0\,\Phi(0,s)[\tilde{B}_i,\tilde{C}](X_s),q\rangle^2\,ds$$

and

$$[QV_1-QV_t](\langle F_1\,\Phi(1,t)\,\tilde{C}(X_t),q\rangle)=\int_t^1\sum_{i=1}^k\langle F_1\,\Phi(1,s)[\tilde{B}_i,\tilde{C}](X_s),q\rangle^2\,ds.$$

Proof. Write

$$Z_s = \langle F_0\, \Phi(0, s)\, \tilde{C}(X_s), q\rangle = q_l\, z_l(s)$$

where

$$z_l(s) = [F_0\, \Phi(0, s)\, \tilde{C}(X_s)]_l.$$

Then

$$\sum_{j=0}^{n-1} (Z_{t_{j+1}} - Z_{t_j})^2 = \sum_{l,m=1}^{d} q_l\, q_m \sum_{j=0}^{n-1} (z_l(t_{j+1}) - z_l(t_j))(z_m(t_{j+1}) - z_m(t_j)).$$

However, from Nualart and Pardoux [16], Theorem 5.4,

$$\sum_{j=0}^{n-1} (z_l(t_{j+1}) - z_l(t_j))(z_m(t_{j+1}) - z_m(t_j))$$

$$\to \sum_{i=1}^{k} \int_0^t [F_0\, \Phi(0, s)[\tilde{B}_i, \tilde{C}](X_s)]_l [F_0\, \Phi(0, s)[\tilde{B}_i, \tilde{C}](X_s)]_m\, ds$$

because $\Phi(0, s)\, \tilde{C}(X_s)$ satisfies (5.13). The proof for $\langle F_1\, \Phi(1, t)\, \tilde{C}(X_t), q\rangle$ is similar. □

Now suppose that $\tilde{C} \in \mathscr{H}$ and let $t_0(\omega) > 0$, $t_1(\omega) < 1$ be such that

$$\langle F_0\, \Phi(0, s)\, \tilde{C}(X_s), q\rangle = 0 \quad \text{for } 0 \leqq s < t_0(\omega),$$

$$\langle F_1\, \Phi(1, s)\, \tilde{C}(X_s), q\rangle = 0 \quad \text{for } t_1(\omega) < s \leqq 1.$$

Then from Lemmas 5.7 and 5.8,

$$\langle F_0\, \Phi(0, s)[\tilde{B}_i, \tilde{C}](X_s), q\rangle = 0 \quad \text{for } 0 \leqq s < t_0(\omega) \text{ a.s.,} \quad 1 \leqq i \leqq k$$

$$\langle F_1\, \Phi(1, s)[\tilde{B}_i, \tilde{C}](X_s), q\rangle = 0 \quad \text{for } t_1(\omega) < s \leqq 1 \text{ a.s.,} \quad 1 \leqq i \leqq k.$$

Thus $[\tilde{B}_i, \tilde{C}] \in \mathscr{H}$ for $1 \leqq i \leqq k$. But then, by definition of the Skorohod integral the stochastic integrals in the expression for $\langle F_0\, \Phi(0, t)\, \tilde{C}(X_t), q\rangle$ (resp. $\langle F_1(1, t)\, \tilde{C}(X_t), q\rangle$) is also identically zero a.s. for $0 \leqq t < t_0(\omega)$ (resp. $t_1(\omega) < t \leqq 1$). It follows that

$$\langle F_0\, \Phi(0, s)[\tilde{A}, \tilde{C}](X_s), q\rangle = 0 \quad \text{for } 0 \leqq s < t_0(\omega) \quad \text{a.s.}$$

$$\langle F_1\, \Phi(1, s)[\tilde{A}, \tilde{C}](X_s), q\rangle = 0 \quad \text{for } t_1(\omega) < s \leqq 1 \quad \text{a.s.}$$

Hence $[A, \tilde{C}] \in \mathscr{H}$ also. This completes the proof that $\mathscr{H}$ is an ideal in $\mathscr{G}$.

Proof of Lemma 5.6. For notational convenience, we consider the case $a = b_1 = \ldots = b_k = 0$. The general case has a similar proof. We define

$$\underline{U}(0+) = \bigcap_{t>0} \operatorname{Span}\{F_0\, \Phi(0, s)\, B_i\, \Phi(s, 0) \mid 0 \leqq s \leqq t,\ 1 \leqq i \leqq k\}$$

and

$$\underline{V}(1-) = \bigcap_{1>t} \operatorname{Span}\{F_1\, \Phi(1, s)\, B_i\, \Phi(s, 1) \mid t \leqq s \leqq 1,\ 1 \leqq i \leqq k\}.$$

By the 0–1 law for Brownian motion, $\underline{U}(0+)$ and $V(1-)$ are each a.s. equal to some fixed subspace. For $x \in \mathbb{R}^d$, let $\underline{U}(0+)(x)=\{Cx \mid C \in \underline{U}(0+)\}$ and $\underline{V}(1-)(x)=\{Cx \mid C \in \underline{V}(1-)\}$. Then, it is clear that $\mathscr{U}(0+)=\underline{U}(0+)(X_0)$ and $\mathscr{V}(1-)=\underline{V}(1-)(X_1)$. We shall show that there is a Borel measurable $\bar{q}$: $BC \to \mathbb{R}^d$ satisfying

$$\begin{aligned} &\text{i) } \bar{q}(x_0, x_1) \perp U(0+)(x_0)+V(1-)(x_1), \forall (x_0, x_1) \in BC \\ &\text{ii) } |\bar{q}(x_0, x_1)|=1 \text{ if } U(0+)(x_0)+V(1-)(x_1) \neq \mathbb{R}^d. \end{aligned} \tag{5.15}$$

$q(\omega)=\bar{q}(X_0(\omega), X_1(\omega))$ will then be a measurable random vector satisfying the requirements of Lemma 5.6.

To prove (5.15) we use

(5.16) Let $(\mathscr{E}, B(\mathscr{E}))$ be a metric space equipped with its Borel σ-field, let S be a complete, separable metric space and $x \mapsto \Gamma_x$ a mapping from $\mathscr{E}$ to closed subsets of S. If for any sequence $\{x_n, z_n\} \subset \mathscr{E} \times S$ such that $z_n \in \Gamma_{x_n} \forall n$, $\lim_{n\to\infty} x_n = x$ implies that $\{z_n\}$ has an accumulation point in Γ_x, then there is a measurable f: $\mathscr{E} \to S$ such that $f(x) \in \Gamma_x$ for every x. (See Ethier and Kurtz [6], Appendix, Sect. 10.)

Let $\mathscr{E}=\{(x_0, x_1) \in BC \mid \underline{U}(0+)(x_0)+\underline{V}(1-)(x_1) \neq \mathbb{R}^d\}$. This is a closed subset of BC with respect to the relative topology. We take $S=\mathbb{R}^d$ and

$$\Gamma_{x_0,x_1}=\{y \mid y \perp \underline{U}(0+)(x_0)+\underline{V}(1-)(x_1), |y|=1\}.$$

Let $(x_0^n, x_1^n) \to (x_0, x_1)$ in $\mathscr{E}$ and let $y_n \in \Gamma_{x_0^n, x^n}$ for every n. Since $\{y_n\} \subset S^{d-1}$, where S^{d-1} denotes the unit sphere, $\{y_n\}$ has an accumulation point $y \in S^{d-1}$. But for every $C \in \underline{U}(0+)$,

$$\langle y, Cx_0 \rangle = \lim_{k\to\infty} \langle y_{n_k}, Cx_0^{n_k} \rangle = 0 \quad \text{where} \quad \lim_{k\to\infty} y_{n_k}=y.$$

Similarly $\langle y, Cx_1 \rangle = 0$ for every $C \in \underline{V}(1-)$. Thus $y \perp \underline{U}(0+)+\underline{V}(1-)$, and so $y \in \Gamma_{x_0 x_1}$. The existence of $\bar{q}$ satisfying (5.15) follows from (5.16) if $\bar{q}(x)$ is defined to be 0 on $BC \backslash \mathscr{E}$. □

(5.17) *Example*

$$dX_t = B_i X_t \circ dW_t^i, \qquad F_0' X_0 = f_0, \qquad F_1' X_0 = f_1$$

where

i) $\mathscr{Q}=$ Lie algebra $\{B_1, \ldots, B_k\}=\mathbb{R}^{d \times d}$

ii) $f_0 \neq 0$, $f_1 \neq 0$.

As usual $F_0' \in \mathbb{R}^{l \times d}$, $F_1' \in \mathbb{R}^{(d-l) \times d}$ and rank $F_0'=l$, rank $F_1'=d-l$. Then

$$\{F_0' C_0 x_0 \mid C_0 \in \mathscr{Q}\}+\{F_1' C_1 x_1 \mid C_1 \in \mathscr{Q}\}=\mathbb{R}^d \qquad \text{for every } (x_0, x_1) \in BC,$$

since $F_0' x_0 = f_0$ and $F_1' x_1 = f_1$ imply $x_0 \neq 0$ and $x_1 \neq 0$. □

5.2 The Co-area Formula and Densities

The analytic properties of the manifolds $e^{tA} G_0$ in which $\Phi(t, 0)$ takes values and of the maps which determine X_t as a function of $\Phi(0, t)$ and $\Phi(1, t)$ suggest that better sufficient conditions than (5.10) or (5.11) are available. In particular, it should be possible to show whether X_t admits a density by testing $\{F_0 C_0 x_0 | C_0 \in \tilde{\mathscr{Q}}\} + \{F_1 C_1 x_1 | C_1 \in \tilde{\mathscr{Q}}\}$ at just one point (x_0, x_1) in certain circumstances. We shall obtain a condition of this sort by using the co-area formula to represent explicitly the probability distribution of X_t and, in fact, our condition will turn out to be necessary for the existence of densities as well. Our theorem shall be developed under the assumption

$$a = b_1 = \ldots = b_k = 0, \tag{A.1}$$

and we assume that this holds for the remainder of this subsection.

The co-area formula is a generalized change of variables formula. We shall state it for the special case of analytic maps on analytic manifolds, which is the case we need; our statement is a corollary of Theorem 3.2.22, Corollary 3.2.32 and Remark 3.2.33 in Federer [7]. Let $N \subset \mathbb{R}^k$ be a smooth manifold of dimension r, and let φ: $N \to \mathbb{R}^m$ be a C^1-function. We need to define a Jacobian of φ at points $p \in N$. For $p \in N$, we shall think of $T_p N$, the tangent space to N at p, as a subspace of $\mathbb{R}^k$ with the inner product $\langle \cdot, \cdot \rangle$ induced from $\mathbb{R}^k$. Let $\mu = \dim[\text{image}\, \partial \varphi(p)]$ where $\partial \varphi$ denotes the differential of φ, and let $e_1, \ldots, e_\mu$ be an orthonormal basis of image $\partial \varphi(p)$ using the Euclidean inner product from $\mathbb{R}^m$. We define

$$J_\mu \varphi(p) = (\det[\langle \partial \varphi(p)^* e_i, \partial \varphi(p)^* e_j \rangle_{1 \le i, j \le \mu}])^{1/2}$$

where $\partial \varphi^*$ denotes the adjoint of $\partial \varphi$. This definition is independent of the choice of basis; it has a basis-free definition in terms of exterior algebra. Let $\mathscr{H}^n$ denote Hausdorff measure of dimension n; for the definition, again see Federer [7]. Note that if $M \subset \mathbb{R}^m$ is a manifold of dimension n, $\mathscr{H}^n$ on M coincides with the canonical surface measure on M.

Proposition 5.9. *Let $N \subset \mathbb{R}^k$ be a connected, analytic, r-dimensional manifold and let φ: $N \to \mathbb{R}^m$ be an analytic map. Set $\mu = \sup\{\dim[\text{image}\, \partial \varphi(p)] | p \in N\}$. Then*

i) *$S = \{p | \text{rank}\, \partial \varphi(p) = \mu\}$ is an open set of N containing $\mathscr{H}^r$-almost all of N; and,*

ii) *If g: $N \to \mathbb{R}$ is $\mathscr{H}^r$-integrable*

$$\int_N g(p)\, d\mathscr{H}^r(p) = \int_{\mathbb{R}^m} \Big[\int_{\varphi^{-1}(z) \cap S} g(u) [J_\mu \varphi(u)]^{-1}\, d\mathscr{H}^{r-\mu}(u) \Big]\, d\mathscr{H}^\mu(z). \quad \square \tag{5.18}$$

(5.18) is called the co-area formula. It is usually stated

$$\int_N g(p) J_\mu \varphi(p)\, d\mathscr{H}^r(p) = \int_{\mathbb{R}^m} \Big[\int_{\varphi^{-1}(z)} g(u)\, d\mathscr{H}^{r-\mu}(u) \Big]\, d\mathscr{H}^\mu(z). \tag{5.19}$$

However, $J_\mu \varphi(p) > 0$ as long as dim image $\partial \varphi(p) = \mu$ and so by Proposition 5.9 i), $J_\mu \varphi(p) > 0$ $\mathscr{H}^r$-almost everywhere. Therefore, for any $\varepsilon > 0$,

$$\int_N g(p) 1_{[J_\mu \varphi(p) > \varepsilon]} d\mathscr{H}^r(p)$$

$$= \int_{\mathbb{R}^m} \Big[\int_{\varphi^{-1}(z)} g(u) 1_{\{J_\mu \varphi(u) > \varepsilon\}} [J_\mu \varphi(u)]^{-1} d\mathscr{H}^{r-\mu}(u) \Big] d\mathscr{H}^\mu(z),$$

and we take $\varepsilon \downarrow 0$ to recover (5.18). Note that, for convenience, we integrate with respect to $\mathscr{H}^\mu$ over all of $\mathbb{R}^m$ in (5.18), but really the integrand can be non-zero only on image $\varphi(z)$, which is $\mathscr{H}^r - \sigma$-finite. Finally, when $r = \mu$, $\mathscr{H}^{r-\mu} = \mathscr{H}^0$ should be interpreted as the counting measure, and then (5.18) is the usual change of variables formula.

When $\mu = m$, corresponding to the case in which $\partial \varphi$ is full rank, the Jacobian takes a simple form. Let us abuse notation and let $\partial \varphi(p)$ denote the matrix representation of the differential w.r. to fixed orthonormal bases of $T_p N$ and $\mathbb{R}^m$. Then, it turns out that

$$J_m \varphi(p) = \sqrt{\det \partial \varphi(p) [\partial \varphi(p)]^T}. \tag{5.20}$$

(5.20) brings out an interesting relationship between the co-area formula and the Malliavin calculus. Since $\mathscr{H}^m$ on $\mathbb{R}^m$ is Lebesgue measure, it is clear from (5.19) that if

$$v(U) = \int_N 1_{\{\varphi(p) \in U\}} J_m \varphi(p) \, d\mathscr{H}^r(p),$$

then v is absolutely continuous w.r.t. Lebesgue measure. Thus, if $J_m \varphi(p) > 0$ $\mathscr{H}^r$-a.e. or equivalently, if $\partial \varphi(p) [\partial \varphi(p)]^T > 0$ $\mathscr{H}^r$-a.e., then $\mathscr{H}^r \circ \varphi^{-1}$ is absolutely continuous w.r.t. Lebesgue measure also

$$\mathscr{H}^r \circ \varphi^{-1}(U) = \int_N 1_{\{\varphi(p) \in U\}} d\mathscr{H}^r(p).$$

The Malliavin covariance matrix $\langle\!\langle D\psi, D\psi \rangle\!\rangle$ for $\psi: \Omega \to \mathbb{R}^m$ is precisely a generalization of $\partial \varphi(p) [\partial \varphi(p)]^T$. The fact that the a.s. positivity of $\langle\!\langle D\psi, D\psi \rangle\!\rangle$ is related to existence of densities generalizes the fact that $J_m \varphi(p) > 0$ $\mathscr{H}^r$-a.e. implies $\mathscr{H}^r \circ f^{-1} \ll$ Lebesgue measure. There is more than analogy here. Bouleau and Hirsch [3] use the co-area formula to deduce existence of densities from a.s. positivity of $\langle\!\langle D\psi, D\psi \rangle\!\rangle$.

We shall employ the co-area formula to compute the density of X_t w.r.t. an appropriate Hausdorff measure, and also to compute conditional densities of $\Phi(0, t)$ and $\Phi(1, t)$ given X_t for use in Sect. 6. Fix $t \in (0, 1)$ and let $N_t = G_0 e^{-tA} \times e^A G_0 e^{-tA}$. Assume $\dim \mathscr{G}_0 = r$, so that $\dim N_t = 2r$. $(\Phi(0, t), \Phi(1, t))$ takes values in N_t, and, moreover, by the analysis of Proposition 4.6, $\Phi(0, t)$ admits a probability density q_t^0 w.r.t. $\mathscr{H}^r$ on $G_0 e^{-tA}$, and $\Phi(1, t)$ admits a probability density q_t^1 w.r.t. $\mathscr{H}^r$ on $e^A G_0 e^{-tA}$; $q_t^0(u) q_t^1(v)$ is the density w.r.t. $\mathscr{H}^{2r}$ on N_t for

$(\Phi(0, t), \Phi(1, t))$. The function φ in the co-area formula will be replaced by ρ: $N_t \to \mathbb{R}^d$, where

$$\rho(U, V) = [F_0 U + F_1 V]^{-1} f.$$

Then $X_t = \rho(\Phi(0, t), \Phi(1, t))$. ρ is defined everywhere on N_t except on the subvariety $S_t^0 = \{(U, V) \in N_t \mid \det[F_0 U + F_1 V] = 0\}$. S_t^0 is a subset of zero $\mathscr{H}^{2r}$-measure because of our standing assumption that $F_0 \Phi(0, t) + F_1 \Phi(1, t)$ be a.s. invertible.

We want to apply the co-area formula to ρ on N_t. However, ρ is analytic not on N_t, as would be required in Proposition 5.9 but on $N_t - S_t^0$. $N_t - S_t^0$ is a union of connected open analytic manifolds, and we apply Proposition 5.9 on each component and add up. For this it is useful to know that the constant μ does not change from component to component.

Lemma 5.10. *Let*

$$\mu = \sup\{\dim\ [\text{image}\, \partial\rho(U, V)] \mid (U, V) \in N_t \backslash S_t^0\}.$$

Then dim [image $\partial\rho(U, V)] = \mu$, *$\mathscr{H}^{2r}$-almost everywhere on $N_t - S_t^0$.*

Proof. We note that $T_{(U,V)} N_t = \{(C_0 U, C_1 V) \mid C_0, C_1 \in \mathscr{Q}\}$ where $\mathscr{Q}$ is defined in Sect. 4. Also, if $\{C_0 U, C_1 V) \in T_{(U,V)} N_t$,

$$\begin{aligned}&\partial\rho(U, V)(C_0 U, C_1 V)\\ &\quad = -\det(F_0 U + F_1 V)^{-1} [F_0 U + F_1 V]^{-1} [F_0 C_0 U + F_1 C_1 V] \overline{[F_0 U + F_1 V]}\, f\end{aligned}$$

where $\overline{F_0 U + F_1 V} = \det(F_0 U + F_1 V) [F_0 U + F_1 V]^{-1}$ can be extended as an everywhere defined analytic function. Now let $E_1, \ldots, E_r$ be a basis of $\mathscr{Q}$. Then $\{E_1 U, 0), \ldots, (E_r U, 0), (0, E_1 V), \ldots, (0, E_r V)\}$ is a basis of $T_{(U,V)} N_t$ depending analytically on (U, V). Combining these facts, we find that for $(U, V) \in N_t \backslash S_t^0$,

$$\begin{aligned}\text{image}\, \partial\rho(U, V) = \text{span}\, \{&(F_0 U + F_1 V)^{-1} F_0 E_i U \overline{[F_0 U + F_1 V]}\, f,\\ &(F_0 U + F_1 V)^{-1} F_1 E_j V \overline{[F_0 U + F_1 V]}\, f \mid 1 \leqq i, j \leqq r\}.\end{aligned}$$

Let $\{\Delta_i(U, V)\}$ be the collection of $\mu \times \mu$ minors of the $d \times 2r$ matrix

$$[F_0 E_1 U [F_0 U + F_1 V] f \vdots \ldots \vdots F_1 E_r V \overline{F_0 U + F_1 V]}\, f].$$

Then dim image $\partial\rho(U, V) < \mu$ iff $\sum_i \Delta_i^2(U, V) = 0$. But $\sum_i \Delta_i^2(U, V)$ is analytic on N_t. By assumption $\sum_i \Delta_i^2(U, V) > 0$ at some point of N_t. Hence $\sum_i \Delta_i^2(U, V) = 0$ only on a subvariety of N_t of lower dimension than $2r$. $\square$

By applying Proposition 5.9 we now obtain the following result; statements (ii) and (iii) are noted for later use.

Proposition 5.11. *Let* $\mu = \sup\{\dim\,[\text{image}\, \partial\rho(U, V)] \mid (U, V) \in N_t \backslash S_t^0\}$ *and let* $S_t = S_t^0 \cup \{(U, V) \in N_t \mid J_\mu\, \rho(U, V) = 0\}$. *Then*

i) X_t admits a probability density with respect to $\mathscr{H}^\mu$ on $\mathbb{R}^d$ given by

$$p(z,t)=\int_{\rho^{-1}(z)\setminus S_t} [J_\mu \rho(U,V)]^{-1} q_t^1(U)\, q_t^2(V)\, d\mathscr{H}^{2r-\mu}(U,V)$$

ii) If g is a bounded, measurable function on $e^{-tA} G_0$ and

$$\varphi_g(z,t)=p^{-1}(z,t) \int_{\rho^{-1}(z)\setminus S_t} g(U)(J_\mu \rho)^{-1}(U,V)\, q_t^1(U)\, q_t^2(V)\, d\mathscr{H}^{2r-\mu}(U,V)$$

where this is defined, then $\varphi_g(X_t,t)=E[g(\Phi(0,t))\,|\,X_t]$.

iii) If h is a bounded, measurable function on $e^A G_0\, e^{-tA}$ and

$$\psi_h(z,t)=p^{-1}(z,t) \int_{\rho^{-1}(z)\setminus S_t} h(V)(J_\mu \rho)^{-1}(U,V)\, q_t^1(U)\, q_t^2(V)\, d\mathscr{H}^{2r-\mu}(U,V)$$

then $\psi_h(X_t,t)=E[h(\Phi(1,t))\,|X_t]$.

Proof. Let g, h, k be bounded measurable functions. Then using the co-area formula and Lemma 5.10, we deduce:

$$\begin{aligned}
&E[k(X_t)\, g(\Phi(0,t))\, h(\Phi(1,t))]\\
&=\int_N k(\rho(U,V))\, g(U)\, h(V)\, q_t^1(U)\, q_t^2(V)\, d\mathscr{H}^{2r}(U,V)\\
&=\int_{\mathbb{R}^d} k(z)\Big\{\int_{\rho^{-1}(z)\setminus S_t} (J_\mu \rho)^{-1}(U,V)\, g(U)\, h(V)\, q_t^1(U)\, q_t^2(V)\, d\mathscr{H}^{2r-\mu}(U,V)\Big\}\, d\mathscr{H}^\mu(z).
\end{aligned} \tag{5.21}$$

For (i) set $g\equiv h\equiv 1$, for (ii) set $h\equiv 1$, and for (iii) set $g\equiv 1$. $\square$

Recall the definition of BC'' from Corollary 5.3. Corollary 5.3 can now be greatly improved.

Proposition 5.12. *Let* (A.1) *hold. Then X_t admits a probability density with respect to Lebesgue measure iff $\exists(x_0, x_1)\in BC''$ such that*

$$\{F_0\, C_0\, x_0\,|\, C_0\in\mathscr{Q}\}+\{F_1\, C_1\, x_1\,|\, C_1\in\mathscr{Q}\}=\mathbb{R}^d. \tag{5.22}$$

Proof. Assume that (5.22) holds. We shall check that

$$\dim \operatorname{image} \partial\rho(u,v)|_{U,TU}=d$$

for any $U\in G_0\, e^{-tA}$. Since $\mathscr{H}^d=$Lebesgue measure in $\mathbb{R}^d$ this will imply that X_t admits a density by Proposition 5.11(i). Let $(x_0, Tx_0)\in BC''$ satisfy (5.22) and let $V=TU$. Then $U^{-1}x_0=U^{-1}[F_0+F_1\, T]^{-1}f=[F_0\, U+F_1\, V]^{-1}f$. Then, using the characterization of $\partial\rho(U,V)$ given in the proof of Lemma 5.10,

$$\begin{aligned}
&\operatorname{image} \partial\rho(U,TU)\\
&\quad=\{(F_0\, U+F_1\, V)^{-1}\, F_0\, C_0\, U(F_0\, U+F_1\, V)^{-1} f,\\
&\qquad (F_0\, U+F_1\, V)^{-1}\, F_1\, C_1\, V(F_0\, U+F_1\, V)^{-1} f\,|\, C_0, C_1\in\mathscr{Q}\}\\
&\quad=\{(F_0\, U+F_1\, V)^{-1}\, F_0\, C_0\, x_0\,|\, C_0\in\mathscr{Q}\}+\{(F_0\, U+F_1\, V)^{-1}\, F_1\, C_1\, Tx_0\,|\, C_1\in\mathscr{Q}\}\\
&\quad=R^d.
\end{aligned}$$

Conversely, if X_t admits a density then $\mu=d$, since otherwise Proposition 5.11 implies that X_t admits a density with respect to $\mathscr{H}^\mu$ for $\mu<d$ and so the Hausdorff dimension of image $\rho=\mu<d$. Thus there is a $(U, V)\in N$ with $F_0 U+F_1 V$ invertible and $\dim \operatorname{image} \partial\rho(U, V)=d$. $(x_0, x_1)=(U[F_0 U+F_1 V]^{-1}f, \quad V[F_0+F_1 V]^{-1}f)\in BC''$ then satisfies (5.22). □

Finally, we wish to demonstrate how the co-area methodology may also be used to derive expressions for the joint densities of $\{X_t\}$. We again assume (A.1) holds and we consider finding the joint density of (X_s, X_t). On $M=G_0 e^{-sA}\times e^A G_0 e^{-At}\times e^{At}\times G_0 e^{-As}$, define $R(U, V, T)$: $M\to\mathbb{R}^{2d}$ by

$$R(U, V, T)=\begin{bmatrix} F_0 U & F_1 V \\ T & -I \end{bmatrix}^{-1}\begin{bmatrix} f \\ 0 \end{bmatrix}.$$

Then

$$\begin{pmatrix} X_s \\ X_t \end{pmatrix}=R(\Phi(0, s), \Phi(1, t), \Phi(t, s)).$$

By repeating the analysis of Proposition 5.11 on R, we get

$$p(z_1, z_2, t)=\int_{R^{-1}((z_1, z_2))\setminus S} (J_\mu R)^{-1}(U, V, T)\, q_s^0(U)\, \dot{q}_t^1(V)\, q_{s,t}^2(T)\, d\mathscr{H}^{3r-\mu}(U, V, T)$$

for the density of (X_s, X_t) with respect to $\mathscr{H}^\mu$ on $\mathbb{R}^{2d}$, where $\mu=\sup\{\dim(\operatorname{image} \partial R(U, V, T))\mid (U, V, T)\in M\}$, $S=\{(U, V, T)\in M\mid R(U, V, T)$ is undefined or $J_\mu R(U, V, T)=0\}$ and $q_{s,t}^2(T)$ is the density of $\Phi(t, s)$ on $e^{At}K_0 e^{-As}$ w.r.t. to $\mathscr{H}^r$.

§ 6. The Markov Property

6.1 The Markov and Markov Field Properties

In this subsection, we assume that $a, b_1, \ldots, b_k$ and f are deterministic, $a\in L^1(0, 1; \mathbb{R}^d)$, $b_1, \ldots, b_k\in L^2(0, 1; \mathbb{R}^d)$ and $f\in\mathbb{R}^d$. We also assume that (4.1)–(4.2) has a unique solution.

For the process $\{X_t, t\in[0, 1]\}$, two notions of Markov property can be considered. Let us recall their definitions.

Definition 6.1. The process $\{X_t, t\in[0, 1]\}$ is said to be a *Markov process* if for any $t\in(0, 1)$, the σ-algebras $\sigma(X_s; 0\leqq s\leqq t)$ and $\sigma(X_s; t\leqq s\leqq 1)$ are conditionally independent, given X_t; i.e., the past and future are conditionally independent, given the present. □

Definition 6.2. The process $\{X_t, t\in[0, 1]\}$ is said to be a *Markov field* if for any $0\leqq s<t\leqq 1$, the σ-algebras $\sigma(X_r; s\leqq r\leqq t)$ and $\sigma(X_r; 0\leqq r\leqq s)\vee\sigma(X_r; t\leqq r\leqq 1)$ are conditionally independent, given (X_s, X_t); i.e., the process outside (s, t) and the process inside (s, t) are conditionally independent, given (X_s, X_t). □

The Markov field property has been discussed in particular by Jamison [10]. He shows that any Markov process $\{X_t\}$ is a Markov field. But the converse needs not be true, as we will see below, except when either X_0 or X_1 is deterministic.

Our solution $\{X_t, t\in[0, 1]\}$ is not going to possess the Markov property in general. Indeed, in the particular case of the periodic boundary condition (2.2″), X_0 and X_1 are not conditionally independent given X_t. One may think that the Markov field property is better suited for our system, since in a sense the flow is running from the two endpoints $t=0$ and $t=1$. Unfortunately, we have not been able to decide whether or not the solution to the system is always a Markov field.

Let us first consider three cases where the solution is a Markov process. We will use the following notation: for $0\leqq s<t\leqq 1$,

$$\mathscr{F}_t^s=\sigma\{W_r-W_s;\ s\leqq r\leqq t\}, \qquad \mathscr{F}_t=\mathscr{F}_t^0.$$

Theorem 6.3. *Suppose that one of the following conditions is satisfied:*

(i) $F_1=0$

(ii) $F_0=0$

(iii) $B_1=\ldots=B_k=0$, and $\operatorname{Im} F_0\cap\operatorname{Im} F_1=\{0\}$. Then $\{X_t, t\in[0, 1]\}$ *is a Markov process.*

Proof. Under either (i) or (ii), the result is well known. Let us consider the condition (iii).

We have to show that for $0\leqq t<r\leqq 1$, X_r is conditionally independent of $\sigma(X_s;\ 0\leqq s\leqq t)$, given X_t.

Since $B_1=\ldots=B_k=0$, $\Phi(t, s)$ is deterministic, and $\{X_r, t\in[0, 1]\}$ is a Gaussian process. The formula:

$$X_t=[F_0\,\Phi(0, t)+F_1\,\Phi(1, t)]^{-1}\left(f+F_0\int_0^t \Phi(0, s)\circ d V_s-F_1\int_t^1 \Phi(1, s)\circ d V_s\right) \tag{6.1}$$

can be rewritten in the form:

$$X_t=c+C(F_0\,\xi_t+F_1\,\eta^t)$$

where $c\in\mathbb{R}^d$, C is a $d\times d$ invertible matrix (of course, both depend on t), ξ_t is a $\mathscr{F}_t$ measurable Gaussian random vector, η^t is a $\mathscr{F}_1^t$ measurable Gaussian random vector. The condition $\operatorname{Im} F_0\cap\operatorname{Im} F_1=\{0\}$ implies that $\sigma(X_t)=\sigma(\bar{\xi}_t, \bar{\eta}^t)$, where $\bar{\xi}_t=F_0\,\xi_t$, $\bar{\eta}^t=F_1\,\eta^t$.

$$X_r=\Phi(r, t)\,X_t+\int_t^r \Phi(r, s)\circ d V_s.$$

Since $\left(\int_t^r \Phi(r, s) \circ dV_s, \bar{\eta}^t\right)$ is a $\mathscr{F}_1^t$ measurable Gaussian random vector,

$$\int_t^r \Phi(r, s) \circ dV_s = C^1 \bar{\eta}^t + v_r^t$$

where v_r^t is $\mathscr{F}_1^t$ measurable and independent of $\bar{\eta}^t$, hence independent of $\mathscr{F}_t \vee \sigma(X_t)$. Since $\bar{\eta}^t$ is $\sigma(X_t)$ measurable, we have written X_r as a function of X_t and v_r^t, where v_r^t is independent of $\sigma(X_s; 0 \leqq s \leqq t)$. The result follows. □

Remark 6.4. As was noted in §2, the condition $\operatorname{Im} F_0 \cap \operatorname{Im} F_1 = \{0\}$ is equivalent to the fact that the boundary condition (4.2) can be rewritten in the form of the two-point boundary condition (2.2′). Thus our result is consistent with that of Russek [18], who studies gaussian solutions of a different class of stochastic boundary value problems.

The following counterexample shows that the condition $\operatorname{Im} F_0 \cap \operatorname{Im} F_1 = \{0\}$ does not imply the Markov property in the non-gaussian case: $A=0$, $k=1$, $B=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$, $a=b=0$, $F_0=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$, $F_1=\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$, $f=\begin{pmatrix}1\\ 1\end{pmatrix}$. In this case, the solution is given by:

$$X_t = \begin{pmatrix} e^{W_t}(1+e^{-W_1}) \\ -e^{2W_t} e^{-W_1} \end{pmatrix}.$$

Note that $\sigma(X_0) = \sigma(X_1) = \sigma(W_1)$. Since $\sigma(W_1)$ is not contained in $\sigma(X_t)$ (indeed, $\sigma(X_t) \subsetneqq \sigma(W_t, W_1 - W_t)$), clearly X_0 and X_1 are not conditionally independent, given X_t. □

Let us recall that for $0 \leqq s < t \leqq 1$,

$$X_t = \Phi(t, s) X_s + \int_s^t \Phi(t, r) \circ dV_r.$$

This formula motivates the following definitions. To any pair s, t with $0 \leqq s < t \leqq 1$, we associate the σ-algebras:

$$\bar{\mathscr{G}}_{s,t} = \sigma\left(\Phi(t, s), \int_s^t \Phi(t, r) \circ dV_r\right)$$

$$\mathscr{G}_{s,t} = \sigma(X_s, X_t) \vee \bar{\mathscr{G}}_{s,t}$$

$$\mathscr{H}_{s,t}^i \bigvee_{\{u, v; s \leqq u < v \leqq t\}} \mathscr{G}_{u,v}$$

$$\mathscr{H}_{s,t}^e = \bigvee_{\{u, v; u < v; u, v \in [0, s] \cup [t, 1]\}} \mathscr{G}_{u,v}.$$

In these definitions, i stands for "interior", e for "exterior".

We have the following kind of extended or weak Markov field property:

Theorem 6.4. *For any* $0 \leqq s < t \leqq 1$, $\mathscr{H}^i_{s,t}$ *and* $\mathscr{H}^e_{s,t}$ *are conditionally independent, given* $\mathscr{G}_{s,t}$.

Proof. It is sufficient to show that for any event $H \in \mathscr{H}^i_{s,t}$

$$P(H/\mathscr{H}^e_{s,t}) = P(H/\mathscr{G}_{s,t}). \tag{6.2}$$

Clearly, for $s \leqq u < v \leqq t$, $\overline{\mathscr{G}}_{u,v} \subset \mathscr{F}^s_t$ and $\sigma(X_u, X_v) \subset \sigma(X_s, X_t) \vee \mathscr{F}^s_t$. Consequently, $\mathscr{H}^i_{s,t} \subset \mathscr{G}_{s,t} \vee \mathscr{F}^s_t$. It then suffices, using the monotone class theorem, to show (6.2) for any $H \in \mathscr{F}^s_t$. From (6.1), we conclude that X_s and X_t are $\overline{\mathscr{G}}_{s,t} \vee \mathscr{F}_s \vee \mathscr{F}^t_1$ measurable, and therefore it is easily checked that:

$$\overline{\mathscr{G}}_{s,t} \subset \mathscr{G}_{s,t} \subset \mathscr{H}^e_{s,t} \subset \overline{\mathscr{G}}_{s,t} \vee \mathscr{F}_s \vee \mathscr{F}^t_1.$$

Since $\overline{\mathscr{G}}_{s,t} \subset \mathscr{F}^s_t$, it follows easily from the independence between $\mathscr{F}^s_t$ and $\mathscr{F}_s \vee \mathscr{F}^t_1$ that for any $H \in \mathscr{F}^s_t$,

$$P(H/\overline{\mathscr{G}}_{s,t} \vee \mathscr{F}_s \vee \mathscr{F}^t_1) = P(H/\overline{\mathscr{G}}_{s,t})$$

(6.2) follows, using the following lemma which is an easy consequence of the definition of conditional expectation:

Lemma 6.5. *Let* $\mathscr{H}_1 \subset \mathscr{H}_2 \subset \mathscr{H}_3 \subset \mathscr{H}_4$ *be sub σ-algebras of* $\mathscr{F}$. *For any* $G \in \mathscr{F}$,

$$P(G/\mathscr{H}_4) = P(G/\mathscr{H}_1) \Rightarrow P(G/\mathscr{H}_3) = P(G/\mathscr{H}_2). \quad \square$$

Using again Lemma 6.5, we immediately deduce from Theorem 6.4 the:

Corollary 6.6. *Suppose that for* $0 \leqq s < t \leqq 1$, $\Phi(t,s)$ *and* $\int_s^t \Phi(t,r) \circ dV_r$ *are* $\sigma(X_s, X_t)$ *measurable. Then* $\{X_t, t \in [0,1]\}$ *is a Markov field.* $\square$

It then follows:

Corollary 6.7. *In each of the following two cases,* $\{X_t, t \in [0,1]\}$ *is a Markov field:*

(i) *(Gaussian case):* $B_1 = \ldots = B_k = 0$

(ii) $a = b_1 = \ldots = b_k = 0$, *and* $\Phi(t,s)$ *is a diagonal matrix, for any* $s, t \in [0,1]$ (the latter holds in particular if $d = 1$). $\square$

Each of the cases (i) or (ii), together with the periodic boundary condition (2.2″), provides an example for a Markov field which is not a Markov process.

Here are two examples of systems with Markov field solutions.

Example 6.1. If $A = 0$, $a = 0$, $k = 1$, B is diagonal and invertible, and b is constant, then the solution $\{X_t\}$ is a Markov field, because Corollary 6.7 applies to the system which $Y_t = X_t + B^{-1} b$ solves.

Example 6.2. Let $\{X_t, t \in [0, 1]\}$ be the two dimensional process which solves

$$dX_t = AX_t\,dt + \sum_1^k B_i X_t \circ dW_t^i$$
$$X_0^1 = 0 \qquad X_1^1 = 1.$$

More precisely, assume that this problem is well-posed. Then $X_t = (Y_1^1)^{-1} Y_t$, where $\{Y_t, t \in [0, 1]\}$ is the solution to the same equation, but with initial values $Y_0^1 = 0$, $Y_0^2 = 1$. Note that Y_t is an adapted process. We wish to show that $\{X_t\}$ is a Markov field; that is, for any $\phi \in C_0(R^2)$ and $0 \leqq r < s < t \leqq 1$, we want to show

$$E[\phi(X_s)/X_u, u \in (r, t)^c] = E[\phi(Xs)/X_r, X_t].$$

Let $C = \Phi(s, r)$ and $D = \Phi(t, r)$. We need two facts; i) for any bounded, Borel ψ on R^4, $E[\psi(C)/Y_r, Y_t]$ is $\sigma(X_r, X_t)$-measurable up to sets of measure 0; ii) $\sigma\{X_u;\ u \in (r, t)^c\} = \sigma\{Y_u;\ u \in (r, t)^c\}$. ii) is a simple consequence of the definition of $\{X_t\}$. To prove i), first note that $E[\psi(C)/Y_r, Y_t] = E[E[\psi(C)/D] \mid Y_r, Y_t]$ because of the independent of Y_r from (C, D). Hence it suffices to prove $E[\rho(D)/Y_r, Y_t]$ is $\sigma(X_r, X_t)$ measurable up to zero measure sets. Define $H(x, y)$ so that $H(x, Dx) = E[\rho(D)/Dx]$. We may assume that $H(\alpha x, \alpha y) = H(x, y)$ for any $\alpha \neq 0$.

However, $H(Y_r, Y_t)$ is a version of $E[\rho(D)/Y_r, Y_t]$ since, from the independence of Y_r and D,

$$\begin{aligned} E[1_{U_1}(Y_r)1_{U_2}(Y_t)\rho(D)] &= E[1_{U_1}(Y_r)E[1_{U_2}(Dx)\rho(D)]|_{x=Y_r}] \\ &= E[1_{U_1}(Y_r)1_{U_2}(Y_t)H(Y_r, Y_t)]. \end{aligned}$$

Since $(X_r, X_t) = (Y_1^1)^{-1}(Y_r, Y_t)$, $H(Y_r, Y_t) = H(X_r, X_t)$ a.e. thereby proving i). Now let ϕ be given and set $G(w) = E[\phi(Cw)/Y_r, Y_t]$. We will show that $E[\phi(X_s)/X_u, u \in (r, t)^c] = G(X_r)$. By i) it follows that $G(X_r)$ is $\sigma(X_r, X_t)$-measurable, and so $\{X_t, t \in [0, 1]\}$ is a Markov field. Thus, to finish, observe from ii) and the Markov field property of $\{Y_t, t \in [0, 1]\}$ that

$$\begin{aligned} E[\phi(X_s)/X_u, u \in (r, t)^c] &= E[\phi(CY_r/Y_1^1)/Y_u, u \in (r, t)^c] \\ &= E[\phi(CY_r/y)/Y_u, u \in (r, t)^c]|_{y=Y_1^1} \\ &= E[\phi(CY_r/y)/Y_r, Y_t]|_{y=Y_1^1} \\ &= E[\phi(Cz/y)/Y_r, Y_t]|_{y=Y_1^1, z=Y_r} \\ &= E[\phi(Cw)/Y_r, Y_t]|_{w=X_r}. \end{aligned}$$

With the notations introduced in the proof of Theorem 6.4, Corollary 6.6 says that a sufficient condition for $\{X_t\}$ to be a Markov field is that $\bar{\mathscr{G}}_{s,t} \subset \sigma(X_s, X_t)$, for any $0 \leqq s < t \leqq 1$. We want now to establish a weaker sufficient condition. For the sake of completeness, we state a similar sufficient condition for the Markov property.

Proposition 6.8. (i) *If for any* $t\in(0,1)$, $\bar{\mathscr{G}}_{0,t}$ *and* $\bar{\mathscr{G}}_{t,1}$ *are conditionally independent given* X_t, *then* $\{X_t, t\in[0,1]\}$ *is a Markov process.*

(ii) *If for any* $0\leqq s<t\leqq 1$, $\bar{\mathscr{G}}_{0,s}\vee\bar{\mathscr{G}}_{t,1}$ *and* $\bar{\mathscr{G}}_{s,t}$ *are conditionally independent, given* (X_s, X_t), *then* $\{X_t, t\in[0,1]\}$ *is a Markov field.*

Proof. We prove only (ii). The proof of (i) is analogous. It suffices to show that for any $0\leqq s<t\leqq 1$, and any pair of events $G_i\in\sigma\{X_u, u\in[s,t]\}$ and $G_e\in\sigma\{X_u, u\in[0,1]\backslash(s,t)\}$, we have,

$$P(G_i\cap G_e/X_s, X_t)=P(G_i/X_s, X_t)\,P(G_e/X_s, X_t). \tag{6.3}$$

Since

$$\sigma\{X_u, u\in[s,t]\}\subset\sigma(X_s, X_t)\vee\mathscr{F}_t^s,\ \sigma\{X_u, u\in[0,1]\backslash(s,t)\}\subset\sigma(X_s, X_t)\vee\mathscr{F}_s\vee\mathscr{F}_1^t.$$

It suffices from the monotone class theorem to show (6.3) for $G_i\in\mathscr{F}_t^s$, $G_e\in\mathscr{F}_s\vee\mathscr{F}_1^t$. Since $\bar{\mathscr{G}}_{s,t}\subset\mathscr{F}_t^s$, $\bar{\mathscr{G}}_{0,s}\vee\bar{\mathscr{G}}_{t,1}\subset\mathscr{F}_s\vee\mathscr{F}_1^t$, and $\mathscr{F}_t^s$ and $\mathscr{F}_s\vee\mathscr{F}_1^t$ are independent, one obtains, with $G_i\in\mathscr{F}_t^s$, $G_e\in\mathscr{F}_s\vee\mathscr{F}_1^t$,

$$\begin{aligned}P(G_i\cap G_e/X_s, X_t)&=E[P(G_i\cap G_e/\bar{\mathscr{G}}_{s,t}\vee\bar{\mathscr{G}}_{0,s}\vee\bar{\mathscr{G}}_{t,1})/X_s, X_t]\\&=E[P(G_i/\bar{\mathscr{G}}_{s,t})\,P(G_e/\bar{\mathscr{G}}_{0,s}\vee\bar{\mathscr{G}}_{t,1})/X_s, X_t]\end{aligned}$$

(6.3) then follows, if $\bar{\mathscr{G}}_{0,s}\vee\bar{\mathscr{G}}_{t,1}$ and $\bar{\mathscr{G}}_{s,t}$ are conditionally independent, given (X_s, X_t). □

We will see below that the sufficient condition of Proposition 6.8 (ii) is not always satisfied.

6.2 Analytical Conditions for Conditional Independence

Throughout this subsection, we assume

$$a=b_1=\ldots=b_k=0. \tag{A.1}$$

If (A.1) holds, Proposition 6.8 implies that $\{X_t\colon t\in[0,1]\}$ is a Markov process if

$$\begin{gathered}\Phi(0,t)\ \text{and}\ \Phi(1,t)\\ \text{are conditionally independent given } X_t\end{gathered} \tag{6.4}$$

and $\{X_t\colon t\in[0,1]\}$) is a Markov field if

$$\begin{gathered}(\Phi(0,s),\Phi(1,t))\ \text{and}\ \Phi(t,s)\\ \text{are conditionally independent given } (X_s, X_t).\end{gathered}$$

In this section we apply the co-area formula developed in § 5.2 to give necessary and sufficient conditions for either (6.4) or (6.5) to hold when we assume

$$\text{Image } F_0\cap\text{Image } F_1=\{0\}. \tag{A.2}$$

Thus, we will assume (A.2) also for the rest of this section. By the remark in § 2, we may and do assume that F_0 and F_1 are in the form

$$F_0=\begin{bmatrix}F_0'\\0\end{bmatrix},\qquad F_1=\begin{bmatrix}0\\F_1'\end{bmatrix}$$

where F_0' is a $l\times d$ matrix of rank l and F_1' is a $(d-l)\times d$ matrix of rank $d-l$. We shall also write $f=\begin{bmatrix}f_0\\f_1\end{bmatrix}$.

Our technique will be to use the co-area formula to compute explicitly the conditional densities of $\Phi(0,t)$ given X_t, of $\Phi(1,t)$ given X_t, etc. We treat in detail the necessary and sufficient conditions for (6.4), as the notation is simpler in this case, and we shall just state the analogous result for (6.5). For background and notation, the reader should consult § 5.2, particularly Proposition 5.11. In addition, we note that for $z\in\mathbb{R}^d$ $\rho^{-1}(z)=L_z^0\times L_z^1$ where $L_z^0=\{U\in G_0\,e^{-At}\,|\,F_0'\,Uz=f_0\}$, $L_z^1=\{V\in e^A\,G_0\,e^{-At}\,|\,F_1'\,Vz=f_1\}$. We let $v_0(z)=\dim L_z^0$ and $v_1(z)=\dim L_z^1$. Finally let P_{X_t} denote the probability distribution of X_t; recall that $dP_{X_t}/d\mathscr{H}^\mu=p(z,t)$ for μ and $p(z,t)$ given in Proposition 5.11.

Theorem 6.9. *Let* $\mu=\sup\{\dim \operatorname{image}\partial\rho(U,V)\,|\,(U,V)\in N_t\backslash S_t^0\}$ *and set* $S_t=S_t^0\cup\{(U,V)\,|\,J_\mu\,\rho(U,V)=0\}$. *Then* $\Phi(0,t)$ *and* $\Phi(1,t)$ *are conditionally independent given* X_t *if and only if there exist functions* $\alpha(u,z)$ *and* $\beta(v,z)$ *such that for* P_{X_t}*-almost every* z

$$(J_\mu\,\rho)^{-1}(U,V)\,1_{S_t^c}(U,V)=\alpha(U,z)\,\beta(V,z)\tag{6.6}$$

$q_t^0(U)\,q_t^1(V)\,d\mathscr{H}^{2r-\mu}(U,V)$*-almost everywhere on* $\rho^{-1}(z)$.

Remark. While t does not appear explicitly in (6.6), $J_\mu\,\rho(u,v)$ depends very much on t because it is defined in terms of $\partial\rho$ on N_t. Thus the condition must be checked at each t.

Proof. Assume that (6.6) is true. We want to show

$$E[g(\Phi(0,t))\,h(\Phi(1,t))\,|\,X_t]=E[g(\Phi(0,t))\,|\,X_t]\,E[h(\Phi(1,t))\,|\,X_t]$$

for arbitrary, bounded, measurable g and h. Let

$$p^0(z)=\int_{L_z^0}\alpha(U,z)\,q_t^0(U)\,d\mathscr{H}^{v_0(z)}(U)$$

$$p^1(z)=\int_{L_z^1}\beta(V,z)\,q_t^1(V)\,d\mathscr{H}^{v_1(z)}(V).$$

From Proposition 5.11, we find $p(z,t)=p^0(z)\,p^1(z)$ and

$$\varphi_g(z,t)=\int_{L_z^0}\alpha(U,z)\,g(U)\,q_t^0(U)\,d\mathscr{H}^{v_0(z)}(U)/p^0(z),$$

$$\psi_h(z,t)=\int_{L_z^1}\beta(V,z)\,h(V)\,q_t^1(V)\,d\mathscr{H}^{v_1(z)}(V)/p^1(z).$$

P_{X_t}-a.e. However, b from (5.21), $\rho^{-1}(z)=L_z^0\times L_z^1$, and $d\mathscr{H}^{2r-\mu}(U,V)=d\mathscr{H}^{\nu_0(z)}(U)\times d\mathscr{H}^{\nu_1(z)}(V)$,

$$\begin{aligned}
&E[g(\Phi(0,t))h(\Phi(1,t))\mid X_t]\\
&\quad=\frac{\int\limits_{L_z^0}\int\limits_{L_z^1} g(U)h(V)\alpha(U,z)\beta(V,z)q_t^0(U)q_t^1(V)\,d\mathscr{H}^{2r-\mu}(U,V)}{p^0(z)\,p^1(z)}\Bigg|_{z=X_1}\\
&\quad=\varphi_g(X_t,t)\,\psi_h(X_t,t)\\
&\quad=E[g(\Phi(0,t))\mid X_t]\,E[h(\Phi(1,t))\mid X_t].
\end{aligned}$$

Conversely, assume that $\Phi(0,t)$ and Φ(t, t) are conditionally independent given X_t. Let

$$\begin{aligned}
\theta_{g,h}(z,t)&=p^{-1}(z,t)\\
&\quad\cdot\int\limits_{\rho^{-1}(z)} g(U)h(V)1_{S^c}(U,V)(J_\mu\rho)^{-1}(U,V)q_t^0(U)q_t^1(V)\,d\mathscr{H}^{2r-\mu}(U,V),
\end{aligned}$$

Then, since

$$E[g(\Phi(0,t))\mid X_t]\,E[h(\Phi(1,t))\mid X_t]=E[g(\Phi(0,t))h(\Phi(1,t))\mid X_t]=\theta_{g,h}(X_t,t)$$

a.s.

$$\varphi_g(z,t)\,\psi_h(z,t)=\theta_{g,h}(z,t) \tag{6.7}$$

for P_{X_t}-almost all z. In fact, we can choose one set $\mathcal{O}$, with $P_{X_t}(\mathcal{O})=1$, such that (6.7) holds for all continuous g and h with compact support and $z\in\mathcal{O}$. Indeed, the set of compactly supported continuous g and h is separable in sup-norm. If $\{g_j,h_i\}$ is a separating set, $\mathcal{O}=\{z\mid(5.8)$ holds for every $g_j,h_i\}$. Fix $z\in\mathcal{O}$. Then

$$\varphi_g(z,t)\,\psi_h(z,t)=p^{-2}(z,t)\int\limits_{L_z^0}\int\limits_{L_z^1} g(U)h(V)\eta_z(U)\lambda_z(V)\,q_t^0(U)\,q_t^1(V)\,d\mathscr{H}^{2r-\mu}(U,V)$$

where

$$\begin{aligned}
\eta_z(U)&=\int\limits_{L_z^0}(J_\mu\rho)^{-1}(U,V)1_{S^c}(U,V)q_t^1(V)\,d\mathscr{H}^{\nu_1(z)}(V)\\
\lambda_z(V)&=\int\limits_{L_z^1}(J_\mu\rho)^{-1}(U,V)1_{S^c}(U,V)q_t^0(U)\,d\mathscr{H}^{\nu_0(z)}(U).
\end{aligned}$$

For continuous and compactly supported g and h, this must equal $\theta_{g,h}(z,t)$ and this can happen only if

$$1_{S^c}(U,V)(J_\mu\rho)^{-1}(U,V)=\eta_z(U)\lambda_z(V)p(z,t)$$

$q_t^0(U)\,q_t^1(V)\,d\mathscr{H}^{2r-\mu}(U,V)$ almost everywhere. □

A similar analysis may be applied to obtain necessary and sufficient conditions for the conditional independence of $\Phi(t,s)$ and $(\Phi(0,t),\Phi(1,t))$ given (X_s,X_t). We shall only state the result since the proof differs only in requiring more complicated notation. Let $M_{s,t}=G_0\, e^{-sA}\times e^{A}\, G_0\, e^{-At}\times e^{At}\, G_0\, e^{-As}$ and define

$$R(U,V,T)=\begin{bmatrix} F_0'U & 0 \\ 0 & F_1'V \\ T & -I \end{bmatrix}^{-1}\begin{bmatrix} f_0 \\ f_1 \\ 0 \end{bmatrix}.$$

If $\dim \mathcal{Q}=r$, $\dim M=3r$. Let $q_s^1(U)\, q_t^2(V)\, q_{s,t}^3(T)$ denote the density w.r.t. $\mathcal{H}^{3r}$ on M of $(\Phi(0,s), \Phi(1,t), \Phi(t,s))$. It is not hard to see that if $F_0+F_1\,\Phi(1,0)$ is invertible a.s., then so is

$$\begin{bmatrix} F_0'\,\Phi(0,s) & 0 \\ 0 & F_1'\,\Phi(1,t) \\ \Phi(t,s) & -I \end{bmatrix}.$$

Hence R is defined $\mathcal{H}^{3r}$-almost everywhere on M. R is chosen precisely so that

$$\begin{bmatrix} X_s \\ X_t \end{bmatrix}=R(\Phi(0,s), \Phi(1,t), \Phi(t,s)).$$

Theorem 6.10. *Let* $\mu=\sup\{\dim \text{image}\, \partial R(U,V,T)\,|\,(U,V,T)\in M\}$ *and let* $S_{s,t}=\{(U,V,T)\in M_{s,t}\,|\,R(U,V,T)$ *is undefined or* $J_\mu R(U,V,T)=0\}$ *Then* $(\Phi(0,s), \Phi(1,t))$ *is conditionally independent of* $\Phi(t,s)$ *given* (X_s, X_t) *iff there are functions* $\alpha(U,V,r)$ *and* $\beta(T,r)$ *such that*

$$1_{S^c}(U,V,T)\,[J_\mu R(U,V,T)]^{-1}=\alpha(U,V,r)\,\beta(T,r)$$

for $q_s^0(U)\, q_t^1(V)\, q_{s,t}^2(T)\, d\mathcal{H}^{3r-\mu}(U,V,T)$ *almost all* $(U,V,T)\in R^{-1}(r)$ *and* P_{X_s,X_t}*-almost all* $r\in\mathbb{R}^{2d}$.

Example 6.11. Let

$$dX_t=\sum_1^k B_i X_t\circ dW_t^i, \quad F_0'X_0=f_0, \quad F_1'X_1=f_1 \tag{6.8}$$

where

i) $\mathcal{Q}=$ Lie algebra $\{B_1,\ldots,B_k\}$,
$=\left\{\text{upper triangular matrices} \begin{bmatrix} * & \ddots & * \\ 0 & & * \end{bmatrix}\right\}$,
ii) $F_0'=[F_{01}'\,F_{02}']$ where $F_{01}'\in\mathbb{R}^{l^2}$ and $\det F_{01}'\neq 0$,
iii) $F_1'=[0\colon F_{12}']$ where $F_{12}'\in\mathbb{R}^{(d-l)^2}$ and $\det F_{12}'\neq 0$, and
iv) If $g:=[F_0+F_1]^{-1}f=(g_1,\ldots,g_d)^T$, $g_d\neq 0$.

We shall apply Theorem 6.9 to show that solutions to (6.8) are Markov. In general these solutions do not satisfy $\sigma(\Phi(t,s))\subset\sigma(X_s,X_t)$, so they give examples of Markov processes, hence Markov fields not satisfying the sufficient condition of Corollary 6.6.

In this case, it can be checked that

$$[J_d\,\rho(U, V)]^{-1}\, 1_{S^c}(U, V) = |\det F'_{01}\, U_1|\,|\det F'_{12}\, V_3|\, \gamma(z)$$

on $\rho^{-1}(z)$ where $\gamma(z)$ is a function only of z, and

$$U = \begin{bmatrix} U_1 & U_2 \\ 0 & U_3 \end{bmatrix} \quad V = \begin{bmatrix} V_1 & V_2 \\ 0 & V_3 \end{bmatrix}.$$

Thus the factorization criterion is met, and X_t is a Markov process.

As a particular example, consider

$$dX_t = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} X_t \circ dW_t^1 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} X_t \circ dW_t^2 \tag{6.9}$$

with $X_0^1 + X_0^2 = 1$ $X_1^2 = 1$.

Then

$$\Phi(t, s) = \begin{pmatrix} e^{W_t^1 - W_s^1} & \alpha_t^s \\ 0 & e^{W_t^2 - W_s^2} \end{pmatrix}$$

and

$$(X_t = e^{W_t^1}(1 - e^{-W_1^2} + \alpha_t^0\, e^{-W_1^2}),\ e^{W_t^2 - W_1^2})$$

where $\alpha_t^s = e^{W_t^1} \int_s^t e^{-W_u^1} e^{W_u^2 - W_s^2}\, dW_u^1$. One can easily see that $\sigma(X_s, X_t)$ does not contain $\sigma(\Phi(t, s))$.

Example 6.12. We shall use Theorem 6.10 to give an example of a two-point boundary value problem such that $(\Phi(0, s), \Phi(1, t))$ and $\Phi(t, s)$ are not conditionally independent. Let X_t solve (6.9), but now with the boundary conditions

$$X_0^1 + X_0^2 = 1 \qquad X_1^1 + X_1^2 = 1.$$

For this problem we calculate the following results; the notation is that of Theorem 6.10

a) $M_{t,s} = G_0 \times G_0 \times G_0$ where

$$G_0 = \left\{ \begin{bmatrix} u_1 & u_2 \\ 0 & U_3 \end{bmatrix} \,\middle|\, u_1, u_3 > 0, u_2 \in \mathbb{R} \right\}$$

b) $R^{-1}(r) = \{U \mid u_1 r_1 + (u_1 + u_3) r_2 = 1,\ u_1, u_3 > 0\}$
$\cdot \{V \mid v_1 r_3 + (v_2 + v_3) r_4 = 1,\ v_1, v_3 > 0\}$
$\cdot \{T \mid t_3 = r_4/r_2,\ t_1 r_1 + t_2 r_3 = r_3,\ t_1 > 0\}$

c) If $R(U, V, T) = r$, $J_4^{-1}\, R(U, V, T)\, 1_{S^c}(U, V, T)$
$= (u_1 - t_1 v_1)\, 1_{\{u_1 - t_1 v_1 \neq 0\}} \sqrt{(r_1^2 + 2r_2^2)(r_1^2 + r_2^2)\, r_2^2 (r_3^2 + 2r_4^2)}.$

Clearly, the factorization criterion of Theorem 6.10 is not met because u_1, t_1, v_1 are independent variables on $R^{-1}(r)$; in fact, by b) $\{(u_1, t_1, v_1) | u_1, t_1, v_1 > 0\}$ parameterizes $R^{-1}(r)$.

Notice that we have not shown that $\{X_t\}$ is not a Markov field. Using the co-area formula, it is possible to compute explicit representations of joint densities of $\{X_t\}$. However, these representations involve the probability densities of $\Phi(s, t)$, which are not known explicitly. Therefore we have not been able to see whether $\{X_t\}$ in this example is a Markov field.

References

1. Adams, M.B., Willsky, A.S., Levy, B.C.: Linear estimation of boundary value stochastic processes. IEEE Trans. Autom. Control, AC-**29**, 803–821 (1984)
2. Bismut, J-M.: Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions. Z. Wahrscheinlichkeitstheor. Verw. Geb. **56**, 469–505 (1981)
3. Bouleau, N., Hirsch, F.: Propriétés d'absolue continuité dans les espaces de Dirichlet et applications aux équations differentielles, in Séminaire de Probabilités XX. (Lect. Notes Math., vol. 1204) Berlin Heidelberg New York: Springer 1986
4. Chay, S.C.: On quasi-Markov random fields. J. Multivariate Anal. **2**, 14–76 (1972)
5. Cinlar, E., Wang, J.G.: Random circles and fields on circles. Preprint
6. Ethier, S.N., Kurtz, T.G.: Markov processes; characterization and convergence. New York: Wiley 1986
7. Federer, H.: Geometric Measure Theory. Berlin Heidelberg New York: Springer 1969
8. Helgason, S.: Differential geometry, Lie groups, and symmetric spaces. New York: Academic Press 1978
9. Jamison, B.: Reciprocal processes, the stationary gaussian case. Ann. Math. Stat. **41**, 1624–1630 (1970)
10. Jamison, B.: Reciprocal processes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **30**, 65–86 (1974)
11. Jeulin, T.: Semi-martingales et grossissement d'une filtration. (Lect. Notes Math., vol. 833) Berlin Heidelberg New York: Springer 1980
12. Jeulin, T., Yor, M. (eds.): Grossissement de filtrations; exemples et applications. Lect. Notes Math. 1118. Berlin Heidelberg New York: Springer 1985
13. Krener, A.: Reciprocal processes and the stochastic realization problem for acausal systems, in C.I. Byrnes and A. Lindquist, eds., Modelling, identification, and robust control. Amsterdam New York: Elsevier-North Holland 1986
14. Kwakernaak, H.: Periodic linear differential stochastic processes. SIAM J. Control Optimization **13**, 400–413 (1975)
15. Meyer, P.-A.: Un cours sur les intégrales stochastiques Séminaire de Probabilités X. (Lect. Notes in Math., vol. 511) Berlin Heidelberg New York: Springer 1976
16. Nualart, D., Pardoux, E.: Stochastic calculus with anticipating integrands. Probab. Th. Rel. Fields **78**, 535–581 (1988)
17. Ocone, D., Párdoux, E.: A generalized Itô-Ventzell formula: Application to a class of anticipating stochastic differential equations. Ann. IHP **25**, 39–71 (1989)
18. Russek, A.: Gaussian n-Markovian processes and stochastic boundary value problems. Z. Wahrscheinlichkeitstheor. Verw. Geb. **53**, 117–122 (1980)

Received November 30, 1987