Stochastics and Stochastics Reports, Vol. 67, pp. 159–167
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BACKWARD STOCHASTIC VARIATIONAL INEQUALITIES

ETIENNE PARDOUX[a,*] and AUREL RĂŞCANU[b]

[a] *LATP, UMR–CNRS 6632, Centre de Mathématiques et d'Informatique, 39, rue F. Joliot Curie, 13453 Marseille cedex 13, France;*
[b] *Facultatea de Matematică, Universitatea "Alexandru Ioan Cuza", 6600 Iaşi, Romania*

(Received 9 October 1997; In final form 1 March 1999)

The aim of this paper is to show that our earlier results in [9] can be extended to Hilbert spaces. We then give examples of backward stochastic partial differential equations which can be solved with our results.

Keywords: Backward stochastic differential equations; variational inequalities; stochastic partial differential equations

AMS Classifications: 60H05, 60H15

1. INTRODUCTION

A new class of stochastic differential equations – called "backward stochastic differential equations", and which consist in fact in solving a certain inverse problem for a forward stochastic differential equation – has become recently the subject of intense study, see in particular Pardoux, Peng [8], Pardoux [7] and the references therein. The motivation from mathematical finance, as well as the connections with stochastic control and nonlinear partial differential equations, are largely responsible for the interest in BSDEs.

Recently, the authors have solved in [9] BSDEs involving the sub-differential of a convex function in their coefficient. The aim of this note is to show that those results can be readily extended to a Hilbert space setting,

*Corresponding author.

thus providing existence and uniqueness results for large classes of backward stochastic partial differential equations. We illustrate the width of our result by giving several examples of such BSPDEs.

Note that Ma, Yong [4, 5] have recently studied certain classes of BSPDEs where the underlying Brownian motion is finite dimensional, while our is infinite dimensional.

A natural conjecture is that BSPDEs should be connected with infinite dimensional PDEs, and stochastic control for SPDEs, hence probably with partially observed stochastic control problems.

The paper is organized as follows. The problem and preliminary results are formulated in Section 1. The existence and uniqueness result is stated at the end of Section 1. Since the proof is identical to that in finite dimension (see the proof of Theorem 1.1 in Pardoux, Rascanu [9]), we do not repeat it. Finally, we give in Section 3 examples of BSPDEs which are covered by our result.

2. FORMULATION OF THE PROBLEM AND STATEMENT OF THE RESULT

Let H and K be two real separable Hilbert spaces, and $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ a probability space equipped with a filtration, such that:

$$\mathcal{F}_t = \sigma(B_s^i,\ 0 \leq s \leq t,\ i \in \mathbb{N}^*) \vee \mathcal{N}$$

where $(B_t^i,\ t \geq 0)_{i=1,2,\dots}$ are mutually independent standard Brownian motions, and $\mathcal{N}$ is the class of P-null sets of $\mathcal{F}$. T will be throughout a fixed positive real. For any Hilbert space $\mathcal{H}$, let $\mathcal{S}^2(\mathcal{H})$ denote the space of continuous and $\mathcal{F}_t$-progressively measurable $\mathcal{H}$-valued processes, $\{X_t; 0 \leq t \leq T\}$, satisfying

$$E\left(\sup_{0\leq t\leq T} \|X_t\|_{\mathcal{H}}^2\right) < \infty;$$

$M^2(\mathcal{H})$ denote the space of $\mathcal{F}_t$-progressively measurable $\mathcal{H}$-valued processes $\{X_t; 0 \leq t \leq T\}$, satisfying

$$E\int_0^T \|X_t\|_{\mathcal{H}}^2 dt < \infty.$$

$\{e_i; i = 1, 2, \ldots\}$ being an arbitrary orthonormal basis of K, let $\{B_t, t \geq 0\}$ be defined formally as:

$$B_t = \sum_{i=1}^{\infty} B_t^i e_i.$$

It is well known that this series does not converge in K, but rather in any larger space $\tilde{K}, K \subset \tilde{K}$, which is such that the injection from K into $\tilde{K}$ is Hilbert–Schmidt. We will however not be concerned with $\tilde{K}$ any further. Rather, we introduce a class of processes with values in the space $\mathcal{L}^2(K, H)$ of Hilbert–Schmidt operators from K into H, *i.e.*, the space of linear operators Λ which are such that

$$\sum_{i=1}^{\infty} (\Lambda\Lambda^* e_i, e_i)_H < \infty.$$

It is well known (see *e.g.*, Métivier [6]) that to any element $Z \in M^2$ $(\mathcal{L}^2(K, H))$ one can associate an H-valued stochastic integral

$$\left\{ \int_0^t Z_s dB_s, \quad 0 \leq t \leq T \right\},$$

which is in particular the mean square limit as $n \to \infty$ of the approximating sequence:

$$\sum_{i=1}^{n} \int_0^t Z_s e_i dB_s^i.$$

This stochastic integral is a continuous $\mathcal{F}_t$-martingale, and the process $\int_0^t Z_s dB_s (\int_0^t Z_s dB_s)^* - \int_0^t Z_s Z_s^* ds, t \geq 0$ is also a martingale (with values in $\mathcal{L}^1(H)$, the space of nuclear operators on H).

The particular choice of our filtration $(\mathcal{F}_t)$ implies the following representation theorem, which extends to Hilbert spaces a well-known result of Itô. We do not give the proof, since it is a word-to-word copy of the well known finite dimensional analogous proof, see *e.g.*, Revuz–Yor [10], proof of Proposition 3.2.

THEOREM 2.1 *Let $\xi \in L^2(\Omega, \mathcal{F}_T, P; H)$. Then there exists a unique $Z \in M^2$ $(\mathcal{L}^2(K; H))$ such that*

$$\xi = \mathbb{E}(\xi) + \int_0^T Z_s dB_s.$$

The second ingredient of the proofs of existence-uniqueness for BSDEs is Itô's formula. We recall a version of that formula in Hilbert space, following Métivier [6].

PROPOSITION 2.2 *Let $\{Y_t, t \in [0,T]\}$ be an H-valued semimartingale which is such that for some $F \in M^2(H), Z \in M^2(\mathcal{L}^2(K;H))$,*

$$Y_t = Y_0 + \int_0^t F_s ds + \int_0^t Z_s dB_s.$$

Then for any $\psi \in C^2(H)$,

$$\psi(Y_t) = \psi(Y_0) + \int_0^t (\psi'(Y_s), F_s)ds + \int_0^t (\psi'(Y_s), Z_s dB_s) + \frac{1}{2}\int_0^t Tr\psi''(Y_s)Z_s Z_s^* ds.$$

In particular, in the case $\psi(y) \equiv \|y\|_H^2$, we have

$$|Y_t|^2 = |Y_0|^2 + \int_0^t (Y_s, F_s)ds + 2\int_0^t (Y_s, Z_s dB_s) + \frac{1}{2}\int_0^t \|Z_s\|_2^2 ds,$$

where $|\cdot| = \|\cdot\|_H, \|\cdot\|_2 = \|\cdot\|_{\mathcal{L}^2(K;H)}$.

We need to introduce one last object, which is the subdifferential of a convex l.s.c. function from H into $\mathbb{R}$. More precisely, let

$$\varphi : H \to]-\infty, +\infty]$$

be a proper (*i.e.*, $\not\equiv +\infty$) l.s.c. convex function. $\partial\varphi$ is a multivalued function from H into itself (*i.e.*, it maps H into subsets of H) which is defined as follows.

For any $u \in H$,

$$\partial\varphi(u) = \{h \in H; (h, v-u) + \varphi(u) \leq \varphi(v), \quad \forall\, v \in H\},$$

we define $\mathrm{Dom}(\partial\varphi)$ as the set of $u \in H$ such that $\partial\varphi(u)$ is not empty. We shall write $(u,v) \in \partial\varphi*$ to mean that $u \in \mathrm{Dom}(\partial\varphi)$ and $v \in \partial\varphi(u)$.

In addition to the above we are given:

- a final condition $\xi \in L^2(\Omega, \mathcal{F}_T, P; H)$, s.t. $E\varphi(\xi) < \infty$.
- a coefficient $F : \Omega \times [0,T] \times H \times \mathcal{L}^2(K;H) \to H$, which is such that there exist $\alpha \in \mathbb{R}, \beta, \gamma > 0$ and $\{\eta_t, t \in [0,T]\}$ a progressively measurable process satisfying $E\int_0^T \eta_t^2 dt < \infty$, and such that for all $y, y' \in H, z, \ z' \in \mathcal{L}^2(K;H)$,

(i) $F(\cdot, y, z)$ is $\mathcal{F}_t$-progressively measurable;
(ii) $y \longrightarrow F(t, y, z)$ is continuous, $dP \times dt$ a.e.;
(iii) $(F(t, y, z) - F(t, y', z), y - y') \leq \alpha |y - y'|^2$;
(iv) $|F(t, y, z) - F(t, y, z')| \leq \beta \|z - z'\|_2$;
(v) $|F(t, y, 0)| \leq \eta_t + \gamma |y|$.

Formally, solving our BSDE in the Hilbert space H consists in finding a pair $(Y, Z) \in S^2(H) \times M^2(\mathcal{L}^2(K; H))$ such that

$$\begin{cases} dY_t + F(t, Y_t, Z_t)dt \in \partial\varphi(Y_t)dt + Z_t dB_t, & 0 \leq t \leq T, \\ Y_t = \xi. \end{cases}$$

More precisely, we are looking for a triple $(Y, Z, U) \in S^2(H) \times M^2(\mathcal{L}^2(K; H)) \times M^2(H)$ such that:

(j) $Y_t + \int_t^T U_s ds = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s$, $0 \leq t \leq T$;
(jj) $Y_t \in \mathrm{Dom}(\partial\varphi)$ and $U_t \in \partial\varphi(Y_t)$, $dP \times dt$ a.e.

Our basic result is the following theorem, whose proof is a copy of the proof of the same result in finite dimension, see Theorem 1.1 in Pardoux, Rascanu [9].

THEOREM 2.3 *Under the above assumptions, in particular* (i), (ii), (iii), (iv) *and* (v), *there exists a unique triple* $(Y, Z, U) \in S^2(H) \times M^2(\mathcal{L}^2(K; H)) \times M^2(H)$ *which satisfies* (j), (jj). *Moreover,* $E \int_0^T \varphi(Y_t)dt < \infty$.

3. EXAMPLES

Let D be an open and bounded subset of $\mathbb{R}^d$ with a sufficiently smooth boundary Γ. By $H^m(D), H_0^m(D), W_0^{m,p}(D)$ we will denote the usual Sobolev spaces on D. The dual of $H_0^m(D)$ is $H^{-m}(D)$. The canonical isomorphism $J : H_0^1(D) \rightarrow H^{-1}(D)$ is $J = -\Delta$. The space $H^{-1}(D)$ is a Hilbert space when equipped with the inner product.

$$((u, v)) = \langle J^{-1}u, v \rangle \quad \forall\, u, v \in H^{-1}(D),$$

where $\langle \cdot, \cdot \rangle$ is the usual paring between $H_0^1(D)$ and $H^{-1}(D)$.

We note that any Hilbert–Schmidt operator on $L^2(D)$ has a square integrable kernel, so that $\mathcal{L}^2(L^2(D))$ can be identified with $L^2(D \times D)$.

We shall choose first $K = L^2(D)$. This choice implies that dB_t/dt is the so-called "space-time white noise", *i.e.*, the generalized random field $\{(h, \dot{B}),$

$h \in L^2(\mathbb{R}_+ \times D)\}$ defined by

$$(h, \dot{B}) = \int_0^\infty (h(t,\cdot), dB_t)_{L^2(D)}$$

is a zero mean Gaussian random field such that

$$E[(h,\dot{B})(k,\dot{B})] = (h,k)_{L^2(\mathbb{R}_+ \times D)}.$$

Finally, let $\beta \subset \mathbb{R} \times \mathbb{R}$ be a maximal monotone operator. One knows (see [1], p. 60, or [2] p. 43) that there exists a proper convex lower-semicontinuous function $j : \mathbb{R} \to]-\infty, +\infty]$ such that $\beta = \partial j$.

Example 3.1 Let $H = L^2(D)$ and $\varphi : H \to]-\infty, +\infty]$ given by

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_D |\text{gradu}(x)|^2 dx + \int_\Gamma j(u) d\sigma, & \text{if } u \in H^1(D),\ j(u) \in L^1(\Gamma), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.1}$$

Then, from [1], p. 63, it follows that:

$$\begin{array}{ll} \text{(a)} & \varphi \text{ is a proper convex l.s.c. function,} \\ \text{(b)} & \partial\varphi(u) = -\Delta u, \quad \forall\, u \in \text{Dom}(\partial\varphi) \\ \text{(c)} & \text{Dom}(\partial\varphi) = \{u \in H^2(D) : -\frac{\partial u}{\partial n} \in \beta(u) \text{a.e. on } \Gamma\} \\ \text{(d)} & \|u\|_{H^2(D)} \leq C_1\|u - \Delta u\|_{L^2(D)} + C_2, \quad \forall\, u \in \text{Dom}(\partial\varphi), \end{array} \tag{3.2}$$

where $(\partial/\partial n)$ is the outward normal derivative and C_1, C_2 are two constants independent of u.

From Theorem 2.3, we deduce that:

PROPOSITION 3.2 *If F satisfies* (i)–(v) *with* $H = K = L^2(D)$,

$$\xi \in L^2(\Omega, \mathcal{F}_T, P; H^1(D)), \quad j(\xi) \in L^1(\Omega \times \Gamma),$$

then the boundary value backward stochastic problem

$$\begin{cases} dY(t,x) + \Delta Y(t,x)dt + F(t, Y(t), Z(t), x)dt & \\ = \int_D Z(t,x,y)\dot{B}(dt,dy) & \text{on } \Omega \times [0,T] \times D, \\ -\frac{\partial Y(t,x)}{\partial n} \in \beta(Y(t,x)) & \text{on } \Omega \times [0,T] \times \Omega, \\ Y(T,x) = \xi(x) & \text{on } \Omega \times D \end{cases} \tag{3.3}$$

has a unique solution $(Y,Z) \in S^2(L^2(D)) \cap M^2(H^2(D)) \times M^2(L^2(D \times D))$ *such that* $-(\partial Y(t,x)/\partial n) \in \beta(Y(t,x))$ *a.e. on* $\Omega \times]0,T[\times \Gamma$.

Moreover, $Y \in L^\infty(0,T; L^2(\Omega H^1(D))$, *and* $j(Y) \in L^\infty(0,T; L^1(\Omega \times \Gamma))$.

Example 3.3 Let $H = L^2(D)$ and $\varphi : H \to]-\infty; +\infty]$ given by

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_D(|\mathrm{grad}u(x)|^2 + j(u(x)))dx, & \text{if } u \in H_0^1(D),\ \ j(u) \in L^1(D), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.4}$$

Then, from [2], p. 203, it follows:

$$\begin{array}{ll} \text{(a)} & \varphi \text{ is a proper convex l.s.c. function,} \\ \text{(b)} & \mathrm{Dom}\,\partial\varphi = \{u \in H_0^1(D) \cap H^2(D) : u(x) \in \mathrm{Dom}\,\beta \text{ a.e. } x \in D\} \\ \text{(c)} & \partial\varphi(u) = \{U \in L^2(D) : U(x) \in \beta(u(x)) - \Delta u(x) \text{ a.e. on } D\}\ \forall\, u \in \mathrm{Dom}(\partial\varphi), \\ \text{(d)} & \|u\|_{H^2(D)} \le C\|U\|_{L^2(D)}\ \ \forall (u, U) \in \partial\varphi, \\ \text{(e)} & \overline{\mathrm{Dom}(\partial\varphi)} = \{u \in L^2(D) : u(x) \in \overline{\mathrm{Dom}\,\beta}, \text{ a.e. } x \in D\}, \end{array} \tag{3.5}$$

and by Theorem 2.3, we have:

PROPOSITION 3.4 *Let* (i)–(v) *be satisfied with* $H = K = L^2(D)$*. If*

$$\xi \in L^2(\Omega, \mathcal{F}_T, P; H_0^1(D)), \quad j(\xi) \in L^1(\Omega \times D),$$

then the boundary value backward stochastic problem

$$\begin{cases} dY(t,x) + \Delta Y(t,x)dt + F(t, Y(t), Z(t), x)dt \in \beta(Y(t,x))dt \\ + \int_D Z(t,x,y)\dot{B}(dt,dy) \ on\ \Omega \times [0,T] \times D, \\ Y(t,x) = 0 \ on\ \Omega \times [0,T] \times \Gamma, \\ Y(T,x) = \xi(x) \ on\ \Omega \times D, \end{cases} \tag{3.6}$$

has a unique solution $(Y, Z) \in S^2(L^2(D)) \cap M^2(H_0^1(D) \cap H^2(D)) \times M^2(L^2(D \times D))$ *such that* $Y(t) \in H_0^1(D) \cap H^2(D) dP \times dt$ *a.e.,* $Y(t,x) \in Dom\ \beta$*,* $dP \times dt \times dx$ *a.e., and*

$$j(Y) \in L^\infty(0, T; L^1(\Omega \times D)).$$

Example 3.5 Let $H = L^2(D), r \ge 2, \varphi : H \to]-\infty, +\infty]$ given by

$$\varphi(u) = \begin{cases} \frac{1}{r}\sum_{i=1}^d \int_D \left|\frac{\partial u}{\partial x_i}(x)\right|^r dx, & \text{if } u \in W_0^{1,r}(D), \\ +\infty, & \text{otherwise.} \end{cases}$$

and $A: W_0^{1,r}(D) \to W_0^{-1,r'}(D)$, where $(1/r)+(1/r')=1$, given by

$$(Au,v) = \sum_{i=1}^{d} \int_D \left|\frac{\partial u}{\partial x_i}\right|^{r-2} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i} dx, u, v \in W_0^{1,r}(D).$$

Then, from [1], p. 204, it follows: φ is proper convex l.s.c. function, Dom $(\partial\varphi) = \{u \in W_0^{1,r}(D); Au \in L^2(D)\}$ and $\partial\varphi(u) = Au = -\sum_{i=1}^{d}(|(\partial u/\partial x_i)|^{r-2}(\partial u/\partial x_i)), \forall u \in \mathrm{Dom}(\partial\varphi)$.

Now, by Theorem 2.3, we have:

PROPOSITION 3.6 *Let* (i)–(v) *be satisfied with* $H = K = L^2(D)$. *If* $r \geq 2$ *and*

$$\xi \in L^1(\Omega, \mathcal{F}_T, P; W_0^{1,r}(D)),$$

then the BSPDE:

$$\begin{cases} dY(t,x) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial Y}{\partial x_i}\right|^{r-2}\frac{\partial Y}{\partial x_i}\right)(t,x)dt + F(t,Y(t),Z(t),x)dt \\ = \int_D Z(t,x,y)\dot{B}(dt,dy), \text{ on } \Omega \times [0,T) \times D, \\ Y(t,x) = 0 \text{ on } \Omega \times [0,T) \times \Gamma, \\ Y(T,x) = \xi(x) \text{ on } \Omega \times D, \end{cases} \tag{3.7}$$

has a unique solution $(Y,Z) \in S^2(L^2(D)) \cap M^r(W_0^{1,r}(D)) \times M^2(L^2(D \times D))$.

In the last example, we shall choose $K = \mathbb{R}^d$, so that $\{B_t, t \geq 0\}$ is now a d-dimensional Brownian motion.

Example 3.7 Assume $j: \mathbb{R} \to \mathbb{R}$ is continuous $\lim_{|r|\to\infty}(j(r)/|r|) = \infty$. Let $H = H^{-1}(D)$ and $\varphi: H \to]-\infty, +\infty]$ be given by

$$\varphi(u) = \begin{cases} \int_D j(u(x))dx, & \text{if } u \in L^1(D), j(u) \in L^1(D) \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, from [1], p. 67, it follows that:

$$\begin{array}{ll} \text{(a)} & \varphi \text{ is proper convex l.s.c. on } H^{-1}(D), \\ \text{(b)} & U \in \partial\varphi(u) \Longleftrightarrow U \in -\Delta\beta(u). \end{array} \tag{3.8}$$

Hence

PROPOSITION 3.8 *Let the assumptions* (i)–(v) *be satisfied with* $H = H^{-1}(D)$, $K = \mathbb{R}^d$. *If* $\xi \in L^1(\Omega \times D) \cap L^2(\Omega, \mathcal{F}_T, P; H), j(\xi) \in L^1(\Omega \times D)$, *then* *the BSPDE*:

$$\begin{cases} dY(t,x) + \Delta\beta(Y(t,x)dt + F(t,Y(t),Z(t),x)dt \ni \sum_{i=1}^{d} Z^i(t,x)dB_t^i \\ on\ \Omega\times[0,T]\times D, \\ Y(T,x) = \xi(x),\ on\ \Omega\times D \\ \beta(Y(t,x)) \ni 0,\ on\ \omega\times]0,T[\times\Gamma. \end{cases} \tag{3.9}$$

has a unique solution

$$(Y,Z)\in S^2(H^{-1}(D))\times M^2((H^{-1}(D))^d).$$

Moreover

$$j(Y)\in L^\infty(0,T;L^1(\Omega\times D))$$
$$\beta(Y)\in M^2(H_0^1(D)).$$

References

[1] Barbu, V. (1976). Nonlinear Semigroups and Differential Equations in Banach Spaces, Ed. Academiei Române and Noordhoff International Publishing.

[2] Brézis, H. (1973). Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert, North-Holland Publ. Co.

[3] Darling, R. W. R. and Pardoux, E. (1997). Backward SDE with random terminal time and applications to semilinear elliptic PDE, *Ann. Probab.*, **25**, 1135–1159.

[4] Ma, J. and Yong, J. (1997). Adapted solution of a degenerate backward SPDE, with applications, *Stoch. Proc. and their Applic.*, **70**, 59–84.

[5] Ma, J. and Yong, J., On linear degenerate backward stochastic partial differential equations, submitted.

[6] Métivier, M. (1982). Semimartingales: a course on stochastic processes, De Gruyter studies in Mathematics, **2**, Berlin, New York, W. de Gruyter.

[7] Pardoux, E. (1998). Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order, in Stochastic Analysis and Related Topics VI: The Geilo Workshop, 1996, Decreusefond, L., Gjerde, J., Øksendal, B. and Üstünel, A. S. (Eds.), Birkhäuser, pp. 79–127.

[8] Pardoux, E. and Peng, S. (1990). Adapted solution of a backward stochastic differential equation, *Systems and Control Letters*, **14**, 55–61.

[9] Pardoux, E. and Răşcanu, A. (1998). Backward stochastic differential equations with subdifferential operator and related variational inequalities, *Stoch. Proc. and their Applic.*, **76**, 191–215.

[10] Revuz, D. and Yor, M. (1994). Continuous martingales and Brownian motion, Grundlehren der Mathematischen Wissenschaften, **293**, 2nd edn., Berlin, Springer.