The speed of propagation for KPP type problems. II - General domains

Henri Berestycki [a], François Hamel [b] and Nikolai Nadirashvili [c]

[a] EHESS, CAMS, 54 Boulevard Raspail, F-75006 Paris, France

[b] Université Aix-Marseille III, LATP, Faculté des Sciences et Techniques

Avenue Escadrille Normandie-Niemen, F-13397 Marseille Cedex 20, France

[c] CNRS, LATP, CMI, 39 rue F. Joliot-Curie, F-13453 Marseille Cedex 13, France

Abstract

This paper is devoted to nonlinear propagation phenomena in general unbounded domains of $\mathbb{R}^N$, for reaction-diffusion equations with Kolmogorov-Petrovsky-Piskunov (KPP) type nonlinearities. This article is the second in a series of two and it is the follow-up of the paper [8] which dealt which the case of periodic domains. This paper is concerned with general domains and we give various definitions of the spreading speeds at large times for solutions with compactly supported initial data. We study the relationships between these new notions and analyze their dependency on the geometry of the domain and on the initial condition. Some a priori bounds are proved for large classes of domains. The case of exterior domains is also discussed in detail. Lastly, some domains which are very thin at infinity and for which the spreading speeds are infinite are exhibited ; the construction is based on some new heat kernel estimates in such domains.

Contents

1 Introduction and main results

1.1 Introduction

This paper is concerned with nonlinear spreading and propagation phenomena for reaction-diffusion equations in general unbounded domains. We consider reaction terms of the Fisher or KPP (for Kolmogorov, Petrovsky, Piskunov) type. Propagation phenomena in a homogeneous framework are well understood and we will recall below the main results. This article is the second in a series of two and it is the follow-up of the article [8] (part I). Both papers deal with *heterogeneous* problems. Part I was concerned with equations with periodic coefficients in domains having periodic structures. The present paper (part II) deals with reaction-diffusion equations with constant coefficients, but in very general domains which are not periodic. We define and analyze various notions of asymptotic spreading speeds for solutions with compactly supported initial data. Before introducing the main notions and stating the main results, let us recall some basic features of the homogeneous framework in $\mathbb{R}^N$ and let us shortly recall some of the results in the periodic framework.

Consider first the Fisher-KPP equation :

$$u_t - \Delta u = f(u) \ \text{ in } \mathbb{R}^N. \tag{1.1}$$

It has been introduced in the celebrated papers of Fisher (1937, [18]) and KPP (1937, [32]) originally motivated by models in biology (u stands for the concentration of a species in such models). The main assumption is that f is say a $C^1(\mathbb{R}_+)$ function satisfying

$$\begin{cases} f(0) = f(1) = 0, \quad f'(1) < 0, \quad f'(0) > 0, \quad f > 0 \text{ in } (0,1), \quad f < 0 \text{ in } (1,+\infty), \\ f(s) \le f'(0)s \ \text{ for all } s \in [0,1]. \end{cases} \tag{1.2}$$

Archetypes of such nonlinearities are $f(s) = s(1-s)$ or $f(s) = s(1-s^2)$.

Two fundamental features of this equation account for its success in representing propagation (or invasion) and spreading. First, this equation has a family of planar travelling fronts. These are solutions of the form $u(t,x) = U(x \cdot e - ct)$ where e is a fixed vector of unit norm which is the direction of propagation, and $c > 0$ is the speed of the front. Here $U : \mathbb{R} \mapsto \mathbb{R}$ is given by

$$-U'' - cU' = f(U) \ \text{ in } \mathbb{R}, \quad U(-\infty) = 1, \ U(+\infty) = 0.$$

In the original paper of Kolmogorov, Petrovsky and Piskunov, it was proved that, under the above assumptions, there is a threshold value $c^* = 2\sqrt{f'(0)} > 0$ for the speed c. Namely, no fronts exist for $c < c^*$, and, for each $c \geq c^*$, there is a unique front U of the previous type. Uniqueness is up to shift in space or time variables.

Another fundamental property of this equation was established mathematically by Aronson and Weinberger (1978, [1]). It deals with the asymptotic speed of spreading. Namely, if u_0 is a nonnegative continuous function in $\mathbb{R}^N$ with compact support and $u_0 \not\equiv 0$, then the solution $u(t,x)$ of (1.1) with initial condition u_0 at time $t = 0$ spreads with the speed c^* in all directions for large times : as $t \to +\infty$,

$$\max_{|x| \leq ct} |u(t,x) - 1| \to 0 \text{ for each } c \in [0, c^*), \quad \text{and} \quad \max_{|x| \geq ct} u(t,x) \to 0 \text{ for each } c > c^*.$$

In Part I [8] and in an earlier paper [5], we introduced a general heterogeneous periodic framework extending (1.1). The types of equations which were considered there were :

$$u_t - \nabla \cdot (A(x)\nabla u) + q(x) \cdot \nabla u = f(x,u) \text{ in } \Omega, \quad \nu \cdot A\nabla u = 0 \text{ on } \partial\Omega, \tag{1.3}$$

where ν denotes the outward unit normal on $\partial\Omega$. Both the coefficients of the equation, namely the diffusion matrix $A(x)$, the drift $q(x)$ and the reaction term $f(x,s)$, as well as the geometry of the underlying domain Ω were assumed to be periodic. More precisely, there are $d \in \{1, \ldots, N\}$ and d positive real numbers $L_1, \ldots, L_d$ such that

$$\left\{ \begin{array}{l} \forall k \in L_1\mathbb{Z} \times \cdots \times L_d\mathbb{Z} \times \{0\}^{N-d}, \quad \Omega + k = \Omega \\ \exists\ C \geq 0,\ \forall\ x = (x_i)_{1 \leq i \leq N} \in \overline{\Omega}, \quad |x_{d+1}| + \cdots + |x_N| \leq C, \end{array} \right. \tag{1.4}$$

and the functions A, q and f are periodic with periods $L_1, \ldots, L_d$ in the variables $x_1, \ldots, x_d$. Given a unit direction $e \in \mathbb{R}^d \times \{0\}^{N-d}$, a pulsating travelling front in the direction e is a solution $u(t,x)$ of the type $u(t,x) = U(x \cdot e - ct, x)$, where $U = U(s,x)$ is periodic in the variables $x_1, \ldots, x_d$ (wtih periods $L_1, \ldots, L_d$) and $U(s,x) \to 1$ as $s \to -\infty$, $U(s,x) \to 0$ as $s \to +\infty$, uniformly with respect to $x \in \overline{\Omega}$ (assuming that $f(x,0) = f(x,1) = 0$). Under some natural assumptions on f (generalizing the hypothesis (1.2)) and on A and q, existence of pulsating fronts for, and only for, speeds $c \geq c^*(e)$ was proved in [5] and [8]. A variational formula for the minimal speed $c^*(e)$, in terms of some periodic eigenvalue problems) was also derived in [8]. These results extended some earlier results in dimension 1 (see e.g. [28, 42]) and in straight infinite cylinders with shear flows [13]. Let us mention here that other types of nonlinearities (combustion type, bistable type, other nonlinearities arising in population dynamics...) were also dealt with in the literature (see [3, 5, 9, 10, 12, 23, 24, 25, 27, 37, 39, 43, 44, 47] for some references on the existence of fronts in homogeneous or periodic media and formulæ for the speeds of propagation). Many papers dealt with the stability of travelling fronts in dimension 1, for equation (1.1) in $\mathbb{R}^N$, or in straight infinite cylinders (see e.g. [1, 11, 14, 17, 30, 32, 33, 38, 36, 40, 41, 46]).

Furthermore, the same type of spreading properties holds in the periodic framework as in the homogeneous one. Namely, for problem (1.3) under the assumption that $0 < f(x,s) \leq f'_s(x,0)s$ for all $s \in (0,1)$ and $x \in \overline{\Omega}$, Gärtner and Freidlin [21] and Freidlin [19] in the case of $\mathbb{R}^N$, and then Weinberger [45] in the general periodic framework described above, proved the existence of an asymptotic spreading speed (or ray speed) $w^*(e) > 0$ such that if

$u(t,x)$ solves (1.3) with a nonnegative, continuous and compactly supported initial condition $u_0 \not\equiv 0$, then,

$$\left\{\begin{array}{lll}\displaystyle\max_{x\in K,\ 0\le s\le ct,\ x+se\ \in\ \overline{\Omega}} |u(t,x+se)-1| & \to\ 0 & \text{if}\ \ 0\le c<w^*(e)\\ \displaystyle\max_{x\in K,\ s\ge ct,\ x+se\ \in\ \overline{\Omega}} u(t,x+ste) & \to\ 0 & \text{if}\ \ c>w^*(e),\end{array}\right. \qquad \text{as } t\to+\infty, \tag{1.5}$$

for any large enough compact set K so that the sets in which the maxima are taken are not empty. Moreover, $w^*(e)$ is given in terms of the minimal speeds of pulsating fronts by the geometrical formula $w^*(e)=\min_{\xi\in\mathbb{R}^d\times\{0\}^{N-d},\ \xi\cdot e>0}\ c^*(\xi)/(e\cdot\xi)$ ([45], see also [1, 17, 29, 30] for other results with other types of nonlinearities in the homogeneous case, and [36, 41] for equations with shear flows in straight infinite cylinders ; other results, including some with more general time-space scalings, were also obtained in [35]). The dependency of $c^*(e)$ and $w^*(e)$ on the coefficients of (1.3) (monotonicity, bounds, asymptotics) is analyzed in Part I [8] (see also [2, 4, 7, 15, 26, 31]).

We also studied in [8] the influence of the geometry of the periodic domain Ω (under assumption (1.4)) on the propagation speeds, for the equation

$$u_t=\Delta u+f(u)\ \text{ in }\Omega,\quad \nu\cdot\nabla u=0\ \text{ on }\partial\Omega$$

under assumption (1.2) for f. More precisely, one of the results was that

$$w^*(e)\le c^*(e)\le 2\sqrt{f'(0)}$$

and $w^*(e)=2\sqrt{f'(0)}$ if and only if Ω is invariant in the direction e (straight cylinder in the direction e, with bounded or unbounded section). Notice that this geometrical condition is also necessary for the equality $c^*(e)=2\sqrt{f'(0)}$ to hold (see [8]). In other words, the presence of holes or of an undulating boundary always hinder the progression or the spreading. Moreover, we proved in [8] that the speeds $c^*(e)$ are not in general monotone with respect to the size of the perforations. The inequality $w^*(e)\le c^*(e)$ always works. The equality $w^*(e)=c^*(e)$ $(=2\sqrt{f'(0)})$ holds in the homogeneous framework (1.1) in $\mathbb{R}^N$, but the inequality $w^*(e)\le c^*(e)$ may be strict in general (see Remark 1.12 in [8]).

1.2 Spreading speeds in general domains and main results

Let us now come back to the general non periodic case and deal with the Cauchy problem for the Fisher-KPP equation

$$\left\{\begin{array}{rcll} u_t &=& \Delta u+f(u) & \text{in }\Omega,\ t>0,\\ \nu\cdot\nabla u &=& 0 & \text{on }\partial\Omega,\ t>0,\\ u(0,x) &=& u_0(x) & \text{in }\Omega,\end{array}\right. \tag{1.6}$$

where Ω is an open connected and locally C^1 subset of $\mathbb{R}^N$, with outward unit normal ν. The initial condition u_0 is continuous, nonnegative, $u_0\not\equiv 0$ in $\overline{\Omega}$ and u_0 is compactly supported in $\overline{\Omega}$. One calls $\mathcal{E}$ the set of such functions u_0. The C^1 function $f\ :\ \mathbb{R}_+\to\mathbb{R}$ is assumed to satisfy (1.2). *This assumption on f is made from now on throughout the paper.* The function $u(t,x)$ is defined as the nondecreasing limit, as $n \to +\infty$, of the functions $u^n(t,x)$ which solve the equation $u^n_t = \Delta u^n + f(u^n)$ in $\Omega \cap B_n$ for $t > 0$, with boundary condition $\nu \cdot \nabla u^n = 0$ on $\partial\Omega \cap B_n$, $u^n = 0$ on $\overline{\Omega} \cap \partial B_n$ and initial condition $u^n(0,\cdot) = u_{0|\overline{\Omega}\cap\overline{B_n}}$. Here, B_r denotes the open euclidean ball of $\mathbb{R}^N$ with centre 0 and radius $r > 0$. Notice that, for all $t > 0$ and $x \in \overline{\Omega}$, $0 < u(t,x) < \max(\max_{\overline{\Omega}} u_0, 1)$ from the maximum principle.

Traveling or pulsating fronts do not exist anymore in this general non periodic framework, even if the notion of fronts can be generalized in very general geometries (see [6]). But the purpose of this paper is rather, first, to understand how we can extend the notions of asymptotic spreading speeds for the solutions of the Cauchy problem (1.6) with a compactly supported initial condition $u_0 \in \mathcal{E}$. Different definitions can be given, which are coherent with the periodic case. We then analyze the relationships between these general new definitions. Some other fundamental questions will then be asked : how do the spreading speeds depend on the initial condition ?, can they be compared to the spreading speed $2\sqrt{f'(0)}$ of the whole space $\mathbb{R}^N$? We will especially see that the answer to this last question is yes for a large class of domains, but is no in some domains for which the spreading speed is infinite. We also analyze in detail the case of exterior domains.

Let us now make more precise the definitions of spreading speeds in unbounded directions of Ω. In all what follows, one calls $B(z,r)$ the open euclidean ball of centre z and radius r in $\mathbb{R}^N$. We also take the convention that, for a function $v : E \subset \mathbb{R}^N \to \mathbb{R}$, $\max_{\emptyset} v = +\infty$.

Definition 1.1 *We say that Ω is unbounded in a direction $e \in \mathbb{S}^{N-1}$ if there exist $R_0 \geq 0$ and $s_0 \in \mathbb{R}$ such that $\overline{B(se,R_0)} \cap \overline{\Omega} \neq \emptyset$ for all $s \geq s_0$. With a slight abuse of notation, we set $\overline{B(y,0)} = \{0\}$ for all $y \in \mathbb{R}^N$. We then define $R(e) \geq 0$ as*

$$R(e) = \inf \{ \ R \geq 0, \ \exists \ s \in \mathbb{R}, \ \forall \ s' \geq s, \quad \overline{B(s'e,R)} \cap \overline{\Omega} \neq \emptyset \ \}.$$

As an example, a periodic domain Ω, satisfying (1.4), is unbounded in any unit direction $e \in \mathbb{R}^d \times \{0\}^{N-d}$.

Since problem (1.6) is well-understood when $N = 1$ (in which case unboundedness in the direction ± 1 means that $\Omega \supset \pm[a,+\infty)$ for some $a \in \mathbb{R}$), one can assume that $N \geq 2$ in the sequel.

Definition 1.2 *Let e be a direction in which Ω is unbounded and let $R(e) \geq 0$ be as in Definition 1.1. Let u be the solution of (1.6) with initial condition $u_0 \in \mathcal{E}$.*

We define the spreading speed of u in the direction e as

$$w^*(e,u_0) = \inf \left\{ c > 0, \ \forall \ A > R(e), \ \limsup_{t \to +\infty, \ s \geq ct,} \ \max_{x \in \overline{B(se,A)} \cap \overline{\Omega}} u(t,x) = 0 \right\}.$$

We set $w^*(e,u_0) = +\infty$ if there is no $c > 0$ such that $\sup_{s \geq ct} \max_{x \in \overline{B(se,A)} \cap \overline{\Omega}} u(t,x) \to 0$ as $t \to +\infty$ for all $A > R(e)$.

The nonnegative real number $w^*(e,u_0)$, if finite, can be viewed as the asymptotic speed of the leading edge of the solution u uniformly with respect to all cylinders along the direction e.

Another related notion, which is more precise in some sense, is that of spreading speed along a half-line.

Definition 1.3 *Under the same assumptions as in Definition 1.2, we define the spreading speed of u along the half-line $z+\mathbb{R}_+e$, for $z\in\mathbb{R}^N$, as*

$$w^*(e,z,u_0)=\inf\left\{c>0,\ \exists\ A>0,\ \limsup_{t\to+\infty,\ s\geq ct}\ \max_{x\in\overline{B(z+se,A)\cap\overline{\Omega}}} u(t,x)=0\right\}.$$

We set $w^*(e,z,u_0)=+\infty$ if for all $c>0$ and $A>0$, $\sup_{s\geq ct}\ \max_{x\in\overline{B(z+se,A)\cap\overline{\Omega}}} u(t,x)\not\to 0$ as $t\to+\infty$.

The nonnegative real number $w^*(e,z,u_0)$, if finite, is the asymptotic spreading speed of u locally along the line $z+\mathbb{R}_+e$ (notice that $w^*(e,z,u_0)=w^*(e,z+se,u_0)$ for all $s\in\mathbb{R}$). We would like to thank S. Luckhaus for pointing us out this other notion of spreading speed.

Remark 1.4 Under the above notations, call

$$R(e,z)=\inf\ \{\ R\geq 0,\ \exists\ s\in\mathbb{R},\ \forall\ s'\geq s,\quad \overline{B(z+s'e,R)}\cap\overline{\Omega}\neq\emptyset\ \}.$$

Notice that $R(e,0)=R(e)$ and that $R(e)-|z-(z\cdot e)e|\leq R(e,z)\leq R(e)+|z-(z\cdot e)e|$ for all $z\in\mathbb{R}^N$. If $R(e,z)>0$ and if there exists $s\in\mathbb{R}$ such that $\overline{B(z+s'e,R(e,z))}\cap\overline{\Omega}\neq\emptyset$ for all $s'\geq s$,[1] then the definition of $w^*(e,z,u_0)$ is equivalent to the following one :

$$w^*(e,z,u_0)=\inf\left\{c>0,\ \limsup_{t\to+\infty,\ s\geq ct}\ \max_{x\in\overline{B(z+se,R(e,z))\cap\overline{\Omega}}} u(t,x)=0\right\}.$$

In the case where $R(e,z)=0$ or if there is no $s\in\mathbb{R}$ such that $\overline{B(z+s'e,R(e,z))}\cap\overline{\Omega}\neq\emptyset$ for all $s'\geq s$, then the definition of $w^*(e,z,u_0)$ is equivalent to the following one :

$$w^*(e,z,u_0)=\inf\left\{c>0,\ \exists\ A>R(e,z),\ \limsup_{t\to+\infty,\ s\geq ct}\ \max_{x\in\overline{B(z+se,A)\cap\overline{\Omega}}} u(t,x)=0\right\}.$$

Furthermore, it immediately follows from the above definitions that

$$\forall\ \gamma>w^*(e,u_0),\ \forall\ A>R(e),\quad \max_{x\in\overline{B(\gamma te,A)\cap\overline{\Omega}}} u(t,x)\to 0 \text{ as } t\to+\infty$$

and that

$$\forall\ \gamma>w^*(e,z,u_0),\ \exists\ A>0,\quad \max_{x\in\overline{B(z+\gamma te,A)\cap\overline{\Omega}}} u(t,x)\to 0 \text{ as } t\to+\infty.$$

[1]Notice that the existence of such a real number s is not guaranteed in general, as the following example shows : in $\mathbb{R}^2$, call $x_k=(k^2,0)$ for $k\in\mathbb{N}$ and set $\Omega=\mathbb{R}^2\setminus\bigcup_{k\in\mathbb{N}}B(x_k,1+1/k)$. For $e=(1,0)$ and $z=(0,0)$, one has $R(e)=R(e,z)=1$ but there is no $s\in\mathbb{R}$ such that $\overline{B(z+s'e,1)}\cap\overline{\Omega}\neq\emptyset$ for all $s'\geq s$.

If Ω is a periodic domain satisfying (1.4), then these new notions of asymptotic spreading speeds are coherent with the previous one $w^*(e)$ characterized by (1.5), namely

$$w^*(e,z,u_0) = w^*(e,u_0) = w^*(e)$$

for all $u_0 \in \mathcal{E}$, for all $z \in \mathbb{R}^N$ and for all unit direction $e \in \mathbb{R}^d \times \{0\}^{N-d}$.

In general non periodic domains, it is clear that the inequality

$$w^*(e,z,u_0) \leq w^*(e,u_0)$$

holds for all $z \in \mathbb{R}^N$. However, the inequality may be strict, as the following theorem shows. We can furthermore make more precise the relationship between $w^*(e,u_0)$ and the $w^*(e,z,u_0)$ when z varies.

Before stating these results, let us introduce the following notation :

Definition 1.5 *Let Ω be unbounded in a direction $e \in \mathbb{S}^{N-1}$. For any y and z in $\mathbb{R}^N$, we say that Ω satisfies Hypothesis $H_{y,z}$ if there exist $s_0 \in \mathbb{R}$ and a bounded open set $\omega \subset \mathbb{R}^N$ such that 1) $\overline{B(y,R(e,y))} \cup \overline{B(z,R(e,z))} \subset \omega$ and 2) $\omega + se \ \cap \ \Omega$ is connected for all $s \geq s_0$ and $\partial(\omega + se \ \cap \ \Omega) \ \cap \ \Omega$ is of class $C^{2,\alpha}$ uniformly with respect to $s \geq s_0$, for some $\alpha > 0$.*

Theorem 1.6 (Dependency on z) *Let $N \geq 2$ and $e \in \mathbb{S}^{N-1}$ be given.*

a) For each domain Ω which is unbounded in the direction e and for each initial condition $u_0 \in \mathcal{E}$, one has

$$\sup_{z \in \mathbb{R}^N} w^*(e,z,u_0) = w^*(e,u_0). \tag{1.7}$$

b) Assume that Ω is unbounded in the direction e and that it satisfies Hypothesis $H_{y,z}$ for some y and z in $\mathbb{R}^N$. Then

$$\forall\ u_0 \in \mathcal{E}, \quad w^*(e,y,u_0) = w^*(e,z,u_0).$$

As a consequence, if Ω satisfies Hypothesis $H_{y,z}$ for all points y and z in $\mathbb{R}^N$, then $w^*(e,z,u_0) = w^*(e,u_0)$ for all $z \in \mathbb{R}^N$ and $u_0 \in \mathcal{E}$.

c) Given $z \in \mathbb{R}^N$, there are some domains Ω which are unbounded in the direction e and such that $w^*(e,z,u_0) < w^*(e,u_0)$ for all $u_0 \in \mathcal{E}$.

Part b) gives a sufficient condition for the spreading speed $w^*(e,z,u_0)$ not to depend on z. This condition is a type of relative connectedness and smoothness assumption in the direction e. It is especially satisfied if Ω is a smooth periodic domain of the type (1.4).

The proof of part c) relies of some precise heat kernel estimates as well as on some lower bounds of $w^*(e,u_0)$ for some domains containing half-spaces (see Remark 1.11 below). We actually prove more than what is stated in part c) : namely, up to translation and rotation, we exhibit some domains Ω for which $w^*(e,u_0) = 2\sqrt{f'(0)}$ for all $u_0 \in \mathcal{E}$ and $w^*(e,z,u_0) = 0$ for all $u_0 \in \mathcal{E}$ and for all $z \in \mathbb{R}^N$ such that $z \cdot e' > h$ (here, $e' \in \mathbb{S}^{N-1}$ is any given direction which is orthogonal to e, and h is any given real number).

Some other fundamental questions concern the possible *a priori* dependency of $w^*(e,u_0)$ or $w^*(e,z,u_0)$ on the initial condition $u_0 \in \mathcal{E}$, as well as some bounds for the spreading speeds. For periodic domains satisfying (1.4), one recalls that the spreading speeds do not depend on u_0 (or on z) and are bounded from above by $2\sqrt{f'(0)}$. We will see that the same properties hold for a general class of domains. This class of domain is defined now : quoting Davies [16], an open subset Ω of $\mathbb{R}^N$ is said to have the *extension property* if, for all $1 \le p \le +\infty$, there exists a bounded linear map E from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^N)$ such that Ef is an extension of f from Ω to $\mathbb{R}^N$ for all $f \in W^{1,p}(\Omega)$. This property is equivalent to the existence of $\varepsilon > 0$, $k \in \mathbb{N}$, $M > 0$ and of a countable sequence of open sets $(U_n)_{n\in\mathbb{N}}$ such that :

(i) if $x \in \partial\Omega$, then the ball with centre x and radius ε is contained in U_n for some n,

(ii) no point in $\mathbb{R}^N$ is contained in more than k distinct sets U_n,

(iii) for each n, there exists an isometry $T_n : \mathbb{R}^N \to \mathbb{R}^N$ and a Lipschitz-continuous function $\phi_n : \mathbb{R}^{N-1} \to \mathbb{R}$ whose Lipschitz norm is bounded by M. Moreover, $U_n \cap \Omega = U_n \cap T_n\Omega_n$, where

$$\Omega_n = \{(z_1, \cdots, z_N) \in \mathbb{R}^N,\ \phi_n(z_1, \cdots, z_{N-1}) < z_N\}.$$

Any smooth bounded or exterior domain satisfies the extension property. So does any smooth periodic domain.

The following theorem provides a general sufficient condition for the spreading speeds $w^*(e, u_0)$ and $w^*(e, z, u_0)$ not to depend on u_0.

Theorem 1.7 (Dependency on u_0) *Let Ω be a connected open subset of $\mathbb{R}^N$ satisfying the extension property, and assume that $\partial\Omega$ is globally of class $C^{2,\alpha}$ for some $\alpha > 0$. Let μ_r^z denote the Lebesgue-measure of $\Omega \cap B(z, r)$. Assume that there exists R_0 such that $\mu_r^z > 0$ for all $z \in \mathbb{R}^N$ and $r \ge R_0$, and that $\mu_{r+1}^z/\mu_r^z \to 1$ as $r \to +\infty$, uniformly in $z \in \mathbb{R}^N$. Let u be the solution of (1.6) with a given initial condition $u_0 \in \mathcal{E}$.*

Then $u(t, x) \to 1$ locally in $x \in \overline{\Omega}$ as $t \to +\infty$. Furthermore, Ω is unbounded in any direction $e \in \mathbb{S}^{N-1}$ and $w^*(e, u_0)$ and $w^*(e, z, u_0)$ do not depend on the initial condition u_0, provided that $u_0 < 1$.

As far as bounds for the spreading speeds are concerned, the speed $2\sqrt{f'(0)}$, which is the spreading speed if $\Omega = \mathbb{R}^N$, bounds from above the spreading speed if Ω is a periodic domain satisfying (1.4). Furthermore, the same property turns out to be true for the large class of domains satisfying the extension property :

Theorem 1.8 (General upper bound) *Let Ω be a locally C^1 connected open subset of $\mathbb{R}^N$ satisfying the extension property. Assume that Ω is unbounded in a direction e. Let u be the solution of (1.6) with a given initial condition $u_0 \in \mathcal{E}$. Then*

$$w^*(e, u_0) \le 2\sqrt{f'(0)} \tag{1.8}$$

and

$$\forall\ c > 2\sqrt{f'(0)}, \qquad \max_{|x|\ge ct,\ x\in\overline{\Omega}} u(t, x) \to 0 \quad \text{as} \quad t \to +\infty. \tag{1.9}$$

Under the assumptions of Theorem 1.8, inequality (1.8) especially yields

$$w^*(e,z,u_0) \le 2\sqrt{f'(0)}$$

for all $z \in \mathbb{R}^N$. Notice that property (1.9) is actually stronger than (1.8). Theorem 1.8 means that, for the large class of domains satisfying the extension property, the minimal speed of planar fronts, $2\sqrt{f'(0)}$, turns out to be an upper bound for the asymptotic spreading speeds in any direction e in which Ω is unbounded, as for periodic domains.

Furthermore, as already underlined, for a periodic domain Ω satisfying (1.4), for any unit vector $e \in \mathbb{R}^d \times \{0\}^{N-d}$ and for any $u_0 \in \mathcal{E}$, inequality (1.8) is an equality if and only if Ω is a cylinder in direction e. However, this property is far from true for general domains, as shows the following theorem for exterior domains :[2]

Theorem 1.9 (Exterior domain) *Let Ω be a connected exterior domain of class C^1. Then,*

$$\forall e \in \mathbb{S}^{N-1},\ \forall z \in \mathbb{R}^N,\ \forall u_0 \in \mathcal{E}, \quad w^*(e,z,u_0) = w^*(e,u_0) = 2\sqrt{f'(0)}.$$

Furthermore, if u solves (1.6) with $u_0 \in \mathcal{E}$, one has

$$\left\{\begin{array}{ll} \forall\ 0 \le c < 2\sqrt{f'(0)}, & \displaystyle\max_{|x|\le ct,\ x\in\overline{\Omega}} |u(t,x)-1| \to 0 \\ \forall\ c > 2\sqrt{f'(0)}, & \displaystyle\max_{|x|\ge ct,\ x\in\overline{\Omega}} u(t,x) \to 0 \end{array}\right. \quad \text{as } t \to +\infty. \tag{1.10}$$

Remark 1.10 The second property is clearly stronger than the first one. Theorem 1.9 actually extends the classical result of Aronson and Weinberger [1] mentionned above which was concerned with the case of the whole space $\mathbb{R}^N$.

Remark 1.11 (Lower bounds for the spreading speeds for domains containing semi-infinite cylinders) The arguments used in the proof of Theorem 1.9 imply that if Ω contains a semi-infinite cylinder in the direction e with large enough section, then $w^*(e,u_0)$ is bounded from below by a constant close to $2\sqrt{f'(0)}$. More precisely, given $\varepsilon > 0$, there exists $R_0 = R_0(\varepsilon) > 0$ such that if

$$\Omega \ \supset\ \mathcal{C}_{e,A,x_0,R} := \{x \in \mathbb{R}^N,\ x\cdot e > A,\ |(x-x_0) - ((x-x_0)\cdot e)e| < R\} \tag{1.11}$$

for some $A \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$ and $R > R_0$, then

$$w^*(e,u_0) \ge 2\sqrt{f'(0)} - \varepsilon \ \text{ and } \ w^*(e,z,u_0) \ge 2\sqrt{f'(0)} - \varepsilon \tag{1.12}$$

for all $u_0 \in \mathcal{E}$ and $z \in \mathbb{R}^N$ such that $|z - x_0 - ((z-x_0)\cdot e)e| < R$. We refer to the end of Section 3 for the proof.

As a consequence, if Ω contains a sequence of semi-infinite cylinders of the type $(\mathcal{C}_{e,A_n,x_{0,n},R_n})_{n\in\mathbb{N}}$ with $A_n \in \mathbb{R}$, $x_{0,n} \in \mathbb{R}^N$ and $R_n \to +\infty$ as $n \to +\infty$, then $w^*(e,u_0) \ge 2\sqrt{f'(0)}$ for all $u_0 \in \mathcal{E}$. Of course, if Ω satisfies the extension property as well, then $w^*(e,u_0) = 2\sqrt{f(0)}$ in this case. Lastly, notice that the property of containing a sequence of

[2]A domain $\Omega \subset \mathbb{R}^N$ is called exterior if $\mathbb{R}^N \backslash \Omega$ is compact.

such semi-infinite cylinders holds especially if Ω contains a "semi-infinite half-space" in the direction e, namely if

$$\Omega \supset \{x \in \mathbb{R}^N,\ x \cdot e > A,\ \pm(x \cdot e' - B) > 0\}$$

for some $(A, B) \in \mathbb{R}^2$ and $e' \in \mathbb{S}^{N-1}$ with $e' \cdot e = 0$. In this last case, one actually has that $w^*(e, z, u_0) \geq 2\sqrt{f'(0)}$ for all $u_0 \in \mathcal{E}$ and z such that $\pm(z \cdot e' - B) > 0$ (see Remark 3.3 below).

As already underlined, any periodic domain Ω satisfying (1.4) is such that $0 < w^*(e, u_0) \leq 2\sqrt{f'(0)}$ for all unit vector $e \in \mathbb{R}^d \times \{0\}^{N-d}$ and for all $u_0 \in \mathcal{E}$. Furthermore, the upper bound holds for a large class of domains (*see* Theorem 1.8). However, the following theorem asserts that the spreading speeds $w^*(e, u_0)$ and $w^*(e, z, u_0)$ may be zero or infinite for some domains.

Theorem 1.12 (Domains with zero or infinite spreading speeds) *a) There are some domains of $\mathbb{R}^2$ which satisfy the extension property and are unbounded in every direction $e \in \mathbb{S}^1$, and such that $w^*(e, z, u_0) = w^*(e, u_0) = 0$ for all $e \in \mathbb{S}^1$, $z \in \mathbb{R}^2$ and $u_0 \in \mathcal{E}$.*

b) For every $N \geq 2$ and $e \in \mathbb{S}^{N-1}$, there are some domains of $\mathbb{R}^N$, which do not satisfy the extension property, and such that $w^*(e, z, u_0) = w^*(e, u_0) = +\infty$ for all $z \in \mathbb{R}^N$ and $u_0 \in \mathcal{E}$.

Therefore, even in the class of domains satisfying the extension property, there are domains for which the asymptotic speeds $w^*(e, z, u_0)$ and $w^*(e, u_0)$ are zero in any direction e (such a phenomenon does not happen under the periodicity condition (1.4). We actually exhibit in the proof of Theorem 1.12 some domains which have the shape of a spiral and for which the asymptotic spreading speeds are zero in all directions.

Furthermore, there is no universal upper bound without the extension property. Some domains with an infinite cusp have infinite spreading speeds (see the proof of Theorem 1.12, part b). For such domains, we prove some new specific lower bounds for the heat kernel (*see* Lemma 4.2 in Section 4.3 below).

1.3 Other related notions

Here, we would like to mention some other notions of spreading speeds. We compare them to the notions introduced in Definitions 1.2 and 1.3 and state their main properties.

First, given a connected C^1 open subset Ω of $\mathbb{R}^N$, given $e \in \mathbb{S}^{N-1}$ and $u_0 \in \mathcal{E}$, we can define the asymptotic spreading speed of the leading edge of the solution u of (1.6) in the direction e, uniformly with respect to the directions which are orthogonal to e, as

$$w^{**}(e, u_0) = \inf \left\{ c > 0,\ \lim_{t \to +\infty} \sup_{x \cdot e \geq ct,\ x \in \overline{\Omega}} u(t, x) = 0 \right\},$$

provided that Ω satisfies

$$\exists\ s \in \mathbb{R},\ \forall\ s' \geq s, \quad \{x \in \mathbb{R}^N,\ x \cdot e \geq s'\} \cap \overline{\Omega} \neq \emptyset. \tag{1.13}$$

Notice that if Ω is unbounded in the direction e in the sense of Definition 1.1, then assumption (1.13) is immediately satisfied. This notion of asymptotic spreading speed $w^{**}(e,u_0)$ is rougher than the previous ones $w^*(e,u_0)$ or $w^*(e,z,u_0)$, and it does not give a precise description of where or in which precise direction the leading edge of the solution u moves. However, we can compare it to the previous notions $w^*(e,u_0)$ and $w^*(e,z,u_0)$ and we can derive some properties of $w^{**}(e,u_0)$ from the above results.

It is immediate to check that if Ω satisfies (1.13) and if it is unbounded in a direction $e'\in\mathbb{S}^{N-1}$ such that $e'\cdot e>0$, then

$$\forall\ u_0\in\mathcal{E},\ \forall\ z\in\mathbb{R}^N,\quad w^{**}(e,u_0)\ge w^*(e',u_0)\times(e'\cdot e)\ (\ \ge w^*(e',z,u_0)\times(e'\cdot e)\ ).$$

It may then happen that $w^{**}(e,u_0)>w^*(e,u_0)$ for all $u_0\in\mathcal{E}$. For instance, in $\mathbb{R}^2$, call $H=\{x\in\mathbb{R}^2,\ x_2-x_1>0\}$, let $(a_n)_{n\in\mathbb{N}^*}$ be a sequence of negative numbers such that $a_n/n\to-\infty$ as $n\to+\infty$, let

$$\Gamma=\bigcup_{n\in\mathbb{N}}\ (\ [2n,2n+1]\times\{0\}\ \cup\ [2n+1,2n+2]\times\{a_{n+1}\}\ \cup\ \{2n+1,2n+2\}\times[a_{n+1},0]\ )$$

and let Ω be a smooth open connected domain satisfying the extension property and such that

$$H\cup\Gamma\quad\subset\quad\Omega\quad\subset\quad\{x\in\mathbb{R}^2,\ d(x,H\cup\Gamma)<1/3\},$$

where $d(x,E)$ denotes the euclidean distance of a point x to a set E. With $e=(1,0)$ and $e'=(1/\sqrt{2},1/\sqrt{2})$, one can check that $w^*(e,u_0)=0$ for all $u_0\in\mathcal{E}$ (by using the same arguments as in the proofs of Theorem 1.8 or Theorem 1.12, part a)), while $w^*(e',u_0)=2\sqrt{f'(0)}$ for all $u_0\in\mathcal{E}$ (because of Theorem 1.8 and Remark 1.11). Thus,

$$\forall\ u_0\in\mathcal{E},\quad w^{**}(e,u_0)\ge\sqrt{2}\sqrt{f'(0)}>0=w^*(e,u_0).$$

Furthermore, with the same arguments as in the proofs of Theorems 1.7, 1.8, 1.9 and 1.12, the following properties hold :

1) if Ω satisfies the general assumptions of Theorem 1.7, then assumption (1.13) is satisfied for all $e\in\mathbb{S}^{N-1}$ and $w^{**}(e,u_0)$ does not depend on $u_0\in\mathcal{E}$, provided that $u_0<1$;

2) if Ω satisfies the assumptions of Theorem 1.8 (extension property), then –because of (1.9)– $w^{**}(e,u_0)\le2\sqrt{f'(0)}$ for all $u_0\in\mathcal{E}$ and for any direction $e\in\mathbb{S}^{N-1}$ such that (1.13) holds ;

3) if Ω is an exterior domain, then –because of (1.10)– $w^{**}(e,u_0)=2\sqrt{f'(0)}$ for all $e\in\mathbb{S}^{N-1}$ and for all $u_0\in\mathcal{E}$;

4) with the same examples as in Theorem 1.12, there are some domains of $\mathbb{R}^2$ satisfying (1.13) for all $e\in\mathbb{S}^1$ and such that $w^{**}(e,u_0)=0$ for all $e\in\mathbb{S}^1$ and for all $u_0\in\mathcal{E}$;

5) given $e\in\mathbb{S}^{N-1}$, there are some domains of $\mathbb{R}^N$ satisfying (1.13) and such that $w^{**}(e,u_0)=+\infty$ for all $u_0\in\mathcal{E}$.

Other notions are these of asymptotic spreading speeds, locally uniformly in the direction e or locally along a line $z+\mathbb{R}_+e$, of the expanding region where u converges to 1. Namely, if Ω is unbounded in a direction $e \in \mathbb{S}^{N-1}$ and if u solves (1.6) with a given initial condition $u_0 \in \mathcal{E}$, we can define, under the same notations as above,

$$w_*(e,u_0) = \sup\Big\{c>0,\quad u(t,x)\to 1 \text{ as } t\to+\infty \text{ locally uniformly in } x\in\overline{\Omega} \text{ and } \forall\ A>R(e),\ \lim_{\tau\to+\infty}\ \limsup_{t\to+\infty,\ \tau\le s\le ct}\ \max_{x\in\overline{B(se,A)}\cap\overline{\Omega}} |u(t,x)-1| = 0\Big\}$$

and, for $z\in\mathbb{R}^N$,

$$w_*(e,z,u_0) = \sup\Big\{c>0,\quad u(t,x)\to 1 \text{ as } t\to+\infty \text{ locally uniformly in } x\in\overline{\Omega} \text{ and } \exists\ A>0,\ \lim_{\tau\to+\infty}\ \limsup_{t\to+\infty,\ \tau\le s\le ct}\ \max_{x\in\overline{B(z+se,A)}\cap\overline{\Omega}} |u(t,x)-1| = 0\Big\}.$$

By convention, we set $w_*(e,u_0)=0$ if $u(t,x)\to 1$ as $t\to+\infty$ locally uniformly in $x\in\overline{\Omega}$ but if there is no $c>0$ such that, for all $A>R(e)$, $\limsup_{t\to+\infty,\ \tau\le s\le ct}\max_{x\in\overline{B(se,A)}\cap\overline{\Omega}}\ |u(t,x)-1|\to 0$ as $\tau\to+\infty$. We set $w_*(e,z,u_0)=0$ if $u(t,x)\to 1$ as $t\to+\infty$ locally uniformly in $x\in\overline{\Omega}$ but if $\limsup_{t\to+\infty,\ \tau\le s\le ct}\ \max_{x\in\overline{B(z+se,A)}\cap\overline{\Omega}}\ |u(t,x)-1|\ \not\to 0$ as $\tau\to+\infty$, for any $c>0$ and $A>0$. Lastly, we set $w_*(e,u_0)=w_*(e,z,u_0)=-\infty$ if $u(t,x)$ does not converge to 1 locally uniformly in $x\in\overline{\Omega}$ as $t\to+\infty$.

It follows immediately from the above definitions that

$$w_*(e,u_0)\le w_*(e,z,u_0)\le w^*(e,z,u_0)\le w^*(e,u_0)$$

for all $z\in\mathbb{R}^N$ and $u_0\in\mathcal{E}$. if Ω is a periodic domain satisfying (1.4), then, because of (1.5), the equality holds for all $e\in\mathbb{R}^d\times\{0\}^{N-d}$, $z\in\mathbb{R}^N$ and $u_0\in\mathcal{E}$. It is an interesting open question to ask if the equality $w_*(e,z,u_0)=w^*(e,z,u_0)$ always hold, or if there are some domains for which the inequality $w_*(e,z,u_0)<w^*(e,z,u_0)$ may be strict.

Furthermore, with the same arguments as the ones used in the next sections, one can prove the following properties :

1) if Ω is unbounded in a direction $e\in\mathbb{S}^{N-1}$, then

$$\forall\ u_0\in\mathcal{E},\quad w_*(e,u_0)=\inf_{z\in\mathbb{R}^N} w_*(e,z,u_0)\ ;$$

2) if Ω is unbounded in a direction $e\in\mathbb{S}^{N-1}$ and satisfies hypothesis $H_{y,z}$ for some points y and z in $\mathbb{R}^N$, and if $u_0\in\mathcal{E}$ is less than 1, then $w_*(e,y,u_0)=w_*(e,z,u_0)$;

3) if Ω satisfies the general assumptions of Theorem 1.7, then $w_*(e,u_0)$ and $w_*(e,z,u_0)$ are nonnegative and do not depend on $u_0\in\mathcal{E}$, provided that $u_0<1$;

4) if Ω satisfies the assumptions of Theorem 1.8 (extension property), then –because of (1.9)– $w_*(e,u_0)\le w_*(e,z,u_0)\le 2\sqrt{f'(0)}$ for all $u_0\in\mathcal{E}$, $z\in\mathbb{R}^N$ and for any direction $e\in\mathbb{S}^{N-1}$ in which Ω is unbounded ;

5) if Ω is an exterior domain, then –because of (1.10)– $w_*(e,u_0)=w_*(e,z,u_0)=2\sqrt{f'(0)}$ for all $e\in\mathbb{S}^{N-1}$, for all $z\in\mathbb{R}^N$ and for all $u_0\in\mathcal{E}$;

6) there are some domains of $\mathbb{R}^2$ which are unbounded in all directions $e\in\mathbb{S}^1$ and such that $w_*(e,u_0)=w_*(e,z,u_0)=0$ for all $e\in\mathbb{S}^1$, for all $z\in\mathbb{R}^2$ and for all $u_0\in\mathcal{E}$ (notice

that such domains are constructed in Section 4.2 so that the assumptions of Theorem 1.7 are satisfied, thus u converges to 1 locally in $\overline{\Omega}$ as $t \to +\infty$ and $0 \le w_*(e,u_0) \le w_*(e,z,u_0))$;

7) given $e \in \mathbb{S}^{N-1}$, there are some domains of $\mathbb{R}^N$ which are unbounded in the direction e and such that $w_*(e,u_0) = w_*(e,z,u_0) = +\infty$ for all $u_0 \in \mathcal{E}$ and $z \in \mathbb{R}^N$.

Outline of the paper. The paper is organized as follows : Section 2 is devoted to the proof of the general properties (Theorems 1.6 (parts a) and b)), 1.7, 1.8), Section 3 is concerned with exterior domains (Theorem 1.9) and Section 4 deals with the construction of some domains for which the spreading speeds $w^*(e,z,u_0)$ really depend on z (Theorem 1.6, part c)). We also exhibit in Section 4 some domains with zero or infinite speeds of propagation (Theorem 1.12).

2 General properties

This section is devoted to the proofs of Theorems 1.6 (parts a) and b)), 1.7, 1.8. More precisely, we prove in Section 2.1 the relationship between the spreading speeds $w^*(e,u_0)$ and $w^*(e,z,u_0)$. In Section 2.2, we study the dependency on u_0. Lastly, in Section 2.3, we prove the general upper bound for the spreading speeds in the large class of domains satisfying the extension property.

2.1 Relationship between $w^*(e,z,u_0)$ and $w^*(e,u_0)$

This section is devoted to the proof of parts a) and b) of Theorem 1.6. The proof of part c) is given in Section 4. Let us begin with the

Proof of formula (1.7) in Theorem 1.6. Let $\Omega \subset \mathbb{R}^N$ be unbounded in a given direction $e \in \mathbb{S}^{N-1}$ and let $u_0 \in \mathcal{E}$ be given. Call $R = R(e)$ the real number defined in Definition 1.1.

As already emphasized, the inequality

$$0 \le w^*(e,z,u_0) \le w^*(e,u_0)$$

follows from Definitions 1.2 and 1.3, for all $z \in \mathbb{R}^N$. Notice also that formula (1.7) is immediate in the case where $w^*(e,u_0) = 0$. One can then assume here that $w^*(e,u_0) > 0$. Fix any $\varepsilon \in (0, w^*(e,u_0))$ and set

$$\gamma = w^*(e,u_0) - \varepsilon.$$

There exists $A > R$ such that

$$\sup_{s \ge \gamma t} \ \max_{x \in \overline{B(se,A)} \cap \overline{\Omega}} u(t,x) \not\to 0 \ \text{ as } \ t \to +\infty.$$

Therefore, there exist some sequences $(t_n)_{n\in\mathbb{N}} \to +\infty$, $(s_n)_{n\in\mathbb{N}}$ such that $s_n \ge \gamma t_n$, and some points $(x_n)_{n\in\mathbb{N}}$ in $\overline{B_A}$ such that $x_n + s_n e \in \overline{\Omega}$ and

$$\liminf_{n\to+\infty} u(t_n, x_n + s_n e) > 0. \tag{2.1}$$

Up to extraction of some subsequence, one can assume that $x_n \to z \in \overline{B_A}$.

We now claim that

$$w^*(e, z, u_0) \geq \gamma.$$

Assume not. Then, owing to Definition 1.3, there is $A' > 0$ such that

$$\sup_{s \geq \gamma t} \max_{x \in \overline{B_{A'}},\ x+z+se\ \in\ \overline{\Omega}} u(t, x + z + se) \to 0 \ \text{ as } \ t \to +\infty.$$

For n large enough, $x_n - z \in \overline{B_{A'}}$. On the other hand, $s_n \geq \gamma t_n$ and $(x_n - z) + z + s_n e = x_n + s_n e \ \in\ \overline{\Omega}$. Thus, $u(t_n, x_n + s_n e) \to 0$ as $n \to +\infty$. This contradicts (2.1).

Therefore, the claim $w^*(e, z, u_0) \geq \gamma$ is proved. Hence,

$$w^*(e, z, u_0) \geq w^*(e, u_0) - \varepsilon$$

for all $\varepsilon > 0$ and formula (1.7) follows. □

Proof of part b) of Theorem 1.6. Assume that Ω satisfies Hypothesis $H_{y,z}$ for some points y and z in $\mathbb{R}^N$. Let the real number s_0 and the open set ω be as in Hypothesis 1.5.

From the definition of $R(e, y)$ and $R(e, z)$, and from the smoothness assumption in Hypothesis 1.5, there exist $\beta > 0$, $\gamma > 0$, $s_1 \geq s_0$ and a map $s \mapsto w_s \in \mathbb{R}^N$ defined in $[s_1, +\infty)$ such that

$$\begin{cases} \forall\ \delta \in [0, \beta], & \overline{B(y + \delta e, R(e, y) + \beta)}\ \cup\ \overline{B(z + \delta e, R(e, z) + \beta)}\ \subset\ \omega, \\ \forall\ s \geq s_1, & \overline{B(y + se, R(e, y) + \beta)} \cap \overline{\Omega} \neq \emptyset, \quad \overline{B(z + se, R(e, z) + \beta)} \cap \overline{\Omega} \neq \emptyset, \\ \forall\ s \geq s_1, & B(w_s, \gamma)\ \subset\ \omega + se\ \cap\ \Omega. \end{cases}$$

Fix any u_0 in $\mathcal{E}$ and let u solve (1.6). If both spreading speeds $w^*(e, y, u_0)$ and $w^*(e, z, u_0)$ are infinite, then the desired conclusion $w^*(e, y, u_0) = w^*(e, z, u_0)$ follows. Assume now that at least one of the spreading speeds, say $w^*(e, z, u_0)$, is finite. Fix any $c > w^*(e, z, u_0)$. From the connectedness and smoothness assumptions in Hypothesis 1.5, Harnack inequality yields the existence of $\eta > 0$ such that

$$\forall\ t \geq 1,\ \forall\ s \geq s_1, \quad \max_{x \in \overline{B(y+se, R(e,y)+\beta)} \cap \overline{\Omega}} u(t, x) \leq \eta \min_{x \in \overline{B(z+se+\beta e, R(e,z)+\beta)} \cap \overline{\Omega}} u(t + \beta/c, x). \tag{2.2}$$

From Definition 1.3, there exists $A > 0$ such that

$$\sup_{s \geq ct} \max_{x \in \overline{B(z+se, A)} \cap \overline{\Omega}} u(t, x) \to 0 \text{ as } t \to +\infty.$$

As already underlined in Remark 1.4, one can assume, even if it means decreasing A, that $A \leq R(e, z) + \beta$.

Let ε be any positive real number. There exists then $t_0 \geq \max(1, s_1/c)$ such that

$$\forall\ t \geq t_0,\ \forall\ s \geq ct, \quad \max_{x \in \overline{B(z+se, A)} \cap \overline{\Omega}} u(t, x) \leq \varepsilon.$$

Choose any $t \geq t_0$ and $s \geq ct$. Observe that $t + \beta/c \geq t_0$ and $s + \beta \geq c(t + \beta/c)$, whence

$$\max_{x \in \overline{B(z+se+\beta e, A)} \cap \overline{\Omega}} u(t + \beta/c, x) \leq \varepsilon. \tag{2.3}$$

Since $t \geq t_0 \geq 1$ and $s \geq ct \geq ct_0 \geq s_1$, and since $A \leq R(e,z) + \beta$, it follows from (2.2) and (2.3) that

$$\max_{x \in \overline{B(y+se, R(e,y)+\beta) \cap \Omega}} u(t,x) \leq \eta \min_{x \in \overline{B(z+se+\beta e, A) \cap \Omega}} u(t+\beta/c, x) \leq \eta\, \varepsilon.$$

Since this is true for all $t \geq t_0$ and $s \geq ct$ and since η is independent of ε, one gets that

$$\sup_{s \geq ct} \max_{x \in \overline{B(y+se, R(e,y)+\beta) \cap \Omega}} u(t,x) \to 0 \text{ as } t \to +\infty.$$

Therefore, $w^*(e,y,u_0) \leq c$. Since this inequality holds for all $c > w^*(e,z,u_0)$, one concludes that $w^*(e,y,u_0)$ is finite and satisfies $w^*(e,y,u_0) \leq w^*(e,z,u_0)$.

By changing the role of y and z, one then concludes that $w^*(e,y,u_0) = w^*(e,z,u_0)$ and the proof of part b) of Theorem 1.6 is complete. □

2.2 Independence of $w^*(e,u_0)$ and $w^*(e,z,u_0)$ from u_0

The proof of Theorem 1.7 is based on some auxiliary results. Let us first introduce a few notations. If D is an open subset of $\mathbb{R}^N$ such that $\overline{\Omega} \cap D \neq \emptyset$, we call

$$\lambda_D = \inf_{\psi \in C^1_c(\overline{\Omega} \cap D),\ \psi \not\equiv 0} \frac{\displaystyle\int_{\Omega \cap D} |\nabla \psi|^2}{\displaystyle\int_{\Omega \cap D} \psi^2},$$

where $C^1_c(\overline{\Omega} \cap D)$ denotes the set of functions which are of class C^1 in $\overline{\Omega} \cap D$ and have a support which is compactly included in $\overline{\Omega} \cap D$. Under the assumptions of Theorem 1.7, for all $r \geq R_0$ and $z \in \mathbb{R}^N$, we denote

$$\lambda^z_r = \lambda_{B(z,r)},$$

where we recall that $B(z,r)$ denotes the open euclidean ball of radius r and centre z.

Lemma 2.1 *Under the assumptions of Theorem 1.7,*

$$\lambda^z_r \to 0 \textit{ as } r \to +\infty \textit{ uniformly in } z \in \mathbb{R}^N.$$

Proof. Fix a family $(\zeta_r)_{r \geq R_0}$ of $C^\infty(\mathbb{R}^N)$ functions such that, for each $r \geq R_0$, the support of ζ_r is included in $B(0,r+1)$ and $\zeta_r = 1$ in $B(0,r)$. One can choose the functions ζ_r so that $\|\zeta_r\|_{C^1(B(0,r+1))} \leq C$, for some constant C independent from $r \geq R_0$.

Let $r \geq R_0$ and z be any point in $\mathbb{R}^N$. Call ζ^z_r the function defined by $\zeta^z_r(x) = \zeta_r(x-z)$ for all $x \in \mathbb{R}^N$. One has

$$0 \leq \lambda^z_{r+1} \leq \frac{\displaystyle\int_{\Omega \cap B(z,r+1)} |\nabla \zeta^z_r|^2}{\displaystyle\int_{\Omega \cap B(z,r+1)} (\zeta^z_r)^2} \leq C^2 \frac{|\Omega \cap (B(z,r+1) \backslash B(z,r))|}{|\Omega \cap B(z,r)|} \leq C^2 \frac{\mu^z_{r+1} - \mu^z_r}{\mu^z_r},$$

where $|E|$ denotes the Lebesgue-measure of a measurable set $E \subset \mathbb{R}^N$. Since $\mu^z_{r+1}/\mu^z_r \to 1$ uniformly in $z \in \mathbb{R}^N$ as $r \to +\infty$, the conclusion of Lemma 2.1 follows. □

It is immediate by definition that $D \mapsto \lambda_D$ is nonincreasing with the respect to the inclusion (in the class of open sets D such that $\overline{\Omega} \cap D \neq \emptyset$), and it is known that if Ω and D are smooth, with outward unit normals ν and ν_D such that $\nu(x) \cdot \nu_D(x) = 0$ for all $x \in \partial\Omega \cap \partial D$, then there is an eigenfunction function $\psi_D \in C^2(\Omega \cap D) \cap C^1(\overline{\Omega} \cap \overline{D})$ such that

$$\left\{\begin{array}{rcll} -\Delta\psi_D & = & \lambda_D \psi_D & \text{in } \Omega \cap D \\ \psi_D & \geq & 0 & \text{in } \overline{\Omega} \cap \overline{D} \\ \psi_D & = & 0 & \text{on } \overline{\Omega} \cap \partial D \\ \partial_\nu \psi_D & = & 0 & \text{on } \partial\Omega \cap D \\ \|\psi_D\|_{L^\infty(\Omega\cap D)} & = & 1. & \end{array}\right. \tag{2.4}$$

Furthermore, if $\Omega \cap D$ is connected, then $\psi_D > 0$ in $\overline{\Omega} \cap D$.

Proposition 2.2 *Let Ω be a domain satisfying the assumptions of Theorem 1.7. Let $g : [0, +\infty) \to \mathbb{R}$ be a C^1 function such that $g(0) = g(1) = 0$, $g'(0) > 0$, $g > 0$ in $(0, 1)$ and $g < 0$ in $(1, +\infty)$. Let u be a classical bounded solution of*

$$\left\{\begin{array}{rl} \Delta u + g(u) = 0 & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{2.5}$$

Then $u \equiv 0$ or $u \equiv 1$.

If compared to Proposition 1.14 in Part I ([8]), the proof of this Proposition 1.14 strongly used the periodicity of the domain but could deal with equations involving gradient terms. Proposition 2.2 is restricted to the case of equation (2.5) without gradient terms but it deals with the case of domains Ω which may or may not be periodic.

Proof of Proposition 2.2. Without loss of generality, one can assume that $u \not\equiv 0$, whence $u > 0$ in $\overline{\Omega}$ from the strong maximum principle and Hopf lemma.

First, from of Lemma 2.1, there exists $R > R_0$ such that

$$0 \leq \lambda^z_R < \frac{g'(0)}{2} \text{ for all } z \in \mathbb{R}^N.$$

Assume now $\inf_{\overline{\Omega}} u = 0$. There exists then a sequence $(z_n)_{n\in\mathbb{N}}$ in $\overline{\Omega}$ such that

$$u(z_n) \to 0 \text{ as } n \to +\infty.$$

From Harnack inequality, it follows that

$$\max_{z\in\overline{\Omega}\cap\overline{B(z_n,R+1)}} u(z) \to 0 \text{ as } n \to +\infty.$$

Therefore, there exists $N \in \mathbb{N}$ such that

$$0 = \Delta u + g(u) > \Delta u + \frac{g'(0)}{2} u \text{ in } \Omega \cap B(z_N, R+1) \tag{2.6}$$

because $u > 0$ in $\overline{\Omega}$ and $g'(0) > 0$. Let now D be a smooth open subset of $\mathbb{R}^N$ such that $B(z_N, R) \subset D \subset B(z_N, R+1)$ and $\nu(x) \cdot \nu_D(x) = 0$ for all $x \in \partial\Omega \cap \partial D$, and let ψ_D solve (2.4). Since u is continuous and positive in $\overline{\Omega}$, there exists $\varepsilon_0 > 0$ such that

$$\varepsilon\psi_D \leq u \text{ in } \overline{\Omega} \cap \overline{D}$$

for all $\varepsilon \in [0, \varepsilon_0]$. Let

$$\varepsilon^* = \sup \{\varepsilon > 0,\ \varepsilon\psi_D \leq u \text{ in } \overline{\Omega} \cap \overline{D}\}.$$

0ne has $0 < \varepsilon_0 \leq \varepsilon^* < +\infty$, and $\varepsilon^*\psi_D \leq u$ in $\overline{\Omega} \cap \overline{D}$. Furthermore, there is a point $x \in \overline{\Omega} \cap \overline{D}$ such that $\varepsilon^*\psi_D(x) = u(x)$. Since $u > 0$ is $\overline{\Omega}$, it follows that $x \in \overline{\Omega} \cap D$. But $\lambda_D \leq \lambda_{B(z_N,R)} = \lambda_R^{z_N} \leq g'(0)/2$, whence $\varepsilon^*\psi_D$ satisfies

$$-\Delta(\varepsilon^*\psi_D) \leq \frac{g'(0)}{2}\varepsilon^*\psi_D \text{ in } \Omega \cap D.$$

If $x \in \Omega \cap D$, it follows from the strong maximum principle that $\varepsilon^*\psi_D \equiv u$ in the connected component of $\Omega \cap D$ containing x. This is impossible because of the strict inequality in (2.6) and the positivity of u and $g'(0)$.

As a consequence, $\varepsilon^*\psi_D < u$ in $\Omega \cap D$ and $x \in \partial\Omega \cap D$. But Hopf lemma then yields $\partial_\nu(\varepsilon^*\psi_D)(x) > \partial_\nu u(x)$, which is again impossible because both quantities are zero.

One has then reached a contradiction. Therefore,

$$m := \inf_{\overline{\Omega}}\ u > 0.$$

Choose now ξ_0 such that

$$0 < \xi_0 < \min(m, 1),$$

and let $\xi(t)$ be the solution of $\dot{\xi}(t) = g(\xi(t))$ with $\xi(0) = \xi_0$. Since $g > 0$ on $(0, 1)$ and $g(1) = 0$, one gets that $\xi'(t) > 0$ for all $t \geq 0$ and $\xi(+\infty) = 1$. On the other hand, since u solves (2.5), the parabolic maximum principle implies that $u(z) \geq \xi(t)$ for all $z \in \overline{\Omega}$ and $t \geq 0$. Thus, $m \geq 1$.

Similarly, using the fact that $g < 0$ in $(1, +\infty)$, one gets that $M := \sup_{\overline{\Omega}} u \leq 1$. As a conclusion, $u \equiv 1$, and the proof of Proposition 2.2 is complete. □

Let us now turn to the

Proof of Theorem 1.7. Under the notations of Theorem 1.7, it follows from the strong parabolic maximum principle that $u(t, x) > 0$ for all $t > 0$ and $x \in \overline{\Omega}$. From Lemma 2.1, there exists $R > R_0$ such that $\lambda_R^0 \leq f'(0)/2$. Let D be a smooth open subset of $\mathbb{R}^N$ such that $B(0, R) \subset D \subset B(0, R+1)$ and $\nu(x) \cdot \nu_D(x) = 0$ for all $x \in \partial\Omega \cap \partial D$, and let ψ_D solve (2.4). By continuity, one has

$$u(1, \cdot) \geq \varepsilon\psi_D \ \text{ in } \overline{\Omega} \cap \overline{D}$$

for some $\varepsilon > 0$ small enough. Since $\lambda_D \le \lambda_R^0 \le f'(0)/2$, one can choose $\varepsilon > 0$ small enough so that $\varepsilon \le 1$ and

$$\Delta(\varepsilon\psi_D) + f(\varepsilon\psi_D) = -\lambda_D\varepsilon\psi_D + f(\varepsilon\psi_D) \ge 0 \text{ in } \Omega \cap D.$$

Therefore,

$$\forall t \ge 1,\ \forall x \in \overline{\Omega}, \quad u(t+1,x) \ge v(t,x), \tag{2.7}$$

where v solves (1.6) with initial condition

$$v(0,x) = \begin{cases} \varepsilon\psi_D(x) & \text{if } x \in \overline{\Omega} \cap \overline{D} \\ 0 & \text{if } x \in \overline{\Omega} \backslash \overline{D}. \end{cases}$$

From the choice of ε, $v(0,\cdot)$ is a subsolution of the corresponding elliptic equation, whence $v(t,x)$ is nondecreasing in t for each $x \in \overline{\Omega}$. Since $0 \le \varepsilon\psi_D \le \varepsilon \le 1$ and $f(1) = 0$, one also has $v(t,x) \le 1$ for all $t \ge 0$ and $x \in \overline{\Omega}$. From standard parabolic estimates, $v(t,x)$ converges locally uniformly in $\overline{\Omega}$ to a classical solution $w = w(x)$ of

$$\begin{cases} \Delta w + f(w) &= 0 \text{ in } \Omega \\ \partial_\nu w &= 0 \text{ on } \partial\Omega \\ 0 \le w &\le 1 \text{ in } \Omega. \end{cases}$$

Furthermore, $w \ge v(0,\cdot)$, whence $w \not\equiv 0$. It follows from Proposition 2.2 that $w \equiv 1$. Inequality (2.7) then yields

$$\liminf_{t\to+\infty} \min_{x\in K} u(t,x) \ge 1$$

for all compact $K \subset \overline{\Omega}$.

On the other hand, u_0 is bounded, whence $u_0 \le M$ for some $M > 0$. Thus, $u(t,x) \le \xi(t)$ for all $t \ge 0$ and $x \in \overline{\Omega}$, where $\xi = \xi(t)$ solves $\dot{\xi} = f(\xi)$ with $\xi(0) = M$. From the choice of f (f is positive in $(0,1)$ and negative in $(1,+\infty)$), one concludes that $\xi(t) \to 1$ as $t \to +\infty$. Hence,

$$\limsup_{t\to+\infty} \max_{x\in K} u(t,x) \le 1$$

for all compact $K \subset \overline{\Omega}$.

One concludes that $u(t,x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \overline{\Omega}$.

Let now u_0 and v_0 be two continuous, nonnegative and nonzero functions which are compactly supported in $\overline{\Omega}$. Assume that u_0 and v_0 are less than 1. Let e be a unit vector in $\mathbb{R}^N$. Notice that the assumptions in Theorem 1.7 immediately imply that Ω is unbounded in the direction e.

Since $\max_{\overline{\Omega}} v_0 < 1$ and v_0 is compactly supported, it follows from the first part of the proof of Theorem 1.7 that $u(t_0,x) \ge v_0(x)$ for all $x \in \overline{\Omega}$, for some $t_0 \ge 0$. Therefore, $u(t+t_0,x) \ge v(t,x)$ for all $t \ge 0$ and $x \in \overline{\Omega}$, whence $w^*(e,u_0) \ge w^*(e,v_0)$.

Changing the roles of u and v leads to the inequality $w^*(e,v_0) \ge w^*(e,u_0)$. Therefore, $w^*(e,u_0) = w^*(e,v_0)$.

The same arguments also imply that

$$w^*(e,z,u_0) = w^*(e,z,v_0)$$

for all $e \in \mathbb{S}^{N-1}$, $z \in \mathbb{R}^N$ and $(u_0,v_0) \in \mathcal{E}^2$ with $u_0, v_0 < 1$ in $\overline{\Omega}$. That completes the proof of Theorem 1.7. □

2.3 Upper bound for domains with the extension property

This section is devoted to the

Proof of Theorem 1.8. As already underlined, it is sufficient to prove property (1.9). Fix a speed $c > 2\sqrt{f'(0)}$ and $u_0 \in \mathcal{E}$. Let then $R_0 > 0$ be such that B_{R_0} contains the support of u_0 and let $C_0 > 4$, $\varepsilon > 0$ and $t_0 > 0$ be such that

$$\forall\ t \geq t_0,\ \forall\ z \in B_{R_0},\ \forall\ |x| \geq ct, \quad \frac{|z-x|^2}{C_0 t} \geq (f'(0)+\varepsilon)t. \tag{2.8}$$

Call $v(t,x)$ the solution of

$$\begin{cases} v_t &= \Delta v \\ \partial_\nu v &= 0 \ \text{ on } \partial\Omega \end{cases}$$

with initial condition u_0. Since $f(s) \leq f'(0)s$ for all $s \geq 0$, the maximum principle yields

$$0 \leq u(t,x) \leq e^{f'(0)t} v(t,x)$$

for all $t \geq 0$ and $x \in \overline{\Omega}$.

The function v can be written as

$$v(t,x) = \int_\Omega p(t,z,x) u_0(z) dz,$$

where p is the heat kernel in Ω with Neumann boundary conditions on $\partial\Omega$. Since Ω satisfies the extension property, it follows from Theorem 2.4.4 by Davies [16] that there exists $C_1 > 0$ such that $0 \leq p(t,z,x) \leq C_1 t^{-N/2}$ in $\Omega \times \Omega$ and for all $0 < t < 1$. Since $p(t,x,x)$ is nonincreasing with respect to t for each x, one gets that

$$\forall\ x \in \Omega,\ \forall\ t > 0, \quad p(t,x,x) \leq \frac{1}{g(t)},$$

where, say, $g(t) = (C_1 t^{-N/2} + C_1)^{-1}$.

Since the function g is "regular" in the sense of [22] and since $C_0 > 4$, it follows from the Gaussian upper bounds by Grigor'yan [22] that there exist two positive constants δ and C_2, which only depends on C_0 and g, such that

$$\forall\ (z,x) \in \Omega \times \Omega,\ \forall\ t > 0, \quad p(t,z,x) \leq \frac{C_2}{g(\delta t)} e^{-\frac{r^2}{C_0 t}},$$

where $r = r(z,x)$ denotes the geodesic distance between z and x in $\overline{\Omega}$.

Since the geodesic distance in $\overline{\Omega}$ is bounded from below by the euclidean distance, it follows from all above estimates that

$$0 \leq u(t,x) \leq e^{f'(0)t} \|u_0\|_\infty \int_{B_{R_0}} \frac{C_2}{g(\delta t)} e^{-\frac{|z-x|^2}{C_0 t}} dz.$$

One concludes from (2.8) that

$$0 \leq u(t,x) \leq \|u_0\|_\infty \frac{C_2}{g(\delta t)} |B_{R_0}| e^{-\varepsilon t}$$

for all $t \geq t_0$ and $|x| \geq ct$, $x \in \overline{\Omega}$. The estimate (1.9) follows and

$$w^*(e, z, u_0) \leq w^*(e, u_0) \leq 2\sqrt{f'(0)}$$

for all $z \in \mathbb{R}^N$ and $u_0 \in \mathcal{E}$. □

3 The case of exterior domains

This section is devoted to the proof of Theorem 1.9. Throughout this section, we say that Ω is an exterior domain if Ω is a connected open subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \backslash \Omega$ is compact and $\partial\Omega$ is of class C^1.

Lemma 3.1 *Let Ω be an exterior domain of $\mathbb{R}^N$, let $u_0 \not\equiv 0$ be nonnegative, continuous, bounded in $\overline{\Omega}$ and let $u(t, x)$ be the solution of (1.6) with initial condition u_0. Assume that $f : \mathbb{R}_+ \to \mathbb{R}$ is C^1, and such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f > 0$ on $(0, 1)$ and $f < 0$ on $(1, +\infty)$. Then $u(t, x) \to 1$ locally uniformly in $x \in \overline{\Omega}$ as $t \to +\infty$.*

If Ω were smoother (of class $C^{2,\alpha}$), then Lemma 3.1 would follow from Theorem 1.7. The proof of Lemma 3.1 will actually be similar but simpler than that of the first part of Theorem 1.7. It is sketched here for the sake of completeness.

Proof of Lemma 3.1. First of all, as in the proof of Theorem 1.7, it follows from the boundedness of u_0 and from the profile of f that $\limsup_{t\to+\infty} \sup_{x\in\overline{\Omega}} u(t, x) \leq 1$.

Choose $R > 0$ large enough so that $\lambda_R < f'(0)$, where (λ_R, ψ_R) is the pair of first eigenvalue and first eigenfunction of problem

$$\begin{cases} -\Delta\psi_R &= \lambda_R\psi_R & \text{in } B_R \\ \psi_R &> 0 & \text{in } B_R \\ \psi_R &= 0 & \text{on } \partial B_R \\ \|\psi_R\|_{L^\infty(B_R)} &= 1. \end{cases} \tag{3.1}$$

This is indeed possible since $\lambda_R \to 0$ as $R \to +\infty$.

Then fix $R_0 > 0$ such that $\mathbb{R}^N \backslash \Omega \subset B_{R_0}$. From the strong parabolic maximum principle, one has $u(t, x) > 0$ for all $t > 0$ and $x \in \overline{\Omega}$. Therefore, by continuity, there exists $\varepsilon > 0$ small enough so that

$$u(1, x) \geq \varepsilon\psi_R(x - x_0) \;\; \text{for all } x \in \overline{B(x_0, R)}$$

and for all $x_0 \in \mathbb{R}^N$ with $|x_0| = R_0 + R$.

As a consequence, $u(1 + t, x) \geq v(t, x)$ for all $t \geq 0$ and for all $x \in \overline{\Omega}$, where v is the solution of (1.6) with initial condition

$$v_0(x) = \begin{cases} 0 \;\; \text{if } x \in \overline{\Omega} \text{ and } (|x| \leq R_0 \text{ or } |x| \geq R_0 + 2R) \\ \displaystyle\max_{|x_0|=R_0+R,\; x\in\overline{B(x_0,R)}} \varepsilon\psi_R(x - x_0) \;\; \text{if } R_0 < |x| < R_0 + 2R. \end{cases}$$

Even if it means decreasing $\varepsilon > 0$, one can assume from the choice of R that

$$\Delta(\varepsilon\psi_R) + f(\varepsilon\psi_R) = -\varepsilon\lambda_R\psi_R + f(\varepsilon\psi_R) \geq 0 \;\; \text{in } B_R$$

and that $\varepsilon \leq 1$ (whence $\varepsilon\psi_R \leq 1$ and $v_0 \leq 1$ in $\overline{\Omega}$). Therefore, v_0 is a subsolution for the associated elliptic equation and $v(t,x)$ is nondecreasing with respect to t. Moreover, $v(t,x) \leq 1$ for all $t \geq 0$ and $x \in \overline{\Omega}$. Hence, standard parabolic estimates imply that $v(t,x)$ converges locally uniformly in $x \in \overline{\Omega}$ as $t \to +\infty$ to a classical solution v_∞ of

$$\left\{ \begin{array}{rcl} \Delta v_\infty + f(v_\infty) & = & 0 \text{ in } \Omega \\ \partial_\nu v_\infty & = & 0 \text{ on } \partial\Omega. \end{array} \right.$$

Furthermore, $0 \leq, \not\equiv v_0 \leq 1$, whence $0 \leq v_0 \leq v_\infty \leq 1$. From the strong elliptic maximum principle, one gets that $v_\infty > 0$ in $\overline{\Omega}$.

Using the same arguments as in the proof of Proposition 2.2, one easily gets that

$$v_\infty(x) \geq \varepsilon\psi_R(x - te) \text{ in } \overline{B(te, R)}$$

for all $e \in \mathbb{S}^{N-1}$ and for all $t \geq R + R_0$. Indeed, $\varepsilon'\psi_R(\cdot - te)$ vanishes on $\partial B(te,R)$ and is a subsolution of $\Delta\phi + f(\phi) \geq 0$ in $B(te,R)$, for each $\varepsilon' \in [0,\varepsilon]$.

Thus, $v_\infty(x) \geq \varepsilon\psi_R(0)$ as soon as $|x| \geq R_0 + R$, whence

$$\inf_{\mathbb{R}^N \setminus B_{R_0+R}} v_\infty > 0.$$

Since v_∞ is continuous and positive in $\overline{\Omega}$, it follows that $m = \inf_{\overline{\Omega}} v_\infty > 0$.

If m is reached at some point $x \in \overline{\Omega}$, the strong elliptic maximum principle and Hopf lemma yield $m \geq 1$, since $f > 0$ in $(0,1)$. Then $v_\infty \equiv 1$ (remember that $v_\infty \leq 1$ in $\overline{\Omega}$). If m is not attained, there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ in $\overline{\Omega}$ such that $|x_n| \to +\infty$ and $v_\infty(x_n) \to m$ as $n \to +\infty$. The functions $w_n(x) = v_\infty(x+x_n)$ then converge locally uniformly in $\mathbb{R}^N$, up to extraction of some subsequence, to a classical solution w_∞ of $\Delta w_\infty + f(w_\infty) = 0$ in $\mathbb{R}^N$ with $m = w_\infty(0) \leq w_\infty \leq 1$ in $\mathbb{R}^N$. One concludes as above that $m = 1$.

Therefore, $v_\infty \equiv 1$ in $\overline{\Omega}$. Since $u(1+t,x) \geq v(t,x)$ for all $t \geq 1$ and $x \in \overline{\Omega}$, it follows that

$$\liminf_{t\to+\infty} \min_{x\in K} u(t,x) \geq 1,$$

for all compact subset $K \subset \overline{\Omega}$. Together with $\limsup_{t\to+\infty} \sup_{x\in\overline{\Omega}} u(t,x) \leq 1$, that completes the proof of Lemma 3.1. □

Lemma 3.2 *Let $u(t,x)$ be a solution of (1.6) with $\Omega = \mathbb{R}^N$ and with an initial condition $u_0 \not\equiv 0$ which is nonnegative, continuous and bounded. Let $g : \mathbb{R}_+ \to \mathbb{R}$ be of class C^1, and such that $g(0) = g(1) = 0$, $g'(0) > 0$, $g > 0$ on $(0,1)$ and $g < 0$ on $(1,+\infty)$. Then, for all $0 \leq c < 2\sqrt{g'(0)}$ and for all $e \in \mathbb{R}^N$ with $|e| = 1$,*

$$u(t, x + ct\, e) \to 1$$

locally uniformly in $x \in \mathbb{R}^N$ as $t \to +\infty$.

This lemma could actually follow from a result by Aronson and Weinberger [1], which was based on the construction of subsolutions involving planar travelling fronts, for the parabolic problem. We present a simpler proof here, which is mainly based on elliptic arguments.

Notice also that the case $c = 0$ is included in Lemma 3.1.

Proof of Lemma 3.2. As in Lemma 3.1, one has that $\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}^N} u(t,x) \le 1$.

Let $e \in \mathbb{R}^N$ be fixed such that $|e| = 1$ and let $0 \le c < 2\sqrt{g'(0)}$. Let $R > 0$ be large enough so that $\lambda_R + c^2/4 < g'(0)$, where (λ_R, ψ_R) is the pair of first eigenvalue and first eigenfunction of problem (3.1) in the ball B_R. Since u is continuous and $u(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}^N$, one can choose $\varepsilon > 0$ small enough so that

$$\forall\, x \in \overline{B_R}, \quad u(1, x + ce) \ge \varepsilon e^{-ce\cdot x/2}\psi_R(x) =: w_0(x).$$

Even if it means decreasing $\varepsilon > 0$, one can assume that $w_0 \le 1$ in B_R and

$$\Delta w_0 + ce\cdot\nabla w_0 + g(w_0) = -\left(\lambda_R + \frac{c^2}{4}\right) w_0 + g(w_0) \ge 0 \ \text{ in } B_R.$$

Since the function $(t,x) \mapsto v(t,x) := u(t, x + ct\, e)$ satisfies the equation

$$\partial_t v = \Delta v + ce\cdot\nabla v + g(v),$$

it follows then that $v(1+t,x) \ge w(t,x)$ for all $t \ge 0$ and $x \in \mathbb{R}^N$, where w satisfies the same equation as v with initial condition $w(0,x) = w_0(x)$ if $x \in B_R$ and $w(0,x) = 0$ if $|x| \ge R$.

Furthermore, from the choice of ε, $w(t,x)$ is nondecreasing in t for all $x \in \mathbb{R}^N$ and converges as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$ to a classical solution w_∞ of

$$\Delta w_\infty + ce\cdot\nabla w_\infty + g(w_\infty) = 0 \ \text{ in } \mathbb{R}^N$$

such that $0 \le w_\infty \le 1$ in $\mathbb{R}^N$ and $w_\infty \ge w_0$ in B_R. It follows from Proposition 1.14 in [8] that $w_\infty \equiv 1$.

Therefore, $\liminf_{t\to+\infty} \min_{x\in K} u(t, x + ct\, e) \ge 1$ for all compact subset $K \subset \mathbb{R}^N$. That completes the proof of Lemma 3.2. □

Let us now turn to the

Proof of Theorem 1.9. As already underlined, one only has to prove formula (1.10). Let u solve (1.6) with an initial condition $u_0 \in \mathcal{E}$. Under the assumptions of Theorem 1.9, the exterior domain Ω satisfies the extension property, whence

$$\max_{|x|\ge ct,\ x\in\overline{\Omega}} u(t,x) \to 0 \text{ as } t \to +\infty,$$

as soon as $c > 2\sqrt{f'(0)}$.

On the other hand, one easily gets as usual that

$$\limsup_{t\to+\infty} \sup_{x\in\overline{\Omega}} u(t,x) \le 1.$$

Therefore, one only has to prove that $\liminf_{t\to+\infty} \min_{|x|\le ct,\ x\in\overline{\Omega}} u(t,x) \ge 1$ if $0 \le c < 2\sqrt{f'(0)}$.

Let c be fixed such that $0 \le c < 2\sqrt{f'(0)}$ and let $\varepsilon \in (0,1)$ be fixed. It follows from Lemma 3.1 that there exists $t_0 > 0$ such that

$$\forall\, t \ge t_0,\ \forall\, x \in \partial\Omega, \quad u(t,x) \ge 1-\varepsilon.$$

Let now g be a C^1 function such that $g \le f$ in $[0,+\infty)$, $g(0) = g(1-\varepsilon) = 0$, $g > 0$ in $(0, 1-\varepsilon)$, $g < 0$ in $(1-\varepsilon, +\infty)$ and $g'(0) = f'(0)$. Let v_0 be a continuous and compactly supported function defined in $\mathbb{R}^N$, such that $0 \le v_0 \le 1-\varepsilon$ and $v_0 \not\equiv 0$. Assume furthermore that v_0 is radially symmetric, nonincreasing with respect to $r = |x|$ and that $u(t_0, x) \ge v_0(x)$ for all $x \in \overline{\Omega}$. Lastly, let $v(t,x)$ be the solution of (1.6) in $\mathbb{R}^N$, with nonlinearity g instead of f, and initial condition v_0.

It follows by construction of g that $v(t,x) \le 1-\varepsilon$ for all $t \ge 0$ and $x \in \mathbb{R}^N$. Therefore, $u(t+t_0, x) \ge 1-\varepsilon \ge v(t,x)$ for all $t \ge 0$ and $x \in \partial\Omega$. The above assumptions on g and v_0 then yield that

$$\forall\, t \ge 0,\ \forall\, x \in \overline{\Omega}, \quad u(t+t_0, x) \ge v(t,x).$$

Thus,

$$\liminf_{t\to+\infty} \min_{|x|\le ct,\ x\in\overline{\Omega}} u(t,x) \ge \liminf_{t\to+\infty} \min_{|x|\le ct+ct_0,\ x\in\overline{\Omega}} v(t,x) \ge \liminf_{t\to+\infty} \min_{|x|\le ct+ct_0,\ x\in\mathbb{R}^N} v(t,x).$$

On the other hand, v stays radially symmetric in $\mathbb{R}^N$ and nonincreasing with respect to $r = |x|$ for all time $t \ge 0$. Therefore,

$$\liminf_{t\to+\infty} \min_{|x|\le ct,\ x\in\overline{\Omega}} u(t,x) \ge \liminf_{t\to+\infty} v(t, c(t+t_0)e)$$

for any given direction $e \in \mathbb{S}^{N-1}$. But,

$$\liminf_{t\to+\infty} v(t, c(t+t_0)e) = 1-\varepsilon,$$

by applying the conclusion of Lemma 3.2 to the function g (remember that $0 \le c < 2\sqrt{f'(0)} = 2\sqrt{g'(0)}$ from the choice of g).

Since $\varepsilon \in (0,1)$ was arbitrary, one concludes that

$$\liminf_{t\to+\infty} \min_{|x|\le ct,\ x\in\overline{\Omega}} u(t,x) \ge 1.$$

Eventually,

$$\lim_{t\to+\infty} \max_{|x|\le ct,\ x\in\overline{\Omega}} |u(t,x) - 1| = 0$$

for all $c \in [0, 2\sqrt{f'(0)})$ and the proof of Theorem 1.9 is complete. □

The same type of arguments as above give a lower bound for the spreading speeds $w^*(e, u_0)$ and $w^*(e, z, u_0)$ in a domain Ω containing a semi-infinite cylinder in the direction e, with large enough section :

Proof of formula (1.12) in domains Ω satisfying (1.11). Fix $\varepsilon \in (0, 2\sqrt{f'(0)}]$ and $R_0 > 0$ large enough so that

$$\forall\ R \geq R_0, \quad \lambda_R + \frac{(2\sqrt{f'(0)} - \varepsilon)^2}{4} < f'(0),$$

where (λ_R, ψ_R) is the pair of first eigenvalue and first eigenfunction of problem (3.1) in the ball B_R. Let Ω satisfy (1.11) for some $A \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$ and $R > R_0$. Fix any R' such that $R_0 \leq R' < R$ and set

$$z_0 = x_0 - (x_0 \cdot e)e + (A + 1 + R')e.$$

The assumption (1.11) implies that

$$\forall\ s \geq 0, \quad \Omega\ \supset\ \overline{B(z_0 + se, R')}.$$

As in the proof of Lemma 3.2, there exists $\eta > 0$ small enough so that

$$\forall\ x \in \overline{B_{R'}}, \quad u(1, x + z_0) \geq \eta e^{-(2\sqrt{f'(0)}-\varepsilon)e\cdot x/2}\psi_{R'}(x) =: w_0(x)$$

and $w_0 \leq 1$ in $\overline{B_{R'}}$. From the choice of R_0, the function w_0 is a subsolution of

$$\Delta w_0 + (2\sqrt{f'(0)} - \varepsilon)e \cdot \nabla w_0 + f(w_0) \geq 0 \ \text{ in } B_{R'}.$$

The function $v(t, x) = u\left(t + 1, x + z_0 + (2\sqrt{f'(0)} - \varepsilon)te\right)$ satisfies

$$\partial_t v = \Delta v + (2\sqrt{f'(0)} - \varepsilon)e \cdot \nabla v + f(v),$$

especially for all $t \geq 0$ and $x \in \overline{B_{R'}}$. Furthermore, $v(t, x) \geq 0$ for all $x \in \partial B_{R'}$. It follows from the maximum principle that

$$v(t, x) \geq w(t, x) \ \text{ for all } t \geq 0 \text{ and for all } x \in \overline{B_{R'}},$$

where w solves the same equation as v in $B_{R'}$, with initial condition $w(0, x) = w_0(x)$ in $B_{R'}$ and boundary condition $w(t, x) = 0$ for all $t \geq 0$ and $x \in \partial B_{R'}$. Furthermore, $0 \leq w(t, x) \leq 1$ for all $t \geq 0$ and $x \in \overline{B_{R'}}$, and w is nondecreasing in t for all $x \in \overline{B_{R'}}$. Standard parabolic estimates imply that $w(t, x) \to w_\infty(x)$ as $x \to +\infty$, where w_∞ satisfies the corresponding elliptic equation and $w_\infty(x) \geq w_0(x)$ for all $x \in \overline{B_{R'}}$.

As a consequence,

$$\forall\ x \in B_{R'}, \quad \liminf_{t\to+\infty}\ u\left(t + 1, x + z_0 + (2\sqrt{f'(0)} - \varepsilon)te\right)\ \geq\ w_0(x) > 0.$$

Thus, $w^*(e, z, u_0) \geq 2\sqrt{f'(0)} - \varepsilon$ for all $u_0 \in \mathcal{E}$ and $z \in \mathbb{R}^N$ such that $|z - z_0 - ((z - z_0) \cdot e)e| < R'$. Since this is true for all $R' \in [R_0, R)$, one concludes that

$$w^*(e, z, u_0) \geq 2\sqrt{f'(0)} - \varepsilon$$

for all $u_0 \in \mathcal{E}$ and $z \in \mathbb{R}^N$ such that $|z - z_0 - ((z - z_0) \cdot e)e| < R$.

Together with Theorem 1.6, that completes the proof of (1.12). □

Remark 3.3 The above arguments imply that if

$$\Omega \supset \{x \in \mathbb{R}^N,\ x \cdot e > A,\ \pm(x \cdot e' - B) > 0\}$$

for some $(A, B) \in \mathbb{R}^2$ and $e' \in \mathbb{S}^{N-1}$ with $e' \cdot e = 0$, then, for all $\varepsilon > 0$, there exists $R_0 > 0$ such that $w^*(e, z, u_0) \geq 2\sqrt{f'(0)} - \varepsilon$ for all $u_0 \in \mathcal{E}$ and

$$z \ \in \bigcup_{R \geq R_0,\ z_0 \in \mathbb{R}^N,\ \pm(z_0 \cdot e' - B) > R} B(z_0, R).$$

Therefore, $w^*(e, z, u_0) \geq 2\sqrt{f'(0)}$ for all $u_0 \in \mathcal{E}$ and z such that $\pm(z \cdot e' - B) > 0$.

4 Domains with zero or infinite spreading speeds, or spreading speeds depending on z

This section is devoted to the construction of some particular domains for which the spreading speeds may be zero, infinite, or may depend on the position z.

4.1 Domains for which $w^*(e, z, u_0)$ depends on z

This subsection is devoted to the
Proof of Theorem 1.6, part c). Up to translation and rotation, one can assume, say, that $e = (1, 0, \dots, 0)$ and $z = (0, 2, 0, \dots, 0)$.

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers such that

$$\frac{a_n}{n} \to +\infty \quad \text{as } n \to +\infty.$$

Let Γ be the subset of $\mathbb{R}^2$ defined by

$$\Gamma = \{(x_1, 0),\ x_1 \geq 0\} \ \cup \ \bigcup_{n \in \mathbb{N}^*} \{n\} \times [0, a_n].$$

Let $\tilde{\Omega}$ be any open subset of $\mathbb{R}^2$ such that

$$\Gamma \ \subset \ \tilde{\Omega} \ \subset \ \left\{x \in \mathbb{R}^2,\ d(x, \Gamma) < \frac{1}{3}\right\}$$

and such that $\Omega_2 := \mathbb{R}^2 \backslash \overline{\tilde{\Omega}}$ is connected and satisfies the extension property defined in Section 1. Here, $d(y, E)$ denotes the euclidean distance of a point $y \in \mathbb{R}^m$ to a subset $E \subset \mathbb{R}^m$.

We then set $\Omega = \Omega_2$ is $N = 2$ and $\Omega = \Omega_2 \times \mathbb{R}^{N-2}$ if $N \geq 3$. The open set Ω is clearly unbounded in the direction e. But such a domain clearly does not satisfy the assumptions of part b) of Theorem 1.6 (more precisely, Ω does not satisfy Hypothesis $H_{y,y'}$, for any y and y' such that, say, $y_2 > 1/3 > -1/3 > y'_2$).

Furthermore,

$$\forall\ u_0 \in \mathcal{E}, \quad w^*(e, u_0) \leq 2\sqrt{f'(0)}$$

from Theorem 1.8. On the other hand, since $\Omega \supset \{x \in \mathbb{R}^N,\ x_2 < -1/3\}$, Remark 1.11 implies that $w^*(e, u_0) \geq 2\sqrt{f'(0)}$ and $w^*(e, z', u_0) \geq 2\sqrt{f'(0)}$ for all $u_0 \in \mathcal{E}$ and $z' \in \mathbb{R}^N$ such that $z'_2 < -1/3$. Hence, $w^*(e, u_0) = w^*(e, z', u_0) = 2\sqrt{f'(0)}$ for all $u_0 \in \mathcal{E}$ and $z' \in \mathbb{R}^N$ such that $z'_2 < -1/3$.

Remember that $z = (0, 2, 0, \ldots, 0)$. Let $\gamma > 0$ be any fixed positive real number and let u_0 be in $\mathcal{E}$. From the construction of Ω, one has that

$$\forall\ s \geq 0, \quad \overline{B(z + se, 1)} \cap \overline{\Omega} \neq \emptyset.$$

Let $C_0 > 4$ be given. The same arguments and notations as in the proof of Theorem 1.8 yield the existence of some positive constants C_1, C_2 and δ such that

$$0 \leq u(t, x) \leq e^{f'(0)t} \|u_0\|_{L^\infty(\Omega)} \int_{\mathrm{supp}(u_0)} C_2 C_1 (\delta^{-1} t^{-1} + 1) e^{-\frac{r^2(y,x)}{C_0 t}} dy$$

for all $t > 0$, $s \geq 0$ and $x \in \overline{B(z + se, 1)} \cap \overline{\Omega}$. Here, $\mathrm{supp}(u_0)$ denotes the support of u_0 and $r(y, y')$ stands for the geodesic distance in $\overline{\Omega}$ between two points y and y' in $\overline{\Omega}$. Since $\mathrm{supp}(u_0)$ is compact, it follows from the construction of Ω (especially the fact that $a_n/n \to +\infty$ as $n \to +\infty$) that

$$\inf_{y\ \in\ \mathrm{supp}(u_0),\ s \geq \gamma t,\ x \in \overline{B(z+se,1)} \cap \overline{\Omega}} \frac{r(y, x)}{t} \ \to\ +\infty \ \text{ as } \ t \to +\infty.$$

Thus, for all $\beta > 0$, there is $t_0 > 0$ such that

$$0 \leq u(t, x) \leq e^{f'(0)t} \|u_0\|_{L^\infty(\Omega)} C_2 C_1 (\delta^{-1} t^{-1} + 1) |\mathrm{supp}(u_0)| e^{-\beta t}$$

for all $t \geq t_0$, $s \geq \gamma t$ and $x \in \overline{B(z + se, 1)} \cap \overline{\Omega}$. Therefore,

$$\limsup_{s \geq \gamma t,\ t \to +\infty} \ \max_{x \in \overline{B(z+se,1)} \cap \overline{\Omega}} u(t, x) = 0.$$

Since this is true for all $\gamma > 0$, one concludes that $w^*(e, z, u_0) = 0$.

Actually, the same type of arguments imply that

$$w^*(e, z', u_0) = 0$$

for all $u_0 \in \mathcal{E}$ and $z' \in \mathbb{R}^N$ such that $z'_2 > 1/2$ (by changing the radius 1 by $1/2 + \varepsilon$ for some small $\varepsilon = \varepsilon(z') > 0$). □

4.2 Domains with zero spreading speeds

Proof of Theorem 1.12, part a). Let us define the curve

$$\Gamma = \{(t \cos t, t \sin t),\ t \geq 0\}$$

and let Ω be a smooth open connected subset of $\mathbb{R}^2$ satisfying the extension property and such that, say, $\Omega \backslash \overline{B_{2\pi}} = \{x,\ d(x, \Gamma) < 1\} \backslash \overline{B_{2\pi}}$. Such a domain Ω is like a spiral. It is clear that Ω is unbounded in every unit direction e of $\mathbb{R}^2$. It is also clear that Ω satisfies the assumptions of Theorem 1.7, and thus $u(t,x) \to 1$ locally in $x \in \overline{\Omega}$ as $t \to +\infty$, for any solution u of (1.6) with initial condition $u_0 \in \mathcal{E}$.

Let $u_0 \not\equiv 0$ be a nonnegative, continuous and compactly supported function in $\overline{\Omega}$. Let $C_0 > 4$, $e \in \mathbb{S}^1$ be given, and let $R > 0$ such that $\overline{\Omega} \cap \overline{B(se,R)} \neq \emptyset$ for all $s \geq 0$. With the same arguments and notations as in the proof of Theorem 1.8, one has

$$\forall\, t > 0,\ \forall\, x \in \overline{\Omega},\quad 0 \leq u(t,x) \leq e^{f'(0)t}\|u_0\|_{L^\infty(\Omega)} \int_{\mathrm{supp}(u_0)} C_2 C_1(\delta^{-1}t^{-1}+1)e^{-\frac{r^2(z,x)}{C_0 t}}\,dz,$$

for some positive constants C_1, C_2 and δ.

Fix any $\gamma > 0$ and $A \geq R$. For all $s \geq 0$ and for all $t > 0$, one has

$$0 \leq \max_{x \in \overline{B(se,A)} \cap \overline{\Omega}} u(t,x) \leq C_1 C_2 \|u_0\|_{L^\infty(\Omega)}(\delta^{-1}t^{-1}+1)e^{f'(0)t}\int_{\mathrm{supp}(u_0)} e^{-\frac{\tilde{r}^2_{z,s}}{C_0 t}}\,dz,$$

where

$$\tilde{r}_{z,s} = \min_{y \in \overline{B(se,A)} \cap \overline{\Omega}} r(z,y).$$

But, owing to the definition of Ω, there exist $\eta > 0$ and $t_0 > 0$ such that

$$\forall\, t \geq t_0,\ \forall\, s \geq \gamma t,\ \forall\, z \in \mathrm{supp}(u_0),\ \tilde{r}_{z,s} = \min_{y \in \overline{B(se,A)} \cap \overline{\Omega}} r(z,y) \geq \eta\, t^2.$$

Thus, for all $t \geq t_0$,

$$0 \leq \sup_{s \geq \gamma t}\ \max_{x \in \overline{B(se,A)} \cap \overline{\Omega}} u(t,x) \leq C_1 C_2 \|u_0\|_{L^\infty(\Omega)}(\delta^{-1}t^{-1}+1)e^{f'(0)t}|\mathrm{supp}(u_0)|e^{-\eta t^3/C_0} \to 0$$

as $t \to +\infty$.

Therefore, $w^*(e,z,u_0) = w^*(e,u_0) = 0$ for all $e \in \mathbb{S}^1$, $z \in \mathbb{R}^N$ and $u_0 \in \mathcal{E}$. □

4.3 Domains with infinite spreading speeds

The proof of part b) of Theorem 1.12 is based on the following Lemmas 4.1 and 4.2. In the remaining part of this section, we fix $N \geq 2$ and we call (x,x') the coordinates in $\mathbb{R}^N$, where $x = x_1$ and $x' = (x_2, \cdots, x_N)$. Let us set $r' = |x'| = \sqrt{x_2^2 + \cdots + x_N^2}$. Let $h : \mathbb{R} \to \mathbb{R}$ be the function defined for all $s \in \mathbb{R}$ by

$$h(s) = e^{-e^s + s}.$$

Set

$$\tilde{\Omega} = \{(x,x') \in \mathbb{R}^N,\ x > A,\ 0 \leq r' < h(x)\},$$

where $A > 0$ is a positive real number to be chosen later, and let Ω be an open connected and locally C^1 domain such that

$$\tilde{\Omega} \quad\subset\quad \Omega \quad\subset\quad \tilde{\Omega} \cup \{A-1 \leq x \leq A,\ 0 \leq r' < 1\}.$$

Such a domain Ω has the shape of an infinite cusp, and it obviously does not satisfy the extension property defined in Section 1.

Lemma 4.1 *Under the above notations, call* $\phi(x,x') = \phi(x,r') = \cos r' - e^{-x}\cos(\sqrt{2}r')$ *for all* $(x,x') \in \mathbb{R}^N$. *Then there exists* $A > 0$ *large enough such that*

$$\begin{cases} \Delta\phi + \phi & \leq & 0 \text{ in } \tilde{\Omega} \\ \partial_\nu \phi & \geq & 0 \text{ on } \partial\tilde{\Omega} \cap \{x > A\} \end{cases}$$

and $1/2 \leq \phi \leq 1$ *in* $\overline{\Omega}$.

Proof. A straighforward calculation gives that the function ϕ is of class C^2 in $\mathbb{R}^N$ and that

$$\Delta\phi + \phi = \frac{N-2}{r'}(-\sin r' \ + \sqrt{2}e^{-x}\sin(\sqrt{2}r'))$$

if $r' > 0$. Therefore, $\Delta\phi + \phi \leq 0$ in $\tilde{\Omega}$ for A large enough.

On the other hand, for $(x,x') \in \partial\tilde{\Omega} \cap \{x > A\}$, one has $r' = h(x)$ and

$$\begin{aligned} \partial_\nu \phi(x,x') &= \frac{1}{\sqrt{h'(x)^2+1}}\Big(-h'(x)e^{-x}\cos(\sqrt{2}h(x)) - \sin h(x) + \sqrt{2}e^{-x}\sin(\sqrt{2}h(x))\Big) \\ &= \frac{h(x)}{\sqrt{h'(x)^2+1}}\big(e^{-x} + O(h^2(x))\big) \\ &\geq 0 \ \text{ for } x \text{ large enough.} \end{aligned}$$

Lastly, the condition $1/2 \leq \phi \leq 1$ in $\overline{\Omega}$ immediately holds if A is large enough. That completes the proof of Lemma 4.1. □

The following lemma provides some lower estimates for the heat kernel in such domains Ω.

Lemma 4.2 *Under the assumptions of Lemma 4.1, let* $p(t,w,z)$ *denote the heat kernel in* Ω *with Neumann boundary conditions on* $\partial\Omega$. *Then, there exists a time* $T > 0$ *such that, for all compact subset* $K \subset \overline{\Omega}$,

$$\inf_{t \geq T,\ w \in K,\ z \in \overline{\Omega}} p(t,w,z) \ > \ 0.$$

Proof. Let us first fix $T_0 > 0$ such that $e^{-T_0} \leq 1/4$. Let K be a compact subset of $\overline{\Omega}$ and let $R > 0$ be such that the open ball B_R contains K and $\overline{\Omega} \cap \{x \leq A\}$. Let $q(t,w,z)$ denote the heat kernel in $\Omega \cap B_R$ with Neumann boundary conditions on $\partial\Omega \cap B_R$ and Dirichlet boundary conditions on $\overline{\Omega} \cap \partial B_R$. One has immediately that $p(t,w,z) \geq q(t,w,z)$ for all $t > 0$ and $(w,z) \in (\overline{\Omega} \cap \overline{B_R})^2$. Therefore, there exists $\eta > 0$ such that, say,

$$\forall\ 1 \leq t \leq 1+T_0,\ \forall\ w \in K,\ \forall\ z = (x,x') \in \overline{\Omega} \cap \{x \leq A\}, \quad p(t,w,z) \geq q(t,w,z) \geq \eta. \quad (4.1)$$

Let η be as above, and let w be any given point in K. Let $\varepsilon > 0$ and $\beta > 0$ be two arbitrary positive real numbers, and let $\overline{u}$ be the function defined for all $t \geq 0$ and $z = (x,x') \in \overline{\Omega}$ by

$$\overline{u}(t,z) = p(1+t,w,z) + \varepsilon e^{\beta x}.$$

One immediately checks that

$$\partial_t \overline{u} - \Delta \overline{u} + \beta^2 \overline{u} = \beta^2 p(1+t,w,z) > 0$$

for all $t \geq 0$ and $z \in \Omega$. Furthermore, for all $z = (x,x') \in \partial\tilde{\Omega} \cap \{x > A\}$, one has

$$\partial_\nu \overline{u} = -\frac{\varepsilon \beta e^{\beta x} h'(x)}{\sqrt{h'(x)^2+1}} \geq 0.$$

Lastly, $\overline{u}(t,\cdot) \geq \eta$ on $\partial\tilde{\Omega} \cap \{x = A\}$ for all $0 \leq t \leq T_0$, because of (4.1).

Call now $\underline{u}$ the function defined for all $t \geq 0$ and $z \in \overline{\Omega}$ by

$$\underline{u}(t,z) = \eta - 2\eta\phi(z)e^{-(1+\beta^2)t} - \beta^2\eta t.$$

From Lemma 4.1, the function $\underline{u}$ satisfies

$$\partial_t \underline{u} - \Delta \underline{u} + \beta^2 \underline{u} = 2\eta(\Delta\phi + \phi)e^{-(1+\beta^2)t} - \beta^4\eta t \leq 0$$

for all $z \in \overline{\tilde{\Omega}}$ and $t \geq 0$. Furthermore,

$$\partial_\nu \underline{u} = -2\eta\partial_\nu\phi \ e^{-(1+\beta^2)t} \leq 0 \ \text{ on } \partial\tilde{\Omega} \cap \{x > A\}$$

from Lemma 4.1, and $\underline{u}(t,\cdot) \leq \eta$ in $\overline{\Omega}$ for all $t \geq 0$. Lastly, since $\phi \geq 1/2$ in $\overline{\Omega}$, one has that

$$\overline{u}(0,\cdot) \geq \varepsilon > 0 \geq \underline{u}(0,\cdot) \ \text{ in } \overline{\tilde{\Omega}}.$$

The parabolic maximum principle yields $\overline{u}(t,z) \geq \underline{u}(t,z)$ for all $0 \leq t \leq T_0$ and $z \in \overline{\tilde{\Omega}}$. In other words,

$$\forall\ 0 \leq t \leq T_0,\ \forall\ z \in \overline{\tilde{\Omega}}, \quad p(1+t,w,z) + \varepsilon e^{\beta x} \geq \eta - 2\eta\phi(z)e^{-(1+\beta^2)t} - \beta^2\eta t.$$

Since $\varepsilon > 0$ and $\beta > 0$ were arbitrary, it follows that

$$\forall\ 0 \leq t \leq T_0,\ \forall\ z \in \overline{\tilde{\Omega}}, \quad p(1+t,w,z) \geq \eta - 2\eta\phi(z)e^{-t}.$$

Since $\phi \leq 1$ in $\overline{\Omega}$, one has $\phi e^{-T_0} \leq e^{-T_0} \leq 1/4$ from the choice of T_0. Therefore,

$$\forall\ z \in \overline{\tilde{\Omega}}, \quad p(1+T_0,w,z) \geq \eta/2.$$

From (4.1), one concludes that $p(1+T_0,w,z) \geq \eta/2$ for all $z \in \overline{\Omega}$. As a consequence,

$$p(t,w,z) \geq \eta/2$$

for all $t \geq T := 1 + T_0$ and for all $z \in \overline{\Omega}$. Since $w \in K$ was arbitrary, the proof of Lemma 4.2 is complete (notice that T does not depend on K). □

Let us now turn to the

Proof of Theorem 1.12, part b). Let Ω be as above and such that the conclusion of

Lemma 4.1 holds. Let $e = e_1 = (1, 0, \cdots, 0)$. It is clear that Ω is unbounded in the direction e. Let $u_0 \not\equiv 0$ be a continuous, nonnegative and compactly supported function in $\overline{\Omega}$, and let $u(t, x)$ be the solution of (1.6) with initial condition u_0.

Let us first observe that

$$\forall\ t \geq 0,\ \forall x \in \overline{\Omega}, \quad u(t, x) \geq v(t, x),$$

where v is the solution of (1.6) with initial condition $v_0 = \min(u_0, 1)$. Since $0 \leq v(t, x) \leq 1$ for all $t \geq 0$ and $x \in \overline{\Omega}$, and since $f \geq 0$ in $[0, 1]$, one gets that

$$\forall\ t \geq 0,\ \forall\ x \in \overline{\Omega}, \quad v(t, x) \geq V(t, x),$$

where V solves the heat equation $V_t = \Delta V$ with Neumann boundary conditions on $\partial\Omega$ and initial condition v_0.

Therefore, under the notations of Lemma 4.2, one has

$$\forall\ t \geq 0,\ \forall\ x \in \overline{\Omega}, \quad u(t, x) \geq V(t, x) = \int_{\mathrm{supp}(v_0)} p(t, w, x) v_0(w) dw.$$

Since $\mathrm{supp}(v_0)(= \mathrm{supp}(u_0))$ is a compact subset of $\overline{\Omega}$, Lemma 4.2 implies that there exist $T > 0$ and $\delta > 0$ such that

$$\forall\ t \geq T,\ \forall\ w \in \mathrm{supp}(u_0),\ \forall\ x \in \overline{\Omega}, \quad p(t, w, x) \geq \delta.$$

Hence,

$$u(t, x) \geq \varepsilon := \delta \int_{\mathrm{supp}(u_0)} v_0(w) dw > 0$$

for all $t \geq T$ and $x \in \overline{\Omega}$.

As a consequence, $u(t+T, x) \geq \zeta(t) > 0$ for all $t \geq 0$ and $x \in \overline{\Omega}$, where ζ solves $\dot{\zeta} = f(\zeta)$ with $\zeta(0) = \varepsilon > 0$. Since $\zeta(t) \to 1$ as $t \to +\infty$ (because of the profile of f), one gets that $\liminf_{t \to +\infty}\ \inf_{x \in \overline{\Omega}}\ u(t, x) \geq 1$.

On the other hand, $u(t, x) \leq \xi(t)$ for all $t \geq 0$ and $x \in \overline{\Omega}$, where ξ solves $\dot{\xi} = f(\xi)$ and $\xi(0) = \max_{\overline{\Omega}} u_0 \in (0, +\infty)$. Since $\xi(t) \to 1$ as $t \to +\infty$, one gets as usual that $\limsup_{t \to +\infty}\ \sup_{x \in \overline{\Omega}}\ u(t, x) \leq 1$.

As a conclusion, $u(t, x) \to 1$ as $t \to +\infty$ uniformly with respect to $x \in \overline{\Omega}$.

Owing to Definitions 1.2 and 1.3, it follows that $w^*(e, z, u_0) = w^*(e, u_0) = +\infty$ for all $z \in \mathbb{R}^N$ and $u_0 \in \mathcal{E}$. That completes the proof of Theorem 1.12, part b). □

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