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Homogenization of a nonlinear random parabolic partial differential equation

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Abstract

The aim of this work is to show how to homogenize a semilinear parabolic second-order partial differential equation, whose coefficients are periodic functions of the space variable, and are perturbed by an ergodic diffusion process, the nonlinear term being highly oscillatory. Our homogenized equation is a parabolic stochastic partial differential equation.

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1. Introduction

The aim of this work is to study the limit as $\varepsilon \to 0$ of the solution u^ε of the second-order semilinear parabolic PDE

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}\right) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t,x)\right),$$

$$(t,x) \in (0,T) \times \mathbb{R}^n, \quad u^\varepsilon(0,x) = u_0(x).$$

The main assumptions are the periodicity (of period one in each direction) of a_{ij} and g with respect to their first variable, the fact that $\{\xi_t, t \geqslant 0\}$ is a d-dimensional

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ergodic diffusion process with a unique invariant measure π, and the following centering condition for g:

$$\int_{\mathbf{T}^n}\int_{\mathbb{R}^d} g(z,y,u)\,\mathrm{d}z\pi(\mathrm{d}y)=0 \quad \forall u\in\mathbb{R}.$$

Here and further below, $\mathbf{T}^n$ denotes the n-dimensional torus, $\mathbf{T}^n=\mathbb{R}^n/\mathbf{Z}^n$. We shall identify periodic functions with functions defined on $\mathbf{T}^n$.

Our equation is a particular model of random homogenization, where the stochastic perturbation fluctuates as time evolves, in contradiction with the more traditional model where the coefficients are time invariant stationary random fields. Note also that the equation is nonlinear and that the nonlinear term is highly oscillating. For the basic results on homogenization of periodic and random equations, we refer, respectively, to Bensoussan et al. (1978), and Jikov et al. (1994).

The same problem, with g linear, has been considered by Campillo et al. (2001). Note also that the same problem, without the appearance of the process $\{\xi_t\}$, has been studied by Pardoux (1999), and without the dependance upon x/ε, by Bouc and Pardoux (1984). It follows clearly from the last quoted work that the limit of u^ε as $\varepsilon\to 0$ should satisfy a stochastic partial differential equation. This is our main result. Note, however, that our approach is not just a combination of the techniques in Pardoux (1999) and Bouc and Pardoux (1984). We need to introduce correctors of a new type, which depend on the whole trajectory of the process $\{\xi_t\}$ after time t/ε^2. The method of proofs in Pardoux (1999) is purely probabilistic, and makes use of backward stochastic differential equations, while here our tools are PDE and SPDE techniques.

2. Setup, assumptions, and statement of the main result

This work is aimed at finding the limit as $\varepsilon\to 0$ of the solution of the following Cauchy problem:

$$\frac{\partial u^\varepsilon}{\partial t}(t,x)=\frac{\partial}{\partial x_i}a_{ij}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2}\right)\frac{\partial u^\varepsilon}{\partial x_j}(t,x)+\frac{1}{\varepsilon}g\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(t,x)\right),$$
$$(t,x)\in(0,T)\times\mathbb{R}^n,\ \ u^\varepsilon(0,x)=u_0(x), \tag{1}$$

where $u_0\in L^2(\mathbb{R}^n)$. $\varepsilon>0$ is a small parameter, $\{\xi_t,t>0\}$ a stationary diffusion process with values in $\mathbb{R}^d$. We denote by L the infinitesimal generator of ξ,

$$L=\frac{1}{2}q_{ij}(y)\frac{\partial^2}{\partial y_i\partial y_j}+b_i(y)\frac{\partial}{\partial y_i},$$

and impose the following conditions on the coefficients of (1) and on the generator of the process ξ:

C1. The functions $a_{ij}(z,y)$ and $g(z,y,u)$ are periodic in z of period 1 in all the coordinate directions; the matrix $\{a_{ij}(z,y)\}$ is uniformly positive definite:

$$0<c^{-1}\mathbf{I}\leqslant a(z,y)\leqslant c\mathbf{I};$$

moreover, the gradient of a_{ij} both with respect to y and z exists and is uniformly bounded:

$$|\nabla_z a_{ij}(z,y)| + |\nabla_y a_{ij}(z,y)| \leqslant c. \tag{2}$$

C2. The following bounds hold for some c, $\mu_1 > 0$:

$$0 < c^{-1}\mathbf{I} \leqslant q(y) \leqslant c\mathbf{I},$$

$$|\nabla q_{ij}(y)| \leqslant c, \quad |b(y)| + |\nabla b(y)| \leqslant c(1+|y|)^{\mu_1},$$

and there exist $M, C > 0$, $\alpha > -1$ such that whenever $|y| > M$,

$$\frac{(b(y)\cdot y)}{|y|} \leqslant -C|y|^{\alpha}; \tag{3}$$

here $(b(y)\cdot y)$ stands for the inner product in $\mathbb{R}^d$.

It follows from these assumptions that the process ξ possesses a unique invariant probability measure $\pi(\mathrm{d}y) = p(y)\,\mathrm{d}y$ whose density decays at infinity faster than any negative power of $|y|$ (see Pardoux and Veretennikov, 2001).

C3. $g(z,y,u)$ satisfies the estimates

$$|g(z,y,u)| \leqslant c|u|, \tag{4}$$

$$|g'_u(z,y,u)| \leqslant c, \tag{5}$$

$$(1+|u|)|g''_{uu}(z,y,u)| \leqslant c; \tag{6}$$

and g, g'_u, g''_{uu} are jointly continuous.

C4. The relation

$$\int_{\mathbf{T}^n}\int_{R^d} g(z,y,u)p(y)\,\mathrm{d}z\,\mathrm{d}y = 0 \tag{7}$$

holds for any $u \in \mathbb{R}$.

Remark 1. If $\{\xi_t\}$ is a diffusion on a compact manifold, only nondegeneracy and some regularity are required, instead of C2.

By our assumptions the diffusion process $\{\xi_t\}$ is a solution of the stochastic equation

$$\mathrm{d}\xi_t = \sigma(\xi_t)\,\mathrm{d}W_t + b(\xi_t)\,\mathrm{d}t, \tag{8}$$

where $\sigma(y) = q^{1/2}(y)$, and $\{W_t\}$ is a standard d-dimensional Wiener processes.

It is convenient to decompose $g(z,y,u)$ as follows:

$$g(z,y,u) = \tilde{g}(z,y,u) + \bar{g}(y,u),$$

where

$$\bar{g}(y,u) = \int_{\mathbf{T}^n} g(z,y,u)\,\mathrm{d}z,$$

so that

$$\int_{\mathbf{T}^n} \tilde{g}(z,y,u)\,\mathrm{d}z = 0, \qquad \forall y \in \mathbb{R}^d,\ u \in \mathbb{R},$$

$$\int_{\mathbb{R}^d} \bar{g}(y,u)p(y)\,\mathrm{d}y = 0 \quad \forall u \in \mathbb{R}. \tag{9}$$

The first relation here implies in a standard way the existence of a vector function $\tilde{G}(z,y,u)$ such that

$$\tilde{g}(z,y,u) = \mathrm{div}_z\, \tilde{G}(z,y,u). \tag{10}$$

Indeed, we choose $\tilde{G} = \nabla v$, where for each $(y,u) \in \mathbb{R}^{d+1}$, $v(\cdot,y,u)$ solves the PDE $\Delta v = \tilde{g}$ on $\mathbf{T}^n$. Then the function $\tilde{G}(z,y,u)$ satisfies estimates (4) and (5). For any $u(x,t)$ we have now

$$\mathrm{div}_x\, \tilde{G}\left(\frac{x}{\varepsilon}, y, u(t,x)\right) = \frac{1}{\varepsilon}\,\tilde{g}\left(\frac{x}{\varepsilon}, y, u(t,x)\right) + \tilde{G}'_u\left(\frac{x}{\varepsilon}, y, u(t,x)\right)\nabla_x u(t,x). \tag{11}$$

According to Pardoux and Veretennikov (2001), under assumptions C2 and C4 the second relation in (9) ensures the solvability of the Poisson equation

$$L\bar{G}(y,u) + \bar{g}(y,u) = 0 \quad \forall u \in \mathbb{R} \tag{12}$$

in the space $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$. Moreover, the solution $G(\cdot,u)$ has polynomial growth in $|y|$ for all $u \in \mathbb{R}$. The solution is unique up to an additive constant, for definiteness we assume that it has zero mean w.r.t. the invariant measure $\pi(\mathrm{d}y) = p(y)\,\mathrm{d}y$.

We define $V_T := L^2(0,T;H^1(\mathbb{R}^n)) \cap C([0,T];L^2(\mathbb{R}^n))$, and let $\tilde{V}_T$ denote the space V_T, equipped with the sup of the weak topology of $L^2(0,T;H^1(\mathbb{R}^n))$, and the topology of the space $C([0,T];L^2_w(\mathbb{R}^n))$, where $L^2_w(\mathbb{R}^n)$ denotes the corresponding L^2 space equipped with its weak topology.

Moreover, χ^k, $k = 1,\dots,d$, and Ψ are defined as stationary solutions of

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial z_i} a_{ij}(z,\xi_\tau)\frac{\partial}{\partial z_j}\right)\chi^k(z,\tau) = -\frac{\partial}{\partial z_i} a_{ik}(z,\xi_\tau), \qquad (z,\tau) \in \mathbf{T}^n \times [0,+\infty),$$

and

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial z_i} a_{ij}(z,\xi_\tau)\frac{\partial}{\partial z_j}\right)\Psi(z,\tau,u) = -\tilde{g}(z,\xi_\tau,u), \qquad (z,\tau) \in \mathbf{T}^n \times [0,+\infty),$$

where u is a real parameter, see Lemma 1 below in Section 4.

This paper is devoted to the proof of

Theorem 1. *Under the above assumptions, the family of laws of the solutions $\{u^\varepsilon\}$ to problem (1) converges weakly, as $\varepsilon \to 0$, in the space $\tilde{V}_T$, for all $T > 0$, to the unique solution of the martingale problem with the drift $\hat{A}(u(s))$, where*

$$\hat{A}(u) = \nabla_x \cdot \langle a(\mathbf{I} + \nabla_z \chi)\rangle \nabla_x u - \nabla_x \cdot \langle \chi g\rangle(u) - \nabla_x \cdot \langle a \nabla_x \Psi\rangle(u)$$
$$+ \langle \Psi_u g\rangle(u) + \langle \bar{G}_u g\rangle(u),$$

and the covariance $R(u(s))$, *where*

$$(R(u)\varphi,\varphi)=\int_{R^d}(q(y)(\nabla_y\bar{G}(y,u),\varphi),(\nabla_y\bar{G}(y,u),\varphi))\,p(y)\,\mathrm{d}y.$$

In the above statement, the notation $\langle a(\mathbf{I}+\nabla_z\chi)\rangle$ stands for

$$\mathbf{E}\int_{\mathbf{T}^n}a(z,\xi_t)(\mathbf{I}+\nabla_z\chi(z,t))\,\mathrm{d}z,$$

which does not depend on t, since (ξ,χ) is stationary. The other uses of the notation $\langle\cdot\rangle$ in the formula for $\hat{A}(u)$ can be made precise in a similar way.

In order to avoid any confusion, we shall use below in Section 4 the notation $\{\langle\langle M\rangle\rangle(t);0\leqslant t\leqslant T\}$ to denote the increasing process associated to the continuous martingale $\{M_t;0\leqslant t\leqslant T\}$, i.e. $t\to\langle\langle M\rangle\rangle(t)$ is continuous and increasing, and $M_t^2-\langle\langle M\rangle\rangle(t)$ is a continuous martingale.

3. A priori estimates and tightness

In this section we obtain uniform a priori estimates for the solution u^ε and then use them in order to show tightness of the distributions of u^ε.

First, considering (9) and (11) one can rewrite the term $g(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(t,x))$ on the right-hand side of (1) in the form

$$\begin{aligned}\frac{1}{\varepsilon}\,g\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(t,x)\right)&=\mathrm{div}_x\,\tilde{G}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(x,t)\right)\\&\quad-\tilde{G}'_u\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(x,t)\right)\nabla_x u^\varepsilon(t,x)+\frac{1}{\varepsilon}\,\bar{g}(\xi_{t/\varepsilon^2},u^\varepsilon(x,t)).\end{aligned}\quad(13)$$

For $u\in L^2(\mathbb{R}^n)$ and $y\in\mathbb{R}^d$ denote

$$\Psi^\varepsilon(u,y)=\frac{1}{2}\|u\|_{L^2}^2+\varepsilon(u,\bar{G}(y,u)).$$

From Itô–Krylov's formula (see Pardoux and Veretennikov, 2001, for justifications), using (1) and (13), we get

$$\begin{aligned}&\mathrm{d}\Psi^\varepsilon(u^\varepsilon(t),\xi_{t/\varepsilon^2})\\&\quad=(A^\varepsilon u^\varepsilon(t),u^\varepsilon(t))\,\mathrm{d}t-\left(\nabla_x u^\varepsilon(t),\tilde{G}\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(t)\right)\right)\,\mathrm{d}t\\&\qquad-\left(\tilde{G}'_u\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(t)\right)\nabla_x u^\varepsilon(t),u^\varepsilon(t)\right)\,\mathrm{d}t+\frac{1}{\varepsilon}(u^\varepsilon(t),\bar{g}(\xi_{t/\varepsilon^2},u^\varepsilon(t)))\,\mathrm{d}t\\&\qquad+\frac{1}{\varepsilon}(u^\varepsilon(t),L\bar{G}(\xi_{t/\varepsilon^2},u^\varepsilon(t)))\,\mathrm{d}t+(u^\varepsilon(t),\nabla_y\bar{G}(\xi_{t/\varepsilon^2},u^\varepsilon(t)))\sigma(\xi_{t/\varepsilon^2})\,\mathrm{d}W_t^\varepsilon\\&\qquad+\varepsilon(A^\varepsilon u^\varepsilon(t),\bar{G}(\xi_{t/\varepsilon^2},u^\varepsilon(t)))\,\mathrm{d}t+\left(g\left(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^2},u^\varepsilon(t)\right),\bar{G}(\xi_{t/\varepsilon^2},u^\varepsilon(t))\right)\,\mathrm{d}t\end{aligned}$$

$$+\,\varepsilon(A^\varepsilon u^\varepsilon(t), \bar{G}'_u(\xi_{t/\varepsilon^2}, u^\varepsilon(t))u^\varepsilon(t))\,\mathrm{d}t$$

$$+\left(g\left(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^2}, u^\varepsilon(t)\right), \bar{G}'_u(\xi_{t/\varepsilon^2}, u^\varepsilon(t))u^\varepsilon(t)\right)\,\mathrm{d}t, \tag{14}$$

where A^ε stands for $(\partial/\partial x_i)a_{ij}(x/\varepsilon, \xi_{t/\varepsilon^2})\partial/\partial x_j$ and W_t^ε is the Wiener process $\varepsilon W_{t/\varepsilon^2}$.

Remark 2. Here and in what follows we use the notation $\nabla_y f(x/\varepsilon, \xi_{t/\varepsilon^2}, u^\varepsilon(t))$ and $\nabla_z f(x/\varepsilon, \xi_{t/\varepsilon^2}, u^\varepsilon(t))$ for $\nabla_y f(x/\varepsilon, y, u^\varepsilon(t))|_{y=\xi_{t/\varepsilon^2}}$ and $\nabla_z f(z, \xi_{t/\varepsilon^2}, u^\varepsilon(t))|_{z=x/\varepsilon}$, respectively. Also, if it does not lead to ambiguity, we omit the arguments x and ω of the solution u^ε.

We first prove the

Proposition 1. *Under our standing assumptions, if moreover $\alpha > 0$ (where α is the exponent in (3)), there exists a constant C such that for all $\varepsilon > 0$,*

$$\mathbf{E}\left(\sup_{0\leqslant t\leqslant T}\|u^\varepsilon(t)\|^2 + \int_0^T \|\nabla_x u^\varepsilon(t)\|^2\,\mathrm{d}t\right) \leqslant C.$$

Proof. It is not hard to see, using standard estimates, that for fixed $\varepsilon > 0$,

$$\mathbf{E}\left(\sup_{0\leqslant t\leqslant T}\|u^\varepsilon(t)\|^2 + \int_0^T \|\nabla_x u^\varepsilon(t)\|^2\,\mathrm{d}t\right) < \infty.$$

Hence, we can take the expectation in (14) integrated from 0 to t. Considering (12) and integrating by parts, one gets

$$\mathbf{E}\Psi^\varepsilon(u^\varepsilon(t), \xi_{t/\varepsilon^2}) + \mathbf{E}\int_0^t \left(a\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}\right)\nabla_x u^\varepsilon(s), \nabla_x u^\varepsilon(s)\right)\,\mathrm{d}s$$

$$= \mathbf{E}\Psi^\varepsilon(u_0(x), \xi_0) - \mathbf{E}\int_0^t \left(\nabla_x u^\varepsilon(s), \tilde{G}\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s)\right)\right)\,\mathrm{d}s$$

$$-\,\mathbf{E}\int_0^t \left(\tilde{G}'_u\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s)\right)\nabla_x u^\varepsilon(s), u^\varepsilon(s)\right)\,\mathrm{d}s$$

$$-\,2\varepsilon\mathbf{E}\int_0^t \left(a\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}\right)\nabla_x u^\varepsilon(s), \bar{G}'_u(\xi_{s/\varepsilon^2}, u^\varepsilon(s))\nabla_x u^\varepsilon(s)\right)\,\mathrm{d}s$$

$$+\,\mathbf{E}\int_0^t \left(g\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s)\right), \bar{G}(\xi_{s/\varepsilon^2}, u^\varepsilon(s)) + \bar{G}'_u(\xi_{t/\varepsilon^2}, u^\varepsilon(s))u^\varepsilon(s)\right)\,\mathrm{d}s$$

$$-\,\varepsilon\mathbf{E}\int_0^t \left(a\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}\right)\nabla_x u^\varepsilon(s), u^\varepsilon(s)\bar{G}''_{uu}(\xi_{s/\varepsilon^2}, u^\varepsilon(s))\nabla_x u^\varepsilon(s)\right)\,\mathrm{d}s. \tag{15}$$

According to Theorem 2 in Pardoux and Veretennikov (2001), under condition C3 the functions $\bar{G}(y;u)$, $\nabla_y\bar{G}(y,u)$, $\bar{G}'_u(y,u)$ and $\bar{G}''_{uu}(y,u)$ admit the following bounds:

$$|\bar{G}(y,u)| \leqslant c(1+|y|)^\mu|u|, \qquad |\nabla_y\bar{G}(y,u)| \leqslant c(1+|y|)^\mu|u|,$$

$$|\bar{G}'_u(y,u)| \leqslant c(1+|y|)^\mu,$$

$$(1+|u|)|\bar{G}''_{uu}(y,u)| \leqslant c(1+|y|)^\mu,$$

where $\mu=\mu(\alpha)$ is equal to 0 or strictly positive depending on whether $\alpha>0$ or $\alpha\leqslant 0$, respectively.

The first two integrals on the r.h.s. of (15) can be estimated as follows:

$$\begin{aligned}
&\left|\mathbf{E}\int_0^t \left(\nabla_x u^\varepsilon(s), \tilde{G}\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s)\right)\right)\,\mathrm{d}s\right.\\
&\qquad\left.+\mathbf{E}\int_0^t \left(\tilde{G}'_u\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^\varepsilon(s)\right)\nabla_x u^\varepsilon(s), u^\varepsilon(s)\right)\,\mathrm{d}s\right|\\
&\quad\leqslant 2c\mathbf{E}\int_0^t \|u^\varepsilon(s)\|\,\|\nabla_x u^\varepsilon(s)\|\,\mathrm{d}s\\
&\quad\leqslant \frac{c}{\gamma}\mathbf{E}\int_0^t \|u^\varepsilon(s)\|^2\,\mathrm{d}s + c\gamma\mathbf{E}\int_0^t \|\nabla_x u^\varepsilon(s)\|^2\,\mathrm{d}s.
\end{aligned}$$

If $\alpha>0$ and thus $\mu=0$, then the two terms involving the factor ε in (15) are dominated by the corresponding terms on the l.h.s., and taking sufficiently small γ, we have by the Gronwall lemma

$$\mathbf{E}\|u^\varepsilon(t)\|^2 + \mathbf{E}\int_0^t \|\nabla_x u^\varepsilon(s)\|^2\,\mathrm{d}s \leqslant C, \qquad t\leqslant T. \tag{16}$$

We then deduce from the Davis–Burkholder–Gundy inequality that

$$\begin{aligned}
&\mathbf{E}\sup_{0\leqslant t\leqslant T}\left|\int_0^t (u^\varepsilon(s), \nabla_y\bar{G}(\xi_{s/\varepsilon^2}, u^\varepsilon(s))\,\mathrm{d}W_s^\varepsilon\right|\\
&\quad\leqslant C\mathbf{E}\left[\left(\int_0^T \|u^\varepsilon(t)\|^4\,\mathrm{d}t\right)^{1/2}\right]\\
&\quad\leqslant \frac{1}{4}\mathbf{E}\left(\sup_{0\leqslant t\leqslant T}\|u^\varepsilon(t)\|^2\right) + C^2\mathbf{E}\int_0^T \|u^\varepsilon(t)\|^2\,\mathrm{d}t.
\end{aligned}$$

The proposition is now easy to deduce from (14) and (16). □

If $\alpha\leqslant 0$ then the function $\bar{G}(y,u)$ admits polynomial growth in y and, as a result, the above method fails to work. In this case the expectation of $\|u^\varepsilon\|$ might explode, as $\varepsilon\to 0$, but we shall control the moments of a slightly different sequence $\{\tilde{u}^\varepsilon\}$.

Let $\theta > 0$ be a constant to be chosen below. For each $\varepsilon > 0$, let

$$\tau_\varepsilon = \inf\left\{0 \leqslant t \leqslant T;\ |\xi_{t/\varepsilon^2}| > \left(\frac{\theta}{\varepsilon}\right)^{1/\mu}\right\},$$

and define $\{\tilde{u}^\varepsilon(t),\ 0 \leqslant t \leqslant T\}$ as the solution of the PDE

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}\right) \frac{\partial \tilde{u}^\varepsilon}{\partial x_j}(t,x) + \mathbf{1}_{[0,\tau_\varepsilon]}(t)\frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^2}, \tilde{u}^\varepsilon(t,x)\right),$$

$$(t,x) \in (0,T) \times \mathbb{R}^n,\ \ \tilde{u}^\varepsilon(0,x) = u_0(x). \tag{17}$$

It follows from Corollary 1 in Pardoux and Veretennikov (2001) that

$$\varepsilon \sup_{0 \leqslant t \leqslant T} |\xi_{t/\varepsilon^2}|^\mu \to 0 \tag{18}$$

in probability, as $\varepsilon \to 0$. Hence, as $\varepsilon \to 0$,

$$\mathbf{P}(\tau_\varepsilon = T) \to 1,$$

and consequently

$$\mathbf{P}(u^\varepsilon(t) = \tilde{u}^\varepsilon(t),\ 0 \leqslant t \leqslant T) \to 1.$$

Hence, tightness (resp. weak convergence to a limit u) of the sequence u^ε is equivalent to tightness (resp. weak convergence to the same limit u) of the sequence $\tilde{u}^\varepsilon$, for any topology.

We now prove the

Proposition 2. *Under our standing assumptions, if $\theta > 0$ is small enough, then there exists another sequence of stopping times $\{S_\varepsilon, \varepsilon > 0\}$, satisfying $\mathbf{P}(S_\varepsilon = T) \to 1$, as $\varepsilon \to 0$, and a constant C such that for all $\varepsilon > 0$,*

$$\mathbf{E}\left(\sup_{0 \leqslant t \leqslant S_\varepsilon} \|\tilde{u}^\varepsilon(t)\|^2 + \int_0^{S_\varepsilon} \|\nabla_x \tilde{u}^\varepsilon(t)\|^2 \,\mathrm{d}t\right) \leqslant C,$$

and also

$$\mathbf{E}\left(\sup_{0 \leqslant t \leqslant S_\varepsilon} \|\tilde{u}^\varepsilon(t)\|^4\right) \leqslant C.$$

Proof. We repeat the argument of the previous proposition, except that we develop by the Itô–Krylov formula the expression

$$\Psi^\varepsilon(t) := \frac{1}{2}\|\tilde{u}^\varepsilon(t)\|^2 + \varepsilon(\bar{G}(\xi_{t\wedge\tau_\varepsilon/\varepsilon^2}, \tilde{u}^\varepsilon(t\wedge\tau_\varepsilon)), \tilde{u}^\varepsilon(t\wedge\tau_\varepsilon))$$

We obtain the identity

$$\Psi^\varepsilon(t) + \int_0^t \left(a\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}\right)\nabla_x \tilde{u}^\varepsilon(s), \nabla_x \tilde{u}^\varepsilon(s)\right) \mathrm{d}s$$

$$= \Psi^\varepsilon(0) + A_\varepsilon(t\wedge\tau_\varepsilon) + B_\varepsilon(t\wedge\tau_\varepsilon) + M_\varepsilon(t\wedge\tau_\varepsilon) + \varepsilon C_\varepsilon(t\wedge\tau_\varepsilon),$$

where

$$A_\varepsilon(t) = -\int_0^t \left(\nabla_x \tilde{u}^{\,\varepsilon}(s), \tilde{G}\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, \tilde{u}^{\,\varepsilon}(s)\right)\right) \mathrm{d}s$$

$$- \int_0^t \left(\tilde{G}'_u\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, \tilde{u}^{\,\varepsilon}(s)\right)\nabla_x \tilde{u}^{\,\varepsilon}(s), \tilde{u}^{\,\varepsilon}(s)\right) \mathrm{d}s,$$

$$B_\varepsilon(t) = \int_0^t \left(g\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, \tilde{u}^{\,\varepsilon}(s)\right), \bar{G}(\xi_{s/\varepsilon^2}, \tilde{u}^{\,\varepsilon}(s)) + \bar{G}'_u(\xi_{s/\varepsilon^2}, \tilde{u}^{\,\varepsilon}(s))\tilde{u}^{\,\varepsilon}(s)\right) \mathrm{d}s,$$

$$M_\varepsilon(t) = \int_0^t (\tilde{u}^{\,\varepsilon}(s), \nabla_y \bar{G}^\varepsilon(s))\sigma(\xi_{s/\varepsilon^2})\,\mathrm{d}W_s^\varepsilon,$$

and

$$C_\varepsilon(t) = -\int_0^t \left(a\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}\right)\nabla_x \tilde{u}^{\,\varepsilon}(s), [2\bar{G}'_u(\xi_{s/\varepsilon^2}, \tilde{u}^{\,\varepsilon}(s))\right.$$

$$\left. +\ \tilde{u}^{\,\varepsilon}(s)\bar{G}''_{uu}(\xi_{s/\varepsilon^2}, \tilde{u}^{\,\varepsilon}(s))]\nabla_x \tilde{u}^{\,\varepsilon}(s)\right) \mathrm{d}s.$$

We now choose θ. There exists $c > 0$ such that the two following estimates hold:

$$\varepsilon(\bar{G}(\xi^\varepsilon_{t\wedge\tau_\varepsilon}, \tilde{u}^{\,\varepsilon}(t\wedge\tau_\varepsilon)), \tilde{u}^{\,\varepsilon}(t\wedge\tau_\varepsilon)) \leqslant \varepsilon c\left(1 + \sup_{0\leqslant s\leqslant t\wedge\tau_\varepsilon} |\xi_{s/\varepsilon^2}|^\mu\right)\sup_{0\leqslant s\leqslant t}\|\tilde{u}^{\,\varepsilon}(s)\|^2$$

$$\leqslant c(\varepsilon+\theta)\sup_{0\leqslant s\leqslant t}\|\tilde{u}^{\,\varepsilon}(t)\|^2,$$

$$\varepsilon C_\varepsilon(t\wedge\tau_\varepsilon) \leqslant \varepsilon c\int_0^{t\wedge\tau_\varepsilon}(1 + |\xi_{s/\varepsilon^2}|^\mu)\|\nabla_x \tilde{u}^{\,\varepsilon}(s)\|^2\,\mathrm{d}s$$

$$\leqslant c(\varepsilon+\theta)\int_0^t \|\nabla_x \tilde{u}^{\,\varepsilon}(s)\|^2\,\mathrm{d}s.$$

We choose θ such that

$$2c\theta = \inf_{X\in\mathbb{R}^n, z\in\mathbf{T}^n, y\in\mathbb{R}} \frac{(a(z,y)X,X)}{|X|^2} \wedge \frac{1}{2}.$$

It then follows, using Schwarz's inequality, that

$$\frac{1}{4}\sup_{0\leqslant s\leqslant t}\|\tilde{u}^{\,\varepsilon}(s)\|^2 + \frac{1}{3}\int_0^t (a^\varepsilon(s)\nabla\tilde{u}^{\,\varepsilon}(s), \nabla\tilde{u}^{\,\varepsilon}(s))\,\mathrm{d}s$$

$$\leqslant \|u_0\|^2 + c\int_0^t \|\tilde{u}^{\,\varepsilon}(s)\|^2\,\mathrm{d}s$$

$$+ c\int_0^t |\xi_{s/\varepsilon^2}|^\mu \|\tilde{u}^{\,\varepsilon}(s)\|^2\,\mathrm{d}s + \sup_{0\leqslant s\leqslant t}|M_\varepsilon(s)|,$$

hence, if v is a stopping time such that $v \leqslant T$,

$$\frac{1}{4}\mathbf{E}\sup_{0\leqslant s\leqslant v}\|\tilde u^{\varepsilon}(s)\|^2 + \frac{1}{3}\mathbf{E}\int_0^v (a^{\varepsilon}(s)\nabla\tilde u^{\varepsilon}(s), \nabla\tilde u^{\varepsilon}(s))\,\mathrm{d}s$$

$$\leqslant \|u_0\|^2 + c\mathbf{E}\int_0^v \|\tilde u^{\varepsilon}(s)\|^2\,\mathrm{d}s$$

$$+c\mathbf{E}\int_0^v |\xi_{s/\varepsilon^2}|^{\mu}\|\tilde u^{\varepsilon}(s)\|^2\,\mathrm{d}s$$

$$+c\mathbf{E}\left[\left(\int_0^v (1+|\xi_{s/\varepsilon^2}|)^{2\mu}\|\tilde u^{\varepsilon}(s)\|^4\,\mathrm{d}s\right)^{1/2}\right]$$

$$\leqslant \|u_0\|^2 + c\mathbf{E}\int_0^v \|\tilde u^{\varepsilon}(s)\|^2\,\mathrm{d}s$$

$$+c\mathbf{E}\left[\sup_{0\leqslant s\leqslant v}\|\tilde u^{\varepsilon}(s)\|^2\left(\int_0^v |\xi_{s/\varepsilon^2}|^{\mu}\,\mathrm{d}s + \sqrt{\int_0^v (1+|\xi_{s/\varepsilon^2}|)^{2\mu}\,\mathrm{d}s}\right)\right].$$

As $\varepsilon \to 0$, $\int_0^t |\xi_{s/\varepsilon^2}|^{\mu}\,\mathrm{d}s \to at$, $\int_0^t (1+|\xi_{s/\varepsilon^2}|)^{2\mu}\,\mathrm{d}s \to bt$ a.s. We choose $r>0$ such that $ar+\sqrt{br} < 1/8c$. Let p be the smallest integer such that $pr \geqslant T$, and define the stopping times

$$S_\varepsilon^1 = \inf\left\{t\leqslant r;\ \int_0^t |\xi_{s/\varepsilon^2}|^{\mu}\,\mathrm{d}s + \sqrt{\int_0^t (1+|\xi_{s/\varepsilon^2}|)^{2\mu}\,\mathrm{d}s} > \frac{1}{8c}\right\},$$

$$S_\varepsilon^3 = \inf\left\{r\leqslant t\leqslant 2r;\ \int_r^t |\xi_{s/\varepsilon^2}|^{\mu}\,\mathrm{d}s + \sqrt{\int_r^t (1+|\xi_{s/\varepsilon^2}|)^{2\mu}\,\mathrm{d}s} > \frac{1}{8c}\right\},$$

$$\vdots$$

$$S_\varepsilon^p = \inf\left\{(p-1)r\leqslant t\leqslant T;\ \int_{(p-1)r}^t |\xi_{s/\varepsilon^2}|^{\mu}\,\mathrm{d}s\right.$$

$$\left.+\sqrt{\int_{(p-1)r}^t (1+|\xi_{s/\varepsilon^2}|)^{2\mu}\,\mathrm{d}s} > \frac{1}{8c}\right\},$$

$$S_\varepsilon = S_\varepsilon^1\mathbf{1}_{\{S_\varepsilon^1<r\}} + S_\varepsilon^2\mathbf{1}_{\{S_\varepsilon^1=r, S_\varepsilon^2<2r\}} + \cdots + S_\varepsilon^p\mathbf{1}_{\{S_\varepsilon^1=r, S_\varepsilon^2=2r,\dots,S_\varepsilon^{p-1}=(p-1)r\}}.$$

It follows from Birkhoff's ergodic theorem and the choice of r that

$$\mathbf{P}(S_\varepsilon = T)\to 1 \quad \text{as } \varepsilon\to 0.$$

We have, choosing $v = t \wedge S_\varepsilon^1$,

$$\frac{1}{8}\mathbf{E}\left(\sup_{s\leqslant t\wedge S_\varepsilon^1} \|\tilde u^{\,\varepsilon}(s)\|^2\right) + \frac{1}{3}\mathbf{E}\int_0^{t\wedge S_\varepsilon^1} (a^\varepsilon(s)\nabla\tilde u^{\,\varepsilon}(s), \nabla\tilde u^{\,\varepsilon}(s))\,\mathrm{d}s$$

$$\leqslant \|u_0\|^2 + c\mathbf{E}\int_0^{t\wedge S_\varepsilon^1} \|\tilde u^{\,\varepsilon}(s)\|^2\,\mathrm{d}s,$$

hence

$$\mathbf{E}\left(\sup_{s\leqslant S_\varepsilon^1} \|\tilde u^{\,\varepsilon}(s)\|^2\right) + \mathbf{E}\int_0^{S_\varepsilon^1} (a^\varepsilon(s)\nabla\tilde u^{\,\varepsilon}(s), \nabla\tilde u^{\,\varepsilon}(s))\,\mathrm{d}s \leqslant C.$$

Moreover, for all $r < t \leqslant 2r$, using the notation

$$\mathbf{E}(X;A) := \mathbf{E}(X\mathbf{1}_A),$$

$$\frac{1}{4}\mathbf{E}\left(\sup_{r\leqslant s\leqslant t\wedge S_\varepsilon^2} \|\tilde u^{\,\varepsilon}(s)\|^2; S_\varepsilon^1 = r\right) + \mathbf{E}\left(\int_r^{t\wedge S_\varepsilon^2} (a^\varepsilon(s)\nabla\tilde u^{\,\varepsilon}(s), \nabla\tilde u^{\,\varepsilon}(s))\,\mathrm{d}s; S_\varepsilon^1 = r\right)$$

$$\leqslant \mathbf{E}\big(\|\tilde u^{\,\varepsilon}(r)\|^2; S_\varepsilon^1 = r\big) + c\mathbf{E}\left(\int_r^{t\wedge S_\varepsilon^2} \|\tilde u^{\,\varepsilon}(s)\|^2\,\mathrm{d}s; S_\varepsilon^1 = r\right)$$

$$+ c\mathbf{E}\left[\sup_{r\leqslant s\leqslant t\wedge S_\varepsilon^2} \|\tilde u^{\,\varepsilon}(s)\|^2 \left(\int_r^t |\xi_{s/\varepsilon^2}|^\mu\,\mathrm{d}s + \sqrt{\int_r^t (1+|\xi_{s/\varepsilon^2}|)^{2\mu}\,\mathrm{d}s}\right); S_\varepsilon^1 = r\right],$$

and we deduce by the same arguments as above that

$$\mathbf{E}\left(\sup_{r\leqslant t\leqslant S_\varepsilon^2} \|\tilde u^{\,\varepsilon}(t)\|^2; S_\varepsilon^1 = r\right) + \mathbf{E}\left(\int_r^{S_\varepsilon^2} \|\nabla\tilde u^{\,\varepsilon}(s)\|^2\,\mathrm{d}s; S_\varepsilon^1 = r\right)$$

$$\leqslant C\mathbf{E}(\|\tilde u^{\,\varepsilon}(r)\|^2; S_\varepsilon^1 = r).$$

Repeating the same argument with $S_\varepsilon^1, S_\varepsilon^2$ replaced by $S_\varepsilon^2, S_\varepsilon^3$, etc., and combining all those estimates, we conclude that there exists a constant C such that

$$\mathbf{E}\left(\sup_{0\leqslant t\leqslant S_\varepsilon} \|\tilde u^{\,\varepsilon}(t)\|^2 + \int_0^{S_\varepsilon} \|\nabla_x\tilde u^{\,\varepsilon}(s)\|^2\,\mathrm{d}s\right) \leqslant C.$$

The second result is proved quite similarly, with the same sequence of stopping times S_ε, starting with the quantity

$$\frac{1}{4}\|\tilde u^{\,\varepsilon}(t)\|^4 + \varepsilon(\bar G(\xi_{t\wedge\tau_\varepsilon/\varepsilon^2}, \tilde u^{\,\varepsilon}(t\wedge\tau_\varepsilon)), \tilde u^{\,\varepsilon}(t\wedge\tau_\varepsilon))\|\tilde u^{\,\varepsilon}(t\wedge\tau_\varepsilon)\|^2,$$

instead of $\Psi^\varepsilon(\tilde u^{\,\varepsilon}, t)$. □

From now on, θ will be chosen as indicated in the above proof. Note that, for the same reasons as above, tightness (resp. convergence) of the sequence $\{\tilde{u}^{\varepsilon}(\cdot \wedge S_{\varepsilon})\}$ is equivalent to tightness (resp. convergence) of the sequence $\{\tilde{u}^{\varepsilon}\}$ (resp. of the sequence $\{u^{\varepsilon}\}$).

We next establish the (here and in the rest of the paper $C_0^{\infty}(\mathbb{R}^n)$ denotes the class of mappings from $\mathbb{R}^n$ into $\mathbb{R}$, which are of class C^{∞}, and have compact support) following proposition:

Proposition 3. *For any $\varphi \in C_0^{\infty}(\mathbb{R}^n)$, the collection of processes $\{(u^{\varepsilon}, \varphi),\ \varepsilon > 0\}$ is tight in $C([0,T])$.*

Proof. Fix $\varphi \in C_0^{\infty}(\mathbb{R}^n)$. We consider the random process

$$\Phi^{\varepsilon,\varphi}(t) = (u^{\varepsilon}(t), \varphi) + \varepsilon(\bar{G}(\xi_{t/\varepsilon^2}, u^{\varepsilon}(t)), \varphi).$$

Applying the Itô–Krylov formula to develop $\Phi^{\varepsilon,\varphi}(t)$, we deduce that

$$(u^{\varepsilon}(t), \varphi) = (u_0, \varphi) + I^{\varepsilon}(t) + J_1^{\varepsilon}(t) + J_2^{\varepsilon}(t) + J_3^{\varepsilon}(t),$$

where

$$I^{\varepsilon}(t) = \int_0^t (\varphi, \nabla_y \bar{G}(\xi_{s/\varepsilon^2}, u^{\varepsilon}(s)))\sigma(\xi_{s/\varepsilon^2})\,\mathrm{d}W_s^{\varepsilon},$$

$$J_1^{\varepsilon}(t) = -\int_0^t \left[\left(a\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}\right) \nabla_x u^{\varepsilon}(s), \nabla_x \varphi\right) + \left(\tilde{G}'_u\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^{\varepsilon}(s)\right) \nabla_x u^{\varepsilon}(t), \varphi\right)\right] \mathrm{d}s,$$

$$J_2^{\varepsilon}(t) = \int_0^t \left[\left(g\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^{\varepsilon}(s)\right), \bar{G}'_u(\xi_{s/\varepsilon^2}, u^{\varepsilon}(s))\varphi\right) - \left(\nabla_x \varphi, \tilde{G}\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}, u^{\varepsilon}(s)\right)\right)\right] \mathrm{d}s,$$

and

$$\begin{aligned} J_3^{\varepsilon}(t) = {} & \varepsilon[(\bar{G}(\xi_0, u^{\varepsilon}(0)), \varphi) - (\bar{G}(\xi_{t/\varepsilon^2}, u^{\varepsilon}(t)), \varphi)] \\ & - \varepsilon \int_0^t \left[\left(a\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}\right) \nabla_x \varphi, \bar{G}'_u(\xi_{s/\varepsilon^2}, u^{\varepsilon}(s))\nabla_x u^{\varepsilon}(s)\right)\right. \\ & \left. + \left(a\left(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^2}\right) \nabla_x u^{\varepsilon}(s), \varphi \bar{G}''_{uu}(\xi_{s/\varepsilon^2}, u^{\varepsilon}(s))\nabla_x u^{\varepsilon}(s)\right)\right] \mathrm{d}s. \end{aligned}$$

We first note that

$$\begin{aligned} |J_3^{\varepsilon}(t)| \leqslant {} & C\varepsilon \left[1 + (1 + |\xi_{t/\varepsilon^2}|)^{\mu}\|u^{\varepsilon}(t)\| + (1 + |\xi_0|)^{\mu}\|u_0\| \right. \\ & \left. + \int_0^t (1 + |\xi_{s/\varepsilon^2}|)^{\mu}(\|\nabla_x u^{\varepsilon}(s)\| + \|\nabla_x u^{\varepsilon}(s)\|^2)\,\mathrm{d}s\right] \\ \leqslant {} & C\varepsilon \sup_{0 \leqslant s \leqslant T} (1 + |\xi_{s/\varepsilon^2}|)^{\mu} \left(\sup_{0 \leqslant s \leqslant T} \|u^{\varepsilon}(s)\| + \int_0^T (1 + \|\nabla_x u^{\varepsilon}(s)\|^2)\,\mathrm{d}s\right). \end{aligned}$$

It then follows from (18) and Proposition 2 that

$$\sup_{0 \leqslant s \leqslant T} |J_3^{\varepsilon}(s)| \to 0 \quad \text{in probability, as } \varepsilon \to 0.$$

We note that for $0 \leqslant s \leqslant t \leqslant T$,

$$|J_1^\varepsilon(t) - J_1^\varepsilon(s)|^2 \leqslant c|t-s| \int_0^T \|\nabla_x u^\varepsilon(r)\|^2 \, dr.$$

But from Proposition 2, for any $\eta, \delta > 0$, one can choose $\theta > 0$ such that for all $\varepsilon > 0$,

$$\mathbf{P}\left(\int_0^T \|\nabla_x u^\varepsilon(r)\|^2 \, dr > \eta^2/c\theta \right) \leqslant \delta,$$

hence for all $\varepsilon > 0$,

$$\mathbf{P}\left(\sup_{|t-s| \leqslant \theta} |J_1^\varepsilon(t) - J_1^\varepsilon(s)| > \eta \right) \leqslant \delta,$$

and the collection of continuous processes $\{J_1^\varepsilon, \ \varepsilon > 0\}$ is tight.

Next we have

$$|J_2^\varepsilon(t) - J_2^\varepsilon(s)| \leqslant c \sup_{0 \leqslant r \leqslant T} \|u^\varepsilon(r)\| \int_s^t (1 + |\xi_{r/\varepsilon^2}|)^\mu \, dr.$$

The tightness of the collection $\{J_2^\varepsilon, \ \varepsilon > 0\}$ now follows from Proposition 2 and the following estimate, for $p \geqslant 1$:

$$\mathbf{E}\left(\left| \int_s^t (1 + |\xi_{r/\varepsilon^2}|)^\mu \, dr \right|^p \right) \leqslant |t-s|^{p-1} C(T, p, \mu),$$

where $C(T, p, \mu) = \int_0^T \mathbf{E}[(1 + |\xi_{r/\varepsilon^2}|)^{p\mu}] \, dr$ is finite and independent of ε, by the stationarity of ξ.

It remains to consider the stochastic integral term. For each $N > 0$, define $\tau_N^\varepsilon := \inf\{t > 0; \|u^\varepsilon(t)\| \geqslant N\}$, $u_N^\varepsilon := u^\varepsilon(t \wedge \tau_N^\varepsilon)$, and

$$I_N^\varepsilon(t) = \int_0^{t \wedge \tau_N^\varepsilon} (\varphi, \nabla_y \bar{G}(\xi_{s/\varepsilon^2}, u^\varepsilon(s))) \sigma(\xi_{s/\varepsilon^2}) \, dW_s^\varepsilon.$$

We have for $p > 2$

$$\begin{aligned} \mathbf{E}\left(\sup_{t_0 \leqslant t \leqslant t_0 + \gamma} |I_N^\varepsilon(t) - I_N^\varepsilon(t_0)|^p \right) &\leqslant CN^p \mathbf{E}\left(\left| \int_{t_0}^{t_0+\gamma} (1 + |\xi_{r/\varepsilon^2}|)^{2\mu} \, dr \right|^{p/2} \right) \\ &\leqslant C(T, p, \mu) N^p \gamma^{p/2-1}, \end{aligned}$$

hence for any $\delta > 0$ we first choose N large enough such that for all $\varepsilon > 0$

$$\mathbf{P}\left(\sup_{0 \leqslant t \leqslant T} |I^\varepsilon(t) - I_N^\varepsilon(t)| > 0 \right) \leqslant \delta/2,$$

and note that

$$\begin{aligned} \mathbf{P}\left(\sup_{t_0 \leqslant t \leqslant t_0 + \gamma} |I_N^\varepsilon(t) - I_N^\varepsilon(t_0)| \geqslant \delta \right) &\leqslant CN^p \frac{\gamma^{p/2-1}}{\delta^p} \\ &\leqslant \gamma\delta/2, \end{aligned}$$

for $p=6$ and $\gamma=\inf(1,\delta^7/2CN^6)$. The tightness of the collection $\{I^\varepsilon,\varepsilon>0\}$ then follows from Theorem 8.3 in Billingsley (1968). The Proposition is established. □

Recall that $V_T := L^2(0,T;H^1(\mathbb{R}^n))\cap C([0,T];L^2(\mathbb{R}^n))$, and $\tilde V_T$ denotes the space V_T, equipped with the sup of the weak topology of $L^2(0,T;H^1(\mathbb{R}^n))$, and the topology of the space $C([0,T];L^2_w(\mathbb{R}^n))$. It follows from the results in Viot (1976), Propositions 2 and 3 the

Proposition 4. *The collection $\{u^\varepsilon,\varepsilon>0\}$ of elements of V_T is tight in $\tilde V_T$.*

4. Passage to the limit

The aim of this section is to pass to the limit, as $\varepsilon\to 0$, in the family of laws of $\{u^\varepsilon\}$ and to determine the limit problem. In view of the tightness result of the preceding section it is sufficient to find the limit distributions of the inner products (φ,u^ε) with $\varphi\in C_0^\infty(\mathbb{R}^n)$, see Viot (1976). To this end we introduce the following two auxiliary parabolic equations:

$$\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial z_i}a_{ij}(z,\xi_\tau)\frac{\partial}{\partial z_j}\right)\chi^k(z,\tau)=-\frac{\partial}{\partial z_i}a_{ik}(z,\xi_\tau),\quad (z,\tau)\in\mathbf{T}^n\times[0,+\infty),\tag{19}$$

and

$$\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial z_i}a_{ij}(z,\xi_\tau)\frac{\partial}{\partial z_j}\right)\Psi(z,\tau,u)=-\tilde g(z,\xi_\tau,u),\quad (z,\tau)\in\mathbf{T}^n\times[0,+\infty),\tag{20}$$

where u is a real parameter. The functions $\chi^k(z,\tau)$ and $\Psi(z,\tau,u)$ are now defined as stationary solutions to these equations.

Lemma 1. *There exist solutions to (19) and (20) such that both pairs $(\chi(\cdot,\tau),\xi_\tau)$ and $(\Psi(\cdot,\tau,\cdot),\xi_\tau)$ are stationary processes, these solutions are ergodic and unique up to an additive constant. Moreover, under the normalization $\int_{\mathbf{T}^n}\chi(z,\tau)\,\mathrm{d}z=0$ and $\int_{\mathbf{T}^n}\Psi(z,\tau,u)\,\mathrm{d}z=0$ the following estimates hold*:

$$\|\chi\|_{L^\infty}+\int_t^{t+1}\|\chi(\cdot,s)\|^2_{H^1(\mathbf{T}^n)}\,\mathrm{d}s\leqslant C,\tag{21}$$

$$\|\Psi\|_{L^\infty}+\left(\int_t^{t+1}\|\Psi(\cdot,s,u)\|^2_{H^1(\mathbf{T}^n)}\,\mathrm{d}s\right)^{1/2}\leqslant C|u|,\tag{22}$$

$$\|\Psi'_u\|_{L^\infty}+\left(\int_t^{t+1}\|\Psi'_u(\cdot,s,u)\|^2_{H^1(\mathbf{T}^n)}\,\mathrm{d}s\right)^{1/2}\leqslant C,\tag{23}$$

$$\|\Psi''_{uu}\|_{L^\infty}+\left(\int_t^{t+1}\|\Psi''_{uu}(\cdot,s,u)\|^2_{H^1(\mathbf{T}^n)}\,\mathrm{d}s\right)^{1/2}\leqslant C/(1+|u|).\tag{24}$$

Proof. The existence and uniqueness of a stationary ergodic solution can be established in the same way as in Lemma 3.5 in Kleptsyna and Piatnitski (2000). Indeed, consider the following family of Cauchy problems:

$$\frac{\partial}{\partial\tau}\,\Psi^N(z,\tau,u)+\frac{\partial}{\partial z_i}\,a_{ij}(z,\xi_\tau)\frac{\partial}{\partial z_j}\,\Psi^N(z,\tau,u)$$
$$= -\mathbf{1}_{\{N\leqslant\tau\leqslant N+1\}}\tilde g(z,\xi_\tau,u),(z,\tau)\in\mathbf{T}^n\times(-\infty,N+1),$$

$$\Psi^N|_{\tau=N+1}=0.$$

By the standard Nash estimate and energy estimate we have

$$\|\Psi^N\|_{L^\infty(\mathbf{T}^n\times(N,N+1))}+\left(\int_N^{N+1}\|\Psi^N(\cdot,s,u)\|^2_{H^1(\mathbf{T}^n)}\,\mathrm{d}s\right)^{1/2}$$
$$\leqslant c_1\|\tilde g\|_{L^\infty}\leqslant c_2|u|, \tag{25}$$

where the constant c_1 only depends on the ellipticity constant in C1 and the dimension n. Due to the assumption C3 the constant c_2 is nonrandom and independent of N.

Now, in the same way as in the proof of Lemma 3.3 in Kleptsyna and Piatnitski (2000) one can show by virtue of the Harnack inequality and maximum principle that

$$\|\Psi^N(\cdot,s,u)\|_{L^\infty(\mathbf{T}^n)}\leqslant c_3\exp(-c_0(N-s))\|\Psi^N(\cdot,N,u)\|_{L^\infty(\mathbf{T}^n)}$$
$$\leqslant c_4|u|\exp(-c_0(N-s)),\qquad s\leqslant N,$$

with nonrandom constants $c_4>0$ and $c_0>0$ which are independent of N. In fact, c_0 only depends on the ellipticity constant in C1 and the dimension n. Summing up the functions Ψ^N over all integer N, we obtain a stationary ergodic solution to problem (20), which is jointly stationary with ξ, and satisfies moreover

$$\|\Psi\|_{L^\infty}\leqslant C|u|.$$

Estimate (22) is now straightforward. Other estimates of the lemma can be justified by the same arguments. Note in particular that estimate (21) can be obtained similarly, also $\partial a/\partial z$ is not an element of $L^\infty(\mathbf{T}^n)$, but of $W^{-1,\infty}(\mathbf{T}^n)$, see Kleptsyna and Piatnitski (2000).

Remark 3. In contrast with $\bar G(\xi_t,u)$, the functions $\chi(z,\tau)$ and $\Psi(z,\tau,u)$ not only depend on the value of ξ at a given time τ but on the whole half-trajectory $\{\xi_s,\ s\geqslant\tau\}$.

Having defined $\bar G(y,u)$, $\chi(z,\tau)$ and $\Psi(z,\tau,u)$, for any arbitrary $\varphi\in C_0^\infty(\mathbb{R}^n)$, we consider the real valued stochastic process $\{\Phi^\varepsilon(t),\ 0\leqslant t\leqslant T\}$ defined as

$$\Phi^\varepsilon(t)=(\tilde u^{\,\varepsilon}(t),\varphi)+\varepsilon(\chi^\varepsilon(t)\tilde u^{\,\varepsilon}(t),\nabla_x\varphi)+\varepsilon(\Psi^\varepsilon(t\wedge\tau_\varepsilon,\tilde u^{\,\varepsilon}(t\wedge\tau_\varepsilon)),\varphi)$$
$$+\varepsilon(\bar G(\xi^\varepsilon_{t\wedge\tau_\varepsilon},\tilde u^{\,\varepsilon}(t\wedge\tau_\varepsilon)),\varphi),$$

where $\chi^\varepsilon(t)$, $\Psi^\varepsilon(t,u)$ and ξ^ε_t stand for $\chi(\cdot/\varepsilon,t/\varepsilon^2)$, $\Psi(\cdot/\varepsilon,t/\varepsilon^2,u)$ and ξ_{t/ε^2}, respectively. Let $\alpha_\varepsilon(t):=\mathbf{1}_{[0,\tau_\varepsilon]}(t)$.

By the Itô formula

$$\begin{aligned}
\mathrm{d}\Phi^\varepsilon(t) = &\left\{\left(\frac{\partial\tilde u^{\,\varepsilon}}{\partial t}(t),\varphi\right)+\varepsilon^{-1}\left(\frac{\partial\chi^\varepsilon}{\partial\tau}(t)\tilde u^{\,\varepsilon}(t),\nabla_x\varphi\right)+\varepsilon\left(\chi^\varepsilon(t)\frac{\partial\tilde u^{\,\varepsilon}}{\partial t}(t),\nabla_x\varphi\right)\right.\\
&+\varepsilon^{-1}\alpha_\varepsilon(t)\left(\frac{\partial\Psi^\varepsilon}{\partial\tau}(t,\tilde u^{\,\varepsilon}(t)),\varphi\right)+\varepsilon\alpha_\varepsilon(t)\left(\frac{\partial\Psi^\varepsilon}{\partial u}(t,\tilde u^{\,\varepsilon}(t))\tilde u^\varepsilon_t(t),\varphi\right)\\
&\left.+\ \varepsilon^{-1}\alpha_\varepsilon(t)(L\bar G(\xi^\varepsilon_t,\tilde u^{\,\varepsilon}(t)),\varphi)+\varepsilon\alpha_\varepsilon(t)(\bar G'_u(\xi^\varepsilon_t,\tilde u^{\,\varepsilon}(t))\tilde u^\varepsilon_t(t),\varphi)\right\}\,\mathrm{d}t\\
&+\alpha_\varepsilon(t)(\nabla_y\bar G(\xi^\varepsilon_t,\tilde u^{\,\varepsilon}(t))\sigma(\xi^\varepsilon_t),\varphi)\,\mathrm{d}W^\varepsilon_t.
\end{aligned}$$

Considering (17), after multiple integration by parts and simple rearrangements, we obtain

$$\begin{aligned}
\mathrm{d}\Phi^\varepsilon(t) = &\big\{(\tilde u^{\,\varepsilon}(t),a^\varepsilon\nabla_x\nabla_x\varphi)+\varepsilon^{-1}(\tilde u^{\,\varepsilon}(t),\nabla_z a^\varepsilon\nabla_x\varphi)+\varepsilon^{-1}\alpha_\varepsilon(t)(\bar g(\xi^\varepsilon_t,\tilde u^{\,\varepsilon}(t)),\varphi)\\
&+\varepsilon^{-1}\alpha_\varepsilon(t)(\tilde g(\xi^\varepsilon_t,\tilde u^{\,\varepsilon}(t)),\varphi)+\varepsilon^{-1}\left(\frac{\partial\chi^\varepsilon}{\partial\tau}(t)\tilde u^{\,\varepsilon}(t),\nabla_x\varphi\right)\\
&+\varepsilon^{-1}(\nabla_z(a^\varepsilon\nabla_z\chi^\varepsilon)(t)\tilde u^{\,\varepsilon}(t),\nabla_x\varphi)+(a^\varepsilon\nabla_z\chi^\varepsilon(t),\tilde u^{\,\varepsilon}(t)\nabla_x\nabla_x\varphi)\\
&-\varepsilon(a^\varepsilon\nabla_x\tilde u^{\,\varepsilon}(t),\chi^\varepsilon(t)\nabla_x\nabla_x\varphi)+\alpha_\varepsilon(t)(\chi^\varepsilon(t)g^\varepsilon(t,\tilde u^{\,\varepsilon}(t)),\nabla_x\varphi)\\
&+\varepsilon^{-1}\alpha_\varepsilon(t)\left(\frac{\partial\Psi^\varepsilon}{\partial\tau}(t,\tilde u^{\,\varepsilon}(t)),\varphi\right)-\alpha_\varepsilon(t)(\nabla_z\Psi^\varepsilon_u(t,\tilde u^{\,\varepsilon}(t))a^\varepsilon\nabla_x\tilde u^{\,\varepsilon}(t),\varphi)\\
&-\varepsilon\alpha_\varepsilon(t)(\Psi^\varepsilon_{uu}a^\varepsilon\nabla_x\tilde u^{\,\varepsilon}(t),\nabla_x\tilde u^{\,\varepsilon}(t)\varphi)-\varepsilon\alpha_\varepsilon(t)(\Psi^\varepsilon_u a^\varepsilon\nabla_x\tilde u^{\,\varepsilon}(t),\nabla_x\varphi)\\
&+\ \alpha_\varepsilon(t)(\Psi^\varepsilon_u g^\varepsilon,\varphi)+\varepsilon^{-1}\alpha_\varepsilon(t)(L\bar G^\varepsilon,\varphi)\big\}\,\mathrm{d}t+\alpha_\varepsilon(t)(\nabla_y\bar G^\varepsilon\sigma(\xi^\varepsilon_t),\varphi)\,\mathrm{d}W^\varepsilon_t\\
&+\alpha_\varepsilon(t)\{-\varepsilon(\bar G^\varepsilon_{uu}a^\varepsilon\nabla_x\tilde u^{\,\varepsilon}(t),\nabla_x\tilde u^{\,\varepsilon}(t)\varphi)-\varepsilon(\bar G^\varepsilon_u a^\varepsilon\nabla_x\tilde u^{\,\varepsilon}(t),\nabla_x\varphi)\\
&+(\bar G^\varepsilon_u g^\varepsilon,\varphi)\}\,\mathrm{d}t.
\end{aligned}$$

In view of (12), (19), (20) and the relation

$$(a^\varepsilon\nabla_z\Psi^\varepsilon_u\nabla_x\tilde u^{\,\varepsilon},\varphi)=-(a^\varepsilon\nabla_z\Psi^\varepsilon,\nabla_x\varphi)-\varepsilon^{-1}(\nabla_z\cdot(a^\varepsilon\nabla_z\Psi^\varepsilon),\varphi),$$

the above expression can be simplified further as follows:

$$\begin{aligned}
\mathrm{d}\Phi^\varepsilon(t) = &\{(\tilde u^{\,\varepsilon}(t)),a^\varepsilon(\mathbf{I}+\nabla_z\chi^\varepsilon(t))\nabla_x\nabla_x\varphi)\\
&+\alpha_\varepsilon(t)(\chi^\varepsilon(t)g^\varepsilon(t,\tilde u^{\,\varepsilon}(t)),\nabla_x\varphi)+\alpha_\varepsilon(t)(a^\varepsilon\nabla_z\Psi^\varepsilon,\nabla_x\varphi)
\end{aligned}$$

$$+ \alpha_\varepsilon(t)(\Psi_u^\varepsilon g^\varepsilon, \varphi) + \alpha_\varepsilon(t)(\bar{G}_u^\varepsilon g^\varepsilon, \varphi)\} \, \mathrm{d}t + \alpha_\varepsilon(t)(\nabla_y \bar{G}\sigma(\xi_t^\varepsilon), \varphi) \, \mathrm{d}W_t^\varepsilon$$

$$- \varepsilon\{(a^\varepsilon \nabla_x \tilde{u}^\varepsilon(t), \chi^\varepsilon(t)\nabla_x\nabla_x\varphi) + \alpha_\varepsilon(t)(\Psi_{uu}^\varepsilon a^\varepsilon \nabla_x \tilde{u}^\varepsilon(t), \nabla_x \tilde{u}^\varepsilon(t)\varphi)$$

$$+ \alpha_\varepsilon(t)(\Psi_u^\varepsilon a^\varepsilon \nabla_x \tilde{u}^\varepsilon(t), \nabla_x\varphi) + \alpha_\varepsilon(t)(\bar{G}_u^\varepsilon a^\varepsilon \nabla_x \tilde{u}^\varepsilon(t), \nabla_x\varphi)$$

$$+ \alpha_\varepsilon(t)(\bar{G}_{uu}^\varepsilon a^\varepsilon \nabla_x \tilde{u}^\varepsilon(t), \nabla_x \tilde{u}^\varepsilon(t)\varphi)\} \, \mathrm{d}t. \tag{26}$$

The following statements will allow us to pass to the limit, as $\varepsilon \to 0$, in the laws of Φ^ε and thus to obtain the desired limiting distribution of (u^ε, φ).

Proposition 5. *Let $v^\varepsilon(x,t)$ converge to $v^0(x,t)$ in $\tilde{V}_T$. Then $v^\varepsilon(x,t)$ converges towards $v^0(x,t)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$. In other words $\tilde{V}_T$ is continuously embedded into $L^2_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$.*

The proof of this and the following four statements will be given in the next and last section.

Proposition 6. *Assume that u^ε converges in law towards some u in the space $\tilde{V}_T$, and that $\Theta : \mathbb{R} \to \mathbb{R}$ is a continuous mapping such that $|\Theta(u)| \leqslant c(1+|u|)$ for some $c > 0$. Then for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ the family $\{\int_0^t (\Theta(u^\varepsilon(s)), \varphi)\, \mathrm{d}s,\ 0 \leqslant t \leqslant T\}$ converges in law towards $\{\int_0^t (\Theta(u(s)), \varphi)\, \mathrm{d}s,\ 0 \leqslant t \leqslant T\}$ in $C([0,T])$.*

Proposition 7. *Suppose that for each $u \in \mathbb{R}$, the random field $\theta(z,\tau,u)$ is periodic in z, stationary and ergodic in τ. Assume, furthermore, that the following bounds hold:*

$$\|\theta(\cdot,\tau,u)\|_{C(\mathbf{T}^n)} \leqslant c\eta(\tau)(1+|u|), \tag{27}$$

$$\|\theta(\cdot,\tau,u_1) - \theta(\cdot,\tau,u_2)\|_{C(\mathbf{T}^n)} \leqslant c\eta(\tau)|u_1 - u_2|, \tag{28}$$

with a stationary process $\eta(\tau)$ subject to the estimate $\mathbf{E}|\eta(\tau)|^p \leqslant c(p)$ for each $p > 1$. Then for any $\varphi \in C_0^\infty$ one has

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^{t \wedge S_\varepsilon} \left(\theta\left(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^2}, \tilde{u}^\varepsilon(s)\right) - \langle\theta\rangle(\tilde{u}^\varepsilon(s)), \varphi \right) \mathrm{d}s \right| \to 0, \quad \text{in } L^1(\Omega), \text{ as } \varepsilon \to 0,$$

where

$$\langle\theta\rangle(u) = \mathbf{E}\int_{\mathbf{T}^n} \theta(z,\tau,u)\, \mathrm{d}z, \qquad u \in \mathbb{R}.$$

The following two statements will allow us to deal with the stochastic integral term, through its quadratic variation.

Proposition 8. *Let $\mathscr{G} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be continuous and satisfy the estimates*

$$|\mathscr{G}(y,u)| \leqslant c(1+|y|)^\mu(1+|u|), \qquad |\mathscr{G}(y,u_1) - \mathscr{G}(y,u_2)| \leqslant c(1+|y|)^\mu|u_1 - u_2|,$$

then for any $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\sup_{t\leqslant T}\left|\int_0^{t\wedge S_\varepsilon} (q(\xi_{s/\varepsilon^2})(\mathscr{G}(\xi_{s/\varepsilon^2},\tilde u^{\,\varepsilon}(s)),\varphi)(\mathscr{G}(\xi_{s/\varepsilon^2},\tilde u^{\,\varepsilon}(s)),\varphi)\right.$$

$$\left. -(\mathscr{R}(\tilde u^{\,\varepsilon}(s))\varphi,\varphi))\,\mathrm{d}s\right| \to 0 \quad \text{in } L^1(\Omega),$$

where

$$(\mathscr{R}(u)\varphi,\varphi) = \int_{R^d} (q(y)(\mathscr{G}(y,u),\varphi),(\mathscr{G}(y,u),\varphi))\,p(y)\,\mathrm{d}y.$$

Proposition 9. *Assume that* u^ε *converges in law towards some* u *in the space* $\tilde V_T$, *then*

$$\int_0^\cdot (\mathscr{R}(u^\varepsilon(s))\varphi,\varphi)\,\mathrm{d}s \to \int_0^\cdot (\mathscr{R}(u(s))\varphi,\varphi)\,\mathrm{d}s$$

in law, in the space $C([0,T])$, *and moreover for some* $C>0$, *all* $t\in[0,T]$,

$$\mathbf{E}\left[\left(\int_0^{t\wedge S_\varepsilon} (\mathscr{R}(\tilde u^{\,\varepsilon}(s))\varphi,\varphi)\,\mathrm{d}s\right)^2\right] \leqslant C.$$

We will need to combine the above propositions with the following result. Recall that $\alpha_\varepsilon(t) = \mathbf{1}_{[0,\tau_\varepsilon]}(t)$.

Lemma 2. *Suppose we have elements* ρ^ε $(\varepsilon>0)$ *and* ρ *of the space* $L^1(\Omega\times(0,T),\mathrm{d}\mathbf{P}\times\mathrm{d}t)$ *which satisfy*

$$\sup_{0\leqslant t\leqslant T}\left|\int_0^t [\rho^\varepsilon(s)-\rho(s)]\,\mathrm{d}s\right| \to 0$$

in $L^1(\Omega)$, *as* $\varepsilon\to 0$. *Then*

$$\sup_{0\leqslant t\leqslant T}\left|\int_0^t [\alpha_\varepsilon(s)\rho^\varepsilon(s)-\rho(s)]\,\mathrm{d}s\right| \to 0$$

in $L^1(\Omega)$, *as* $\varepsilon\to 0$.

Proof. It suffices to note that

$$\sup_{0\leqslant t\leqslant T}\left|\int_0^t [\alpha_\varepsilon(s)\rho^\varepsilon(s)-\rho(s)]\,\mathrm{d}s\right|$$

$$\leqslant \sup_{0\leqslant t\leqslant T}\left|\int_0^t \alpha_\varepsilon(s)[\rho^\varepsilon(s)-\rho(s)]\,\mathrm{d}s\right| + \int_0^T |1-\alpha_\varepsilon(s)|\times|\rho(s)|\,\mathrm{d}s$$

$$\leqslant \sup_{0\leqslant t\leqslant T}\left|\int_0^t [\rho^\varepsilon(s)-\rho(s)]\,\mathrm{d}s\right| + \int_0^T |1-\alpha_\varepsilon(s)|\times|\rho(s)|\,\mathrm{d}s,$$

and that $1\geqslant 1-\alpha_\varepsilon(s)\downarrow 0$ a.s., for all s. $\square$

Now it is natural to rewrite (26) as follows:

$$
\begin{aligned}
(\tilde u^{\varepsilon}(t),\varphi) &= (u_0,\varphi) + \int_0^t (\tilde u^{\varepsilon}(s), \langle a(\mathbf{I}+\nabla_z\chi)\rangle \nabla_x\nabla_x\varphi)\,\mathrm{d}s\\
&\quad + \int_0^t \{(\langle \chi g\rangle(\tilde u^{\varepsilon}(s)),\nabla_x\varphi) + (\langle a\nabla_x\Psi\rangle(\tilde u^{\varepsilon}(s)),\nabla_x\varphi)\}\,\mathrm{d}s\\
&\quad + \int_0^t \{(\langle \Psi'_u g\rangle(\tilde u^{\varepsilon}(s)),\varphi) + (\langle \bar G'_u g\rangle(\tilde u^{\varepsilon}(s)),\varphi)\}\,\mathrm{d}s\\
&\quad + \int_0^t (\nabla_y \bar G\sigma(\xi^{\varepsilon}_t),\varphi)\,\mathrm{d}W^{\varepsilon}_s + K^{\varepsilon}(t),
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
K^{\varepsilon}(t) &= -\varepsilon\{(\chi^{\varepsilon}(t)\tilde u^{\varepsilon}(t),\nabla_x\varphi) + (\Psi^{\varepsilon}(t\wedge\tau_{\varepsilon},\tilde u^{\varepsilon}(t\wedge\tau_{\varepsilon})),\varphi)\\
&\quad + (\bar G(\xi^{\varepsilon}_{t\wedge\tau_{\varepsilon}},\tilde u^{\varepsilon}(t\wedge\tau_{\varepsilon})),\varphi)\}\\
&\quad + \varepsilon\{(\chi^{\varepsilon}(0)u_0,\nabla_x\varphi) + (\Psi^{\varepsilon}(0,u_0),\varphi) + (\bar G(\xi^{\varepsilon}_0,u_0),\varphi)\}\\
&\quad + \int_0^t (\tilde u^{\varepsilon}(s), [a^{\varepsilon}(\mathbf{I}+\nabla_z\chi^{\varepsilon}(s)) - \langle a(\mathbf{I}+\nabla_z\chi)\rangle]\nabla_x\nabla_x\varphi)\,\mathrm{d}s\\
&\quad + \int_0^t \{\alpha_{\varepsilon}(s)(\chi^{\varepsilon}(s)g^{\varepsilon}(s,\tilde u^{\varepsilon}(s)) - \langle \chi g\rangle(\tilde u^{\varepsilon}(s)),\nabla_x\varphi)\\
&\quad + \alpha_{\varepsilon}(s)(a^{\varepsilon}\nabla_z\Psi^{\varepsilon}(s,\tilde u^{\varepsilon}(s)) - \langle a\nabla_z\Psi^{\varepsilon}\rangle(\tilde u^{\varepsilon}(s)),\nabla_x\varphi)\}\,\mathrm{d}s\\
&\quad + \int_0^t \{(\alpha_{\varepsilon}(s)\Psi^{\varepsilon}_u(s,\tilde u^{\varepsilon}(s))g^{\varepsilon}(s,\tilde u^{\varepsilon}(s)) - \langle \Psi'_u g\rangle(\tilde u^{\varepsilon}(s)),\varphi)\\
&\quad + (\alpha_{\varepsilon}(s)\bar G^{\varepsilon}_u g^{\varepsilon} - \langle \bar G'_u g\rangle(\tilde u^{\varepsilon}(s)),\varphi)\}\,\mathrm{d}s\\
&\quad - \varepsilon\int_0^t \{(a^{\varepsilon}\nabla_x\tilde u^{\varepsilon}(s),\chi^{\varepsilon}(s)\nabla_x\nabla_x\varphi) + \alpha_{\varepsilon}(s)(\Psi^{\varepsilon}_{uu}a^{\varepsilon}\nabla_x\tilde u^{\varepsilon}(s),\nabla_x\tilde u^{\varepsilon}(s)\varphi)\\
&\quad + \alpha_{\varepsilon}(s)(\Psi^{\varepsilon}_u a^{\varepsilon}\nabla_x\tilde u^{\varepsilon}(s),\nabla_x\varphi) + \alpha_{\varepsilon}(s)(\bar G^{\varepsilon}_u a^{\varepsilon}\nabla_x\tilde u^{\varepsilon}(s),\nabla_x\varphi)\\
&\quad + \alpha_{\varepsilon}(s)(\bar G^{\varepsilon}_{uu}a^{\varepsilon}\nabla_x\tilde u^{\varepsilon}(s),\nabla_x\tilde u^{\varepsilon}(s)\varphi)\}\,\mathrm{d}s.
\end{aligned}
$$

We rewrite (29) as

$$
\begin{aligned}
F_{\varphi}(t,\tilde u^{\varepsilon}) &= \int_0^t \alpha_{\varepsilon}(s)(\nabla_y\bar G(\xi^{\varepsilon}_s,\tilde u^{\varepsilon}(s))\sigma(\xi^{\varepsilon}_s),\varphi)\,\mathrm{d}W^{\varepsilon}_s + K^{\varepsilon}(t)\\
&= M^{\varepsilon}_{\varphi}(t) + K^{\varepsilon}(t),
\end{aligned}
$$

where, for $u\in V_T$,

$$F_\varphi(t,u):=(u(t),\varphi)-(u_0,\varphi)-\int_0^t(u(s),\langle a(\mathbf{I}+\nabla_z\chi)\rangle\nabla_x\nabla_x\varphi)\,\mathrm{d}s$$
$$-\int_0^t\{(\langle\chi g\rangle(u(s)),\nabla_x\varphi)+(\langle a\nabla_x\Psi\rangle(u(s)),\nabla_x\varphi)\}\,\mathrm{d}s$$
$$-\int_0^t\{(\langle\Psi_u g\rangle(u(s)),\varphi)+(\langle\bar G_u g\rangle(u(s)),\varphi)\}\,\mathrm{d}s,$$

and the quadratic variation of the martingale M_φ^ε is given by

$$\langle\langle M_\varphi^\varepsilon\rangle\rangle(t)=\int_0^t\alpha_\varepsilon(s)(\nabla_y\bar G\sigma(\xi_s^\varepsilon),\varphi)^2\,\mathrm{d}s.$$

By C3, Proposition 7 and Lemma 1, $K^\varepsilon(t\wedge S_\varepsilon)$ tends to zero uniformly in t, in $L^1(\Omega)$, as $\varepsilon\to 0$.

Let $0\leqslant s<t$, and Θ_s^ε be any continuous (in the sense of the topology of $\tilde V_T$) and bounded functional of $\{\tilde u^\varepsilon(r),\ 0\leqslant r\leqslant s\}$.

We have that

$$\mathbf{E}[(F_\varphi(t\wedge S_\varepsilon,\tilde u^\varepsilon)-F_\varphi(s\wedge S_\varepsilon,\tilde u^\varepsilon))\Theta_s^\varepsilon]=\mathbf{E}[(K^\varepsilon(t\wedge S_\varepsilon)-K^\varepsilon(s\wedge S_\varepsilon))\Theta_s^\varepsilon],$$
$$\mathbf{E}[(M_\varphi^\varepsilon(t\wedge S_\varepsilon)-M_\varphi^\varepsilon(s\wedge S_\varepsilon))^2\Theta_s^\varepsilon]=\mathbf{E}[(\langle\langle M_\varphi^\varepsilon\rangle\rangle(t\wedge S_\varepsilon)-\langle\langle M_\varphi^\varepsilon\rangle\rangle(s\wedge S_\varepsilon))\Theta_s^\varepsilon].$$

Let $u\in\tilde V_T$ a.s. be any accumulation point of the sequence $\tilde u^\varepsilon$, as $\varepsilon\to 0$. Taking the limit along the corresponding subsequence in the two last identities, using weak convergence and uniform integrability, see Proposition 2, we conclude with the help of Propositions 6, 8 and 9 that $\{F_\varphi(t,u),\ 0\leqslant t\leqslant T\}$ is a square integrable martingale with respect to the natural filtration of u, with the associated quadratic variation process given by

$$\int_0^t(R(u(s))\varphi,\varphi)\,\mathrm{d}s,$$

where

$$(R(u)\varphi,\varphi)=\int_{R^d}((q(y)(\nabla_y\bar G(y,u),\varphi),(\nabla_y\bar G(y,u),\varphi))p(y))\,\mathrm{d}y.$$

We have shown that the law Q^0 of any accumulation point of the sequence u^ε solves the following martingale problem, which we shall denote problem (MP). For all $\varphi\in C_0^\infty(\mathbb{R}^n)$,

$$F_\varphi(t,u):=(u(t),\varphi)-(u_0,\varphi)-\int_0^t(\hat A(u(s)),\varphi)\,\mathrm{d}s,\qquad t\geqslant 0,$$

where

$$\hat A(v)=\nabla_x\cdot\langle a(\mathbf{I}+\nabla_z\chi)\rangle\nabla_x v-\nabla_x\cdot\langle\chi g\rangle(v)$$
$$-\nabla_x\cdot\langle a\nabla_x\Psi\rangle(v)+\langle\Psi_u g\rangle(v)+\langle\bar G_u g\rangle(v)$$

is a martingale with the associated quadratic variation process

$$\langle\langle F_\varphi(\cdot,u)\rangle\rangle(t) = \int_0^t (R(u(s))\varphi,\varphi)\,\mathrm{d}s.$$

We have already used the following statement, which is a consequence of Proposition 2 and Fatou's lemma:

Lemma 3.

$$\mathbf{E}\left(\sup_{0\leqslant t\leqslant T}\|u(t)\|^2 + \int_0^T \|\nabla_x u(t)\|^2\,\mathrm{d}t\right) < \infty.$$

We finally establish

Lemma 4. *The martingale problem* (*MP*) *has a unique solution.*

Proof. The usual argument of Yamada–Watanabe establishes that uniqueness of the martingale problem is a consequence of pathwise uniqueness for a corresponding SDE (see Viot (1976) for the adaptation of that argument to SPDEs). Now we state our SPDE. There is a certain freedom in defining such a SPDE, our choice is as follows. Let B_t be a standard cylindrical Brownian motion in $(L^2(R^d))^d$, i.e. for any $\phi\in(L^2(\mathbb{R}^d))^d$, (B_t,ϕ) is a real valued Brownian motion with covariance $t\|\phi\|_{(L^2(\mathbb{R}^d))^d}$. For each $v\in L^2(\mathbb{R}^n)$, denote by $H(v)$ the operator $H(v):(L^2(\mathbb{R}^d))^d\to L^2(\mathbb{R}^n)$ given by

$$[H(v)\psi(\cdot)](x) = \int_{R^d}(\Sigma(y)\nabla_y\bar G(y,v(x)),\psi(y))\,\mathrm{d}y,$$

where $\Sigma(y)$ stands for the symmetric square root of the matrix $p(y)q(y)$. Consider the SPDE in $L^2(\mathbb{R}^n)$

$$\mathrm{d}u(t) = \hat A(u(t))\,\mathrm{d}t + H(u(t))\,\mathrm{d}B_t,\qquad u(0)=u_0, \tag{30}$$

or, in the weak form,

$$(u(t),\varphi) = (u_0,\varphi) + \int_0^t(\hat A(u(s)),\varphi)\,\mathrm{d}s$$

$$+\int_0^t(\Sigma(\cdot)(\nabla_y\bar G(\cdot,u(s)),\varphi)_{L^2(\mathbb{R}^n)},\mathrm{d}B_t)_{(L^2(\mathbb{R}^d))^d},\qquad \forall\varphi\in C_0^\infty(\mathbb{R}^n).$$

According to Pardoux and Veretennikov (2001) (see also Campillo et al., 2001), under assumptions (4) and (5) the gradient in y of the function $\bar G(u,y)$ defined by (12), admits the bound

$$|\nabla_y\bar G(u,y)| \leqslant C(1+|y|)^\mu|u|,\qquad |\nabla_y\bar G'_u(u,y)|\leqslant C(1+|y|)^\mu. \tag{31}$$

Taking into account the fast decay of $\Sigma(y)$ at infinity, we conclude that all the terms in the above SPDE make sense. Moreover, as by Da Prato and Zabczyk (1992) this equation does have a solution in the space $V:=\bigcup_{T>0}V_T$, such that for all $T>0$,

$$\mathbf{E}\left(\sup_{0\leqslant t\leqslant T}\|u(t)\|^2 + \int_0^T \|\nabla_x u(t)\|^2\,\mathrm{d}t\right) < \infty. \tag{32}$$

We introduce the following notation for the coefficients of the operator $\hat{A}(u)$:

$$\hat{A}(u) = \nabla_x \cdot \bar{a}\nabla_x u + \nabla_x \cdot \bar{F}^1(u) + \bar{F}^0(u),$$

with $\bar{a} = \{\bar{a}_{ij}\}$ and $\bar{F}^1(u) = (\bar{F}^1_1, \dots, \bar{F}^1_n)$.

We now establish uniqueness. Assume that there are two solutions $v_1(t)$ and $v_2(t)$ with the same initial condition. It follows from the argument in Lemma 3 that both satisfy estimate (32). Apply Itô's formula to the expression $\|v_1 - v_2\|^2$, where $\|\cdot\|$ stands for the norm in $L^2(\mathbb{R}^n)$. This gives

$$\begin{aligned}
\|v_1(t) - v_2(t)\|^2 = &-2\int_0^t \bar{a}\nabla_x(v_1(s) - v_2(s)) \cdot \nabla_x(v_1(s) - v_2(s))\,\mathrm{d}s \\
&-2\int_0^t (F^1(v_1(s)) - F^1(v_2(s)), \nabla_x(v_1(s) - v_2(s)))\,\mathrm{d}s \\
&+2\int_0^t (F^0(v_1(s)) - F^0(v_2(s)), v_1(s) - v_2(s))\,\mathrm{d}s + \mathscr{M}_t \\
&+\int_0^t \int_{R^d} p(y)q(y)(\nabla_y \bar{G}(y, v_1(s)) - \nabla_y \bar{G}(y, v_2(s))) \\
&\times(\nabla_y \bar{G}(y, v_1(s)) - \nabla_y \bar{G}(y, v_2(s)))\,\mathrm{d}y\,\mathrm{d}s,
\end{aligned}$$

where $\mathscr{M}_t$ is a martingale. Taking the expectation on both sides of this identity, making use of the inequality

$$\begin{aligned}
&|(F^1(v_1(s)) - F^1(v_2(s)), \nabla_x(v_1(s) - v_2(s)))| \\
&\quad\leqslant C\|v_1(s) - v_2(s)\|\,\|\nabla_x(v_1(s) - v_2(s))\| \\
&\quad\leqslant C\left(\frac{1}{\gamma}\|v_1(s) - v_2(s)\|^2 + \gamma\|\nabla_x(v_1(s) - v_2(s))\|^2\right)
\end{aligned}$$

and considering the Lipschitz properties in v of all the functions involved, we obtain after simple rearrangements

$$\begin{aligned}
&\mathbf{E}\|v_1(t) - v_2(t)\|^2 + 2\mathbf{E}\int_0^t (\bar{a} - \gamma\mathbf{I})\nabla_x(v_1(s) - v_2(s)) \cdot \nabla_x(v_1(s) - v_2(s))\,\mathrm{d}s \\
&\quad\leqslant C(\gamma)\mathbf{E}\int_0^t \|v_1(s) - v_2(s)\|^2\,\mathrm{d}s.
\end{aligned}$$

For sufficiently small γ by the Gronwall lemma then implies

$$\mathbf{E}\|v_1(t) - v_2(t)\|^2 = 0$$

for any $t \geqslant 0$. This completes the proof of uniqueness.

The proof of Theorem 1 is now complete. □

5. Proof of the technical statements

Here we prove the technical propositions formulated in the preceding section.

Proof of Proposition 5. It suffices to show that for any $r > 0$ the sequence v^ε converges to v^0 in $L^2((0,T)\times B_r)$ with $B_r = [-r,+r]^n$. Let $\phi(x)$ be a cutoff function such that $\phi = 1$ in B_r and $\phi = 0$ in $\mathbb{R}^n \backslash B_{r+1}$, and denote by $\{e_i(x)\}_{i=1}^\infty$ an orthonormal basis in $H_0^1(B_{r+1})$ that consists of orthogonal in $L^2(B_{r+1})$ functions. If we define $\delta_N = \sup_{i > N+1} |e_i|_{L^2(B_{r+1})}$, then $\delta_N \to 0$ as $N \to \infty$ due to the compactness of the embedding of $H_0^1(B_{r+1})$ into $L^2(B_{r+1})$. Writing down the first N coefficients of $v^\varepsilon(t)\phi$ in the basis $\{e_i\}$ for each $t \leqslant T$, we get

$$v^\varepsilon(t,x)\phi(x) = \sum_{i=1}^N y_i^\varepsilon(t) e_i(x) + \tilde{v}^{\varepsilon,N}(t,x), \qquad y_i^\varepsilon(t) = \frac{(v^\varepsilon(t)\phi, e_i)_{L^2(B_{r+1})}}{|e_i|^2_{L^2(B_{r+1})}}.$$

Considering the uniform boundedness of $v^\varepsilon \phi$ in $L^2(0,T;H_0^1(B_{r+1}))$ one easily shows that

$$|\tilde{v}^{\varepsilon,N}|_{L^2((0,T)\times B_{r+1})} \leqslant C\delta_N,$$

with C independent of $\varepsilon \geqslant 0$. Moreover, for every $i \leqslant N$ one has

$$y_i^\varepsilon(t) \to \frac{(v^0(t)\phi, e_i)_{L^2(B_{r+1})}}{|e_i|^2_{L^2(B_{r+1})}}$$

as $\varepsilon \to 0$ in the metric of uniform convergence. It remains to pass to the limit first as $\varepsilon \to 0$ and then as $N \to \infty$ and the desired assertion follows. □

Proof of Proposition 6. For any $u \in V_T$, let

$$\Phi_\varphi(u)(t) := \int_0^t (\Theta(u(s)), \varphi)\, \mathrm{d}s.$$

In view of Proposition 5, it suffices to show that whenever $u^n \to u$ in $L^2((0,T)\times\Lambda(\varphi))$, $\Phi_\varphi(u^n) \to \Phi_\varphi(u)$ in $C([0,T])$. But

$$\sup_{0\leqslant t\leqslant T} |\Phi_\varphi(u^n)(t) - \Phi_\varphi(u)(t)| \leqslant \sqrt{T}\|\varphi\|_{L^2(\mathbb{R}^n)}\|\Theta(u^n) - \Theta(u)\|_{L^2((0,T)\times\Lambda(\varphi))},$$

and the convergence in $L^2((0,T)\times\Lambda(\varphi))$ of $\Theta(u^n)$ to $\Theta(u)$ follows from convergence in measure (since Θ is continuous), and uniform integrability of $\Theta(u^n)^2$, which follows from the L^2 convergence of the sequence $\{u^n\}$, and the linear growth of Θ. The proposition is established. □

Proof of Proposition 7. Let us note that under the assumptions of Proposition 7 we have

$$|\langle\theta(u)\rangle| \leqslant c(1+|u|), \quad |\langle\theta(u_1)\rangle - \langle\theta(u_2)\rangle| \leqslant c|u_1 - u_2|. \tag{33}$$

Denote $\Lambda = \mathrm{supp}(\varphi)$. By Proposition 5 the family $\{u^\varepsilon\}$ is tight in $L^2((0,T)\times\Lambda)$ and thus by the Prokhorov theorem for any $\delta>0$ there is a compact subset $K^\delta \subset L^2((0,T)\times\Lambda)$ such that $\mathbf{P}\{u^\varepsilon \mathbf{1}_\Lambda \notin K^\delta\} < \delta$. Since K^δ is a compact set, for any $\delta_1 > 0$ there exists a finite δ_1-net $q_1(t,x),\dots,q_N(t,x)$, $q_i \in L^2((0,T)\times\Lambda)$. Without loss of generality we assume that all the functions q_i are piecewise constant and moreover that the corresponding values are supported by a finite number of rectangular parallelepipeds. One can represent the space V as the union of disjoint sets $\mathscr{H}_0, \mathscr{H}_1, \dots, \mathscr{H}_N$ with $\mathscr{H}_0 = \{u \in V : \mathbf{1}_\Lambda u \notin K^\delta\}$ and $\mathscr{H}_j \subset \{u \in V : \|\mathbf{1}_\Lambda u - q_j\| \leqslant \delta_1\}$, $j = 1,2,\dots,N$, and introduce the events $\Omega_j^\varepsilon = \{u^\varepsilon \in \mathscr{H}_j\}$, $j = 0,1,\dots,N$. By the definition of K^δ, the probability of Ω_0^ε is not greater than δ; furthermore we have

$$\int_0^t \left(\theta\left(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^2}, u^\varepsilon\right) - \langle\theta\rangle(u^\varepsilon), \varphi\right) \mathrm{d}s = A_j(\varepsilon,t) + B_j(\varepsilon,t),$$

where

$$A_j(\varepsilon,t) = \int_0^t \left(\theta\left(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^2}, u^\varepsilon\right) - \theta\left(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^2}, q_j\right), \varphi\right) \mathrm{d}s + \int_0^t (\langle\theta\rangle(u^\varepsilon) - \langle\theta\rangle(q_j), \varphi)\, \mathrm{d}s,$$

$$B_j(\varepsilon,t) = \int_0^t \left(\theta\left(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^2}, q_j\right) - \langle\theta\rangle(q_j), \varphi\right) \mathrm{d}s.$$

By (28), (33) and the Schwartz inequality, we have

$$|A_j(\varepsilon,t)| \leqslant c \int_0^t \|u^\varepsilon(s) - q_j(s)\|_{L^2(\Lambda)} \left(1 + \eta\left(\frac{s}{\varepsilon^2}\right)\right) \mathrm{d}s.$$

Therefore

$$\mathbf{E}\left(\mathbf{1}_{\Omega_j^\varepsilon} \sup_{t\leqslant T} |A_j(\varepsilon,t)|\right) \leqslant cT\delta_1,$$

and for any $\alpha > 0$,

$$\begin{aligned}
&\mathbf{P}\left(\max_{t\leqslant T}\left|\int_0^t \left(\theta\left(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^2}, u^\varepsilon\right) - \langle\theta\rangle(u^\varepsilon), \varphi\right) \mathrm{d}s\right| > \alpha\right)\\
&\quad\leqslant \mathbf{P}(\Omega_0^\varepsilon) + \sum_{j=1}^N \mathbf{P}\left(\Omega_j^\varepsilon \cap \left\{\sup_{t\leqslant T} |A_j(\varepsilon,t)| > \alpha/2\right\}\right)\\
&\qquad+ \sum_{j=1}^N \mathbf{P}\left(\Omega_j^\varepsilon \cap \left\{\sup_{t\leqslant T} |B_j(\varepsilon,t)| > \alpha/2\right\}\right)\\
&\quad\leqslant \delta + 2cT\delta_1/\alpha + \sum_{j=1}^N \mathbf{P}\left(\sup_{t\leqslant T} |B_j(\varepsilon,t)| > \alpha/2\right).
\end{aligned}$$

Since δ and δ_1 can be made arbitrarily small, convergence in probability will follow from the fact that the last term on the right tends to 0, as $\varepsilon \to 0$. Convergence to 0 in probability of $B_j(\varepsilon,t)$ as $\varepsilon \to 0$ for each j and each fixed t follows from the Birkhoff ergodic theorem. Hence, convergence of the $\sup_{t\leqslant T}$ follows from the tightness in $C([0,T])$ of the collection of random processes $\{B_j(\varepsilon,\cdot),\ \varepsilon > 0\}$, which is a consequence of Theorem 8.3 in Billingsley (1968) like in the proof of Proposition 3, and the estimate (for $0 \leqslant s \leqslant t \leqslant T,\ p \geqslant 1$):

$$\mathbf{E}(|B_j(\varepsilon,t) - B_j(\varepsilon,s)|^p) \leqslant c(p)|t-s|^p,$$

where we have used condition (27), and the assumption on the moments of η.

Convergence to 0 in $L^1(\Omega)$ of

$$\max_{t \leqslant T} \left| \int_0^{t \wedge S_\varepsilon} \left(\theta\left(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^2}, \tilde{u}^{\varepsilon} \right) - \langle \theta \rangle (\tilde{u}^{\varepsilon}), \varphi \right) \mathrm{d}s \right|$$

then follows from uniform integrability, which is a consequence of Proposition 2.

Proof of Proposition 8. For $u, v \in \mathbb{R}$, let us denote by $\mathscr{R}(u,v)$ the quantity

$$\mathscr{R}(u,v) = \int_{\mathbb{R}^d} (q(y)\mathscr{G}(y,u), \mathscr{G}(y,v))\, p(y)\, \mathrm{d}y.$$

The assumptions of Proposition 8 imply the bounds

$$\mathscr{R}(u,v) \leqslant C(1+|u|)(1+|v|),$$

$$|\mathscr{R}(u',v) - \mathscr{R}(u'',v)| \leqslant C|u'-u''|(1+|v|),$$

$$|\mathscr{R}(u,v') - \mathscr{R}(u,v'')| \leqslant C|v'-v''|(1+|u|).$$

Then we follow the same scheme as in the proof of Proposition 7: we introduce Λ, a compact subset K^δ of $L^2((0,T)\times\Lambda)$ and a finite δ_1-net $q_1, \dots, q_N$ in the same way as above. According to Proposition 2 we can also assume without loss of generality that for all $u^\varepsilon \in K^\delta$ and all $q_j,\ j = 1, \dots, N(\delta_1)$, the following bounds hold:

$$\sup_{0 \leqslant t \leqslant T} \|u^\varepsilon(t)\|_{L^2(\Lambda)} \leqslant k, \qquad \sup_{0 \leqslant t \leqslant T} \|q_j(t)\|_{L^2(\Lambda)} \leqslant k$$

with a constant $k = k(\delta)$ which does not depend on ε and δ_1. Now one can construct disjoint sets $\mathscr{H}_j,\ j = 0, 1, \dots, N$, such that $V_T = \bigcup_{j=0}^N \mathscr{H}_j$ and

$$\mathscr{H}_0 = \{u \in V_T : \mathbf{1}_\Lambda u \notin K^\delta\}, \qquad \mathscr{H}_j \subset \{u \in V_T : \|\mathbf{1}_\Lambda u - q_j\| \leqslant \delta_1\}.$$

For $u^\varepsilon \in \mathscr{H}_j$ we have

$$\begin{aligned}
|J^\varepsilon(t)| &= \bigg| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} q(\xi_t^\varepsilon) \{ \mathscr{G}(\xi_t^\varepsilon, u^\varepsilon(t,x'))\mathscr{G}(\xi_t^\varepsilon, u^\varepsilon(t,x'')) \\
&\quad - \mathscr{G}(\xi_t^\varepsilon, q_j(t,x'))\mathscr{G}(\xi_t^\varepsilon, q_j(t,x''))\} \varphi(x')\varphi(x'')\, \mathrm{d}x'\, \mathrm{d}x'' \bigg| \\
&\leqslant \bigg| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} q(\xi_t^\varepsilon) \{ \mathscr{G}(\xi_t^\varepsilon, u^\varepsilon(t,x'))\mathscr{G}(\xi_t^\varepsilon, u^\varepsilon(t,x'')) \\
&\quad - \mathscr{G}(\xi_t^\varepsilon, q_j(t,x'))\mathscr{G}(\xi_t^\varepsilon, u^\varepsilon(t,x''))\} \varphi(x')\varphi(x'')\, \mathrm{d}x'\, \mathrm{d}x'' \bigg| \\
&\quad + \bigg| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} q(\xi_t^\varepsilon) \{ \mathscr{G}(\xi_t^\varepsilon, q_j(t,x'))\mathscr{G}(\xi_t^\varepsilon, u^\varepsilon(t,x''))
\end{aligned}$$

$$
\begin{aligned}
&\quad -\mathscr{G}(\xi^\varepsilon_t, q_j(t,x'))\mathscr{G}(\xi^\varepsilon_t, q_j(t,x''))\}\varphi(x')\varphi(x'')\,\mathrm{d}x'\,\mathrm{d}x''|\\
&\leqslant C(1+|\xi^\varepsilon_t|)^{2\mu}|u^\varepsilon(t)-q_j(t)|_{L^2(\Lambda)}(1+|u^\varepsilon(t)|_{L^2(\Lambda)}+|q_j(t)|_{L^2(\Lambda)}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\left|\int_0^t J^\varepsilon(s)\,\mathrm{d}s\right|\\
&\quad\leqslant C\int_0^T (1+|\xi^\varepsilon_s|)^{2\mu}|u^\varepsilon(s)-q_j(s)|_{L^2(\Lambda)}(1+|u^\varepsilon(s)|_{L^2(\Lambda)}+|q_j(s)|_{L^2(\Lambda)})\,\mathrm{d}s\\
&\quad\leqslant \|u^\varepsilon-q_j\|_{L^2((0,T)\times\Lambda)}\left(\int_0^T (1+|\xi^\varepsilon_s|)^{4\mu}(1+|u^\varepsilon(s)|^2_{L^2(\Lambda)}+|q_j(s)|^2_{L^2(\Lambda)})\,\mathrm{d}s\right)^{1/2}\\
&\quad\leqslant C\delta_1\sqrt{1+2k^2}\left(\int_0^T(1+|\xi^\varepsilon_s|)^{4\mu}\,\mathrm{d}s\right)^{1/2}.
\end{aligned}
$$

By the Birkhoff theorem the integral on the r.h.s. converges a.s., as $\varepsilon\to 0$, to the constant $T\mathbf{E}(1+|\xi_0|)^{4\mu}$. Similarly, whenever $u^\varepsilon\in\mathscr{H}_j$,

$$
\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\{\mathscr{R}(u^\varepsilon(s,x'),u^\varepsilon(s,x''))\right.\\
&\quad -\mathscr{R}(q_j(s,x'),q_j(s,x''))\}\varphi(x')\varphi(x'')\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}s| \leqslant C\sqrt{1+2k^2}\delta_1.
\end{aligned}
$$

To complete the proof it suffices to use the same arguments as those in the proof of Proposition 7. □

Proof of Proposition 9. As in the proof of Proposition 6, the first statement follows from the fact that whenever $u^n\to u$ in $\tilde{V}_T$, then

$$
\int_0^\cdot (\mathscr{R}(u^n(s))\varphi,\varphi)\,\mathrm{d}s\to\int_0^\cdot(\mathscr{R}(u(s))\varphi,\varphi)\,\mathrm{d}s,
$$

in $C([0,T])$. But this follows from the fact that

$$
(\mathscr{R}(u^n(s))\varphi,\varphi)\to(\mathscr{R}(u(s))\varphi,\varphi)
$$

in ds measure, since $u\to(\mathscr{R}(u)\varphi,\varphi)$ is continuous from $L^2(\Lambda)$ into $\mathbb{R}$, and is uniformly integrable on $[0,T]$. Indeed, for some $c>0$,

$$
|(\mathscr{R}(u^n(s))\varphi,\varphi)|\leqslant c\|u^n(s)\|^2,
$$

and the right-hand side of the last inequality in uniformly integrable, since it converges in $L^1(0,T)$.

The second statement is an immediate consequence of the second estimate in Proposition 2.

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