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Stochastic Calculus with Anticipating Integrands

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Summary. We study the stochastic integral defined by Skorohod in [24] of a possibly anticipating integrand, as a function of its upper limit, and establish an extended Itô formula. We also introduce an extension of Stratonovich's integral, and establish the associated chain rule. In all the results, the adaptedness of the integrand is replaced by a certain smoothness requirement.

1. Introduction

In the standard theory of integration, the measurability requirement on the integrand is essentially less restrictive than the integrability condition, which imposes a certain bound on its absolute value. One might say that with the Itô stochastic integral, the situation is reversed. Clearly the measurability condition which prescribes that the integrand should be independent of future increments of the Brownian integrator, is a very restrictive one. Whereas it is a natural condition in many situations, where the filtration represents the evolution of the available information, it is in many cases a limitation which has been felt quite restrictive, both for developing the theory, as well as in applications of stochastic calculus.

There have been many attempts, in particular during the last twelve years, to weaken the adaptedness requirement for the integrand of Itô's stochastic integral, such as in the theory of "enlargment of a filtration", which allows some anticipativity of the integrand. A completely different approach has been initiated by Skorohod in 1975 [24]. The two main aspects of Skorohod's integral are its total symmetry with respect to time reversal – it generalizes both the Itô forward and the Itô backward integrals – and the fact that no restriction whatsoever is put on the possible dependence of the integrand upon the future increments of the Brownian integrator. The price that has to be paid for that generality is some smoothness requirement upon the integrand, in a sense which will be made precise below. Also, we are restricted to define the integral in Wiener space, or at least on a space where the derivation can be defined as in Sect. 2 below. The ideas of Skorohod have been subsequently developed by Gaveau and Trauber [4] and Nualart and Zakai [16].

Our aim in this paper is threefold. First, we give intuitive approximations of Skorohod's integral, for several classes of integrands. Second, we study some properties of the process obtained by integrating from 0 to t, and establish a generalized Itô formula. Third, we define a "Stratonovich version" of Skorohod's integral, and establish a chain rule of Stratonovich type.

After most of this work was completed, we learned the existence of the work of Sevljakov [22] and Sekiguchi and Shiota [21], as well as that of Ustunel [25]. The intersection of these papers with our is the generalized Itô formula. While our Itô formula is slightly more general than the others, we feel that our proof is more direct than that of the first two other papers. On the other hand, our proof, which is very much like the proof of the usual Itô formula, is very different from that of Ustunel [25], which has more a functional analysis flavour.

Let us finally mention that our Stratonovich-Skorohod integral has strong similarities with some of the other existing generalized stochastic integrals, which include those of Berger and Mizel [1], Kuo and Russek [10], Ogawa [18] and Rosinski [20].

Finally, we want to point out that this work owes very much to the previous works of both authors on the same subject. Therefore, we want to thank Moshe Zakai and Philip Protter, with whom many ideas which where at the origin of this paper have been discussed by one of us, and appear in [16, 19].

The paper is organized as follows. In Sect. 2 we define the gradient operator on Wiener space, and in section three we define Skorohod's integral. In Sect. 4, we study some approximations of Skorohod's integral, and prove additional properties. In Sect. 5, we study some properties of Skorohod's integral as a process. In Sect. 6, we prove the generalized Itô rule. In Sect. 7, we define a "Stratonovich-Skorohod" integral, and establish a chain rule of Stratonovich type. Section 8 is concerned with the particular case of what we call the "two-sided integral", which is a direct generalization of the work of Pardoux and Protter [19]. Most of the results have been announced in [15].

2. Definition and Some Properties of the Derivation on Wiener Space

In this section, we define the derivative of functions defined on Wiener space, and introduce the associated Sobolev spaces. This is part of the machinery which is used in particular in the Malliavin calculus, see Malliavin [13], Ikeda and Watanabe [5, 6], Shigekawa [23], Zakai [28]. We refer to Watanabe [26], Kree [11] and Kree and Kree [12] for other expositions.

Let $\{W(t), t \in [0,1]\}$ be a d-dimensional standard Wiener process defined on the canonical probability space $(\Omega, \mathscr{F}, P)$. That means $\Omega = C([0,1], \mathbb{R}^d)$, P is the Wiener measure, $\mathscr{F}$ is the completion of the Borel σ-algebra of Ω with respect to P, and $W_t(\omega) = \omega(t)$. The Borel σ-algebra and the Lebesgue measure on $[0,1]$ will be denoted, respectively, by $\mathscr{B}$ and λ.

For each $t \in [0,1]$ we denote by $\mathscr{F}_t$ and $\mathscr{F}^t$, respectively, the σ-algebras generated by the families of random vectors $\{W(s), 0 \leqq s \leqq t\}$ and $\{W(1) - W(s), t \leqq s \leqq 1\}$, completed with respect to P.

Let $C_b^\infty(\mathbb{R}^k)$ be the set of C^∞ functions $f:\mathbb{R}^k\to\mathbb{R}$ which are bounded and have bounded derivatives of all orders. A smooth functional will be a random variable $F:\Omega\to\mathbb{R}$ of the form $F=f(W(t_1),\dots,W(t_n))$, where the function $f(x^{11},\dots,x^{d1};\dots;x^{1n},\dots,x^{dn})$ belongs to $C_b^\infty(\mathbb{R}^{dn})$ and $t_1,\dots,t_n\in[0,1]$. The class of smooth functionals will be denoted by $\mathcal{S}$.

The derivative of a smooth functional F can be defined as the d-dimensional stochastic process given by

$$(DF)_t^j=\sum_{i=1}^n \frac{\partial f}{\partial x^{ji}}(W(t_1);\dots;W(t_n))1_{[0,t_i]}(t),$$

for $t\in[0,1]$ and $j=1,\dots,d$.

The derivative DF can be regarded as a random variable taking values in the Hilbert space $H=L^2([0,1];\mathbb{R}^d)$. More generally, the N-th derivative of F, D^NF will be the $H^{\otimes N}$-valued random variable

$$(D^NF)_{s_1,\dots,s_N}^{j_1,\dots,j_N}=\sum_{i_1,\dots,i_N=1}^n \frac{\partial^N f}{\partial x^{j_1i_1}\dots\partial x^{j_Ni_N}}(W(t_1);\dots;W(t_n))$$

$$\cdot 1_{[0,t_{i_1}]}(s_1)\dots 1_{[0,t_{i_N}]}(s_N),$$

where $s_1,\dots,s_N\in[0,1]$ and $j_1,\dots,j_N=1,\dots,d$.

We write also D_t^jF for $(DF)_t^j$. Notice that with this notation,

$$(D^NF)_{s_1,\dots,s_N}^{j_1,\dots,j_N}$$

coincides with the iterated derivative

$$D_{s_1}^{j_1}D_{s_2}^{j_2}\dots D_{s_N}^{j_N}F.$$

For any integer $N\geqq 1$ and any real number $p>1$ we introduce the seminorm on $\mathcal{S}$

$$\|F\|_{p,N}=\|F\|_p+\|\,\|D^NF\|_{HS}\|_p$$

where $\|\,.\,\|_{HS}$ denotes the Hilbert-Schmidt norm in $H^{\otimes N}$, that means,

$$\|D^NF\|_{HS}^2=\sum_{j_1,\dots,j_N=1}^d \int_{[0,1]^N}[(D^NF)_{s_1,\dots,s_N}^{j_1,\dots,j_N}]^2ds_1\dots ds_N.$$

In case $N=1$, we will denote by $\|\,.\,\|$ the norm in H.

Then $\mathbb{D}_{p,N}$ will denote the Banach space which is the completion of $\mathcal{S}$ with respect to the norm $\|F\|_{p,N}$.

Consider the orthogonal Wiener-Chaos decomposition (see Itô [7]) $L^2(\Omega,\mathcal{F},P)=\bigoplus_{n=0}^\infty \mathbf{H}_n$, and denote by J_n the orthogonal projection on $\mathbf{H}_n$.

Any random variable of $\mathbf{H}_n$ can be expressed as a multiple Itô integral $I_n(f_n)$ of some symmetric kernel $f_n\in L^2([0,1]^n;\mathbb{R}^{dn})=H^{\otimes n}$, i.e., $f_n(t_1,\dots,t_n)^{j_1,\dots,j_n}$ is symmetric in the n variables $(t_1,j_1),\dots,(t_n,j_n)$.

Then it holds that

$$D_t^j(I_n(f_n))=nI_{n-1}(f_n(\cdot,t)^{\cdot j}), \tag{2.1}$$

(note that $I_0(f_1(t)^j)=f_1(t)^j$)

and the space $\mathbb{D}_{2,1}$ coincides with the set of square integrable random variables F such that

$$E(\|DF\|_{HS}^2)=\sum_{n=1}^{\infty} nE(|J_nF|^2)<\infty.$$

The derivation operator D (also called the gradient operator) is a closed linear operator defined in $\mathbb{D}_{2,1}$ and taking values on $L^2([0,1]\times\Omega;\mathbb{R}^d)$.

Notice that our d-dimensional Wiener process can be regarded as a particular example of a Gaussian orthogonal measure on the measure space $\mathbf{T}=[0,1]\times\{1,\dots,d\}$. In this sense we can use the results of Nualart-Zakai [16].

Following [16], for any square integrable random variable $F=\sum_{n=0}^{\infty} I_n(f_n)$ and any $h\in H$ we define

$$D_hF=\sum_{n=1}^{\infty}\sum_{j=1}^{d}\int_0^1 nI_{n-1}(f_n(.,t)^{\cdot,j})h^j(t)dt, \tag{2.2}$$

provided that the series converges in $L^2(\Omega)$.

We denote by $\mathbb{D}_{2,h}$ the domain of D_h. Equipped with the norm $(\|F\|_2^2+\|D_hF\|_2^2)^{1/2}$, $\mathbb{D}_{2,h}$ is a Hilbert space, and clearly, $\mathbb{D}_{2,1}\subset\mathbb{D}_{2,h}$. Conversely, if $F\in\mathbb{D}_{2,h}$ for all $h\in H$ and the linear map $h\to D_hF$ defines a square integrable H-valued random variable, then F belongs to $\mathbb{D}_{2,1}$ and $D_hF=\langle DF,h\rangle_H$. From now on we use the notation $u.v$ to denote the scalar product of $u,v\in\mathbb{R}^d$.

Lemma 2.1. *D_h is a closed operator and for any $F\in\mathbb{D}_{2,h}$ we have*

$$E(D_hF)=E\left(F\int_0^1 h(t).dW_t\right). \tag{2.3}$$

Proof. Suppose that $F=\sum_{n=0}^{\infty} I_n(f_n)$ belongs to $\mathbb{D}_{2,h}$ and put $G=I_n(g)$. Then

$$\begin{aligned}E((D_hF)G)&=E\left(\sum_{j=1}^{d}\int_0^1 (n+1)(I_n(f_{n+1}(.,t)^{\cdot,j}I_n(g))h^j(t)dt\right)\\&=(n+1)!\langle f_{n+1},g\otimes h\rangle_{L^2([0,1]^{n+1};\mathbb{R}^{d(n+1)})}\\&=E(FI_{n+1}(g\otimes h)),\end{aligned} \tag{2.4}$$

where

$$(g\otimes h)(t_1,\dots,t_{n+1})^{j_1,\dots,j_{n+1}}=g(t_1,\dots,t_n)^{j_1,\dots,j_n}h^{j_{n+1}}(t_{n+1}).$$

As a consequence, if $F_n\to 0$ in $L^2(\Omega)$, $F_n\in\mathbb{D}_{2,h}$, and $D_hF_n\to G_1$ in $L^2(\Omega)$, we deduce that $G_1=0$, and D_h is closed. Finally, taking $n=0$ and $G=1$ in (2.4) we obtain the equality (2.3). □

We recall the following fact (see [16], Proposition 2.2) which allows to interpret the operator D_h as a directional derivative.

Proposition 2.2. *Let F be a square integrable random variable. Suppose that the limit*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(F(\omega. + \varepsilon \int_0^x h(s) ds) - F(\omega) \right)$$

exists in $L^2(\Omega)$. Then F belongs to $\mathbb{D}_{2,h}$ and this limit coincides with $D_h F$. □

The next result is the chain rule for the derivation.

Proposition 2.3. *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continously differentiable function with bounded partial derivatives. Suppose that $F = (F^1, \ldots, F^m)$ is a random vector whose components belong to $\mathbb{D}_{2,1}$. Then $\varphi(F) \in \mathbb{D}_{2,1}$ and*

$$D\varphi(F) = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x^i} (F) DF^i. \quad \square$$

A similar differentiation formula is true for the directional derivative D_h.

Let A be a Borel subset of $[0,1]$ and denote by $\mathscr{F}_A$ the σ-algebra generated by the random vectors

$$W(G) = \int_0^1 1_G dW, \quad G \subset A, \quad G \in \mathscr{B}.$$

Then we have the following basic result.

Lemma 2.4. *Let F be a square integrable random variable.*

(i) *If F is $\mathscr{F}_A$-measurable and $h \in L^2([0,1]; \mathbb{R}^d)$ vanishes on A, then F belongs to $\mathbb{D}_{2,h}$ and $D_h F = 0$.*

(ii) *If $F \in \mathbb{D}_{2,1}$, then $E(F/\mathscr{F}_A) \in \mathbb{D}_{2,1}$ and $D_t(E(F/\mathscr{F}_A)) = E(D_t F/\mathscr{F}_A) 1_A(t)$, a.e. in $[0,1] \times \Omega$.*

Proof. It suffices to assume $d = 1$ and $F = I_n(f_n)$, and in this case, the lemma follows easily from (2.1), (2.2) and the equality

$$E[I_n(f_n)/\mathscr{F}_A] = I_n(g_n)$$

where $g_n(t_1, \ldots, t_n) = f_n(t_1, \ldots, t_n) 1_A(t_1) \ldots 1_A(t_n)$. □

In particular, for any $F \in \mathbb{D}_{2,1}$ and $r < s$, we have

$$D_t(E(F/\mathscr{F}_r \vee \mathscr{F}^s)) = E(D_t F/\mathscr{F}_r \vee \mathscr{F}^s) 1_{[r,s]^c}(t),$$

a.e. in $[0,1] \times \Omega$.

Lemma 2.5. *For any $p \geqq 2$, there exists a constant c_p such that $\forall F \in \mathbb{D}_{2,1}$,*

$$E(|F|^p) \leqq c_p \left(|E(F)|^p + E \int_0^1 |D_t F|^p dt \right).$$

Proof. It follows from Ocone's version of a well known representation theorem – see Ocone [17], or Corollary A.2 in the Appendix A – that:

$$F = E(F) + \int_0^1 E(D_t F/\mathscr{F}_t) \,.\, dW_t.$$

The result then follows from Burkholder and Jensen's inequalities. □

Suppose that C is the operator corresponding to the product by the factor $-\sqrt{n}$ on any Wiener-Chaos. From (2.1) it follows easily that the domain of C is $\mathbb{D}_{2,1}$ and $E(|CF|^2)=E(\|DF\|_{HS}^2)$ for any F in $\mathbb{D}_{2,1}$.

The following inequalities due to Meyer (see [14], Theorem 2) provide the equivalence of norms between the powers of the operators D and C:

For any real $p>1$ and any integer $N\geqq 1$ there exist positive constants $a_{p,N}$ and $A_{p,N}$ such that

$$a_{p,N}E(\|D^N F\|_{HS}^p)\leqq E(|C^N F|^p)\leqq A_{p,N}[E(\|D^N F\|_{HS}^p)+E(|F|^p)], \tag{2.5}$$

for any smooth functional F.

We now state a result, whose proof will be given at the end of the next section, which says that the derivation is a local operator.

Lemma 2.6. *Let $F\in\mathbb{D}_{2,1}$. Then $1_{\{F=0\}}D_t F=0$ $dt\times dP$ a.e. on $[0,1]\times\Omega$.*

It will be clear from the proof below that the same result is true for D_h, $h\in H$.

Let us now state:

Definition 2.7. A random variable F will be said to belong to the class $\mathbb{D}_{2,1,\mathrm{loc}}$ if there exists a sequence of measurable subsets of $\Omega: \Omega_k\uparrow\Omega$ a.s. and a sequence $\{F_k, k\in\mathbb{N}\}\subset\mathbb{D}_{2,1}$ such that:

$$F|_{\Omega_k}=F_k|_{\Omega_k} \quad \text{a.s.,} \quad \forall k\in\mathbb{N}.$$

In that case, we will say that F is localized by the sequence $\{(\Omega_k, F_k), k\in\mathbb{N}\}$. □

Clearly, $D_{p,N,\mathrm{loc}}$ can be defined analogously, for any $p\geqq 1$, $N\in\mathbb{N}$.

Thanks to Lemma 2.6, the following definition is consistent:

Definition 2.8. Let F be an element of $\mathbb{D}_{2,1,\mathrm{loc}}$ which is localized by a sequence $\{(\Omega_k, F_k), k\in\mathbb{N}\}$. We then define DF to be the unique equivalence class of $dt\times dP$ a.e. equal d-dimensional processes which satisfies:

$$DF|_{\Omega_k}=DF_k|_{\Omega_k}. \quad \square$$

We can now generalize Proposition 2.3.

Proposition 2.9. *Let $\varphi:\mathbb{R}^m\to\mathbb{R}$ be of class C^1. Suppose that $F=(F^1,\dots,F^m)$ is a random vector whose components belong to $\mathbb{D}_{2,1,\mathrm{loc}}$. Then $\varphi(F)\in\mathbb{D}_{2,1,\mathrm{loc}}$ and:*

$$D\varphi(F)=\sum_{i=1}^m \frac{\partial\varphi}{\partial x_i}(F)DF^i.$$

Proof. The result follows easily from Definitions 2.7, 2.8 and Proposition 2.3. □

An immediate corollary of the above is that whenever $F, G\in\mathbb{D}_{2,1}$ (or only $\mathbb{D}_{2,1,\mathrm{loc}}$), then $FG\in\mathbb{D}_{2,1,\mathrm{loc}}$ and:

$$D(FG)=FDG+GDF.$$

3. Definition of the Skorohod Integral

Let $u \in L^2([0,1] \times \Omega; \mathbb{R}^d)$ be a square integrable d-dimensional process. By means of the Wiener-Chaos decomposition, we can decompose u into an orthogonal series

$$u_t = \sum_{m=0}^{\infty} I_m(f_m(.\,,t)), \tag{3.1}$$

where $f_m(s_1, \dots, s_m, t)^{j_1, \dots, j_m, j} \in L^2([0,1]^{m+1}; \mathbb{R}^{d(m+1)})$ is a symmetric function of the m couples $(s_1, j_1), \dots, (s_m, j_m)$ for each fixed (t, j). Denote by $\tilde{f}_m$ the symmetrization of f_m in the $m+1$ couples (s_i, j_i), $1 \leqq i \leqq m$, (t, j), that means,

$$\tilde{f}_m(s_1, \dots, s_m, t)^{j_1, \dots, j_m, j} = \frac{1}{m+1} \Big[(f_m(s_1, \dots, s_m, t)^{j_1, \dots, j_m, j} + \sum_{i=1}^{m} f_m(s_1, \dots, s_{i-1}, t, s_{i+1}, \dots, s_m, s_i)^{j_1, \dots, j_{i-1}, j, j_{i+1}, \dots, j_m, j_i}) \Big].$$

Then, the Skorohod integral of u (see Skorohod [24]) is defined by

$$\delta(u) = \sum_{m=0}^{\infty} I_{m+1}(\tilde{f}_m), \tag{3.2}$$

$$= \sum_{m=0}^{\infty} \sum_{j_1, \dots, j_m, j=1}^{d} \int_{[0,1]^{m+1}} \tilde{f}_m(s_1, \dots, s_m, t)^{j_1, \dots, j_m, j} dW_{s_1}^{j_1} \dots dW_{s_m}^{j_m} dW_t^j,$$

provided that this series converges in $L^2(\Omega)$. We will also represent the Skorohod integral of u by

$$\int_0^1 u_t \,.\, dW_t,$$

and the set of Skorohod integrable processes will be denoted $\operatorname{Dom} \delta$.

Note that we are integrating a d-dimensional process with respect to a d-dimensional Wiener process. The result is a real valued random variable, and

$$\int_0^1 u_t \,.\, dW_t,$$

is a short notation for

$$\sum_{i=1}^{d} \int_0^1 u_t^i dW_t^i.$$

As before, if we consider W as a Gaussian orthogonal measure on $\mathbf{T} = [0,1] \times \{1, \dots, d\}$, this definition can be viewed as a particular case of the situation considered in Nualart and Zakai [16]. In [4] Gaveau and Trauber have proved that the Skorohod integral coincides with the dual operator of the derivation D. More precisely we can state the next result.

Proposition 3.1. *Let $u \in L^2([0,1] \times \Omega; \mathbb{R}^d)$. Then u is Skorohod integrable if and only if there exists a constant c such that*

$$\left| E\left(\int_0^1 u_t \, . \, D_t F dt \right) \right| \leqq c \|F\|_2$$

for any $F \in \mathbb{D}_{2,1}$ and, in this case, we have

$$E\left(\int_0^1 u_t \, . \, D_t F dt \right) = E(F\delta(u)) \, . \quad \square \tag{3.3}$$

Formula (3.3) is the general version of the integration by parts formula of Bismut [2]. Notice that δ is a closed operator because δ is the adjoint of D and $\mathbb{D}_{2,1}$ is dense in $L^2(\Omega)$.

Let $j = 1, \ldots, d$ be a fixed index. We will say that a one-dimensional process $u \in L^2([0,1] \times \Omega)$ is Skorohod integrable with respect to W^j if $ue_j \in \operatorname{Dom} \delta$, where $e_j = (0, \ldots, 1, \ldots, 0)$ (1 being the jth component of this vector). The class of these processes will be denoted by $\operatorname{Dom} \delta_j$, and we will write $\int_0^1 u_t dW_t^j$ or $\delta_j(u)$ for $\delta(ue_j)$. The random variable $\delta_j(u)$ is determined by the duality formula

$$E(F\delta_j(u)) = E\left(\int_0^1 u_t D_t^j F dt \right),$$

for all $F \in \mathbb{D}_{2,1}$.

If a d-dimensional process u is such that $u^j \in \operatorname{Dom} \delta_j$ for all $j = 1, \ldots, d$, then $u \in \operatorname{Dom} \delta$ and

$$\delta(u) = \sum_{j=1}^{d} \delta_j(u^j) \, .$$

Let us first establish a basic and essential property of the Skorohod integral.

Theorem 3.2. *Let $u \in \operatorname{Dom} \delta$ and $F \in \mathbb{D}_{2,1}$. Then*

$$\int_0^1 F u_t \, . \, dW_t = F \int_0^1 u_t \, . \, dW_t - \int_0^1 u_t \, . \, D_t F dt \tag{3.4}$$

in the sense that $Fu \in \operatorname{Dom} \delta$ if and only if the right hand side of (3.4) *is in $L^2(\Omega)$.*

Proof. To simplify, suppose $d = 1$. For any smooth functional $G = g(W(t_1), \ldots, W(t_n))$ in the space $\mathscr{S}$, we have

$$\int_0^1 E(F u_t D_t G) dt = \int_0^1 E[u_t(D_t(FG) - G D_t F)] dt$$

$$= E\left[\left(F\delta(u) - \int_0^1 u_t D_t F dt \right) G \right],$$

and the result follows from Proposition 3.1. $\square$

The set $\operatorname{Dom} \delta$ is not easy to handle and it is more convenient to deal with processes belonging to some subset of $\operatorname{Dom} \delta$.

Definition 3.3. Let $\mathbb{L}^{2,1}$ denote the class of scalar processes $u \in L^2([0,1]\times\Omega)$ such that $u_t \in \mathbb{D}_{2,1}$ for a.a.t and there exists a measurable version of $D_s u_t$ verifying

$$E\int_0^1\int_0^1 |D_s u_t|^2 \, ds\, dt < \infty .$$

In terms of the Wiener-Chaos expansion this is equivalent to saying that

$$\sum_{m=1}^{\infty} m m! \, \|f_m\|^2_{L^2([0,1]^{m+1};\mathbb{R}^{dm+1})} < \infty$$

if u is given by (3.1).

Let $\mathbb{L}^{2,2}$ denote the set of processes $u \in L^2([0,1]\times\Omega)$ such that $u_t \in \mathbb{D}_{2,2}$ for a.a.t and there exists a measurable version of $D_r D_s u_t$ verifying

$$E\int_0^1\int_0^1\int_0^1 |D_r D_s u_t|^2 \, dr\, ds\, dt < \infty .$$

This is equivalent to saying that

$$\sum_{m=2}^{\infty} m(m-1)m! \, \|f_m\|^2_{L^2([0,1]^{m+1};\mathbb{R}^{md+1})} < \infty ,$$

if u is given by (3.1).

Finally, $\mathbb{L}^{2,1}_d$ (resp. $\mathbb{L}^{2,2}_d$) is defined as the set of d-dimensional processes whose components are in $\mathbb{L}^{2,1}$ (resp. in $\mathbb{L}^{2,2}$). □

Then, $\mathbb{L}^{2,1} \subset \operatorname{Dom}\delta_j$ for all $j=1,\dots,d$ and $\mathbb{L}^{2,1}_d \subset \operatorname{Dom}\delta$. $\mathbb{L}^{2,1}$ and $\mathbb{L}^{2,1}_d$ are Banach spaces (in fact Hilbert spaces) with the norm

$$\|u\| = \left(E\int_0^1 |u_t|^2 \, dt\right)^{1/2} + \left(E\int_0^1\int_0^1 \|D_s u_t\|^2 \, ds\, dt\right)^{1/2},$$

where $\|D_s u_t\|$ denotes a norm of the matrix $(D_s^i u_t^j)$.

For a process $u \in \mathbb{L}^{2,1}_d$ we have the following isometric property (cf. Nualart and Zakai [16], Proposition 3.1)

$$E\left(\int_0^1 u_t \, . \, dW_t\right)^2 = E\left[\int_0^1 |u_t|^2 \, dt + \int_0^1\int_0^1 \sum_{i,j=1}^{d} D_s^i u_t^j D_t^j u_s^i \, ds\, dt\right]. \tag{3.5}$$

Note that Skorohod [24] has defined his integral only for integrands in $\mathbb{L}^{2,1}$.

The next result together with (3.3) and (3.4) will constitute a practical tool in what follows.

Proposition 3.4. *Let $u \in \mathbb{L}^{2,1}_d$ such that for all $i=1,\dots,d$ and for all t a.e. the process $\{D_t^i u_s, 0 \leqq s \leqq 1\}$ belongs to* Dom δ *and there is a version of*

$$\left\{\int_0^1 D_t^i u_s \, . \, dW_s, \quad 0 \leqq t \leqq 1\right\}$$

in $L^2([0,1]\times\Omega)$. Then $\delta(u) \in \mathbb{D}_{2,1}$, and

$$D_t^i\left(\int_0^1 u_s \, . \, dW_s\right) = \int_0^1 D_t^i u_s \, . \, dW_s + u_t^i . \tag{3.6}$$

Proof. Suppose $d=1$. Consider a process $v\in \mathbb{L}^{2,1}$. Using the isometric property (3.5) and the integration by parts formula (3.3) we obtain

$$E(\delta(u)\delta(v)) = E\left(\int_0^1 u_t v_t dt + \int_0^1\int_0^1 D_t u_s D_s v_t ds dt\right)$$
$$= E\left(\int_0^1 u_t v_t dt + \int_0^1 \left(\int_0^1 D_t u_s dW_s\right) v_t dt\right).$$

Finally we may conclude by a duality argument because $\mathbb{L}^{2,1}$ is dense in $L^2([0,1]\times\Omega)$. □

Note that the Proposition applies in particular when $u\in\mathbb{L}_d^{2,2}$. For a proof of (3.6) using the Wiener-Chaos expansion we refer to Proposition 3.4 of Nualart and Zakai [16]. Another proof will be given in the next section.

The following L^p inequalities will be useful in proving the path continuity of the indefinite Skorohod integral.

Proposition 3.5. *Let $u\in\mathbb{L}_d^{2,1}$. Then, for any $p\geqq 2$ there exists a positive constant c_p such that*

$$\left\|\int_0^1 u_t\,.\,dW_t\right\|_p \leqq c_p\left[\left(\int_0^1 |E(u_t)|^2 dt\right)^{1/2} + \left\|\left(\int_0^1\int_0^1 \|D_s u_t\|^2 ds dt\right)^{1/2}\right\|_p\right]. \quad \square \tag{3.7}$$

This result is a consequence of Meyer's inequalities. We refer to Watanabe [26], for a general proof of the continuity properties of the operator δ. For a sake of completeness we have included a proof of (3.7) in the Appendix B.

Let us point out that the operator δ can be extended to the whole space $L^2([0,1]\times\Omega;\mathbb{R}^d)$. But $\delta(u)$, for $u\notin \mathrm{Dom}\,\delta$ is no longer a square integrable random variable, and is rather an element of a Sobolev space with negative index, i.e. a "distribution" over Wiener space, see Watanabe [26].

We now turn to the:

Proof of Lemma 2.6. In order to simplify the notations, let us assume that $d=1$. For any $\varepsilon>0$, we define the mappings φ_ε, $\psi_\varepsilon:\mathbb{R}\to\mathbb{R}$ by:

$$\varphi_\varepsilon(x)=\begin{cases}1+x/\varepsilon & \text{if}\quad -\varepsilon\leqq x\leqq 0\\ 1-x/\varepsilon & \text{if}\quad 0\leqq x\leqq\varepsilon\\ 0 & \text{otherwise}\end{cases}$$

$$\psi_\varepsilon(x)=\int_{-\infty}^x \varphi_\varepsilon(y)dy$$

It follows from Proposition 2.3 that $\psi_\varepsilon(F)\in\mathbb{D}_{2,1}$ and $D_t\psi_\varepsilon(F)=\varphi_\varepsilon(F)D_tF$. If now $u\in\mathbb{L}^{2,1}$ we have:

$$E\left[\int_0^1 D_r\psi_\varepsilon(F)u_r dr\right]=E[\psi_\varepsilon(F)\delta(u)]$$

$$\left|E\left[\int_0^1 D_r\psi_\varepsilon(F)u_r dr\right]\right|\leqq\varepsilon E(|\delta(u)|).$$

On the other hand, from Lebesgue dominated convergence, as $\varepsilon \to 0$,

$$E\left(\varphi_\varepsilon(F) \int_0^1 D_r F u_r dr\right) \to E\left(1_{\{F=0\}} \int_0^1 D_r F u_r dr\right).$$

Then

$$E\left(1_{\{F=0\}} \int_0^1 D_r F u_r dr\right) = 0, \qquad \forall u \in \mathbb{L}^{2,1}.$$

Since $\mathbb{L}^{2,1}$ is dense in $L^2([0,1] \times \Omega)$, the result follows. □

We finally state the following definition:

Definition 3.6. We will say that a measurable process $u \in (\mathrm{Dom}\,\delta)_{\mathrm{loc}}$ whenever there exists a sequence $\{\Omega_k, k \in \mathbb{N}\} \subset \mathscr{F}$ and a sequence $\{u_k, k \in \mathbb{N}\} \subset \mathrm{Dom}\,\delta$ s.t.:

(i) $\Omega_k \uparrow \Omega$ a.s.

(ii) $u = u_k$ on Ω_k a.s.

(iii) $\delta(u_k) = \delta(u_l)$ on Ω_k a.s., whenever $k < l$. In that case, we will say that u is localized by $\{(\Omega_k, u_k)\}$. □

We suspect that δ is a local operator, and that (iii) follows from (i) and (ii). In fact, we will show that property of δ, when restricted to some subclasses of $\mathrm{Dom}\,\delta - \mathbb{L}^{2,1}$ being one of them – in the next section.

Definition 3.7. Let $u \in (\mathrm{Dom}\,\delta)_{\mathrm{loc}}$ be localized by $\{(\Omega_k, u_k)\}$. We then define $\delta(u)$ as the unique equivalence class of a.s. equal random variables s.t.:

$$\delta(u)|_{\Omega_k} = \delta(u_k)|_{\Omega_k} \quad \text{a.s.} \quad \square$$

Note that $\delta(u)$ in Definition 3.6 may depend on the localizing sequence $\{(\Omega_k, u_k)\}$.

4. Approximation of the Skorohod Integral by Riemann Sums, and Additional Properties

We will show that for several subsets of $\mathrm{Dom}\,\delta$ one can approximate the Skorohod integral by Riemann sums.

Let $h \in L^2(0,1)$. From (2.3) and Proposition 3.1, it follows that $h \in \mathrm{Dom}\,\delta_j$ and $\delta_j(h) = \int_0^1 h(t) dW_t^j$, $1 \leqq j \leqq d$. Again for $h \in L^2(0,1)$ we denote by h_i the element of H given by: $h_i(t) = (0, \ldots, 0, h(t), 0, \ldots, 0)'$ where $h(t)$ is the i-th component of the above vector.

Our fundamental tool in the sequel will be the next lemma which, for convenience of the reader, we first state in dimension one.

Lemma 4.1. ($d=1$). *Let $h, k \in L^2(0,1)$, and $F \in \mathbb{D}_{2,h}$, $G \in \mathbb{D}_{2,k}$. Then hF and $kG \in \mathrm{Dom}\,\delta$,*

$$\delta(hF) = F\delta(h) - D_h F$$

and similarly for kG. If $F, G \in \mathbb{D}_{2,h} \cap \mathbb{D}_{2,k}$, *then:*

$$E[\delta(hF)\delta(kG)] = \langle h, k\rangle E(FG) + E[D_k F D_h G].$$

Lemma 4.1. ($d>1$). *Let* $h \in L^2(0,1)$ *and* $F \in L^2(\Omega; \mathbb{R}^d)$, *s.t.* $F^i \in \mathbb{D}_{2,h_j} \forall 1 \leqq i, j \leqq d$. *Then* $hF \in \operatorname{Dom}\delta$ *and:*

$$\delta(hF) = \sum_{i=1}^{d} (F^i \delta_i(h) - D_{h_i} F^i) \tag{4.1}$$

$$E[\delta(hF)^2] = |h|^2 E(|F|^2) + \sum_{i,j=1}^{d} E(D_{h_j} F^i D_{h_i} F^j). \tag{4.2}$$

If moreover $k \in L^2(0,1)$ *and* $G \in L^2(\Omega; \mathbb{R}^d)$, *s.t.* $G^i \in \mathbb{D}_{2,k_j}$, $F^i \in \mathbb{D}_{2,k_j}$, $G^i \in \mathbb{D}_{2,h_j} \forall 1 \leqq i, j \leqq d$. *Then:*

$$E[\delta(hF)\delta(kG)] = \langle h, k\rangle E[F.G] + \sum_{i,j=1}^{d} E(D_{k_j} F^i D_{h_i} G^j). \tag{4.3}$$

Proof. Suppose first that $F^i \in \mathscr{S}$, $1 \leqq i \leqq d$. For any $J \in \mathbb{D}_{2,1}$,

$$F^i D_{h_i} J = D_{h_i}(F^i J) - J D_{h_i} F^i. \tag{4.4}$$

From (2.3),

$$E[D_{h_i}(F^i J)] = E[F^i J \delta_i(h)].$$

Therefore

$$|E[F. D_h J]| \leqq c \|J\|_2$$

and from Proposition 3.1, $hF \in \operatorname{Dom}\delta$ and (4.1) follows from (3.3), (4.4) and (2.3). By a similar argument, $hF^i \in \operatorname{Dom}\delta_i$, $\forall i$. Using again (2.3), we obtain:

$$\begin{aligned} E[\delta_i(hF^i)\delta_j(hF^j)] &= E[(F^i\delta_i(h) - D_{h_i}F^i)(F^j\delta_j(h) - D_{h_j}F^j)] \\ &= |h|^2 \delta_{ij} E[(F^i)^2] + E[D_{h_i} D_{h_j}(F^i F^j) \\ &\quad - D_{h_i}(F^i D_{h_j} F^j) - D_{h_j}(F^j D_{h_i} F^i) + D_{h_i} F^i D_{h_j} F^j] \\ &= |h|^2 \delta_{ij} E[(F^i)^2] + E(D_{h_i} F^j D_{h_j} F^i). \end{aligned}$$

(4.2) now follows by summing up with respect to i and j.

Given now

$$F^i \in \bigcap_j \mathbb{D}_{2,h_j}, \qquad 1 \leqq i \leqq d,$$

there exists a sequence $\{F_n^i, n \in \mathbb{N}\} \subset \mathscr{S}$ such that $F_n^i \to F^i$ in $\mathbb{D}_{2,h_j}$, $\forall 1 \leqq j \leqq d$. It follows easily from (4.2) and the fact that δ is closed that $hF \in \operatorname{Dom}\delta$, and (4.1), (4.2) hold. The proof of (4.3) is similar to that of (4.2). □

For four subsets of $L^2([0,1] \times \Omega; \mathbb{R}^d)$, we are going to construct a sequence $u^n \in \operatorname{Dom}\delta$, for which the expression for $\delta(u^n)$ follows from Lemma 4.1, and such that $u^n \to u$ in $L^2([0,1] \times \Omega; \mathbb{R}^d)$. We will then show that $\{\delta(u^n)\}$ is a Cauchy sequence in $L^2(\Omega)$. This will be done in the three first cases by showing that

$$\lim_{n,m\to\infty} E(\delta(u^n)\delta(u^m))$$

exists; let us call that limit χ. Clearly the above implies:

$$E(|\delta(u^n)-\delta(u^m)|^2)\to\chi-2\chi+\chi=0$$

which gives the Cauchy property. It will then follow from the fact that δ is closed that $u\in\operatorname{Dom}\delta$ and $\delta(u)=\lim\delta(u^n)$. Moreover, in addition to the obvious relation $E\delta(u)=0$ (choose $F=1$ in (3.3)), we will obtain $E[\delta(u)^2]=\chi$.

In order to construct the approximations, we will use a sequence $\{\Pi^n, n\in\mathbb{N}\}$ of partitions of $[0,1]$, of the form:

$$0=t_{0,n}<t_{1,n}<\ldots<t_{n,n}=1$$

such that

$$|\Pi^n|=\sup_{0\leq k\leq n-1}(t_{k+1,n}-t_{k,n})\to 0 \quad \text{as} \quad n\to\infty\,.$$

Notice that the convergence (in probability or in L^p, $p\geq 1$) of the approximating sums to a fixed limit for any sequence of partitions of the above type is equivalent to the convergence along the set of all partitions when the norm $|\Pi|$ tends to zero. Given $u\in L^2([0,1]\times\Omega;\mathbb{R}^d)$, we define:

$$\bar{u}_{k,n}=\frac{1}{t_{k+1,n}-t_{k,n}}\int_{t_{k,n}}^{t_{k+1,n}}u_s ds \quad \text{for} \quad 0\leq k<n-1$$

and $\bar{u}_{-1,n}=\bar{u}_{n,n}=0$.

4.1. The Forward Itô Integral

Suppose $u\in L^2([0,1]\times\Omega;\mathbb{R}^d)$, and moreover u_t is $\mathscr{F}_t$ measurable t a.e. We then define:

$$u_t'^n=\sum_{k=0}^{n-1}\bar{u}_{k-1,n}1_{[t_{k,n},t_{k+1,n}[}(t)\,,$$

where we suppose here that $t_{k,n}=k/n$.

Clearly, $u'^n\to u$ in $L^2([0,1]\times\Omega;\mathbb{R}^d)$. Indeed, $u'^n=P_n u$, where P_n is a linear operator in $L^2(0,1;L^2(\Omega;\mathbb{R}^d))$ with norm bounded by one, and $P_n u\to u$ whenever $u\in C([0,1],L^2(\Omega;\mathbb{R}^d))$. The above convergence then follows. On the other hand, u_{k-1} is $\mathscr{F}t_{k,n}$ measurable, and from Lemma 2.4 (i), we can apply Lemma 4.1, so that $u'^n\in\operatorname{Dom}\delta$ and:

$$\delta(u'^n)=\sum_{k=0}^{n-1}\bar{u}_{k-1,n}\cdot(W_{t_{k+1,n}}-W_{t_{k,n}})\,. \tag{4.5}$$

Using the adaptedness of u, we obtain:

$$E[\delta(u'^n)\,\delta(u'^m)]=\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}E(\bar{u}_{k-1,n}\bar{u}_{l-1,m})\,(t_{k+1,n}\wedge t_{l+1,m}-t_{k,n}\vee t_{l,m})^+\,.$$

Finally, it is not hard to show that

$$E[\delta(u'^n)\,\delta(u'^m)]\to E\int_0^1|u_t|^2 dt\,.$$

In this case, $\delta(u)$ is the usual forward Itô integral.

Moreover, if u is a d-dimensional measurable process such that u_t is $\mathscr{F}_t$ measurable t a.e. and $u \in L^2(0,1;\mathbb{R}^d)$ a.s., then $u \in (\operatorname{Dom}\delta)_{\text{loc}}$. This follows from usual arguments concerning Itô's integral. $\delta(u)$ does not depend on the localizing sequence $\{(\Omega_k, u_k)\}$, provided u_k is $\mathscr{F}_t$ adapted $\forall k$, since $\delta(u)$ is the limit in probability of the sequence $\{\delta(u'^n)\}$, where u'^n is again defined as above.

4.2. *The Backward Itô Integral*

Suppose now that $u \in L^2([0,1]\times\Omega;\mathbb{R}^d)$, and u_t is $\mathscr{F}^t$ measurable a.e. We then define:

$$u_t''^n = \sum_{k=0}^{n-1} \bar{u}_{k+1,n} 1_{[t_{k,n}, t_{k+1,n}[}(t)$$

where we suppose again that $t_{k,n} = k/n$.

For reasons which are very similar to the above ones, $u''^n \in \operatorname{Dom}\delta$ and:

$$\delta(u''^n) = \sum_{k=0}^{n-1} \bar{u}_{k+1,n} \cdot (W_{t_{k+1,n}} - W_{t_{k,n}})$$

$$E[\delta(u''^n)\,\delta(u''^m)] = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} E[\bar{u}_{k+1,n} \cdot \bar{u}_{l+1,m}]\,(t_{k+1,n} \wedge t_{l+1,m} - t_{k,n} \vee t_{l,m})^+$$

and again we obtain:

$$E[\delta(u''^n)\,\delta(u''^m)] \to E\int_0^1 |u_t|^2\,dt\,.$$

In this case, $\delta(u)$ is the backward Itô integral, i.e. the forward Itô integral of u_{1-t} with respect to $W_{1-t} - W_1$; see Kunita [9], Pardoux and Protter [19].

Finally, if u is a d-dimensional measurable process such that u_t is $\mathscr{F}^t$ adapted a.e. and $u \in L^2(0,1;\mathbb{R}^d)$, a.s., then $u \in (\operatorname{Dom}\delta)_{\text{loc}}$, and $\delta(u''^n) \to \delta(u)$ in probability, where $\delta(u)$ is defined by any localizing sequence $\{(\Omega_k, u_k)\}$ s.t. u_k is $\mathscr{F}^t$ adapted $\forall k$.

4.3. *The Skorohod Integral of an Element of* $\mathbb{L}_d^{2,1}$

Let now $u \in \mathbb{L}_d^{2,1}$, according to the definition given in Sect. 3. Let us define two approximating sequences:

$$u^n = \sum_{k=0}^{n-1} \bar{u}_{k,n} 1_{[t_{k,n}, t_{k+1,n}[}$$

$$\tilde{u}^n = \sum_{k=0}^{n-1} \tilde{u}_{k,n} 1_{[t_{k,n}, t_{k+1,n}[}$$

where

$$\tilde{u}_{k,n} = E(\bar{u}_{k,n}/\mathscr{F}_{t_{k,n}} \vee \mathscr{F}^{t_{k+1,n}})\,.$$

We have:

Lemma 4.2. *$u^n \to u$ and $\tilde{u} \to u$ in $\mathbb{L}_d^{2,1}$.*

Proof. The first convergence is immediate. In order to prove the second one, let us define for each $n \in N$ the σ-algebra $\mathscr{G}^n$ of subsets of $[0,1]\times\Omega$ generated by the sets

$[t_{k,n}, t_{k+1,n}[\times F_{k,n}$ where $0 \leqq k \leqq n-1$ and

$$F_{k,n} \in \mathscr{F}_{t_{k,n}} \vee \mathscr{F}^{t_{k+1,n}} .$$

$\tilde{u}^n$ is the conditional expectation of u given $\mathscr{G}^n$, which respect to the measure $\lambda \times P$ on $[0,1] \times \Omega$. Therefore, in order to establish the convergence in $L^2([0,1] \times \Omega; \mathbb{R}^d)$, it suffices to show that any square-integrable process $v \in L^2([0,1] \times \Omega)$ orthogonal to all the $\mathscr{G}^n$ must be zero. Such a process verifies

$$\int 1_F 1_I v_t dt dP = 0$$

$$\forall F \in \mathscr{F}_{t_{k,n}} \vee \mathscr{F}^{t_{k+1,n}}$$

and $I \in \Pi^m$, $m \geqq n$ with $I \subset [t_{k,n}, t_{k+1,n}]$. Consequently $E[v_t/\mathscr{F}_{t_{k,n}} \vee \mathscr{F}^{t_{k+1,n}}] = 0$ a.s., t a.e. in $[t_{k,n}, t_{k+1,n}]$. Since $\bigcup_n \Pi^n$ contains a countable number of intervals, the above holds true for any k, n s.t. $t \in [t_{k,n}, t_{k+1,n}]$. This clearly implies that $v = 0$ $\lambda \times P$ a.e. The convergence of the derivative follows from the same argument, once we have used Lemma 2.4. (ii) to compute $D_t \tilde{u}_s^n$. □

The fact that $\tilde{u}^n \in \operatorname{Dom} \delta$ follows from the same argument as those used above, and:

$$\delta(\tilde{u}^n) = \sum_{k=0}^{n-1} \tilde{u}_{k,n} \cdot (W_{t_{k+1,n}} - W_{t_{k,n}}) .$$

The fact that $u^n \in \operatorname{Dom} \delta$ follows from Lemma 4.1, using the fact that $u \in \mathbb{L}_d^{2,1}$, and moreover:

$$\delta(u^n) = \sum_{k=0}^{n-1} \bar{u}_{k,n} \cdot (W_{t_{k+1,n}} - W_{t_{k,n}}) - \sum_{k=0}^{n-1} \frac{1}{t_{k+1,n} - t_{k,n}} \int_{t_{k,n}}^{t_{k+1,n}} \int_{t_{k,n}}^{t_{k+1,n}} D_t \,.\, u_s ds dt$$

where $D_t \,.\, u_s$ stands for

$$\sum_{i=1}^{d} D_t^i u_s^i .$$

Proposition 4.3. *Both sequences* $E(\delta(u^n)\delta(u^m))$ and $E(\delta(\tilde{u}^n)\delta(\tilde{u}^m))$ converge, as $n, m \to \infty$, to

$$E \int_0^1 |u|^2 dt + \sum_{i,j=1}^{d} E \int_0^1 \int_0^1 D_s^i u_t^j D_t^j u_s^i ds dt .$$

Proof. For the sake of notational simplicity, let us replace k, n by k and l, m by l. Define

$$\alpha_{kl} = (t_{k+1} \wedge t_{l+1} - t_k \vee t_l)^+ = \lambda([t_k, t_{k+1}] \cap [t_l, t_{l+1}]) .$$

It follows from (4.3):

$$E[\delta(u^n)\delta(u^m)] = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left[\alpha_{kl} E(\bar{u}_k \bar{u}_l) + \sum_{i,j=1}^{d} E \int_{t_k}^{t_{k+1}} \int_{t_l}^{t_{l+1}} D_r^j \bar{u}_k^i D_s^i \bar{u}_l^j dr ds\right] .$$

The convergence is immediate from Lemma 4.2. The other sequence is treated analogously. □

Remark 4.4. (i) We have established again the isometric identity (3.5). Note that the fact that δ is a linear map from $\mathbb{L}_d^{2,1}$ into $L^2(\Omega)$ such that $E(\delta(u))=0$ and (3.5) are satisfied does completely characterize the random variable $\delta(u)$ for $u\in\mathbb{L}_d^{2,1}$. Indeed, it follows from (3.5) that $\forall u,\ v\in\mathbb{L}_d^{2,1}$,

$$E[\delta(u)\delta(v)]=E\int_0^1 u_t\,.\,v_t dt+\sum_{i,j}\int_0^1\int_0^1 D_s^i u_t^j D_t^j v_s^i ds dt\ . \tag{4.6}$$

For any $h\in H$, define for $t\in[0,1]$

$$X_t(h)=\exp\left(\int_0^t h(s)\,.\,dW_s-\frac{1}{2}\int_0^t |h(s)|^2 ds\right)\ .$$

Then:

$$X_1(h)=1+\int_0^1 X_t(h)h(t)\,.\,dW_t\ .$$

The last integral is a Skorohod integral, since it is an Itô integral (see Sect. 4.1), and moreover $X_{.}(h)h(.)\in\mathbb{L}_d^{2,1}$. Therefore from (4.6), $\forall u\in\mathbb{L}_d^{2,1}$,

$$E[\delta(u)X_1(h)]=E\int_0^1 u_t\,.\,h(t)X_t(h)dt+\sum_{i,j}\int_0^1\int_0^t D_t^i u_s^j h^i(t)h^j(s)X_t(h)ds dt\ .$$

Then the scalar product in $L^2(\Omega)$ of $\delta(u)$ with each $X_1(h)$ is uniquely determined. Since $\{X_1(h), h\in H\}$ is total in $L^2(\Omega)$, this determines $\delta(u)$.

(ii) If $u\in L^2([0,1]\times\Omega;\mathbb{R}^d)$ is either $\mathscr{F}_t$ adapted or $\mathscr{F}^t$ adapted, then $\delta(\tilde{u}^n)\to\delta(u)$ in $L^2(\Omega)$ as well, by the same argument as those used above. It is interesting to note that the same approximating sequence $\delta(\tilde{u}^n)$ converges to $\delta(u)$, in the three cases $u\ \mathscr{F}_t$ adapted, $u\ \mathscr{F}^t$ adapted, and $u\in\mathbb{L}_d^{2,1}$. Moreover, for any $u\in L^2([0,1]\times\Omega;\mathbb{R}^d)$, if $\delta(\tilde{u}^n)$ converge in $L^2(\Omega)$, then $u\in\operatorname{Dom}\delta$ and $\delta(u)=\lim\delta(\tilde{u}^n)$, since δ is a closed operator. □

We now establish the *local property* of the Skorohod integral, when restricted to $\mathbb{L}_d^{2,1}$.

Proposition 4.5. *Let* $u\in\mathbb{L}_d^{2,1}$ *and* $A\in\mathscr{F}$ *such that* $u_t(\omega)=0$, $dt\times dP$ *a.e. on* $[0,1]\times A$. *Then*

$$\int_0^1 u_t\,.\,dW_t=0\quad a.s.\ on\quad A\ .$$

Proof. It suffices to show that $\delta(u^n)=0$ a.s. on A, $\forall n\in\mathbb{N}$, which follows easily from Lemma 2.6. □

Definition 4.6. Let $\mathbb{L}_{d,\mathrm{loc}}^{2,1}$ denote the class of d-dimensional measurable processes u which have the property that there exists a sequence $\Omega_k\uparrow\Omega$ a.s. and a sequence $\{u_k, k\in\mathbb{N}\}\subset\mathbb{L}_d^{2,1}$, such that:

$$u|_{\Omega_k}=u_k|_{\Omega_k}\quad \text{a.s.},\quad \forall k\ .$$

We will then say that u is localized by the sequence $\{(\Omega_k,u_k), k\in\mathbb{N}\}$. □

It follows from Proposition 4.5 that $\mathbb{L}^{2,1}_{d,\mathrm{loc}} \subset (\mathrm{Dom}\,\delta)_{\mathrm{loc}}$, and that for $u \in \mathbb{L}^{2,1}_{d,\mathrm{loc}}$ $\delta(u)$ does not depend on the localizing sequence $\{(\Omega_k, u_k)\}$, provided $u_k \in \mathbb{L}^{2,1}_d, \forall k$.

We finally study some stability properties of $\mathbb{L}^{2,1}_{d,\mathrm{loc}}$ and $\mathbb{L}^{2,1}_d$ under composition with functions.

We will say that a measurable function $\Phi: [0,1] \times \mathbb{R}^{dm} \to \mathbb{R}^d$ belongs to class Λ if $z \to \Phi(t,z)$ is of class C^1 t a.e., and moreover $\Phi(t,z)$ and $\Phi'_z(t,z)$ are bounded on bounded subsets of $[0,1] \times \mathbb{R}^{dm}$.

Proposition 4.7. *Let $\{u^i_t\}$, $1 \leqq i \leqq m$ be continuous processes belonging to $\mathbb{L}^{2,1}_{d,\mathrm{loc}}$, and $\Phi \in \Lambda$. Then $v_t = \Phi(t, u_t)$ belongs to $\mathbb{L}^{2,1}_{d,\mathrm{loc}}$.*

Proof. For $k \geqq 1$, define

$$A_k = \left\{ \sup_{0 \leqq t \leqq 1} |u_t| \leqq k \right\} .$$

For each i, u^i is localized by $\{(\Omega^i_k, u^i_k)\}$. Define $\Omega_k = A_k \cap \Omega^1_k \cap \ldots \cap \Omega^m_k$. Clearly, $\Omega_k \uparrow \Omega$ a.s. Let $f = \mathbb{R}^{dm} \to [0,1]$ be a smooth function with compact support, such that $f(x) = 1$ whenever $|x| \leqq 1$; and $f_k(x) = f(x/k)$. We define:

$$v_k(t) = \Phi(t, u_k(t)) f_k(u_k(t)) .$$

Clearly $v_k = v$ on Ω_k, and since $\Phi(t,z) f_k(z)$ is bounded with bounded derivative with respect to z, it will follow from the next proposition that $v_k \in \mathbb{L}^{2,1}_d$. □

Proposition 4.8. *Let $u^i \in \mathbb{L}^{2,1}_d$, $1 \leqq i \leqq m$, and $\Phi \in \Lambda$. Each of the following conditions implies that $v_t = \Phi(t, u_t)$ is an element of $\mathbb{L}^{2,1}_d$:*

(i) *Φ and Φ'_z are bounded*

(ii) *$\exists a \geqq 1$, $p > 1$ and $K > 0$ s.t.:*

(ii$_1$) $|\Phi(t,z)| + |\Phi'_z(t,z)| \leqq K(1 + |z|^a)$

(ii$_2$) $E \int_0^1 |u_t|^{2ap} dt < \infty$

(ii$_3$) $E \int_0^1 \left(\int_0^1 |D_s u_t|^2 ds \right)^q dt < \infty$, *where* $1/p + 1/q = 1$.

Proof. The fact that $v \in L^2([0,1] \times \Omega; \mathbb{R}^d)$ follows from either (i) or (ii$_1$) + (ii$_2$). The fact that t a.e. $v_t \in \mathbb{D}_{2,1}$ and

$$E \int_0^1 \int_0^1 \|D_s v_t\|^2 ds\, dt < \infty$$

follows easily under condition (i). Under condition (ii), using Proposition 2.9, we obtain, restricting ourself for simplicity to the case $d = 1$,

$$E \int_0^1 \int_0^1 |D_s v_t|^2 ds\, dt = E \int_0^1 \int_0^1 \left| \sum_{i=1}^m \Phi'_{z_i}(t, u_t) D_s u^i_t \right|^2 ds\, dt$$

$$\leqq 2^{m-1} K^2 \sum_{i=1}^m E \int_0^1 (1 + |u_t|^a)^2 \int_0^1 |D_s u^i_t|^2 ds\, dt .$$

The fact that the last quantity is finite follows readily from Hölder's inequality, $(\mathrm{ii}_2)+(\mathrm{ii}_3)$. □

4.4. Another Class of Skorohod Integrable Processes

In order to simplify the notations, we will restrict ourselves in this subsection to the case $d=1$. Let us now indicate our motivation for what follows. Suppose we have a process $u_t(x)$, parametrized by $x \in \mathbb{R}^p$, which belongs to $L^2([0,1]\times\Omega)$ and is $\mathscr{F}_t$-adapted, $\forall x \in \mathbb{R}^p$. We then can define the forward Itô integral

$$\int_0^1 u_t(x)dW_t\ , \qquad \forall x \in \mathbb{R}^p\ .$$

Suppose now that the resulting random field is a.s. continuous w.r. to x, and let θ be a p-dimensional random vector. We then can "evaluate the stochastic integral at $x=\theta$", i.e. consider the random variable:

$$\int_0^1 u_t(x)dW_t|_{x=\theta}\ .$$

A natural question, which was raised to us by P. Priouret is then: under which conditions is the (non-adapted) process $\{u_t(\theta)\}$ Skorohod integrable, and does then $\delta(u(\theta))$ coincide with the above random variable?

We will now show that provided u is C^1 in x, and θ belongs to a certain Sobolev space, we do not need any smoothness of $u(.,.,x)$ for fixed x, in order for $u(\theta)$ to belong to $\mathrm{Dom}\,\delta$, and we will compare $\delta(u(\theta))$ with $\delta(u(x))|_{x=\theta}$.

We first suppose that $u(t,\omega,x)$ is a real valued measurable function defined on $[0,1]\times\Omega\times D$, where D is a given open and bounded subset of $\mathbb{R}^p$. We make the following hypotheses:

(H1) $(t,\omega)\to u(t,\omega,x)$ is $\mathscr{F}_t$ progressively measurable, $\forall x$.

(H2) $x\to u(t,\omega,x)$ is of class C^1, $\forall t,\omega$.

As usual, we will from now on omit the variable ω, and write $u(t,x)$ for $u(t,\omega,x)$. We will write $u'(t,x)$ for the gradient of u with respect to x.

(H3) $E\int_0^1 \sup_{x\in D} |u'(t,x)|^4 dt<\infty$

(H4) $\exists q>p$ s.t. $q\geqq 2$ and $E\int_D\int_0^1 (|u(t,x)|^q+|u'(t,x)|^q)dt\,dx<\infty$

(Recall that p is the dimension of x)

We are finally given a D-valued random vector θ, s.t. $\theta^i \in \mathbb{D}_{4,1}$, $1\leqq i\leqq p$.

We define as before, assuming again that $t_{k,n}=k/n$:

$$\bar{u}_{k,n}(x)=\frac{1}{t_{k+1,n}-t_{k,n}}\int_{t_{k,n}}^{t_{k+1,n}} u_s(x)ds$$

for $0\leqq k\leqq n-1$, $\bar{u}_{-1,n}(x)=0$, and:

$$u_t^n(x)=\sum_{i=0}^{n-1} \bar{u}_{k-1,n}(x)\,1_{[t_{k,n},t_{k+1,n}[}(t)\ .$$

It follows from (H2) and (H3) that $x \to \bar{u}_{k,n}(x)$ is of class C^1; we will denote by $\bar{u}'_{k,n}(x)$ its gradient with respect to x. Let us define

$$h_k = 1_{[t_{k,n}, t_{k+1,n}[} \ .$$

Lemma 4.9. $\forall k \leq n-1$,

$$\bar{u}_{k-1,n}(\theta) \in \mathbb{D}_{2,h_k}$$

and moreover

$$D_{h_k}(\bar{u}_{k-1,n}(\theta)) = \bar{u}'_{k-1,n}(\theta) \,.\, D_{h_k}\theta \ ,$$

where "." means the scalar product between the gradient of $\bar{u}$ *and the vector* $(D_h\theta^1, \ldots, D_h\theta^p)'$.

Proof. For simplicity, we drop the indices k, n, and we assume that $p=1$. Let us first suppose that $\theta \in \mathscr{S}$, which implies that:

$$\varepsilon^{-1}\left[\theta\left(\omega + \varepsilon \int_0^{\cdot} h ds\right) - \theta(\omega)\right] \to D_h\theta \quad \text{in } L^4(\Omega) \ .$$

On the other hand, since $\bar{u}(x)$ is $\mathscr{F}_{t_{k,n}}$-measurable $\forall x$,

$$\bar{u}\left(\omega + \varepsilon \int_0^{\cdot} h ds,\ \theta\left(\omega + \varepsilon \int_0^{\cdot} h ds\right)\right) = \bar{u}\left(\omega, \theta\left(\omega + \varepsilon \int_0^{\cdot} h ds\right)\right)$$

and using again (H3), we obtain that

$$\varepsilon^{-1}\left[\bar{u}(\theta)\left(\omega + \varepsilon \int_0^{\cdot} h ds\right) - \bar{u}(\theta)(\omega)\right] \to \bar{u}'(\theta) D_h\theta \quad \text{in } L^2(\Omega) \ ,$$

which implies the lemma in the particular case where $\theta \in \mathscr{S}$. In the general case, let $\{\theta_m\}$ be a sequence in $\mathscr{S}$ which converges to θ in $\mathbb{D}_{4,1}$. Then

$$\bar{u}(\theta_m) \to \bar{u}(\theta) \quad \text{in} \quad L^2(\Omega)$$

$$\bar{u}(\theta_m) D_h\theta_m \to \bar{u}(\theta) D_h\theta \quad \text{in} \quad L^2(\Omega) \ .$$

The result follows from the fact that D_h is a closed operator. □

It follows from Lemma 4.1 and Lemma 4.9 that $u^n(\theta) \in \operatorname{Dom}\delta$ and, dropping the index n on the right side, with the convention $t_{-1}=0$:

$$\delta(u^n((\theta)) = \sum_{k=0}^{n-1} \bar{u}_{k-1}(\theta)(W_{t_{k+1}} - W_{t_k}) - \sum_{k=0}^{n-1} (t_k - t_{k-1})^{-1} \int_{t_k}^{t_{k+1}} \int_{t_{n-1}}^{t_k} u'(s,\theta) \,.\, D_r\theta ds dr \ .$$

Let us first prove:

Lemma 4.10. *The random fields* $\delta(u^n(x))$ *and* $\delta(u(x))$ *have* a.s. *continuous modifications which satisfy:*

$$\sup_{x \in D} |\delta(u(x)) - \delta(u^n(x))| \to 0 \quad \text{in} \quad L^2(\Omega) \ .$$

Proof. We use the technique and results in Kunita ([8], Sect. 6). Let q be the index appearing in (H4). We denote by $W^{q,1}(D)$ the usual Sobolev space of real-valued functions defined on D which, together with their first-order distributional derivatives, belong to $L^q(D)$. Since $q>p$, it follows that $W^{q,1}(D)\subset C(D)$, and moreover $\exists c$ s.t. $\forall f\in W^{q,1}(D)$,

$$\sup_{x\in D}|f(x)|\leqq c\|f\|_{q,1}$$

where

$$\|f\|_{q,1}=\|f\|_{L^q(D)}+\sum_{i=1}^{p}\|\partial f/\partial x_i\|_{L^q(D)}\ .$$

(H4) means that $u\in L^q([0,1]\times\Omega;\, W^{q,1}(D))$, and clearly the same is true for u^n, and $u^n\to u$ in $L^q([0,1]\times\Omega;\, W^{q,1}(D))$. Then Lemma 6.4 in Kunita [8] implies that $\delta(u(x))$ and $\delta(u^n(x))$ have modifications which belong to $L^q(\Omega;\, W^{q,1}(D))$, and

$$E(\|\delta(u(.))-\delta(u^n(.))\|_{q,1}^q)\to 0\ ,\qquad \text{as}\quad n\to\infty\,.$$

The result follows from Sobolev's embedding theorem. □

We can now prove:

Proposition 4.11. *$\delta(u^n(\theta))$ converges in $L^2(\Omega)$ to*

$$\int_0^1 u(t,x)dW_t|_{x=\theta}-\int_0^1 u'(t,\theta)\,.\,D_t\theta dt\,.$$

Proof. The convergence of the first term follows from Lemma 4.10. We now establish the convergence of the second term.

$$E\left(\left|\int_0^1[(u^n)'(t,\theta)-u'(t,\theta)]\,.\,D_t\theta dt\right|^2\right)$$
$$\leqq\left\{E\left[\left(\int_0^1|D_t\theta|^2dt\right)^2\right]\int_0^1 E[|u'(t,\theta)-(u^n)'(t,\theta)|^4]dt\right\}^{1/2}\,.$$

Clearly, the right side tends to zero as $n\to\infty$. □

Let us now localize the result which we have just proved.

Proposition 4.12. *Suppose u is a real valued measurable function defined on $[0,1]\times\Omega\times\mathbb{R}^p$ which satisfies* (H1) *for any $x\in\mathbb{R}^p$,* (H2) *and* (H3)–(H4) *for any bounded open subset D in $\mathbb{R}^p$. Let θ be a d-dimensional random vector s.t. $\theta^i\in\mathbb{D}_{4,1,\text{loc}}$, $1\leqq i\leqq p$.*

Then $u(\theta)\in(\text{Dom}\,\delta)_{\text{loc}}$ and $\delta(u(\theta))$ can be defined as:

$$\delta(u(\theta))=\int_0^1 u(t,x)dW_t|_{x=\theta}-\int_0^1 u'(t,\theta)\,.\,D_t\theta dt\,.$$

Proof. Let us assume $p=1$ for notational convenience. In the case where $\theta\in\mathbb{D}_{4,1}$ and takes values is some bounded open set D, the result follows from Proposition 4.11 and the fact that δ is closed. Suppose now that $\{(\Omega_k,\theta_k)\}$ localizes θ. For $k\geqq 1$, let φ_k be a smooth mapping from $\mathbb{R}$ into $\mathbb{R}$, such that $\varphi_k(x)=x$ whenever $|x|\leqq k$ and $\varphi_k(x)=0$ whenever $|x|\geqq k+1$. Define $\bar{\Omega}_k=\Omega_k\cap\{|\theta_k|\leqq k\}$, $\bar{\theta}_k=\varphi_k(\theta_k)$. Then

$\{(\bar{\Omega}_k, \bar{\theta}_k)\}$ localizes θ, and moreover $\bar{\theta}_k$ takes values in a bounded set. We then know that:

$$\delta(u(\bar{\theta}_k)) = \delta(u(x))|_{x=\bar{\theta}_k} - \int_0^1 u'(t, \bar{\theta}_k) D_t \bar{\theta}_k dt .$$

It follows from this relation that whenever $l<k$,

$$\delta(u(\bar{\theta}_k))|_{\bar{\Omega}_l} = \delta(u(\bar{\theta}_l))|_{\bar{\Omega}_l} .$$

The result follows from this, and again our last equality. □

5. The Skorohod Integral as a Process

We will restrict ourselves in this section to integrands belonging to $\mathbb{L}_d^{2,1}$, see Definition 3.3. Note that if $u \in \mathbb{L}_d^{2,1}$, resp. $\mathbb{L}_{d,\text{loc}}^{2,1}$, $t \in [0,1]$, then $u 1_{[0,t]} \in \mathbb{L}_d^{2,1}$, resp. $\mathbb{L}_{d,\text{loc}}^{2,1}$. We then define the process

$$\left\{\int_0^t u_s . dW_s,\ t \in [0,1]\right\}$$

by:

$$\int_0^t u_s . dW_s = \delta(u 1_{[0,t]}) .$$

This process is clearly mean-square continuous and then measurable. It does not have any type of martingale property, for lack of adaptedness. Nevertheless, it has the following property:

Proposition 5.1. *Let $u \in \mathbb{L}_d^{2,1}$ and $0 \leqq s < t \leqq 1$. Then we have:*

(i) $E\left(\int_s^t u_r . dW_r / \mathscr{F}_s \vee \mathscr{F}^t\right) = 0$

(ii) $E\left[\left(\int_s^t u_r . dW_r\right)^2 / \mathscr{F}_s \vee \mathscr{F}^t\right] = E\left[\int_s^t |u_r|^2 dr + \sum_{i,j=1}^d \int_s^t \int_s^t D_\alpha^i u_r^j D_r^j u_\alpha^i dr\, d\alpha / \mathscr{F}_s \vee \mathscr{F}^t\right].$

Proof. For simplicity, we suppose that $d=1$. We first prove (i). For any $F \in \mathbb{D}_{2,1}$ which is $\mathscr{F}_s \vee \mathscr{F}^t$ measurable, $D_r F = 0$ for almost all $r \in [s,t]$. For such an F, using Proposition 3.1, we obtain:

$$E\left(F \int_s^t u_r dW_r\right) = E \int_s^t u_r D_r F dr = 0 .$$

We now prove (ii). It suffices to prove (ii) for $u \in \mathbb{L}^{2,2}$, which we now assume. It then follows that

$$\int_s^t u_r dW_r \in \mathbb{D}_{2,1}$$

and we can use Proposition 3.4 in order to compute its derivative. Let now F be an $\mathscr{F}_s \vee \mathscr{F}^t$-measurable element of $\mathscr{S}$. We then have, using repeatedly Proposition 3.1:

$$E\left[F\left(\int_s^t u_r dW_r\right)^2\right] = E\left[F\int_s^t u_r D_r\left(\int_s^t u_\alpha dW_\alpha\right)dr\right]$$

$$= E\left[F\left(\int_s^t u_r^2 dr + \int_s^t u_r\left(\int_s^t D_r u_\alpha dW_\alpha\right)dr\right)\right]$$

$$= E\left[F\left(\int_s^t u_r^2 dr + \int_s^t\int_s^t D_\alpha u_r D_r u_\alpha dr\, d\alpha\right)\right],$$

which proves the result. □

We now give a sufficient condition for the existence of an a.s. continuous modification:

Theorem 5.2. *Let $u \in \mathbb{L}_d^{2,1}$. Then each one of the following conditions implies that the process*

$$\left\{\int_0^t u_s . dW_s,\ t \in [0,1]\right\}$$

has an a.s. continuous modification:

(i) $\exists p > 1$ *s.t.* $\sup_{t\in[0,1]} E\left[\left(\int_0^1 \|D_s u_t\|^2 ds\right)^p\right] < \infty$.

(ii) $\exists p > 2$ *s.t.* $E\int_0^1\left(\int_0^1 \|D_s u_t\|^2 ds\right)^p dt < \infty$.

Proof. Clearly, the process

$$\int_0^t E(u_s) . dW_s$$

has a continuous modification. Let us define $v_t = u_t - E(u_t)$. Since obviously $D_s v_t = D_s u_t$, it follows from (3.7) and Hölder's inequality that for $q \geqq 2$:

$$E\left(\left|\int_s^t v_r . dW_r\right|^q\right) \leqq c_q E\left[\left(\int_s^t\int_0^1 \|D_\alpha u_r\|^2 d\alpha\, dr\right)^{q/2}\right]$$

$$\leqq c_q (t-s)^{\frac{q}{2}-1}\int_s^t E\left[\left(\int_0^1 \|D_\alpha u_r\|^2 d\alpha\right)^{q/2}\right]dr$$

$$\leqq c_q (t-s)^{q/2}\sup_{r\in[0,1]} E\left[\left(\int_0^1 \|D_\alpha u_r\|^2 d\alpha\right)^{q/2}\right].$$

Clearly, either (i) or (ii) permits us to use Kolmogorov's lemma in order to conclude that

$$\left\{\int_0^t v_s . dW_s,\ t \in [0,1]\right\}$$

possesses a continuous modification. □

Let us now prove the same result under slightly different hypotheses. Recall the Definition 3.3 of the space $\mathbb{L}_d^{2,2}$.

Theorem 5.3. *Let* $u \in \mathbb{L}_d^{2,2}$ *satisfy:*

(i) $\sup_{s,t \in [0,1]} \left[\|E(D_s u_t)\| + E \int_0^1 \|D_s D_r u_t\|^2 dr \right] < \infty$

as well as either

(ii) $\exists p > 2$ s.t. $\sup_{t \in [0,1]} E(|u_t|^p) < \infty$

or

(ii′) $\exists p > 4$ *s.t.* $E \int_0^1 |u_t|^p dt < \infty$.

Then the process

$$\left\{ \int_0^t u_s . dW_s ,\ t \in [0,1] \right\}$$

has an a.s. continuous modification.

Proof. We assume again for simplicity that $d=1$. First note that from Lemma 2.5, (i) implies that:

$$\sup_{s,t} E(|D_s u_t|^2) < \infty .$$

We have the following decomposition:

$$\int_s^t u_r dW_r = \int_s^t E(u_r / \mathscr{F}_s \vee \mathscr{F}^t) dW_r$$

$$+ \int_s^t [u_r - E(u_r / \mathscr{F}_s \vee \mathscr{F}^t)] dW_r = \xi + \theta .$$

ξ being an ordinary Itô integral, we obtain from Bukholder-Gundy, Hölder and Jensen's inequalities,

$$E(|\xi|^p) \leqq c_p (t-s)^{\frac{p}{2}-1} E \int_s^t |u_r|^p dr .$$

It then follows both from (ii) or from (ii′) that there exists $p > 2$ and $\varepsilon > 0$ s.t.:

$$E(|\xi|^p) \leqq c'_p (t-s)^{1+\varepsilon} . \tag{5.1}$$

On the other hand, from Proposition A.1,

$$\theta = \int_s^t \int_s^t E(D_\alpha u_r / \mathscr{F}_\alpha \vee \mathscr{F}^t) dW_\alpha dW_r$$

$$E(\theta^2) = \int_s^t \int_s^t E[E(D_\alpha u_r / \mathscr{F}_\alpha \vee \mathscr{F}^t)^2] d\alpha dr$$

$$+ E \int_s^t \int_s^t v(r, \beta) v(\beta, r) d\beta dr$$

where

$$v(r,\beta)=\int_\beta^t E(D_\beta D_\alpha u_r/\mathscr{F}_\alpha \vee \mathscr{F}^t)dW_\alpha + E(D_\beta u_r/\mathscr{F}_\beta \vee \mathscr{F}^t).$$

It then follows from (i) that:

$$E(\theta^2) \leqq c(t-s)^2 \tag{5.2}$$

The result follows from (5.1) and (5.2), using the same argument as in Pardoux and Protter ([19], Theorem 4.3). □

We compute next the quadratic variation of the process

$$\left\{\int_0^t u_s . dW_s,\ 0 \leqq t \leqq 1\right\}.$$

Let again $\{\Pi^n\}$ denote a sequence of partitions of $[0,1]$ such that $|\Pi^n| \to 0$ as$\to\infty$, as defined at the beginning of Sect. 4.

Theorem 5.4. *Let $u \in \mathbb{L}^{2,1}_{\text{loc}}$. Then $\forall 1 \leqq i,j \leqq d$,*

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} u_s dW_s^i \int_{t_k}^{t_{k+1}} u_s dW_s^j \to \delta_{ij} \int_0^1 u_s^2 ds$$

in probability, as $n \to \infty$.

Proof. Let us first consider the case $i=j$, and drop the index i. Let $u, v \in \mathbb{L}^{2,1}$. Then:

$$\begin{aligned} &E\sum_k \left|\left(\int_{t_k}^{t_{k+1}} u_s dW_s\right)^2 - \left(\int_{t_k}^{t_{k+1}} v_s dW_s\right)^2\right| \\ &\quad \leqq \left(E\sum_k \left(\int_{t_k}^{t_{k+1}} (u_s - v_s) dW_s\right)^2\right)^{1/2} \left(E\sum_k \left(\int_{t_k}^{t_{k+1}} (u_s + v_s) dW_s\right)^2\right)^{1/2} \\ &\quad \leqq \|u-v\|_{L^{2,1}} \times \|u+v\|_{L^{2,1}}. \end{aligned}$$

It follows from this estimate that it suffices to prove the result in case $u \in \mathbb{L}^{2,1} \cap L^4([0,1]\times\Omega)$.

Choosing now $v=u^n$ in the last estimate (u^n is defined as in Sect. 4.3), we conslude that:

$$E\sum_k \left|\left(\int_{t_k}^{t_{k+1}} u_s dW_s\right)^2 - \left(\int_{t_k}^{t_{k+1}} u_s^n dW_s\right)^2\right| \to 0$$

as $n \to \infty$. It then remains to consider:

$$\begin{aligned} \sum_k \left(\int_{t_k}^{t_{k+1}} u_s^n dW_s\right)^2 &= \sum_k \left\{\frac{W_{t_{k+1}} - W_{t_k}}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} u_s ds - \frac{1}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} D_r u_s ds dr\right\}^2 \\ &= \sum_k (a_{k,n}^2 + b_{k,n}^2 - 2a_{k,n} b_{k,n}) \end{aligned}$$

where

$$a_{k,n}=\frac{W_{t_{k+1}}-W_{t_k}}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}} u_s ds$$

$$b_{k,n}=\frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} D_r u_s ds dr$$

$$\sum_k b_{k,n}^2 \leqq \sum_k \int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} |D_r u_s|^2 ds dr$$

and the last term tends to zero in $L^1(\Omega)$, as $n\to\infty$.

$$\sum_k a_{k,n}^2=\sum_{k=0}^{n-1}\frac{(W_{t_{k+1}}-W_{t_k})^2}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}} (u_s^n)^2 ds .$$

Since $(u^n)^2\to u^2$ in $L^2([0,1]\times\Omega)$, it follows from an obvious modification of the first part of Lemma C1 (see Appendix C) that:

$$\sum_k a_{k,n}^2 \to \int_0^1 u_s^2 ds \quad \text{in} \quad L^1(\Omega).$$

Finally

$$\left|\sum_k a_{k,n} b_{k,n}\right| \leqq \left(\sum_k a_{k,n}^2\right)^{1/2}\left(\sum_k b_{k,n}^2\right)^{1/2}$$

and the latter tends to zero in $L^1(\Omega)$, as $n\to\infty$.

The proof for $i\neq j$ is similar, the only serious difference being the use of the second part of Lemma C1, instead of its first part. □

It follows readily from Proposition 3.4:

Proposition 5.5. *Let $u\in \mathbb{L}_d^{2,2}$. Then for any $0\leqq\alpha<\beta\leqq 1$,*

$$\int_\alpha^\beta u_s . dW_s \in \mathbb{D}_{2,1},$$

and:

$$D_t^i \int_\alpha^\beta u_s . dW_s = \int_\alpha^\beta D_t^i u_s . dW_s + u_t^i 1_{[\alpha,\beta]}(t), \ t \text{ a.e.} \quad \square$$

6. The Itô Formula

The aim of this section is to prove a chain rule which generalizes the Itô formula. For the sake of clarity, we first state and prove a one-dimensional result. Recall the Definition 3.3 of $\mathbb{L}_d^{2,2}$.

Theorem 6.1. *Suppose $d=1$. Let $\Phi=\mathbb{R}^2\to\mathbb{R}$ be a continuous function, such that the derivatives Φ'_x, Φ'_y, Φ''_{yx} and Φ''_{yy} exist and are continuous, and moreover let*

(i) $\left[\begin{array}{l} u \text{ be an element of } \mathbb{L}^{2,2} \text{ s.t. there exists } p>4 \text{ with} \\ \int_0^1\int_0^1 |E(D_s u_t)|^p ds dt + E\int_0^1\int_0^1\int_0^1 |D_r D_s u_t|^p dr ds dt<\infty \end{array}\right.$

(ii) $\left[\begin{array}{l}\{V_t,\ t\in[0,1]\} \text{ be a continuous process with a.s. finite variation belonging to } \mathbb{L}^{2,1}, \text{ s.t.}\\ E\int_0^1\int_0^1 (D_s V_t)^4 ds dt<\infty \text{ and the mapping } t\to D_s V_t \text{ is continuous}\\ \text{with values in } L^4(\Omega), \text{ uniformly with respect to } s.\end{array}\right.$

Then for any $t\in[0,1]$, *with the notation*

$$U_t=\int_0^t u_s dW_s$$

we have the following:

$$\Phi(V_t, U_t)=\Phi(V_0,0)+\int_0^t \Phi_x'(V_s, U_s)dV_s$$
$$+\int_0^t \Phi_y'(V_s, U_s)u_s dW_s + \frac{1}{2}\int_0^t \Phi_{yy}''(V_s, U_s)u_s^2 ds$$
$$+\int_0^t [\Phi_{yx}''(V_s, U_s)D_s V_s+\Phi_{yy}''(V_s, U_s)\int_0^s D_s u_r dW_r]u_s ds.$$

Corollary 6.2. *The conclusion of Theorem 6.1 remains valid if, the hypotheses concerning* Φ *remaining unchanged, and assuming that the process* $\{U_t, t\in[0,1]\}$ *is a.s. continuous, we replace* (i) *by:*

(i′) $u\in\mathbb{L}^{2,2}\cap L^\infty([0,1]\times\Omega)$

and (ii) *by either*

(ii′) $\{V_t, t\in[0,1]\}$ *is a continuous process with a.s. finite variation belonging to* $\mathbb{L}^{2,1}$, *and s.t.* $t\to D_s V_t$ *is continuous with values in* $L^2(\Omega)$, *uniformly with respect to s*

or

(ii″) V_t *is a.s. absolutely continuous,* $V_0\in\mathbb{D}_{2,1}$ *and* $\frac{dV_t}{dt}\in\mathbb{L}^{2,1}$.

Moreover, we may drop the requirement that $\{U_t\}$ *is a.s. continuous, provided we assume that the derivatives of* Φ *are bounded.* □

Remark 6.3. (i) A new term appears in the Itô formula. Note that this term does cancel when both V and u are $\mathcal{F}_t$ adapted. Indeed, in that case, $D_s u_r=0$ for $s>r$, and $D_s V_s=0$, since $D_s V_r=0$ for $s>r$, and $r\to D_s V_r$ is continuous.

(ii) The hypotheses under which the chain rule is proved in Sekiguchi and Shiota [21] are those at the end of our Corollary 6.2. □

Proof of Theorem 6.1. From Lemma 2.5, (i) implies that

$$E\int_0^1\int_0^1 |D_s v_t|^p ds dt<\infty,$$

where $v_t=u_t-E(u_t)$, which implies that $\{U_t\}$ has an a.s. continuous modification, which we will choose from now on. It is easily seen that the hypotheses of the theorem imply that the Itô formula makes sense; in particular the integrands of the Skorohod integrals belong to $\mathbb{L}^{2,1}_{\text{loc}}$.

Using the localization argument, it suffices to establish the Itô formula for functions Φ such that Φ and the derivatives Φ'_x, Φ'_y, Φ''_{yx}, Φ''_{yy} are bounded. Let $\{\Pi^n, n\in\mathbb{N}\}$ be a refining sequence of partitions of $[0,t]$ of the form $\Pi^n=\{0=t_{0,n}<t_{1,n}<\ldots<t_{n,n}=t\}$, with $|\Pi^n|=\sup(t_{i+1,n}-t_{i,n})\to 0$, as $n\to\infty$. As usual, we write t_i for $t_{i,n}$.

$$\begin{aligned}\Phi(V_t,U_t)&=\Phi(V_0,U_0)+\sum_{i=0}^{n-1}[\Phi(V_{t_{i+1}},U_{t_{i+1}})-\Phi(V_{t_i},U_{t_i})]\\&=\Phi(V_0,U_0)+\sum_{i=0}^{n-1}[\Phi(V_{t_{i+1}},U_{t_{i+1}})-\Phi V_{t_i},U_{t_{i+1}})]\\&\quad+\sum_{i=0}^{n-1}[\Phi(V_{t_i},U_{t_{i+1}})-\Phi(V_{t_i},U_{t_i})]\ .\end{aligned}$$

We can write

$$\sum_{i=0}^{n-1}[\Phi(V_{t_{i+1}},U_{t_{i+1}})-\Phi(V_{t_i},U_{t_{i+1}})]=\sum_{i=0}^{n-1}\Phi'_x(\bar V_i,U_{t_{i+1}})(V_{t_{i+1}}-V_{t_i}),$$

where $\bar V_i$ is a random intermediate point between V_{t_i} and $V_{t_{i+1}}$.

It follows easily from the continuity of Φ'_x and that of the processes V_t and U_t that:

$$\sum_{i=0}^{n-1}[\Phi(V_{t_{i+1}},U_{t_{i+1}})-\Phi(V_{t_i},U_{t_{i+1}})]\to\int_0^t\Phi'_x(V_s,U_s)dV_s,\tag{6.1}$$

a.s., as $n\to\infty$.

On the other hand we have

$$\begin{aligned}\sum_{i=0}^{n-1}[\Phi(V_{t_i},U_{t_{i+1}})-\Phi(V_{t_i},U_{t_i})]&=\sum_{i=0}^{n-1}\Phi'_y(V_{t_i},U_{t_i})(U_{t_{i+1}}-U_{t_i})\\&\quad+\frac{1}{2}\sum_{i=0}^{n-1}\Phi''_{yy}(V_{t_i},\bar U_i)(U_{t_{i+1}}-U_{t_i})^2,\end{aligned}$$

where $\bar U_i$ is a random intermediate point between U_{t_i} and $U_{t_{i+1}}$.

It follows immediately from the continuity of Φ''_{yy} and the quadratic variation result (Theorem 5.3), using Lemma C2 in Appendix C that:

$$\sum_{i=0}^{n-1}\Phi''_{yy}(V_{t_i},U_i)(U_{t_{i+1}},U_{t_i})^2\to\int_0^t\Phi''_{yy}(V_s,U_s)u_s^2ds,\tag{6.2}$$

in probability, as $n\to\infty$.

Now, from Proposition 3.2 and 5.5 we have

$$\begin{aligned}\Phi'_y(V_{t_i},U_{t_i})\int_{t_i}^{t_{i+1}}u_sdW_s&=\int_{t_i}^{t_{i+1}}\Phi'_y(V_{t_i},U_{t_i})u_sdW_s\\&\quad+\int_{t_i}^{t_{i+1}}D_s[\Phi'_y(V_{t_i},U_{t_i})]u_sds,\end{aligned}$$

and

$$\int_{t_i}^{t_{i+1}} D_s[\Phi'_y(V_{t_i}, U_{t_i})]u_s ds$$

$$= \int_{t_i}^{t_{i+1}} \Phi''_{yx}(V_{t_i}, U_{t_i}) D_s V_{t_i} u_s ds + \int_{t_i}^{t_{i+1}} \Phi''_{yy}(V_{t_i}, U_{t_i}) D_s U_{t_i} u_s ds$$

$$= \int_{t_i}^{t_{i+1}} \Phi''_{yx}(V_{t_i}, U_{t_i}) D_s V_{t_i} u_s ds + \int_{t_i}^{t_{i+1}} \Phi''_{yy}(V_{t_i}, U_{t_i}) \left(\int_0^{t_i} D_s u_r dW_r\right) u_s ds .$$

Moreover:

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \Phi''_{yy}(V_{t_i}, U_{t_i}) \left(\int_0^{t_i} D_s u_r dW_r\right) u_s ds$$

$$\to \int_0^t \Phi''_{yy}(V_s, U_s) \left(\int_0^s D_s u_r dW_r\right) u_s ds, \tag{6.3}$$

in probability, as $n \to \infty$. Indeed,

$$\left|\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[\Phi''_{yy}(V_{t_i}, U_{t_i}) \int_0^{t_i} D_s u_r dW_r - \Phi''_{yy}(V_s, U_s) \int_0^s D_s u_r dW_r\right] u_s ds\right|$$

$$\leqq \left|\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \Phi''_{yy}(V_{t_i}, U_{t_i}) \left(\int_{t_i}^{s} D_s u_r dW_r\right) u_s ds\right|$$

$$+ \left|\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[\Phi''_{yy}(V_{t_i}, U_{t_i}) - \Phi''_{yy}(V_s, U_s) \left(\int_0^s D_s u_r dW_r\right)\right] u_s ds\right|$$

$$\leqq \|\Phi''_{yy}\|_\infty \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left|\int_{t_i}^{s} D_s u_r dW_r\right| |u_s| ds$$

$$+ \sup_i \sup_{s \in [t_i, t_{i+1}]} \left|\Phi''_{yy}(V_{t_i}, U_{t_i}) - \Phi''_{yy}(V_s, U_s)\right| \int_0^t \left|u_s \int_0^s D_s u_r dW_r\right| ds. \tag{6.4}$$

The mathematical expectation of the first term in (6.4) is bounded by

$$\|\Phi''_{yy}\|_\infty \left\{\left(E \int_0^1 u_s^2 ds\right) \left(E \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} |D_s u_r|^2 dr ds \right.\right.$$

$$\left.\left. + E \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \int_{t_i}^{s} |D_\theta D_s u_r|^2 dr d\theta ds\right)\right\}^{1/2}$$

which tends to zero as $n \to \infty$, because $u \in \mathbb{L}^{2,2}$. The second summand of (6.4) converges a.s. to zero by continuity.

Using a similar argument we can prove that

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \Phi''_{yx}(V_{t_i}, U_{t_i}) D_s V_{t_i} u_s ds$$

$$\to \int_0^t \Phi''_{yx}(V_s, U_s) D_s V_s u_s ds, \tag{6.5}$$

in probability as $n \to \infty$.

Indeed, we have

$$\left|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}[\Phi''_{yx}(V_{t_i},U_{t_i})D_sV_{t_i}-\Phi''_{yx}(V_s,U_s)D_sV_s]u_s ds\right|$$

$$\leqq \|\Phi''_{yx}\|_\infty \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|D_sV_{t_i}-D_sV_s|\,|u_s|ds$$

$$+\left[\sup_i \sup_{s\in[t_i,t_{i+1}]}|\Phi''_{yx}(V_{t_i},U_{t_i})-\Phi''_{yx}(V_s,U_s)|\right]\int_0^t |D_sV_s|\,|u_s|ds,$$

and we use (ii) to obtain the desired convergence.

It remains finally to show that

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\Phi'_y(V_{t_i},U_{t_i})u_s dW_s\to\int_0^t \Phi'_y(V_s,U_s)u_s dW_s, \tag{6.6}$$

in $L^2(\Omega)$, as $n\to\infty$.

In fact we will show that

$$u_s\sum_{i=0}^{n-1}\Phi'_y(V_{t_i},U_{t_i})1_{]t_i,t_{i+1}]}(s)\to u_s\Phi'_y(V_s,U_s)$$

as $n\to\infty$ in $\mathbb{L}^{2,1}$. Obviously the convergence holds in $L^2([0,1]\times\Omega)$. Then it suffices to show

$$(D_ru_s)\sum_{i=0}^{n-1}\Phi'_y(V_{t_i},U_{t_i})1_{]t_i,t_{i+1}]}(s)\to(D_ru_s)\Phi'_y(V_s,U_s) \tag{6.7}$$

$$u_s\sum_{i=0}^{n-1}\Phi''_{yx}(V_{t_i},U_{t_i})D_rV_{t_i}1_{]t_i,t_{i+1}]}(s)\to u_s\Phi''_{yx}(V_s,U_s)D_rV_s \tag{6.8}$$

$$u_s\sum_{i=0}^{n-1}\Phi''_{yy}(V_{t_i},U_{t_i})\left(\int_0^s D_ru_\theta dW_\theta\right)1_{]t_i,t_{i+1}]}(s)$$

$$\to u_s\Phi''_{yy}(V_s,U_s)\left(\int_0^s D_ru_\theta dW_\theta\right) \tag{6.9}$$

and

$$u_s\sum_{i=0}^{n-1}\Phi''_{yy}(V_{t_i},U_{t_i})u_r1_{[0,t_i[}(r)1_{]t_i,t_{i+1}]}(s)$$

$$\to u_s\Phi''_{yy}(V_s,U_s)u_r1_{\{r\leqq s\}} \tag{6.10}$$

in $L^2([0,1]^2\times\Omega)$.

(6.7) follows easily from the fact that $u\in\mathbb{L}^{2,1}$. To show (6.8) we first remark that $u_sD_rV_s$ belongs to $L^2([0,1]^2\times\Omega)$. So, by Lebesgue dominated convergence theorem we have

$$u_s\sum_{i=0}^{n-1}\Phi''_{yx}(V_{t_i},U_{t_i})D_rV_s1_{]t_i,t_{i+1}]}(s)\to u_s\Phi''_{yx}(V_s,U_s)D_rV_s,$$

in $L^2([0,1]^2 \times \Omega)$. In addition,

$$E\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^1 u_s^2 \Phi''_{yx}(V_{t_i}, U_{t_i})^2 [D_r V_{t_i} - D_r V_s]^2 dr\, ds\right]$$
$$\leqq c\left(E\int_0^1 u_s^4 ds\right)^{1/2} \left(\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^1 E(|D_r V_{t_i} - D_r V_s|^4) dr\, ds\right)^{1/2}$$

which tends to zero as $n \to \infty$, from (ii).

The proof of (6.9) is similar. Hypotheses (i) and (ii) imply that $u_s \int_0^s D_r u_\theta dW_\theta$ belongs to $L^2([0,1]^2 \times \Omega)$:

$$E\int_0^1\int_0^1 u_s^2 \left(\int_0^s D_r u_\theta dW_\theta\right)^2 ds\, dr \leqq \left[E\left(\int_0^1 u_s^4 ds\right) \int_0^1\int_0^1 E\left(\int_0^s D_r u_\theta dW_\theta\right)^4 ds\, dr\right]^{1/2}$$
$$\leqq c\left[E\left(\int_0^1 u_s^4 ds\right)\left\{\int_0^1\int_0^1 E(|D_r u_\theta|^4) dr\, d\theta + \int_0^1\int_0^1\int_0^1 E(|D_\zeta D_r u_\theta|^4) d\zeta\, dr\, d\theta\right\}\right]^{1/2}.$$

Here we have applied the L^p inequality of Proposition 3.5 for $p=4$. Then, to complete the proof of (6.9) we have to verify that the following expectation tends to zero:

$$E\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^1 u_s^2 \Phi''_{yy}(V_{t_i}, U_{t_i})^2 \left(\int_{t_i}^s D_r u_\theta dW_\theta\right)^2 dr\, ds\right]$$
$$\leqq \|\Phi''_{yy}\|_\infty^2 \left(E\int_0^1 u_s^4 ds\right)^{1/2} \left[\sum_i E\int_{t_i}^{t_{i+1}} \int_0^1 \left(\int_{t_i}^s D_r u_\theta dW_\theta\right)^4 dr\, ds\right]^{1/2}.$$

Using the same L^4 estimate as above, we deduce that the last factor tends to zero, as $n \to \infty$. (6.10) is immediate and the proof is complete. □

Proof of Corollary 6.2. The proof of the chain rule under the first set of hypotheses follows exactly the same steps as the proof of the theorem. The L^∞ bound on u permits to avoid using any fourth order moment.

Once we have the chain rule under this first set of hypotheses, the result will follow under the second set of hypotheses by a limiting argument (which uses the fact that the derivatives of Φ are bounded, and we have only dt and dW_t integrals) once we show that there exists a sequence $\{u_n, n \in \mathbb{N}\}$ such that each u_n satisfies condition (i),

$$u_n \to u \quad \text{in} \quad \mathbb{L}^{2,2}$$

and

$$\sup_n \|u_n\|_{L^\infty([0,1]\times\Omega)} < \infty .$$

We now construct such a sequence. Let $\{\Pi^n\}$ be a sequence of partitions with $|\Pi^n| \to 0$ as $n \to \infty$. We define:

$$v_n^i = \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} u_s ds .$$

Clearly, $v_n^i \in \mathbb{D}_{2,2} \cap L^\infty(\Omega)$, and if we define:

$$v_n = \sum_{i=0}^{n-1} v_n^i 1_{[t_i, t_{i+1}[}$$

$$v_n \to u \quad \text{in} \quad \mathbb{L}^{2,2}.$$

Now $\forall i, n$, there exists a sequence $\{{}^p v_n^i, p \in \mathbb{N}\}$ in $\mathscr{S}$, such that:

$$\sup_p \|{}^p v_n^i\|_{L^\infty(\Omega)} < \infty$$

$${}^{p'} v_n^i \to v_n^i \quad \text{in} \quad \mathbb{D}_{2,2}.$$

Finally, there exists a sequence of integers $\{p(n), n \in \mathbb{N}\}$ such that:

$$u_n = \sum_{i=0}^{n-1} {}^{p(n)} v_n^i 1_{[t_i, t_{i+1}[}$$

converges to u in $\mathbb{L}^{2,2}$. □

We now state the multidimensional analogues of Theorem 6.1 and Corollary 6.2. We use below the convention of summation upon repeated indices.

Theorem 6.4. *Let $\Phi = \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function, such that the derivatives Φ'_{x_i}, Φ'_{y_j}, $\Phi''_{y_j x_i}$, $\Phi''_{y_j y_k}$ exist and are continuous for $1 \leqq i \leqq M$, $1 \leqq j, k \leqq N$.*

Let $\{u^{ij}; 1 \leqq i \leqq N, 1 \leqq j \leqq d\}$ be a set of processes, each of which satisfies (i) *in Theorem 6.1, and $\{V^i; 1 \leqq i \leqq M\}$ be another set of processes, each of which satisfies* (ii) *in Theorem 6.1.*

For $t \in [0,1]$, we denote by $U_t = \int_0^t u_s dW_s$ the N-dimensional process defined by $U_t^i = \int_0^t u_s^{ij} dW_s^j$. We then have:

$$\Phi(V_t, U_t) = \Phi(V_0, 0) + \int_0^t \Phi'_{x_i}(V_s, U_s) dV_s^i$$

$$+ \int_0^t \Phi'_{y_k}(V_s, U_s) u_s^{kj} dW_s^j + 1/2 \int_0^t \Phi''_{y_k y_l}(V_s, U_s) u_s^{kj} u_s^{lj} ds$$

$$+ \int_0^t \left[\Phi''_{y_k x_i}(V_s, U_s) D_s^j V_s^i + \Phi''_{y_k y_l}(V_s, U_s) \int_0^s D_s^j u_r^{lh} dW_r^h \right] u_s^{kj} ds. \quad \square$$

Corollary 6.5. *Suppose that the process $\{U_t, t \in [0,1]\}$ is a.s. continuous, and we replace* (i) *by* (i′) *and* (ii) *by* (ii′) *or* (ii″) *in Theorem 6.4. Then its conclusion remains true. Moreover, we may drop the requirement that $\{U_t\}$ be continuous, provided we assume that the derivatives of Φ are bounded.* □

Remark 6.6. Suppose that the assumptions of Theorem 6.4 or Corollary 6.5 are satisfied, and moreover that $M = N$. Let us define:

$$X_t = V_t + U_t.$$

It is easily seen that $s \to D_t X_s$ is mean square continuous on $[0,1]-\{t\}$, and we then can define,

$$(\nabla_+^i X)_t = \lim_{\varepsilon\to 0, \varepsilon>0} (D_t^i X_{t+\varepsilon} + D_t^i X_{t-\varepsilon})$$

$$(\nabla_-^i X)_t = \lim_{\varepsilon\to 0, \varepsilon>0} (D_t^i X_{t+\varepsilon} - D_t^i X_{t-\varepsilon}).$$

Note that $(\nabla_-^i X)_t = u_t^i$.

With these notations, the Itô formula takes the form:

$$\Phi(X_t) = \Phi(X_0) + \int_0^t \langle \Phi'(X_s), dX_s\rangle + \frac{1}{2}\sum_{i=1}^d \int_0^t \langle \Phi''(X_s)(\nabla_+^i X)_s, (\nabla_-^i X)_s\rangle ds,$$

which is more concise. □

7. A Stratonovich Type Integral and the Associated Chain Rule

7.1. Definition of the Stratonovich Integral

Let $\{\Pi^n, n\in\mathbb{N}\}$ denote again a sequence of partitions of $[0,1]$, $\Pi^n = \{0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = 1\}$, with $|\Pi^n| \to 0$, as $n\to\infty$. Let $\{u_t, t\in[0,1]\}$ be a d-dimensional measurable process defined on $(\Omega, \mathscr{F}, P)$, s.t. $\int_0^1 |u_t|^2 dt < \infty$ a.s. We then associated to each $\{\Pi^n\}$ the process:

$$u^n = \sum_{k=0}^{n-1} \bar{u}_{k,n} 1_{[t_{k,n}, t_{k+1,n}[}$$

where again

$$\bar{u}_{k,n} = \frac{1}{t_{k+1,n} - t_{k,n}} \int_{t_{k,n}}^{t_{k+1,n}} u_s ds.$$

Definition 7.1. A d-dimensional measurable process $\{u_t, t\in[0,1]\}$ defined on a probability space $(\Omega, \mathscr{F}, P)$ is said to be *Stratonovich integrable* if the sequence:

$$\sum_{k=0}^{n-1} \bar{u}_{k,n} \cdot (W_{t_{k+1,n}} - W_{t_{k,n}})$$

converges in probability as $n\to\infty$, and if moreover the limit does not depend on the choice of the sequence of partitions $\{\Pi^n\}$.

Whenever $\{u_t\}$ is Stratonovich integrable, we denote by

$$\int_0^1 u_t \circ dW_t$$

the above limit, which will be called the Stratonovich or the Stratonovich-Skorohod integral of $\{u_t\}$. □

Note that, in case $d=1$, if $\{X_t\}$ is a continuous $\mathscr{F}_t$ semi-martingale, and $f \in C^1(\mathbb{R})$ then $u_t = f(X_t)$ is Stratonovich integrable, and:

$$\int_0^1 u_t \circ dW_t = \int_0^1 u_t \,.\, dW_t + \frac{1}{2} \int_0^1 f'(X_t) d\langle X, W\rangle_t. \tag{7.1}$$

On the other hand, if Y_t is a continuous backward $\mathscr{F}^t$ semi martingale, and $f \in C^1(\mathbb{R})$ then $v_t = f(Y_t)$ is again Stratonovich integrable, but now:

$$\int_0^1 v_1 \circ dW_t = \int_0^1 v_t \,.\, dW_t - \frac{1}{2} \int_0^1 f'(Y_t) d\langle Y, W\rangle_t. \tag{7.2}$$

Clearly, $u_t + v_t$ is again Stratonovich integrable, but the correction term between its Stratonovich and its Itô integral cannot be expressed in terms of its joint quadratic variation with W_t.

Definition 7.2. A process $\{u_t, t \in [0,1]\}$ will be said to belong to the class $\mathbb{L}^{2,1}_{d,C}$ whenever $u \in \mathbb{L}^{2,1}_d$, and moreover there exists a neighbourhood V in $[0,1]^2$ of the diagonal of $[0,1]^2$ such that:

(i) $\{D_s u_t\}$ has one version for which $t \to D_s u_t$ is continuous with values in $L^2(\Omega)$ uniformly with respect to s, on $V \cap \{s \leqq t\}$.

(ii) $\{D_s u_t\}$ has a (possibly different) version for which $t \to D_s u_t$ is continuous with values in $L^2(\Omega)$ uniformly with respect to s, on $V \cap \{s \geqq t\}$.

(iii) $\operatorname{ess\,sup}_{(s,t) \in V} E(\|D_s u_t\|^2) < \infty$

In the case $d=1$, we delete the index d, as above. □

If $u \in \mathbb{L}^{2,1}_{d,C}$, then we can define:

$$D_t^+ \,.\, u_t = \lim_{s \to t, s > t} \sum_{i=1}^d D_t^i u_s^i$$

$$D_t^- \,.\, u_t = \lim_{s \to t, s < t} \sum_{i=1}^d D_t^i u_s^i$$

as elements of $L^2([0,1] \times \Omega)$.

Theorem 7.3. *Let $u \in \mathbb{L}^{2,1}_{d,C}$. Then u is Stratonovich integrable, and*

$$\int_0^1 u_t \circ dW_t = \delta(u) + \frac{1}{2} \int_0^1 [D_t^+ \,.\, u_t + D_t^- u_t] dt. \tag{7.3}$$

Proof. From the analysis in Sect. 4 (see Proposition 4.3 and its consequences) it suffices to show that:

$$\sum_{k=0}^{n-1} \frac{1}{t_{k+1,n} - t_{k,n}} \int_{t_{k,n}}^{t_{k+1,n}} \int_{t_{k,n}}^{t_{k+1,n}} D_t \,.\, u_s ds dt$$

$$\to \frac{1}{2} \int_0^1 [D_t^+ \,.\, u_t + D_t^- \,.\, u_t] dt$$

in probability, as $n \to \infty$. This follows easily from (i), (ii) and (iii) in Definition 7.2. □

Definition 7.4. A process u will be said to belong to class $\mathbb{L}^{2,1}_{d,C,\mathrm{loc}}$ whenever $u \in \mathbb{L}^{2,1}_{d,\mathrm{loc}}$ and possesses a localizing sequence $\{(\Omega_k, u_k), k \in \mathbb{N}\}$ such that u_k satisfies the conditions (i), (ii) and (iii) in Definition 7.2, $\forall k \in \mathbb{N}$. □

It is easily seen that Theorem 7.3 still holds true with $u \in \mathbb{L}^{2,1}_{d,C,\mathrm{loc}}$.

Proposition 7.5. *Suppose that* $u \in \mathbb{L}^{2,1}_{d,C}$, *is mean-square continuous, and satisfies* (i), (ii) *and* (iii) *in Definition 7.2 with* $V = [0,1]^2$. *Then* $\int_0^1 u_t \circ dW_t$ *is the* $L^2(\Omega)$*-limit, as* $n \to \infty$, *of the sequence:*

$$\frac{1}{2} \sum_{k=0}^{n-1} (u_{t_k} + u_{t_{k+1,n}}) \cdot (W_{t_{k+1,n}} - W_{t_{k,n}}) .$$

Proof.

$$\frac{1}{2} \sum_{k=0}^{n-1} (u_{t_k} + u_{t_{k+1}}) \cdot (W_{t_{k+1}} - W_{t_k})$$

$$= \int_0^1 \left[\frac{1}{2} \sum_{k=0}^{n-1} (u_{t_k} + u_{t_{k+1}}) 1_{[t_k, t_{k+1}[}(t) \right] \cdot dW_t$$

$$+ \frac{1}{2} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (D_s \cdot u_{t_k} + D_s \cdot u_{t_{k+1}}) ds .$$

The hypotheses imply that the sequence

$$u_n = \frac{1}{2} \sum_{k=0}^{n-1} (u_{t_k} + u_{t_{k+1}}) 1_{[t_k, t_{k+1}[}(t)$$

converges to u in $\mathbb{L}^{2,1}_d$, and moreover:

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} D_s \cdot u_{t_k} ds \to \int_0^1 D_t^- \cdot u_t dt$$

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} D_s, u_{t_{k+1}} ds \to \int_0^1 D_t^+ \cdot u_t dt$$

in mean square. □

In order to compare the correction term between Stratonovich's and Skorohod's integral with the classical one, let us establish:

Theorem 7.6. *Suppose u satisfies the hypotheses of Proposition 7.5. Then:*

$$\sum_{k=0}^{n-1} (u_{t_{k+1,n}} - u_{t_{k,n}}) \cdot (W_{t_{k+1,n}} - W_{t_{k,n}}) .$$

$$\to \int_0^1 (D_t^+ \cdot u_t - D_t^- \cdot u_t) dt$$

in mean square.

Proof.

$$\sum_{k=0}^{n-1} (u_{t_{k+1}} - u_{t_k}) . (W_{t_{k+1}} - W_{t_k})$$

$$= \int_0^1 \left[\sum_{k=0}^{n-1} (u_{t_{k+1}} - u_{t_k}) 1_{[t_k, t_{k+1}[}(t) \right] . dW_t + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathrm{D}_s . (u_{t_{k+1}} - u_{t_k}) ds .$$

The first term on the right tends to zero, since:

$$\sum_{k=0}^{n-1} (u_{t_{k+1}} - u_{t_k}) 1_{[t_k, t_{k+1}[}(t) \to 0$$

in $\mathbb{L}_d^{2,1}$. The result follows from the last convergences in Proposition 7.5. □

We note that the derivative of u is discontinuous across the diagonal of $[0,1]^2$ if and only if the joint quadratic variation of u and W is non zero. This is consistent with the hypothesis concerning the bounded variation process $\{V_t\}$ is Theorem 6.1. Moreover, if V_t is both $\mathscr{F}_t$ adapted and of bounded variation, then $D_t V_t = 0$.

For the comparison of the Itô-Stratonovich correction terms, let us consider the case $d=1$ for simplicity. We note that

$$\langle u, W \rangle_1 = \lim_{n\to\infty} \sum_{k=0}^{n-1} (u_{t_{k+1,n}} - u_{t_{k,n}}) (W_{t_{k+1,n}} - W_{t_{k,n}}) .$$

If $u \in \mathbb{L}_C^{2,1}$, and u is $\mathscr{F}_t$ adapted, then $D_t^- u_t = 0$, and

$$\frac{1}{2} \langle u, W \rangle_1 = \frac{1}{2} \int_0^1 \mathrm{D}_t^+ u_t dt$$

$$= \frac{1}{2} \int_0^1 (\mathrm{D}_t^+ u_t + D_t^- u_t) dt .$$

If now $u \in \mathbb{L}_C^{2,1}$ and is $\mathscr{F}^t$ adapted, then $D_t^+ u_t = 0$, and

$$-\frac{1}{2} \langle u, W \rangle_1 = \frac{1}{2} \int_0^1 \mathrm{D}_t^- u_t dt$$

$$= \frac{1}{2} \int_0^1 (\mathrm{D}_t^+ u_t + D_t^- u_t) dt .$$

From these two relations, we see that (7.3) is in agreement both with (7.1) and with (7.2).

7.2. *Another Class of Stratonovich-Integrable Processes*

We now consider the Stratonovich integral of processes of the type introduced in Sect. 4.4. Again, we restrict ourselves to the case $d=1$.

Let D be a bounded open subset of $\mathbb{R}^p$, and $u: [0,1] \times \Omega \times D \to \mathbb{R}$ be a measurable function, which satisfies (H1), (H2), (H3) and (H4) in Sect. 4.4, and moreover:

(H5) $t \to u(t,x)$ and $t \to u'(t,x)$ are continuous in $L^q(\Omega)$, uniformly with respect to x.

(H6) There exists a measurable function $a: [0,1] \times \Omega \times D \to \mathbb{R}$ such that:

$$\sum_{k=0}^{n-1} (u(t_{k,n} + \alpha(t_{k+1,n} - t_{k,n}), x) - u(t_{k,n}, x))(W_{k+1,n} - W_{k,n}) \to \alpha \int_0^1 a(t,x)dt$$

in probability, uniformly with respect to $\alpha \in [0,1]$, $x \in D$; for any sequence of partitions $\Pi^n = \{0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = 1\}$ with $|\Pi^n| \to 0$ as $n \to \infty$.

It is easy to give explicit sufficient conditions for (H6), see e.g. Yor [27].

Proposition 7.7 *Let u satisfy* (H1)...(H6) *and θ be a D-valued random vector. Then $\{u(t,\theta), t \in [0,1]\}$ is Stratonovich integrable, as well as $\{u(t,x), t \in [0,1]\}$, $\forall x \in D$ and:*

$$\int_0^1 u(t,\theta) \circ dW_t = \int_0^1 u(t,x) \circ dW_t|_{x=\theta}.$$

Proof. Note that $u(t,x)$ is clearly Stratonovich integrable,

$$\int_0^1 u(t,x) \circ dW_t = \int_0^1 u(t,x)dW_t + \frac{1}{2} \int_0^1 a(t,x)dt$$

and from the results in Sect. 4.4 it makes sense to "evaluate $\int_0^1 u(t,x)dW_t$ at $x = \theta$". It then makes sense to evaluate $\int_0^1 u(t,x) \circ dW_t$ at $x = \theta$.

Now:

$$\begin{aligned}
&\sum_{k=0}^{n-1} \left(\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} u(s,\theta)ds \right)(W_{t_{k+1}} - W_{t_k}) \\
&= \sum_{k=0}^{n-1} \left(\frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} u(s,\theta)ds \right)(W_{t_{k+1}} - W_{t_k}) \\
&+ \sum_{k=0}^{n-1} \left(\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} [u(s,\theta) - u(t_k,\theta)]ds \right)(W_{t_{k+1}} - W_{t_k}) \\
&+ \sum_{k=0}^{n-1} \left(\frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} [u(t_k,\theta) - u(s,\theta)]ds \right)(W_{t_{k+1}} - W_{t_k}) \\
&= A_n + B_n + C_n.
\end{aligned}$$

From Lemma 4.10,

$$A_n \to \int_0^1 u(t,x)dW_t|_{x=\theta}$$

in $L^2(\Omega)$, as $n \to \infty$. From (H6) integrated over $\alpha \in [0,1]$,

$$B_n \to \frac{1}{2} \int_0^1 a(t,\theta)dt$$

in probability, as $n \to \infty$. It remains to show that $C_n \to 0$ in probability.

This will follow from:

$$\sup_{x\in D}\left|\sum_{k=0}^{n-1}\left(\frac{1}{t_k-t_{k-1}}\int_{t_{k-1}}^{t_k}[u(t_k,x)-u(s,x)]ds\right)(W_{t_{k+1}}-W_{t_k})\right|\to 0$$

in $L^2(\Omega)$, as $n\to\infty$ which is a consequence of the fact that $u\in L^q([0,1]\times\Omega; W^{q,1}(D))$ for a $q>p$ (see the proof of Lemma 4.10) and (H5). □

By a localization procedure, we obtain:

Proposition 7.8. *The statement of Proposition 7.7 is still true if u and θ satisfy the hypotheses of Proposition 4.13, and u satisfies* (H5) *and* (H6) *where D is replaced by $\mathbb{R}^p$ and the assumed convergences are uniform for x in any compact subset of $\mathbb{R}^p$.* □

7.3. The Chain Rule of Stratonovich Type

We first state and prove a one dimensional result.

Theorem 7.9. *Suppose $d=1$. Let $\Phi:\mathbb{R}^2\to\mathbb{R}$ be a continuous function, such that the derivatives Φ'_x, Φ'_y, Φ''_{yx} and Φ''_{yy} exist and are continuous, and moreover let:*

(i) *u be an element of $\mathbb{L}^{2,2}\cap\mathbb{L}^{2,1}_C$ s.t. there exists $p>4$ with*

$$\int_0^1\int_0^1\|E(D_su_t)\|^p dsdt+E\int_0^1\int_0^1\int_0^1\|D_rD_su_t\|^p drdsdt<\infty,$$

the process $\{D_t^+u_t+D_t^-u_t,\ t\in[0,1]\}$ belongs to $\mathbb{L}^{2,1}$ and moreover:

$$\sup_{t\in[0,1]} E\int_0^1|D_t(D_s^+u_s+D_s^-u_s)|^4ds<\infty$$

(ii) *$\{V_t, t\in[0,1]\}$ be a continuous process with a.s. finite variation belonging to $\mathbb{L}^{2,1}$, s.t. $E\int_0^1\int_0^1(\mathrm{D}_sV_t)^4dsdt<\infty$ and the mapping $t\to D_sV_t$ is continuous with values in $L^4(\Omega)$, uniformly with respect to s.*

Then for any $t\in[0,1]$, with the notation $\tilde{U}_t=\int_0^t u_s\circ dW_s$, we have the following:

$$\Phi(V_t,\tilde{U}_t)=\Phi(V_0,0)+\int_0^t\Phi'_x(V_s,\tilde{U}_s)dV_s+\int_0^t\Phi'_y(V_s,\tilde{U}_s)u_s\circ dW_s.$$

Proof.

$$\tilde{U}_t=U_t+\tilde{V}_t,\quad\text{where}$$

$$U_t=\int_0^t u_sdW_s$$

$$\tilde{V}_t=\frac{1}{2}\int_0^t(D_s^+u_s+D_s^-u_s)ds.$$

So that $\Phi(V_t, \tilde{U}_t) = \tilde{\Phi}_t(V_t, \tilde{V}_t, U_t)$, and we can apply Theorem 6.4., which yields:

$$\begin{aligned}\Phi(V_t, \tilde{U}_t) = {} & \Phi(V_0, 0) + \int_0^t \Phi'_x(V_s, \tilde{U}_s)\, dV_s \\ & + \frac{1}{2} \int_0^t \Phi'_y(V_s, \tilde{U}_t)(D_s^+ u_s + D_s^- u_s)\, ds \\ & + \int_0^t \Phi'_y(V_s, \tilde{U}_s) u_s\, dW_s + \frac{1}{2} \int_0^t \Phi''_{yy}(V_s, \tilde{U}) u_s^2\, ds \\ & + \int_0^t \Big\{ \Phi''_{yx}(V_s, \tilde{U}_s) D_s V_s + \Phi''_{yy}(V_s, \tilde{U}_s) \Big[\int_0^s D_s u_r\, dW_r \\ & + \frac{1}{2} \int_0^s D_s(D_r^+ u_r + D_r^- u_r)\, dr \Big] \Big\} u_s\, ds .\end{aligned}$$

And it is easily seen that the sum of the four last terms is equal to:

$$\int_0^t \Phi'_y(V_s, \tilde{U}_s) u_s \circ dW_s .$$

Note that $\Phi'_y(V_t, \tilde{U}_t) u_t$ is in $\mathbb{L}^{2,1}_{C,\mathrm{loc}}$. □

We note that both the usual and the new "additional" terms in the Itô formula disappear in the Stratonovich chain rule.

We finally state the multi-dimensional version of the Stratonovich chain rule, using the convention of summation upon repeated indices:

Theorem 7.10. *Let $\Phi = \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function, such that the derivatives Φ'_{x_i}, Φ'_{y_j}, $\Phi''_{y_j x_i}$, $\Phi''_{y_j y_k}$ exist and are continuous, for $1 \leqq i \leqq M$, $1 \leqq j, k \leqq N$.*

Let $\{u^{ij}; 1 \leqq i \leqq N, 1 \leqq j \leqq d\}$ be a set of processes, each of which satisfies (i) *in Theorem 7.9; and $\{V^i, 1 \leqq i \leqq M\}$ another set of processes, each of which satisfies* (ii) *in Theorem 7.9.*

For $t \in [0,1]$, we denote by $\tilde{U}_t = \int_0^t u_s \circ dW_s$ the N dimensional process defined by $U_t^i = \int_0^t u_s^{ij} \circ dW_s^j$.

We then have:

$$\begin{aligned}\Phi(V_t, \tilde{U}_t) = {} & \Phi(V_0, 0) + \int_0^t \Phi'_{x_i}(V_s, \tilde{U}_s)\, dV_s^i \\ & + \int_0^t \Phi'_{y_k}(V_s, \tilde{U}_s) u_s^{kj} \circ dW_s^j .\end{aligned}$$ □

8. The Two-Sided Integral

In this section, we specialize our results to a particular class of integrands, and thus obtain direct generalizations of the results in Pardoux and Protter [19].

Let $\Phi=[0,1]\times\mathbb{R}^M\times\mathbb{R}^N\to\mathbb{R}^d$ be a measurable function s.t. $(x,y)\to\Phi(t,x,y)$ is of class C^1 t a.e., and moreover $\Phi(t,x,y)$, $\Phi'_x(t,x,y)$ and $\Phi'_y(t,x,y)$ are bounded on bounded subsets of $[0,1]\times\mathbb{R}^M\times\mathbb{R}^N$.

Let $\{X_t^i, t\in[0,1], 1\leqq i\leqq M\}$ be continuous $\mathscr{F}_t$ adapted processes which belong to $\mathbb{L}^{2,1}_{\text{loc}}$; and $\{Y_t^j, t\in[0,1], 1\leqq j\leqq N\}$ be continuous $\mathscr{F}^t$ adapted processes which belong to $\mathbb{L}^{2,1}_{\text{loc}}$.

All the above hypotheses are supposed to hold throughout this section.

It follows from Proposition 4.7 that $\Phi(.,X.,Y.)$ belongs to $\mathbb{L}^{2,1}_{d,\text{loc}}$.

Proposition 8.1. *Suppose that $\{D_s^l X_t^i; t\in[s,1]\}$ and $\{D_s^l Y_t, t\in[0,s]\}$ have modifications which are continuous functions of t with values in $L^2(\Omega)$, uniformly with respect to s; $1\leqq i\leqq M$, $1\leqq j\leqq N$, $1\leqq l\leqq d$.*

Then, for any sequence $\{\Pi^n\}$ of partitions of $[0,1]$, with $|\Pi^n|\to 0$ as $n\to\infty$,

$$\sum_{k=0}^{n-1}\Phi_k(X_{t_k}, Y_{t_{k+1}}).(W_{t_{k+1}}-W_{t_k})\to\int_0^1\Phi(t,X_t,Y_t).dW_t$$

in probability, as $n\to\infty$, where

$$\Phi_k(x,y)=\frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}}\Phi(t,x,y)dt.$$

Proof. By the usual localization argument, it suffices to consider the case where X^i, $Y^j\in\mathbb{L}^{2,1}$ and Φ, Φ'_{x_i}, Φ'_{y_j} are bounded, $1\leqq i\leqq M$, $1\leqq j\leqq N$, which we suppose from now on. We also assume for simplicity that $d=1$.

Define

$$u_t^n=\sum_{k=0}^{n-1}\Phi_k(X_{t_k}, Y_{t_{k+1}})1_{[t_k,t_{k+1}[}(t)$$

$$u_t=\Phi(t,X_t,Y_t).$$

Clearly, since

$$\Phi_k(X_{t_k}, Y_{t_{k+1}})\quad\text{is}\quad\mathscr{F}_{t_k}\vee\mathscr{F}^{t_{k+1}}$$

measurable,

$$\delta(u^n)=\sum_{k=0}^{n-1}\Phi_k(X_{t_k}, Y_{t_{k+1}})(W_{t_{k+1}}-W_{t_k}).$$

It then suffices to show that $u^n\to u$ in $\mathbb{L}^{2,1}$. The convergence in $L^2([0,1]\times\Omega)$ is immediate. It thus remains to show that:

$$\sum_{k=0}^{n-1}(\Phi_k)'_x(X_{t_k}, Y_{t_{k+1}})D_sX_{t_k}1_{[t_k,t_{k+1}[}(t)\to\Phi'_x(t,X_t,Y_t)D_sX_t$$

$$\sum_{k=0}^{n-1}(\Phi_k)'_y(X_{t_k}, Y_{t_{k+1}})D_sY_{t_{k+1}}1_{[t_k,t_{k+1}[}(t)\to\Phi'_y(t,X_t,Y_t)D_sY_t$$

in $L^2([0,1]^2\times\Omega)$. This follows from the hypothesis of the proposition. □

We now want to see the particular form which takes the Itô formula for a process of the type $\Phi(t,X_t,Y_t)$. For that sake, we need to particularize the situation. We now suppose that $\{X_t^i, 1\leqq i\leqq M\}$ are continuous $\mathscr{F}_t$ semi-martingales, and

$\{Y_t^j, 1 \leqq j \leqq N\}$ are continuous $\mathscr{F}^t$ semi-martingales, with canonical representations:

$$X_t^i = x^i + A_t^i + \sum_{k=1}^{d} \int_0^t \sigma_s^{ik} dW_s^k; \quad i = 1, \dots, M \tag{8.1}$$

$$Y_t^j = y^j + B_t^j + \sum_{k=1}^{d} \int_t^1 \gamma_s^{jk} dW_s^k; \quad j = 1, \dots, N. \tag{8.2}$$

We first suppose that A^i and B^j have a.s. bounded variations, $\{A_t^i\}$ and $\{\sigma_t^{ik}\}$ being $\mathscr{F}_t$ adapted, $\{B_t^j\}$ and $\{\gamma_t^{jk}\}$ $\mathscr{F}^t$ adapted. We suppose further that A^i, σ^{ik}, B^j, γ^{jk} are elements of $\mathbb{L}_{\mathrm{loc}}^{2;1}$ which can be localized by processes having the same adaptedness property as themselves, and which are bounded together with their derivatives, the processes which localize A^i and B^j being continuous with a.s. bounded variation. We note than, from Lemma 2.4 and Proposition 3.4,

$$D_r^l X_t^i = \left[D_r^l A_t^i + \sigma_r^{il} + \sum_{k=1}^{d} \int_r^t (D_r^l \sigma_s^{ik}) dW_s^k \right] 1_{\{r \leqq t\}}$$

$$D_r^l Y_t^j = \left[D_r^l B_t^j + \gamma_r^{jl} + \sum_{k=1}^{d} \int_t^r (D_r^l \gamma_s^{jk}) dW_s^k \right] 1_{\{r \geqq t\}}.$$

Proposition 8.2. *Let $\Phi = [0,1] \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function, which is once continuously differentiable with respect to t, and twice continuously differentiable with respect to x and y. Let $\{X_t\}$ and $\{Y_t\}$ be respectively an $\mathbb{R}^M$ valued continuous $\mathscr{F}_t$ semi-martingale and an $\mathbb{R}^N$ valued continuous $\mathscr{F}^t$ semi-martingale of the forms* (8.1) *and* (8.2), *A, B, σ, γ satisfying the above hypotheses, and moreover*:

(H1) *$\{D_s^l A_t^i; t \in [s,1]\}$ and $\{D_s^l B_t^j; t \in [0,s]\}$ have modifications which are continuous functions of t which values in $L^2(\Omega)$, uniformly with respect to s; $1 \leqq i \leqq M$, $1 \leqq j \leqq N$, $1 \leqq l \leqq d$.*

We then have, $\forall 0 \leqq s \leqq t \leqq 1$,

$$\begin{aligned}\Phi(t, X_t, Y_t) = \Phi(s, X_s, Y_s) &+ \int_s^t \Phi_t'(r, X_r, Y_r) dr \\ &+ \int_s^t \Phi_x'(r, X_r, Y_r) . dX_r + \frac{1}{2} \int_s^t \mathrm{Tr}\, [\Phi_{xx}''(r, X_r, Y_r) \sigma_r \sigma_r^*] dr \\ &+ \int_s^t \Phi_y'(r, X_r, Y_r) . dY_r - \frac{1}{2} \int_s^t \mathrm{Tr}\, [\Phi_{yy}''(r, X_r, Y_r) \gamma_r \gamma_r^*] dr .\end{aligned}$$

Proof. For simplicity, let us suppose that $N = M = d = 1$. For the proof, we may and do assume that Φ, Φ_t', Φ_x', Φ_y', Φ_{xx}'', Φ_{yy}'' are bounded, $A, \sigma, B, \gamma \in \mathbb{L}^{2,1}$, these processes being bounded as well as their derivatives.

Let $\{\Pi^n\}$ denote a sequence of subdivisions of $[s,t]$, with $|\Pi^n| \to 0$ as $n \to \infty$.

$$\begin{aligned}\Phi(t, X_t, Y_t) - \Phi(s, X_s, Y_s) &= \sum_{i=0}^{n-1} \Phi(t_{i+1}, X_{t_{i+1}}, Y_{t_{i+1}}) - \Phi(t_i, X_{t_i}, Y_{t_i}) \\ &= \sum_{i=0}^{n-1} [\Phi(t_{i+1}, X_{t_{i+1}}, Y_{t_{i+1}}) - \Phi(t_i, X_{t_{i+1}}, Y_{t_{i+1}})]\end{aligned}$$

$$+\sum_{i=0}^{n-1}[\Phi(t_i, X_{t_{i+1}}, Y_{t_{i+1}})-\Phi(t_i, X_{t_i}, Y_{t_{i+1}})]$$

$$+\sum_{i=0}^{n-1}[\Phi(t_i, X_{t_i}, Y_{t_{i+1}})-\Phi(t_i, X_{t_i}, Y_{t_i})]$$

$$=\alpha_n+\beta_n+\gamma_n .$$

$$\alpha_n \to \int_s^t \Phi_t'(r, X_r, Y_r)dr \quad \text{a.s., as} \quad n\to\infty .$$

$$\beta_n=\sum_{i=0}^{n-1}\Phi_x'(t_i, X_{t_i}, Y_{t_{i+1}})(A_{t_{i+1}}-A_{t_i})$$

$$+\sum_{i=0}^{n-1}\Phi_x'(t_i, X_{t_i}, Y_{t_{i+1}})\int_{t_i}^{t_{i+1}}\sigma_s dW_s$$

$$+\frac{1}{2}\sum_{i=0}^{n-1}\Phi_{xx}''(t_i, \bar{X}_i, Y_{t_{i+1}})(X_{t_{i+1}}-X_{t_i})^2$$

where $\bar{X}_i$ is a random intermediate point between X_{t_i} and $X_{t_{i+1}}$.

$$\sum_{i=0}^{n-1}\Phi_x'(t_i, X_{t_i}, Y_{t_{i+1}})(A_{t_{i+1}}-A_{t_i})\to\int_s^t \Phi_x'(r, X_r, Y_r)dA_r \quad \text{a.s.}$$

and from Lemma C2 in Appendix C,

$$\sum_{i=0}^{n-1}\Phi_{xx}''(t_i, \bar{X}_i, Y_{t_{i+1}})(X_{t_{i+1}}-X_{t_i})^2\to\int_s^t \Phi_{xx}''(r, X_r, Y_r)\sigma_r^2 dr \quad \text{a.s.}$$

Define

$$u_r^n=\sum_{i=0}^{n-1}\Phi_x'(t_i, X_{t_i}, Y_{t_{i+1}})1_{[t_i,t_{i+1}[}(r).$$

Then from Theorem 3.2:

$$\sum_{i=0}^{n-1}\Phi_x'(t_i, X_{t_i}, Y_{t_{i+1}})\int_{t_i}^{t_{i+1}}\sigma_r dW_r=\int_s^t u_r^n\sigma_r dW_r .$$

We then need to check that: $u^n\sigma\to\Phi_x'(X, Y)\sigma$ in $\mathbb{L}^{2,1}$, as $n\to\infty$ which follows from the hypotheses. γ_n is treated analogously. □

We note that Proposition 8.2 could be formally deduced from Theorem 6.4, choosing $V_t=(t, x+A_t, y+B_1-B_t+\int_0^1 \gamma_s dW_s)$, $u_t=(\sigma_t, \gamma_t)$, and $F(x_1, x_2, x_3, y_1, y_2)=\Phi(x_1, x_2+y_1, x_3-y_2)$. However, in order to apply Theorem 6.4, we would have needed more hypotheses on the processes. Indeed, and this is the interesting aspect of the particular case considered in this section, the additional terms in the Itô formula of Theorem 6.4 cancel here, and we dont't need to require the corresponding regularity.

Appendix A

The aim of this appendix is to establish:

Proposition A.1. *Let* $G \in \mathbb{D}_{2,1}$. *Then* $\forall 0 \leqq s \leqq t \leqq 1$,

$$G = E(G/\mathscr{F}_s \vee \mathscr{F}^t) + \int_s^t E(D_r G/\mathscr{F}_r \vee \mathscr{F}^t) \,.\, dW_r \,.$$

Choosing $s=0$ and $t=1$ in Proposition A.1, we obtain Ocone's representation theorem as a particular case:

Corollary A.2. *Let* $G \in \mathbb{D}_{2,1}$. *Then:*

$$G = E(G) + \int_0^1 E(D_t G/\mathscr{F}_t) \,.\, dW_t \,.$$

Proof of Proposition A.1. For simplicity, we restrict ourselves to the case $d=1$. Let us write the Wiener chaos decomposition of G:

$$G = \sum_{m=0}^{\infty} I_m(g_m) \,.$$

We then have:

$$D_r G = \sum_{m=1}^{\infty} m I_{m-1}(g_m(\dots, r))$$

$$E(D_r G/\mathscr{F}_r \vee \mathscr{F}^t) = \sum_1^{\infty} m I_{m-1}(g_m(\dots, r) h_{m,r})$$

where

$$h_{m,r}(t_1, \dots, t_{m-1}) = \prod_{i=1}^{m-1} 1_{[r,t[^c}(t_i) \,.$$

Denote by $f_m(t_1, \dots, t_m)$ the function obtained by symmetrizing:

$$h_{m,t_m}(t_1, \dots, t_{m-1}) \, 1_{[s,t[}(t_m) \,.$$

We then have that:

$$m f_m = 1_{A_m}$$

where

$$A_m = \bigcup_{i=1}^{m} \{(t_1, \dots, t_m) \in [0,1]^m;\ t_i \in [s, t[\} \,.$$

We then have:

$$\int_s^t E(D_r G/\mathscr{F}_r \vee \mathscr{F}^t) dW_r = \sum_{m=1}^{\infty} I_m(g_m 1_{A_m})$$

$$= \sum_{m=0}^{\infty} I_m(g_m) - \sum_{m=0}^{\infty} I_m(g_m 1_{A_m^c})$$

$$= G - E(G/\mathscr{F}_s \vee \mathscr{F}^t) \,. \quad \square$$

Appendix B

In this appendix, we prove the estimate (3.7).

Suppose $d=1$ and $u \in \mathbb{L}^{2,1}$. Let G be a polynomial functional. We have

$$E\left(G \int_0^1 u_t dW_t\right) = E\left(\int_0^1 u_t D_t G\, dt\right) = \int_0^1 \sum_{n=0}^{\infty} E(J_n u_t J_n D_t G)\, dt$$

$$\leqq \left\| \left[\int_0^1 \left(\sum_{n=0}^{\infty} \sqrt{n+1}\, J_n u_t \right)^2 dt \right]^{1/2} \right\|_p \cdot \left\| \left[\int_0^1 \left(\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} J_n D_t G \right)^2 dt \right]^{1/2} \right\|_q, \tag{B.i}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Let F be such that $CF=G-J_0G$. Then, using Meyer's inequalities we have

$$\left\| \left[\int_0^1 \left(\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} J_n D_t G \right)^2 dt \right]^{1/2} \right\|_q = \left\| \left[\int_0^1 (D_t F)^2 dt \right]^{1/2} \right\|_q$$

$$\leqq c_q \|CF\|_q = c_q \|G - J_0 G\|_q \leqq c'_q \|G\|_q .$$

For the first factor of (B.i) we can write

$$\sum_{n=0}^{\infty} \sqrt{n+1}\, J_n u_t = -RCu_t + E(u_t),$$

where R denotes the multiplication operator by $\sqrt{1+\frac{1}{n}}$.

Let $\{e_i(t); 0 \leqq t \leqq 1\}$ be an orthonormal basis in $L^2([0,1])$. Using Khintchin's inequality we obtain

$$E\left(\int_0^1 (RCu_t)^2 dt \right)^{p/2} = E\left(\sum_{i=1}^{\infty} \left(\int_0^1 RCu_t e_i(t)\, dt \right)^2 \right)^{p/2}$$

$$\leqq C_p E \int_0^1 \left| \sum_{i=1}^{\infty} \int_0^1 RCu_t e_i(t) \gamma_i(\theta)\, dt \right|^p d\theta, \tag{B.ii}$$

where $\{\gamma_i(\theta); 0 \leqq \theta \leqq 1, i \geqq 1\}$ is a sequence of Rademacher functions.

Now we apply Meyer's inequality in the form

$$E(|CF|^p) \leqq C_p E\left(\int_0^1 (D_t F)^2 dt \right)^{p/2},$$

which is true for all $p \geqq 2$ and $F \in \mathbb{D}_{2,1}$ (this can be deduced from the opposite inequality for polynomial functionals, using a duality argument, see Watanabe [26]).

As a consequence, and applying again Khintchin's inequality, we obtain that (B.ii) is bounded by

$$C_p E \int_0^1 \left| \sum_{j=1}^{\infty} \left| \int_0^1 D_s \left[\sum_{i=1}^{\infty} \int_0^1 (R-J_0)u_t e_i(t) r_i(\theta) dt \right] e_j(s) ds \right|^2 \right|^{p/2} d\theta$$

$$\leqq C_p E \int_0^1 \int_0^1 \left| \sum_{i,j=1}^{\infty} \int_0^1 \int_0^1 \hat{R}(D_s u_t) e_j(s) e_i(t) r_i(\theta) r_j(\theta') dt ds \right|^p d\theta d\theta'$$

$$\leqq C_p E \left(\int_0^1 \int_0^1 (D_s u_t)^2 ds dt \right)^{p/2},$$

where $\hat{R}$ is the multiplication operator by $(n+2/n+1)^{1/2}$, which is bounded in L^p.

As usual, C_p denotes a constant depending only on p that may be different from one formula to another one. □

Appendix C

The aim of this Appendix is to extend slightly a lemma due to Föllmer [3]. See also Pardoux and Protter [19].

The processes below are defined on an arbitrary probability space (E, ξ, Q).

Lemma C.1. *Let $\{W_t, t \in [0,1]\}$ and $\{V_t, t \in [0,1]\}$ denote two mutually independent standard Wiener processes. Let $\{X_t, t \in [0,1]\}$ denote a measurable process and $p > 1$.*

Suppose that $X \in L^p(0,1)$ a.s.

Then

$$\sum_{k=0}^{n-1} \left(\frac{1}{t_{k+1,n} - t_{k,n}} \int_{t_{k,n}}^{t_{k+1,n}} X_s ds \right) (W_{t_{k+1,n}} - W_{t_{k,n}})^2 \to \int_0^1 X_s ds \tag{C.i}$$

$$\sum_{k=0}^{n-1} \left(\frac{1}{t_{k+1,n} - t_{k,n}} \int_{t_{k,n}}^{t_{k+1,n}} X_s ds \right) (W_{t_{k+1,n}} - W_{t_{k,n}}) (V_{t_{k+1,n}} - V_{t_{k,n}}) \to 0 \tag{C.ii}$$

in probability, as $n \to \infty$, where $\Pi^n = \{0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = 1\}$ satisfies $|\Pi^n| \to 0$ as $n \to \infty$.

If moreover $X \in L^p([0,1] \times \Omega)$, then the above convergences hold in $L^1(\Omega)$.

Proof. It clearly suffices to show that (C.i) and (C.ii) hold in $L^1(\Omega)$ whenever $X \in L^p([0,1] \times \Omega)$. Let us first prove (C.i).

Define

$$X^l = \sum_{i=0}^{l-1} \left(\frac{1}{t_{i+1,l} - t_{i,l}} \int_{t_{i,l}}^{t_{i+1,l}} X_s ds \right) 1_{[t_{i,l}, t_{i+1,l}[},$$

$$\alpha_n(X) = \sum_{k=0}^{n-1} \left(\frac{1}{t_{k+1,n} - t_{k,n}} \int_{t_{k,n}}^{t_{k+1,n}} X_s ds \right) (W_{t_{k+1,n}} - W_{t_{k,n}})^2,$$

and $\alpha_n(X^l)$ similarly. It follows from Hölder's inequality that if $1/p+1/q=1$,

$$E|\alpha_n(X)|\leqq\left\{E\sum_k \frac{|W_{t_{k+1,n}}-W_{t_{k,n}}|^{2q}}{(t_{k+1,n}-t_{k,n})^{q-1}}\right\}^{1/q}\left\{E\sum_k \frac{\left(\int_{t_{k,n}}^{t_{k+1,n}}|X_s|ds\right)^p}{(t_{k+1,n}-t_{k,n})^{p/q}}\right\}^{1/p}$$

$$\|\alpha_n(X)\|_{L^1(\Omega)}\leqq C_p\|X\|_{L^p([0,1]\times\Omega)}. \tag{C.iii}$$

It then follows:

$$\begin{aligned}E\left|\alpha_n(X)-\int_0^1 X_t dt\right| &\leqq E|\alpha_n(X-X^l)|\\ &+E\left|\alpha_n(X^l)-\int_0^1 X_s^l ds\right|+E\int_0^1|X_s-X_s^l|ds\\ &\leqq E\left|\alpha_n(X^l)-\int_0^1 X_s^l ds\right|+(C_p+1)\|X-X^l\|_{L^p([0,1]\times\Omega)}.\end{aligned}$$

Clearly, $X^l\to X$ in $L^p([0,1]\times\Omega)$. On the other hand, for fixed l, $\alpha_n(X^l)\to\int_0^1 X_s^l ds$ in $L^1(\Omega)$ as $n\to\infty$. Indeed, the convergence in probability is immediate, and a slight modification in the proof of (C.iii) yields that $\forall p'\in]1,p[$,

$$\|\alpha_n(X^l)\|_{L^{p'}(\Omega)}\leqq C(p,p')\|X^l\|_{L^p([0,1]\times\Omega)},$$

so that the sequence $\{\alpha_n(X^l), n\in\mathbb{N}\}$ is uniformly integrable. The result now follows.

The proof of (C.ii) is quite similar. □

Lemma C.2. *Let*

$$X_t=\begin{pmatrix}x_t^1\\ x_t^2\end{pmatrix}$$

denote a two-dimensional continuous process, such that

$$\sum_{\{k;t_{k+1,n}\leqq t\}}(X_{t_{k+1,n}}^i-X_{t_{k,n}}^i)(X_{t_{k+1,n}}^j-X_{t_{k,n}}^j)\to\int_0^1 a_s^{ij}ds \tag{C.iv}$$

in probability, as $n\to\infty$, with $i,j=1,2$; where $\{a_t^{ij}, t\in[0,1]; i,j=1,2\}$ are measurable processes s.t.

$$\int_0^1|a_t^{ij}|dt<\infty p.s.;\quad i,j=1,2.$$

Let $\{Y_t, t\in[0,1]\}$ be a continuous process, and $\{Y_t^n, t\in[0,1]\}$ be measurable processes which converge a.s. to $\{Y_t\}$ as $n\to\infty$, uniformly with respect to $t\in[0,1]$.

Then $\forall i,j\in\{1,2\}$,

$$\sum_{k=0}^{n-1}Y_{t_{k,n}}^n(X_{t_{k+1,n}}^i-X_{t_{n,n}}^i)(X_{t_{k+1,n}}^j-X_{t_{k,n}}^j)\to\int_0^1 Y_t a_t^{ij}dt$$

in probability, as $n\to\infty$; $i,j=1,2$.

Proof. By a classical subsequence argument, it suffices to prove the result under the hypothesis that the convergence in (C.iv) holds a.s. For simplicity, we write t_k instead of $t_{k,n}$. Let first $\{Z_t, t \in [0,1]\}$ be a process which is constant on each element of a finite partition of $[0,1]$. Then from (C.iv):

$$\sum_{k=0}^{n-1} Z_{t_k}(X^i_{t_{k+1}} - X^i_{t_k})(X^j_{t_{k+1}} - X^i_{t_k}) \to \int_0^1 Z_t a_t^{ij}\, dt. \tag{C.v}$$

Let now $\{Z_t^p, t \in [0,1]; p \in \mathbb{N}\}$ be a sequence of processes, each possessing the properties of $\{Z_t\}$, s.t. $Z_t^p \to Y_t$ a.s., uniformly with respect to $t \in [0,1]$, as $p \to \infty$

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} (Y^n_{t_n} - Z^p_{t_k})(X^i_{t_{k+1}} - X^i_{t_k})(X^j_{t_{k+1}} - X^j_{t_k})\right| \\
&\quad \leqq \left(\sup_{t \in [0,1]} |Y_t^n - Z_t^p|\right)\left(\sum_{k=0}^{n-1} (X^i_{t_{k+1}} - X^i_{t_k})^2\right)^{1/2}\left(\sum_{k=0}^{n-1} (X^j_{t_{k+1}} - X^j_{t_k})^2\right)^{1/2} \times \\
&\quad \times \overline{\lim_{n\to\infty}} \left|\sum_{k=0}^{n-1} (Y^n_{t_k} - Z^p_{t_k})(X^i_{t_{k+1}} - X^i_{t_k})(X^j_{t_{k+1}} - X^j_{t_k})\right| \\
&\quad \leqq \left(\sup_{t \in [0,1]} |Y_t - Z_t^p|\right)\left(\int_0^1 a_t^{ii}\, dt\right)^{1/2}\left(\int_0^1 a_t^{jj}\, dt\right)^{1/2}.
\end{aligned} \tag{C.vi}$$

The result follows from (C.v) and (C.vi). □

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