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Probability Theory and Related Fields

A Two-Sided Stochastic Integral and its Calculus

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Summary. Let X be a forward diffusion and Y a backward diffusion, both defined on $[0, 1]$, X_t and Y^t being respectively adapted to the past of a Wiener process $W(\cdot)$, and to its future increments. We construct a "two-sided" stochastic integral of the form.

$$\int_0^t \Phi(u, X_u, Y^u)\, dW(u)$$

which generalizes the backward and forward Itô integrals simultaneously. Our construction is quite intuitive, and leads to a generalized stochastic calculus. It is also shown that for each fixed t, our integral coincides with that defined by Skorohod in [18].

1. Introduction

The Itô integral defines a process

$$X_t := \int_0^t \varphi_s\, dW_s, \qquad t \geqq 0$$

where $\{W_t\}$ is a standard real valued Wiener process defined on a probability space with filtration (Ω, F, F_t, P), and $\{\varphi_t\}$ satisfies

(i) $\varphi_{\cdot}$ is a measurable process and φ_t is an F_t-measurable random variable, $\forall t \geqq 0$.

(ii) $\int_0^T \varphi_t^2\, dt < \infty$ a.s., $\quad \forall T > 0$.

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Clearly, the second part of condition (i), which means that the process is "adapted to $\{W_t\}$" or in other words "non anticipative" (i.e. φ_t is independent of future increments of $\{W_\cdot\}$ after time t), is by far the most restrictive one.

It has been a challenging problem, and has become important for applications, to be able to relax condition (i), i.e. to define stochastic integrals with anticipative integrands. There have been several important results in that direction, using at least three different kinds of methods. The first method consists of replacing $\{F_t\}$ by a larger filtration $\{G_t\}$, with respect to which $\{W_t\}$ is no longer a Wiener process, but might still be a semi-martingale, i.e. in this case the sum of a Wiener process and a process with bounded variation, which still is a possible integrator. This idea was proposed by K. Itô himself in [6], and it led to the theory of "grossissement d'une filtration", of which a rather complete account may be found in [7]. The second method allows the integration of a process of the type $\varphi_t(X)$, where $\varphi_t(x)$ is an adapted random field, and X is an anticipative random vector. The idea is to consider the stochastic integral $\int_0^t \varphi_s(x)\,dW_s$, which depends on the parameter x. Provided one can show that it has a modification which is an a.s. continuous function of x, one can then "evaluate it at $x=X$". This kind of technique has been used in connection with the theory of flows by Bismut [1]. The third method consists in expanding the integrand into a series of multiple Itô-Wiener integrals, and then defining the integral through its series expansion. This last method has been used by Skorohod [18], Berger-Mizel [2], Kuo-Russek [9], Rosinski [17]. A related approach is used by Ogawa [14]. For an account of Skorohod's integral and its relation to the Malliavin Calculus, we refer to Nualart-Zakaï [13]. The last method seems to be the most general, but apparently little is known about the resulting integral.

The aim of the present paper is to construct via an elementary and very intuitive method (i.e. a variation of Itô's original construction of the stochastic integral) the integral of a particular class of anticipative integrands.

Suppose $\{X_t,\ t\in[0,1]\}$ and $\{Y^t,\ t\in[0,1]\}$ are real valued processes, which solve respectively the following forward and backward stochastic differential equations:

$$X_t=\bar{x}+\int_0^t b(X_s)\,ds+\int_0^t \sigma(X_s)\,dW(s),$$

$$Y^t=\bar{y}+\int_t^1 c(Y^s)\,ds+\int_t^1 \gamma(Y^s)\,dW(s)$$

where the last integral is a backward Itô integral (see the definition below in §2). It then follows that at each instant t, X_t is $\sigma(W(s),\ 0\leqq s\leqq t)$ measurable, and Y^t is $\sigma(W(s)-W(1),\ t\leqq s\leqq 1)$ measurable, and we want to integrate with respect to $dW(t)$ a function of both X_t and Y^t, say $\Phi(X_t, Y^t)$. Our aim is in fact to get a stochastic calculus for C^2 functions of both X_t and Y^t. Our chief motivation was the pair of forward and backward stochastic PDEs that arise in nonlinear filtering theory, see Pardoux [15]. Nevertheless, we will treat here only the case of a pair of finite dimensional SDEs, and we will give some simple applications of our results in § 6.

The paper is organized as follows. Section 2 is concerned with preliminaries, notations and a technical Lemma which will be very useful later. In Sect. 3, we construct our two sided Itô integral as a limit of sums. In Sect. 4, we prove the path continuity of our integral, and compute its quadratic variation. In Sect. 5, we study the continuity of the integral with respect to the integrand. In Sect. 6, we prove a chain rule of Itô type, define a two sided Stratonovich integral and prove a Stratonovich version of the chain rule. In Sect. 7, we compare our results with the other approaches described above. In particular, we check that our integral is a particular case of Skorohod's integral as was indicated to us by Nualart [12]. We also discuss possible extensions.

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2. Notation and Preliminaries

2.1. *Preliminaries*

Let $\{W(t),\ t\in[0,1]\}$ be a D-dimensional standard Wiener process satisfying $W(o)=0$, defined on a probability space (Ω, F, P); i.e. $W(t) = (W_1(t), W_2(t), \ldots, W_D(t))'$.

To each $t\in[0,1]$, we now associate two σ-algebras

$$F_t=\sigma(W(s),\ 0\leqq s\leqq t)$$

and

$$F^t=\sigma(W(s)-W(1);\ t\leqq s\leqq 1).$$

Then $\{F_t\}$ is a forward filtration (i.e. $F_t\uparrow$ as $t\uparrow$), and $\{F^t\}$ is a backward filtration (i.e. $F^t\uparrow$ as $t\downarrow$). We will use the notation with subscript $\{X_t\}$ to denote an F_t-adapted process, and the notation with superscript $\{Y^t\}$ to denote an F^t-adapted process. The reason for the notation $\{W(t)\}$ is that $\{W(t), t\uparrow\}$ is an F_t Wiener process, and $\{W(t)-W(1), t\downarrow\}$ is an F^t Wiener process, both having the same differential $dW(t)$.

Let us now recall the definitions of forward and backward stochastic integrals. Below, $w(t)$ stands for any of the $W_i(t)$ $1\leqq i\leqq D$. Let $\{X_t,\ t\in[0,1]\}$ be an F_t-adapted continuous process (i.e. with a.s. continuous paths) with values in $\mathbb{R}^N$, and $\Phi\in C(\mathbb{R}^N)$. Let $\{\pi^n,\ n\in\mathbb{N}\}$ denote any sequence of partitions:

$$\pi^n=\{0=t_0^n<t_1^n<\ldots<t_n^n=1\}.$$

Such that $|\pi^n|:=\sup\limits_{0\leqq k\leqq n-1}(t_{k+1}^n-t_k^n)\to 0$ as $n\to\infty$. We will in fact write t_k instead of t_k^n, for notational convenience. Then the forward Itô integral of $\Phi(X_t)$ with respect to $dw(t)$ can be defined as

$$\int_0^t \Phi(X_s)\,dw(s):=P\text{-}\lim_{n\to\infty}\sum_{k=0}^{n-1}\Phi(X_{t_k})(w(t_{k+1}\wedge t)-w(t_k\wedge t))$$

and it is well known (see, e.g., [11]) that the resulting process is a continuous forward F_t local martingale.

Let now $\{Y^t,\ t\in[0,1]\}$ be an F^t-adapted continuous process with values in $\mathbb{R}^M$, and $\psi\in C(\mathbb{R}^M)$. Then the backward Itô integral of $\psi(Y^t)$ with respect to $dw(t)$ can be defined as:

$$\int_t^1 \psi(Y^s)\,dw(s):=P\text{-}\lim_{n\to\infty}\sum_{k=0}^{n-1}\psi(Y^{t_{k+1}})(w(t_{k+1}\vee t)-w(t_k\vee t))$$

and the resulting process is a continuous backward F^t local martingale, as is readily checked by reversing the usual construction and properties of the forward integral. Note that the operation of backward Itô integration does definitely differ from that of forward Itô integration, as well as their associated chain rules (see Sect. 6). In fact, the backward Itô integral of $\psi(Y^s)$ with respect to $W(s)$ may be understood as the forward integral of $\psi(Y^{1-s})$ with respect to $W(1-s)-W(1)$.

We nevertheless avoid any specific distinct notation in order to avoid complications, since we are using different notation for F_t and F^t adapted processes.

Suppose now that $\{X_t\}$ is a forward continuous F_t semi-martingale, and $\{Y^t\}$ is a backward continuous F^t semi-martingale. We moreover assume that $\Phi\in C^1(\mathbb{R}^N)$, and $\Psi\in C^1(\mathbb{R}^M)$. We can define the forward Stratonovich integral of $\Phi(X_t)$ with respect to $dw(t)$ as:

$$\int_0^t \Phi(X_s)\circ dw(s)=\int_0^t \Phi(X_s)\,dw(s)+\tfrac{1}{2}\int_0^t \Phi'(X_s)\cdot d\langle X,w\rangle_s$$

$\left(\text{note that } \int_0^t \Phi'(X_s)\cdot d\langle X,w\rangle_s:=\sum_{i=1}^N\int_0^t \Phi'_{x_i}(X_s)\,d\langle X_i,w\rangle_s,\right.$ the . denoting scalar product$\Big)$ or also as:

$$\begin{aligned}\int_0^t \Phi(X_s)\circ dw(s)&=P\text{-}\lim\sum_{i=0}^{n-1}\frac{\Phi(X_{t_k})+\Phi(X_{t_{k+1}})}{2}(w(t_{k+1}\wedge t)-w(t_k\wedge t))\\&=P\text{-}\lim\sum_{k=0}^{n-1}\Phi(X^{\frac{t_k+t_{k+1}}{2}})(w(t_{k+1}\wedge t)-w(t_k\wedge t))\end{aligned}$$

see [11]. Note that the validity of the second definition is restricted to integration with respect to a Wiener process. Similarly, we can define the backward Stratonovich integral of $\Psi(Y^t)$ with respect to $dw(t)$ as

$$\int_t^1 \Psi(Y^s)\circ dw(s)=\int_t^1 \Psi(Y^s)\,dw(s)-\tfrac{1}{2}\int_t^1 \Psi'(Y^s)\cdot d\langle Y,w\rangle_s$$

where – as usual – $\langle Y,w\rangle_t$ denotes the joint quadratic variation of Y and w over the interval $[0,t]$.

Again, we also have:

$$\int_t^1 \Psi(Y^s) \circ dw(s) = P\text{-}\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{\Psi(Y^{t_k}) + \Psi(Y^{t_{k+1}})}{2} (w(t_{k+1} \vee t) - w(t_k \vee t))$$
$$= P\text{-}\lim_{n\to\infty} \sum_{k=0}^{n-1} \Psi(Y^{\frac{t_k+t_{k+1}}{2}}) (w(t_{k+1} \vee t) - w(t_k \vee t)).$$

Clearly, there is no need for a distinction between forward and backward Stratonovich integration, and both associated chain rules coincide with the usual one (see Sect. 6).

Let us introduce now some notation that we will be using constantly below. If x is a vector, x_i will denote its i-th component. If a is a matrix, a_i will denote its i-th row. Let f be a real-valued function. f'_x means the (partial) derivative of f with respect to x whenever x is a real variable, or the gradient of f with respect to x if x is a vector. If x varies in $\mathbb{R}^d$, and f takes values in $\mathbb{R}^k$, f'_x denotes the $k \times d$ matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$.

Let us finally indicate that δ_t will stand for the Dirac measure at t, and $\delta_{ij} \triangleq \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$

2.2. *Our Framework and First Assumptions*

Suppose we are given functions:

$$b\colon\ [0,1] \times \mathbb{R}^M \to \mathbb{R}^M,$$
$$\sigma\colon\ [0,1] \times \mathbb{R}^M \to \mathbb{R}^{M \times D},$$
$$c\colon\ [0,1] \times \mathbb{R}^N \to \mathbb{R}^N,$$
$$\gamma\colon\ [0,1] \times \mathbb{R}^N \to \mathbb{R}^{N \times D}.$$

We assume that each of these functions is measurable in (t,x) [resp. in (t,y)]; that $b(t,o)$, $\sigma(t,o)$, $c(t,o)$ and $\gamma(t,o)$ are bounded functions of t, $t \in [0,1]$; and that $\forall t \in [0,1]$, $x \to (b(t,x), \sigma(t,x))$ and $y \to (c(t,y), \gamma(t,y))$ are functions of class C^1, each first order partial derivative being a bounded function of (t,x) [resp. of (t,y)].

Given $\bar{x} \in \mathbb{R}^M$ and $\bar{y} \in \mathbb{R}^N$, we define $\{X_t,\ t \in [0,1]\}$ as the unique solution of the Itô forward stochastic differential equation

$$X_t = \bar{x} + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW(s)$$

and $\{Y^t,\ t \in [0,1]\}$ as the unique solution of the Itô backward stochastic differential equation

$$Y^t = \bar{y} + \int_t^1 c(s, Y^s)\, ds + \int_t^1 \gamma(s, Y^s)\, dW(s).$$

Note that both $E(|X_t|^p)$ and $E(|Y^t|^p)$ are bounded functions of $t\in[0,1]$, for any $p\in\mathbb{N}$.

Associated with the above SDEs are two stochastic flows, one running forward and the other backward. More precisely, for $s\leqq t$, we denote by

$$x\to\varphi(t;s,x) \tag{2.2.1}$$

the mapping from $\mathbb{R}^M$ into the set of M dimensional random vectors, which is specified by the fact that for fixed $s\in[0,1]$, $\{\varphi(t;s,x),\ s\leqq t\leqq 1\}$ solves the SDE:

$$X_t=x+\int_s^t b(u,X_u)\,du+\int_s^t \sigma(u,X_u)\,dW(u).$$

We will also use the notation $X_t^{s,x}$ for $\varphi(t;s,x)$.

Again for $s\leqq t$, we denote by

$$y\to\psi(s;t,y) \tag{2.2.2}$$

the mapping from $\mathbb{R}^N$ into the set of N dimensional random vectors, which is specified by the fact that for fixed $t\in[0,1]$, $\{\psi(s;t,y),\ 0\leqq s\leqq t\}$ solves the SDE

$$Y^s=y+\int_s^t c(u,Y^u)\,du+\int_s^t \gamma(u,Y^u)\,dW(u).$$

We will also use the notation $Y_{t,y}^s$ for $\psi(s;t,y)$.

We are not going to use any of the recently discovered properties of stochastic flows, and we do not make the corresponding hypotheses.

We will only use the following result:

Lemma 2.1. *For any* $0\leqq s\leqq t\leqq 1$, *the mappings*

$$x\to\varphi(t;s,x)\quad\textit{and}\quad y\to\psi(s;t,y)$$

*are mean-square differentiable and for any F_s measurable M-*dim *random vector ξ [resp. for any F^t measurable N-*dim *random vector η] the norm of the $M\times M$ matrix valued process $\varphi'_x(t;s,\xi)$ [resp. the $N\times N$ matrix valued process $\psi'_y(s;t,\eta)$] has a moment of order p which is bounded for $0\leqq s\leqq t\leqq 1$, $\forall p\in N$. Finally, the processes $\varphi'_x(t;s,X_s)$ and $\psi'_y(s;t,Y^t)$ are* a.s. *continuous in (s,t) on $0\leqq s\leqq t\leqq 1$.*

Proof. The mean-square differentiability is proved in Gihman-Skorohod ([4], p. 59).

Let us write $X_t^{s,\xi}$ for $\varphi_x(t;s,\xi)$, and $Z_{s,t}^i$ for $\varphi'_{x_i}(t;s,\xi)$. Then $\{Z_{s,t}^i,\ t\geqq s\}$ solves

$$Z_{s,t}^i=e_i+\int_s^t b'_x(u,X_u^{s,\xi})\,Z_{s,u}^i\,du+\sum_{j=1}^D\int_s^t(\sigma_j)'_x(u,X_u^{s,\xi})\cdot Z_{s,u}^i\,dW_j(u).$$

Where e_i denotes the vector in $\mathbb{R}^M$ whose i-th component is one, and the others zero; b'_x is the matrix $\dfrac{\partial b_i}{\partial x_j}$, and similarly for $(\sigma_j)'_x$. The fact that all moments of $Z^i_{s,t}$ are bounded follows from the boundedness of the derivative of b and σ. The existence of a modification $\varphi'_x(t; s, X_s)$ which is a.s. jointly continuous in (s,t) follows from Kolmogorov's and Gronwall's Lemmas. $\square$

We let

$$\Phi\colon [0,1] \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$$

be a measurable mapping such that $\forall (t,y) \in [0,1] \times \mathbb{R}^N$, $x \to \Phi(t,x,y)$ is of class C^1, and $\forall (t,x) \in [0,1] \times \mathbb{R}^M$, $y \to \Phi(t,x,y)$ is of class C^1, and moreover

$$\text{(H1)} \quad \begin{cases} \Phi, \Phi'_x, \Phi'_y \text{ are continuous with} \\ \text{respect to } (x,y) \text{, uniformly in } t \in [0,1] \end{cases}$$

and either

$$\text{(H2)} \quad \begin{cases} \exists K>0 \text{ and } d \in \mathbb{N} \text{ such that:} \\ |\Phi(t,x,y)| + |\Phi'_x(t,x,y)| + |\Phi'_y(t,x,y)| \leqq K(1+|x|^d+|y|^d) \\ \qquad\qquad\qquad \forall (t,x,y) \in [0,1] \times \mathbb{R}^M \times \mathbb{R}^N \end{cases}$$

or

$$\text{(H3)} \quad \begin{cases} \forall C, \ \exists K_C \text{ s.t.:} \\ |\Phi(t,x,y)| + |\Phi'_x(t,x,y)| + |\Phi'_y(t,x,y)| \leqq K_C \\ \qquad \forall (x,y) \in \mathbb{R}^{M+N} \text{ s.t. } |x| \leqq C,\ |y| \leqq C. \end{cases}$$

Our first goal is to define a "two sided Itô stochastic integral".

$$\int_s^t \Phi(u, X_u, Y^u)\, dW(u)$$

such that, when Φ does not depend on y, we get the usual forward Itô integral, and when Φ does not depend on x, we get the backward Itô integral. We will then study the properties of the above process, as a function of s and t, define a two-sided Stratonovich integral, and establish chain rules. But before doing that, let us establish a lemma, which will be a useful and practical tool in much of what follows.

2.3. *A Föllmer-Type Lemma*

The main step in the classical proof of Itô's formula consists in showing that if $\{Z_t\}$ is an adapted continuous and bounded process, then we have the following convergence in $L^2(\Omega)$:

$$\sum_{k=0}^{n-1} Z_{t^n_k}(w(t^n_{k+1}) - w(t^n_k))^2 \underset{n\to\infty}{\longrightarrow} \int_0^1 Z_t\, dt.$$

While the classical arguments use in a crucial way the adaptedness of $\{Z_t\}$, Föllmer [3] has remarked that the above convergence holds a.s., for any continuous process $\{Z_t\}$, since the random measures:

$$\mu_n = \sum_{k=0}^{n-1} (w(t_{k+1}^n) - w(t_k^n))^2 \, \delta_{t_k^n}$$

converge a.s. weakly to Lebesgue measure on $[0,1]$. The latter follows easily from the a.s. convergence: $\mu_n([0,t]) \to t$, $\forall t \in [0,1]$.

Let us now generalise Föllmer's idea. We will consider random signed measures on $[0,1]^k$, with $k=1$ or 2. For $t=(t_1,\ldots,t_k) \in [0,1]^k$, we denote by $[0,t]$ the set $\{s=(s_1,\ldots,s_k);\ 0 \leqq s_i \leqq t_i,\ i=1,\ldots,k\}$. In the sequel, for any signed measure μ, $|\mu|$ will denote the total variation of μ.

Lemma 2.2. *Let $\{\mu^n,\ n \in \mathbb{N}\}$ and μ be random signed measures on $([0,1]^k, B([0,1]^k))$, such that*

(i) $\mu^n([0,t]) \to \mu([0,t])$ *in probability*, $\forall t \in [0,1]^k$,

(ii) $\sup_n P(|\mu^n|([0,1]^k) > M) \to 0$, *as* $M \to +\infty$.

Then for any continuous process $Z = (Z(t))_{t \in [0,1]^k}$, $\mu^n(Z) \to \mu(Z)$ in probability, as $n \to \infty$; where $\mu(Z) := \int_{[0,1]^k} Z(t)\, \mu(dt)$.

Proof. Each partition π^n (as defined in Sect. 2.1) induces the following partition of $[0,1]$:

$$[0, t_1^n] \cup]t_1^n, t_2^n] \cup \ldots \cup]t_{n-1}^n, 1];$$

which in turn induces a partition $\tilde{\pi}^n$ of $[0,1]^k$.

We again assume that $|\pi^n| \to 0$, as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary. First choose $K > 0$ s.t.

$$\operatorname*{Sup}_n P(|\mu^n|([0,1]^k) + |\mu|([0,1]^k) > K) \leqq \varepsilon/2.$$

There exists $p \in \mathbb{N}$ and a random field $(Z^p(t))_{t \in [0,1]^k}$ such that

(a) $Z^p(t,\omega)$ remains constant, as t remains in a partition element of $\tilde{\pi}^p$.

(b) $P\left(\operatorname*{Sup}_{t \in [0,1]^k} |Z^p(t,\omega) - Z(t,\omega)| > \dfrac{\varepsilon}{2K}\right) \leqq \dfrac{\varepsilon}{2}$.

Clearly $\mu^n(Z^p) \to \mu(Z^p)$ in probability as $n \to \infty$, as a consequence of (i) and (a). Moreover:

$$\begin{aligned} |\mu(Z) - \mu^n(Z)| &\leqq |\mu(Z) - \mu(Z^p)| + |\mu(Z^p) - \mu^n(Z^p)| + |\mu^n(Z^p) - \mu^n(Z)| \\ &\leqq |\mu(Z^p) - \mu^n(Z^p)| + (\operatorname*{Sup}_{t \in [0,1]^k} |Z(t) - Z^p(t)|)(|\mu|([0,1]^k) + |\mu^n|([0,1]^k)), \end{aligned}$$

$$\begin{aligned} P(|\mu(Z) - \mu^n(Z)| \geqq \varepsilon) &\leqq P(|\mu(Z^p) - \mu^n(Z^p)| \geqq \varepsilon/2) \\ &\quad + P(|\mu|([0,1]^k) + |\mu^n|([0,1]^k) > K) + P\left(\operatorname*{Sup}_{t \in [0,1]^k} |Z^p(t) - Z(t)| > \frac{\varepsilon}{2K}\right), \end{aligned}$$

$$\overline{\lim_{n \to \infty}}\, P(|\mu(Z) - \mu^n(Z)| \geqq \varepsilon) \leqq \varepsilon.$$

And this last inequality holds $\forall \varepsilon > 0$. □

Lemma 2.3. *Let $\{\lambda^n,\ n\in\mathbb{N};\ \lambda\}$, $\{\mu^m,\ m\in\mathbb{N};\ \mu\}$ be random signed measures on $[0,1]$; $\{\bar\lambda^n,\ n\in\mathbb{N};\ \bar\lambda\}$, $\{\bar\mu^m,\ m\in\mathbb{N};\ \bar\mu\}$ be random finite measures on $[0,1]$, such that each of these four sets satisfy the hypotheses of Lemma 2.2 and moreover $|\lambda^n|\leqq\bar\lambda^n$ a.s., $|\mu^m|\leqq\bar\mu^m$ a.s. If $\bar\lambda\times\bar\mu(\Delta)=0$ a.s., where Δ denotes the diagonal of $[0,1]^2$, then for any random field $(Z(t))_{t\in[0,1]^2}$ which is a.s. bounded and continuous on $[0,1]^2-\Delta$, $\lambda^n\times\mu^m(Z)\to\lambda\times\mu(Z)$ in probability as $n,m\to\infty$.*

Proof. We first show that $|\lambda|\leqq\bar\lambda$; a similar proof shows that $|\mu|\leqq\bar\mu$. Let $\mathscr{B}$ denote the Borel σ-field over $[0,1]$. It suffices to show that $\mathscr{C}=\{A\in\mathscr{B};\ |\lambda(A)|\leqq\bar\lambda(A)\}$ equals $\mathscr{B}$. It follows from the hypotheses that $\mathscr{C}$ contains all intervals of the form $]s,t]$; then it contains all finite unions of disjoint subintervals of $[0,1]$. But $\mathscr{C}$ is a monotone class. Then $\mathscr{C}=\mathscr{B}$. Let now h be any smooth function from $\mathbb{R}^2$ into $[0,1]$, which is zero on a neighborhood of Δ.

$$\begin{aligned}P(|\lambda^m\times\mu^m(Z)-\lambda\times\mu(Z)|>\varepsilon)&\leqq P(|\lambda^n\times\mu^m(Z(1-h))|>\varepsilon/3)\\&\quad+P(|\lambda^n\times\mu^m(Zh)-\lambda\times\mu(Zh)|>\varepsilon/3)+P(|\lambda\times\mu(Z(1-h))|>\varepsilon/3).\end{aligned}$$

By Lemma 2.2 the middle term on the right side of the above inequality tends to zero. We obtain

$$\begin{aligned}\overline{\lim_{n,m}}\,P(|\lambda^n\times\mu^m(Z)-\lambda\times\mu(Z)|>\varepsilon)&\leqq\overline{\lim_{n,m}}\,P([\sup_t|Z(t)|]\,\bar\lambda^n\times\bar\mu^n(1-h)>\varepsilon/3)\\&\quad+P([\sup_t|Z(t)|]\,\bar\lambda\times\bar\mu(1-h)>\varepsilon/3)\leqq 2P([\sup_t|Z(t)|]\,\bar\lambda\times\bar\mu(1-h)\geqq\varepsilon/3).\end{aligned}$$

But since $\bar\lambda\times\bar\mu(\Delta)=0$ a.s., we can choose h such that the last term is as small as we want. □

3. Definition of the Two-Sided Integral

We first construct and characterize our two-sided Itô integral on the fixed interval $[0,1]$. For clarity, we first state and prove our result in the case $D=1$, and then in the case $D>1$.

Our first class of integrands, which we will denote by $\mathscr{L}^2$, is the set of processes $\{\Phi(t,X_t,Y^t),\ t\in[0,1]\}$, where X and Y are given as in Sect. 2.2, and Φ satisfies assumptions (H1) and (H2).

Lemma 3.1. *$\mathscr{L}^2$ is a vector space, and for $\rho\in L^\infty(0,1)$, if*

$$\bar X_t=\exp\left(\int_0^t\rho(s)\,dW(s)-\tfrac12\int_0^t\rho^2(s)\,ds\right),\qquad \bar X_\cdot\rho(\cdot)\in\mathscr{L}^2.$$

Proof. Let $\Phi(X,Y)$ and $\bar\Phi(\bar X,\bar Y)\in\mathscr{L}^2$. Then, if we define $\tilde X=(X\bar X)'$, $\tilde Y=(Y\bar Y)'$, clearly $\Phi(X,Y)+\bar\Phi(\bar X,\bar Y)=\tilde\Phi(\tilde X,\tilde Y)$, with $\tilde\Phi(\tilde X,\tilde Y)\in\mathscr{L}^2$. $\bar X\rho\in\mathscr{L}^2$ follows from the fact that $\bar X_t$ is the solution of the stochastic differential equation

$$\bar X_t=1+\int_0^t\bar X_s\rho(s)\,dW(s).\quad\square$$

Proposition 3.2. *Suppose $D=1$; $\{X_t\}$, $\{Y^t\}$ and Φ are defined as in Sect. 2.2, and Φ satisfies assumptions* (H1) *and* (H2).

Suppose moreover

(H4) Φ, Φ'_x *and* Φ'_y *are jointly continuous* in (t, x, y).

Let $\{\pi^n, n\in\mathbb{N}\}$ be any refining sequence of partitions of the interval $[0,1]$, *such that $|\pi^n|\to 0$, as $n\to +\infty$. For any $n\in\mathbb{N}$, define:*

$$\xi_n(\Phi)=\sum_{i=0}^{n-1} \Phi(t_i^n, X_{t_i^n}, Y^{t_{i+1}^n})(W(t_{i+1}^n)-W(t_i^n)).$$

Then $\{\xi_n(\Phi), n\in\mathbb{N}\}$ is a Cauchy sequence in $L^2(\Omega)$.

Remark. Note that in case Φ does not depend on Y, $\xi_n(\Phi)$ converges to the forward Itô integral; and in case Φ does not depend on X, $\xi_n(\Phi)$ converges to the backward Itô integral. □

Before proceeding to the proof of Proposition 3.2 let us state its main consequence.

We will use below the notations defined in (2.2.1) and (2.2.2). In the following statement, as well as in all similar expressions below Φ'_x and Φ'_y are understood as row vectors.

Theorem 3.3. *There exists a unique linear mapping $\Phi(X,Y)\to\xi[\Phi(X,Y)]$ from $\mathscr{L}^2$ into $L^2(\Omega, F_1, P)$ such that*

(i) $E[\xi(\Phi)]=0$,

(ii) $$E[\xi^2(\Phi)]=E\int_0^1 \Phi^2(t, X_t, Y^t)\,dt$$
$$+2E\int_0^1\int_0^t (\Phi'_y(s, X_s, Y^s)\,\psi'_y(s; t, Y^t)\,\gamma(t, Y^t))$$
$$(\Phi'_x(t, X_t, Y^t)\,\varphi'_s(t; s, X_s)\,\sigma(s, X_s))\,ds\,dt.$$

Moreover, if Φ satisfies (H4), *$\xi(\Phi)$ is the $L^2(\Omega)$-limit of the sequence $\{\xi_n(\Phi), n\in\mathbb{N}\}$ defined in Proposition* 3.1.

Proof of Proposition 3.2. We write $\Delta^i W$ for $W(t_{i+1}^n)-W(t_i^n)$, so that

$$\xi_n=\sum_{i=0}^{n-1} \Phi(t_i, X_{t_i}, Y^{t_{i+1}})\,\Delta^i W.$$

The proposition will follow from

$$(*)\qquad \lim_{n,m\to\infty} E(\xi_n\,\xi_m)=\chi$$

where χ is the right side of (ii).

Let us suppose without loss of generality that $n\leqq m$; i.e. π^m in a refinement of π^n. Note that the hypothesis that $\{\pi^n\}$ be a refining sequence is not essential, but does simplify the proof. We will write t_i for t_i^n and t_j for t_j^m.

$$\xi_n \xi_m = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \alpha_{ij}(n,m);$$

where

$$\alpha_{ij}(n,m) = \Phi(t_i, X_{t_i}, Y^{t_{i+1}}) \Phi(t_j, X_{t_j}, Y^{t_{j+1}}) \Delta^i W \Delta^j W$$

$$\xi_n \xi_m = A_{nm} + B_{nm} + C_{nm},$$

where

$$A_{nm} = \sum_{\{i,j;\, t_{j+1} \leqq t_i\}} \alpha_{ij}(n,m),$$

$$B_{nm} = \sum_{\{i,j;\, t_i \leqq t_j < t_{j+1} \leqq t_{i+1}\}} \alpha_{ij}(n,m),$$

$$C_{nm} = \sum_{\{i,j;\, t_{i+1} \leqq t_j\}} \alpha_{ij}(n,m).$$

Let us first compute the limit of $E(B_{nm})$. Conditioning upon $F_{t_j} \vee F^{t_{j+1}}$, one easily checks that

$$E(B_{nm}) = \sum_{\{t_i \leqq t_j < t_{j+1} \leqq t_{i+1}\}} E[\Phi(t_i, X_{t_i}, Y^{t_{i+1}}) \Phi(t_j, X_{t_j}, Y^{t_{j+1}})(t_{j+1} - t_j)].$$

It easily follows from the continuity of Φ that

$$E(B_{nm}) \to E \int_0^1 \Phi^2(t, X_t, Y^t)\, dt.$$

Let us next compute the limit of $E(C_{nm})$.

Consider $E(\alpha_{ij})$, for $t_{i+1} \leqq t_j$. With the notation introduced in Sect. 2.2, we can rewrite X_{t_j} as $\varphi(t_j; t_{i+1}, X_{t_{i+1}})$ and $Y^{t_{i+1}}$ as $\psi(t_{i+1}; t_j, Y^{t_j})$. Suppose we replace X_{t_j} by $X^i_{t_j} := \varphi(t_j; t_{i+1}, X_{t_i})$ [resp. $Y^{t_{i+1}}$ by $Y_j^{t_{i+1}} := \psi(t_{i+1}; t_j, Y^{t_{j+1}})$]; then $\Delta^i W$ [resp. $\Delta^j W$] becomes independent of all other terms in the such modified α_{ij}.

It then follows that

$$E(\alpha_{ij}) = E\{[\Phi(t_i, X_{t_i}, Y^{t_{i+1}}) - \Phi(t_i, X_{t_i}, Y_j^{t_{i+1}})] \\ \times [\Phi(t_j, X_{t_j}, Y^{t_{j+1}}) - \Phi(t_j, X^i_{t_j}, Y^{t_{j+1}})] \Delta^i W \Delta^j W\}.$$

Applying the mean value theorem twice, we obtain

$$E(\alpha_{ij}) = E\{\Phi'_y(t_i, X_{t_i}, \bar{Y}^{t_{i+1}}) \cdot (Y^{t_{i+1}} - Y_j^{t_{i+1}}) \\ \times \Phi'_x(t_j, \bar{X}_{t_j}, Y^{t_{j+1}}) \cdot (X_{t_j} - X^i_{t_j}) \Delta^i W \Delta^j W\}$$

where $\bar{Y}^{t_{i+1}}(\omega)$ lies on the segment joining $Y^{t_{i+1}}(\omega)$ and $Y_j^{t_{i+1}}(\omega)$ in $\mathbb{R}^N$. It follows from our hypotheses and the proof of the mean value theorem that one can choose $\{\bar{Y}^{t_{i+1}}(\omega)\}$ in such a way that $\bar{Y}^{t_{i+1}}$ is an $F_{t_i} \vee F^{t_{i+1}}$ measurable random vector. We could also argue exactly as we do below with the introduction of the function f. Similarly, $\bar{X}_{t_j}$ is an $F_{t_j} \vee F^{t_{j+1}}$ measurable random vector, s.t. $\bar{X}_{t_j}(\omega)$ lies on the segment joining $X_{t_j}(\omega)$ and $X^i_{t_j}(\omega)$ in $\mathbb{R}^M$.

We would like next to apply again the mean value theorem to

$$Y^{t_{i+1}} - Y_j^{t_{i+1}} = \psi(t_{i+1}; t_j, Y^{t_j}) - \psi(t_{i+1}; t_j, Y^{t_{j+1}})$$

and

$$X_{t_j} - X_{t_j}^i = \varphi(t_j; t_{i+1}, X_{t_{i+1}}) - \varphi(t_j; t_{i+1}, X_{t_i}).$$

Unfortunately, under our standing assumptions, the flows are only mean-square differentiable, so that the mean-value theorem cannot be applied directly. For $s, t \in [0, 1]$, define

$$f(t,s) = E\{\Phi'_y(t_i, X_{t_i}, \bar{Y}^{t_{i+1}}) \cdot \psi(t_{i+1}; t_j, Y^{t_j} + t(Y^{t_{j+1}} - Y^{t_j})) \times \Phi'_x(t_j, \bar{X}_{t_j}, Y^{t_{j+1}}) \cdot \varphi(t_j; t_{i+1}, X_{t_i} + s(X_{t_{i+1}} - X_{t_i}))\, \Delta^i W \Delta^j W\}.$$

Clearly

$$E(\alpha_{ij}) = f(0,1) - f(1,1) - f(0,0) + f(1,0)$$

and f is C^1 in t and f'_t is C^1 in s.

Therefore, $\exists (u,v) \in]0,1[X]0,1[$ such that

$$\begin{aligned} E(\alpha_{ij}) &= -f''_{ts}(u,v) \\ &= -E[(\Phi'_y(t_i, X_{t_i}, \bar{Y}^{t_{i+1}})\, \psi'_y(t_{i+1}; t_j, \bar{\bar{Y}}^j)\, \Delta^j Y)\, \Delta^j W \\ &\quad \times (\Phi'_x(t_j, \bar{X}_{t_j}, Y^{t_{j+1}})\, \varphi'_x(t_j; t_{i+1}, \bar{\bar{X}}_i)\, \Delta^i X)\, \Delta^i W)] \end{aligned}$$

where

$$\bar{\bar{Y}}^j = Y^{t_j} + u(Y^{t_{j+1}} - Y^{t_j}), \qquad \Delta^j Y = Y^{t_{j+1}} - Y^{t_j},$$
$$\bar{\bar{X}}_i = X_{t_i} + v(X_{t_{i+1}} - X_{t_i}), \qquad \Delta^i X = X_{t_{i+1}} - X_{t_i}.$$

Now define

$$(**) \qquad \begin{aligned} \bar{\alpha}_{ij}(n,m) = &-(\Phi'_y(t_i, X_{t_i}, Y^{t_i})\, \psi'_y(t_i; t_j, Y^{t_j})\, \Delta^j Y)\, \Delta^j W \\ &\times (\Phi'_x(t_j; X_{t_j}, Y^{t_j})\, \varphi'_x(t_j; t_i, X_{t_i})\, \Delta^i X)\, \Delta^i W. \end{aligned}$$

It is easily seen that as n and $m \to \infty$,

$$E|\alpha_{ij}(n,m) - \bar{\alpha}_{ij}(n,m)| = o([t^n_{i+1} - t^n_i]\,[t^m_{j+1} - t^m_j]).$$

On the other hand, it follows from Lemma 3.4 below that

$$\begin{aligned} C_{n,m} \to \int_0^1 \int_0^t &(\Phi'_y(s, X_s, Y^s)\, \psi'_y(s; t, Y^t)\, \gamma(Y^t)) \\ &\times (\Phi'_x(t, X_t, Y^t)\, \varphi'_x(t; s, X_s)\, \sigma(X_s))\, ds\, dt \end{aligned}$$

in probability, as n and $m \to \infty$. Uniform integrability, and hence the convergence of $E(C_{n,m})$, follows from hypothesis (ii) on Φ and Lemma 2.1. Since the fact that $n \leq m$ has not been used in the computation of $\lim E(C_{n,m})$, clearly $\lim_{n,m\to\infty} E(A_{nm}) = \lim_{n,m\to\infty} E(C_{nm})$ and $(*)$ is proved. $\square$

Lemma 3.4. *For* $0 \leq s \leq t \leq 1$, $1 \leq k \leq M$ *and* $1 \leq l \leq N$ *define*

$$Z_{kl}(s,t) = (\Phi'_y(s; X_s, Y^s)\, \psi'_y(s; t, Y^t))_l\, (\Phi'_x(t, X_t, Y^t)\, \phi'_x(t; s, X_s))_k.$$

Then

$$\sum_{\{i,j;\,t_{i+1}^n \leqq t_j^m\}} \bar{\alpha}_{ij}(n,m) \to \sum_{k,l} \int_0^1 \int_0^t Z_{kl}(s,t)\,\sigma_k(X_s)\,\gamma_l(Y^t)\,ds\,dt$$

in probability, as n and $m \to \infty$; *where* $\bar{\alpha}_{ij}(n,m)$ *has been defined by* (**) *in the preceding proof.*

Proof. Define $\bar{\alpha}_{ij}^{kl}(n,m) = -Z_{kl}(t_i^n, t_j^n)\,\Delta^i X_k \Delta^i W \Delta^i Y_l \Delta^j W$ and the signed measures

$$\lambda_k^n = \sum_{i=0}^{n-1} \Delta^i X_k \Delta^i W \delta_{t_i^n},$$

$$\mu_l^m = -\sum_{j=0}^{m-1} \Delta^j Y_l \Delta^j W \delta_{t_j^m}.$$

Note that

$$\sum_{\{i,j;\,t_{i+1}^n \leqq t_j^m\}} \alpha_{ij}^{kl}(n,m) = -\sum_{\{i,j;\,t_i^n < t_j^m < t_{i+1}^n\}} \bar{\alpha}_{ij}^{kl}(n,m) + \int_0^1 \int_0^1 \chi(s,t)\,Z_{kl}(s,t)\,d\lambda_k^n(s)\,d\mu_l^m(t)$$

with $\chi(s,t)=1$ if $s<t$, and 0 otherwise.

It is easily seen that the first term in the right side of the last equality tends to zero in $L^1(\Omega)$.

Define $\lambda_k(ds) = \sigma_k(s, X_s)\,ds$, $\mu_l(dt) = \gamma_l(t, Y^t)\,dt$,

$$\bar{\lambda}^n = \tfrac{1}{2} \sum_{i=0}^{n-1} [(\Delta^i X_k)^2 + (\Delta^i W)^2]\,\delta_{t_i^n},$$

$$\bar{\mu}_l^n = \tfrac{1}{2} \sum_{j=0}^{m-1} [(\Delta^j Y_l)^2 + (\Delta^j W)^2]\,\delta_{t_j^m},$$

$$\bar{\lambda}_k(ds) = \tfrac{1}{2}(1+\sigma_k^2(s, X_s))\,ds, \qquad \bar{\mu}_k(dt) = \tfrac{1}{2}(1+\gamma_l^2(t, Y^t))\,dt.$$

It is easily seen that we can apply Lemma 2.3, which yields

$$\lambda_k^n \times \mu_l^m(\chi Z) \to \lambda_k \times \mu_l(\chi Z)$$

in probability, as $n, m \to \infty$. □

Proof of Theorem 3.3

(a) Existence. Suppose first that (H4) holds. Using Proposition 3.1, we then define

$$\xi(\Phi) = L^2 - \lim_{n\to\infty} \xi_n(\Phi).$$

But obviously $E\xi_n(\Phi) = 0$, $\forall n$, and we have shown in the proof of Proposition 3.1 that $E[\xi_n^2(\Phi)] \to \chi$ where χ denotes the right side of (ii). It then follows that $\xi(\Phi)$ satisfies (i) and (ii).

Suppose now Φ satisfies (H1) and (H2), but not (H4). All we need to do is find a sequence of Φ_n''s satisfying (H4), s.t.

(α) $\{\xi(\Phi_n), n\in\mathbb{N}\}$ is Cauchy in $L^2(\Omega)$,

(β) the right side of $(\mathrm{ii})_n$ converges to the right side of (ii).

(α) being checked by applying (ii) to $\xi(\Phi_n-\Phi_m)=\xi(\Phi_n)-\xi(\Phi_m)$, only ($\beta$) needs to be proved.

Let $\{\rho_n, n\in\mathbb{N}\}$ be a sequence of smooth functions from $\mathbb{R}$ into $\mathbb{R}$, such that $\rho_n\geqq 0$, $\int\rho_n(t)\,dt=1$ and $\mathrm{supp}(\rho_n)\subset\left[-\frac{1}{n},\frac{1}{n}\right]$. We define

$$\Phi_n(t,x,y)=(\rho_n*\bar{\Phi}(\cdot,x,y))(t)$$

for $(t,x,y)\in[0,1]\times\mathbb{R}^M\times\mathbb{R}^N$, where:

$$\bar{\Phi}(t,x,y)=\begin{cases}\Phi(t,x,y) & \text{if } t\in[0,1]\\ \Phi(0,x,y) & \text{if } t<0\\ \Phi(1,x,y) & \text{if } t>1.\end{cases}$$

It is easy to check that Φ_n is jointly continuous, and to verify (β) with this sequence.

We note that the linearity of ξ follows immediately from the construction.

(b) Uniqueness. Choose $\rho\in L^\infty(0,1)$, $\{\bar{X}_t, 0\leqq t\leqq 1\}$ solution of the SDE

$$\bar{X}_t=1+\int_0^t\bar{X}_s\rho(s)\,dW(s),$$

$\bar{Y}_t\equiv 0$, and $\bar{\Phi}(t,x,y)=\rho(t)x$. Then $\bar{\Phi}(\bar{X},\bar{Y})\in\mathscr{L}^2$ – see Lemma 3.1. $\int_0^t\bar{X}_s\rho(s)\,dW(s)$, which is a forward Itô integral, coïncides with $\xi(\bar{\Phi}(\bar{X},\bar{Y}))$, and

$$E(\xi(\Phi(X,Y))\bar{X}_1)=E(\xi(\Phi))+E[\xi(\Phi)\,\xi(\bar{\Phi})].$$

But $E(\xi(\Phi))=0$ and

$$E[\xi(\Phi)\,\xi(\bar{\Phi})]=\tfrac{1}{2}[E(\xi^2(\Phi+\bar{\Phi}))-E(\xi^2(\Phi))-E(\xi^2(\bar{\Phi}))].$$

Using (ii), we obtain

$$\begin{aligned}E(\xi(\Phi(X,Y)),\bar{X}_1)=&\,E\int_0^1\Phi(t,X_t,Y_t)\,\rho(t)\,\bar{X}_t\,dt\\&+E\int_0^1\int_0^t(\Phi_y'(s,X_s,Y_s)\,\psi_y'(s;t,Y^t)\,\gamma(t,Y^t))\,\bar{X}_t\,\rho(t)\,\rho(s)\,ds\,dt.\end{aligned}$$

Thus $E(\xi(\Phi(X,Y))\bar{X}_1)$ is completely determined, $\forall\rho\in L^\infty(0,1)$. But as ρ varies in $L^\infty(0,1)$, $\bar{X}_1$ describes a total set in $L^2(\Omega,F_1,P)$. $\square$

We have already proved a particular case of the following immediate.

Corollary 3.5. *Let $D=1$; $\Phi(X,Y)$, $\bar{\Phi}(\bar{X},\bar{Y})\in\mathscr{L}^2$. Then*

$$E[\xi(\Phi(X,Y))\,\xi(\bar{\Phi}(\bar{X},\bar{Y}))]=E\int_0^1\Phi(t,X_t,Y^t)\,\bar{\Phi}(t,\bar{X}_t,\bar{Y}^t)\,dt$$

$$+E\int_0^1\int_0^t(\Phi'_y(s,X_s,Y^s)\,\psi'_y(s;t,Y^t)\,\gamma(t,Y^t))\,(\bar{\Phi}'_y(t,\bar{X}_t,\bar{Y}^t)\,\bar{\varphi}'_x(t;s,\bar{X}_s)\,\bar{\sigma}(s,\bar{X}_s))\,ds\,dt$$

$$+E\int_0^1\int_0^t(\bar{\Phi}'_y(s,\bar{X}_s,\bar{Y}^s)\,\bar{\psi}'_y(s;t,\bar{Y}^t)\,\bar{\gamma}(t,\bar{Y}^t))\,(\Phi'_x(t,X_t,Y^t)\,\varphi'_x(t;s,X_s)\,\sigma(s,X_s))\,ds\,dt. \quad \square$$

We now generalize the above results in the case $D>1$. We only state the results, since the proofs are obvious variations of the above ones.

Theorem 3.6. *There exists a unique linear mapping $\Phi(X,Y)\to\xi[\Phi(X,Y)]$ from $\mathscr{L}^2$ into $L^2(\Omega,F_1,P;\mathbb{R}^D)$ such that*

(i) $E[\xi(\Phi)]=0$,

(ii) $$E[\xi_i(\Phi)\,\xi_j(\Phi)]=\delta_{ij}E\int_0^1\Phi^2(t,X_t,Y^t)\,dt$$
$$+E\int_0^1\int_0^t(\Phi'_y(s,X_s,Y^s)\,\psi'_y(s;t,Y^t)\,\gamma_i(t,Y^t))$$
$$\times(\Phi'_x(t,X_t,Y^t)\,\varphi'_x(t;s,X_s)\,\sigma_j(s,X_s))\,ds\,dt$$
$$+E\int_0^1\int_0^t(\Phi'_y(s,X_s,Y^s)\,\psi'_y(s;t,Y^t)\,\gamma_j(t,Y^t))$$
$$\times(\Phi'_x(t,X_t,Y^t)\,\varphi'_x(t;s,X_s)\,\sigma_i(s,X_s))\,ds\,dt.$$

Moreover, if Φ satisfies (H4). $\{\pi^n\}$ is a refining sequence of partitions of $[0,1]$ such that $|\pi^n|\to 0$ as $n\to\infty$, and if

$$\xi_n(\Phi)=\sum_{i=0}^{n-1}\Phi(t_i^n,X_{t_i^n},Y^{t_{i+1}^n})\,(W(t_{i+1}^n)-W(t_i^n))$$

then $\xi_n(\Phi)\to\xi(\Phi)$ in $L^2(\Omega,F_1,P;\mathbb{R}^D)$, as $n\to\infty$.

Corollary 3.7. *Let $\Phi(X,Y)$, $\bar{\Phi}(\bar{X},\bar{Y})\in\mathscr{L}^2$. Then*

$$E[\xi_i(\Phi)\,\xi_j(\bar{\Phi})]=\delta_{ij}E\int_0^1\Phi(t,X_t;Y^t)\,\bar{\Phi}(t,\bar{X}_t,\bar{Y}^t)\,dt$$

$$+E\int_0^1\int_0^t(\Phi'_y(s,X_s,Y^s)\,\psi'_y(s;t,Y^t)\,\gamma_j(t,Y^t))\,(\bar{\Phi}'_x(t,\bar{X}_t,\bar{Y}^t)\,\bar{\varphi}'_x(t;s,\bar{X}_s)\,\bar{\sigma}_i(s,\bar{X}_s))\,ds\,dt$$

$$+E\int_0^1\int_0^t(\bar{\Phi}'_y(s,\bar{X}_s,\bar{Y}^s)\,\bar{\psi}'_y(s;t,\bar{Y}^t)\,\bar{\gamma}_i(t,\bar{Y}^t))\,(\Phi'_x(t,X_t,Y^t)\,\varphi'_x(t;s,X_s)\,\sigma_j(s,X_s))\,ds\,dt. \quad \square$$

Let us finally construct the integral in case Φ satisfies (H1) and (H3). We denote by $\mathscr{L}$ the set of processes $\{\Phi(t,X_t,Y^t),\ t\in[0,1]\}$, where X and Y are given as in Sect. 2.2, and Φ satisfies (H1) and (H3).

Let $f\in C^\infty(\mathbb{R}^{M+N})$ have compact support, and satisfy $f(x,y)=1$ on the set $\{(x,y);\ |x|\leqq 1 \text{ and } |y|\leqq 1\}$. For any $k\in\mathbb{N}^*$, we define $f_k(x,y):=f\left(\frac{x}{k},\frac{y}{k}\right)$. If

$\Phi(X, Y) \in \mathscr{L}$, we define for each $k \in \mathbb{N}^*$ $\Phi_k(X, Y) \in \mathscr{L}^2$ by:

$$\Phi_k(t, X_t, Y^t) := \Phi(t, X_t, Y^t)\, f_k(X_t, Y^t).$$

Finally, we denote

$$\Omega_k := \{\omega;\ \sup_{t \in [0,1]} |X_t(\omega)| \leq k,\ \sup_{t \in [0,1]} |Y^t(\omega)| \leq k\}.$$

Theorem 3.8. *There exists a unique linear mapping: $\Phi(X, Y) \to \xi(\Phi)$ from $\mathscr{L}$ into the set of classes of a.s. equal F_1-measurable random vectors s.t. $\forall k \in \mathbb{N}^*$ $\xi(\Phi) = \xi(\Phi_k)$ a.s. on Ω_k.*

Proof. Since $\bigcup_k \Omega_k = \Omega$ a.s., it suffices to check that for $l > k$, $\xi(\Phi_l)$ coincides a.s. with $\xi(\Phi_k)$ on Ω_k, which follows easily from the constructions of $\xi(\Phi_k)$ and $\xi(\Phi_l)$. □

It is worthwhile to verify the following uniqueness result.

Proposition 3.9. *Suppose $\Phi(X, Y) \in \mathscr{L}$, and moreover*

$$\Phi(t, X_t, Y^t) = 0 \qquad dt \times dP \ \text{a.e.}$$

Then $\int_0^1 \Phi(t, X_t, Y^t)\, dW(t) = 0$ a.s.

Proof. In view of Theorem 3.6(ii), it suffices to show that $\forall i \leq D$, *either*

$$(*) \qquad \Phi'_y(s, X_s, Y^s)\, \psi'_y(s; t, Y^t)\, \gamma_i(t, Y^t) = 0 \qquad 1_{\{s \leq t\}}\, ds\, dt\, dP \ \text{a.e.}$$

or else

$$(**) \qquad \Phi'_x(t, X_t, Y^t)\, \varphi'_x(t; s, X_s)\, \sigma_i(s, X_s) = 0 \qquad 1_{\{s \leq t\}}\, ds\, dt\, dP \ \text{a.e.}$$

Let us for instance establish $(**)$. The proof of $(*)$ would be analogous. Let $\{X_t^\varepsilon, 0 \leq t \leq 1\}$ be the solution of

$$X_t^\varepsilon = x + \int_0^t [b(u, X_u^\varepsilon) + 1_{[s-\varepsilon, s]}(u)\, \sigma_i(u, X_u^\varepsilon)]\, du + \int_0^t \sigma(u, X_u^\varepsilon)\, dW(u).$$

It follows from Girsanov's Lemma that the laws of $X_t{}^\varepsilon$ and X_t are equivalent. Since each of these random vectors is independent of Y^t, it follows from the hypothesis that

$$\Phi(t, X_t^\varepsilon, Y^t) = 0 \qquad dt\, dP \ \text{a.e.}$$

Moreover, $X_t^\varepsilon = \varphi(t; s, X_s^\varepsilon)$, and $X_s^\varepsilon = X_s + \int_{(s-\varepsilon)^+}^s \sigma_i(u, X_u)\, du + \eta_\varepsilon$ with η_ε given by

$$\eta_\varepsilon = \int_{(s-\varepsilon)^+}^s [b(u, X_u^\varepsilon) + \sigma_i(u, X_u^\varepsilon) - b(u, X_u) - \sigma_i(u, X_u)]\, du$$
$$+ \int_{(s-\varepsilon)^+}^s [\sigma(u, X_u^\varepsilon) - \sigma(u, X_u)]\, dW(u).$$

It is easy to show that $\frac{1}{\varepsilon}\|\eta_\varepsilon\|_{L^2(\Omega)}\to 0$. We then have

$$\frac{1}{\varepsilon}\left[\Phi\left(t,\varphi\left(t;s,X_s+\int\limits_{(s-\varepsilon)^+}^{s}\sigma_i(u,X_u)\,du+\eta_\varepsilon\right),Y^t\right)-\Phi(t,X_t,Y^t)\right]=0$$

$$1_{\{s\leqq t\}}\,ds\,dt\,dP \quad \text{a.e.}$$

(**) then follows by taking the limit in probability along a particular sequence $\varepsilon_n\to 0$, provided we show that for almost all $s\in[0,1]$,

$$\frac{1}{\varepsilon_n}\int\limits_{s-\varepsilon_n}^{s}\sigma_i(u,X_u)\,du\to\sigma_i(s,X_s)$$

in probability, for a certain sequence $\varepsilon_n\to 0$. This will follow if we show that

$$\int\limits_0^1 E\left|\frac{1}{\varepsilon}\int\limits_{s-\varepsilon}^{s}\sigma_i(u,X_u)\,du-\sigma_i(s,X_s)\right|ds\to 0.$$

But

$$\int\limits_0^1\left|\frac{1}{\varepsilon}\int\limits_{s-\varepsilon}^{s}\sigma_i(u,X_u)\,du-\sigma_i(s,X_s)\right|ds\to 0 \quad \text{a.s.}$$

and this last sequence is uniformly integrable with respect to dP. □

Remark 3.10. It is easily seen that the converse of Proposition 3.9 is not true. Choose

$$X_t=\exp\left(W(t)-\frac{t}{2}\right),\qquad Y^t=\exp\left(W(1)-W(t)-\frac{1-t}{2}\right).$$

$X-Y\in\mathscr{L}^2$, from Lemma 3.1; and $X_t-Y^t\neq 0$ $dP\times dt$ a.e. But, from both the forward and the backward Itô calculus, we have

$$\int\limits_0^1(X_t-Y^t)\,dW(t)=0 \quad \text{a.s.} \quad \square$$

4. The Two-Sided Integral as a Process

Let now $0\leqq s<t\leqq 1$. If $\Phi(X,Y)\in\mathscr{L}$, we can define

$$\xi(\Phi)_t^s:=\xi(1_{]s,t]}\Phi)=\int\limits_s^t\Phi(u,X_u,Y^u)\,dW(u).$$

Proposition 4.1. *Let (X,Y) and $(\bar{X},\bar{Y})\in\mathscr{L}^2$.*

We then have

(i) $E^{F_s\vee F^t}[\xi(\Phi)_t^s]=0,$

$$\text{(ii)}\quad E^{F_s\vee F^t}[\xi_i(\Phi)_t^s\,\xi_j(\Phi)_t^s]=\delta_{ij}E^{F_s\vee F^t}\int_s^t \Phi(u,X_u,Y^u)\,\bar{\Phi}(u,\bar{X}_u,\bar{Y}^u)\,du$$
$$+E^{F_s\vee F^t}\int_s^t\int_s^u(\Phi'_y(v,X_v,Y^v)\,\psi'_y(v;u,Y^u)\,\gamma_j(u,Y^u))\,(\bar{\Phi}'_x(u,\bar{X}_u,\bar{Y}^u)\,\bar{\varphi}'_x(u;v,\bar{X}_v)$$
$$\times\bar{\sigma}_i(v,X_v)\,dv\,du$$
$$+E^{F_s\vee F^t}\int_s^t\int_s^u(\bar{\Phi}'_y(v,\bar{X}_v,\bar{Y}^v)\,\bar{\psi}'_y(v;u,\bar{Y}^u)\,\bar{\gamma}_i(u,\bar{Y}^u))\,(\Phi'_x(u,X_u,Y^u)\,\varphi'_x(u;v,X_v)$$
$$\times\sigma_j(v,X_v)\,dv\,du.\quad\square$$

Remark 4.2. Under additional regularity assumptions on Φ and the coefficients b, σ, c, γ, it is possible to obtain an estimate of the form

$$E\left(\left|\int_s^t \Phi(u,X_u,Y^u)\,dW(u)\right|^4\right)\leqq c(t-s)^2.$$

Then from Kolmogorov's Lemma the process $\left\{\int_s^t \Phi\,dW;\ 0\leqq s<t\leqq 1\right\}$ possesses a continuous modification.

We will now prove, however, this result in greater generality with a less tedious method.

Theorem 4.3. *Let $\Phi(X,Y)\in\mathscr{L}$. Then the process $\{\xi(\Phi)_t^s,\ 0\leqq s\leqq t\leqq 1\}$ possesses a modification which is almost surely continuous.*

Proof. In order to simplify the notation, we restrict ourself to the case $D=1$. From the argument in Theorem 3.8, it is enough to prove the theorem in case $\Phi(X,Y)\in\mathscr{L}^2$, which we now assume. On the other hand, it suffices to show that $\{\xi(t):=\xi(\Phi)_t^0,\ 0\leqq t\leqq 1\}$ has an a.s. continuous modification. This will follow if we show that

$$(*)\qquad\begin{cases}\exists c,\ \gamma>0,\ \alpha>0 \text{ such that } \forall 0\leqq s\leqq t\leqq 1,\\ P(|\xi(t)-\xi(s)|>(t-s)^\gamma)\leqq C(t-s)^{1+\alpha}.\end{cases}$$

Indeed, one way of proving Kolmogorov's Lemma consists in first establishing (*) and then showing that the existence of an a.s. continuous modification follows from (*) (see, e.g., Loève [10]).

We now prove (*)

$$\xi(t)-\xi(s)=\theta+\eta,$$

where

$$\theta=\int_s^t[\Phi(u,X_u,Y^u)-\Phi(u,X_s,Y^t)]\,dW(u)$$

$$\eta=\int_s^t\Phi(u,X_s,Y^t)\,dW(u).$$

Since (X_s,Y^t) is independent of $\{W(v)-W(u);\ s\leqq u,v\leqq t\}$ η is in fact a usual Itô-Wiener integral. It then follows from (H2) and the bounds on all moments of $|X_s|$ and $|Y^t|$ that $\exists c_1$ s.t.

$$E(\eta^4)\leqq c_1(t-s)^2.$$

One easily sees that (ii) of Proposition 4.1 makes it possible to compute $E(\theta^2)$, yielding

$$E(\theta^2)=E\int_s^t [\Phi(u,X_u,Y^u)-\Phi(u,X_s,Y^t)]^2\,du$$
$$+2E\int_s^t\int_s^u (\Phi_y'(v,X_v,Y^v)\,\psi_y'(v,u,Y^u)\,\gamma(u,Y^u))$$
$$\times(\Phi_x'(u,X_u,Y^u)\,\varphi_x'(u;v,X_v)\,\sigma(v,X_v))\,dv\,du.$$

Clearly, the second term on the right side is bounded by $c_2(t-s)^2$. On the other hand from the mean value theorem,

$$E\int_s^t [\Phi(u,X_u,Y^u)-\Phi(u,X_s,Y^t)]^2\,du$$
$$=E\int_s^t [\Phi_x'(u,\bar{X}_u,\bar{Y}^u)\cdot(X_u-X_s)+\Phi_y'(u,\bar{X}_u,\bar{Y}^u)\cdot(Y^u-Y^t)]^2\,du$$
$$\leqq c_3\int_s^t [(E|X_u-X_s|^4)^{1/2}+(E|Y^u-Y^t|^4)^{1/2}]\,du$$
$$\leqq \bar{c}_3(t-s)^2.$$

Finally, for $\gamma\in(0,1/4)$,

$$P[|\xi(t)-\xi(s)|>(t-s)^\gamma]\leqq P\left[|\theta|>\frac{(t-s)^\gamma}{2}\right]+P\left[|\eta|>\frac{(t-s)^\gamma}{2}\right]$$
$$\leqq\frac{2^2}{(t-s)^{2\gamma}}E(\theta^2)+\frac{2^4}{(t-s)^{4\gamma}}E(\eta^4)$$
$$\leqq c(t-s)^{2-4\gamma}$$

where c does not depend on s,t, and $(*)$ follows. $\square$

Henceforth, $\{\xi(\Phi)_t^s,\ 0\leqq s\leqq t\leqq 1\}$ stands for its a.s. continuous modification. It follows readily from the continuity and Proposition 4.1.

Proposition 4.4. *Let Φ satisfy* (H1) *and* (H2).

Then $\{\xi(\Phi)_t^s,\ 0\leqq s\leqq t\leqq 1\}$ is the unique continuous process such that $\forall\rho\in L^2(0,1;\mathbb{R}^D)$, $\forall 0\leqq s\leqq t\leqq 1$, $\forall i\leqq D$,

$$E[\xi_i(\Phi)_t^s\bar{X}_1]=E\left[\bar{X}_1^t\int_s^t\Phi(u,X_u,Y^u)\,\rho_i(u)\,\bar{X}_u\,du\right]$$
$$+\sum_{j=1}^D E\left[\bar{X}_1^t\int_s^t\int_s^u\Phi_y'(\theta,X_\theta,Y^\theta)\,\psi_y'(\theta;u,Y^u)\,\gamma_j(u,Y^u)\,\rho_j(u)\,\bar{X}_u\,\rho_i(\theta)\,d\theta\,du\right]$$

where $\bar{X}_t^s=\exp\left\{\int_s^t\rho(u)\cdot dW(u)-\tfrac{1}{2}\int_s^t|\rho(u)|^2\,du\right\}$, $\bar{X}_t=\bar{X}_t^0$. $\square$

We could have given another formula in Proposition 4.4, had we considered $\bar{X}_1^t$ as a backward diffusion.

We now compute the quadratic variation of the process $\xi(t)$.

Theorem 4.5. *$\forall 0 \leqq s < t \leqq 1$, let $\{\pi^n, n \in \mathbb{N}\}$ be a sequence of partitions of $[s,t]$, of the form*

$$\pi^n = \{s = t_0^n < t_1^n < \ldots < t_n^n = t\}$$

where $|\pi^n| = \max_{0 \leqq k \leqq n-1} (t_{k+1}^n - t_k^n) \to 0$, as $n \to \infty$. Then, if $\Phi(X, Y) \in \mathscr{L}$,

$$\sum_{k=0}^{n-1} [\xi_i(t_{k+1}^n) - \xi_i(t_k^n)] [\xi_j(t_{k+1}^n) - \xi_j(t_k^n)] \to \delta_{ij} \int_s^t \Phi^2(u, X_u, Y^u)\, du$$

in probability, as $n \to \infty$.

In other words, we can associate to $\{\xi(t);\ 0 \leqq t \leqq 1\}$ its quadratic variation as a $d \times d$ matrix valued process $\{\langle\!\langle \xi \rangle\!\rangle(t), 0 \leqq t \leqq 1\}$ which is given by

$$\langle\!\langle \xi \rangle\!\rangle(t) = \left(\int_0^t \Phi^2(s, X_s, Y^s)\, ds \right) I.$$

Proof. Again, it suffices to establish the result in case Φ satisfies (H1) and (H2), which we suppose from now on. The proof is split into two steps.

(a) First suppose that Φ satisfies (H4). It follows from Lemma 2.2 that

$$\sum_{k=0}^{n-1} \Phi^2(t_k, X_{t_k}, Y^{t_{k+1}})\, \Delta^k W_i\, \Delta^k W_j \to \delta_{ij} \int_s^t \Phi^2(u, X_u, Y^u)\, du$$

in probability, as $n \to \infty$.

It then suffices to show that

$$\alpha_n := \sum_k \int_{t_k}^{t_{k+1}} \Phi(u)\, dW_i(u) \int_{t_k}^{t_{k+1}} \Phi(u)\, dW_j(u) - \sum_k \Phi_k^2\, \Delta^k W_i\, \Delta^k W_j \to 0$$

in probability, as $n \to \infty$; where

$$\Phi(u) := \Phi(u, X_u, Y^u),$$

$$\Phi_k := \Phi(t_k, X_{t_k}, Y^{t_{k+1}}),$$

$$2\alpha_n = \sum_k \int_{t_k}^{t_{k+1}} (\Phi(u) - \Phi_k)\, dW_i \int_{t_k}^{t_{k+1}} (\Phi(u) + \Phi_k)\, dW_j$$

$$+ \sum_k \int_{t_k}^{t_{k+1}} (\Phi(u) - \Phi_k)\, dW_j \int_{t_k}^{t_{k+1}} (\Phi(u) + \Phi_k)\, dW_i.$$

Using Schwarz's inequality, we then get

$$2E|\alpha_n| \leqq \left(\sum_k E\left[\left(\int_{t_k}^{t_{k+1}} (\Phi(u) - \Phi_k)\, dW_i \right)^2 \right] \right)^{1/2} \left(\sum_k E\left[\left(\int_{t_k}^{t_{k+1}} (\Phi(u) + \Phi_k)\, dW_j \right)^2 \right] \right)^{1/2}$$

$$+ \left(\sum_k E\left[\left(\int_{t_k}^{t_{k+1}} (\Phi(u) - \Phi_k)\, dW_j \right)^2 \right] \right)^{1/2} \left(\sum_k E\left[\left(\int_{t_k}^{t_{k+1}} (\Phi(u) + \Phi_k)\, dW_i \right)^2 \right] \right)^{1/2}.$$

It then suffices to show

(i) $\sum_k E\left[\left(\int_{t_k}^{t_{k+1}} (\Phi(u)-\Phi_k)\,dW_i\right)^2\right] \to 0$, as $n\to\infty$,

(ii) $\exists c$ s.t. $\sum_k E\left[\left(\int_{t_k}^{t_{k+1}} (\Phi(u)+\Phi_k)\,dW_i\right)^2\right] \leqq c$, $\forall n$.

Let us prove (i), (ii) being proved exactly in the same way. By the formula already used to compute $E(\theta^2)$ in the proof of Theorem 4.3, we obtain

$$\sum_k E\left[\left(\int_{t_k}^{t_{k+1}} (\Phi(u)-\Phi_k)\,dW_i\right)^2\right] = \sum_k E\int_{t_k}^{t_{k+1}} |\Phi(u)-\Phi_k|^2\,du$$

$$+2\sum_k E\int_s^t\int_s^u g_k(v,u)\,\Phi'_y(v)\,\psi'_y(v;u,Y^u)\,\gamma_i(Y^u)\,\Phi'_x(u)\,\varphi'_x(u;v,X_v)\,\sigma_i(X_v)\,dv\,du$$

where

$$g_k(v,u)=\begin{cases}1 & \text{if } t_k\leqq v\leqq u\leqq t_{k+1}\\ 0 & \text{otherwise.}\end{cases}$$

Boths terms of the above right side tend to zero as $n\to\infty$: for the first term, use the continuity of Φ; for the second, use the fact that

$$\sum_{k=0}^{n-1} g_k(v,u)\to 0 \quad dv\cdot du \text{ a.e.}$$

(b) We now suppose that Φ satisfies only (H1) *and* (H2). We associate to Φ the sequence $\{\Phi_p,\ p\in\mathbb{N}\}$ defined in the proof of Theorem 3.3 (where the index n was used instead of p). Define

$$\beta_p^n=\sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}}\Phi(s)\,dW_i\int_{t_k}^{t_{k+1}}\Phi(s)\,dW_j-\int_{t_k}^{t_{k+1}}\Phi_p(s)\,dW_i\int_{t_k}^{t_{k+1}}\Phi_p(s)\,dW_j\right).$$

It follows from arguments very similar to those used in the proof of $E|\alpha_n|\to 0$ that

$$E|\beta_p^n|\to 0, \qquad \text{as } p\to\infty, \textit{ uniformly in } n.$$

On the other hand, $\forall\varepsilon>0$,

$$P\left(\left|\sum_k\int_{t_k}^{t_{k+1}}\Phi(u)\,dW_i\int_{t_k}^{t_{k+1}}\Phi(u)\,dW_j-\delta_{ij}\int_s^t\Phi^2(u)\,du\right|>\varepsilon\right)$$

$$\leqq P\left(\left|\sum_k\int_{t_k}^{t_{k+1}}\Phi_p(u)\,dW_i\int_{t_k}^{t_{k+1}}\Phi_p(u)\,dW_j-\delta_{ij}\int_s^t\Phi_p^2(u)\,du\right|>\varepsilon/3\right)$$

$$+P(|\beta_p^n|>\varepsilon/3)+P\left(\left|\int_s^t[\Phi^2(u)-\Phi_p^2(u)]\,du\right|>\varepsilon/3\right).$$

Let us fix p such that each of the two last terms in the above right hand side is less than $\varepsilon/3$, $\forall n\in\mathbb{N}$. We can then find, using the result of Part (a), n_ε s.t. $\forall n\geqq n_\varepsilon$, the first term of the right hand side is less than $\varepsilon/3$.

We have shown that $\forall \varepsilon$, $\exists n_\varepsilon$ s.t. $\forall n \geqq n_\varepsilon$,

$$P\left(\left|\sum_k \int_{t_k}^{t_{k+1}} \Phi(u)\,dW_i \int_{t_k}^{t_{k+1}} \Phi(u)\,dW_j - \delta_{ij}\int_s^t \Phi^2(u)\,du\right| > \varepsilon\right) \leqq \varepsilon.$$

The result follows. □

Corollary 4.6. *Let* $\{A(t),\ t\in[0,1]\}$ *be a process of bounded variation, and suppose*

$$\forall t\in[0,1], \qquad A(t)+\int_0^t \Phi(s, X_s, Y^s)\,dW_i(s)=0 \quad a.s.$$

Then $A(t)=0$ *a.s.,* $\forall t\in[0,1]$, *and*

$$P\left(\exists t\in[0,1],\ s.t.\ \int_0^t \Phi(s, X_s, Y^s)\,dW_i(s)\neq 0\right)=0.$$

Proof. It follows from the assumed identity that $\{A(t)\}$ possess an a.s. continuous modification. Since it is of bounded variation, its quadratic variation is zero, as well as the joint quadratic variation of $A(\cdot)$ and $\int_0^{\cdot} \Phi(s, X_s, Y^s)\,dW_i(s)$. We then infer from the assumed identity and Theorem 4.5

$$\int_0^1 \Phi^2(t, X_t, Y^t)\,dt=0 \quad \text{a.s.}$$

The result then follows from Proposition 3.9 $\left(\text{whose conclusion holds as well for } \int_s^t \Phi(u, X_u, Y^u)\,dW(u)\right)$ and Theorem 4.3. □

5. Continuity of the Two-Sided Integral with Respect to its Integrand

We have already established a convergence result of the type $\xi(\Phi_n)\to\xi(\Phi)$ in the proof of Theorem 3.3. Here we want to have the coefficients b, σ, c, γ of Sect. 2.2 varying as well, which of course means the forward and backward diffusion X and Y will vary also. We will restrict ourselves to establishing a convergence result in $L^2(\Omega)$. However, this result can clearly be "localized".

Let $\{\bar{x}^n, \bar{y}^n;\ n\in\mathbb{N}\}$ be a sequence of initial conditions, and $\{{}^n b, {}^n\sigma, {}^n c, {}^n\gamma;\ n\in\mathbb{N}\}$ sequences of coefficients, while all possess the same regularity properties as b, σ, c, γ. We assume

(H5) $\bar{x}^n\to\bar{x}$ and $\bar{y}^n\to\bar{y}$,

(H6) $$\sup_{n,t,x,y}\{|{}^n b'(t,0)|+|{}^n b'_x(t,x)|+|{}^n\sigma(t,0)|+|{}^n\sigma'_x(t,x)| + |{}^n c(t,0)|+|{}^n c'_y(t,y)|+|{}^n\gamma(t,0)|+|{}^n\gamma'_y(t,y)|\}<\infty.$$

For almost all $t\in[0,1]$, and all $K>0$,

$$\text{(H7)}\quad \begin{aligned}&\sup_{|x|\leqq K}\{|{}^n b(t,x)-b(t,x)|+|{}^n b'_x(t,x)-b'_x(t,x)|\\ &\qquad +|{}^n\sigma(t,x)-\sigma(t,x)|+|{}^n\sigma'_x(t,x)-\sigma'_x(t,x)|\}\to 0 \quad \text{as } n\to\infty\\ &\sup_{|y|\leqq K}\{|{}^n c(t,y)-c(t,y)|+|{}^n c'_y(t,y)-c'_y(t,y)|\\ &\qquad +|{}^n\gamma(t,x)-\gamma(t,x)|+|{}^n\gamma'_y(t,x)-\gamma'_y(t,y)|\}\to 0 \quad \text{as n}\to\infty.\end{aligned}$$

Let $\{{}^nX_t\}$ and $\{{}^nY^t\}$ be the solutions of

$$ {}^nX_t=\bar{x}^n+\int_0^t {}^nb(s,{}^nX_s)\,ds+\int_0^t {}^n\sigma(s,{}^nX_s)\,dW(s),$$

$$ {}^nY^t=\bar{y}^n+\int_t^1 {}^nc(s,{}^nY^s)\,ds+\int_t^1 {}^n\gamma(s,{}^nY^s)\,dW(s).$$

We then have

Lemma 5.1. *Under* (H5), (H6) *and* (H7), $\forall p\in\mathbb{N}$, *as* $n\to\infty$

$$\sup_{t\in[0,1]} E|{}^nX_t-X_t|^p\to 0,$$

$$\sup_{t\in[0,1]} E|{}^nY_t-Y_t|^p\to 0,$$

$$\forall s\in[0,1],\qquad \sup_{t\in[s,1]} E|{}^n\varphi'_x(t;s,{}^nX_s)-\varphi'_x(t;s,X_s)|^p\to 0,$$

$$\forall t\in[0,1],\qquad \sup_{s\in[0,t]} E|{}^n\psi'_y(s;t,{}^nY^t)-\psi'_y(s;t,Y^t)|^p\to 0$$

where φ *and* ψ *are the flows defined by* (2.2.1) *and* (2.2.2).

Proof. We only prove the result concerning $\{{}^nX\}$ and $\{{}^n\varphi'_x\}$, the other proofs being similar. It suffices to prove the result for $p\geqq 2$.

(a) Convergence of $\{{}^nX\}$. Using the decompositions

$$b(X)-{}^nb({}^nX)=b(X)-{}^nb(X)+{}^nb(X)-{}^nb({}^nX),$$

$$\sigma(X)-{}^n\sigma({}^nX)=\sigma(X)-{}^n\sigma(X)+{}^n\sigma(X)-{}^n\sigma({}^nX)$$

and (H5), it is easy to establish

$$E(|X_t-{}^nX_t|^2)\leqq C_p\theta_n+C_p\int_0^t E(|X_s-{}^nX_s|^2)\,ds$$

with

$$\theta_n=E\int_0^t|b(s,X_s)-{}^nb(s,X_s)|^p\,ds+E\int_0^t|\sigma(s,X_s)-{}^n\sigma(s,X_s)|^p\,ds.$$

It follows from (H6) that $\theta_n\to 0$. The result then follows using Gronwall's Lemma.

(b) Convergence of $\{{}^n\varphi'_x\}$. We fix $s\in[0,1]$, and define

$$Z^i_t := \varphi'_{x_i}(t; s, X_s), \qquad {}^nZ^i_t := {}^n\varphi'_{x_i}(t; s, {}^nX_s).$$

We have

$$(*) \qquad Z^i_t - {}^nZ^i_t = \xi^n_t + \int_s^t {}^nb'_x(u, {}^nX_u)\,(Z^i_u - {}^nZ^i_u)\,du + \sum_{j=1}^{D} \int_s^t {}^n(\sigma_j)'_x(u, {}^nX_u)\,(Z^i_u - {}^nZ^i_u)\,dW_j(u)$$

where

$$\xi^n_t = \int_s^t [b'_x(u, {}^nX_u) - {}^nb'_x(u, {}^nX_u)]\, Z^i_u\,du + \sum_{j=1}^{D} \int_s^t [(\sigma_j)'_x(u, X_u) - ({}^n\sigma_j)'_x(u, {}^nX_u)]\, Z^i_u\,dW_j(u)$$

from

$$|b'_x(u, X_u) - {}^nb'_x(u, {}^nX_u)| \leqq |b'_x(u, X_u) - b'_x(u, {}^nX_u)| + |b'_x(u, {}^nX_u) - {}^nb'_x(u, {}^nX_u)|$$

and a similar decomposition for $(\sigma_j)'_x$, one gets, using (H7) and the first part of the proof

$$\sup_{t\in[0,1]} E(|\xi_t - \xi^n_t|^p) \to 0, \qquad \text{as } n\to\infty.$$

The result then follows from $(*)$, using (H6) and Gronwall's Lemma. $\square$

Let now $\{{}^n\Phi;\, n\in\mathbb{N}\}$ be a sequence of mappings from $[0,1]\times\mathbb{R}^M\times\mathbb{R}^N$ into $\mathbb{R}$, each one having the same regularity as Φ and satisfying (H1). We suppose moreover

$\exists K>0$ and $d\in\mathbb{N}$ such that:

$$\text{(H8)} \qquad \begin{cases} |{}^n\Phi(t,x,y)| + |{}^n\Phi'_x(t,x,y)| + |{}^n\Phi'_y(t,x,y)| \leqq K(1+|x|^d+|y|^d) \\ \qquad \forall (t,x,y)\in[0,1]\times\mathbb{R}^M\times\mathbb{R}^N, \ \forall n\in\mathbb{N}. \end{cases}$$

For all $t\in[0,1]$, and all $K>0$, as $n\to\infty$

$$\text{(H9)} \quad \begin{cases} \sup\limits_{|x|,\,|y|\leqq K} \{|{}^n\Phi(t,x,y) - \Phi(t,x,y)| + |{}^n\Phi'_x(t,x,y) - \Phi'_x(t,x,y)| \\ \qquad + |{}^n\Phi'_y(t,x,y) - \Phi'_y(t,x,y)|\} \to 0. \end{cases}$$

We finally define

$$\xi(\Phi)^s_t = \int_s^t \Phi(u, X_u, Y^u)\,dW(u),$$

$$\xi({}^n\Phi)^s_t = \int_s^t {}^n\Phi(u, {}^nX_u, {}^nY^n)\,dW(u).$$

Theorem 5.2. *Suppose* $\Phi(X,Y)\in\mathscr{L}^2$, *and moreover that* (H5), (H6), (H7), (H8) *and* (H9) *hold. Then*

$$\sup_{0\leqq s<t\leqq 1} E(|\xi(\Phi)^s_t - \xi({}^n\Phi)^s_t|^2) \to 0, \qquad \textit{as } n\to\infty.$$

Proof. To simplify the notation, we suppose that $D=1$. Considering that Φ and ${}^n\Phi$ are functions of both the $2M$ dimensional forward diffusion $\binom{X_t}{{}^nX_t}$ and the $2N$ dimensional backward diffusion $\binom{Y^t}{{}^nY^t}$, the expression for $E(|\xi(\Phi)_t^s - \xi({}^n\Phi)_t^s|^2)$ is given by Proposition 4.1, and all we have to show is that the following goes to zero as $n\to\infty$

$$
\begin{aligned}
&E\int_0^1 |\Phi(t,X_t,Y^t) - {}^n\Phi(t,{}^nX_t,{}^nY^t)|^2\,dt\\
&\quad + 2E\int_0^1\int_0^t |\Phi_y'(s,X_s,Y^s)\,\psi_y'(s;t,Y^t)\,\gamma(t,Y^t)\\
&\quad - {}^n\Phi_y'(s,{}^nX_s,{}^nY^s)\,{}^n\psi_y'(s;t,{}^nY^t)\,{}^n\gamma(t,{}^nY^t)|\\
&\quad \times|\Phi_x'(t,X_t,Y^t)\,\varphi_x'(t;s,X_s)\,\sigma(s,X_s)\\
&\quad - {}^n\Phi_x'(t,{}^nX_t,{}^nY^t)\,{}^n\varphi_x'(t;s,{}^nX_s)\,{}^n\sigma(s,{}^nX_s)|\,ds\,dt.
\end{aligned}
$$

In other words, we need only check

(*) $\quad {}^n\Phi(t,{}^nX_t,{}^nY^t)\to\Phi(t,X_t,Y^t)\quad$ in $L^2(dt\,dP)$,

(**) $\quad {}^n\Phi_y'(s,{}^nX_s,{}^nY^s)\,{}^n\psi_y'(s;t,{}^nY^t)\,{}^n\gamma(t,{}^nY^t)\to\Phi_y'(s,X_s,Y^s)\,\psi_y'(s;t,Y^t)\,\gamma(t,Y^t)$ in $L^2(1_{\{s\leqq t\}}\,ds\,dt\,dP)$,

(***) $\quad {}^n\Phi_x'(t,{}^nX_t,{}^nY^t)\,{}^n\varphi_x'(t;s,{}^nX_s)\,{}^n\sigma(s,{}^nX_s)\to\Phi_x'(t,X_t,Y^t)\,\varphi_x'(t;s,X_s)\,\sigma(s,X_s)$ in $L^2(1_{\{s\leqq t\}}\,ds\,dt\,dP)$.

These follow easily from Lemma 5.1, (H8) and (H9). Note that we use Lemma 5.3 below to take the limit in probability of ${}^n\Phi(\cdot)$, ${}^n\Phi_x'(\cdot)$ and ${}^n\Phi_y'(\cdot)$; and (H8) plus Lemma 5.1 to get the uniform integrability. □

Lemma 5.3. *Let $\{Z_n,\ n\in\mathbb{N};\ Z\}$ be k-dimensional random variables, and $\{f_n,\ n\in\mathbb{N};\ f\}\subset C(\mathbb{R}^k)$. If $Z_n\to Z$ in probability, and $f_n\to f$ uniformly on compact sets, then $f_n(Z_n)\to f(Z)$ in probability.*

Proof. Since convergence in probability of a sequence of r.v. is equivalent to the fact that from any subsequence one can extract a further subsequence which converges a.s., it is in fact sufficient to show that $Z_n\to Z$ a.s. $\Rightarrow f_n(Z_n)\to f(Z)$ a.s. This follow from the decomposition

$$f(Z)-f_n(Z_n)=f(Z)-f(Z_n)+f(Z_n)-f_n(Z_n)$$

and the fact that

$$\{Z_n(\omega)\}\text{ converges}\Rightarrow\{Z_n(\omega)\text{ remains in a compact subset of }\mathbb{R}^k\}.\quad\square$$

6. Differential Calculus

6.1. A Chain Rule of Itô Type

Theorem 6.1. *Let Φ: $[0,1]\times\mathbb{R}^M\times\mathbb{R}^N\to\mathbb{R}$ be once continuously differentiable with respect to t, and twice continuously differentiable with respect both to x and to y, Φ, Φ'_t, Φ'_x, Φ''_{xx}, Φ'_y and Φ''_{yy} being jointly continuous in (t,x,y). We then have*

$$\forall 0\leqq s<t\leqq 1,$$

$$\begin{aligned}\Phi(t,X_t,Y^t)=\Phi(s,X_s,Y^s)&+\int_s^t\Phi'_u(u,X_u,Y^u)\,du+\int_s^t\Phi'_x(u,X_u,Y^u)\,b(u,X_u)\,du\\&+\int_s^t\Phi'_x(u,X_u,Y^u)\,\sigma(u,X_u)\,dW(u)+\tfrac{1}{2}\int_s^t\mathrm{Tr}[\Phi''_{xx}(u,X_u,Y^u)\,\sigma\sigma^*(u,X_u)]\,du\\&-\int_s^t\Phi'_y(u,X_u,Y^u)\,c(u,Y^u)\,du-\int_s^t\Phi'_y(u,X_u,Y^u)\,\gamma(u,Y^u)\,dW(u)\\&-\tfrac{1}{2}\int_s^t\mathrm{Tr}[\Phi''_{yy}(u,X_u,Y^u)\,\gamma\gamma^*(u,Y^u)]\,du\quad a.s.\end{aligned}$$

which we also write in more concise form as

$$\begin{aligned}\Phi(t,X_t,Y^t)=\Phi(s,X_s,Y^s)&+\int_s^t\Phi'_u(u,X_u,Y^u)\,du+\int_s^t\Phi'_x(u,X_u,Y^u)\,dX_u\\&+\tfrac{1}{2}\int_s^t\mathrm{Tr}[\Phi''_{xx}(u,X_u,Y^u)\,\sigma\sigma^*(u,X_u)]\,du+\int_s^t\Phi'_y(u,X_u,Y^u)\,dY^u\\&-\tfrac{1}{2}\int_s^t\mathrm{Tr}[\Phi''_{yy}(u,X_u,Y^u)\,\gamma\gamma^*(u,Y^u)]\,du\quad a.s.\end{aligned}$$

Proof. We first remark that the formula makes sense, in particular since the coefficients of the two-sided stochastic integrals belong zo $\mathscr{L}$.

Since it suffices to show the formula on each $\Omega_n:=\{\omega;\ |X_t(\omega)|\leqq n, |Y^t(\omega)|\leqq n,\ \forall t\in[0,1]\}$, we assume without loss of generality that Φ, Φ'_t, Φ'_x, Φ''_{xx}, Φ'_y and Φ''_{yy} are bounded.

Since by Theorem 5.2 we can approximate σ and γ by sequences of jointly continuous coefficients in such a way that we can take the limits in all the terms of the formula to be proved, we further assume that σ, γ, σ'_x and γ'_y are jointly continuous in (t,x) [resp. (t,y)]. Also, we will prove the case $N=M=D=1$, its multidimensional version being exactly the same, except for vector and matrix notation.

Let $\{\pi^n, n\in\mathbb{N}\}$ be a refining sequence of partitions of $[s,t]$, of the form

$$\pi^n=\{s=t_0^n<t_1^n<\ldots<t_n^n=t\}$$

and such that $|\pi^n|=\sup_{0\leqq i\leqq n-1}(t_{i+1}^n-t_i^n)\to 0$, as $n\to\infty$. As usual, we write t_i instead of t_i^n.

$$\Phi(t, X_t, Y^t) - \Phi(s, X_s, Y^s) = \sum_{i=0}^{n-1} [\Phi(t_{i+1}, X_{t_{i+1}}, Y^{t_{i+1}}) - \Phi(t_i, X_{t_i}, Y^{t_i})]$$

$$= \sum_{i=0}^{n-1} [\Phi(t_{i+1}, X_{t_{i+1}}, Y^{t_{i+1}}) - \Phi(t_i, X_{t_{i+1}}, Y^{t_{i+1}})]$$

$$+ \sum_{i=0}^{n-1} [\Phi(t_i, X_{t_{i+1}}, Y^{t_{i+1}}) - \Phi(t_i, X_{t_i}, Y^{t_{i+1}})]$$

$$+ \sum_{i=0}^{n-1} [\Phi(t_i, X_{t_i}, Y^{t_{i+1}}) - \Phi(t_i, X_{t_i}, Y^{t_i})] = A_n + B_n + C_n.$$

Now

$$A_n = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \Phi_t'(s, X_{t_{i+1}}, Y^{t_{i+1}})\, ds,$$

and it follows easily from the continuity of Φ_t' with respect to x and y and from the continuity of the paths of $\{X_t\}$ and $\{Y^t\}$ that:

$$A_n \to \int_s^t \Phi_u'(u, X_u, Y^u)\, du \text{ a.s.,} \qquad \text{as } n \to \infty.$$

$$B_n = \sum_{i=0}^{n-1} \Phi_x'(t_i, X_{t_i}, Y^{t_{i+1}})(X_{t_{i+1}} - X_{t_i}) + \tfrac{1}{2} \sum_{i=0}^{n-1} \Phi_{xx}''(t_i, \bar{X}_i, Y^{t_{i+1}})(X_{t_{i+1}} - X_{t_i})^2$$

where $\bar{X}_i$ is a random intermediate point between X_{t_i} and $X_{t_{i+1}}$.

$$B_n = B_n^1 + B_n^2 + \tfrac{1}{2} B_n^3,$$

with

$$B_n^1 = \sum_{i=0}^{n-1} \Phi_x'(t_i, X_{t_i}, Y^{t_{i+1}}) \int_{t_i}^{t_{i+1}} b(u, X_u)\, du,$$

$$B_n^2 = \sum_{i=0}^{n-1} \Phi_x'(t_i, X_{t_i}, Y^{t_{i+1}}) \int_{t_i}^{t_{i+1}} \sigma(u, X_u)\, dW(u),$$

$$B_n^3 = \sum_{i=0}^{n-1} \Phi_{xx}''(t_i, \bar{X}_i, Y^{t_{i+1}})(X_{t_{i+1}} - X_{t_i})^2.$$

One easily checks that

$$B_n^1 \to \int_s^t (\Phi_x'(u, X_u, Y^u)\, b(u, X_u)\, du \text{ a.s.,} \qquad \text{as } n \to \infty.$$

On the other hand, if we define

$$\mu_n = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2\, \delta_{t_i}$$

it follows easily from Lemma 2.2

$$\tilde{B}_n^3 := \sum_{i=0}^{n-1} \Phi_{xx}''(t_i, X_{t_i}, Y^{t_i})(X_{t_{i+1}} - X_{t_i})^2 \to \int_s^t \Phi_{xx}''(u, X_u, Y^u)\, \sigma^2(u, X_u)\, du$$

in probability, as $n\to\infty$. But, from uniform continuity,

$$|B_n^3-\tilde{B}_n^3|\leqq\sup_i|\Phi''_{xx}(t_i,\bar{X}_i,Y^{t_{i+1}})-\Phi''_{xx}(t_i,X_{t_i},Y^{t_i})|\left(\sum_{i=0}^{n-1}|X_{t_{i+1}}-X_{t_i}|^2\right)$$

and the latter tends to zero a.s., as $n\to\infty$.

From Proposition 3.2, we know that

$$\sum_{i=0}^{n-1}\Phi'_x(t_i,X_{t_i},Y^{t_{i+1}})\,\sigma(t_i,X_{t_i})\,(W_{t_{i+1}}-W_{t_i})\to\int_s^t\Phi'_x(u,X_u,Y^u)\,\sigma(u,X_u)\,dW(u)$$

in probability, as $n\to\infty$.

To establish the desired convergence of the sequence B_n, it remains to show that

$$\sum_{i=0}^{n}\Phi'_x(t_i,X_{t_i},Y^{t_{i+1}})\int_{t_i}^{t_{i+1}}[\sigma(u,X_u)-\sigma(t_i,X_{t_i})]\,dW(u)\to 0$$

in probability, as $n\to\infty$. We use again Lemma 2.2. Indeed, let

$$\mu_n:=\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}[\sigma(u,X_u)-\sigma(t_i,X_{t_i})]\,dW(u)\right)\delta_{t_i}$$

$\{\mu_n,\ n\in\mathbb{N}\}$ satisfies the hypotheses of Lemma 2.2, with 0 as its limit.

The sequence C_n is treated in exactly the same way as B_n. □

Example 6.2. We suppose here that $M=N$. Let $A, B_1, \ldots, B_D$ be $M\times M$ matrices (which might as well depend on t), and let $\{X_t\}$, $\{Y^t\}$ be the solutions of:

$$X_t=\bar{x}+\int_0^t AX_s\,ds+\sum_{i=1}^{D}\int_0^t B_iX_s\,dW_i(s),$$

$$Y^t=\bar{y}+\int_t^1 A^*Y^s\,ds+\sum_{i=1}^{D}\int_t^1 B_i^*Y^s\,dW_i(s).$$

It is known (for the corresponding result for stochastic PDES, see Pardoux [15], Krylov and Rozovskii [8]) that the scalar valued process $\{(X_t,Y^t),\ t\in[0,1]\}$ is a.s. constant. With the aid of our Itô formula, we can prove it directly.

$$\begin{aligned}(X_t,Y^t)=(X_s,Y^s)&+\int_s^t(AX_u,Y^u)\,du+\sum_i\int_s^t(B_iX_u,Y^u)\,dW_i(u)\\&-\int_s^t(X_u,A^*Y^u)\,du-\sum_i\int_s^t(X_u,B_i^*Y^u)\,dW_i(u)=(X_s,Y^s).\end{aligned}$$

We have used the linearity of the two-sided stochastic integral. □

6.2. A Chain Rule of Stratonovich Type

We begin by defining the two-sided Stratonovich integral.

Let first Φ denote a functional which satisfies (H1) and (H3), and does not depend on t. $\{\pi^n\}$ again denotes a refining sequence of partitions of $[0,1]$.

Consider the sequence

$$\eta_n := \sum_{i=0}^{n-1} \tfrac{1}{2}[\Phi(X_{t_i}, Y^{t_i}) + \Phi(X_{t_{i+1}}, Y^{t_{i+1}})]\, \Delta^i W,$$

$$\eta_n = \sum_{i=0}^{n-1} \Phi(X_{t_i}, Y^{t_{i+1}})\, \Delta^i W + \tfrac{1}{2}\sum_{i=0}^{n-1} [\Phi(X_{t_{i+1}}, Y^{t_{i+1}}) - \Phi(X_{t_i}, Y^{t_{i+1}})\, \Delta^i W$$
$$+ \tfrac{1}{2}\sum_{i=0}^{n-1} [\Phi(X_{t_i}, Y^{t_i}) - \Phi(X_{t_i}, Y^{t_{i+1}})]\, \Delta^i W.$$

Using again the mean value theorem and Lemma 2.2, we obtain

$$\eta_n \to \int_0^1 \Phi(X_s, Y^s)\, dW(s) + \tfrac{1}{2}\int_0^1 (\Phi'_x(s, X_s, Y^s)\, \sigma(s, X_s))^*\, ds$$
$$+ \tfrac{1}{2}\int_0^1 (\Phi'_y(s, X_s, Y^s)\, \gamma(s, Y^s))^*\, ds$$

in probability, as $n \to \infty$; where $*$ denotes transpose. Note that the sequence

$$\eta'_n := \sum_{i=0}^{n-1} \Phi\left(\frac{X_{t_i} + X_{t_{i+1}}}{2}, \frac{Y^{t_i} + Y^{t_{i+1}}}{2}\right) \Delta^i W$$

also converges to the same limit as $\{\eta_n\}$.

Motivated by these considerations, we give the following

Definition 6.3. Let Φ satisfy (H1) and (H3), and $0 \leqq s \leqq t \leqq 1$. We define the two-sided Stratonovich stochastic integral of $\Phi(u, X_u, Y^u)$ with respect to $dW(u)$ over the interval $[s,t]$ as

$$\int_s^t \Phi(u, X_u, Y^u) \circ dW(u) := \int_s^t \Phi(u, X_u, Y^u)\, dW(u)$$
$$+ \tfrac{1}{2}\int_s^t (\Phi'_x(u, X_u, Y^u)\, \sigma(u, X_u))^*\, du + \tfrac{1}{2}\int_s^t (\Phi'_y(u, X_u, Y^u)\, \gamma(u, Y^u))^*\, du. \quad \square$$

Using the connection between the Itô forward [resp. backward] and the Stratonovich forward [resp. backward] integrals, we can rewrite the equations for $\{X_t\}$ and $\{Y^t\}$ in Stratonovich form as follows

$$X_t = \bar{x} + \int_0^t \tilde{b}(s, X_s)\, ds + \int_0^t \sigma(s, X_s) \circ dW(s)$$

where $\tilde{b}(s,x)=b(s,x)-\frac{1}{2}\sum_{i=1}^{D}[(\sigma_i)'_x\sigma_i](s,x)$, $(\sigma_i)'_x$ denoting the $N\times N$ matrix whose element of the j-th row and k-th column is $\frac{\partial\sigma_{ij}}{\partial x_k}$.

$$Y^t=\bar{y}+\int_t^1 \tilde{c}(s,Y^s)\,ds+\int_t^1 \gamma(s,Y^s)\circ dW(s)$$

where $\tilde{c}(s,y)=c(s,y)-\frac{1}{2}\sum_{i=1}^{D}[(\gamma_i)'_y\gamma_i](s,y)$.

Theorem 6.4. *Let Φ: $[0,1]\times\mathbb{R}^M\times\mathbb{R}^N\to\mathbb{R}$ be once continuously differentiable with respect to t, twice continuously differentiable with respect to (x,y), Φ, Φ'_t, Φ'_x, Φ''_{xx}, Φ'_y, Φ''_{yy} and Φ''_{xy} being jointly continuous with respect to (t,x,y). We then have*

$$\forall 0\leqq s<t\leqq 1,$$

$$\begin{aligned}\Phi(t,X_t,Y^t)=\Phi(s,X_s,Y^s)&+\int_s^t \Phi'_u(u,X_u,Y^u)\,du+\int_s^t \Phi'_x(u,X_u,Y^u)\cdot\tilde{b}(u,X_u)\,du\\&+\int_s^t \Phi'_x(u,X_u,Y^u)\,\sigma(u,X_u)\circ dW(u)-\int_s^t \Phi'_y(u,X_u,Y^u)\cdot\tilde{c}(u,Y^u)\,du\\&-\int_s^t \Phi'_y(u,X_u,Y^u)\,\gamma(u,Y^u)\circ dW(u)\quad a.s.\end{aligned}$$

which we also write in more concise form as

$$\begin{aligned}\Phi(t,X_t,Y^t)=\Phi(s,X_s,Y^s)&+\int_s^t \Phi'_u(u,X_u,Y^u)\,du+\int_s^t \Phi'_x(u,X_u,Y^u)\circ dX_u\\&+\int_s^t \Phi'_y(u,X_u,Y^u)\circ dY^u\quad a.s.\end{aligned}$$

Proof. From Theorem 6.1, it suffices to show that:

$$\begin{aligned}&\int_s^t(\Phi'_x(u,X_u,Y^u)\,\sigma(u,X_u)\circ dW(u))-\tfrac{1}{2}\sum_{i=1}^{D}\int_s^t \Phi'_x(u,X_u,Y^u)\cdot[(\sigma'_i)_x\sigma_i](u,X_u)\,du\\&\quad-\int_s^t(\Phi'_y(u,X_u,Y^u)\,\gamma(u,Y^u)\circ dW(u))+\tfrac{1}{2}\sum_{i=1}^{D}\int_s^t \Phi'_y(u,X_u,Y^u)\cdot[(\gamma_i)'_y\gamma_i](u,Y^u)\,du\\&=\int_s^t(\Phi'_x(u,X_u,Y^u)\,\sigma(u,X_u)\,dW(u))+\tfrac{1}{2}\int_s^t \mathrm{Tr}[\Phi''_{xx}(u,X_u,Y^u)\,\sigma\sigma^*(u,X_u)]\,du\\&\quad-\int_s^t(\Phi'_y(u,X_u,Y^u)\,\gamma(u,Y^u)\,dW(u))-\tfrac{1}{2}\int_s^t \mathrm{Tr}[\Phi''_{yy}(u,X_u,Y^u)\,\gamma\gamma^*(u,Y^u)]\,du.\end{aligned}$$

But this equality follows from Definition 6.3. □

Example 6.5. Let $A(t)$, $B_1(t),\ldots,B_D(t)$ again denote $M\times M$ matrix valued bounded and measurable functions of t. We consider the following stochastic differential equation written in Stratonovich form

$$dX_t = A(t)\, X_t\, dt + \sum_{i=1}^{D} B_i(t)\, X_t \circ dW_i(t).$$

We associate to this equation its fundamental solution, i.e. the process $\Phi(t,s)$ which takes values in the set of $M \times M$ matrices, and solves, $\forall s$ fixed

$$(*) \qquad d\Phi(t,s) = A(t)\, \Phi(t,s)\, dt + \sum_{i=1}^{D} B_i(t)\, \Phi(t,s) \circ dW_i(t)$$

together with the boundary condition $\Phi(s,s) = I$. We can consider $(*)$ either as a forward SDE for $t \geqq s$, or as a backward SDE for $t \leqq s$, so that $\Phi(t,s)$ is defined for all $s, t \in \mathbb{R}$. We want to prove that

$$\Phi(t,s) = \Phi^{-1}(s,t) \text{ a.s.}, \qquad \forall s, t \in \mathbb{R}$$

which is in fact a particular case of general results on stochastic flows, and a generalization of a well-known fact on O.D.E.s. Let us choose $s < t$, and consider the following process

$$\{\Phi(u,t)^{-1}\, \Phi(u,s),\ u \in [s,t]\}.$$

It is a function of both the forward diffusion

$$\Phi(u,s) = I + \int_s^u A(\theta)\, \Phi(\theta,s)\, d\theta + \sum_i \int_s^u B_i(\theta)\, \Phi(\theta,s) \circ dW_i(\theta),$$
$$(s \leqq u \leqq t)$$

and the backward diffusion

$$\Phi^{-1}(u,t) = I + \int_u^t \Phi^{-1}(\theta,t)\, A(\theta)\, d\theta + \sum_i \int_u^t \Phi^{-1}(\theta,t)\, B_i(\theta) \circ dW_i(\theta),$$
$$(s \leqq u \leqq t).$$

It now follows from Theorem 6.4 that

$$d_u[\Phi(u,t)^{-1}\, \Phi(u,s)] = 0.$$

Hence $\Phi(t,s) = \Phi(s,t)^{-1}$ a.s. □

7. Comparison with Other Approaches and Possible Extensions

7.1. Comparison with the Filtration Enlargment Approach

It is not hard to show that Y^t is $F_t \vee \sigma\ (Y^0)$-measurable. Therefore we now define: $G_t = F_t \vee \sigma(Y^0)$. The filtration $\{G_t\}$ is obtained from $\{F_t\}$ by an initial enlargment, i.e. we enlarge F_0 to $G_0 = F_0 \vee \sigma(Y^0)$, and then define $G_t = F_t \vee G_0$.

The question now is whether or not $W(t)$ is a G_t-semi-martingale. It follows from the result in Pardoux [16] that, provided in addition to the hypotheses in Sect. 2.2

$$
\text{(H10)} \begin{cases} \text{(i) } \forall t<1, \text{ the law of } Y^t \text{ has a density } p(t,\cdot) \text{ and there exists} \\ \qquad k\in\mathbb{N} \text{ s.t. } p(t,\cdot)\in L^2(\mathbb{R}^N; (1+|x|^k)^{-1}dx), \\ \text{(ii) } \dfrac{\partial^2(\gamma\gamma^*)_{ij}}{\partial x_i \partial x_j}\in L^\infty(]0,1[\times\mathbb{R}^N) \end{cases}
$$

then $\{W(t),\ t\in[0,1[\}$ is a G_t semi-martingale. It then follows that $\forall t\in]0,1[$, we can define the forward Itô integral of the process $\{\Phi(t, X_t, Y^t)\}$, which is G_t-adapted, with respect to the semi-martingale $W(t)$

$$\int_0^t \Phi(s, X_s, Y^s)\cdot dW(s).$$

If in addition Φ satisfies (H4), then the above integral is the limit of

$$\sum_{i=0}^{n-1} \Phi(t_i^n, X_{t_i^n}, Y^{t_i^n})(W(t_{i+1}^n)-W(t_i^n)) \quad \text{if} \quad \pi^n=0=t_0^n<t_1^n<\ldots<t_n^n=t,$$

and $|\pi^n|\to 0$. It follows that

$$\int_0^t \Phi(s, X_s, Y^s)\cdot dW(s)=\int_0^t \Phi(s, X_s, Y^s)\,dW(s)+\int_0^t \Phi_y'(s, X_s, Y^s)\,\gamma(s, Y^s)\,ds.$$

Clearly, the filtration enlargment approach is feasible only under additional restrictions. Of course, we could interchange the roles of X and Y. In any case, the symmetry of the two-sided integral is lost.

7.2. *Comparison with the "Random Field Approach"*

Suppose the coefficients c and γ are continuous in (t, y). Then $\psi(s; t, y)$ has a modification which is a.s. continuous in (s, t, y), and such that moreover

$$y\to\psi(s; t, y)$$

is a.s. an onto homeomorphism. Denote by $\psi_{t,s}^{-1}$ its inverse. The process to be integrated can be written as

$$\Phi(t, X_t, \psi_{t,0}^{-1}(Y^0)).$$

Let $y\in\mathbb{R}^M$, then the process $\Phi(t, X_t, \psi_{t,0}^{-1}(y))$ is F_t-adapted, and we can define the forward Itô integral

$$I(y)=\int_0^1 \Phi(t, X_t, \psi_{t,0}^{-1}(y))\,dW(t).$$

Let $p>0$. It follows from Burkholder-Davis-Gundy's inequality (see Ikeda and Watanabe [5]) that

$$E(|I(y)-I(z)|^p) \leqq C_p \int_0^1 E(|\Phi(t, X_t, \psi_{t,0}^{-1}(y)) - \Phi(t, X_t, \psi_{t,0}^{-1}(z))|^p)\, dt.$$

The existence of an a.s. continuous modification of $\{I(y),\ y \in \mathbb{R}^M\}$ will follow from Kolmogorov's Lemma if we can estimate the above quantity by $C|x-y|^p$, provided that $p>M$. Such an estimate can be obtained under slightly more restrictive conditions than our conditions in Sect. 2.2. Provided $I(y)$ is a.s. continuous, we can define $I(Y^0)$, and we have again

$$I(Y^0) = \int_0^1 \Phi(t, X_t, Y^t)\, dW(t) + \int_0^1 \Phi_y'(t, X_t, Y^t)\, \gamma(t, Y^t)\, dt.$$

In addition to the fact that it does break the symmetry with respect to time reversal, the present approach is not extendable to infinite dimensional situations. Indeed Kolmogorov's Lemma would not apply. Moreover if we replace the SDEs for X and Y by stochastic partial differential equations of parabolic type, then the associated flows do not possess smooth inverses.

7.3. Comparison with Skorohod's Integral

In [18] Skorohod defined a stochastic integral of a large class of anticipative integrands with respect to a Wiener process, over a fixed time-interval. Unfortunately, this work seems not be well known. Only at the very end of our research did we learn about it. We would like to thank D. Nualart and M. Zakaï as well as E. Wong, who drew our attention to Skorohod's integral. We now prove a result, which was first suggested to us by E. Wong, and elaborated upon by D. Nualart [12] (we restrict ourself for simplicity to the case $D=1$).

Theorem 7.1. *Suppose that the hypotheses of Sect. 2.2 are in force, and in particular that Φ satisfies* (H1) *and* (H2). *Then the Skorohod integral of $\Phi(t, X_t, Y^t)$ over the interval* $[0, 1]$ *exists and coïncides with the two-sided integral*

$$\int_0^1 \Phi(t, X_t, Y^t)\, dW(t).$$

Proof. The result is a direct consequence of Proposition 3.1 in Nualart and Zakaï [13]. Indeed, from Theorem 3.3, all we need to show is that any element in $\mathscr{L}^2$ is integrable in the sense of Skorohod, and that the Skorohod integral is linear and satisfies (i) and (ii) in Theorem 3.3.

On the other hand, Proposition 3.1 in [13] says that any measurable process u such that

$$E\int_0^1 u^2(t)\, dt + E\int_0^1 \int_0^1 |D_s u(t)|^2\, ds\, dt < \infty$$

is Skorohod integrable, its Skorohod integral has mean zero and variance equal to

$$E\int_0^1 u^2(t)\,dt + E\int_0^1\int_0^1 D_s u(t)\, D_t u(s)\, ds\, dt$$

where $\{D_s u(t),\ 0 \leqq s \leqq 1\}$ denotes the Malliavin derivative of the random variable $u(t)$. Let us compute the latter in our case. We use well-known facts about Malliavin derivatives, which can be found e.g. in [13].

$$\begin{aligned} &D_s \Phi(t, X_t, Y^t) = \Phi'_x(t, X_t, Y^t)\, D_s X_t + \Phi'_y(t, X_t, Y^t)\, D_s Y^t,\\ &D_s X_t = 1_{\{s \leqq t\}}\, \varphi'_x(t; s, X_s)\, \sigma(s, X_s),\\ &D_s Y^t = 1_{\{t \leqq s\}}\, \psi'_y(t; s, Y^s)\, \gamma(s, Y^s). \end{aligned}$$

The result follows immediately. □

The same result would be true for integrals over the interval $[s, t]$, $\forall 0 \leqq s \leqq t \leqq 1$. Note that there exists up to now no result concerning the general Skorohod integral as a process.

7.4. *Possible Extensions*

Clearly, our approach could be adapted to the case of a pair of diffusion processes with values in an infinite dimensional space, e.g. to the case of a pair of stochastic partial differential equations. It could also be adapted to the case of "diffusions with jumps".

In fact, the comparison with Skorohod's integral suggest that it might be possible to adapt our results to a pair of forward and backward semi-martingales, which would not necessarily be Markov processes.

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