*Stochastics*, 1979, Vol. 3, pp. 127–167
0309-1929/79/0302-0127 $04.50/0


# Stochastic Partial Differential Equations and Filtering of Diffusion Processes

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*(Received April 15, 1979; in final form August 5, 1979)*

We establish basic results on existence and uniqueness for the solution of stochastic PDE's. We express the solution of a backward linear stochastic PDE in terms of the conditional law of a partially observed Markov diffusion process. It then follows that the adjoint forward stochastic PDE governs the evolution of the "unnormalized conditional density".



## INTRODUCTION

Let $X_t$ be a Markov diffusion process with generator $L$, and whose initial law has the density $Po$. Suppose we observe the process:

$$Y_t = \int_0^t h(X_s)ds + W_t$$

where $W_t$ is a standard Brownian motion, independent of $X$. Zakai [17] has shown, under rather strong conditions, that the so-called "unnormalized conditional density" of $X_t$, given $(Y_s, s \leqq t)$, satisfies the following stochastic partial differential equation:

$$\left.\begin{aligned} du(t) &= L^*u(t)dt + hu(t)dY_t \\ u(o) &= Po \end{aligned}\right\} \qquad (0.1)$$

The aim of this paper is twofold. First to present some results on stochastic PDE's, which enable us to study existence and uniqueness of the solution of (0.1). This is what Part I is about.

†This work was done while the author was with the Centre National de la Recherche Scientifique (France).

Second, to relate the unique solution of (0.1) to the filtering problem, under rather mild hypotheses—in particular we allow correlation between the signal and the observation noise.

The idea is to associate to (0.1) a backward stochastic PDE:

$$\begin{cases} dv(t)+Lv(t)dt+hv(t)dY_t=0,\ 0\leqq t\leqq T \\ v(T)=f \end{cases} \tag{0.2}$$

The solution of (0.2) is expressible in terms of the conditional law of $X_t$ in a way that generalizes the classical Feynman–Kac formula for second-order parabolic (deterministic!) PDE's. Besides, (0.1) and (0.2) are adjoint one to the other, in the sense that the trajectories of $(u(t), v(t))$ are constants. The fact that $u$ is the "unnormalized conditional density" then follows immediately.

Similar results on Eq. (0.1) have been obtained by Krylov–Rosovskii [6], with different methods. Our exposition is self-contained, and does not use previous results in filtering theory.

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# PART I: SOME RESULTS ON STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

## §0. ORIENTATION

We present some results of Pardoux [11] on stochastic PDE's, restricting our attention to the type of equations that will be useful in Part II. In particular, we limit ourself to equations with linear operators.

Our method consists in extending the variational method of Lions [9] to stochastic equations of Ito type. We first recall a few results from the theories of (deterministic) parabolic PDE's and Hilbert space valued stochastic integrals. We then establish an Ito formula which is necessary for our particular purpose, and finally prove existence and uniqueness for a class of stochastic PDE's. Our existence proof uses a Galerkin approximation scheme.

Similar results have been obtained on the same type of equations, mostly using semi-group theory, by several authors, among others Balakrishnan [2], Curtain [3], Dawson [4], Ichikawa [5] and Krylov–Rosovskii [7].

We then present in an appendix, without proofs, more general results from Pardoux [11] on non-linear stochastic PDE's.



## §1. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF STOCHASTIC PDE's

### §1.1. Notations and hypotheses

Let $(\Omega, \mathscr{F}, P, \mathscr{F}_t, W_t)$ be a $R^d$ valued standard Wiener process. $[.,.]$ will denote the scalar product in $R^d$. We want to study equations of the type:

$$\begin{cases} du(t) + Au(t)dt = [Bu(t), dW_t] \\ \qquad\qquad u(o) = u_0 \end{cases} \tag{1.1}$$

where $A$ and $B$ are partial differential operators. In this first paragraph, we will consider $A$ and $B$ as unbounded operators in a Hilbert Space.

More precisely, let $V$ and $H$ be two separable Hilbert spaces, such that: $V$ is included and dense in $H$, the injection being continuous. We identify $H$ with its dual space, and denote by $V'$ the dual of $V$. We have then:

$$V \subset H \subset V'.$$

We will denote by $||.||$, $|.|$ and $||.||^*$ the norms in $V$, $H$ and $V'$ respectively; by $\langle .,. \rangle$ the duality product between $V$ and $V'$, and by $(.,.)$ the scalar product in $H$. Let us fix $T>0$.

We are given two families of linear operators $A(t)$, $B(t)$, $t \in [0,T]$, satisfying

$$A(.) \in L^\infty(0,T; \mathscr{L}(V,V')) \tag{1.2}$$

$$B(.) \in L^\infty(0,T; \mathscr{L}(V; H^d)) \tag{1.3}$$

and we will make the following coercivity hypothesis:

$$\left.\begin{array}{l} \exists\, \alpha>0 \text{ and } \lambda \text{ s.t. } \forall u \in V, \text{ p.p.t:} \\ 2\langle A(t)u,u\rangle + \lambda |u|^2 \geqq \alpha ||u||^2 + |B(t)u|^2 \end{array}\right\} \tag{1.4}$$

As an abuse of notation, we also use $|\ |$ for the following norm in $H^d$:

$$|u|_{H^d} = \left(\sum_{i=1}^{d} |u_i|_H^2\right)^{1/2}.$$

## §1.2. Two basic results for PDE's

The following two Lemmas are proved in Lions [9]:

LEMMA 1.1 *Let $u$ be absolutely continuous from $[0,T]$, with values in $V'$. Suppose moreover that:*

$$\left\{\begin{array}{l} u \in L^2(0,T; V) \\ \dfrac{du}{dt} \in L^2(0,T; V') \end{array}\right.$$

*then*

i) $u \in C(0,T; H)$†

ii) $\dfrac{d}{dt}|u(t)|^2 = 2\left\langle u, \dfrac{du}{dt}\right\rangle$ a.e. in $t$.

LEMMA 1.2 *Suppose* (1.2) *and* (1.4) *are satisfied (with $B=0$). Let $u_0 \in H$ and $f \in L^2(0,T; V')$. Then the equation:*

$$\left\{\begin{array}{l} u \in L^2(0,T; V) \\ \dfrac{du}{dt} + Au = f \\ u(o) = u_0 \end{array}\right.$$

†For simplicity we will always use the notation $C(0,T;X)$ for $C([0,T];X)$.

*has a unique solution. Moreover the function $f \to u$ is continuous from $L^2(0, T; V')$, with values in $L^2(0, T; V)$.*

## §1.3 Hilbert space valued stochastic integrals

The theory of stochastic integrals in Hilbert space is well understood, cf. Metivier [10], Pardoux [12], and the bibliographies therein. Here we will use it only in a very particular case, where it can be built very easily from the real case.

Let $M^2(0, T; H)$ denote the space of $H$-valued measurable processes which satisfy:

i) $\varphi(t)$ is $\mathscr{F}_t$ measurable, a.e. in $t$

ii) $E\int_0^T |\varphi(t)|^2 dt < +\infty$.

We define similarly $M^2(0, T; X)$, for $X = R^d$, $H^d$, $V^d$, $V$ and $V'$. It is easy to check that $M^2(0, T; X)$ is a closed subspace of $L^2(\Omega x ]o, T[, dP \otimes dt; X)$.

If $\varphi \in M^2(0, T; H^d)$ and $h \in H$,

$$h \to \int_0^t [(h, \varphi(s)), dW_s]$$

is a linear map from $H$ into $L^2(\Omega)$. It follows that we can define the $H$-valued random variable $\int_0^t [\varphi(s), dW_s]$ by

$$(h, \int_0^t [\varphi(s), dW_s]) = \int_0^t [(h, \varphi(s)), dW_s], \forall h \in H.$$

It is easy to check using a basis of $H$, and taking the limit on the finite dimensional results, that

$M_t = \int_0^t [\varphi(s), dW_s]$ is a continuous $H$-valued martingale, which satisfies:

$$|M_t|^2 = 2\int_0^t [(M_s, \varphi(s)), dW_s] + \int_0^t |\varphi(s)|^2 ds \tag{1.5}$$

$$E|M_t|^2 = E\int_0^t |\varphi(s)|^2 ds \tag{1.6}$$

We will make use of the following Burkholder–Gundy inequality, in the case $\varphi \in M^2(0, T; R^d)$:

$$E\left(\sup_{t \leq T} |\mathrm{M}_t|\right) \leq 3E(\int_0^T |\varphi(t)|^2 dt)^{1/2} \tag{1.7}$$

If $u(t)$ and $du/dt(t)$ are continuous $H$-valued processes, adapted to $\mathscr{F}_t$, then:

$$d(u(t), M_t) = \left(\frac{du}{dt}(t), M_t\right)dt + [(u(t), \varphi(t)), dW_t]. \tag{1.8}$$

If $u(t)$ is a.s. absolutely continuous with values in $V'$, and $u \in M^2(0,T; V) du/dt \in M^2(0,T; V')$, it follows from Lemma 1.1 that $u \in C(0,T; H)$ a.s. If moreover $\varphi \in M^2(0,T; V^d)$, then one easily proves from (1.8):

$$d(u(t), M_t) = \left\langle \frac{du}{dt}, M_t \right\rangle dt + [(u(t), \varphi(t)), dW_t]. \tag{1.9}$$

Let $\psi : H \to R$ be twice differentiable. The first derivative, evaluated at a point $v$, $\psi'(v)$, is an element of $H' = H$. $\psi''(v)$ is a bilinear continuous form on $H$, which we can identify with an element of $\mathscr{L}(H)$. We denote by $\mathscr{L}^1(H)$ the Banach space of trace-class operators, and by $\operatorname{Tr} Q$ the trace of $\varphi \in \mathscr{L}^1(H)$. We then have the following Ito formula (cf. Pardoux [12]).

LEMMA 1.3 *Let $\psi$ be a functional on $H$, which is twice differentiable at each point, and satisfies:*

i) $\psi$, $\psi'$ and $\psi''$ are locally bounded.

ii) $\psi$ and $\psi'$ are continuous on $H$.

iii) $\forall Q \in \mathscr{L}^1(H), \operatorname{Tr}[Q o \psi'']$ is a continuous functional on $H$.

Then if $V_t$ is an $H$-valued adapted process with bounded variation, and

$$M_t = \int_0^t [\varphi(s), dW_s],$$

$$\begin{aligned}\psi(V_t + M_t) = \psi(Vo) &+ \int_0^t (\psi'(V_s + M_s), dV_s) \\ &+ \int_0^t [(\psi'(V_s + M_s), \varphi(s)); dW_s] \\ &+ \frac{1}{2} \sum_{i=1}^d \int_0^t (\psi''(V_s + M_s)\varphi_i(s), \varphi_i(s)) ds \quad \square\end{aligned}$$

*Remark 1.1* Lemma 1.3 is proved in [12] for more general functions $\psi$, and with $M_t$ a general sample continuous $H$-valued local martingale. □

*Remark 1.2* Ito formula is of course an essential tool in the study of stochastic PDE's, mainly for the particular case $\psi(u) = |u|^2$. But Lemma 1.3 will not be applicable to the solution of Eq. (1.1). In view of Lemma 1.2, we will look for a solution $u$ of (1.1) in the space $M^2(0,T; V)$. $u$ will then be the sum of a process with bounded variations in $V'$ (and not $H$!), and an $H$-valued martingale. We then need to adapt Lemma 1.3 to this class of processes. This will be done in the next section, by studying a first class of stochastic PDE's. □

## §1.4. An Ito formula

We first study the following equation:

$$\begin{cases} u \in M^2(0,T;\ V) \\ du(t)+A(t)u(t)dt=[\varphi(t), dW_t] \\ u(o)=u_0 \end{cases} \tag{1.10}$$

where $A$ satisfies (1.2) and (1.4), $u_0 \in H$.

LEMMA 1.4 Let $\varphi \in M^2(0,T;\ V^d)$. Then (1.10) has a unique solution $u$, which belongs moreover to $L^2(\Omega;\ C(0,T;\ H))$, and satisfies:

$$|u(t)|^2+2\int_0^t \langle Au, u\rangle ds=|u_0|^2+2\int_0^t [(u,\varphi), dW_s]+\int_0^t |\varphi(s)|^2 ds. \tag{1.11}$$

*Proof* Define $M_t=\int_0^t [\varphi(s), dW_s]$. Then $M \in M^2(0,T;V)$.
Consider:

$$\begin{cases} \dfrac{dv}{dt}(t)+A(t)v(t)=-A(t)M_t \\ v(o)=u_0 \end{cases} \tag{1.12}$$

For each $\omega$, (1.12) has a unique solution in $L^2(0,T;\ V)$, and the operator which maps the right-hand side to the solution is continuous (see Lemma 1.2). So (1.12) has a solution $v$ as random element of $L^2(0,T;\ V)$. Moreover, by Lemma 1.1:

$$|v(t)|^2+2\int_0^t \langle Av, v\rangle ds=|u_0|^2-2\int_0^t \langle AM, v\rangle ds \tag{1.13}$$

Using (1.4), we get from (1.13) and Gronwall's inequality:

$$v \in M^2(0,T;\ V) \cap L^2(\Omega;\ C(0,T;\ H))$$

(the adaptedness is easy to check).

Define $u(t)=v(t)+M_t$; then $u \in M^2(0,T;\ V) \cap L^2(\Omega;\ C(0,T;\ H))$, and satisfies Eq. (1.10). (1.11) is an easy consequence of (1.13), (1.5) and (1.9).

It remains to prove uniqueness. Let $v$ be a solution of (1.10). Then:

$$\begin{cases} \dfrac{d}{dt}(u-v)+A(u-v)=o \qquad \text{a.e. in } t \\ u(o)-v(o)=o \end{cases} \tag{1.14}$$

From (1.14), Lemma 1.1, (1.4) and Gronwall's inequality, we conclude that:

$$\int_0^T \|u-v\|^2 dt = o \quad \text{a.s.} \quad \square$$

We can now prove:

THEOREM 1.1 *Let* $\varphi \in M^2(0, T; H^d)$. *Then* (1.10) *has a unique solution* $u$, *which belongs to* $L^2(\Omega; C(0, T; H))$, *and satisfies* (1.11).

*Proof* Uniqueness is proved exactly as in Lemma 1.4. It remains to prove existence.

Let $\varphi^n \in M^2(0, T; V^d)$, such that:

$$\varphi^n \to \varphi \text{ in } M^2(0, T; H^d). \tag{1.15}$$

And let $u^n \in M^2(0, T; V)$ be the solution of:

$$\begin{cases} du^n(t) + A(t)u^n(t)dt = [\varphi^n(t), dW_t] \\ u^n(o) = u_0 \end{cases} \tag{1.16}$$

and $u^m$ be the solution corresponding to $\varphi^m$. It follows from (1.11) that

$$|u^n(t) - u^m(t)|^2 + 2\int_0^t \langle A(u^n - u^m), u^n - u^m \rangle ds$$
$$= 2\int_0^t [(u^n - u^m, \varphi^n - \varphi^m), dW_s] + \int_0^t |\varphi^n - \varphi^m|^2 ds$$

$$E\left(\sup_{s \leq t} |u^n(s) - u^m(s)|^2\right) + 2E\int_0^t \langle A(u^n - u^m), u^n - u^m \rangle ds$$
$$\leqq 4E\left(\sup_{s \leq t} \left|\int_0^s [(u^n - u^m, \varphi^n - \varphi^m), dW_\theta]\right|\right)$$
$$+ 2E\int_0^t |\varphi^n - \varphi^m|^2 ds$$

Making use of (1.7), we get:

$$E\left(\sup_{s \leq t} |u^n(s) - u^m(s)|^2\right) + 2E\int_0^t \langle A(u^n - u^m), u^n - u^m \rangle ds$$
$$\leqq 12E[(\int_0^t |(u^n - u^m, \varphi^n - \varphi^m)|^2 ds)^{1/2}]$$
$$+ 2E\int_0^t |\varphi^n - \varphi^m|^2 ds \leqq \tfrac{1}{2}E\left(\sup_{s \leq t} |u^n(s) - u^m(s)|^2\right)$$
$$+ cE\int_0^t |\varphi^n - \varphi^m|^2 ds$$

It then follows from (1.4) that:

$$\begin{cases} E\left(\sup\limits_{s\leq t}|u^n(s)-u^m(s)|^2\right) \\ \qquad + 2\alpha E\int_0^t \|u^n-u^m\|^2 ds \leqq 2\lambda E\int_0^t |u^n-u^m|^2 ds \\ \qquad\qquad + 2cE\int_0^t |\varphi^n-\varphi^m|^2 ds \end{cases} \tag{1.17}$$

We first deduce from (1.17):

$$E(|u^n(t)-u^m(t)|^2) \leqq c_1 E\int_0^t |\varphi^n-\varphi^m|^2 ds + c_2 E\int_0^t |u^n-u^m|^2 ds$$

which implies, by Gronwall's inequality:

$$E|u^n(t)-u^m(t)|^2 \leqq c_1 E\int_0^t |\varphi^n-\varphi^m|^2 e^{c_2(t-s)} ds \tag{1.18}$$

It follows from (1.15), (1.17) and (1.18) that $u^n$ forms a Cauchy sequence in $M^2(0,T;\ V)\cap L^2(\Omega;\ C(0,T;\ H))$. The existence follows by taking the limit in (1.16), and $u$ satisfies (1.11) because $u^n$ does, and we can take the limit as $n\to+\infty$. □

*Remark 1.3* The same method gives a similar result for the equation:

$$\begin{cases} u\in M^2(0,T;\ V) \\ du(t)+(A(t)u(t)+f(t))dt=[\varphi(t), dW_t] \\ u(o)=u_0 \end{cases}$$

where $f$ is given in $M^2(0,T;\ V')$. □

A solution of Eq. (1.1) can be viewed as a solution of equation (1.10). Then we have proved an Ito formula for our class of processes, and the functional $\psi(u)=|u|^2$. Let us prove a more general Ito formula:

THEOREM 1.2 *Suppose:*

$$u\in M^2(0,T;\ V)$$

$$u_0\in H$$

$$v\in M^2(0,T;\ V')$$

$$\varphi\in M^2(0,T;\ H^d), \text{ with:}$$

$$u(t)=u_0+\int_0^t v(s)ds+\int_0^t [\varphi(s), dW_s].$$

Let $\psi$ be a twice differentiable functional on $H$, which satisfies assumptions (i), (ii) and (iii) of Lemma 1.3 and moreover:

iv) If $u \in V$, $\psi'(u) \in V$; $u \to \psi'(u)$ is continuous form $V$ (with the strong topology), into $V$ endowed with the weak topology.

v) $\exists\, k$ s.t. $\|\psi'(u)\| \leqq k(1+\|u\|)$, $\forall u \in V$.

Then:

$$\begin{cases} \psi(u(t)) = \psi(u_0) + \int_0^t \langle v, \psi'(u)\rangle ds + \int_0^t [(\psi'(u), \varphi), dW_s] \\ \qquad\qquad + \frac{1}{2} \sum_{i=1}^{d} \int_0^t (\psi''(u)\varphi_i, \varphi_i) ds \end{cases} \tag{1.19}$$

*Proof* (a) Consider first the case where $\varphi \in M^2(0, T;\ V^d)$. Define once again

$$M_t = \int_0^t [\varphi(s), dW_s].$$

Then $M \in M^2(0, T;\ V)$, and if $\tilde{u} = u - M$, $\tilde{u} \in M^2(0, T;\ V)$

$$\frac{d\tilde{u}}{dt} = v \in M^2(0, T;\ V')$$

Let $\tilde{u}^n$ be a sequence in $M^2(0, T;\ V)$, such that $\tilde{u}^n \in C^1(0, T;\ H)$ a.s., and:

$$\tilde{u}^n(o) = u_0$$

$$\tilde{u}^n \to \tilde{u} \text{ in } M^2(0, T;\ V)$$

$$v^n = \frac{d\tilde{u}^n}{dt} \to v \text{ in } M^2(0, T;\ V').$$

It follows from Lemma 1.3:

$$\psi(\tilde{u}^n(t) + M_t) = \psi(u_0) + \int_0^t \langle \psi'(\tilde{u}^n + M), v\rangle ds + \int_0^t [(\psi'(\tilde{u}^n + M), \varphi), dW]$$
$$+ \frac{1}{2} \sum_{i=1}^{d} \int_0^t (\psi''(\tilde{u}^n + M)\varphi_i, \varphi_i) ds \tag{1.19$^n$}$$

It follows from Lemma 1.1 and the above convergences that:

$$\tilde{u}^n \to \tilde{u} \text{ in } L^2(\Omega;\ C(0, T;\ H)).$$

It is then easy to see that we can take the limit in $(1.19)^n$, yielding (1.19).

b) Let $\varphi^n \to \varphi$ in $M^2(0, T;\ H^d)$ with $\varphi^n \in M^2(0, T;\ V^d)$. Define $u_n$ by:

$$\begin{cases} du_n + Au_n dt = (v + Au)dt + [\varphi, dW] \\ \qquad u_n(o) = u_0 \end{cases}$$

This equation has a unique solution in $M^2(0, T; V) \cap C(0, T; H))$, from the result in Remark 1.3.

Exactly as in the Proof of Theorem 1.1, we get:

$$u_n \to u \text{ in } M^2(0, T; V) \cap L^2(\Omega; C(0, T; H))$$

and consequently $Au_n \to Au$ in $M^2(0, T; V)$.

This permits us to take the limit on the result obtained in (a) yielding (1.19). □

## §1.5. Existence and uniqueness

We consider finally the equation:

$$\begin{cases} u \in M^2(0, T; V) \\ du(t) + A(t)u(t)dt = [B(t)u(t), dW_t] \\ u(o) = u_0 \end{cases} \tag{1.20}$$

where $A$ and $B$ are supposed throughout this section to satisfy (1.2), (1.3) and (1.4), and $u_0 \in H$.

Because $B$ maps $V$ (and not $H$) into $H^d$, it is not possible to get our existence result from Theorem 1.1 by a Picard type iterative scheme. Therefore we will use a Galerkine finite dimensional approximation. One can also use a time discretisation, as indicated in the Proof of Theorem 3.1 of Part II.

Let us prove:

THEOREM 1.3 *Equation* (1.20) *has a unique solution* $u$, *which satisfies moreover:*

i) $u \in L^2(\Omega; C(0, T; H))$

ii) $|u(t)|^2 + 2\int_0^t \langle Au, u\rangle ds = |u_0|^2 + 2\int_0^t [(Bu, u), dW_s]$
$+ \int_0^t |Bu|^2 ds$, a.s.

*Proof* *Proof of* (i) *and* (ii)

Let $u$ be a solution of (1.20). Then $Bu \in M^2(0, T; H^d)$. And from Theorem 1.1 there exists a unique $v \in M^2(0, T; V)$, solution of

$$\begin{cases} dv(t) + A(t)v(t)dt = [B(t)u(t), dW_t] \\ u(o) = u_0 \end{cases}$$

But $u$ is such a solution, then $u = v$ and (i) and (ii) follow from Theorem 1.1.

*Uniqueness*

Let $u$ and $v$ be two solutions of (1.20). Then $u-v$ is a solution of (1.20), with $u_0=o$. But we may now use (ii), yielding:

$$E(|u(t)-v(t)|^2+2E\int_0^t\langle A(u-v),u-v\rangle ds=E\int_0^t|Bu-Bv|^2ds$$

From (1.4), we get:

$$E(|u(t)-v(t)|^2\leqq\lambda E\int_0^t|u(s)-v(s)|^2ds$$

It follows from Gronwall's inequality that

$$E(|u(t)-v(t)|^2)=o,\ \forall\, t\leqq o$$

*Existence*

Let $v_1, v_2, \ldots, v_n \ldots$ be a Hilbert basis of $V$, which is orthonormal as a basis of $H$.

Define $V_n=S_p\{v_1,v_2,\ldots,v_n\}$, choose for each $n$ $u_{0n}\in V_n$, such that:

$$u_{0n}\to u_0 \text{ in } H \tag{1.21}$$

Define $u_n(t)=\sum_{i=1}^n g_{ni}(t)v_i$, where $g_n(t)=(g_{n1}(t)g_{n2}(t)\ldots g_{nn}(t))$ is the solution of the following Ito equation in $R^n$:

$$\begin{cases} d(u_n(t),v_i)+\langle Au_n(t),v_i\rangle dt=[(Bu_n(t),v_i),dW_t], \quad i=1\ldots n \\ u_n(o)=u_{0n} \end{cases} \tag{1.22}$$

It follows from Ito formula:

$$E|u_n(t)|^2+2E\int_0^t\langle Au_n,u_n\rangle ds=|u_{0n}|^2+E\int_0^t\sum_{i=1}^n|(Bu_n,v_i)|^2ds$$
$$\leqq|u_{0n}|^2+E\int_0^t|Bu_n|^2ds$$

Using (1.4) and Gronwall's inequality in the same way as above, we get:

$$E\int_0^t||u_n(t)||^2dt\leqq c.$$

It follows that there exists a subsequence $u_\mu$ such that:

$$u_\mu\rightharpoonup u \text{ in } M^2(0,T;\,V) \text{ weakly.}$$

Let $\varphi$ be an absolutely continuous function from $[0,T]$ into $R$, with $\varphi'\in L^2(0,T)$, and $\varphi(T)=o$. Define $\varphi_i(t)=\varphi(t)v_i$.

Multiplying (1.22) by $\varphi_i(t)$, and using Ito formula, we get:

$$\int_0^T \langle Au_\mu(t), \varphi_i(t)\rangle dt = (u_{0\mu}, \varphi_i(o)) + \int_0^T (u_\mu(t), \varphi_i'(t))dt$$
$$+ \int_0^T [(Bu_\mu(t), \varphi_i(t)), dW_t].$$

We can take the limit in $L^2(\Omega)$ weakly in each term of the preceding equality. Indeed, the mapping:

$$\psi \to \int_0^T [\psi(s), dW_s]$$

is linear and continuous from $M^2(o,T;\ R^d)$ into $L^2(\Omega)$; it is then continuous for the weak topologies.

It follows that

$$\int_0^T \langle Au(t), \varphi_i(t)\rangle dt = (u_0, \varphi_i(o)) + \int_0^T (u(t), \varphi_i'(t))dt$$
$$+ \int_0^T [(Bu(t), \varphi_i(t)), dW_t].$$

The last equality is true $\forall\, i$, then:

$$\int_0^T \langle Au(t), v\rangle \varphi(t)dt = (u_0, v)\varphi(o) + \int_0^T (u(t), v)\varphi'(t)dt$$
$$+ \int_0^T \varphi(t)[(Bu(t), v), dW_t],\ \forall\, v \in V.$$

Choose $\varphi_n$ defined by:

$$\varphi_n(s) = \begin{cases} 1 \text{ if } s \leqq t - \frac{1}{2}n \\ \frac{1}{2} + n(t-s), \text{ if } t - \frac{1}{2}n < s < t + \frac{1}{2}n \\ o \text{ if } s \geqq t + \frac{1}{2}n \end{cases}$$

$$\begin{cases} n\int_{t-1/2n}^{t+1/2n} (u(s), v)ds + \int_0^T \langle Au(s), v\rangle \varphi_n(s)ds = (u_0, v) \\ \qquad + \int_0^T \varphi_n(t)[(Bu(t), v), dW_t]. \end{cases} \tag{1.23}$$

It follows from a well-known Theorem of Lebesgue that we can take the limit in the first term of (1.23), for almost all $t \in ]0, T[$. Then:

$$(u(t), v) + \int_0^t \langle Au, v\rangle ds = (u_0, v)$$
$$+ \int_0^t [(Bu, v), dW_s] \text{ a.e. in } t, \forall\, v \in V.$$

Using the separability of $V$, we get:

$$u(t) + \int_0^t Au\, ds = u_0 + \int_0^t [Bu, dW_s], \text{ a.e. in } t.$$

C

Then $u$ is a.e. equal to a continuous process with values in $V'$, which we define to be $u$, and which satisfies Eq. (1.20). $\square$

We will need a slight generalization of Theorem 1.3. Define:

$$f \in M^2(0, T; V') \tag{1.24}$$

$$g \in M^2(0, T; H^d) \tag{1.25}$$

and consider the equation:

$$\begin{cases} u \in M^2(0, T; V) \\ du(t) + (A(t)u(t) + f(t))dt = [B(t)u(t) + g(t), dW_t] \\ u(o) = = u_0. \end{cases} \tag{1.26}$$

where $u_0$ is given in $H$.

THEOREM 1.4 *Equation* (1.26) *has a unique solution, which satisfies moreover:*

i) $u \in L^2(\Omega; C(0, T; H))$.

ii) $|u(t)|^2 + 2\int_0^t \langle Au + f, u\rangle ds = |u_0|^2 + \int_0^t [(Bu + g, u), dW_s]$
$+ \int_0^t |Bu + g|^2 ds.$ $\square$

The proof follows the same lines as that of Theorem 1.3, using the generalization of Theorem 1.1—see Remark 1.3.

## §2. EXAMPLES OF STOCHASTIC PDE's

We will restrict ourselves to stochastic PDE's in $R^n$, as we will in the second part of this paper restrict our attention to filtering of diffusion processes without boundaries. For the case of boundary conditions, see Pardoux [13] and [14].

### §2.1. Application of the abstract results

We will, for short, write $H^1$ for $H^1(R^n) = \{u \in L^2(R^n);\ \partial u/\partial x \in L^2(R^n), i = 1 \ldots N\}$.

Define:

$$a_{ij}, a_i, c_{ki}, d_k \in L^\infty(]0, T[xR^n);\ i,j=1\dots N;\ k=1\dots d \tag{2.1}$$

and, for $u, v \in H^1$:

$$\langle A(t)u, v\rangle = \sum_{i,j=1}^{N} \int_{R^N} a_{ij}(t,x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}dx + \sum_{i=1}^{N} \int_{R^N} a_i(t,x)\frac{\partial u}{\partial x_i}(x)v(x)dx$$

$$(B_k(t)u)(x) = \sum_{i=1}^{N} c_{ki}(t,x)\frac{\partial u}{\partial x_i}(x) + d_k(t,x)u(x).$$

$A$ and $B$ satisfy (1.2) and (1.3), from (2.1).

We suppose moreover that $\exists\, \alpha > o$ such that:

$$\sum_{i,j=1}^{N} (2a_{ij}(t,x) - \sum_{k=1}^{d} c_{ki}(t,x)c_{kj}(t,x))\xi_i\xi_j \geqq \alpha|\xi|^2$$
$$\forall\, \xi \in R^N, \text{ a.e. in } (t,x). \tag{2.2}$$

The coercivity condition (1.4) is easy to check from (2.2). In this example, Eq. (1.20) has a unique solution in $M^2(0,T;H^1) \cap L^2(\Omega; C(0,T; L^2(R^N)))$.

Equation (1.20) can be interpreted in the following way:

$$du(t,x) - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_i}(t,x)\right)dt + \sum_{i=1}^{N} a_i(t,x)\frac{\partial u}{\partial x_i}(t,x)dt$$
$$= \sum_{k=1}^{d}\left(\sum_{i=1}^{N} c_{ki}(t,x)\frac{\partial u}{\partial x_i}(t,x) + d_k u(t,x)\right)dW_t^k \text{ a.e.}$$
$$u(o,x) = u_0(x) \tag{2.3}$$

*Remark 2.1* One can easily check that the same result holds if we replace $A(t)$ by its adjoint $A^*(t)$, or if we add in the expression of $\langle A(t)u, v\rangle$ a zero order term

$$\int a_0(t,x)u(x)v(x)dx. \quad \square$$

### §2.2. A regularity result

We now prove a regularity result, which will be useful in the sequel. We will denote

$$H^n = H^n(R^N) = \left\{ u \in L^2(R^N);\ \frac{\partial^k u}{\partial x_{i_1} \dots \partial x_{i_k}} \in L^2(R^N),\ \forall\, k \leqq n \right\}$$

Let us consider Eq. (1.26) with $g = o$.

THEOREM 2.1 *Let $n$ be any integer $\geqq 1$. Suppose, in addition to* (2.1) *and* (2.2), *that all coefficients $a_{ij}$, $a_i$, $c_{ki}$ and $d_k$ have bounded partial derivatives in $x$ up to order $n$, and that moreover:*

$$u_0 \in H^n, f \in M^2(0, T;\ H^{n-1}). \tag{2.3}$$

Then $u$, solution of Eq. (1.26) (where $g = o$) belongs to $M^2(0, T;\ H^{n+1}) \cap L^2(\Omega;\ C(0, T;\ H^n))$.

*Proof* It suffices to prove that all partial derivatives of $u$ in $x$, up to order $n$, belong to $M^2(0, T;\ H^1) \cap L^2(\Omega;\ C(0, T;\ L^2(R^N)))$.

Let us give the proof for $\partial u/\partial x_1$. If $\rho(x) = \rho(x_1, x_2, \dots, x_n)$, call:

$$\rho^h(x) = \rho(x_1 + h, x_2, \dots, x_n)$$

$$\tau_h \rho = 1/h(\rho^h - \rho).$$

Choose $v \in H^1$, and multiply Eq. (1.26) by $\tau_{-h} v$:

$$(u(t), \tau_{-h} v) + \int_0^t \langle A(s)u(s) + f(s), \tau_{-h} v\rangle ds = (u_0, \tau_{-h} v) + \int_0^t [(B(s)u(s), \tau_{-h} v), dW_s]$$

$$\langle Au, \tau_{-h} v\rangle = \sum_{i,j} \int_{R^N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial(\tau_{-h} v)}{\partial x_j} dx + \sum_i \int_{R^N} a_i \frac{\partial u}{\partial x_i} \tau_{-h} v\, dx$$

$$= -\sum_{i,j} \int_{R^N} a_{ij} \frac{\partial}{\partial x_i}(\tau_h u) \frac{\partial v}{\partial x_j} dx - \sum_i \int_{R^N} a_i \frac{\partial}{\partial x_i}(\tau_h u) v\, dx$$

$$-\sum_{i,j} \int_{R^N} (\tau_h a_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_i \int_{R^N} (\tau_h a_i) \frac{\partial u^h}{\partial x_i} v\, dx$$

$$\langle f, \tau_{-h} v\rangle = \langle \tau_h f, v\rangle$$

$$(u_0, \tau_{-h} v) = (\tau_h u_0, v)$$

$$(B_k u, \tau_{-h} v) = -\sum_i \int_{R^N} c_{ki} \frac{\partial(\tau_h u)}{\partial x_i} v dx - \int_{R^N} d_k \tau_h u v dx$$
$$-\sum_i \int_{R^N} \tau_h c_{ki} \frac{\partial u^h}{\partial x_i} v dx - \int_{R^N} \tau_h d_k u^h v dx$$

We conclude that $\tau_h u$ satisfies the following equation:

$$\begin{aligned} d(\tau_h u) + (A(\tau_h u) + f_h)dt &= [B(\tau_h u) + g_h, dW_t] \\ (\tau_h u)(o) &= \tau_h u_0 \end{aligned} \tag{2.4}$$

where:

$$\langle f_h, v\rangle = \sum_{i,j} \int_{R^N} (\tau_h a_{ij}) \frac{\partial u^h}{\partial x_i} \frac{\partial v}{\partial x_j} dx$$

$$+\sum_i \int_{R^N} (\tau_h a_i) \frac{\partial u^h}{\partial x_i} v dx + \langle \tau_h f, v\rangle$$

$$g_h^k = \sum_i (\tau_h c_{ki}) \frac{\partial u^h}{\partial x_i} + (\tau_h d_k) u^h.$$

It follows from the hypotheses we made that $f_h$ is bounded in $M^2(0, T; H^{-1})$,† $g_h$ is bounded in $M^2(0, T; L^2(R^N))$, and $\tau_h u_0$ is bounded in $L^2(R^N)$. We can conclude from Theorem 1.4, using standard estimates, that $\tau_h u$ remains in a bounded subset of $M^2(0, T; H^1)$. There exists a subsequence $\tau_l u$ such that:

$$\tau_l u \to \chi \text{ in } M^2(0, T; H^1) \text{ weakly.}$$

But $\tau_l u \to \partial u/\partial x_1$ a.s. in the sense of distributions. It follows that $\chi = \partial u/\partial x_1$, and taking the limit in Eq. (2.4), and using Theorem 1.4, we conclude that $\partial u/\partial x_1 \in M^2(0, T; H^1) \cap L^2(\Omega; C(0, T; L^2(R^N)))$. □

*Remark 2.1* Similar results, with $H^n$ replaced by $W^{n,p}(R^N)$ [the Sobolev space defined as $H^n$, where $L^p(R^N)$ replaces $L^2(R^N)$], and $\forall p > 1$, are given for the same type of equation in Krylov–Rosovskii [7]. □

Following an idea in Bensoussan–Lions [1], we now deduce from Theorem 2.1:

THEOREM 2.2 *Let $u$ be the solution of Eq. (1.20), with initial condition $u_0 \in L^2(R^N)$.*

*We suppose, as in Theorem 2.1, that (2.1) and (2.2) hold, and all coefficients $a_{ij}$, $a_i$, $c_{ki}$ and $d_k$ have bounded partial derivatives up to order $2k$.*

† $H^{-1} \triangleq H^{-1}(R^N) = (H^1(R^N))'$.

*Then there exists a constant c, independent of $u_0$, such that:*

$$t^k E\|u(t)\|_H^2 2k \leqq c|u_0|^2, \ \forall t \in [o, T].$$

*Proof* Define $u_1 = tu$. It is easy to verify that:

$$du_1(t) = t\,du(t) + u(t)dt$$

Then $u_1$ is the solution of:

$$\begin{cases} du_1(t) + (A(t)u_1(t) - u(t))dt = [B(t)u_1(t), dW_t] \\ u_1(o) = o \end{cases}$$

But $u \in M^2(0, T; H^1)$. It then follows from Theorem 2.1:

$$u_1 \in M^2(0, T; H^3) \cap L^2(\Omega; C(0, T; H^2)).$$

By recurrence, it is easy to check that $t^k u \in L^2(\Omega; C(0, T; H^{2k}))$, and it is easy to verify that, $\forall t \in [0, T]$, $u_0 \to t^k u$ is a continuous linear mapping from $L^2(R^N)$ into $C(0, T; L^2(\Omega; H^{2k}))$. ☐

## §2.3. Remark on the coercivity condition

We want to show here why the coercivity condition (2.2) is crucial. Therefore, we will give an explicit solution in the case where the coefficients are functions of $t$ only, and satisfy, contrary to (2.2):

$$2a_{ij}(t) - \sum_{k=1}^{d} c_{ki}(t)c_{kj}(t) = o \text{ a.e. in } ]0, T[, \ \forall i, j = 1, \ldots, N. \tag{2.5}$$

We suppose:

$$u_0 = \varphi \in C^2(R^N) \cap H^1. \tag{2.6}$$

$$a_{ij}, a_i, c_{ki}, d_k \in L^\infty(]0, T[); \ i, j = 1 \ldots N; \ k = 1 \ldots d.$$

Define:

$$b_i(t) = -a_i(t) - \sum_{k=1}^{d} c_{ki}(t)d_k(t), \ i = 1 \ldots N.$$

$$X_t = \int_0^t b(s)ds + \int_0^t c^*(s)dW_s. \tag{2.8}$$

$$\varphi(t, x) = \varphi(x + X_t)\exp\{\int_0^t [d(s), dW_s] - \tfrac{1}{2}\int_0^t |d(s)|^2 ds\}. \tag{2.9}$$

Once again, we choose $V = H^1$, and $A$, $B$ are given as in Section 2.1. We have the following.

THEOREM 2.3 *Under the hypotheses* (2.5), (2.6) *and* (2.7), $\phi$, *defined by* (2.8) *and* (2.9) *is the unique solution of* (1.20).

*Proof* It is easy to check, using (2.6), that $\phi \in M^2(0, T; H^1)$. The fact that $\phi$ satisfies (1.20) is a consequence of Ito formula.

It remains to show that if $u$ is a solution of (1.20) with $u_0 = o$, then $u \equiv o$. Let us apply Theorem 1.2 to such a $u$:

$$E|u(t)|^2 + 2E\int_0^t \langle Au, u\rangle ds = E\int_0^t |Bu|^2 ds.$$

Using (2.5), we deduce:

$$E|u(t)|^2 + 2\sum_i \int_0^t ds\,(a_i - \sum_k d_k c_{ki}) \times \int_{R^N} \frac{\partial u}{\partial x_i} u\,dx = \sum_k E\int_0^t ds\, d_k^2 \int_{R^N} u^2 dx. \tag{2.10}$$

But it is easy to check that any $v$ in $H^1$ satisfies:

$$\int_{R^N} v \frac{\partial v}{\partial x_i} dx = o.$$

The result then follows from (2.10) and Gronwall's Lemma. $\square$

Theorem 2.3 shows that when the coercivity condition is no longer valid, Eq. (1.20) degenerates exactly in the same way as a parabolic PDE degenerates to a first-order hyperbolic PDE, whose solution is given by a method of characteristics. In particular, the solution given by Theorem 2.3 has the property that the regularity in $x$ is the same for all times $t$, whereas the coercivity has a regularizing effect.

*Remark 2.2* The question of what happens to the solution when the left-hand side of (2.5) is negative is, as far as we know, an open problem. $\square$

## §3. APPENDIX: NON LINEAR STOCHASTIC PDE's

Our method applies to the more general case where the operators $A$ and $B$ are non-linear, satisfying a condition of monotonicity. We present here some results of Pardoux [11], without proofs.

We use the same triple as above (with the same notations for the norms):

$$V \subset H \subset V'$$

where here $V$ is supposed only to be a reflexive Banach space.

Let $A(t,.)$ be a family of operators from $V$ into $V'$ and $p>1$, such that:

$$\exists \beta \text{ s.t. } ||A(t,u)||_* \leqq \beta ||u||^{p-1}, \forall u \in V, \text{ a.e.t.} \tag{3.1}$$

$$\theta \to \langle A(t,u+\theta v), w\rangle \text{ is continuous, } \forall u, v, w, \in V, \text{ a.e.t.} \tag{3.2}$$

$$t \to A(t,u) \text{ is Lebesgue measurable from } ]0,T[ \text{ into } V', \forall u \in V. \tag{3.3}$$

Let $K$ be a Hilbert space, and $(\Omega, \mathscr{F}, \mathscr{F}_t, P, W_t)$ be a $K$-valued Wiener process, such that

$$E[(W_t, h)(W_t, k)] = (Qh, k), \forall h, k \in K,$$

where $Q$ is a given nuclear operator on $K$.

Let $B(t,.)$ be a family of operators from $V$ into $\mathscr{L}(K,H)$ such that:

$$\begin{aligned} &\forall h \in H,\ k \in K,\ N \in R_+,\ \exists L \text{ s.t.} \\ &|(h, B(t,u)k) - (h, B(t,v)k| \leqq L||u-v|| \\ &\forall u, v \in V \text{ s.t. } ||u||, ||v|| \leqq N. \end{aligned} \tag{3.4}$$

$$t \to B(t,u) \text{ is Lebesgue measurable from } ]0,T[ \text{ into } \mathscr{L}(K;\ H), \quad \forall u \in V. \tag{3.5}$$

We suppose moreover:

Coercivity $\exists \alpha > o, \lambda$ and $\gamma$ s.t. $\forall u \in V$, a.e.t.,

$$2\langle A(t,u), u\rangle + \lambda |u|^2 + \gamma \geqq \alpha ||u||^p + ||B(t,u)Q^{1/2}||_2^2. \tag{3.6}$$

Monotonicity $\forall u, v \in V$, a.e.t.,

$$2\langle A(t,u) - A(t,v), u-v\rangle + \lambda|u-v|^2 \geqq ||(B(t,u) - B(t,v))Q^{1/2}||_2^2 \tag{3.7}$$

where $||.||_2$ denotes the norm in $\mathscr{L}^2(K;\ H)$ the space of Hilbert–Schmidt operators from $K$ into $H$.

Define:

$$u_0 \in L^2(\Omega, \mathscr{F}_0, P;\ H)$$

$$f \in M^{P'}(0, T;\ V')\dagger \left(\frac{1}{P} + \frac{1}{P'} = 1\right)$$

$M_t$ a continuous square integrable $H$-valued martingale.

---

$\dagger M^q(0,T;X)$ is defined in a similar way as $M^2(0,T;X)$, with the condition

$$E\int_0^T ||f||_X^q \, dt \langle +\infty \text{ instead of } E\int_0^T ||f||_X^2 \, dt < \infty.$$

Now consider the equation:

$$\begin{cases} u \in M^p(0, T; V) \\ du(t) + (A(t, u(t)) + f(t))dt = B(t, u(t))dW_t + dM_t \\ u(o) = u_0. \end{cases} \tag{3.8}$$

THEOREM 3.1 *Under the above hypotheses, Eq.* (3.8) *has a unique solution, which satisfies moreover:*

$$u \in L^2(\Omega; C(0, T; H)). \quad \square$$

One has an Ito formula for the solution of (3.8).

Suppose:

$$V \text{ and } V' \text{ are uniformly convex.} \tag{3.9}$$

Let $\psi$ be a functional on $H$, which satisfies the hypotheses of Theorem 1.2.

Let $u \in M^p(o, T; V)$, $u \in L^2(\Omega, \mathscr{F}_0, P; H)$, $v \in M^{p'}(o, T; V')$, and $M_t$ be a continuous square integrable $H$-valued martingale, such that:

$$u(t) = u_0 + \textstyle\int_0^t v(s)ds + M_t, t \in [o, T].$$

Then the following holds:

THEOREM 3.2

$$\forall t \in [o, T], \psi(u(t)) = \psi(u_0) + \textstyle\int_0^t \langle \psi'(u(t)), v(s) \rangle ds$$
$$+ \textstyle\int_0^t (\psi'(u(s)), dM_s) + \frac{1}{2} T_r \int_0^t \psi''(u(s)) d\langle\langle M \rangle\rangle_s$$

*where* $\langle\langle M \rangle\rangle_t$ *is the unique continuous increasing process with values in the space of nuclear operators on H, such that* $(M_t, h)(M_t, k) - (\langle\langle M \rangle\rangle_t h, k)$ *is a martingale,* $\forall h, k \in H$, *cf.* [10], [12]. $\square$

*Remark 3.1* In the case $\psi(u) = |u|^2$, we need only to assume, instead of (3.9), that $V'$ is strictly convex. This is always true (perhaps after an equivalent change of norms) because $V'$ is reflexive. $\square$

# PART II. FILTERING OF DIFFUSION PROCESSES

## §0. ORIENTATION

In this section, we will see that the stochastic PDE's we have defined in the first section are closely related to the non-linear filtering problem. We will associate to a non-linear filtering problem two stochastic PDE's which play the role of the backward and forward Kolmogrov equations associated with unconditioned diffusions.

Our main result is a kind of Feynman–Kac formula for a backward stochastic PDE (Theorem 2.1 below)†. Our formula expresses the solution of the stochastic PDE in terms of the conditional law of a Markov-diffusion process, given observations corrupted by noise; whereas the Feynman–Kac formula expresses the solution of a (deterministic) PDE in terms of the unconditioned law of a Markov-diffusion process.

The next step is then to express the relation between the two stochastic PDE's we consider: the backward and the forward one.

The difficulty in proving these two results is that they involve differential calculus with both processes adapted to the past, and to the future increments of a Wiener process. There is no differential rule in this context, and we have to discretize time, and to "kill" (before passing to the limit) the terms which would not make sense in the limit.

Our technique can be applied to other filtering problems, where the signal is a Markov process, and where a Girsanov transformation can be applied to the observation process. It has already been done in Pardoux [14] in the case where the signal is a diffusion with boundary condition, and the observation noise is independent of the signal; and in Pardoux [15] in the case where the signal is a diffusion, and the observation is a Poisson process whose intensity is a given function of the signal.

## §1. THE FILTERING PROBLEM—NOTATION AND HYPOTHESES

Define:

$\sigma_{ij}(t,x)$, continuous and bounded on $[o,T]\times R^N$, $\forall\, T>o$; $i,j=1\ldots N$.

$b_i(t,x), h_k(t,x)$ Borel measurable and bounded on $[o,T]\times R^N$, $\forall\, T>o$; $i=1,\ldots,N$, $k=1,\ldots,d$.

$g_{ki}(t), \tilde{g}_{k1}(t)$ continuous on $R_+$; $k,1=1\ldots d$, $i=1,\ldots,N$.

We assume the following:

$$g(t)g^*(t)+\tilde{g}(t)\tilde{g}^*(t)\equiv I\ddagger. \tag{1.1}$$

†After having proved the result, we discovered that, in the case of state-independent observation noise, it had been stated formally by Kushner in [8 bis]—and in an earlier paper referenced therein.

‡ * denotes the transpose.

$$\exists \beta > o \text{ s.t. } (\tilde{g}(t)\tilde{g}^*(t)\xi, \xi) \geqq \beta|\xi|^2, \forall \xi \in R^d, \forall t \geqq o. \tag{1.2}$$

$$\exists \alpha > o \text{ s.t. } (a(t,x)\xi, \xi) \geqq \alpha|\xi|^2, \forall \xi \in R^N, \forall (t,x) \in R_+ \times R^N \tag{1.3}$$

where $a = \sigma\sigma^*$. We define $c = g\sigma^*$.

Following the work of Stroock–Varadhan [16], we consider the martingale problems associated with the following systems of stochastic differential equations:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ dY_t = h(t, X_t)dt + g(t)dW_t + \tilde{g}(t)d\tilde{W}_t \end{cases} \tag{1.4}$$

$$\begin{cases} dX_t = [b(t, X_t) - c^*(t, X_t)h(t, X_t)]dt + \sigma(t, X_t)dW_t \\ dY_t = g(t)dW_t + \tilde{g}(t)d\tilde{W}_t \end{cases} \tag{1.5}$$

where $W_t$ and $\tilde{W}_t$ are two independent standard Wiener processes, with values in $R^N$ and $R^d$ respectively.

*Remark 1.1* (1.4) is our filtering problem, where $X_t$ is the signal process, and $Y_t$ the observation process. The system (1.5) will be used for technical reasons. $\square$

*Remark 1.2* Hypothesis (1.1) is a normalisation hypothesis. If $g(t)g^*(t) + \tilde{g}(t)\tilde{g}^*(t) = M(t)$, where $o < \gamma I \leqq M(t) \leqq \delta I$, it is always possible to reformulate the problem, choosing a new observation process

$$y'_t = \int_0^t M^{-1}(s)dy_s,$$

such that (1.1) is satisfied. $\square$

Define $\Omega = C(R_+; R^{N+d})$;

$$\begin{pmatrix} X_t(\omega) \\ Y_t(\omega) \end{pmatrix} = \omega(t), \ \mathscr{G}_t^s = \sigma\{\omega(\theta), s \leqq \theta \leqq t\}, \ \mathscr{G}^s = \bigvee_{t \geqq s} \mathscr{G}_t^s.$$

We write $\mathscr{G}_t$ for $\mathscr{G}_t^0$ and $\mathscr{G}$ for $\mathscr{G}^0$.

According to Stroock–Varadhan [16], $\forall s \geqq o$, $x \in R^N$, there exists a unique probability measure $P_{sx}$[resp. $\tilde{P}_{sx}$] on $(\Omega, \mathscr{G}^s)$, solution of the martingale problem associated with (1.4) [resp. (1.5)], and such that $P_{sx}(X_s = x, Y_s = o) = 1$ [resp. $\tilde{P}_{sx}(X_s = x, Y_s = o) = 1$].

This means that $\forall f \in C_b^{1,2}(R_+ x R^{N+d})$ the following process is a $P_{sx}$ martingale:

$$f(t, X_t, Y_t) - \int_s^t (f'_\theta + L_\theta f + M_\theta f)(\theta, X_\theta, Y_\theta)d\theta$$

where $L_\theta$ and $M_\theta$ are the following operators:

$$L_\theta = \tfrac{1}{2}\sum_{i,j} a_{ij}(\theta,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(\theta,x)\frac{\partial}{\partial x_i}.$$

$$M_\theta = \tfrac{1}{2}\sum_k \frac{\partial^2}{\partial y_k^2} + \sum_{i,k} c_{ki}(\theta,x)\frac{\partial^2}{\partial y_k \partial x_i} + \sum_k h_k(\theta,x)\frac{\partial}{\partial y_k}.$$

And there is a similar formulation for $\tilde{P}_{sx}$.

Let $p_0 \in L^2(R^N)$ be the initial density of $X_0$ [then $p_0(x) \geqq o$ a.e. and $\int_{R^N} p_0(x)dx = 1$].

We define the probability measures $P$ and $\tilde{P}$ on $(\Omega, \mathscr{G})$ by:

$$\forall B \text{ Borel subset of } R^{N+d},\ \forall t \geqq o,$$

$$P[(X_t, Y_t)\in B] = \int_{R^N} p_0(x) P_{0x}[(X_t, Y_t)\in B]dx$$

$$\tilde{P}[(X_t, Y_t)\in B] = \int_{R^N} p_0(x) \tilde{P}_{0x}[(X_t, Y_t)\in B]dx.$$

Define:

$$Z_t^s = \exp\{\textstyle\int_s^t [h(\theta, X_\theta), dY_\theta] - \frac{1}{2}\int_s^t |h(\theta, X_\theta)|^2 d\theta\}$$

and write $Z_t$ for $Z_t^0$.

We deduce from the well-known Cameron–Martin formula (see [16]):

$$\left.\frac{dP}{d\tilde{P}}\right|_{\mathscr{G}_t} = Z_t,\ \left.\frac{dP_{sx}}{d\tilde{P}_{sx}}\right|_{\mathscr{G}_t^s} = Z_t^s,$$

We will write $E$ [resp. $E_{sx}, \tilde{E}, \tilde{E}_{sx}$] for the expectation with respect to $P$ [resp. $P_{sx}, \tilde{P}, \tilde{P}_{sx}$]. We will write $\Omega$ for $(\Omega, \mathscr{G}, P)$ and $\tilde{\Omega}$ for $(\Omega, \mathscr{G}, \tilde{P})$.

Define $\mathscr{F}_t^s = \sigma\{Y_\theta - Y_s, s \leqq \theta \leqq t\}$, and write $\mathscr{F}_t$ for $\mathscr{F}_t^0$.

The aim of filtering theory is to characterize at each time $t$ the law of the signal $X_t$, conditioned on the observation $\sigma$-field $\mathscr{F}_t$, i.e. quantities of the form $E[f(X_t)|\mathscr{F}_t]$.

We will prove in the rest of this section that this law has a density with respect to Lebesgue measure in $R^N$, which, up to a normalizing factor, is the solution of a stochastic PDE of the kind we studied in the first section.

We will make use of the following formula, which is well known and easy to verify:

$$E(f(X_t)|\mathscr{F}_t) = \frac{\tilde{E}(f(X_t)Z_t|\mathscr{F}_t)}{\tilde{E}(Z_t|\mathscr{F}_t)} \tag{1.6}$$

where $f$ is any bounded measurable function.

Finally let us make the following hypothesis:

$$\frac{\partial}{\partial x_i}\sigma_{ij} \in L^\infty(]o, T[xR^N);\ \forall\, T > o;\ i,j = 1\ldots N. \tag{1.7}$$

## §2. STUDY OF A BACKWARD STOCHASTIC PDE

Let $L_t$ be the infinitesimal generator of the $X_t$ process, i.e.:

$$L_t = \tfrac{1}{2}\sum_{i,j=1}^{N} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(t,x)\frac{\partial}{\partial x_i}.$$

Using the hypotheses introduced in Section 1, we can consider $L_t$ as a family of elements of $\mathscr{L}(H^1, H^{-1})$, defined in the following way:

$$\langle L_t u, v\rangle = -\tfrac{1}{2}\sum_{i,j}\int_{R^N} a_{ij}(t,x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx + \sum_{i=1}^{N}\int_{R^N} a_i(t,x)\frac{\partial u}{\partial x_i}v\,dx$$

where

$$a_i = b_i - \tfrac{1}{2}\sum_{j=1}^{N}\frac{\partial a_{ij}}{\partial x_j}.$$

Let $T > o$. Consider the following stochastic PDE:

$$\begin{cases} dv(t) + L_t v(t)dt + [h(t)v(t) + c(t).\nabla v(t), dY_t] = o,\ o \leqq t \leqq T \\ v(T) = f \end{cases} \tag{2.1}$$

Where $f$ is a given Borel measurable function from $R^N$ into $R$ satisfying:

$$f \in L^\infty(R^N) \cap L^2(R^N). \tag{2.2}$$

We have to consider (2.1) as a backward equation because $-L_t$ (and not $L_t$!) is coercive. Our probability space in this and the next sections will be $\tilde{\Omega} \triangleq (\Omega, \mathscr{G}, \tilde{P})$. We remark that $Y_t$ is a $\mathscr{F}_t - \tilde{P}$ standard Wiener process with values in $R^d$.

Define $\tilde{Y}_\theta = Y_T - Y_{T-\theta}$.

$\tilde{Y}_\theta$ is $\mathscr{F}_T^{T-\theta} - \tilde{P}$ Wiener process, and setting $\tilde{v}(\theta) = v(T-\theta)$, we see that Eq. (2.1) is equivalent to:

$$\begin{cases} d\tilde{v}(\theta) - L_{T-\theta}\tilde{v}(\theta)d\theta = [h(T-\theta)\tilde{v}(\theta) + c(T-\theta).\nabla\tilde{v}(\theta), d\tilde{Y}_\theta],\ o \leqq \theta \leqq T \\ \tilde{v}(o) = f \end{cases}$$

It follows from (1.1), (1.2), (1.3) and the fact $gg^*$ and $g^*g$ have the same largest eigenvalue:

$$\exists \gamma > o \text{ s.t. } ([a(t,x) - c^*c(t,x)]\xi, \xi) \geqq \gamma |\xi|^2 \qquad \forall (t,x) \in R_+ \times R^N, \forall \xi \in R^N. \tag{2.3}$$

Define:

$$B(t)u = h(t)u + c(t).\nabla u, u \in H^1.$$

It follows from (2.3) that the pair of operators $(-L_t, B_t)$ satisfy the coercivity condition, and all the conditions of Section 2.1 of Part I, yielding a unique solution $\tilde{v}(\theta)$ of the last equation, $\mathscr{F}_T^{T-\theta}$—adapted.

Then Eq. (2.1) has a unique solution:

$$v \in L^2(\tilde{\Omega} \times ]o, T[; H^1) \cap L^2(\tilde{\Omega}; C(o, T; L^2(R^N)))$$

where $v$ is $\mathscr{F}^t_T$ adapted.

The rest of this section will be devoted to the proof of the following theorem, which gives a sort of Feynman–Kac formula for the stochastic PDE (2.1):

THEOREM 2.1 $\forall t \in [o, T]$, the following equality holds $d\tilde{P} \times ds$ a.e.:

$$v(t,x) = \tilde{E}_{tx}(f(X_T)Z^t_T | \mathscr{F}^t_T). \tag{2.4}$$

*Remark 2.1* The right-hand side term of (2.4) can be also written:

$$\tilde{E}(f(X_T)Z^t_T | \mathscr{F}^t_T, X_t = x).$$

It then follows that—$t$ being fixed—it is a measurable function defined on $(\Omega \times R^N, \mathscr{F}^t_T \otimes \mathscr{R}^N)$. □

We will actually prove the result under additional regularity assumptions:

LEMMA 2.1 *Suppose in addition to the above hypotheses, that* $b_i, \sigma_{ij}, h_k, g_{ki}$ $(i,j = 1 \ldots N;\ k = 1 \ldots d)$, *have continuous and bounded partial derivatives in* $t$ *and* $x$ *of any order, and* $f \in \cap_n H^n$.

Then equality (2.3) holds $\forall (t,x)$, a.s.

We first assume Lemma 2.1 is true, and proceed to:

*Proof of Theorem 2.1* Let us first suppose that $f$ is continuous, with compact, support. Let $b_i^n, \sigma^n_{i,j}, h^n_k, f^n$ be a sequence of smooth coefficients and final conditions, such that:

i) $|b_i^n|$, $|\sigma^n_{ij}|$, $|\partial a^n_{ij}/\partial x_j|$, $|h^n_k|$ and $|f_n|$ are all uniformly bounded by a constant independent of $n$, and $a^n$ satisfies (1.3) with $\alpha$ independent of $n$.

ii) $\sigma_{ij}^n \to \sigma_{ij}$ and $f^n \to f$ uniformly on each compact set of $[o, T] \times R^N$ (resp. of $R^N$).

iii) $\partial a_{ij}^n/\partial x_j \to \partial a_{ij}/\partial x_j$, $b_i^n \to b_i$ and $h_k^n \to h_k$ in measure on each compact set of $[o, T] \times R^N$.

Denote by $\tilde{P}_{tx}^n$, $v_n(t,x)$ and ${}^nZ_T^t$ the corresponding objects associated with $b^n$, $T^n$, $h^n$ and $f^n$ instead of $b$, $T$, $h$ and $f$. It follows from Lemma 2.1:

$$v^n(t,x) = \tilde{E}_{tx}^n(f^n(X_t)\,{}^nZ_T^t/\mathscr{F}_T^t). \tag{2.5}$$

Let

$$u \in C_K^1(R^N),\dagger \text{ s.t.:}$$

$$u(x) \geqq o, \int_{R^N} u(x)dx = 1.$$

Denote by $\tilde{P}_{tu}^n$ the solution of the martingale problem associated with $(1.5)^n$, satisfying the initial condition at time $t$:

$$\tilde{E}_{tu}^n[g(X_t)] = \int_{R^N} g(x)u(x)ds.$$

We define similarly $P_{tu}^n$, replacing $(1.5)^n$ by $(1.4)^n$, and also $\tilde{P}_{tu}$ and $P_{tu}$.

$$\left.\frac{dP_{tu}^n}{d\tilde{P}_{tu}^n}\right|_{\mathscr{G}_T^t} = {}^nZ_T^t.$$

It follows from (2.5):

$$(u, v^n(t)) = \tilde{E}_{tu}^n(f^n(X_T)\,{}^nZ_T^t/\mathscr{F}_T^t). \tag{2.6}$$

Let $\varphi$ be a continuous and bounded application from $\Omega$ into $R$, which is $\mathscr{F}_T^t$ measurable. It follows from (2.6):

$$\tilde{E}_{tu}^n[\varphi(u, v^n(t))] = E_{tu}^n[\varphi f^n(X_T)].$$

But $\varphi \,.\, (u, v^n(t))$ is $\mathscr{F}_T^t$ measurable, and the restriction of $\tilde{P}_{tu}^n$ to $\mathscr{F}_T^t$ does not depend on $n$:

$$\begin{aligned} \tilde{E}_{tu}^n[\varphi(u, v^n(t))] &= \tilde{E}_{tu}[\varphi(u, v^n(t))] \\ \tilde{E}_{tu}[\varphi(u, v^n(t))] &= E_{tu}^n[\varphi f^n(X_T)]. \end{aligned} \tag{2.7}$$

Let us now take the limit for $n \to \infty$ in (2.7). From the results of Stroock–Varadhan [16], $P_{tx}^n \to P_{tx}$ weakly, uniformly on each compact set. The support of $u$ being compact, $P_{tu}^n \to P_{tu}$ weakly. The limit in the right-hand side then follows from (ii).

$\dagger C_K^1(R^N)$ is the space of $C^1$ functions having compact support.

It follows easily from (i) that $v^n(t)$ is bounded in $L^2(\tilde{\Omega}\times R^N)$, and $v^n$ is bounded in $L^2(\tilde{\Omega}\times ]t,T[;\ H^1(R^N))$. Then there exists a subsequence $v^\mu$, such that:

$$v^\mu(t)\rightharpoonup\xi \text{ in } L^2(\tilde{\Omega}\times R^N) \text{ weakly}$$

$$v^\mu\rightharpoonup\chi \text{ in } L^2(\tilde{\Omega}\times ]t,T[ ]H_0^1(R^N)) \text{ weakly.}$$

It remains to show that $\chi=v$, and $\xi=v(t)$. This will prove that the whole sequence $v^n(t)$ converges to $v(t)$ in $L^2(\tilde{\Omega}\times R^N)$ weakly, and (2.4) will follow from the limit in (2.7), and the freedom of choice of $\varphi$ and $u$.

Let $\theta\in C^1([t,T])$. It follows from $(2.1)^\mu$:

$$\begin{cases}(f^\mu,u)\theta(T)+\int_t^T\theta\langle L^\mu v^\mu,u\rangle ds+\int_t^T\theta[h^\mu v^\mu+c^\mu.v^\mu,u),dy_s]\\ \qquad =(v^\mu(t),u)\theta(t)+\int_t^T\theta'(v^\mu,u)ds.\end{cases}\tag{2.8}$$

Using (ii), (iii), and the fact that $u$ has a compact support, it is easy to check that the following convergences hold in $L^2(]t,T[\times R^N)$ strongly:

$$a^\mu\frac{\partial u}{\partial x_i}\to a\frac{\partial u}{\partial x_i},\quad i=1\ldots N;$$

$$b^\mu u\to bu;\ h^\mu u\to hu;\ c^\mu u\to cu.$$

We then can take the weak limit in $L^2(\tilde{\Omega})$ of (2.8), yielding:

$$\begin{cases}(f,u)\theta(T)+\int_t^T\theta\langle L_s\chi,u\rangle ds+\int_t^T\theta[(h\chi+c.\chi,u),dy_s]\\ \qquad =(\xi,u)\theta(t)+\int_t^T\theta'(\chi,u)ds,\ \forall\,\theta\in C^1([t,T]),\ \forall\,u\in C_K^1(R^N)\end{cases}$$

It is then easy to conclude from the uniqueness of the solution of (2.1) that $\chi=v$, and $\xi=v(t)$.

Finally, it is easy to generalize (2.4) to any bounded and Borel measurable $f$, s.t. $f\in L^2(R^N)$. □

We now prove Lemma 2.1 in the particular case where the observation noise is independent of the signal:

LEMMA 2.2 *Suppose that all the hypotheses of Lemma* 2.1 *are satisfied, and in addition* $g(t)\equiv o$.

Then (2.4) is satisfied.

*Proof* It follows from the hypotheses and Theorem 2.1 of Section I that each trajectory of $v$ belongs to $\cap_n C(o,T;\ H^n)$, so that $\forall\,t, v(t,.)\in C_b^\infty(R^N)$ a.s.

In order to simplify the notations, let us prove (2.3) for $t=o$, write $\tilde{E}_x^T(\circ)$ for $\tilde{E}_{0x}(.|\mathscr{F}_T)$ and $Z_T$ for $Z_T^0$. We will suppose that $d=1$. The more general case is handled exactly in the same way.

Let $o=t_0<t_1<t_2<\dots t_n=T$ be a mesh with $t_{i+1}-t_i=T/.\,k$.

Define:

$$\Delta_i=\tilde{E}_x^T[Z_{t_{i+1}}v(t_{i+1}X_{t_{i+1}})-Z_{t_i}v(t_i,X_{t_i})]$$

$$\tilde{E}_n^T(f(X_T)Z_T)-v(o,x)=\sum_{i=1}^{n-1}\Delta_i.$$

In order to prove (2.4), it suffices to show that:

$$\sum_{i=0}^{n-1}\Delta_i\to o \text{ in } L^1(\Omega,\tilde{P}_x)$$

when $n\to+\infty$.

$$\Delta_i=\tilde{E}_x^T[Z_{t_{i+1}}v(t_{i+1},X_{t_{i+1}})-Z_{t_i}v(t_{i+1},X_{t_i})] \\ +\tilde{E}_x^T[Z_{t_i}v(t_{i+1},X_{t_i})-Z_{t_i}v(t_i,X_{t_i})].$$

We express the first term in $\Delta_i$ by means of Ito formula, applied to $Z_sv(t_{i+1},X_s)$. Because $g=o$, $Y_t$ and $W_t$ are $\tilde{P}_x$-independent, and $\tilde{E}_x^T\int_a^b\Psi_sdW_s=o$, as soon as $\Psi\in L^2(\Omega\times]o,T[)$, and $\Psi_s$ is independent of $\sigma\{w_\theta-w_s,\ s\leqq\theta\leqq b\}$. We express the second term in $\Delta_i$ using Eq. (2.1) at $x=X_{t_i}$, which makes sense because of the regularity of the solution.

It follows:

$$\Delta_i=\tilde{E}_x^T\int_{t_i}^{t_{i+1}}Z_sv(t_{i+1},X_s)h(s,X_s)dY_s+\tilde{E}_x^T\int_{t_i}^{t_{i+1}}Z_sL_sv(t_{i+1},X_s)ds \\ -\tilde{E}_x^T\int_{t_i}^{t_{i+1}}Z_{t_i}hv(s,X_{t_i})dY_s-\tilde{E}_x^T\int_{t_i}^{t_{i+1}}Z_{t_i}L_sv(s,X_{t_i})ds \tag{2.9}$$

The first stochastic integral in (2.9) is forward, the integral being adapted to $\mathscr{F}_s\vee\mathscr{F}_T^{t_{i+1}}$. The second one is backward, where the integrand is adapted to $\mathscr{F}_{t_i}\vee\mathscr{F}_T^s$.

It is easy to show, by Lebesgue dominated convergence theorem, that:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\tilde{E}_x^T\int_{t_i}^{t_{i+1}}Z_sL_sv(t_{i+1},X_s)ds \\ =\lim_{n\to\infty}\sum_{i=0}^{n-1}\tilde{E}_x^T\int_{t_i}^{t_{i+1}}Z_{t_i}L_sv(s,X_{t_i})ds=\int_0^TZ_sL_sv(s,X_s)ds.$$

It is not possible to take the limit in the same way in the stochastic integrals, because what should be the common limit do not make sense.

$$\int_{t_i}^{t_{i+1}} Z_s v(t_{i+1}, X_s) h(s, X_s) dY_s - \int_{t_i}^{t_{i+1}} Z_{t_i} hv(s, X_{t_i}) dY_s = \alpha_i + \beta_i + \gamma_i$$

where:

$$\alpha_i = \int_{t_i}^{t_{i+1}} [Z_s v(t_{i+1}, X_s) h(s, X_s) - Z_{t_i} v(t_{i+1}, X_{t_i}) h(t_i, X_{t_i})] dY_s$$

$$\beta_i = \int_{t_i}^{t_{i+1}} Z_{t_i} [hv(t_{i+1}, X_{t_i}) - hv(s, X_{t_i})] dY_s$$

$$\gamma_i = Z_{t_i} v(t_{i+1}, X_{t_i}) [h(t_i, X_{t_i}) - h(t_{i+1}, X_{t_i})] (Y_{t_{i+1}} - Y_{t_i}).$$

Again, $\alpha_i$ is a forward stochastic integral, and $\beta_i$ is a backward one. $v$ and $\partial h/\partial t$ being bounded,

$$\begin{aligned} \tilde{E}_x \left| \sum_{i=0}^{n-1} \gamma_i \right| &\leqq \sum_{i=0}^{n-1} \tilde{E}_x |\gamma_i| \\ &\leqq ck \sum_i \tilde{E}_x (Z_{t_i} | Y_{t_{i+1}} - Y_{t_i}|) \\ &\leqq ck^{3/2} \sum_i (\tilde{E}_x Z_{t_i}^2)^{1/2} \\ &\leqq \sqrt{\frac{T}{n}} . cT. \sup_i (\tilde{E}_x Z_{t_i}^2)^{1/2}. \end{aligned}$$

It follows that, as $n \to +\infty \sum_{i=0}^{n-1} \tilde{E}_x^T \gamma_i \to o$ in $L^1(\Omega, \tilde{P}_x)$.

It remains to show that $\tilde{E}_x^T \sum_{i=0}^{n-1} \alpha_i$ and $\tilde{E}_x^T \sum_{i=0}^{n-1} \beta_i$ converge to 0 in $L^1(\Omega; \tilde{P}_x)$. Let us first consider $\tilde{E}_x^T \sum_i \alpha_i$. It follows from Ito formula:

$$\begin{aligned} &\tilde{E}_x^T [Z_s v(t_{i+1}, X_s) h(s, X_s) - Z_{t_i} v(t_{i+1}, X_{t_i}) h(t_i, X_{t_i})] \\ &= \tilde{E}_x^T \int_{t_i}^s Z_\theta \{h'_\theta(\theta, X_\theta) v(t_{i+1}, X_\theta) + L_\theta [h(\theta, .) v(t_{i+1}, .)](X_\theta)\} d\theta \\ &\quad + \tilde{E}_x^T \int_t^s Z_\theta h^2(\theta, X_\theta) v(t_{i+1}, X_\theta) dY_\theta \\ &= \tilde{E}_x^T \int_{t_i}^s Z_\theta \rho_i(\theta, X_\theta) d\theta + \tilde{E}_x^T \int_{t_i}^s Z_\theta \eta_i(\theta, X_\theta) dY_\theta. \end{aligned}$$

It follows:

$$\tilde{E}_x^T \alpha_i = \tilde{E}_x^T \alpha_i^{(1)} + \tilde{E}_x^T \alpha_i^{(2)};$$

with

$$\tilde{E}_x |\alpha_i^{(1)}| = \tilde{E}_x \left| \int_{t_i}^{t_{i+1}} dY_s \int_{t_i}^s Z_\theta \rho_i(\theta, X_\theta) d\theta \right|$$

$$\tilde{E}_x |\alpha_i^{(1)}| \leqq ||\rho||_{L^\infty} \left( \tilde{E}_x \left\{ \sup_{0 \leqq \theta \leqq T} Z_\theta^2 \right\} \right)^{1/2} . \frac{k^{3/2}}{\sqrt{3}}. \tag{2.10}$$

$$\begin{aligned} \alpha_i^{(2)} = \int_{t_i}^{t_{i+1}} dY_s \int_{t_i}^s Z_\theta \eta_i(\theta, X_\theta) dY_\theta &= \tfrac{1}{2} Z_{t_i} \eta_i(t_i, X_{t_i}) [(Y_{t_{i+1}} - Y_{t_i})^2 - k] \\ &+ \int_{t_i}^{t_{i+1}} dY_s \int_{t_i}^s [Z_\theta \eta_i(\theta, X_\theta) - Z_{t_i} \eta_i(t_i, X_{t_i})] dY_\theta = \alpha'_i + \alpha''_i \end{aligned}$$

with

$$\tilde{E}_x|\alpha_i''| \leqq \left(\tilde{E}_x\left\{\sup_{t_i \leqq \theta \leqq t_{i+1}} [Z_\theta \eta_i(\theta, X_\theta) - Z_{t_i}\eta_i(t_i, X_{t_i})]^2\right\}\right)^{1/2} \cdot \frac{k}{\sqrt{2}}.$$

We then deduce, using the Ito formula applied to $Z_\theta \eta_i(\theta, X_\theta)$ and the classical inequalities for martingales:

$$\tilde{E}_x|\alpha_i''| \leqq Ck^{3/2}. \tag{2.11}$$

It follows from (2.10) and (2.11):

$$\tilde{E}_x\left|\sum_i \alpha_i\right| \leqq \tilde{E}_x\left|\sum_i \alpha_i'\right| + c'\sqrt{k}.$$

And the desired convergence follows from the fact that:

$$\sum_i \alpha_i' \to o \text{ in } L^1(\Omega, \tilde{P}_x).$$

Consider now $\sum_i \beta_i$. Here we use Eq. (2.1), instead of Ito formula, and we get:

$$\beta_i = \beta_i^{(1)} + \beta_i^{(2)},$$

where

$$\beta_i^{(1)} = \int_{t_i}^{t_{i+1}} Z_{t_i} v(s, X_{t_i})(h(t_{i+1}, X_{t_i}) - h(s, X_{t_i}))dY_s$$

$$\beta_i^{(2)} = \int_{t_i}^{t_{i+1}} Z_{t_i}(h(t_{i+1}, X_{t_i})(v(t_{i+1}, X_{t_i}) - v(s, X_{t_i}))dY_s$$

$$E|\beta_i^{(1)}| \leqq C(E(Z_{t_i})^2)^{1/2}. k^{3/2}.$$

Using Eq. (2.1), we get:

$$\beta_i^{(2)} = -\beta_i' - \beta_i'' + \beta_i''',$$

where

$$\beta_i' = Z_{t_i} h(t_{i+1}, X_{t_i}) \int_{t_i}^{t_{i+1}} dY_s \int_s^{t_{i+1}} Lv(\theta, X_{t_i})d\theta$$

$$\beta_i'' = Z_{t_i} h^2(t_{i+1}, X_{t_i}) v(t_{i+1}, X_{t_i}) \tfrac{1}{2}[(Y_{t_{i+1}} - Y_{t_i})^2 - k]$$

$$\beta_i''' = Z_{t_i} h(t_{i+1}, X_{t_i}) \int_{t_i}^{t_{i+1}} dY_s \int_s^{t_{i+1}} [hv(t_{i+1}, X_{t_i}) - hv(\theta, X_{t_i})]dY_\theta$$

$$E|\beta_i'| \leqq c(E(Z_{t_i})^2)^{1/2}. k^{3/2}$$

$$\sum_i \beta_i'' \to o \text{ in } L^1(\Omega, \tilde{P}_x)$$

$$E|\beta_i'''| \leqq ck(E\{\sup_{t_i \leqq s \leqq t_{i+1}} [hv(t_{i+1}, X_{t_i}) - hv(s, X_{t_i})]^2\})^{1/2}. \tag{2.12}$$

It is easy to see from (2.12), using the regularity of $h$ and Eq. (2.1), that:

$$E|\beta_i'''| \leqq c' k^{3/2}. \quad \square$$

Let us finally proceed to:

*Proof of Lemma 2.1* We will show how one can adapt the idea of the proof of Lemma 2.2 to this more general case, where $g \not\equiv o$. Once again, we suppose for simplicity that $d=1$.

Using again Ito formula, we get:

$$\begin{aligned} Z_{t_{i+1}} v(t_{i+1}, X_{t_{i+1}}) - Z_{t_i} v(t_{i+1}, X_{t_i}) &= \int_{t_i}^{t_{i+1}} Z_s L_s v(t_{i+1}, X_s) ds \\ &- \int_{t_i}^{t_{i+1}} Z_s hc(s, X_s) . \nabla v(t_{i+1}, X_s) ds \\ &+ \int_{t_i}^{t_{i+1}} Z_s v(t_{i+1}, X_s) h(s, X_s) dY_s \\ &+ \int_{t_i}^{t_{i+1}} Z_s [\nabla v(t_{i+1}, X_s), \sigma(s, X_s) dW_s] \\ &+ \int_{t_i}^{t_{i+1}} Z_s hc(s, X_s) . \nabla v(t_{i+1}, X_s) ds. \end{aligned}$$

On the other hand, it follows from Eq. (2.1):

$$\begin{aligned} Z_{t_i} v(t_{i+1}, X_{t_i}) - Z_{t_i} v(t_i, X_{t_i}) = &- \int_{t_i}^{t_{i+1}} Z_{t_i} L v(s, X_{t_i}) ds \\ &- \int_{t_i}^{t_{i+1}} Z_{t_i} (hv + c . \nabla v)(s, X_{t_i}) dY_s. \end{aligned}$$

$\tilde{E}_x^T$ of the stochastic integral with respect to $W_t$ is no longer zero, but we have:

$$\begin{aligned} &\tilde{E}_x \{ \int_{t_i}^{t_{i+1}} Z_s [\nabla v(t_{i+1}, X_s) \sigma(s, X_s) dW_s] \int_0^{t_{i+1}} \varphi_s dY_s \} \\ &= \tilde{E}_x \{ \int_{t_i}^{t_{i+1}} Z_s \varphi_s c(s, X_s) . \nabla v(t_{i+1}, X_s) ds \} \\ &= \tilde{E}_x \{ \int_{t_i}^{t_{i+1}} Z_s c(s, X_s) . \nabla v(t_{i+1}, X_s) dY_s \int_0^{t_{i+1}} \varphi_s dY_s \} \end{aligned}$$

$\forall \varphi$ measurable, bounded and $\mathscr{F}_t \vee \mathscr{F}_T^{t_{i+1}}$ adapted.

And this suffices to show that:

$$\tilde{E}_x^T \int_{t_i}^{t_{i+1}} Z_s [\nabla v(t_{i+1}, X_s) \sigma(s, X_s) dW_s] = \tilde{E}_x^T \int_{t_i}^{t_{i+1}} Z_s c(s, X_s) . \nabla v(t_{i+1}, X_s) dY_s.$$

It follows that:

$$\begin{aligned} \Delta_i = &\tilde{E}_x^T \int_{t_i}^{t_{i+1}} Z_s L_s v(t_{i+1}, X_s) ds + \tilde{E}_x^T \int_{t_i}^{t_{i+1}} Z_s [h(s, X_s) v(t_{i+1}, X_s) \\ &+ c(s, X_s) . \nabla v(t_{i+1}, X_s)] dY_s \\ &- \tilde{E}_x^T \int_{t_i}^{t_{i+1}} Z_{t_i} L_s v(s, X_{t_i}) ds - \tilde{E}_x^T \int_{t_i}^{t_{i+1}} Z_{t_i} [hv(s, X_{t_i}) \\ &+ c . \nabla v(s, X_{t_i})] dY_s. \end{aligned}$$

The end of the proof is then similar to that of Lemma 2.2, where here we apply Ito formula to $\nabla v(t_{i+1}, X_s)$ as well as to $v(t_{i+1}, X_s)$, and we use Eq. (2.1) differentiated once or twice in $x$. □

## §3. EQUATION FOR THE UNNORMALIZED CONDITIONAL DENSITY

Let us now consider the following forward stochastic PDE:

$$\begin{cases} du(t)+A_t u(t)dt=[\bar{h}(t)u(t)-c(t).\nabla u(t),dY_t] \\ u(o)=p_0 \end{cases} \tag{3.1}$$

where $A_t=-L_t^*$, $p_0$ has been defined in Section 1 as the density of the law of $X_0$, and $\bar{h}(t)$ is given by:

$$\bar{h}_k(t)=h_k(t)-\sum_{i=1}^{N}\frac{\partial c_{ki}(t)}{\partial x_i}. \tag{3.2}$$

Again we can apply the results of Section I to Eq. (3.1), yielding the existence of a unique solution $u\in L^2(\bar{\Omega}\times]o,T[,H^1)\cap L^2(\bar{\Omega};\ C([o,T],L^2(R^N)))$ where $u(t)$ is adapted to $\mathscr{F}_t$.

Indeed (3.1) is adjoint to (2.1):

THEOREM 3.1 *Almost all trajectories of the process $R_t=(u(t),\ v(t))$, $t\in[o,T]$, are constant.*

*Proof* Recall that $(.,.)$ denotes the scalar product in $L^2(R^N)$. As $R_t$ has a.s. continuous paths, it suffices to show that $\forall s,t\in]o,T[$, $R_s=R_t$ a.s.

Again, as in the proof of Lemma 2.2, we cannot differentiate $R_t$, because its differential would involve terms which do not make sense.

Let $o<s<t<T$, and $s=t_0<t_1<\ldots<t_n=t$ be a mesh, with $t_{i+1}-t_i=t-s/n=k$. We suppose that $k<s\Lambda(T-t)$. Define:

$$v^n=\frac{1}{k}\int_t^{t+k}v(\theta)d\theta \tag{3.3}$$

$$u^0=\frac{1}{k}\int_{s-k}^{s}u(\theta)d\theta \tag{3.4}$$

and consider the following time discretized approximations of (2.1) and (3.1):

$$v^{i+1}-v^i+kL_iv^i=-[h^iv^{i+1}+c^i.\nabla v^{i+1},\Delta_iY] \qquad i=n-1,n-2,\ldots,o \tag{3.5}$$

$$u^{i+1}-u^i+kA_{i+1}u^{i+1}=[\bar{h}^iu^i-c^i.\nabla u^i,\Delta_iY] \qquad i=0,1,\ldots,n-1 \tag{3.6}$$

where $L_i$, $A_i$, $h^i$, $\bar{h}^i$ and $c^i$ are the mean values of $L$, $A$, $h$, $\bar{h}$ and $c$ respectively on $[t_i, t_{i+1}]$; $\Delta_i Y = Y_{t_{i+1}} - Y_{t_i}$.

It is easy to check that when $k$ is small enough such that $(I+kA_i)$ satisfies:

$$\langle (I+kA_i)u, u\rangle \geqq \gamma \|u\|^2$$

for some $\gamma > o$, $\forall i$, $\forall u \in H^1$, then (3.3), (3.5) [resp. (3.4), (3.6)] define a unique sequence $v^i$; $i=n,\ldots,o$ [resp. $u^i$; $i=o,\ldots,n$] in $L^2(\Omega; H^1)$, where $v^i$ is $\mathscr{F}^{t_i}_T$ measurable and $u^i$ is $\mathscr{F}_{t_i}$ measurable; $i=o,\ldots,n$.

Moreover, if we multiply (3.5) by $u^i$, and (3.6) by $v^{i+1}$, sum the two relations from $i=o$ to $i=n-1$, we get:

$$(u^0, v^0) + k\langle A_0 u^0, v^0\rangle = (u^n; v^n) + k\langle A_n u^n, v^n\rangle. \tag{3.7}$$

Define $v_n(t)$ and $u_n(t)$ by:

$$v_n(\theta) = v^i, \text{ for } \theta \in [t_i, t_{i+1}[;\ i=0,1,\ldots,n$$

$$u_n(\theta) = u^i, \text{ for } \theta \in [t_i, t_{i+1}[;\ i=0,1,\ldots,n$$

where $t_{n+1} = t_n + k$.

LEMMA 3.1 *$u_n$ [resp. $v_n$] remains in a bounded subset of $L^2(\tilde{\Omega} \times ]s,t[; H^1)$; $u_n(t)$ [resp. $v_n(s)$] remains in a bounded subset of $L^2(\tilde{\Omega}; L^2(R^N))$.*

*Proof* Let us indicate the proof for $u_n$. Take the square in (3.6), and then multiply (3.6) by $2u^i$:

$$|u^{i+1} - u^i|^2 + 2k\langle A_{i+1}u^{i+1}, u^{i+1} - u^i\rangle$$
$$\leqq |\bar{h}^i u^i - c^i . \nabla u^i|^2 . |\Delta_i Y|^2. \tag{3.8}$$

$$2(u^{i+1} - u^i, u^i) + 2k\langle A_{i+1}u^{i+k}, u^i\rangle$$
$$= 2(\bar{h}^i u^i - c^i . \nabla u^i, u^i) . \Delta_i Y. \tag{3.9}$$

Taking the sum of (3.8) and (3.9), and the expectation yields:

$$E|u^{i+1}|^2 - E|u^i|^2 + 2kE\langle A_{i+1}u^{i+1}, u^{i+1}\rangle \leqq kE|\bar{h}^i u^i - c^i . \nabla u^i|^2$$
$$E|u^n|^2 - E|u^0|^2 + 2E\int_{s+k}^{t+k} \langle A_n(\theta)u_n(\theta), u_n(\theta)\rangle d\theta$$
$$\leqq E\int_s^t |B_n(\theta)u_n(\theta)|^2 d\theta \tag{3.10}$$

where $A_n(\theta) = A_i$ and $B_n(\theta) = \bar{h}^i I - c^i . \nabla$, for $\theta \in [t_i, t_{i+1}[$.

$$E|u^n|^2 + 2E\int_s^t \langle A_n(\theta)u_n(\theta), u_n(\theta)\rangle d\theta \leqq E|u^0|^2$$
$$+ 2E\int_s^{s+k} \langle A_n(\theta)u_n(\theta), u_n(\theta)\rangle d\theta + E\int_s^t |B_n(\theta)u_n(\theta)|^2 d\theta.$$

It follows from $A \in L^r(o, T; \mathscr{L}(H^1, H^{-1}))$;

$$2E\int_s^{s+k} \langle A_n(\theta)u_n(\theta), u_n(\theta)\rangle d\theta \leqq c\,.\,k\|u^0\|^2$$

$$\leqq \frac{c}{k}\left(\int_{s-k}^{s} \|u(\theta)\| d\theta\right)^2$$

$$\leqq c\int_{s-k}^{s} \|u(\theta)\|^2 d\theta.$$

It then follows from (1.1), (1.2), (1.3):

$$E|u_n(t)|^2 + \gamma E\int_s^t \|u_n(\theta)\|^2 d\theta \leqq \frac{1}{k}E\int_{s-k}^{s} |u(\theta)|^2 d\theta$$

$$+ cE\int_{s-k}^{s} \|u(\theta)\|^2 d\theta + \lambda E\int_s^t |u_n(\theta)|^2 d\theta.$$

The result follows from (3.11) by standard estimates. □

Let us go back to (3.7).

$$kE|\langle A_0 u^0, v^0\rangle| \leqq ckE\{\|u^0\|\,.\,\|v^0\|\}$$

$$\leqq \frac{c}{k}E\left\{\int_{s-k}^{s} \|u(\theta)\| d\theta\,.\int_s^{s+k} \|v_n(\theta)\| d\theta\right\}$$

$$\leqq c\left(E\int_{s-k}^{s} \|u(\theta)\|^2 d\theta\right)^{1/2}\left(E\int_s^{s+k} \|v_n(\theta)\|^2 d\theta\right)^{1/2}.$$

It then follows from Lemma 3.1 that $k\langle A_0 u^0, v^0\rangle \to o$ in $L^1(\tilde{\Omega})$, as $k \to o$. The same holds for $k\langle A_n u^n, v^n\rangle$. Moreover, again from Lemma 3.1, there exists a subsequence $n'$ such that: $v_{n'}(s) \rightharpoonup \zeta$, and $u_{n'}(t) \rightharpoonup \xi$ in $L^2(\tilde{\Omega}; L^2(R))$ weakly.

The theorem will then follow from (3.7), if we prove that $\xi = u(t)$ and $\zeta = v(s)$. Let us indicate the proof for $u$.

Suppose that the subsequence has been chosen such that moreover: $u_n \rightharpoonup \chi$ in $L^2(\tilde{\Omega} \times ]s, t[; H^1)$ weakly.

But $u_n \in M^2(s, t; H^1)$. Then $\chi \in M^2(s, t; H^1)$. Let

$$\rho \in C^1(R), \rho^i = \frac{1}{k}\int_{t_i}^{t_{i+1}} \rho(s) ds;$$

and $\theta \in H^1$. It follows from (3.6):

$$\rho^i(u^{i+1},\theta)-\rho^i(u^i,\theta)+k\rho^i\langle u^{i+1},A^*_{i+1}\theta\rangle = \rho^i[(u^i,h^i\theta+c^i.\nabla\theta),\Delta_i Y].$$

$$\rho^{n-1}(u^n,\theta)-\sum_{i=1}^{n-1}(\rho^i-\rho^{i+1})(u^i,\theta) + k\sum_{i=0}^{n-1}\rho^i\langle u^{i+1},A^*_{i+1},\theta\rangle=\rho^0(u^0,\theta) + \sum_{k=0}^{n-1}\rho^i[(u^i,h^i\theta+c^i.\nabla\theta),\Delta_i Y]. \tag{3.12}$$

Taking the weak limit in (3.12) yields:

$$\rho(t)(\xi,\theta)-\int_s^t(\chi,\theta)\rho'(v)dv=\rho(s)(u(s),\theta) + \int_s^t\rho(v)[\bar{h}\chi-c.\nabla\chi,dYv]. \tag{3.13}$$

It is easy to prove from (3.13) that $(\chi(v),\theta)$ is (almost everywhere equal to) a continuous function, that $\chi$ is the restriction to $[s,t]$ of the unique solution of Eq. (3.1), and that $\chi(t)=\xi$. It then follows that $\xi=u(t)$. □

*Remark 3.1* The above proof contains an alternate proof of existence in Theorem 1.3 of Part I. □

It follows from Theorems 2.1 and 3.1:

COROLLARY 3.1 $(u(T),f)=\tilde{E}(f(X_T)Z_T/\mathscr{F}_T)\,\forall f\in L^\infty(R^N)\cap L^2(R^N)$.

*Proof* Apply Theorem 2.1, and then Theorem 3.1:

$$(u(T),v(T))=(u(o),v(o))$$

$$(u(T),f)=\int_{R^N}p_0(x)\tilde{E}_x(f(X_T)Z_T/\mathscr{F}_T)dx.$$

Let $\varphi$ be any bounded $\mathscr{F}_T$ measurable random variable.

$$\tilde{E}_x[(u(T),f)\varphi]=\int_{R^N}p_0(x)\tilde{E}_x[f(X_T)Z_T\varphi]dx.$$

It then follows from the fact that $\tilde{P}_x/_{\mathscr{F}_T}=\tilde{P}/_{\mathscr{F}_T}$, and definition of $\tilde{P}$:

$$\tilde{E}[(u(T),f)\varphi]=\tilde{E}[f(X_T)Z_T\varphi]. \quad \square$$

But $T$ in Corollary 3.1 is arbitrary. It then follows:

COROLLARY 3.2 (i) $\forall t\geqq o, u(t,x)\geqq o$ a.e., a.s.

(ii) $(u(t),f)=\tilde{E}(f(X_t)Z_t/\mathscr{F}_t)\,\forall f\in L^\infty(R^N)$.

*Proof* (i) follows from the fact that $(u(t), f) \geqq o$, $\forall f \in L^\infty(R^N) \cap L^2(R^N)$, $f \geqq o$.

Let $f$ be a bounded Borel measurable and positive function defined on $R^N$. Let $f_n \in L^\infty(R^N) \cap L^2(R^N)$; $f_{n+1}(x) \geqq f_n(x) \geqq o$, and $f_n(x) \uparrow f(x)$ as $n \to +\infty$, $\forall x \in R^N$. We then can take the limit in:

$$(u(t), f_n) = \tilde{E}(f_n(X_t)Z_t/\mathscr{F}_t)$$

and the limit is a.s. finite. Then if $f$ is bounded and Borel measurable, we defined:

$$(u(t), f) = (u(t), f^+) - (u(t), f^-).$$

But if $f$ and $g$ coincide $dx$ a.e., $(u(t), f) = (u(t), g)$ a.s. □

In particular,

$$(u(t), 1) = \tilde{E}(Z_t/\mathscr{F}_t).$$

But:

$$Z_t = 1 + \int_0^t Z_s[h(X_s), dY_s]$$

$$\tilde{E}(Z_t/\mathscr{F}_t) = 1 + \int_0^t [\tilde{E}(Z_s h(X_s)/\mathscr{F}_s), dY_s]$$

$$(u(t), 1) = 1 + \int_0^t [(u(s), h(s)), dY_s]. \tag{3.14}$$

It follows from (3.14) and the properties of $Z_t$ that $(u(t), 1)$ is a continuous process which does not reach $o$ nor $+\infty$ in finite time. We then can define:

$$p(t,x) = (u(t), 1)^{-1} u(t,x). \tag{3.15}$$

COROLLARY 3.3 *$\forall f \in L^\infty(R^N) \cap L^2(R^N)$, $(p(t), f)$, $t \geqq o$ is a continuous version of $E(f(X_t)/\mathscr{F}_t)$.*

*Proof* It follows from (1.6) and Corollary 3.2

$$\forall t \geqq o, E(f(X_t)/\mathscr{F}_t) = \frac{(u(t), f)}{(u(t), 1)} \text{ a.s.}$$

But if $f \in L^2(R^N)$, $(u(t), f)$ is a continuous process, as well as $(u(t), 1)$. □

We can now derive the classical Kushner–Statonovitch equation for $p(t,x)$—see Kushner [8].

Let $v \in H^1$.

$$d(u(t), v) + \langle Au(t), v \rangle dt = [(\bar{h}u(t) - c.\nabla u(t), v), dY_t]$$

$$d(u(t), 1) = [(u(t), h), dY_t].$$

It then follows from Ito formula:

$$d(p(t),v)+\langle Ap(t),v\rangle dt=[(\bar{h}p(t)-c.\nabla p(t),v)$$
$$-(p(t),h)(p(t),v),dY_t-(p(t),h)dt]\ \forall v\in H^1.$$

$$\begin{cases} dp(t)+Ap(t)dt=[(\bar{h}-(p(t),h))p(t)-c.\nabla p(t),dY_t-(p(t),h)dt] \\ p(o)=p_0 \end{cases} \tag{3.16}$$

*Remark 3.2* $p(t,x)$ could be characterized as the solution of (3.16), But the equation for $u$ is both easier to study theoretically, and easier to approximate numerically. □

## §4. REMARK ON THE COERCIVITY CONDITION

The coercivity is implied by (1.3) and (1.2)–(1.1). The later means that the "proportion" of the noise independent of the signal, in the observation noise, is uniformly positive. It is well known that this is a crucial regularity assumption in filtering, even for the linear Kalman–Bucy theory.

In order to understand what happens when $\tilde{g}(t)\equiv o$, let us choose a "singular" filtering problem, which will lead to stochastic PDE's of the type considered in Section 2.2 of Part I.

Assume:

$$\tilde{g}(t)\equiv o,\ g(t)\equiv I. \tag{4.1}$$

$b$, $\sigma$ and $h$ are bounded functions of $t$ only. (4.2)

Then (1.4) reduces to:

$$\begin{cases} X_t=X_0+\int_0^t b(s)ds+\int_0^t\sigma(s)dW_s \\ Y_t=\int_0^t h(s)ds+W_t \end{cases}$$

It follows:

$$X_t=X_s+\int_s^t[b(\theta)-\sigma(\theta)h(\theta)]d\theta+\int_0^t\sigma(\theta)dY_\theta=X_s+X_t^s$$

where $X_t^s$ is $\mathscr{F}_t^s$ measurable.

It is easy to check:

$$p(t,x)=p_0(x-X_t^0)$$
$$u(t,x)=p_0(x-X_t^0)\exp\{\textstyle\int_0^t[h(s),dY_s]-\frac{1}{2}\int_0^t|h(s)|^2ds\}. \tag{4.3}$$

Suppose that $p_0 \in C^2(R^N) \cap H^1$. Then, according to Theorem 2.3 of Part I, Eq. (3.1) has a unique solution. The solution is given by (4.3), as can be verified by Ito formula.

The backward Eq. (2.1) must also have a unique solution, provided that $f \in C^2(R^N) \cap H^1$. Let us guess this solution, with the help of (2.4).

$$v(t,x) = \tilde{E}_{tx}(f(X_T)Z_T^t/\mathscr{F}_T^t)$$
$$= f\left(x + \int_t^T (b - \sigma h)ds + \int_t^T \sigma dY_s\right) \circ \exp\left\{\int_t^T [h, dY_s] - \tfrac{1}{2}\int_t^T |h|^2 ds\right\}. \quad (4.4)$$

Once again, using Ito (backward!) calculus, we can check that (4.4) is the unique solution of (2.1).

It is here clear why the parabolic equation (3.1) degenerates: the dynamics of $X_t$ is completely observed. The only unknown is $X_0$. In particular, if the law of $X_0$ is a Dirac measure, there is no density at any time for the conditional law.

Let us now see that the contrary holds in the coercive case. Let $n_0 = [N/2] + 1$. Then $\delta_x \in H^{-n_0}$. Suppose that all coefficients $a_{ij}$, $b_i$, $\sum_j \partial aij/\partial xj$, $h_k$ and $c_{ki}$ have bounded partial derivatives up to order $n_0$ and that (1.1), (1.2) and (1.3) hold. Let $t \in R_+$, $f \in L^2(R^N)$, and consider:

$$\begin{cases} dv(s) + L_s v(s)ds + [h(s)v(s) + c(s).\nabla v(s), dY_s] = o \\ v(t) = f \end{cases}$$

Let $k_0$ be an integer s.t. $2k_0 \geqq n_0$. It follows from Theorem 2.2 of Part I that:

$$t^{k0}(\tilde{E}_{0x}\|v(o)\|_{H^{n0}}^2)^{1/2} \leqq c|f|_{L^2(R^N)}$$

Fix $x \in R^N$. There exists a constant $c'$ (depending possibly on $x$, but not on $f$) such that:

$$|v(o,x)| \leqq c'\|v(o)\|_{H^{n0}}.$$

Then:

$$(\tilde{E}_0|v(o,x)|^2)^{1/2} \leqq \frac{c''(x)}{t^{k0}}|f|_{L^2}$$

or alternatively:

$$(\tilde{E}_{0x}|\tilde{E}_{0x}(f(X_t)Z_t/\mathscr{F}_t)|^2)^{1/2} \leqq c''(t,x)|f|_{L^2}$$

But if $\delta_x$ is the law of $X_0$,

$$\begin{aligned} E|E(f(X_t)/\mathscr{F}_t)| &= E_{0x}|E_{0x}(f(X_t)/\mathscr{F}_t)| \\ &= \tilde{E}_{0x}\left(Z_t \frac{|v(o,x)|}{\tilde{E}_{0x}(Z_t/\mathscr{F}_t)}\right) \\ &\leqq (\tilde{E}_{0x}|v(o,x)|^2)^{1/2} \\ &\leqq \frac{c''(x)}{t^{k0}}|f|_{L^2} \end{aligned}$$

Then, if $t > o$, $f \to E\ (f(X_t)/\mathscr{F}_t)$ is a linear continuous mapping from $L^2(R^N)$ into $L^1(\Omega, \mathscr{F}_t, P_{0x})$. There exists a Linear Random Functional $p(t)$ on $L^2(R^N)$, such that:

$$(p(t), f) = E(f(X_t)/\mathscr{F}_t).$$

Here we have a density for $t > o$, which is a L.R.F. on $L^2(R^N)$. We were not able to prove that it is a $L^2(R^N)$-valued process.



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