Probab. Theory Relat. Fields 98, 209–227 (1994)

Probability Theory and Related Fields

Backward doubly stochastic differential equations and systems of quasilinear SPDEs

Etienne Pardoux [1,*], **Shige Peng** [2,**]

[1] Laboratoire APT, URA 225, Université de Provence, F-13331 Marseille Cedex 3, France
[2] Institute of Mathematics, Shandong University, Jinan, People's Republic of China

Received: 30 December 1992 / In revised form: 6 October 1993

Summary. We introduce a new class of backward stochastic differential equations, which allows us to produce a probabilistic representation of certain quasilinear stochastic partial differential equations, thus extending the Feynman-Kac formula for linear SPDE's.

Mathematics Subject Classification: 60H10, 60H15, 60H30

Introduction

A new kind of backward stochastic differential equations (in short BSDE), where the solution is a pair of processes adapted to the past of the driving Brownian motion, has been introduced by the authors in [6]. It was then shown in a series of papers by the second and both authors (see [8, 7, 9, 10]), that this kind of backward SDEs gives a probabilistic representation for the solution of a large class of systems of quasi-linear parabolic PDEs, which generalizes the classical Feynman-Kac formula for linear parabolic PDEs.

On the other hand, the classical Feynman-Kac formula has been generalized by the first author in [4, 5] to provide a probabilistic representation for solutions of linear parabolic stochastic partial differential equations; see also Krylov and Rozovskii [1], Rozovskii [11] and Ocone and Pardoux [3] for further extensions. The aim of this paper is to combine the two above types of results, and relate a new class of backward stochastic differential equations, which we call "doubly stochastic" for reasons which will become clear below, to a class of systems of quasilinear parabolic SPDEs. Hence we shall give a probabilistic representation of solutions of such systems of quasilinear SPDEs, and use it to prove an existence and uniqueness result of such SPDEs.

* The research of this author was partially supported by DRET under contract 901636/A000/DRET/DS/SR
** The research of this author was supported by a grant from the French "Ministère de la Recherche et de la Technologie", which is gratefully acknowledged

Let us be more specific. Let $\{W_t, t \geq 0\}$ and $\{B_t, t \geq 0\}$ be two mutually independent standard Brownian motions, with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^l$. For each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, let $\{X_s^{t,x}; t \leq s \leq T\}$ be the solution of the SDE:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})\,dr + \int_t^s \sigma(X_r^{t,x})\,dW_r, \qquad t \leq s \leq T.$$

We next want to find a pair of processes $\{(Y_s^{t,x}, Z_s^{t,x}); t \leq s \leq T\}$ with values in $\mathbb{R}^k \times \mathbb{R}^{k \times l}$ such that for each $s \in [t, T]$ $(Y_s^{t,x}, Z_s^{t,x})$ is $\sigma(W_r; t \leq r \leq s) \vee \sigma(B_r - B_s; s \leq r \leq T)$ measurable and

$$\begin{aligned} Y_s^{t,x} = h(X_T^{t,x}) &+ \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,dr \\ &+ \int_s^T g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,dB_r - \int_s^T Z_r^{t,x}\,dW_r, \qquad t \leq s \leq T \end{aligned}$$

where the dW integral is a forward Itô integral and the dB integral is a backward Itô integral. We shall show that, under appropriate conditions on f and g, the above "backward doubly stochastic differential equation" has a unique solution.

We finally will show that under rather strong smoothness conditions on b, σ, f and g, $\{Y_t^{t,x}; (t, x) \in [0, T] \times \mathbb{R}^d\}$ is the unique solution of the following system of backward stochastic partial differential equations:

$$\begin{aligned} u(t, x) = h(x) &+ \int_s^T [\mathscr{L}_s u(s, x) + f(x, u(s, x), (\nabla u \sigma)(s, x))]\,ds \\ &+ \int_t^T g(x, u(s, x), (\nabla u \sigma)(s, x))\,dB_s, \qquad 0 \leq t \leq T \end{aligned}$$

where u takes values in $\mathbb{R}^k$,

$$(\mathscr{L} u)_i(t, x) = (L u_i)(t, x), \qquad 1 \leq i \leq k$$

and

$$L = \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^*)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}.$$

The paper is organised as follows. In Sect. 1, we study existence and uniqueness of the solution to a backward doubly stochastic differential equation, and estimate the moments of the solution. In Sect. 2, we consider both a forward and a backward SDE, as introduced above, and study the regularity of the solution of the latter with respect to x, the initial condition of the former. Finally in Sect. 3 we relate our BSDE to a system of quasilinear stochastic partial differential equations.

Notation. The Euclidean norm of a vector $x \in \mathbb{R}^k$ will be denoted by $|x|$, and for a $d \times d$ matrix A, we define $\|A\| = \sqrt{\operatorname{Tr} A A^*}$.

1 Backward doubly stochastic differential equations

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and $T>0$ be fixed throughout this paper. Let $\{W_t, 0 \leqq t \leqq T\}$ and $\{B_t, 0 \leqq t \leqq T\}$ be two mutually independent standard Brownian motion processes, with values respectively in $\mathbb{R}^d$ and in $\mathbb{R}^l$, defined on $(\Omega, \mathscr{F}, P)$. Let $\mathscr{N}$ denote the class of P-null sets of $\mathscr{F}$. For each $t \in [0, T]$, we define

$$\mathscr{F}_t \triangleq \mathscr{F}_t^W \vee \mathscr{F}_{t,T}^B$$

where for any process $\{\eta_t\}$, $\mathscr{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s;\, s \leqq r \leqq t\} \vee \mathscr{N}$, $\mathscr{F}_t^\eta = \mathscr{F}_{0,t}^\eta$.

Note that the collection $\{\mathscr{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a filtration.

For any $n \in \mathbb{N}$, let $M^2(0, T; \mathbb{R}^n)$ denote the set of (classes of $dP \times dt$ a.e. equal) n dimensional jointly measurable random processes $\{\varphi_t;\, t \in [0, T]\}$ which satisfy:

(i) $E \int_0^T |\varphi_t|^2 \, dt < \infty$

(ii) φ_t is $\mathscr{F}_t$ measurable, for a.e. $t \in [0, T]$.

We denote similarly by $S^2([0, T]; \mathbb{R}^n)$ the set of continuous n dimensional random processes which satisfy:

(i) $E(\sup_{0 \leqq t \leqq T} |\varphi_t|^2) < \infty$

(ii) φ_t is $\mathscr{F}_t$ measurable, for any $t \in [0, T]$.

Let

$$f\colon\ \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k$$
$$g\colon\ \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^{k \times l}$$

be jointly measurable and such that for any $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$f(\cdot, y, z) \in M^2(0, T; \mathbb{R}^k)$$
$$g(\cdot, y, z) \in M^2(0, T; \mathbb{R}^{k \times l}).$$

We assume moreover that there exist constants $c>0$ and $0<\alpha<1$ such that for any $(\omega, t) \in \Omega \times [0, T]$, $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times l}$,

$$\text{(H.1)} \qquad \begin{aligned} &|f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leqq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2) \\ &\|g(t, y_1, z_1) - g(t, y_2, z_2)\|^2 \leqq c\,|y_1 - y_2|^2 + \alpha\, \|z_1 - z_2\|^2. \end{aligned}$$

Given $\xi \in L^2(\Omega, \mathscr{F}_T, P; \mathbb{R}^k)$, we consider the following backward doubly stochastic differential equation:

$$\text{(1.1)} \qquad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s)\, dB_s - \int_t^T Z_s \, dW_s, \qquad 0 \leqq t \leqq T.$$

We note that the integral with respect to $\{B_t\}$ is a "backward Itô integral" and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral, see Nualart and Pardoux [2].

The main objective of this section is to prove the:

Theorem 1.1 *Under the above conditions, in particular* (H.1), *Eq.* (1.1) *has unique solution*

$$(Y, Z) \in S^2([0, T]; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d}).$$

Before we start proving the theorem, let us establish the same result in case f and g do not depend on Y and Z. Given $f \in M^2(0, T; \mathbb{R}^k)$ and $g \in M^2(0, T; \mathbb{R}^{k \times l})$ and ξ as above, consider the SDE:

$$Y_t = \xi + \int_t^T f(s)\, ds + \int_t^T g(s)\, dB_s - \int_t^T Z_s\, dW_s, \qquad 0 \leqq t \leqq T. \tag{1.2}$$

Proposition 1.2 *There exists a unique pair*

$$(Y, Z) \in S^2([0, T]; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})$$

which solves Eq. (1.2).

Proof. Uniqueness is immediate, since if $(\bar{Y}, \bar{Z})$ is the difference of two solutions,

$$\bar{Y}_t + \int_t^T \bar{Z}_s\, dW_s = 0, \qquad 0 \leqq t \leqq T.$$

Hence by orthogonality

$$E(|\bar{Y}_t|^2) + E \int_t^T T_r[\bar{Z}_s \bar{Z}_s^*]\, ds = 0,$$

and $\bar{Y}_t \equiv 0$ P a.s., $\bar{Z}_t = 0$ $dt\, dP$ a.e.

We now prove existence. We define the filtration $(\mathcal{G}_t)_{0 \leqq t \leqq T}$ by

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B$$

and the $\mathcal{G}_t$-square integrable martingale

$$M_t = E^{\mathcal{G}_t}\left[\xi + \int_0^T f(s)\, ds + \int_0^T g(s)\, dB_s\right], \qquad 0 \leqq t \leqq T.$$

An obvious extension of Itô's martingale representation theorem yields the existence of $\mathcal{G}_t$-progressively measurable process $\{Z_t\}$ with values in $\mathbb{R}^{k \times d}$ such that

$$E \int_0^T |Z_t|^2\, dt < \infty$$

$$M_t = M_0 + \int_0^t Z_s\, dW_s, \qquad 0 \leqq t \leqq T.$$

Hence

$$M_T = M_t + \int_0^T Z_s\, dW_s.$$

Replacing M_T and M_t by their defining formulas and subtracting $\int_0^t f(s)\,ds + \int_0^t g(s)\,dB_s$ from both sides of the equality yields

$$Y_t=\xi+\int_t^T f(s)\,ds+\int_t^T g(s)\,dB_s-\int_t^T Z_s\,dW_s,$$

where

$$Y_t \triangleq E^{\mathscr{G}_t}\left(\xi+\int_t^T f(s)\,ds+\int_t^T g(s)\,dB_s\right).$$

It remains to show that $\{Y_t\}$ and $\{Z_t\}$ are in fact $\mathscr{F}_t$-adapted. For Y_t, this is obvious since for each t,

$$Y_t=E(\Theta/\mathscr{F}_t \vee \mathscr{F}_t^B)$$

where Θ is $\mathscr{F}_T^W \vee \mathscr{F}_{t,T}^B$ measurable. Hence $\mathscr{F}_t^B$ is independent of $\mathscr{F}_t \vee \sigma(\Theta)$, and

$$Y_t=E(\Theta/\mathscr{F}_t).$$

Now

$$\int_t^T Z_s\,dW_s=\xi+\int_t^T f(s)\,ds+\int_t^T g(s)\,dB_s-Y_t,$$

and the right side is $\mathscr{F}_T^W \vee \mathscr{F}_{t,T}^B$ measurable.

Hence, from Itô's martingale representation theorem, $\{Z_s, t<s<T\}$ is $\mathscr{F}_s^W \vee \mathscr{F}_{t,T}^B$ adapted. Consequently Z_s is $\mathscr{F}_s^W \vee \mathscr{F}_{t,T}^B$ measurable, for any $t<s$, so it is $\mathscr{F}_s^W \vee \mathscr{F}_{s,T}^B$ measurable. □

We shall need the following extension of the well-known Itô formula.

Lemma 1.3 *Let* $\alpha\in S^2([0,T];\mathbb{R}^k)$, $\beta\in M^2(0,T;\mathbb{R}^k)$, $\gamma\in M^2(0,T;\mathbb{R}^{k\times l})$, $\delta\in M^2(0,T;\mathbb{R}^{k\times d})$ *be such that:*

$$\alpha_t=\alpha_0+\int_0^t \beta_s\,ds+\int_0^t \gamma_s\,dB_s+\int_0^t \delta_s\,dW_s, \qquad 0\leqq t\leqq T.$$

Then

$$|\alpha_t|^2=|\alpha_0|^2+2\int_0^t (\alpha_s,\beta_s)\,ds+2\int_0^t (\alpha_s,\gamma_s\,dB_s)$$

$$+2\int_0^t (\alpha_s,\delta_s\,dW_s)-\int_0^t \|\gamma_s\|^2\,ds+\int_0^t \|\delta_s\|^2\,ds$$

$$E|\alpha_t|^2=E|\alpha_0|^2+2E\int_0^t (\alpha_s,\beta_s)\,ds-E\int_0^t \|\gamma_s\|^2\,ds+E\int_0^t \|\delta_s\|^2\,ds.$$

More generally, if $\phi \in C^2(\mathbb{R}^k)$,

$$\phi(\alpha_t) = \phi(\alpha_0) + \int_0^t (\phi'(\alpha_s), \beta_s)\, ds + \int_0^t (\phi'(\alpha_s), \gamma_s\, dB_s) + \int_0^t (\phi'(\alpha_s), \delta_s\, dW_s)$$

$$-\tfrac{1}{2}\int_0^t \operatorname{Tr}[\phi''(\alpha_s)\gamma_s\gamma_s^*]\, ds + \tfrac{1}{2}\int_0^t \operatorname{Tr}[\phi''(\alpha_s)\delta_s\delta_s^*]\, ds.$$

Proof. The first identity is a combination of Itô's forward and backward formulae, applied to the process $\{a_t\}$ and the function $x \to |x|^2$. We only sketch the proof.

Let $0 = t_0 < t_1 < \ldots < t_n = t$.

$$|\alpha_{t_{i+1}}|^2 - |\alpha_{t_i}|^2 = 2(\alpha_{t_{i+1}} - \alpha_{t_i}, \alpha_{t_i}) + |\alpha_{t_{i+1}} - \alpha_{t_i}|^2$$

$$= 2\left(\int_{t_i}^{t_{i+1}} \beta_s\, ds, \alpha_{t_i}\right) + 2\left(\int_{t_i}^{t_{i+1}} \gamma_s\, dB_s, \alpha_{t_{i+1}}\right) + 2\left(\int_{t_i}^{t_{i+1}} \delta_s\, dW_s, \alpha_{t_i}\right)$$

$$-2\left(\int_{t_i}^{t_{i+1}} \gamma_s\, dB_s, \alpha_{t_{i+1}} - \alpha_{t_i}\right) + |\alpha_{t_{i+1}} - \alpha_{t_i}|^2$$

$$= 2\int_{t_i}^{t_{i+1}} (\alpha_{t_i}, \beta_s)\, ds + 2\int_{t_i}^{t_{i+1}} (\alpha_{t_{i+1}}, \gamma_s\, dB_s) + 2\int_{t_i}^{t_{i+1}} (\alpha_{t_i}, \delta_s\, dW_s)$$

$$-\left|\int_{t_i}^{t_{i+1}} \gamma_s\, dB_s\right|^2 + \left|\int_{t_i}^{t_{i+1}} \delta_s\, dW_s\right|^2 + \rho_i,$$

where $\sum_{i=0}^{n-1} \rho_i \to 0$ in probability, as $\sup_i t_{i+1} - t_i \to 0$. The rest of the proof is standard.

The second identity follows from the first, provided the stochastic integrals have zero expectation. This will follow from

$$E\left(\sup_{0 \leq t \leq T}\left|\int_t^T (\alpha_s, \gamma_s\, dB_s)\right| + \sup_{0 \leq t \leq T}\left|\int_0^t (\alpha_s, \delta_s\, dW_s)\right|\right) < \infty,$$

which is a consequence of Burkholder-Davis-Gundy's inequality and the assumptions made on α, γ and δ. Indeed, considering e.g. the forward integral, we have:

$$E\left(\sup_{0 \leq t \leq T}\left|\int_0^t (\alpha_s, \delta_s\, dW_s)\right|\right) \leq c\, E \sqrt{\int_0^T |\alpha_t|^2\, \|\delta_t\|^2\, dt}$$

$$\leq \frac{c}{2}\left(E\left(\sup_{0 \leq t \leq T} |\alpha_t|^2\right) + E\int_0^T \|\delta_t\|^2\, dt\right).$$

The last identity is proved in a way very similar to the first one. □

We can now turn to the

Proof of Theorem 1.1 Uniqueness. Let $\{(Y_t^1, Z_t^1)\}$ and $\{(Y_t^2, Z_t^2)\}$ be two solutions. Define

$$\bar{Y}_t = Y_t^1 - Y_t^2, \quad \bar{Z}_t = Z_t^1 - Z_t^2, \quad 0 \leqq t \leqq T.$$

Then

$$\bar{Y}_t = \int_t^T [f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2)]\, ds + \int_t^T [g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)]\, dB_s$$
$$- \int_t^T \bar{Z}_s\, dW_s.$$

Applying Lemma 1.3 to $\bar{Y}$ yields:

$$E(|\bar{Y}_t|^2) + E\int_t^T \|\bar{Z}_s\|^2\, ds = 2E\int_t^T (f(s, Y_s^1, Z_s) - f(s, Y_s^2, Z_s^2), \bar{Y}_s)\, ds$$
$$+ E\int_t^T \|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)\|^2\, ds.$$

Hence from (H.1) and the inequality $ab \leqq \frac{1}{2(1-\alpha)} a^2 + \frac{1-\alpha}{2} b^2$,

$$E(|\bar{Y}_t|^2) + E\int_t^T \|\bar{Z}_s\|^2\, ds \leqq c(\alpha) E\int_t^T |\bar{Y}_t|^2\, ds + \frac{1-\alpha}{2} E\int_t^T \|\bar{Z}_s\|^2\, ds + \alpha E\int_t^T \|\bar{Z}_s\|^2\, ds,$$

where $0 < \alpha < 1$ is the constant appearing in (H.1). Consequently

$$E(|\bar{Y}_t|^2) + \frac{1-\alpha}{2} E\int_t^T \|\bar{Z}_s\|^2\, ds \leqq c(\alpha) E\int_t^T \|\bar{Y}_s\|^2\, ds.$$

From Gronwall's lemma, $E(|\bar{Y}_t|^2) = 0$, $0 \leqq t \leqq T$, and hence $E\int_0^T \|\bar{Z}_t\|^2\, ds = 0$.

Existence. We define recursively a sequence $\{(Y_t^i, Z_t^i)\}_{i=0,1,\ldots}$ as follows. Let $Y_t^0 \equiv 0$, $Z_t^0 \equiv 0$. Given $\{(Y_t^i, Z_t^i)\}$, $\{(Y_t^{i+1}, Z_t^{i+1})\}$ is the unique solution, constructed as in Proposition 1.2, of the following equation:

$$Y_t^{i+1} = \xi + \int_t^T f(s, Y_s^i, Z_s^i)\, ds + \int_t^T g(s, Y_s^i, Z_s^i)\, dB_s - \int_t^T Z_s^{i+1}\, dW_s.$$

Let $\bar{Y}_t^{i+1} \triangleq Y_t^{i+1} - Y_t^i$, $\bar{Z}_t^{i+1} \triangleq Z_t^{i+1} - Z_t^i$, $0 \leqq t \leqq T$. The same computations as in the proof of uniqueness yield:

$$E(|\bar{Y}_t^{i+1}|^2) + E\int_t^T \|\bar{Z}_t^{i+1}\|^2\, ds = 2E\int_t^T (f(s, Y_s^i, Z_s^i) - f(s, Y_s^{i-1}, Z_s^{i-1}), \bar{Y}_s^{i+1})\, ds$$
$$+ E\int_t^T \|g(s, Y_s^i, Z_s^i) - g(s, Y_s^{i-1}, Z_s^{i-1})\|^2\, ds.$$

Let $\beta \in \mathbb{R}$. By integration by parts, we deduce

$$E(|\bar{Y}_t^{i+1}|^2)\, e^{\beta t} + \beta E \int_t^T |\bar{Y}_s^{i+1}|^2\, e^{\beta s}\, ds + E \int_t^T \|\bar{Z}_s^{i+1}\|^2\, e^{\beta s}\, ds$$

$$= 2E \int_t^T (f(s, Y_s^i, Z_s^i) - f(s, Y_s^{i-1}, Z_s^{i-1}), \bar{Y}_s^{i+1})\, e^{\beta s}\, ds$$

$$+ E \int_t^T \|g(s, Y_s^i, Z_s^i) - g(s, Y_s^{i-1}, Z_s^{i-1})\|^2\, e^{\beta s}\, ds.$$

There exists $c, \gamma > 0$ such that

$$E(|\bar{Y}_t^{i+1}|^2)\, e^{\beta s} + (\beta - \gamma) E \int_t^T |\bar{Y}_s^{i+1}|^2\, e^{\beta s}\, ds + E \int_t^T \|\bar{Z}_s^{i+1}\|^2\, e^{\beta s}\, ds$$

$$\leqq E \int_t^T \left(c\, |\bar{Y}_s^i|^2 + \frac{1+\alpha}{2} \|\bar{Z}_s^i\|^2 \right) e^{\beta s}\, ds.$$

Now choose $\beta = \gamma + \frac{2c}{1+\alpha}$, and define $\bar{c} = \frac{2c}{1+\alpha}$.

$$E(|\bar{Y}_t^{i+1}|^2)\, e^{\beta t} + E \int_t^T (\bar{c}\, |\bar{Y}_s^{i+1}|^2 + \|\bar{Z}_s^{i+1}\|^2)\, e^{\beta s}\, ds \leqq \frac{1+\alpha}{2} E \int_t^T (c\, |\bar{Y}_s^i|^2 + \|\bar{Z}_s^i\|^2)\, e^{\beta s}\, ds.$$

It follows immediately that

$$E \int_t^T (\bar{c}\, |\bar{Y}_s^{i+1}|^2 + \|\bar{Z}_s^{i+1}\|^2)\, e^{\beta s}\, ds \leqq \left(\frac{1+\alpha}{2}\right)^i E \int_t^T (\bar{c}\, |Y_s^1|^2 + \|Z_s^1\|^2)\, e^{\beta s}\, ds$$

and, since $\frac{1+\alpha}{2} < 1$, $\{(Y_t^i, Z_t^i)\}_{i=0,1,2,\ldots}$ is a Cauchy sequence in $M^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})$. It is then easy to conclude that $\{Y_t^i\}_{i=0,1,2,\ldots}$ is also Cauchy in $S^2([0, T]; \mathbb{R}^k)$, and that

$$\{(Y_t, Z_t)\} = \lim_{i \to \infty} \{(Y_t^i, Z_t^i)\}$$

solves Eq. (1.1). $\square$

We next establish higher order moment estimates for the solution of Eq. (1.1). For that sake, we need an additional assumption on g.

(H.2) $\begin{cases} \text{There exists } c \text{ such that for all } (t, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}, g\, g^*(t, y, z) \\ \leqq z\, z^* + c(\|g(t, 0, 0)\|^2 + |y|^2)\, I. \end{cases}$

Theorem 1.4 *Assume, in addition to the conditions of Theorem* 1.1, *that* (H.2) *holds and for some* $p>2$, $\xi \in L^p(\Omega, \mathscr{F}_T, P; \mathbb{R}^k)$ *and*

$$E\int_0^T (|f(t,0,0)|^p + \|g(t,0,0)\|^p)\,dt < \infty.$$

Then

$$E\left(\sup_{0\leqq t\leqq T} |Y_t|^p + \left(\int_0^T \|Z_t\|^2\,dt\right)^{p/2}\right) < \infty.$$

Proof. We apply Lemma 1.3 with $\varphi(x)=|x|^p$, yielding

$$|Y_t|^p + \frac{p}{2}\int_t^T |Y_s|^{p-2}\,\|Z_s\|^2\,ds + \frac{p}{2}(p-2)\int_t^T |Y_s|^{p-4}(Z_s Z_s^* Y_s, Y_s)\,ds$$

$$= |\xi|^p + p\int_t^T |Y_s|^{p-2}(f(s,Y_s,Z_s), Y_s)\,ds + p\int_t^T |Y_s|^{p-2}(Y_s, g(s,Y_s,Z_s)\,dB_s)$$

$$+ \frac{p}{2}\int_t^T |Y_s|^{p-2}\,\|g(s,Y_s,Z_s)\|^2\,ds$$

$$+ \frac{p}{2}(p-2)\int_t^T |Y_s|^{p-4}(g g^*(s,Y_s,Z_s)\,Y_s, Y_s)\,ds - p\int_t^T |Y_s|^{p-2}(Y_s, Z_s\,dW_s).$$

Also we do not know a priori that the above stochastic integrals have zero expectation, arguing as in the proof of Lemma 2.1 in Pardoux and Peng [7], we obtain that

$$E(|Y_t|^p) + \frac{p}{2}E\int_t^T |Y_s|^{p-2}\,\|Z_s\|^2\,ds + \frac{p}{2}(p-2)E\int_t^T |Y_s|^{p-4}(Z_s Z_s^* Y_s, Y_s)\,ds$$

$$\leqq E(|\xi|^p) + pE\int_t^T |Y_s|^{p-2}(f(s,Y_s,Z_s), Y_s)\,ds + \frac{p}{2}E\int_t^T |Y_s|^{p-2}\,\|g(s,Y_s,Z_s)\|^2\,ds$$

$$+ \frac{p}{2}(p-2)E\int_t^T |Y_s|^{p-4}(g g^*(s,Y_s,Z_s)\,Y_s, Y_s)\,ds.$$

Note that we can conclude from (H.1) that for any $\alpha<\alpha'<1$, there exists $c(\alpha')$ such that

$$\|g(t,y,z)\|^2 \leqq c(\alpha')(|y|^2 + \|g(t,0,0)\|^2) + \alpha'\,\|z\|^2.$$

From the last two inequalities, (H.1) and (H.2), and using Hölder's and Young's inequalities, we deduce that there exists $\theta>0$ and c such that for $0\leqq t\leqq T$,

$$E(|Y_t|^p) + \theta E\int_t^T |Y_s|^{p-2}\,\|Z_s\|^2\,ds$$

$$\leqq E(|\xi|^p) + cE\int_t^T (|Y_t|^p) + |f(s,0,0)|^p + \|g(s,0,0)\|^p)\,ds.$$

It then follows, using Gronwall's lemma, that

$$\sup_{0\leqq t\leqq T} E(|Y_t|^p)+E\int_0^T |Y_t|^{p-2}\,\|Z_t\|^2\,dt<\infty.$$

Applying the same inequalities we have already used to the first identity of this proof, we deduce that

$$|Y_t|^p\leqq|\xi|^p+c\int_t^T (|Y_s|^p)+|f(s,0,0)|^p+\|g(s,0,0)\|^p)\,ds$$
$$+p\int_t^T |Y_s|^{p-2}(Y_s, g(s,Y_s,Z_s)\,dB_s)-p\int_t^T |Y_s|^{p-2}(Y_s, Z_s\,dW_s).$$

Hence, from Burkholder-Davis-Gundy's inequality,

$$E(\sup_{0\leqq t\leqq T}|Y_t|^p)\leqq E(|\xi|^p)+c\,E\int_0^T (|Y_t|^p+|f(t,0,0)|^p+\|g(t,0,0)\|^p)\,dt$$
$$+c\,E\sqrt{\int_0^T |Y_t|^{2p-4}(g\,g^*(t,Y_t,Z_t)\,Y_t, Y_t)\,dt}$$
$$+c\,E\sqrt{\int_0^T |Y_t|^{2p-4}(Z_t Z_t^*\,Y_t, Y_t)\,dt}.$$

We estimate the last term as follows:

$$E\sqrt{\int_0^T |Y_t|^{2p-4}(Z_t Z_t^*\,Y_t, Y_t)\,dt}\leqq E\left(Y_t^{p/2}\sqrt{\int_0^T |Y_t|^{p-2}\,\|Z_t\|^2\,dt}\right)$$
$$\leqq\tfrac{1}{3}E(\sup_{0\leqq t\leqq T}|Y_t|^p)+\tfrac{1}{4}E\int_0^T |Y_t|^{p-2}\,\|Z_t\|^2\,dt.$$

The next to last term of the above inequality can be treated analogously, and we deduce that

$$E(\sup_{0\leqq t\leqq T}|Y_t|^p)<\infty.$$

Now we have

$$\int_0^T \|Z_t\|^2\,dt=|\xi|^2-|Y_0|^2+2\int_0^T (f(t,Y_t,Z_t), Y_t)\,dt+2\int_0^T (Y_t, g(t,Y_t,Z_t)\,dB_t)$$
$$+\int_0^T \|g(t,Y_t,Z_t)\|^2\,dt-2\int_0^T (Y_t, Z_t\,dW_t).$$

Hence for any $\delta>0$,

$$\left(\int_0^T \|Z_t\|^2\,dt\right)^{p/2} \leqq (1+\delta)\left(\int_0^T \|g(t,Y_t,Z_t)\|^2\,dt\right)^{p/2}$$
$$+c(\delta,p)\left[|\xi|^p+|Y_0|^p+\left|\int_0^T (f(t,Y_t,Z_t),Y_t)\,dt\right|^{p/2}\right.$$
$$\left.+\left|\int_0^T (Y_t, g(t,Y_t,Z_t)\,dB_t)\right|^{p/2}+\left|\int_0^T (Y_t, Z_t\,dW_t)\right|^{p/2}\right]$$

$$E\left[\left(\int_0^T \|Z_t\|^2\,dt\right)^{p/2}\right]$$
$$\leqq (1+\delta)^2\alpha E\left[\left(\int_0^T \|Z_t\|^2\,dt\right)^{p/2}\right]+c'(\delta,p)$$
$$+c(\delta,p)E\left[\left(\int_0^T |Y_t|\,\|Z_t\|\,dt\right)^{p/2}\right]+c(\delta,p)E\left[\left(\int_0^T |Y_t|^2\,\|Z_t\|^2\,dt\right)^{p/4}\right]$$
$$\leqq (1+\delta)^2\alpha E\left[\left(\int_0^T \|Z_t\|^2\,dt\right)^{p/2}\right]+c'(\delta,p)$$
$$+c(\delta,p)E\left\{\left(\sup_{0\leqq t\leqq T}|Y_t|^{p/2}\right)\left[\left(\int_0^T \|Z_t\|\,dt\right)^{p/2}+\left(\int_0^T \|Z_t\|^2\,dt\right)^{p/4}\right]\right\}$$
$$\leqq [(1+\delta)^2\alpha+(1+\delta)]\,E\left[\left(\int_0^T \|Z_t\|^2\,dt\right)^{p/2}\right]+c''(\delta,p).$$

The second part of the result now follows, if we choose $\delta>0$ small enough such that

$$(1+\delta)^2\alpha+(1+\delta)<1$$

(recall that $\alpha<1$). □

2 Regularity of the solution of the BDSDE

Let us first repeat some notations from Pardoux and Peng [7].

$C^k(\mathbb{R}^p;\mathbb{R}^q)$, $C^k_{l,b}(\mathbb{R}^p;\mathbb{R}^q)$, $C^k_p(\mathbb{R}^p;\mathbb{R}^q)$ will denote respectively the set of functions of class C^k from $\mathbb{R}^p$ into $\mathbb{R}^q$, the set of those functions of class C^k whose partial derivatives of order less than or equal to k are bounded (and hence the function itself grows at most linearly at infinity), and the set of those functions of class C^k which, together with all their partial derivatives of order less than or equal to k, grow at most like a polynomial function of the variable x at infinity.

We are given $b \in C^3_{l,b}(\mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C^3_{l,b}(\mathbb{R}^d, \mathbb{R}^{d\times d})$, and for each $t \in [0, T)$, $x \in \mathbb{R}^d$, we denote by $\{X^{t,x}_s, t \leqq s \leqq T\}$ the unique strong solution of the following SDE:

$$dX^{t,x}_s = b(X^{t,x}_s)\,ds + \sigma(X^{t,x}_s)\,dW_s, \qquad t \leqq s \leqq T \tag{2.1}$$
$$X^{t,x}_s = x.$$

It is well known that the random field $\{X^{t,x}_s;\ 0 \leqq t \leqq s \leqq T,\ x \in \mathbb{R}^d\}$ has a version which is a.s. of class C^2 in x, the function and its derivatives being a.s. continuous with respect to (t, s, x).

Moreover, for each (t, x),

$$\sup_{t \leqq s \leqq T} (|X^{t,x}_s| + |\nabla X^{t,x}_s| + |D^2 X^{t,x}_s|) \in \bigcap_{p \geqq 1} L^p(\Omega),$$

where $\nabla X^{t,x}_s$ denotes the matrix of first order derivatives of $X^{t,x}_s$ with respect to x and $D^2 X^{t,x}_s$ the tensor of second order derivatives.

Now the coefficients of the BDSDE will be of the form (with an obvious abuse of notations):

$$f(s, y, z) = f(s, X^{t,x}_s, y, z)$$
$$g(s, y, z) = g(s, X^{t,x}_s, y, z)$$

where

$$f\colon\ [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$$
$$g\colon\ [0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^{k\times l}.$$

We assume that for any $s \in [0, T]$, $(x, y, z) \to (f(s, x, y, z), g(s, x, y, z))$ is of class C^3, and all derivatives are bounded on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$.

We assume again that (H.1) and (H.2) hold, together with

(H.3)

$$g'_z(t, x, y, z)\,\theta\theta^*\, g'_z(t, x, y, z)^* \leqq \theta\theta^*, \qquad \forall t \in [0, T],\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^k,\ z, \theta \in \mathbb{R}^{k\times d}.$$

Let $h \in C^3_p(\mathbb{R}^d; \mathbb{R}^k)$. For any $t \in [0, T]$, $x \in \mathbb{R}^d$, let $\{(Y^{t,x}_s, Z^{t,x}_s);\ t \leqq s \leqq T\}$ denote the unique solution of the BDSDE:

$$Y^{t,x}_s = h(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)\,dr + \int_s^T g(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)\,dB_r - \int_s^T Z^{t,x}_r\,dW_r, \qquad t \leqq s \leqq T. \tag{2.2}$$

We shall define $X^{t,x}_s$, $Y^{t,x}_s$ and $Z^{t,x}_s$ for all $(s, t) \in [0, T]^2$ by letting $X^{t,x}_s = X^{t,x}_{s\vee t}$, $Y^{t,x}_s = Y^{t,x}_{s\vee t}$, and $Z^{t,x}_s = 0$ for $s < t$.

Theorem 2.1 *$\{Y^{t,x}_s;\ (s, t) \in [0, T]^2,\ x \in \mathbb{R}^d\}$ has a version whose trajectories belong to $C^{0,0,2}([0, T]^2 \times \mathbb{R}^d)$.*

Before proceeding to the proof of this theorem, let us state an immediate corollary:

Corollary 2.2 *There exists a continuous version of the random field* $\{Y_t^{t,x};\, t\in[0,T],\, x\in\mathbb{R}^d\}$ *such that for any* $t\in[0,T]$, $x\to Y_t^{t,x}$, *is of class* C^2 a.s., *the derivatives being* a.s. *continuous in* (t,x).

Proof of Theorem 2.1 We first note that we can deduce from Theorem 1.4 applied to the present situation that, for each $p\geqq 2$, there exist c_p and q such that

$$E\left(\sup_{t\leqq s\leqq T}|Y_s^{t,x}|^p+\left(\int_t^T \|Z_s^{t,x}\|^2\,ds\right)^{p/2}\right)\leqq c_p(1+|x|^q).$$

Next for $t\vee t'\leqq s\leqq T$,

$$\begin{aligned}Y_s^{t,x}-Y_s^{t',x'}=&\left[\int_0^1 h'(X_T^{t',x'}+\lambda(X_T^{t,x}-X_T^{t',x'}))\,d\lambda\right](X_T^{t,x}-X_T^{t',x'})\\&+\int_s^T(\varphi_r(t,x;t',x')[X_r^{t,x}-X_r^{t',x'}]+\psi_r(t,x;t',x')[Y_r^{t,x}-Y_r^{t',x'}]\\&+\chi_r(t,x;t',x')[Z_r^{t,x}-Z_r^{t',x'}])\,dr+\int_s^T(\bar\varphi_r(t,x;t',x')[X_r^{t,x}-X_r^{t',x'}]\\&+\bar\psi_r(t,x;t',x')[Y_r^{t,x}-Y_r^{t',x'}]+\bar\chi_r(t,x;t',x')[Z_r^{t,x}-Z_r^{t',x'}])\,dB_r\\&-\int_s^T(Z_r^{t,x}-Z_r^{t',x'})\,dW_r\end{aligned}$$

where

$$\varphi_r(t,x;t',x')=\int_0^1 f_x'(\Sigma_{r,\lambda}^{t,x;t',x'})\,d\lambda$$

$$\psi_r(t,x;t',x')=\int_0^1 f_y'(\Sigma_{r,\lambda}^{t,x;t',x'})\,d\lambda$$

$$\chi_r(t,x;t',x')=\int_0^1 f_z'(\Sigma_{r,\lambda}^{t,x;t',x'})\,d\lambda$$

$\bar\varphi_r$, $\bar\psi_r$ and $\bar\chi_r$ are defined analogously, with f replaced by g, and

$$\Sigma_{r,\lambda}^{t,x;t',x'}=(r,\,X_r^{t',x'}+\lambda(X_r^{t,x}-X_r^{t',x'}),\,Y_r^{t',x'}+\lambda(Y_r^{t,x}-Y_r^{t',x'}),\,Z_r^{t',x'}+\lambda(Z_r^{t,x}-Z_r^{t',x'})).$$

Combining the argument of Theorem 1.4 with the estimate:

$$E(\sup_{0\leqq s\leqq T}|X_s^{t,x}-X_s^{t',x'}|^p)\leqq c_p(1+|x|^p+|x'|^p)(|x-x'|^p+|t-t'|^{p/2}),$$

we deduce that for all $p\geqq 2$, there exists c_p and q such that

$$\begin{aligned}&E\left(\sup_{0\leqq s\leqq T}|Y_s^{t,x}-Y_s^{t',x'}|^p+\left(\int_t^T\|Z_s^{t,x}-Z_s^{t',x'}\|^2\,ds\right)^{p/2}\right)\\&\leqq c_p(1+|x|^q+|x'|^q)(|x-x'|^p+|t-t'|^{p/2}).\end{aligned}$$

Note that (H.3) is used in the proof; it plays the same role as (H.2) in the proof of Theorem 1.4. Note also that (H.1) implies that $\|\bar\chi_r\|\leqq\alpha<1$. We conclude from the last estimate, using Kolmogorov's lemma, that $\{Y_s^{t,x};\, s,t\in[0,T],\, x\in\mathbb{R}^d\}$ has an a.s. continuous version.

Next we define

$$\Delta_h^i X_s^{t,x}\triangleq(X_s^{t,x+he_i}-X_s^{t,x})/h,$$

where $h \in \mathbb{R} \setminus \{0\}$, $\{e_1, \ldots, e_d\}$ is an orthonormal basis of $\mathbb{R}^d$. $\Delta_h^i Y_s^{t,x}$ and $\Delta_h^i Z_s^{t,x}$ are defined analogously. We have

$$\begin{aligned}
\Delta_h^i Y_s^{t,x} &= \int_0^t h'(X_T^{t,x} + \lambda h \Delta_h^i X_T^{t,x}) \Delta_h^i X_T^{t,x} \, d\lambda \\
&\quad + \int_s^T \int_0^1 [f'_x(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i X_r^{t,x} + f'_y(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Y_r^{t,x} + f'_z(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Z_r^{t,x}] \, d\lambda \, dr \\
&\quad + \int_s^T \int_0^1 [g'_x(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i X_r^{t,x} \, g'_y(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Y_r^{t,x} + g'_z(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Z_r^{t,x}] \, d\lambda \, dB_r \\
&\quad - \int_s^T \Delta_h^i Z_r^{t,x} \, dW_r,
\end{aligned}$$

where $\Xi_{r,\lambda}^{t,x,h} = (r, X_r^{t,x} + \lambda h \Delta_h^i X_r^{t,x}, Y_r^{t,x} + \lambda h \Delta_h^i Y_r^{t,x}, Z_r^{t,x} + \lambda h \Delta_h^i Z_r^{t,x})$.

We note that for each $p \geqq 2$, there exists c_p such that

$$E(\sup_{0 \leqq s \leqq T} |\Delta_h^i X_s^{t,x}|^p) \leqq c_p.$$

The same estimates as above yields

$$E\left(\sup_{t \leqq s \leqq T} |\Delta_h^i Y_s^{t,x}|^p + \left(\int_t^T \|\Delta_h^i Z_s^{t,x}\|^2 \, ds\right)^{p/2}\right) \leqq c_p(1 + |x|^q + |h|^q).$$

Finally, we consider

$$\begin{aligned}
\Delta_h^i Y_s^{t,x} - \Delta_{h'}^i Y_s^{t',x'} &= \int_0^1 h'(X_T^{t,x} + \lambda h \Delta_h^i X_T^{t,x}) \Delta_h^i X_T^{t,x} \, d\lambda \\
&\quad - \int_0^1 h'(X_T^{t',x'} + \lambda h' \Delta_h^i X_T^{t',x'}) \Delta_{h'}^i X_T^{t',x'} \, d\lambda \\
&\quad + \int_s^T \int_0^1 [f'_x(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i X_r^{t,x} - f'_x(\Xi_{r,\lambda}^{t',x',h'}) \Delta_{h'}^i X_r^{t',x'}] \, d\lambda \, dr \\
&\quad + \int_s^T \int_0^1 [f'_y(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Y_r^{t,x} - f'_y(\Xi_{r,\lambda}^{t',x',h'}) \Delta_{h'}^i Y_r^{t',x'}] \, d\lambda \, dr \\
&\quad + \int_s^T \int_0^1 [f'_z(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Z_r^{t,x} - f'_z(\Xi_{r,\lambda}^{t',x',h'}) \Delta_{h'}^i Z_r^{t',x'}] \, d\lambda \, dr \\
&\quad \int_s^T \int_0^1 [g'_x({}_{r,\lambda}^{t,x,h}) \Delta_h^i X_r^{t,x} - g'_x(\Xi_{r,\lambda}^{t',x',h'}) \Delta_{h'}^i X_r^{t',x'}] \, d\lambda \, dB_r \\
&\quad + \int_s^T \int_0^1 [g'_y(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Y_r^{t,x} - g'_y(\Xi_{r,\lambda}^{t',x',h'}) \Delta_{h'}^i Y_r^{t',x'}] \, d\lambda \, dB_r \\
&\quad + \int_s^T \int_0^1 [g'_x(\Xi_{r,\lambda}^{t,x,h}) \Delta_h^i Z_r^{t,x} - g'_z(\Xi_{r,\lambda}^{t',x',h'}) \Delta_{h'}^i Z_r^{t',x'}] \, d\lambda \, dB_r \\
&\quad - \int_s^T [\Delta_h^i Z_r^{t,x} - \Delta_{h'}^i Z_r^{t',x'}] \, dW_r.
\end{aligned}$$

We note that

$$E\left(\sup_{0\leq s\leq T}|\Delta_h^i X_s^{t,x}-\Delta_{h'}^i X_s^{t',x'}|^p\right)\leq c_p(1+|x|^p)(|x-x'|^p+|h-h'|^p+|t-t'|^{p/2})$$

and

$$\begin{aligned}|\Xi_{r,\lambda}^{t,x,h}-\Xi_{r,\lambda}^{t',x',h'}|\leq(&|X_r^{t,x}-X_r^{t',x'}|+|X_r^{t,x+he_i}-X_r^{t',x'+h'e_i}|\\&+|Y_r^{t,x}-Y_r^{t',x'}|+|Y_r^{t,x+he_i}-Y_r^{t',x'+h'e_i}|\\&+\|Z_r^{t,x}-Z_r^{t',x'}\|+\|Z_r^{t,x+he_i}-Z_r^{t',x'+h'e_i}\|).\end{aligned}$$

Using similar arguments as those in Theorem 1.4, combined with those of Theorem 2.9 in Pardoux and Peng [7], we show that

$$\begin{aligned}E\Bigg(\sup_{0\leq s\leq T}|\Delta_h^i Y_s^{t,x}-\Delta_{h'}^i Y_s^{t',x'}|^p+\Bigg(\int_{t\wedge t'}^T\|\Delta_h^i Z_s^{t,x}-\Delta_{h'}^i Z_s^{t',x'}\|^2\,ds\Bigg)^{p/2}\Bigg)\\\leq c_p(1+|x|^q+|x'|^q+|h|^q+|h'|^q)\times(|x-x'|^p+|h-h'|^p+|t-t'|^{p/2}).\end{aligned}$$

The existence of a continuous derivative of $Y_s^{t,x}$ with respect to x follows easily from the above estimate, as well as the existence of a mean-square derivative of $Z_s^{t,x}$ with respect to x, which is mean square continuous in (s,t,x). The existence of a continuous second derivative of $Y_s^{t,x}$ with respect to x is proved in a similar fashion. □

It is easy to deduce, as in Pardoux and Peng [7], that $\left\{\left(\nabla Y_s^{t,x}=\frac{\partial Y_s^{t,x}}{\partial x}, \nabla Z_s^{t,x}=\frac{\partial Z_s^{t,x}}{\partial x}\right)\right\}$ is the unique solution of the BDSDE:

$$\begin{aligned}\nabla Y_s^{t,x}=h'(X_T^{t,x})\nabla X_T^{t,x}+\int_s^T[&f_x'(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\nabla X_r^{t,x}\\&+f_y'(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\nabla Y_r^{t,x}+f_z'(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\nabla Z_r^{t,x}]\,dr\\&+\int_s^T[g_x'(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\nabla X_r^{t,x}+g_y'(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\nabla Y_r^{t,x}\\&+g_z'(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\nabla Z_r^{t,x}]\,dB_r-\int_s^T\nabla Z_r^{t,x}\,dW_r.\end{aligned}$$

We shall need below a formula relating Z with the gradients of Y and X:

Proposition 2.3 *The random field* $\{Z_s^{t,x};0\leq t\leq s\leq T, x\in\mathbb{R}^d\}$ *has an* a.s. *continuous version which is given by:*

$$Z_s^{t,x}=\nabla Y_s^{t,x}(\nabla X_s^{t,x})^{-1}\,\sigma(X_s^{t,x})$$

and in particular

$$Z_t^{t,x}=\nabla Y_t^{t,x}\,\sigma(x).$$

Proof. We only indicate the main ideas, the details being obvious adaptations of those leading to Lemma 2.5 in Pardoux and Peng [7].

For any random variable F of the form $F=f(W(h_1),\dots,W(h_n);B(k_1),\dots,B(k_p))$ with $f\in C_b^\infty(\mathbb{R}^{n+p})$, $h_1,\dots,h_n\in L^2([0,T],\mathbb{R}^d)$, $k_1,\dots,k_p\in L^2([0,T],\mathbb{R}^l)$, where

$$W(h_i)\triangleq\int_0^T (h_i(t),dW_t),\qquad B(k_j)\triangleq\int_0^T (k_j(t),dB_t),$$

we let

$$D_t F\triangleq\sum_{i=1}^n f_i'(W(h_1),\dots,W(h_n);B(k_1),\dots,B(k_p))\,h_i(t),\qquad 0\leqq t\leqq T.$$

For such an F, we define its 1, 2-norm as:

$$\|F\|_{1,2}=\left(E\left[F^2+\int_0^T |D_t F|^2\,dt\right]\right)^{1/2}.$$

$\mathbb{S}$ denoting the set of random variables of the above form, we define the Sobolev space:

$$\mathbb{D}^{1,2}\triangleq\bar{\mathbb{S}}^{\|\cdot\|_{1,2}}.$$

The "derivation operator" $D_\cdot$ extends as an operator from $\mathbb{D}^{1,2}$ into $L^2(\Omega;L^2([0,T],\mathbb{R}^d))$. It turns out that under the assumptions of Theorem 2.1, the components of $X_s^{t,x}$, $Y_s^{t,x}$ and $Z_s^{t,x}$ take values in $\mathbb{D}^{1,2}$, and the pair $\{(D_\theta Y_s^{t,x}, D_\theta Z_s^{t,x};\ t\leqq\theta\leqq s\leqq T\}$ satisfies for each fixed θ the same equation as $\{(\nabla Y_s^{t,x},\nabla Z_s^{t,x})\}$, but where $\nabla X_s^{t,x}$ has been replaced by $D_\theta X_s^{t,x}$. Now since for $t\leqq\theta<s$,

$$D_\theta X_s^{t,x}=\nabla X_s^{t,x}(\nabla X_\theta^{t,x})^{-1}\,\sigma(X_\theta^{t,x})$$

and moreover the mapping

$$D_\theta X_\cdot^{t,x}\to(D_\theta Y_\cdot^{t,x}, D_\theta Z_\cdot^{t,x})$$

is the same linear mapping as

$$\nabla X_\cdot^{t,x}\to(\nabla Y_\cdot^{t,x},\nabla Z_\cdot^{t,x}),$$

it follows that

$$D_\theta Y_s^{t,x}=\nabla Y_s^{t,x}(\nabla X_\theta^{t,x})^{-1}\,\sigma(X_\theta^{t,x}).$$

Now $D_\theta Y_s^{t,x}=0$ for $\theta>s$, and

$$\begin{aligned}D_\theta Y_\theta^{t,x}&\triangleq\lim_{s\downarrow\downarrow\theta} D_\theta Y_s^{t,x}\\&=Z_\theta^{t,x},\ \theta\text{ a.e.}\end{aligned}$$

This gives the first part of the proposition. The second part follows. $\square$

3 BDSDEs and systems of quasilinear SPDEs

We now relate our BDSDE to the following system of quasilinear backward stochastic partial differential equations:

$$(3.1)\qquad u(t,x)=h(x)+\int_t^T [\mathscr{L}u(s,x)+f(s,x,u(s,x),(\nabla u\sigma)(s,x))]\,ds$$
$$+\int_t^T g(s,x,u(s,x),(\nabla u\sigma)(s,x))\,dB_s,\qquad 0\leqq t\leqq T;$$

where $u\colon \mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^k$,

$$\mathscr{L}u=\begin{pmatrix}Lu_1\\ \vdots\\ Lu_k\end{pmatrix},$$

with $L=\frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i(t,x)\frac{\partial}{\partial x_i}$.

Theorem 3.1 *Let f and g satisfy the assumptions of Sect.* 1 *and h be of class C^2. Let $\{u(t,x);\ 0\leqq t\leqq T,\ x\in\mathbb{R}^d\}$ be a random field such that $u(t,x)$ is $\mathscr{F}^B_{t,T}$-measurable for each (t,x), $u\in C^{0,2}([0,T]\times\mathbb{R}^d;\mathbb{R}^k)$* a.s., *and u satisfies Eq.* (3.1).

Then $u(t,x)=Y^{t,x}_t$, where $\{(Y^{t,x}_s,Z^{t,x}_s);\ t\leqq s\leqq T\}_{t\geqq 0,x\in\mathbb{R}^d}$ is the unique solution of the BDSDE (2.2).

Proof. It suffices to show that $\{(u(t,X^{t,x}_s),(\nabla u\sigma)(s,X^{t,x}_s);\ 0\leqq s\leqq t\}$ solves the BDSDE (2.2).

Let $t=t_0<t_1<t_2<\ldots<t_n=T$

$$\sum_{i=0}^{n-1}[u(t_i,X^{t,x}_{t_i})-u(t_{i+1},X^{t,x}_{t_{i+1}})]$$
$$=\sum_i[u(t_i,X^{t,x}_{t_i})-u(t_i,X^{t,x}_{t_{i+1}})]+\sum_i[u(t_i,X^{t,x}_{t_{i+1}})-u(t_{i+1},X^{t,x}_{t_{i+1}})]$$
$$=-\int_{t_i}^{t_{i+1}}\mathscr{L}u(t_i,X^{t,x}_s)\,ds-\int_{t_i}^{t_{i+1}}(\nabla u\sigma)(t_i,X^{t,x}_s)\,dW_s$$
$$+\int_{t_i}^{t_{i+1}}[\mathscr{L}(u(s,X^{t,x}_{t_{i+1}}))+f(s,X^{t,x}_{t_{i+1}},u(s,X^{t,x}_{t_{i+1}}),(\nabla u\sigma)(s,X^{t,x}_{t_{i+1}}))]\,ds$$
$$+\int_{t_i}^{t_{i+1}}g(s,X^{t,x}_{t_{i+1}},u(s,X^{t,x}_{t_{i+1}}),(\nabla u\sigma)(s,X^{t,x}_{t_{i+1}}))\,dB_s,$$

where we have used the Itô formula and the equation satisfied by u. It finally suffices to let the mesh size go to zero in order to conclude. □

We have also a converse to Theorem 3.1:

Theorem 3.2 *Let f, g and h satisfy the assumptions of Sects.* 1 *and* 2. *Then* $\{u(t,x) \triangleq Y_t^{t,x};\ 0 \leqq t \leqq T,\ x \in \mathbb{R}^d\}$ *is the unique classical solution of the system of backward* SPDEs (3.1).

Proof. We prove that $\{Y_t^{t,x}\}$ is a solution. Uniqueness will then follow from Theorem 3.2. We first note that $Y_{t+h}^{t,x} = Y_{t+h}^{t+h, X_{t+h}^{t,x}}$. Hence

$$\begin{aligned} u(t+h,x)-u(t,x) &= u(t+h,x)-u(t+h,X_{t+h}^{t,x})+u(t+h,X_{t+h}^{t,x})-u(t,x)\\ &= -\int_t^{t+h} \mathscr{L}u(t+h,X_s^{t,x})\,ds - \int_t^{t+h} (\nabla u\,\sigma)(t+h,X_s^{t,x})\,dW_s\\ &\quad -\int_t^{t+h} f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})\,ds - \int_t^{t+h} g(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})\,dB_s\\ &\quad +\int_t^{t+h} Z_s^{t,x}\,dW_s. \end{aligned}$$

We can then finish the proof exactly as in Theorem 3.2 of Pardoux and Peng [7]. □

Remark 3.3 Condition (H.1), with $\alpha<1$, is a very natural condition for (3.1) to be well posed. Indeed, in the case where g is linear with respect to its last argument, and does not depend on y, g is of the form:

$$g(s,x,z)=c(s,x)\,z$$

i.e. the stochastic integral term in (3.1) reads:

$$\int_t^T c(s,x)(\nabla u\,\sigma)(s,x)\,dB_s.$$

Condition (H.1) for g, in this case, reduces to $|c(s,x)| \leqq \alpha < 1$. This is a well known condition (see e.g. Pardoux [5]) for the SPDE (3.1) to be a well-posed stochastic parabolic equation. □

Remark 3.4 Our result generalizes the stochastic Feynman-Kac formula of Pardoux [4] for linear SPDEs. Indeed, if $k=1$, f and g are linear in y and do not depend on z, the BDSDE becomes

$$Y_s^{t,x}=h(X_T^{t,x})+\int_s^T a(r,X_r^{t,x})\,Y_r^{t,x}\,dr+\int_s^T b(r,X_r^{t,x})\,Y_r^{t,x}\,dB_r-\int_s^T Z_r^{t,x}\,dW_r$$

and it has an explicit solution given by:

$$\begin{aligned} Y_s^{t,x} &= \exp\left(\int_s^T a(r,X_r^{t,x})\,dr+\int_s^T b(r,X_r^{t,x})\,dB_r-\tfrac{1}{2}\int_s^T |b(r,X_r^{t,x})|^2\,dr\right) h(X_T^{t,x})\\ &\quad -\int_s^T \exp\left(\int_s^r a(\theta,X_\theta^{t,x})\,d\theta+\int_s^r b(\theta,X_\theta^{t,x})\,dB_\theta-\tfrac{1}{2}\int_s^r |b^2(\theta,X_\theta^{t,x})|^2\,d\theta\right) Z_r^{t,x}\,dW_r \end{aligned}$$

and because $Y_t^{t,x}$ is $\mathscr{F}_{t,T}^B$ measurable,

$$Y_t^{t,x}=E\left[h(X_T^{t,x})\exp\left(\int_t^T a(r,X_r^{t,x})\,dr+\int_t^T b(r,X_r^{t,x})\,dB_r\right.\right.$$
$$\left.\left.-\frac{1}{2}\int_t^T |b(r,X_r^{t,x})|^2\,dr\right)\Big/\mathscr{F}_{t,T}^B\right],$$

which is the formula in Pardoux [4] (where only the case $a\equiv 0$ is considered). Note however that in [4] B and W are allowed to be correlated. This does not seem possible here, unless we allow the stochastic integrals in the BDSDE to be of a non adapted nature. □

References

1. Krylov, N.V., Rozovskii, B.L.: Stochastic evolution equations. J. Sov. Math. **16**, 1233–1277 (1981)
2. Nualart, D., Pardoux, E.: Stochastic calculus with anticipating integrands. Probab. Theory Relat. Fields **78**, 535–581 (1988)
3. Ocone, D., Pardoux, E.: A stochastic Feynman-Kac formula for anticipating SPDEs, and application to nonlinear smoothing. Stochastics **45**, 79–126 (1993)
4. Pardoux, E.: Un résultat sur les équations aux dérivés partielles stochastiques et filtrage des processus de diffusion. Note C.R. Acad. Sci., Paris Sér. A **287**, 1065–1068 (1978)
5. Pardoux, E.: Stochastic PDEs, and filtering of diffusion processes. Stochastics **3**, 127–167 (1979)
6. Pardoux, E., Peng, S.: Adapted solution of a backward stochastic differential equation. Syst. Control Lett. **14**, 55–61 (1990)
7. Pardoux, E., Peng, S.: Backward stochastic differential equations and quasilinear parabolic partial differential equations. In: Rozuvskii, B.L., Sowers, R.B. (eds.) Stochastic partial differential equations and their applications. (Lect. Notes Control Inf. Sci., vol. 176, pp. 200–217) Berlin Heidelberg New York: Springer 1992
8. Peng, S.: Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. Stochastics **37**, 61–74 (1991)
9. Peng, S.: A generalized dynamic programming principle and Hamilton-Jacobi-Bellmann equation. Stochastics **38**, 119–134 (1992)
10. Peng, S.: A non linear Feynman-Kac formula and applications. In: Chen, S.P., Yong, J.M. (eds.) Proc. of Symposium on system science and control theory, pp. 173–184. Singapore: World Scientific 1992
11. Rozovskii, B.: Stochastic evolution systems. Dordrecht: Reidel 1991