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Homogenization of Linear and Semilinear Second Order Parabolic PDEs with Periodic Coefficients: A Probabilistic Approach

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We study the limit of the solution of linear and semilinear second order PDEs of parabolic type, with rapidly oscillating periodic coefficients, singular drift, and singular coefficient of the zeroth order term. Our method of proof is fully probabilistic and builds upon the arguments in earlier work. In the linear case, we use the Feynman Kac formula to represent the solution of the parabolic PDE, and in the semilinear case we use an associated backward stochastic differential equation. © 1999 Academic Press

1. INTRODUCTION

It is known from Chapter 3 of Bensoussan *et al.* [1] (see also the earlier work of Freidlin [5]) that the homogenization of linear second order partial differential operators can be proved by probabilistic methods, based on the ergodic theorem and the functional central limit theorem.

The object of this paper is to study homogenization of second order parabolic PDEs with periodic coefficients by probabilistic methods. Our basic tool is the approach given in Chapter 3 of [1]. In the case of linear parabolic PDEs, we both weaken slightly the assumptions on the coefficients, and we consider the case of a highly oscillating coefficient in the zeroth order term. In the semilinear case, we similarly consider a highly oscillating nonlinear term, and we use weak convergence of backward stochastic differential equations in order to prove the result. Our results seem to be completely new. There apparently does not exist to date analytic proofs of similar results, except in the linear case, where our result is a particular case of the results in Campillo *et al.* [4] (except for the fact that our operator is not taken in divergence form). For homogenization results of nonlinear PDEs proved by analytic techniques, we refer the

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reader to the work of Bensoussan *et al.* [2]. Note that other homogenization results for semilinear second order parabolic PDE's have been proved using probabilistic techniques; see Buckdahn *et al.* [3] and Gaudron and Pardoux [7]. Moreover our result in the linear case, but for a PDE operator in divergence form, has been recently established by Lejay [12], also using a probabilistic approach.

The paper is organized as follows. Section 2 is devoted to the proof of preliminary results, consisting mainly in a variant of the ergodic theorem for an ergodic diffusion on the d-dimensional torus, the proof of which relies essentially on an estimate of a spectral gap. Section 3 studies the homogenization of linear parabolic equations, and Section 4 homogenization of semilinear parabolic equations.

2. PRELIMINARY RESULTS

In the next sections, we study the asymptotic behavior, as $\varepsilon \to 0$, of the process $\{X_t^\varepsilon; t \geqslant 0\}$ solution of the SDE,

$$X_t^\varepsilon = x + \int_0^t c\left(\frac{X_s^\varepsilon}{\varepsilon}\right) ds + \frac{1}{\varepsilon}\int_0^t b\left(\frac{X_s^\varepsilon}{\varepsilon}\right) ds + \int_0^t \sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right) dB_s, \tag{2.1}$$

under a centering condition on the drift b, where $\{B_t; t \geqslant 0\}$ is a standard d-dimensional Brownian motion. We assume that $c, b\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable, bounded and periodic functions of x, of period 1 in each direction. We assume that $a(x) \triangleq \sigma\sigma^*(x)$ is continuous and satisfies, for some $\alpha > 0$,

$$a(x) \geqslant \alpha I, \qquad x \in \mathbb{R}^d; \tag{2.2}$$

$$\sum_{j=1}^d \frac{\partial a_{ij}}{\partial x_j}(x) \in L^\infty(\mathbb{R}^d), \qquad i = 1, ..., d. \tag{2.3}$$

It follows from a classical result of Stroock and Varadhan [20] that under the above assumptions the solution of (2.1) is unique in law. Let $\tilde{X}_t^\varepsilon \triangleq \varepsilon^{-1} X_{\varepsilon^2 t}^\varepsilon$. It is easily checked that for each $\varepsilon > 0$ there exists a standard Brownian motion $\{B_t^\varepsilon; t \geqslant 0\}$ such that

$$\tilde{X}_t^\varepsilon = \frac{x}{\varepsilon} + \varepsilon \int_0^t c(\tilde{X}_s^\varepsilon)\, ds + \int_0^t b(\tilde{X}_s^\varepsilon)\, ds + \int_0^t \sigma(\tilde{X}_s^\varepsilon)\, dB_s^\varepsilon.$$

The aim of this section is to study the ergodic properties of the process $\{\tilde{X}_t^\varepsilon\}$ for $\varepsilon \geqslant 0$; considered as a process taking values in the d-dimensional torus $\mathbf{T}^d$. Note that from (2.3) the infinitesimal generator of $\tilde{X}^\varepsilon$ can be written in divergence form, as well as its adjoint. Clearly, the process $\tilde{X}^\varepsilon$ possesses an invariant measure μ_ε, which is unique thanks to condition (2.2). We have moreover the

PROPOSITION 2.1. *For each $\varepsilon \geqslant 0$, the invariant measure μ_ε is absolutely continuous, its density p_ε satisfies $p_\varepsilon \in H^1(\mathbf{T}^d)$ and for some $c > 0$*

$$\frac{1}{c} \leqslant p_\varepsilon(x) \leqslant c, \qquad x \in \mathbf{T}^d, \quad \varepsilon \geqslant 0.$$

Moreover $p_\varepsilon \to p_0$ in $L^2(\mathbf{T}^d)$, as $\varepsilon \to 0$.

Proof. We drop the index ε for notational simplicity. Let $\{a^n, b^n, c^n\}$ be a sequence of smooth coefficients which converge a.e. to $\{a, b, c\}$, are uniformly bounded, together with $\sum_{j=1}^d (\partial a^n_{ij}/\partial x_j)$, $i = 1, ..., d$, and such that a^n satisfies (2.2) with the same constant $\alpha > 0$ for each $n \geqslant 1$. It follows from the results of the Malliavin calculus that the invariant measure μ_n corresponding to (a^n, b^n, c^n) has a smooth density p_n, which solves the elliptic equation

$$\frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a^n_{ij} \frac{\partial p_n}{\partial x_j} \right)(x) + \sum_{i=1}^d \frac{\partial}{\partial x_i} \left[\left(\frac{1}{2} \sum_{j=1}^d \frac{\partial a^n_{ij}}{\partial x_j}(x) - b^n_i(x) - \varepsilon c^n_i(x) \right) p_n \right](x) = 0.$$

Taking the scalar product of the above in $L^2(\mathbf{T}^d)$ with p_n, integrating by parts and exploiting the assumptions on the coefficients (in particular (2.2)), we deduce that

$$\sup_n \|p_n\|^{-1}_{L^2(\mathbf{T}^d)} \|\nabla p_n\|_{L^2(\mathbf{T}^d;\, \mathbb{R}^d)} < \infty. \tag{2.4}$$

On the other hand, from the Nash inequality (see, e.g., Stroock [19, Lemma I.1.1], whose second proof is easily adapted to $\mathbf{T}^d$), there exists a constant c_1 such that

$$\|p_n\|^{2+4/d}_{L^2(\mathbf{T}^d)} \leqslant c_1 \|p_n\|^{4/d}_{L^1(\mathbf{T}^d)} \|\nabla p_n\|^2_{L^2(\mathbf{T}^d;\, \mathbb{R}^d)}. \tag{2.5}$$

It follows from (2.4), (2.5), and the identity $\|p_n\|_{L^1(\mathbf{T}^d)} = 1$ that

$$\sup_n \|p_n\|_{H^1(\mathbf{T}^d)} < \infty. \tag{2.6}$$

It is not hard to show that $\mu_n \Rightarrow \mu$, as $n \to \infty$. It then follows from (2.6) that $p \in H^1(\mathbf{T}^d)$; now the upper and lower bounds on p_ε follow from the Harnack inequality; see, e.g., Gilbarg, and Trudinger [8, p. 189]. Finally, as $\varepsilon \to 0$, p_ε converges in $H^1(\mathbf{T}^d)$ weakly, hence in $L^2(\mathbf{T}^d)$ by compactness. The limit of p_ε is the density of μ_0, the weak limit of μ_ε. ∎

It is not hard to show that $p_\varepsilon \to p_0$ in $H^1(\mathbf{T}^d)$ strongly, but we shall not need that result.

We now prove a uniform spectral gap property of the semigroup associated to the process $\{\tilde{X}^\varepsilon_t, t \geqslant 0\}$.

PROPOSITION 2.2. *There exists $p > 0$ such that for all $\varepsilon \geqslant 0$, $f \in L^2(\mathbf{T}^d; \mu_\varepsilon)$ such that $\int_{\mathbf{T}^d} f(x)\, p_\varepsilon(x)\, dx = 0$,*

$$\|\mathbb{E}.\, f(\tilde{X}^\varepsilon_t)\|_{L^2(\mathbf{T}^d;\, \mu_\varepsilon)} \leqslant \|f\|_{L^2(\mathbf{T}^d;\, \mu_\varepsilon)}\, e^{-\rho t}.$$

Proof. We denote by L_ε the infinitesimal generator of the process $\tilde{X}^\varepsilon$,

$$L_\varepsilon = \frac{1}{2} \sum_{i,\, j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{d} (b_i(x) + \varepsilon c_i(x)) \frac{\partial}{\partial x_i},$$

and $\{P^\varepsilon_t; t \geqslant 0\}$ the associated semigroup,

$$(P^\varepsilon_t f)(x) = \mathbb{E}_x f(\tilde{X}^\varepsilon_t), \qquad t \geqslant 0, \quad x \in \mathbf{T}^d,$$

$f\colon \mathbf{T}^d \to \mathbb{R}$. It follows from standard calculations that for any $u \in H^1(\mathbf{T}^d)$,

$$\begin{aligned}
&-\int_{\mathbf{T}^d} (L_\varepsilon u)(x)\, u(x)\, p_\varepsilon(x)\, dx \\
&\quad = -\int_{\mathbf{T}^d} u(x)\, L^*_\varepsilon(u p_\varepsilon)(x)\, dx \\
&\quad = \frac{1}{2} \sum_{i,\, j=1}^{d} \int_{\mathbf{T}^d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x)\, p_\varepsilon(x)\, dx.
\end{aligned}$$

Hence for any $f \in L^2(\mathbf{T}^d, \mu_\varepsilon)$,

$$\begin{aligned}
&\frac{d}{dt} \|P^\varepsilon_t f\|^2_{L^2(\mathbf{T}^d,\, \mu_\varepsilon)} \\
&\quad = -\sum_{i,\, j=1}^{d} \int_{\mathbf{T}^d} a_{ij}(x) \frac{\partial (P^\varepsilon_t f)}{\partial x_i}(x) \frac{\partial (P^\varepsilon_t f)}{\partial x_j}(x)\, p_\varepsilon(x)\, dx. \qquad (2.7)
\end{aligned}$$

It now follows from Proposition 2.1, (2.2), and the Poincaré inequality that for some constant $c>0$ whose value may vary from line to line,

$$\begin{aligned}\int_{\mathbf{T}^d}\left(f(x)-\int f(y)\,p_\varepsilon(y)\,dy\right)^2 p_\varepsilon(x)\,dx\\ &=\inf_{\beta\in\mathbb{R}}\int_{\mathbf{T}^d}(f(x)-\beta)^2\,p_\varepsilon(x)\,dx\\ &\leqslant c\inf_{\beta\in\mathbb{R}}\int_{\mathbf{T}^d}(f(x)-\beta)^2\,dx\\ &=c\int_{\mathbf{T}^d}\left(f(x)-\int f(y)\,dy\right)^2 dx\\ &\leqslant c\int_{\mathbf{T}^d}|\nabla f(x)|^2\,dx\\ &\leqslant c\int_{\mathbf{T}^d}(a\nabla f,\nabla f)(x)\,p_\varepsilon(x)\,dx. \end{aligned}\tag{2.8}$$

It follows from (2.7) and (2.8) that there exists $\rho>0$ such that for any $f\in L^2(\mathbf{T}^d,\mu_\varepsilon)$ satisfying $\int_{\mathbf{T}^d} f(x)\,p_\varepsilon(x)\,dx=0$ (hence also $\int_{\mathbf{T}^d}(P_t^\varepsilon f)(x)\,p_\varepsilon(x)\,dx=0$),

$$\frac{d}{dt}\,\|P_t^\varepsilon f\|^2_{L^2(\mathbf{T}^d,\mu_\varepsilon)}\leqslant -2\rho\,\|P_t^\varepsilon f\|^2_{L^2(\mathbf{T}^d,\mu_\varepsilon)},$$

and the result follows. ∎

In the next statement, the constant ρ is that from Proposition 2.2, and the process $\tilde{X}^\varepsilon$, starts from an arbitrary initial condition.

COROLLARY 2.3. *There exists $c>0$ such that for all $f\in L^\infty(\mathbf{T}^d)$ satisfying $\int f(x)\,p_\varepsilon(x)\,dx=0$, $0\leqslant s<t$,*

$$|\mathbb{E}(f(\tilde{X}_t^\varepsilon)/\tilde{X}_s^\varepsilon)|\leqslant c\,\|f\|_{L^\infty(\mathbf{T}^d)}\,e^{-\rho(t-s)}.$$

Proof. It suffices to prove the result for $s=0$, with $\tilde{X}_0^\varepsilon$ an arbitrary random variable. First the result holds clearly for $0<t\leqslant 1$, provided $c\geqslant e^\rho$. Let now $t>1$. We have, with the notation of the preceding proof,

$$\begin{aligned}\|P_t^\varepsilon f\|_\infty &\leqslant \|P_1^\varepsilon\|_{2\to\infty}\times\|P_{t-1}^\varepsilon f\|_2\\ &\leqslant \|P_1^\varepsilon\|_{2\to\infty}\times\|P_{t-1}^\varepsilon\|_{2\to 2}\times\|f\|_\infty,\end{aligned}$$

where $\|\cdot\|_{p\to q}$ denotes the norm of the corresponding operator, considered as an operator from $L^p(\mathbf{T}^d;\mu_\varepsilon)$ into $L^q(\mathbf{T}^d;\mu_\varepsilon)$. But from Stroock [19, p. 320], Nash's inequality implies that there exists a universal constant C, which depends only on the dimension d and the ellipticity constant of a, such that

$$\|P_1^\varepsilon\|_{2\to\infty}\leqslant C.$$

Moreover, Proposition 2.2 implies that

$$\|P_{t-1}^\varepsilon\|_{2\to 2}\leqslant e^{-\rho(t-1)}. \quad \blacksquare$$

We conclude this section by an ergodic theorem. Again, $\{X_t^\varepsilon, t\geqslant 0\}$ denotes the solution (unique in law) of the SDE (2.1).

Proposition 2.4. *Let $f\in L^\infty(\mathbf{T}^d)$. Then for any $t>0$,*

$$\int_0^t f\left(\frac{X_s^\varepsilon}{\varepsilon}\right) ds \to t\int_{\mathbf{T}^d} f(x)\, p(x)\, dx$$

in probability, as $\varepsilon\to 0$.

Proof. Since $p_\varepsilon\to p$ in $L^2(\mathbf{T}^d)$, it suffices to show that

$$\int_0^t \bar{f}_\varepsilon\left(\frac{X_s^\varepsilon}{\varepsilon}\right) ds \to 0$$

in probability, as $\varepsilon\to 0$, where

$$\bar{f}_\varepsilon(x)=f(x)-\int_{\mathbf{T}^d} f(x)\, p_\varepsilon(x)\, dx.$$

Now $\varepsilon^{-1}X_s^\varepsilon=\tilde{X}^\varepsilon_{s/\varepsilon^2}$. Consequently

$$\int_0^t \bar{f}_\varepsilon\left(\frac{X_s^\varepsilon}{\varepsilon}\right) ds=\varepsilon^2\int_0^{t/\varepsilon^2}\bar{f}_\varepsilon(\tilde{X}_u^\varepsilon)\, du.$$

From the Markov property of $\tilde{X}^\varepsilon$ and Corollary 2.3, we deduce that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t \bar{f}_\varepsilon(\tilde{X}_u^\varepsilon)\, du\right)^2\right]&=2\mathbb{E}\int_0^t\int_0^s \bar{f}_\varepsilon(\tilde{X}_s^\varepsilon)\,\bar{f}_\varepsilon(\tilde{X}_u^\varepsilon)\, ds\, du\\ &\leqslant 2\,\|\bar{f}_\varepsilon\|^2_{L^\infty(\mathbf{T}^d)}\int_0^t\int_0^s e^{-\rho(s-u)}\, ds\, du\\ &=2\,\|\bar{f}_\varepsilon\|^2_{L^\infty(\mathbf{T}^d)}\,\rho^{-2}(1+\rho t-e^{-\rho t}).\end{aligned}$$

But

$$\|\bar{f}_\varepsilon\|_{L^\infty(\mathbf{T}^d)} \leqslant 2\, \|f\|_{L^\infty(\mathbf{T}^d)}.$$

Consequently

$$E\left[\left(\varepsilon^2 \int_0^{t/\varepsilon^2} \bar{f}_\varepsilon(\tilde{X}_u^\varepsilon)\, du\right)^2\right] \leqslant 2c\, \|f\|^2_{L^\infty(\mathbf{T}^d)}\, \rho^{-2}(\varepsilon^4 + \rho\varepsilon^2 t - \varepsilon^4 e^{-\rho t/\varepsilon^2})$$
$$\to 0,$$

as $\varepsilon \to 0$. ▮

3. LINEAR PARABOLIC EQUATION

Let $\varepsilon > 0$, and $\{u^\varepsilon(t, x);\, t \geqslant 0,\, x \in \mathbb{R}^d\}$ denote the solution of the linear second order parabolic PDE,

$$\begin{aligned}\frac{\partial u^\varepsilon}{\partial t}(t, x) = {} & \frac{1}{2} \sum_{i,\, j=1}^d a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial^2 u^\varepsilon}{\partial x_i\, \partial x_j}(t, x) + \sum_{i=1}^d \left(\frac{1}{\varepsilon} b_i\left(\frac{x}{\varepsilon}\right) + c_i\left(\frac{x}{\varepsilon}\right)\right) \\ & \times \frac{\partial u^\varepsilon}{\partial x_i}(t, x) + \left(\frac{1}{\varepsilon} e\left(\frac{x}{\varepsilon}\right) + f\left(\frac{x}{\varepsilon}\right)\right) u^\varepsilon(t, x) \qquad (3.1)\\ u^\varepsilon(0, x) = {} & g(x).\end{aligned}$$

We assume that $g \in C(\mathbb{R}^d)$ with at most polynomial growth at infinity, and all coefficients are measurable, bounded and periodic of period one in each direction. We assume that the matrix $a = \sigma\sigma^*$ is continuous and satisfies (2.2) and (2.3).

It remains to state a centering condition which must be satisfied by the singular coefficients b and e.

Let $\{\tilde{X}_t;\, t \geqslant 0\}$ denote the $\mathbf{T}^d$-valued diffusion process with generator

$$L = \frac{1}{2} \sum_{i,\, j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i},$$

which has been studied in Section 2. We denote by $\mu(dx) = p(x)\, dx$ its invariant probability measure. We assume that

$$\begin{aligned}&\int_{\mathbf{T}^d} b_i(x)\, \mu(dx) = 0, \qquad i = 1, ..., d,\\ &\int_{\mathbf{T}^d} e(x)\, \mu(dx) = 0. \end{aligned} \qquad (3.2)$$

In order to state our result, we need to define the solution of the following Poisson equations. Let $\hat{b}_i$, $i=1, ..., d$, and $\hat{e}$ denote the periodic functions of x which belong to $W^{2,p}(\mathbf{T}^d)$, have zero integral with respect to μ over $\mathbf{T}^d$, and satisfy the Poisson equations

$$
\begin{aligned}
L\hat{b}_i(x) + b_i(x) &= 0, \qquad x \in \mathbf{T}^d, \quad i=1, ..., d; \\
L\hat{e}(x) + e(x) &= 0, \qquad x \in \mathbf{T}^d.
\end{aligned}
$$

Equivalently,

$$
\hat{b}_i(x) = \int_0^\infty \mathbb{E}_x b_i(\tilde{X}_t)\, dt,
$$

$$
\hat{e}(x) = \int_0^\infty \mathbb{E}_x e(\tilde{X}_t)\, dt,
$$

see Pardoux and Veretennikov [18], where similar equations are treated on $\mathbb{R}^d$. The adaptation of those results to the compact manifold $\mathbf{T}^d$ is obvious.

For fixed $\varepsilon > 0$, $x \in \mathbb{R}^d$, let $\{X_t^\varepsilon; t \geqslant 0\}$ be the solution of the SDE (2.1), i.e.,

$$
X_t^\varepsilon = x + \int_0^t \left(\frac{1}{\varepsilon} b\left(\frac{X_s^\varepsilon}{\varepsilon}\right) + c\left(\frac{X_s^\varepsilon}{\varepsilon}\right)\right) ds + \int_0^t \sigma\left(\frac{X_s^\varepsilon}{\varepsilon}\right) dB_s, \qquad t \geqslant 0.
$$

We let finally $\{Y_t^\varepsilon; t \geqslant 0\}$ denote the scalar-valued process given by

$$
Y_t^\varepsilon \triangleq \int_0^t \left(\frac{1}{\varepsilon} e\left(\frac{X_s^\varepsilon}{\varepsilon}\right) + f\left(\frac{X_s^\varepsilon}{\varepsilon}\right)\right) ds.
$$

For us the solution of the linear second order parabolic PDE (3.1) is the quantity defined by the Feynman–Kac formula

$$
u^\varepsilon(t, x) = \mathbb{E}[g(X_t^\varepsilon) \exp(Y_t^\varepsilon)]. \tag{3.3}
$$

The limiting PDE is the following equation with constant coefficients,

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \sum_{i,j=1}^d A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^d (C_i + V_i) \frac{\partial u}{\partial x_i}(t, x) \\
+ \left(\frac{R}{2} + D\right) u(t, x) \qquad (3.4)\\
u(0, x) = g(x),
\end{aligned}
$$

where

$$A = \int_{\mathbf{T}^d} (I + \nabla \hat{b})\, a (I + \nabla \hat{b})^* \,(x)\, \mu(dx),$$

$$C = \int_{\mathbf{T}^d} (I + \nabla \hat{b})\, c(x)\, \mu(dx),$$

$$V = \int_{\mathbf{T}^d} (I + \nabla \hat{b})\, a\, \nabla \hat{e}(x)\, \mu(dx),$$

$$R = \int_{\mathbf{T}^d} \nabla \hat{e}^*\, a\, \nabla \hat{e}(x)\, \mu(dx),$$

$$D = \int_{\mathbf{T}^d} (f(x) + \langle \nabla \hat{e}, c \rangle (x))\, \mu(dx),$$

whose solution is the quantity

$$u(t, x) = \mathbb{E}[\, g(x + (C + V)\, t + A^{1/2} B_t)]\, e^{((R/2) + D)\, t}.$$

The aim of this section is to prove the

THEOREM 3.1. *For each $t \geqslant 0$, $x \in \mathbb{R}^d$,*

$$u_\varepsilon(t, x) \to u(t, x), \qquad \textit{as} \quad \varepsilon \to 0.$$

Proof. It follows from the Itô–Krylov formula, see Krylov [10], that with the notation $\bar{X}^\varepsilon_s \triangleq X^\varepsilon_s / \varepsilon$,

$$\hat{X}^\varepsilon_t \triangleq X^\varepsilon_t + \varepsilon \left(\hat{b}(\bar{X}^\varepsilon_t) - \hat{b}\left(\frac{x}{\varepsilon} \right) \right)$$

$$= x + \int_0^t (I + \nabla \hat{b})\, c(\bar{X}^\varepsilon_s)\, ds + \int_0^t (I + \nabla \hat{b})\, \sigma(\bar{X}^\varepsilon_s)\, dB_s,$$

$$\hat{Y}^\varepsilon_t \triangleq Y^\varepsilon_t + \varepsilon \left(\hat{e}(\bar{X}^\varepsilon_t) - \hat{e}\left(\frac{x}{\varepsilon} \right) \right)$$

$$= \int_0^t (f + \nabla \hat{e} c)(\bar{X}^\varepsilon_s)\, ds + \int_0^t (\nabla \hat{e} \sigma)(\bar{X}^\varepsilon_s)\, dB_s.$$

Since $\hat{b}$ and $\hat{e}$ are bounded on $\mathbb{R}^d$ (they are periodic and continuous), $u^\varepsilon(t, x)$ has the same asymptotic behavior as

$$\hat{u}^\varepsilon(t, x) = \mathbb{E}[\, g(\hat{X}^\varepsilon_t) \exp(\, \hat{Y}^\varepsilon_t)].$$

We now rewrite $\hat{u}^\varepsilon$ using Girsanov's theorem. Let $\widetilde{\mathbb{P}}$ denote a new probability measure such that

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathscr{F}_t} = \exp\left(\int_0^t \nabla\hat{e}\sigma(\bar{X}^\varepsilon_s)\, dB_s - \frac{1}{2}\int_0^t \nabla\hat{e}a\, \nabla\hat{e}^*(\bar{X}^\varepsilon_s)\, ds\right).$$

Then

$$\hat{u}^\varepsilon(t, x) = \widetilde{\mathbb{E}}\left[\, g(\hat{X}^\varepsilon_t) \exp\int_0^t (f + \nabla\hat{e}c + \tfrac{1}{2}\nabla\hat{e}a\, \nabla\hat{e}^*)(\bar{X}^\varepsilon_s)\, ds\right],$$

and

$$\hat{X}^\varepsilon_t = x + \int_0^t (I + \nabla\hat{b})(c + a\, \nabla\hat{e})(\bar{X}^\varepsilon_s)\, ds + \int_0^t (I + \nabla\hat{b})\, \sigma(\bar{X}^\varepsilon_s)\, d\widetilde{B}_s,$$

where $(\widetilde{B}_t;\, t \geqslant 0)$ is a $\widetilde{\mathbb{P}}$-Brownian motion.

LEMMA 3.1. *The following convergences hold in $\widetilde{\mathbb{P}}$ probability when $\varepsilon \to 0$:*

(i) $$\sup_{0 \leqslant s \leqslant t}\left|\int_0^s (I + \nabla\hat{b})(c + a\, \nabla\hat{e}^*)(\bar{X}^\varepsilon_r)\, dr - (C + V)\, s\right| \to 0$$

(ii) $$\int_0^t (I + \nabla\hat{b})\, a(I + \nabla\hat{b})^*\, (\bar{X}^\varepsilon_s)\, ds \to At$$

(iii) $$\int_0^t \left(f + \nabla\hat{e}^* c + \frac{1}{2}\nabla\hat{e}^* a\, \nabla\hat{e}\right)(\bar{X}^\varepsilon_s)\, ds \to \left(\frac{R}{2} + D\right) t.$$

Proof. Parts (ii) and (iii) follow clearly from Proposition 2.4, as well as the fact that

$$C^\varepsilon_t \triangleq \int_0^t (I + \nabla\hat{b})(c + a\, \nabla\hat{e})(\bar{X}^\varepsilon_s)\, ds \to (C + V)\, t \tag{3.5}$$

in probability as $\varepsilon \to 0$, for any $t > 0$. Now let for $p \in \mathbb{N}$, $C_t^{\varepsilon, p} = C^\varepsilon_{[t/p]}$,

$$\sup_{s \leqslant t} |C_s^\varepsilon - (C+V)\, s| \leqslant \frac{2}{p} \|\tilde{c}\|_\infty + \sup_{s \leqslant t} \left| C_s^{\varepsilon, p} - (C+V) \left[\frac{s}{p}\right] p \right|,$$

where $\tilde{c} \triangleq (I + \nabla \hat{b})(c + a\, \nabla \hat{e}^*)$. The second term of the above right-hand side tends to 0 in probability as $\varepsilon \to 0$, from (3.5) and the fact that the sup is over a finite set. ∎

The fact that $\hat{X}^\varepsilon$ converges weakly—under $\tilde{\mathbb{P}}$—to the Gaussian process

$$\{x + (C+V)\, t + A^{1/2} B_t,\, t \geqslant 0\}$$

follows, e.g., from Theorem VIII-2-17 in Jacod and Shyriaev [9]. Then our theorem follows clearly from (iii), the boundedness of the coefficients, and the fact that for any $p > 0$,

$$\sup_\varepsilon \mathbb{E}(|\hat{X}_t^\varepsilon|^p) < \infty,$$

hence the random variables $g(\hat{X}_t^\varepsilon)$ are uniformly integrable. ∎

Remark. It is not hard to show that if $g \in L^p(\mathbb{R}^d)$, u_ε (resp. u) coincides with the unique solution in $W_p^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ of Eq. (3.1) (resp. (3.4)). Hence our result translates truly into a statement for PDEs.

4. SEMILINEAR PARABOLIC EQUATIONS

We now consider the semilinear parabolic equation

$$\begin{aligned}\frac{\partial u^\varepsilon}{\partial t}(t, x) = {} & \frac{1}{2} \sum_{i,j=1}^d a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial^2 u^\varepsilon}{\partial x_i\, \partial x_j}(t, x) + \sum_{i=1}^d \left(\frac{1}{\varepsilon} b_i\left(\frac{x}{\varepsilon}\right) + c_i\left(\frac{x}{\varepsilon}\right)\right) \\ & \times \frac{\partial u^\varepsilon}{\partial x_i}(t, x) + \frac{1}{\varepsilon} e\left(\frac{x}{\varepsilon}, u^\varepsilon(t, x)\right) + f\left(\frac{x}{\varepsilon}, u^\varepsilon(t, x)\right), \qquad (4.1) \\ u^\varepsilon(0, x) = {} & g(x).\end{aligned}$$

The assumptions on a, b, c and g are exactly those in the previous section, including the centering condition

$$\int_{\mathbf{T}^d} b_i(x)\, \mu(dx) = 0, \qquad i = 1, ..., d.$$

We assume that e and f are measurable mappings from $\mathbb{R}^d \times \mathbb{R}$ into $\mathbb{R}$, which are periodic, of period one in each direction, in the first argument, continuous in y uniformly with respect to x and that for all $y \in \mathbb{R}$,

$$\int_{\mathbf{T}^d} e(x, y)\, \mu(dx) = 0, \tag{4.2}$$

and e is twice continuously differentiable in y, uniformly with respect to x.

Moreover, for some $\mu \in \mathbb{R}$, all $x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$,

$$(f(x, y) - f(x, y'))(y - y') \leqslant \mu\, |y - y'|^2. \tag{4.3}$$

We finally assume that $e(x, y) = e_0(x, y) + e_1(x)\, y$, and that there exists a constant K s.t.

$$|e_1(x)| + |e_0(x, y)| + \left|\frac{\partial e_0}{\partial y}(x, y)\right| + \left|\frac{\partial^2 e_0}{\partial y^2}(x, y)\right| \leqslant K, \qquad \forall x \in \mathbf{T}^d, \quad y \in \mathbb{R}, \tag{4.4}$$

and

$$|f(x, y)| \leqslant K(1 + y^2). \tag{4.5}$$

It follows from (4.2) that for each $y \in \mathbb{R}$, we can solve the Poisson equation

$$L\hat{e}(x, y) + e(x, y) = 0, \qquad x \in \mathbf{T}^d, \quad y \in \mathbb{R},$$

whose solution is given by

$$\hat{e}(x, y) = \int_0^\infty \mathbb{E}_x e(\tilde{X}_t, y)\, dt$$

and satisfies $\hat{e} \in C^{0,2}(\mathbf{T}^d \times \mathbb{R})$, $\hat{e}(\cdot, y)$, $(\partial \hat{e}_0/\partial y)(\cdot, y)$, $(\partial \hat{e}_0/\partial y^2)(\cdot, y) \in W^{2,p}(\mathbf{T}^d)$, for any $p \geqslant 1$, $y \in \mathbb{R}$, and for some $K' > 0$,

$$\|\hat{e}_1\|_{W^{2,p}(\mathbf{T}^d)} + \|\hat{e}_0(\cdot, y)\|_{W^{2,p}(\mathbf{T}^d)} + \left\|\frac{\partial \hat{e}_0}{\partial y}(\cdot, y)\right\|_{W^{2,p}(\mathbf{T}^d)} + \left\|\frac{\partial^2 \hat{e}_0}{\partial y^2}(\cdot, y)\right\|_{W^{2,p}(\mathbf{T}^d)} \leqslant K';$$

see Pardoux and Veretennikov [18].

We can now formulate the limiting equation, which is a semilinear parabolic PDE with constant coefficients,

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\sum_{i,j=1}^{d} A_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}(t,x)\\ &\quad+\sum_{i=1}^{d} C_i(u(t,x))\frac{\partial u}{\partial x_i}(t,x)+D(u(t,x)), \qquad (4.6)\\ u(0,x) &= g(x),\end{aligned}$$

where

$$A=\int_{\mathbf{T}^d}(I+\nabla\hat{b})\,a(I+\nabla\hat{b})^*(x)\,\mu(dx),$$

$$C(y)=\int_{\mathbf{T}^d}\left[(I+\nabla\hat{b})\left(c+a\frac{\partial^2\hat{e}}{\partial x\,\partial y}(\cdot,y)\right)\right](x)\,\mu(dx),$$

$$\begin{aligned}D(y)=\int_{\mathbf{T}^d}\bigg[&\left\langle\frac{\partial\hat{e}}{\partial x}(\cdot,y),c\right\rangle-\frac{\partial\hat{e}}{\partial y}(\cdot,y)\,e(\cdot,y)\\ &+\frac{\partial^2\hat{e}^*}{\partial x\,\partial y}(\cdot,y)\,a\frac{\partial\hat{e}}{\partial x}(\cdot,y)+f(\cdot,y)\bigg](x)\,\mu(dx).\end{aligned}$$

Remark 4.1. In the case where $e(x,y)=e(x)\,y$ and $f(x,y)=f(x)\,y$, the PDE (4.6) reduces to (3.5). Indeed, in this case

$$\begin{aligned}-\int_{\mathbf{T}^d}\frac{\partial\hat{e}}{\partial y}(x,y)\,e(x,y)\,\mu(dx) &= -y\int_{\mathbf{T}^d}\hat{e}(x)\,e(x)\,\mu(dx)\\ &= y\int_{\mathbf{T}^d}L\hat{e}(x)\,\hat{e}(x)\,p(x)\,dx\\ &= -\frac{y}{2}\int(a\nabla\hat{e},\nabla\hat{e})(x)\,p(x)\,dx,\end{aligned}$$

as noted in the proof of Proposition 2.2. The identification is now clear.

The aim of this section is to prove:

THEOREM 4.1. *For all* $t\geqslant 0$, $x\in\mathbb{R}^d$,

$$u_\varepsilon(t,x)\to u(t,x),\qquad \textit{as}\quad \varepsilon\to 0,$$

where u_ε *denotes the solution of Eq.* (4.1), *and* u *the solution of* (4.6).

As in the linear case considered in Section 3, we will in fact prove the convergence of probabilistic formulas for the above quantities. We now introduce the probabilistic formula for u_ε.

First recall that $\{X_s^\varepsilon, s \geqslant 0\}$ denotes the solution of the following SDE, starting from $x \in \mathbb{R}^d$,

$$X_s^\varepsilon = s + \frac{1}{\varepsilon}\int_0^s b\left(\frac{X_r^\varepsilon}{\varepsilon}\right) dr + \int_0^s c\left(\frac{X_r^\varepsilon}{\varepsilon}\right) dr + \int_0^s \sigma\left(\frac{X_r^\varepsilon}{\varepsilon}\right) dB_r. \tag{4.7}$$

For each $\varepsilon > 0$, let $\{(Y_s^\varepsilon, Z_s^\varepsilon), 0 \leqslant s \leqslant t\}$ be the $\mathbb{R} \times \mathbb{R}^d$-valued progressively measurable process solution of the BSDE

$$Y_s^\varepsilon = g(X_t^\varepsilon) + \frac{1}{\varepsilon}\int_s^t e\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\right) dr + \int_s^t f\left(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon\right) dr - \int_s^t Z_r^\varepsilon\, dB_r,$$

$$0 \leqslant s \leqslant t, \tag{4.8}$$

satisfying

$$\mathbb{E}\left(\sup_{0 \leqslant s \leqslant t} |Y_s^\varepsilon|^2 + \int_0^t \|Z_s^\varepsilon\|^2\, ds\right) < \infty;$$

see Pardoux and Peng [15] and Pardoux [14]. Note that $Y_\cdot^\varepsilon$ depends both on x, the starting point of $X_\cdot^\varepsilon$, and t, the final time in (4.8), and that from the progressive measurability, Y_0^ε is deterministic. It is well known (see, e.g., Pardoux [14]) that

$$u_\varepsilon(t, x) = Y_0^\varepsilon,$$

both in the sense that any classical solution of the PDE (4.1) is equal to Y_0^ε, and Y_0^ε is—at least in the case where all coefficients are continuous—a viscosity solution of the PDE (4.1). Moreover a similar identification as in the previous section holds if $g \in L^p(\mathbb{R}^d)$.

We shall now study the limit of Y_0^ε, as $\varepsilon \to 0$. The proof will be divided into several steps.

Step 1. *Transformation of the system* (4.7), (4.8). We first employ the same method as in the preceding section, in order to get rid of the ε^{-1} terms. It follows from the Itô–Krylov formula (see Krylov [10], and for a precise justification of the second application of that formula, Pardoux and Veretennikov [18]) (recall that $\bar{X}_s^\varepsilon = X_s^\varepsilon/\varepsilon$),

$$X_t^\varepsilon+\varepsilon\left(\hat{b}(\bar{X}_t^\varepsilon)-\hat{b}\left(\frac{x}{\varepsilon}\right)\right)$$
$$=x+\int_0^t (I+\nabla b)\, c(\bar{X}_s^\varepsilon)\, ds+\int_0^t (I+\nabla\hat{b})\, \sigma(\bar{X}_s^\varepsilon)\, dB_s \tag{4.9}$$
$$Y_s^\varepsilon+\varepsilon(\hat{e}(\bar{X}_t^\varepsilon, Y_t^\varepsilon)-\hat{e}(\bar{X}_s^\varepsilon, Y_s^\varepsilon))$$
$$=g(X_t^\varepsilon)+\int_0^t \left(\langle\nabla_x\hat{e}, c\rangle-\frac{\partial\hat{e}}{\partial y}\, e+f-\varepsilon\,\frac{\partial\hat{e}}{\partial y}\, f\right)(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr$$
$$+\int_s^t \frac{\partial^2\hat{e}}{\partial x\,\partial y}\,(\bar{Y}_r^\varepsilon, Y_s^\varepsilon)\, \sigma(\bar{X}_r^\varepsilon)\, Z_r^\varepsilon\, dr$$
$$+\int_s^t (\nabla_x\hat{e}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, \sigma(\bar{X}_r^\varepsilon)-Z_r^\varepsilon)\, dB_r$$
$$+\varepsilon\int_s^t \frac{\partial\hat{e}}{\partial y}\,(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, Z_r^\varepsilon\, db_r-\varepsilon\int_s^t \frac{\partial^2\hat{e}}{\partial y^2}\,(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, |Z_r^\varepsilon|^2\, dr. \tag{4.10}$$

We now define

$$\tilde{Z}_s^\varepsilon=Z_s^\varepsilon-\nabla_x\hat{e}(\bar{X}_s^\varepsilon, Y_s^\varepsilon)\, \sigma(\bar{X}_s^\varepsilon), \qquad 0\leqslant s\leqslant t,$$

and note that the difference between Z^ε and $\tilde{Z}^\varepsilon$ is bounded by $C\times|Y_\cdot^\varepsilon|$.
It then follows from (4.10) that

$$Y_s^\varepsilon+\varepsilon(\hat{e}(\bar{X}_t^\varepsilon, Y_t^\varepsilon)-\hat{e}(\bar{X}_s^\varepsilon, Y_s^\varepsilon))$$
$$=g(X_t^\varepsilon)+\int_s^t \left[\langle\nabla_x\hat{e}, c\rangle-\frac{\partial\hat{e}}{\partial y}\, e+f-\varepsilon\,\frac{\partial\hat{e}}{\partial y}\, f+\frac{\partial^2\hat{e}^*}{\partial y\,\partial x}\, a\,\nabla_x\hat{e}\right]$$
$$\times(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr-\int_s^t \tilde{Z}_r^\varepsilon\left(dB_r-\left(\sigma^*\,\frac{\partial^2\hat{e}}{\partial x\,\partial y}\right)(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr\right)$$
$$+\varepsilon\int_s^t \frac{\partial\hat{e}}{\partial y}\,(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, Z_r^\varepsilon\, dB_r+\frac{\varepsilon}{2}\int_s^t \frac{\partial^2\hat{e}}{\partial y^2}\,(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, |Z_r^\varepsilon|^2\, dr.$$

We next let

$$\tilde{B}_s=B_s-\int_0^s \left(\sigma^*\,\frac{\partial^2\hat{e}}{\partial x\,\partial y}\right)(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr.$$

It follows from Girsanov's theorem that there exists a new probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$, under which $\{\tilde{B}_s; 0\leqslant s\leqslant t\}$ is a Brownian motion.

We now have

$$
\begin{aligned}
X_s^\varepsilon &+ \varepsilon\left(\hat{b}(\bar{X}_s^\varepsilon) - \hat{b}\left(\frac{x}{\varepsilon}\right)\right) \\
&= x + \int_0^s (I + \nabla \hat{b}) \left(c + a\,\frac{\partial^2 \hat{e}}{\partial x\,\partial y}\right)(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\,dr \\
&\quad + \int_0^s (I + \nabla \hat{b})\,\sigma(\bar{X}_r^\varepsilon)\,r\tilde{B}_r
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
Y_s^\varepsilon &+ \varepsilon(\hat{e}(\bar{X}_t^\varepsilon, Y_t^\varepsilon) - \hat{e}(\bar{X}_s^\varepsilon, Y_s^\varepsilon)) \\
&= g(X_t^\varepsilon) + \int_s^t \left[\langle \nabla_x \hat{e}, x\rangle - \frac{\partial \hat{e}}{\partial y}\,e + f - \varepsilon\,\frac{\partial \hat{e}}{\partial y}\,f + \frac{\partial^2 \hat{e}^*}{\partial y\,\partial x}\,a\,\nabla_x \hat{e}\right] \\
&\qquad (\bar{X}_r^\varepsilon, Y_r^\varepsilon)\,dr - \int_s^t \tilde{Z}_r^\varepsilon\,d\tilde{B}_r \\
&\quad + \varepsilon \int_s^t \frac{\partial \hat{e}}{\partial y}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\,Z_r^\varepsilon\left[d\tilde{B}_r + \left(\sigma^*\,\frac{\partial^2 \hat{e}}{\partial x\,\partial y}\right)(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\,dr\right] \\
&\quad + \frac{\varepsilon}{2}\int_s^t \frac{\partial^2 \hat{e}}{\partial y^2}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\,|Z_r^\varepsilon|^2\,dr.
\end{aligned}
\tag{4.12}
$$

The fact that the sequence $\{X^\varepsilon_\cdot\,;\,\varepsilon > 0\}$ is tight, as a random element of $C([0, t]; \mathbb{R}^d)$ equipped with the topology of uniform convergence, is clear, in particular since $\partial^2 \hat{e}^*/\partial x\,\partial y$ is bounded.

Moreover, for any $p > 0$,

$$\sup_\varepsilon \tilde{\mathbb{E}}(|X_t^\varepsilon|^p) < \infty,$$

hence for any $k > 0$,

$$\sup_\varepsilon \tilde{\mathbb{E}}(|g(X_t^\varepsilon)|^k) < \infty.$$

Step 2. A priori estimates for $(Y^\varepsilon, Z^\varepsilon)$. We need to bound appropriate moments of Y^ε and Z^ε under $\tilde{\mathbb{P}}$. We first go back to (4.8), which we rewrite with the new Brownian motion $\tilde{B}$,

$$
\begin{aligned}
Y_s^\varepsilon = g(X_t^\varepsilon) &+ \int_s^t \left[\frac{1}{\varepsilon}\,e(\bar{X}_r^\varepsilon, Y_r^\varepsilon) + f(\bar{X}_r^\varepsilon, Y_r^\varepsilon) - Z_r^\varepsilon\left(\frac{\partial^2 \hat{e}}{\partial y\,\partial x}\,\sigma\right)(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\right] dr \\
&- \int_s^t Z_r^\varepsilon\,d\tilde{B}_r.
\end{aligned}
$$

Next, we note that from Itô's formula

$$A_s^\varepsilon \triangleq \frac{1}{\varepsilon}\int_0^s e_1(\bar{X}_r^\varepsilon)\,dr - \int_0^s \nabla\hat{e}_1\sigma(\bar{X}_r^\varepsilon)\,dB_r$$

$$= \varepsilon\left(\hat{e}_1(\bar{X}_s^\varepsilon) - \hat{e}_1\left(\frac{x}{\varepsilon}\right)\right) + \int_0^s \nabla\hat{e}_1 c(\bar{X}_r^\varepsilon)\,dr.$$

Hence there exists $c > 0$ s.t.

$$|A_s^\varepsilon| < c \qquad \text{a.s.,} \quad \forall\varepsilon \geqslant 0, \quad 0 \leqslant s \leqslant t.$$

Consequently, if

$$V_s^\varepsilon = \exp(A_s^\varepsilon),$$

$$e^{-c} \leqslant V_s^\varepsilon \leqslant e^c \qquad \text{a.s.,} \quad \forall\varepsilon > 0, \quad 0 \leqslant s \leqslant t.$$

Let now $\check{Y}_s^\varepsilon \triangleq V_s^\varepsilon Y_s^\varepsilon$, $\check{Z}_s^\varepsilon \triangleq V_s^\varepsilon Z_s^\varepsilon$. We have

$$\check{Y}_s^\varepsilon = Z_t^\varepsilon g(X_t^\varepsilon) + \int_s^t \left(\frac{1}{\varepsilon}\check{e}_0(r, \bar{X}_r^\varepsilon, \check{Y}_r^\varepsilon) + \check{f}(r, \bar{X}_r^\varepsilon, \check{Y}_r^\varepsilon) + (\check{Z}_r^\varepsilon, C_r^{1,\varepsilon}) + \check{Y}_r^\varepsilon C_r^{2,\varepsilon}\right) dr$$

$$- \int_s^t (\check{Z}_r^\varepsilon - \check{Y}_r^\varepsilon C_r^{1,\varepsilon})\,d\tilde{B}_r,$$

where $C_s^{1,\varepsilon}$ and $C_s^{2,\varepsilon}$ are bounded and progressively measurable processes, and respectively $\mathbb{R}^d$ and $\mathbb{R}$-valued, $\check{e}_0(s, x, y) = V_s^\varepsilon e_0(x, y/V_s^\varepsilon)$, $\check{f}(s, x, y) = V_s^\varepsilon f(x, y/V_s^\varepsilon)$. $\check{e}_0$ and $\check{f}$ satisfy (4.3), (4.4), and (4.5) with the same constants μ, K as e_0 and f.

Applying once more Itô's formula we obtain

$$e^{\nu s}|\check{Y}_s^\varepsilon|^3 + \int_s^t e^{\nu r}(3|\check{Y}_r^\varepsilon| \times |\check{Z}_r^\varepsilon - \check{Y}_r^\varepsilon C_r^{1,\varepsilon}|^2 + \nu|\check{Y}_r^\varepsilon|^3)\,dr$$

$$= e^{\nu t}|V_t^\varepsilon g(X_t^\varepsilon)|^3 + \frac{3}{\varepsilon}\int_s^t e^{\nu r}|\check{Y}_r^\varepsilon|\ \check{Y}_r^\varepsilon\check{e}_0(r, \bar{X}_r^\varepsilon, \check{Y}_r^\varepsilon)\,dr$$

$$+ 3\int_s^t e^{\nu r}|\check{Y}_r^\varepsilon|\ \check{Y}_r^\varepsilon\check{f}(r, \bar{X}_r^\varepsilon, \check{Y}_r^\varepsilon)\,dr$$

$$+ 3\int_s^t e^{\nu r}C_r^{2,\varepsilon}|\check{Y}_r^\varepsilon|^3\,dr + 3\int_s^t e^{\nu r}|\check{Y}_r^\varepsilon|\ \check{Y}_r^\varepsilon(\check{Z}_r^\varepsilon, C_r^{1,\varepsilon})\,dr$$

$$- 3\int_s^t e^{\nu r}|\check{Y}_r^\varepsilon|\ \check{Y}_r^\varepsilon(\check{Z}_r^\varepsilon - \check{Y}_r^\varepsilon C_r^{1,\varepsilon}, dB_r).$$

It follows from an argument in Pardoux and Peng [16] that the expectation of the above stochastic integral is zero. Moreover, from (4.3) and (4.5)

$$|Y_r^\varepsilon|\ Y_r^\varepsilon \check{f}(r, \bar{X}_r^\varepsilon, Y_r^\varepsilon) \leqslant \mu\, |Y_r^\varepsilon|^3 + K\, |Y_r^\varepsilon|^2$$
$$\leqslant (\mu+1)\, |Y_r^\varepsilon|^3 + c.$$

Since $\check{e}_0$ is bounded, we now deduce by standard inequalities, provided v is large enough (we can as well drop the $\check{}$'s),

$$\tilde{\mathbb{E}} \int_s^t e^{vr}\, |Y_r^\varepsilon| \times |Z_r^\varepsilon|^2\, dr \leqslant c \left(1 + \frac{1}{\varepsilon}\, \tilde{\mathbb{E}} \int_s^t e^{vr}\, |Y_r^\varepsilon|^2\, dr\right),$$

or equivalently

$$\varepsilon \tilde{\mathbb{E}} \int_s^t |Y_r^\varepsilon| \times |Z_r^\varepsilon|^2\, dr \leqslant c \left(\varepsilon + \tilde{\mathbb{E}} \int_s^t |Y_r^\varepsilon|^2\, dr\right). \tag{4.13}$$

We now go back to (4.12); let $\hat{Y}_s^\varepsilon = Y_s^\varepsilon - \varepsilon \hat{e}(\bar{X}_s^\varepsilon, Y_s^\varepsilon)$ and deduce from Itô's formula that

$$|\hat{Y}_s^\varepsilon|^2 + \int_s^t \left| \tilde{Z}_r^\varepsilon - \varepsilon \frac{\partial \hat{e}}{\partial y} (\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, Z_r^\varepsilon \right|^2 dr$$
$$= |g(X_t^\varepsilon) - \varepsilon \hat{e}(\bar{X}_t^\varepsilon, g(X_t^\varepsilon))|^2$$
$$+ 2 \int_s^t \hat{Y}_r^\varepsilon \left[\langle \nabla_x \hat{e}, c\rangle - \frac{\partial \hat{e}}{\partial y}\, e + \left(1 - \varepsilon \frac{\partial \hat{e}}{\partial y}\right) f + \frac{\partial^2 \hat{e}^*}{\partial y\, \partial x}\, a\, \nabla_x \hat{e} \right]$$
$$(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr - 2 \int_s^t \hat{Y}_r^\varepsilon \tilde{Z}_r^\varepsilon\, d\tilde{B}_r$$
$$+ 2\varepsilon \int_s^t \hat{Y}_r^\varepsilon \frac{\partial \hat{e}}{\partial y} (\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, Z_r^\varepsilon \left[d\tilde{B}_r + \left(\sigma^* \frac{\partial^2 \hat{e}}{\partial x\, \partial y}\right) (\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr \right]$$
$$+ \varepsilon \int_s^t \frac{\partial^2 \hat{e}}{\partial y^2} (\bar{X}_r^\varepsilon, Y_r^\varepsilon)\ \hat{Y}_r^\varepsilon\, |Z_r^\varepsilon|^2\, dr.$$

Exploiting (4.13), (4.3) together with the fact that $1 - \varepsilon(\partial \hat{e}/\partial y)\,(\bar{X}_r^\varepsilon, Y_r^\varepsilon) \geqslant 0$ for ε small enough, and standard inequalities, we deduce that

$$\tilde{\mathbb{E}}\, |Y_s^\varepsilon|^2 + \tfrac{1}{2} \tilde{\mathbb{E}} \int_s^t |\tilde{Z}_s^\varepsilon|^2\, ds \leqslant C \left(1 + \tilde{\mathbb{E}} \int_s^t |Y_r^\varepsilon|^2\, dr\right).$$

Hence from Gronwall's theorem

$$\sup_{0\leqslant s\leqslant T} \tilde{\mathbb{E}}(|Y^\varepsilon_s|^2)+\tilde{\mathbb{E}}\int_0^t |\tilde{Z}^\varepsilon_r|^2\,dr\leqslant c,$$

and finally from this last inequality, the above identity and the Davis, Burkholder, and Gundy inequality,

$$\sup_{\varepsilon>0}\tilde{\mathbb{E}}\left(\sup_{0\leqslant s\leqslant t}|Y^\varepsilon_s|^2+\int_0^t|\tilde{Z}^\varepsilon_s|^2\,ds\right)<\infty. \tag{4.14}$$

Step 3. *Convergence in law*. We rewrite (4.12) in the form

$$Y^\varepsilon_s=g(X^\varepsilon_t)+V^\varepsilon_t-V^\varepsilon_s+M^\varepsilon_t-M^\varepsilon_s+N^\varepsilon_t-N^\varepsilon_s,$$

where

$$V^\varepsilon_s=\int_0^s\left[\langle\nabla_x\hat{e},c\rangle-\frac{\partial\hat{e}}{\partial y}e+f+\frac{\partial^2\hat{e}^*}{\partial y\,\partial x}a\nabla_x\hat{e}\right](\bar{X}^\varepsilon_r,Y^\varepsilon_r)\,dr,$$

$$M^\varepsilon_s=-\int_0^s\tilde{Z}^\varepsilon_r\,dB_r,$$

$$\begin{aligned}N^\varepsilon_s={}&-\varepsilon\hat{e}(\hat{X}^\varepsilon_s,Y^\varepsilon_s)-\varepsilon\int_0^s\left(\frac{\partial\hat{e}}{\partial y}f\right)(\bar{X}^\varepsilon_r,Y^\varepsilon_r)\,dr\\&+\varepsilon\int_0^s\frac{\partial\hat{e}}{\partial y}(\bar{X}^\varepsilon_r,Y^\varepsilon_r)\,Z^\varepsilon_r\left[d\tilde{B}_r+\left(\sigma^*\frac{\partial^2\hat{e}}{\partial x\,\partial y}\right)(\bar{X}^\varepsilon_r,Y^\varepsilon_r)\,dr\right]\\&+\frac{\varepsilon}{2}\int_0^s\frac{\partial^2\hat{e}}{\partial y^2}(\bar{X}^\varepsilon_r,Y^\varepsilon_r)\,|Z^\varepsilon_r|^2\,dr.\end{aligned}$$

It is easy to check that

$$\tilde{\mathbb{E}}(\sup_{0\leqslant s\leqslant t}|N^\varepsilon_s|)\to 0,$$

hence $\sup_{0\leqslant s\leqslant t}|N^\varepsilon_s|$ tends to zero in $\tilde{\mathbb{P}}$ probability, or equivalently in law.

In order to treat the other terms, we adopt the point of view of Meyer and Zheng [13] (see also Kurtz [11], Pardoux [14]), which gives the following criteria ensuring tightness of laws of quasi-martingales on $D([0,t])$ (the space of functions which are right continuous and have limits from the left on $[0,t]$) equipped with the topology of convergence in ds-measure: the sequence of quasi-martingales $\{U^n_s; 0 \leqslant s \leqslant t\}$ defined on the filtered probability space $\{\Omega; \mathscr{F}_s, 0 \leqslant s \leqslant t; \mathbb{P}\}$ is tight whenever

$$\sup_n \big[\sup_{0 \leqslant s \leqslant t} \mathbb{E}\,|U^n_s| + CV^0_t(U^n)\big] < \infty,$$

where $CV^0_t(U^n)$, the "conditional variation of U^n on $[0, t]$" is defined as

$$CV^0_t(U^n) = \sup \mathbb{E}\left(\sum_i |\mathbb{E}(U^n_{t_{i+1}} - U^n_{t_i}/\mathscr{F}_{t_i})|\right),$$

with "sup" meaning that the supremum is taken over all partitions of the interval $[0, t]$.

It follows from (4.14) that both V^ε and M^ε satisfy the Meyer–Zheng criteria. Then $\{(Y^\varepsilon, M^\varepsilon)\}$ is tight in the sense of Meyer–Zheng, under $\tilde{\mathbb{P}}$, since from (4.11), $\{X^\varepsilon\}$ is tight "in the usual sense." Hence there exists a subsequence (which we still denote $(X^\varepsilon, Y^\varepsilon, M^\varepsilon)$) such that

$$(X^\varepsilon, Y^\varepsilon, M^\varepsilon) \Rightarrow (X, Y, M),$$

on $C([0, t]; \mathbb{R}^d) \times (D([0, t]))^2$, where the first factor is equipped with the topology of uniform convergence, and he second with the topology of convergence in ds measure.

Let us admit for a moment the following

LEMMA 4.2. *Let $h: \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be measurable, periodic of period one in each direction with respect to its first argument, continuous with respect to its second argument, uniformly with respect to the first. Then*

$$\sup_{0 \leqslant s \leqslant t} \left| \int_0^s h(\bar{X}^\varepsilon_r, Y^\varepsilon_r)\, dr - \int_0^s \bar{h}(Y_r)\, dr \right| \to 0$$

in $\tilde{\mathbb{P}}$ probability as $\varepsilon \to 0$, where $\bar{h}(y) = \int_{\mathbf{T}^d} h(x, y)\, \mu(dx)$.

We can now take the limit in (4.11), (4.12), yielding

$$X_s = x + \int_0^s C(Y_r)\, dr + M^X_s,$$

where $\{M^X_s\}$ is a non-standard Brownian motion whose quadratic variation is given as

$$\langle\!\langle M^X \rangle\!\rangle_s = As,$$

and

$$Y_s = g(X_t) + \int_s^t D(Y_r)\, dr + M_t - M_s.$$

Moreover, both M^X and M are $\mathscr{F}_s^{X,Y}$-martingales, and it follows from the argument in Pardoux and Veretennikov [17, Sect. 4c] (see also Pardoux [14, Proof of Theorem 6.1, step 4]) that Y and M are continuous,

$$M_s = \int_0^s U_r\, dM_r^X,$$

and $Y_0^\varepsilon \to Y_0$, where $Y_0 = u(t, x)$.

It remains to conclude with the:

Proof of Lemma 4.2. We follow the proof of Lemma 5 in Pardoux and Veretennikov [17], where the assumptions are slightly different from ours.

It follows from (4.14) that we can w.l.o.g. assume that h vanishes outside $\mathbb{R}^d \times [-M, M]$, for some $M > 0$. The result for the sup over s will follow from the result with fixed s exactly as in the proof of Lemma 3.1. Let

$$\tilde{h}(x, y) = h(x, y) - \bar{h}(y).$$

It suffices to prove that for any $0 < s \leqslant t$,

$$\int_0^s \tilde{h}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr \to 0$$

in $\tilde{\mathbb{P}}$ probability.

It follows from Lemma 4 in Pardoux and Veretennikov [17] that $\forall \delta > 0$, $\exists N \in \mathbb{N}$ and $y^1, ..., y^N$ step functions from $[0, t]$ into $\mathbb{R}$, such that if

$$A_\varepsilon \triangleq \bigcap_{k=1}^N \{\lambda\{0 \leqslant s \leqslant t; |Y_s^\varepsilon - y^k(s)| > \delta\} > \delta\},$$

where λ denotes Lebesgue's measure,

$$\tilde{\mathbb{P}}(A_\varepsilon) < \delta.$$

Now

$$A_\varepsilon^c = \bigcup_{k=1}^N B_k^\varepsilon,$$

where

$$B_k^\varepsilon = \{\lambda\{0 \leqslant s \leqslant t;\, |Y_s^\varepsilon - y^k(s)| > \delta\} \leqslant \delta\}.$$

Consequently

$$\begin{aligned}
&\left|\int_0^s \tilde{h}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr\right| \\
&\quad \leqslant \left|\int_0^s \tilde{h}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr\right| \mathbf{1}_{A_\varepsilon} + \sum_{k=1}^N \left|\int_0^s \tilde{h}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr\right| \mathbf{1}_{B_k^\varepsilon} \\
&\quad \leqslant sK_M \mathbf{1}_{A_\varepsilon} + \sum_{k=1}^N \left|\int_0^s \tilde{h}(\bar{X}_r^\varepsilon, y^k(r))\, dr\right| \mathbf{1}_{B_k^\varepsilon} \\
&\qquad + \sum_{k=1}^N \left|\int_0^s [\tilde{h}(\bar{X}_r^\varepsilon, Y_r^\varepsilon) - \tilde{h}(\bar{X}_r^\varepsilon, y(r)^k)]\, dr\right| \mathbf{1}_{B_k^\varepsilon} \\
&\quad \leqslant sK_M \mathbf{1}_{A_\varepsilon} + \omega_M(\delta)\, s + 2\delta K_M + \sum_{k=1}^N \left|\int_0^s \tilde{h}(\bar{E}_r^\varepsilon, y^k(r))\, dr\right|,
\end{aligned}$$

where ω_M is the modulus for continuity of $\tilde{h}$ in its second argument on $[-M, +M]$, and

$$K_M = \sup_{\mathbb{R}^d \times [-M, M]} |\tilde{h}(x, y)|.$$

Now, for each k,

$$\int_0^s \tilde{h}(\bar{X}_r^\varepsilon, y^k(r))\, dr \to 0$$

in $\mathbb{P}$-probability from Proposition 2.4, hence also in $\tilde{\mathbb{P}}$-probability. Finally, for any $\eta > 0$, let δ be small enough such that

$$(s+2)\, K_M \delta + s\omega_M(\delta) \leqslant \eta/2.$$

Then

$$\tilde{\mathbb{P}}\left(\left|\int_0^s \tilde{h}(\bar{X}_r^\varepsilon, Y_r^\varepsilon)\, dr\right| > \eta\right) \leqslant \sum_{k=1}^N \tilde{\mathbb{P}}\left(\left|\int_0^s \tilde{h}(\bar{X}_r^\varepsilon, y^k(r))\, dr\right| > \frac{\eta}{2N}\right),$$

and this tends to zero as $\varepsilon \to 0$. ∎

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