

Parallel and circular flows for the two-dimensional Euler equations*

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Abstract

We consider steady flows of ideal incompressible fluids in two-dimensional domains. These flows solve the Euler equations with tangential boundary conditions. If such a flow has no stagnation point in the domain or at infinity, in the sense that the infimum of its norm over the domain is positive, then it inherits the geometric properties of the domain, for some simple classes of domains. Namely, if the domain is a strip or a half-plane, then such a flow turns out to be parallel to the boundary of the domain. If the domain is the plane, the flow is then a parallel flow, that is, its trajectories are parallel lines. If the domain is an annulus, then the flow is circular, that is, the streamlines are concentric circles. The results are based on qualitative properties and classification results for some semilinear elliptic equations satisfied by the stream function.

1 Steady flows of a two-dimensional ideal fluid

We consider the incompressible Euler equations

$$\begin{cases} v \cdot \nabla v + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

in smooth two-dimensional connected subsets (domains) of \mathbb{R}^2 . The vector field v is denoted by

$$v = (v_1, v_2)$$

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and the scalar function p stands for the pressure. Both v and p are assumed to be (at least) of class $C^1(\overline{\Omega})$ and the equations (1.1) are actually satisfied in $\overline{\Omega}$. The flow v is assumed to satisfy the boundary conditions

$$v \cdot n = 0 \quad \text{on } \partial\Omega, \quad (1.2)$$

where n is the outward unit normal to Ω on $\partial\Omega$. In other words, v is tangential on $\partial\Omega$.

Due to the boundary conditions and the incompressibility condition $\operatorname{div} v = 0$, the vector field v admits a stream function u satisfying

$$v = \nabla^\perp u = \left(-\frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_1} \right). \quad (1.3)$$

The function u , which is (at least) of class $C^2(\overline{\Omega})$, is well-defined and unique up to an additive constant.

By definition, u is constant along the streamlines of the flow. The streamlines are the trajectories parametrized by the family of ordinary differential equations $\dot{\xi}(t) = v(\xi(t))$. More precisely, for $x \in \overline{\Omega}$, let us denote ξ_x the solution of

$$\begin{cases} \dot{\xi}_x(t) &= v(\xi_x(t)), \\ \xi_x(0) &= x. \end{cases}$$

The function ξ_x is of class C^1 and it is defined in a maximal open interval $I_x \subset \mathbb{R}$.

Throughout the paper, one assumes that the flow v has no stagnation point in Ω or at infinity, in the sense that

$$\inf_{\Omega} |v| > 0, \quad (1.4)$$

where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^2 .

The goal of this review paper is to see that the smooth solutions v of (1.1)-(1.2) inherit the geometrical properties of the domains Ω , for some simple classes of domains Ω , as soon as they satisfy assumption (1.4) and possibly some further boundedness hypotheses. Namely, on the one hand, the flows turn out to be parallel flows in domains which are invariant by translation, such as strips, half-planes and the whole plane itself. On the other hand, the flows are circular in annular domains.

2 Parallel flows in strips

We deal in this section with two-dimensional strips (with bounded cross sections). Up to rotation and scaling, we only consider the strip $\Omega_2 \subset \mathbb{R}^2$ defined by

$$\Omega_2 = \mathbb{R} \times (0, 1) = \{x = (x_1, x_2) \in \mathbb{R}^2; 0 < x_2 < 1\} \quad (2.1)$$

The flow v is called a shear or a parallel flow if there is a unit vector $e = (e_1, e_2)$ such that v is parallel to e . Thanks to the incompressibility condition $\operatorname{div} v = 0$, this means that v can be written as

$$v(x) = V(x \cdot e^\perp) e,$$

where $V : \mathbb{R} \rightarrow \mathbb{R}$ is a scalar function and

$$e^\perp = (-e_2, e_1).$$

It is easy to see that the flow v is a shear flow if and only if the pressure p is a constant.

The first main result says that the flow is parallel to the boundary in the whole domain under the assumption (1.4).

Theorem 2.1 [14] *Let $\Omega = \Omega_2$ be the strip defined by (2.1) and let $v \in C^2(\overline{\Omega_2})$ solve the Euler equations (1.1) in $\Omega = \Omega_2$ together with the boundary conditions (1.2) on $\partial\Omega_2$. If v satisfies the assumption (1.4) in Ω_2 , then v is a parallel flow, that is,*

$$v(x) = (v_1(x_2), 0) \text{ in } \overline{\Omega_2}. \quad (2.2)$$

The conclusion states that all streamlines of the flow are parallel lines which are parallel to $\partial\Omega_2$ and therefore parallel to the x_1 -axis. Equivalently, this means that the stream function u defined in (1.3) depends only on the x_2 variable.

Theorem 2.1 means that any $C^2(\overline{\Omega_2})$ non-parallel flow which is tangential on $\partial\Omega_2$ must have a stagnation point in $\overline{\Omega_2}$ or at infinity. These stagnation points may well be in $\overline{\Omega_2}$ or only at infinity. For instance, on the one hand, for any $\alpha \neq 0$, the non-parallel cellular flow of the type

$$v(x) = \nabla^\perp(\sin(\alpha x_1) \sin(\pi x_2)) = (-\pi \sin(\alpha x_1) \cos(\pi x_2), \alpha \cos(\alpha x_1) \sin(\pi x_2)), \quad (2.3)$$

which solves (1.1)-(1.2) with pressure given by

$$p(x) = \frac{\pi^2}{4} \cos(2\alpha x_1) + \frac{\alpha^2}{4} \cos(2\pi x_2),$$

has a countable number of stagnation points in $\overline{\Omega_2}$. On the other hand, the non-parallel flow

$$v(x) = \nabla^\perp(\sin(\pi x_2) e^{x_1}) = (-\pi \cos(\pi x_2) e^{x_1}, \sin(\pi x_2) e^{x_1}),$$

which solves (1.1)-(1.2) with

$$p(x) = -\frac{\pi^2}{2} e^{2x_1}$$

has no stagnation point in $\overline{\Omega_2}$ ($|v| > 0$ in $\overline{\Omega_2}$), but $\inf_{\Omega_2} |v| = 0$.

The condition (1.4) is then a simple condition for a flow solving (1.1)-(1.2) to be a parallel flow in a two-dimensional strip. We should however keep in mind that this condition is obviously not equivalent to being a shear flow, since any parallel flow $v(x) = (v_1(x_2), 0)$ in $\overline{\Omega_2}$ for which v_1 does not have a constant strict sign does not satisfy the condition $\inf_{\Omega_2} |v| > 0$ (however, under the conditions of Theorem 2.1, the first component $v_1(x) = v_1(x_2)$ in (2.2) has a constant strict sign in $\overline{\Omega_2}$).

The C^2 smoothness of the flow v is a technical assumption which plays a role in the proof. It implies that the stream function u defined by (1.3) is of class $C^3(\overline{\Omega_2})$ and that the function f appearing in the elliptic equation

$$\Delta u + f(u) = 0$$

satisfied by u is itself of class C^1 and therefore locally Lipschitz-continuous, see (6.2) below. This smoothness property of f is used to guarantee that u satisfies some qualitative monotonicity and symmetry properties based on the maximum principle. We refer to Section 6 for further details.

Remark 2.2 Notice that in Theorem 2.1 the flow v is not assumed to be a priori bounded in $\overline{\Omega}_2$. However, since v is (at least) continuous in $\overline{\Omega}_2$ and the interval $[0, 1]$ is bounded, the conclusion of Theorem 2.1 implies that v is necessarily bounded.

Remark 2.3 Theorem 2.1 does not hold in dimension 3. More precisely, consider the cylinder

$$\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3; x_2^2 + x_3^2 < 1\}$$

together with the vector field

$$v(x) = (1, -x_3, x_2)$$

and the scalar function $p(x) = (x_2^2 + x_3^2)/2$ defined in $\overline{\Omega}$. Then the pair (v, p) solves the Euler equations (1.1)-(1.2) with

$$1 \leq |v| \leq \sqrt{2} \quad \text{in } \overline{\Omega},$$

but the flow v is not a parallel flow.

3 Parallel flows in half-planes

Up to rotation and translation, we consider in this section the case of the half-plane $\Omega = \mathbb{R}_+^2$ defined by

$$\mathbb{R}_+^2 = \mathbb{R} \times (0, +\infty) = \{x = (x_1, x_2) \in \mathbb{R}^2; x_2 > 0\}.$$

As in the strip Ω_2 , the flows v solving (1.1)-(1.2) turn out to be parallel to the boundary of the half-plane \mathbb{R}_+^2 under the assumption (1.4), here together with an additional uniform boundedness.

Theorem 3.1 [14] Let $v \in C^2(\overline{\mathbb{R}_+^2})$ solve the Euler equations (1.1) in $\Omega = \mathbb{R}_+^2$ together with the boundary conditions (1.2) on $\partial\mathbb{R}_+^2$. If v satisfies the assumption (1.4) in \mathbb{R}_+^2 and if $v \in L^\infty(\mathbb{R}_+^2)$, then v is a parallel flow, that is,

$$v(x) = (v_1(x_2), 0) \quad \text{in } \overline{\mathbb{R}_+^2},$$

and the first component v_1 has a constant strict sign in $\overline{\mathbb{R}_+^2}$.

The conclusion implies that all streamlines of the flow are parallel lines which are parallel to $\partial\mathbb{R}_+^2$ and therefore parallel to the x_1 -axis. Equivalently, this means that the stream function u defined in (1.3) depends only on the x_2 variable.

Some comments are now in order on the conditions (1.4) and $v \in L^\infty(\mathbb{R}_+^2)$, namely,

$$0 < \inf_{\mathbb{R}_+^2} |v| \leq \sup_{\mathbb{R}_+^2} |v| < +\infty. \quad (3.1)$$

None of the two strict inequalities in (3.1) can be dropped for the conclusion to hold in general. More precisely, first, any cellular flow v of the type (2.3), which solves (1.1)-(1.2) with $\Omega = \mathbb{R}_+^2$, satisfies $\sup_{\mathbb{R}_+^2} |v| < +\infty$ and $\inf_{\mathbb{R}_+^2} |v| = 0$, and it is not a parallel flow. Second, the flow

$$v(x) = \nabla^\perp(x_2 \cosh(x_1)) = (-\cosh(x_1), x_2 \sinh(x_1)),$$

which solves (1.1)-(1.2) with

$$p(x) = -\frac{\cosh(2x_1)}{4} + \frac{x_2^2}{2},$$

satisfies $\inf_{\mathbb{R}_+^2} |v| > 0$ and $\sup_{\mathbb{R}_+^2} |v| = +\infty$, and it is not a parallel flow.

Question 3.2 *An interesting open question would be to see whether the conclusion of Theorem 3.1 still holds if condition (3.1) is replaced by the following weaker one*

$$\forall A > 0, \quad 0 < \inf_{\mathbb{R} \times (0,A)} |v| \leq \sup_{\mathbb{R} \times (0,A)} |v| < +\infty.$$

4 Parallel flows in the plane

Theorems 2.1 and 3.1 show that, in strips and in half-planes, the C^2 flows solving the Euler equations (1.1) and the boundary conditions (1.2), together with the no-stagnation-point assumption (1.4) (and a boundedness assumption in half-planes), are automatically parallel flows whose streamlines are all parallel lines to the boundary. Actually, in these two geometrical configurations, since the flows are assumed to be tangential on the boundary and to satisfy (1.4), the boundary $\partial\mathbb{R}_+^2$ (for Theorem 3.1) and each connected component of $\partial\Omega_2$ (for Theorem 2.1) are streamlines of the flow. The conclusions of Theorems 2.1 and 3.1 mean that all other streamlines are parallel to these obvious streamlines.

Let us now consider in this section flows solving the Euler equations (1.1) in the whole plane

$$\Omega = \mathbb{R}^2.$$

In this case, there is no boundary and there is no obvious streamline. However, the following result implies that, under the only conditions (3.1) in the plane \mathbb{R}^2 , the $C^2(\mathbb{R}^2)$ solutions of (1.1) are still parallel flows, which are parallel to some non-zero vector e (the vector e depends on the flow v , and it can be now any vector unlike in strips or half-planes).

Theorem 4.1 [15] *Let v be a $C^2(\mathbb{R}^2)$ flow solving (1.1) with $\Omega = \mathbb{R}^2$. If v satisfies the assumption (1.4) in \mathbb{R}^2 and if $v \in L^\infty(\mathbb{R}^2)$, then v is a parallel flow, that is, there exist a non-zero vector e and a function $V : \mathbb{R} \rightarrow \mathbb{R}$ with constant strict sign such that*

$$v(x) = V(x \cdot e^\perp) e \quad \text{in } \mathbb{R}^2.$$

The conclusion means that all streamlines of the flow are parallel lines which are parallel to the direction e . In other words, the stream function u defined in (1.3) depends only on the orthogonal variable $x \cdot e^\perp$. Like Theorems 2.1 and 3.1, Theorem 4.1 can also be viewed

as a Liouville-type rigidity result since the conclusion says that the argument of the flow is actually constant (the argument of the flow is any continuous function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying

$$\frac{v(x)}{|v(x)|} = (\cos \phi(x), \sin \phi(x)) \quad \text{in } \mathbb{R}^2,$$

see (6.3) in Section 6 for further details), and that the pressure p is constant as well.

Let us now comment the two main assumptions (1.4) and $v \in L^\infty(\mathbb{R}^2)$ made in Theorem 4.1. Firstly, the assumption (1.4) means that the flow v has no stagnation point in \mathbb{R}^2 or at infinity. In other words, Theorem 4.1 implies that any $C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ flow which is not a parallel flow must have a stagnation point in \mathbb{R}^2 or at infinity, that is, it must satisfy $\inf_{\mathbb{R}^2} |v| = 0$). Without the condition (1.4), the conclusion of Theorem 4.1 does not hold in general. For instance, for any $(\alpha, \beta) \in \mathbb{R}^* \times \mathbb{R}^*$, the $C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ cellular flow v defined in \mathbb{R}^2 by

$$v(x_1, x_2) = \nabla^\perp(\sin(\alpha x_1) \sin(\beta x_2)) = (-\beta \sin(\alpha x_1) \cos(\beta x_2), \alpha \cos(\alpha x_1) \sin(\beta x_2)),$$

which solves (1.1) with

$$p(x) = \frac{\beta^2}{4} \cos(2\alpha x_1) + \frac{\alpha^2}{4} \cos(2\beta x_2),$$

has countably many stagnation points in \mathbb{R}^2 , and it is not a parallel flow. However, as for Theorems 2.1 and 3.1, the sufficient condition (1.4) is obviously not equivalent to being a shear flow. Indeed, any continuous shear flow $v(x) = V(x \cdot e^\perp) e$ for which V changes sign (or more generally if $\inf_{\mathbb{R}} |V| = 0$) does not satisfy the condition (1.4).

Secondly, without the boundedness of v , the conclusion of Theorem 4.1 does not hold either in general. For instance, the $C^2(\mathbb{R}^2)$ flow v defined in \mathbb{R}^2 by

$$v(x) = \nabla^\perp(x_2 \cosh(x_1)) = (-\cosh(x_1), x_2 \sinh(x_1)),$$

which solves (1.1) with

$$p(x) = -\frac{\cosh(2x_1)}{4} + \frac{x_2^2}{2},$$

satisfies $\inf_{\mathbb{R}^2} |v| = 1 > 0$ but it is not bounded in \mathbb{R}^2 , and it is not a parallel flow.

Theorem 4.1 has some immediate consequences regarding non-vanishing periodic flows and the stability of non-vanishing parallel flows under small L^∞ perturbations. These consequences are stated in the following corollary. In the statement, we say that a flow v is periodic if there is a basis (e_1, e_2) of \mathbb{R}^2 such that $v(x) = v(x + k_1 e_1 + k_2 e_2)$ in \mathbb{R}^2 for all $x \in \mathbb{R}^2$ and $(k_1, k_2) \in \mathbb{Z}^2$.

Corollary 4.2 (i) *Let v be a $C^2(\mathbb{R}^2)$ periodic flow solving (1.1) in $\Omega = \mathbb{R}^2$. If $|v(x)| \neq 0$ for all $x \in \mathbb{R}^2$, then v is a parallel flow. In other words, if v is not a parallel flow, then it has stagnation points.*

(ii) *Let v be a bounded parallel flow solving (1.1) in $\Omega = \mathbb{R}^2$ and satisfying (1.4). There is $\varepsilon > 0$ such that, if v' is a $C^2(\mathbb{R}^2)$ flow solving (1.1) in $\Omega = \mathbb{R}^2$ and satisfying*

$$\|v' - v\|_{L^\infty(\mathbb{R}^2)} \leq \varepsilon,$$

then v' is a parallel flow.

Remark 4.3 *Other Liouville-type and rigidity theorems are known for the Navier-Stokes equations in the plane \mathbb{R}^2 , namely any bounded solution of the Navier-Stokes equations in the plane is constant, see [18].*

5 Circular flows in two-dimensional annuli

Let us finally consider the case of two-dimensional annuli defined by

$$\Omega_{a,b} = \{x \in \mathbb{R}^2; a < |x| < b\}, \quad (5.1)$$

where $a < b$ are two positive real numbers. Let us also define

$$e_r(x) = \frac{x}{|x|} \quad \text{and} \quad e_\theta(x) = e_r(x)^\perp = \left(-\frac{x_2}{|x|}, \frac{x_1}{|x|}\right)$$

for any $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. In the circular domains $\Omega_{a,b}$, one still considers the Euler equations (1.1) together with the tangential boundary conditions (1.2), meaning that $v \cdot e_r = 0$ on $\partial\Omega_{a,b}$.

Here, the notion of parallel flows does not make sense anymore. A meaningful notion is that of circular flows. Namely, a flow v in $\overline{\Omega_{a,b}}$ is called a circular flow if $v(x)$ is parallel to the vector $e_\theta(x)$ for every $x \in \overline{\Omega_{a,b}}$, that is, $v \cdot e_r = 0$ in $\overline{\Omega_{a,b}}$. Together with the incompressibility assumption, this means that v can be written as

$$v(x) = V(|x|) e_\theta(x) \quad \text{in } \overline{\Omega_{a,b}},$$

for some function $V : [a, b] \rightarrow \mathbb{R}$.

Theorem 5.1 [16] *Let $v \in C^2(\overline{\Omega_{a,b}})$ solve the Euler equations (1.1) in $\Omega = \Omega_{a,b}$ together with the boundary conditions (1.2) on $\partial\Omega_{a,b}$. If $|v| > 0$ in $\overline{\Omega_{a,b}}$, then v is a circular flow, and the function $v \cdot e_\theta$ has a constant strict sign in $\overline{\Omega_{a,b}}$.*

The conclusion means that all streamlines of the flow are concentric circles. In other words, the stream function u defined in (1.3) is radially symmetric and, thanks to the condition $|v| > 0$ and the continuity of v in $\overline{\Omega_{a,b}}$, the function u is also strictly monotone with respect to the radial variable $|x|$.

The assumption $|v| > 0$ in $\overline{\Omega_{a,b}}$ means that the flow v has no stagnation point in $\overline{\Omega_{a,b}}$. Theorem 5.1 then implies that any $C^2(\overline{\Omega_{a,b}})$ flow which is tangential on $\partial\Omega_{a,b}$ and is not a circular flow must have at least one stagnation point in $\overline{\Omega_{a,b}}$.

Without the assumption $|v| > 0$ in $\overline{\Omega_{a,b}}$, the conclusion of Theorem 5.1 does not hold in general. Indeed, for every given $C^1(\mathbb{R})$ function f , any non-radial solution $u \in C^3(\overline{\Omega_{a,b}})$ of

$$\Delta u + f(u) = 0 \quad \text{in } \overline{\Omega_{a,b}} \quad (5.2)$$

which is constant on each connected component of $\partial\Omega_{a,b}$ and which has a critical point gives rise to a non-circular solution

$$v = \nabla^\perp u \in C^2(\overline{\Omega_{a,b}})$$

of (1.1)-(1.2) with a stagnation point (notice that v defined as above solves (1.1) for some pressure $p \in C^2(\overline{\Omega_{a,b}})$ because $v \cdot \nabla v$ is curl-free thanks to (5.2) and because the integral of $(v \cdot \nabla v) \cdot e_\theta$ over the circle $\{x \in \mathbb{R}^2; |x| = a\}$ is equal to zero). For instance, let $\lambda \in \mathbb{R}$ and $\varphi \in C^\infty([a, b])$ denote the principal eigenvalue and eigenfunction of the eigenvalue problem

$$-\varphi''(r) - r^{-1}\varphi'(r) + r^{-2}\varphi(r) = \lambda \varphi(r) \quad \text{in } [a, b]$$

with $\varphi > 0$ in (a, b) and Dirichlet boundary condition $\varphi(a) = \varphi(b) = 0$. The function $u \in C^\infty(\overline{\Omega_{a,b}})$ defined by

$$u(x) = \varphi(|x|) \frac{x_1}{|x|}$$

(that is, $u(x) = \varphi(r) \cos \theta$ in the usual polar coordinates) satisfies

$$\Delta u + \lambda u = 0 \quad \text{in } \overline{\Omega_{a,b}}$$

and has some critical points in $\overline{\Omega_{a,b}}$ (since $\min_{\overline{\Omega_{a,b}}} u < 0 < \max_{\overline{\Omega_{a,b}}} u$ and $u = 0$ on $\partial\Omega_{a,b}$). Then the flow $v = \nabla^\perp u \in C^2(\overline{\Omega_{a,b}})$ is a non-circular flow solving (1.1)-(1.2) and it has some stagnation points in $\overline{\Omega_{a,b}}$.

However, the sufficient condition $|v| > 0$ in $\overline{\Omega_{a,b}}$ is obviously not equivalent to being a circular flow, in the sense that there are circular flows with stagnation points. The trivial flow $v = 0$ is an obvious example! More generally speaking, any $C^2(\overline{\Omega_{a,b}})$ circular flow

$$v(x) = V(|x|) e_\theta(x)$$

solving (1.1)-(1.2) with $\Omega = \Omega_{a,b}$ and for which $V \in C^2([a, b])$ does not have a constant strict sign, is then such that $\min_{\overline{\Omega_{a,b}}} |v| = 0$. For instance, if $\mu \in \mathbb{R}$ and $\phi \in C^\infty([a, b])$ denote the principal eigenvalue and eigenfunction of the eigenvalue problem

$$-\phi''(r) - r^{-1}\phi'(r) = \mu \phi(r) \quad \text{in } [a, b]$$

with $\phi > 0$ in (a, b) and Dirichlet boundary condition $\phi(a) = \phi(b) = 0$, then the vector field v defined by

$$v = \nabla^\perp u, \quad \text{with } u(x) = \phi(|x|),$$

is a $C^2(\overline{\Omega_{a,b}})$ non-trivial circular flow solving (1.1)-(1.2) with infinitely many stagnation points in $\overline{\Omega_{a,b}}$: more precisely, if $r^* \in (a, b)$ denotes the real number such that $\phi(r^*) = \max_{[a,b]} \phi$ (from elementary arguments, r^* turns out to be the only critical point of ϕ in $[a, b]$), then the set of stagnation points of the flow v is the whole circle $\{x \in \mathbb{R}^2; |x| = r^*\}$, since

$$v(x) = \phi'(|x|) e_\theta(x) \quad \text{in } \overline{\Omega_{a,b}}.$$

Remark 5.2 *Further results similar to Theorem 5.1 are proved in [16] for the solutions v of (1.1)-(1.2) with $\Omega = \Omega_{a,b}$ when $a = 0 < b < \infty$ ($\Omega_{0,b}$ is a punctured disc, and no boundary condition are imposed on the set $\{0\} \subset \partial\Omega_{0,b}$) and when $0 < a < b = \infty$ ($\Omega_{a,\infty}$ is the complement of a closed disc), together with some a priori bounds on the growth of $|v(x)|$ as $|x| \rightarrow 0$ or as $|x| \rightarrow \infty$, respectively. We also mention other rigidity results for the stationary solutions of (1.1), such as the analyticity of the streamlines under a condition of the type $v_1 > 0$ in the unit disc [17], and the local correspondence between the vorticities of the solutions of (1.1) and the co-adjoint orbits of the vorticities for the non-stationary version of (1.1) in annular domains [5].*

6 Semilinear elliptic equations satisfied by the stream function

In this section, we explain the main lines of the proofs of Theorems 2.1, 3.1, 4.1 and 5.1. They are based on the study of the geometric properties of the streamlines of the flow v , that is, on the qualitative planar or radial symmetry properties of the stream function u defined in (1.3). The streamlines are indeed the connected components of the level sets of u . In the strip Ω_2 defined in (2.1) and in the half-plane \mathbb{R}_+^2 , the goal is to show that the stream function u depends only on the variable x_2 . In the whole plane \mathbb{R}^2 , Theorem 4.1 means that the function u is one-dimensional. Lastly, in the annulus $\Omega_{a,b}$ defined in (5.1), the goal is to show that u is radially symmetric.

To achieve this goal, one uses the trajectories of the gradient flow, which are parametrized by the solutions of the equations

$$\begin{cases} \dot{\sigma}_x(t) &= \nabla u(\sigma_x(t)), \\ \sigma_x(0) &= x, \end{cases}$$

with $x \in \overline{\Omega}$ and $\sigma_x(t) \in \overline{\Omega}$. These trajectories are orthogonal to the streamlines, since $v = \nabla^\perp u$. An important observation is that, due to (1.4), the function

$$g : t \mapsto u(\sigma_x(t))$$

is increasing on its interval of definition, and furthermore

$$g'(t) \geq (\inf_{\Omega} |v|)^2 > 0.$$

When Ω is the strip Ω_2 , it is shown in [14] that the stream function u is bounded in Ω_2 and that each function σ_x is defined in a compact interval. The same property holds if Ω is the annulus $\Omega_{a,b}$, see [16]. When Ω is the half-plane \mathbb{R}_+^2 , each function σ_x is defined in a closed semi-infinite interval. Finally, if Ω is the whole plane \mathbb{R}^2 , then each function σ_x is defined in the whole interval \mathbb{R} , see [15].

Furthermore, in all these four cases, it is shown that the streamlines foliate the set $\overline{\Omega}$ in a monotone way, in the sense that

$$\overline{\Omega} = \bigcup_{y \in \Sigma_x} \Gamma_y, \tag{6.1}$$

where x is any point in $\overline{\Omega}$, Σ_x is the trajectory of the gradient flow in $\overline{\Omega}$ and containing x , and Γ_y is the streamline containing y . The above formula, which is a key-point in the proofs, is based on a continuation argument and on the property that, due to (1.4), the streamlines of two close points stay close to each other in the sense of the Hausdorff distance. It follows from these arguments that, when Ω is the strip Ω_2 or the half-plane \mathbb{R}_+^2 , then the projection on the x_1 -axis of each streamline is equal to that whole axis and furthermore that each streamline is bounded in the x_2 direction when $\Omega = \mathbb{R}_+^2$. However, this property does not imply yet that each streamline is a line parallel to the x_1 -axis.

Another key-observation is the fact that, due to (1.1), the vorticity

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \Delta u$$

is constant along any streamline of the flow v . Since u is strictly monotone along any trajectory Σ_x of the gradient flow, it then follows from (6.1) that the stream function u satisfies a semilinear elliptic equation of the type

$$\Delta u + f(u) = 0 \quad \text{in } \overline{\Omega}, \tag{6.2}$$

where f is a C^1 function defined on the range of u in $\overline{\Omega}$ (the range of u in $\overline{\Omega}$ is a compact interval when $\Omega = \Omega_2$ or $\Omega_{a,b}$, it is a closed semi-infinite interval when $\Omega = \mathbb{R}_+^2$ and it is equal to \mathbb{R} when $\Omega = \mathbb{R}^2$).

Regarding the case of the strip $\Omega_2 = \mathbb{R} \times (0, 1)$, each trajectory Σ_x of the gradient flow in $\overline{\Omega_2}$ ends on the two connected components of the boundary. Since u is strictly monotone along Σ_x and is constant along the two connected components of $\partial\Omega_2$, it follows that u is bounded from above and below in Ω_2 by its two constant values on the two connected components of $\partial\Omega_2$. Therefore, up to changing v into $-v$ and up to normalization of u , Theorem 2.1 reduces to the following Liouville type result.

Theorem 6.1 [14] *Let $c > 0$, let $f : [0, c] \rightarrow \mathbb{R}$ be a Lipschitz continuous function and let u be a $C^2(\overline{\Omega_2})$ solution of the equation (6.2) in the strip $\overline{\Omega_2}$, such that*

$$u = 0 \text{ on } \{x_2 = 0\}, \quad u = c \text{ on } \{x_2 = 1\},$$

and

$$0 < u < c \quad \text{in } \Omega_2.$$

Then u depends only on the variable x_2 , that is, there is a function $U : [0, 1] \rightarrow \mathbb{R}$ such that

$$u(x_1, x_2) = U(x_2) \quad \text{in } \overline{\Omega_2},$$

and $U' > 0$ in $(0, 1)$.

As a matter of fact, under the assumptions of Theorem 6.1, it follows from [3, Theorem 1.1'] applied to u , resp. to $c - u(x_1, 1 - x_2)$, that

$$u_{x_2} = \frac{\partial u}{\partial x_2} > 0 \quad \text{in } \{(x_1, x_2) \in \mathbb{R}^2; 0 < x_2 < 1/2\},$$

resp.

$$u_{x_2} > 0 \quad \text{in } \{(x_1, x_2) \in \mathbb{R}^2, 1/2 < x_2 < 1\}$$

(notice that such a monotonicity is known only in dimension 2 without any further assumptions on f). Therefore, $u_{x_2} \geq 0$ in $\overline{\Omega_2}$. The new result in Theorem 6.1 is the fact that the monotonicity property $u_{x_2} \geq 0$ in $\overline{\Omega_2}$ implies that u is one-dimensional, that is, u depends

only on x_2 . The proof of this property is based on the maximum principle and on a sliding method [4], namely one can show that

$$u(x) \leq u^{\lambda\tau}(x) := u(x_1 + \lambda\tau_1, x_2 + \lambda\tau_2)$$

for every $(\tau_1, \tau_2) \in \mathbb{R} \times (0, +\infty)$, $\lambda \in (0, 1/\tau_2)$ and $x \in \mathbb{R} \times [0, 1 - \lambda\tau_2]$. By letting $\tau_2 \rightarrow 0^+$, it then immediately follows that u is both non-decreasing and non-increasing with respect to the variable x_1 in $\overline{\Omega_2}$, that is, u depends only on x_2 . As a consequence, the flow v is parallel to the x_1 -axis.

Actually, it turns out that the last part of the argument based on the sliding method holds in any dimension $n \geq 2$:

Theorem 6.2 [14] *Let $n \geq 2$, let $c > 0$, let $f : [0, c] \rightarrow \mathbb{R}$ be a Lipschitz continuous function and let u be a $C^2(\overline{\Omega_n})$ solution of the equation (6.2) in the n -dimensional slab Ω_n defined by*

$$\Omega_n = \mathbb{R}^{n-1} \times (0, 1) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, 0 < x_n < 1\},$$

such that

$$u = 0 \text{ on } \{x_n = 0\} \text{ and } u = c \text{ on } \{x_n = 1\}.$$

Suppose also that u is non-decreasing with respect to the variable x_n , namely

$$u_{x_n} = \frac{\partial u}{\partial x_n} \geq 0 \text{ in } \overline{\Omega_n}.$$

Then u depends only on x_n , that is, there is a function $U : [0, 1] \rightarrow \mathbb{R}$ such that

$$u(x_1, \dots, x_n) = U(x_n) \text{ in } \overline{\Omega_n},$$

and $U'(x_n) > 0$ for all $0 < x_n < 1$.

In the case of the half-plane $\Omega = \mathbb{R}_+^2$, one can show from (6.1) that, up to changing v into $-v$ and up to normalization of u , the function u vanishes on $\partial\mathbb{R}_+^2$ and is positive in \mathbb{R}_+^2 , see [14]. To conclude the proof of Theorem 3.1, we use some monotonicity and one-dimensional symmetry results [3, 10] for positive solutions with bounded gradient of semilinear elliptic equations (6.2) in \mathbb{R}_+^2 with Dirichlet boundary conditions: namely, u then depends only on x_2 and the flow v is parallel to the x_1 -axis.

Regarding Theorem 4.1 in the case of the whole plane \mathbb{R}^2 , there is no boundary and therefore no boundary condition for the stream u . As a consequence, u is not constant along a given obvious line, unlike in the case of the strip Ω_2 or the half-plane \mathbb{R}_+^2 . One then uses a completely different method. Namely, one considers the $C^2(\mathbb{R}^2)$ argument ϕ of the flow, defined by

$$\frac{v(x)}{|v(x)|} = (\cos \phi(x), \sin \phi(x)) \text{ for all } x \in \mathbb{R}^2.$$

The argument ϕ is uniquely defined in a continuous way, up to an additive multiple of 2π . From the equation (6.2) satisfied by the stream function u , it turns out that the argument ϕ satisfies the equation

$$\operatorname{div}(|v|^2 \nabla \phi) = 0 \text{ in } \mathbb{R}^2. \tag{6.3}$$

Notice that this equation is uniformly elliptic thanks to the assumption (1.4) and the boundedness of v in Theorem 4.1. The difficult key-point consists in proving that the argument ϕ grows at most as $\ln R$ in balls of large radius R , see [15]. Finally, one uses a compactness argument and some results of Moser [19] on the at-least-algebraic growth of the oscillations of the non-constant solutions of uniformly elliptic equations of the type (6.3) to conclude that the argument ϕ is actually constant. This immediately means that the flow v is a parallel flow.

Remark 6.3 *If, in addition to the condition $\inf_{\mathbb{R}^2} |v| > 0$, one assumes that*

$$v \cdot e > 0 \text{ in } \mathbb{R}^2$$

for some non-zero vector e , then the end of the proof of Theorem 4.1 would be much simpler: indeed, in that case, the argument ϕ of the flow is automatically bounded and the results of Moser [19] imply that ϕ is constant. One can also conclude with another argument which does not use (6.3). Namely, the assumption $v \cdot e > 0$ in \mathbb{R}^2 implies that the stream function u is monotone in the direction e^\perp . Since u satisfies the semilinear elliptic equation of the type

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^2$$

with $f \in C^1(\mathbb{R})$, it then follows that u is one-dimensional, as in the proof of a related conjecture of De Giorgi [6] in dimension 2 (see [3, 11] and see also [1, 2, 7, 8, 9, 20] for further references in that direction). Finally, since u is one-dimensional, the vector field v is a parallel flow.

Finally, regarding Theorem 5.1 in the case of the annuli $\Omega_{a,b}$ defined in (5.1), the stream function u satisfies a semilinear elliptic equation of the type (6.2) in $\overline{\Omega_{a,b}}$ and it is equal to two different constants on the two connected components of the boundary $\partial\Omega_{a,b}$. By using a moving plane method as in [12, 13], one then concludes that u is radially symmetric and strictly monotone with respect to $|x|$, meaning that v is a circular flow. We refer to [16] for further details and further results in punctured discs and complements or discs.

References

- [1] G. Alberti, L. Ambrosio, X. Cabré, *On a long-standing conjecture of E. De Giorgi: old and recent results*, Acta Appl. Math. **65** (2001), 9-33.
- [2] L. Ambrosio, X. Cabré, *Entire solutions of semilinear elliptic equations in \mathbb{R}^3 and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13** (2000), 725-739.
- [3] H. Berestycki, L. Caffarelli, L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1997), 69-94.
- [4] H. Berestycki, L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Bras. Mat. **22** (1991), 1-37.

- [5] A. Choffrut, V. Šverák, *Local structure of the set of steady-state solutions to the 2D incompressible Euler equations*, *Geom. Funct. Anal.* **22** (2012), 136-201.
- [6] E. De Giorgi, *Convergence problems for functionals and operators*, In: Proc. Int. Meeting on Recent Methods in Nonlinear Analysis, Rome, 1978, Pitagora, 1979, 131-188.
- [7] M. Del Pino, M. Kowalczyk, J. Wei, *On the De Giorgi conjecture in dimension $N \geq 9$* , *Ann. Math.* **174** (2011), 1485-1569.
- [8] A. Farina, *Liouville-type theorems for elliptic equations*, *Handbook of Differential Equations: Stationary Partial Differential Equations*, vol. 4, M. Chipot ed., 2007, Elsevier, 61-116.
- [9] A. Farina, E. Valdinoci, *The state of the art for a conjecture of De Giorgi and related problems*, In: Recent progress on reaction-diffusion systems and viscosity solutions, World Sci. Publ., Hackensack, NJ, 2009, 74-96.
- [10] A. Farina, E. Valdinoci, *Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems*, *Arch. Ration. Mech. Anal.* **195** (2010), 1025-1058.
- [11] N. Ghoussoub, C. Gui, *On a conjecture of De Giorgi and some related problems*, *Math. Ann.* **311** (1998), 481-491.
- [12] L.E. Fraenkel, *An introduction to maximum principles and symmetry in elliptic problems*, Cambridge Univ. Press, 2000.
- [13] B. Gidas, W.N. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, *Comm. Math. Phys.* **68** (1979), 209-243.
- [14] F. Hamel, N. Nadirashvili, *Shear flows of an ideal fluid and elliptic equations in unbounded domains*, *Comm. Pure Appl. Math.* **70** (2017), 590-608.
- [15] F. Hamel, N. Nadirashvili, *A Liouville theorem for the Euler equations in the plane*, preprint hal.archives-ouvertes.fr/hal-01491806.
- [16] F. Hamel, N. Nadirashvili, *Circular flows for the Euler equations in two-dimensional annular domains*, in preparation.
- [17] H. Koch, N. Nadirashvili, *Partial analyticity and nodal sets for nonlinear elliptic systems*, arxiv.org/pdf/1506.06224.pdf.
- [18] G. Koch, N. Nadirashvili, G.A. Seregin, V. Šverák, *Liouville theorems for the Navier-Stokes equations and applications*, *Acta Math.* **203** (2009), 83-105.
- [19] J. Moser, *On Harnack's theorem for elliptic differential equations*, *Comm. Pure Appl. Math.* **14** (1961), 577-591.
- [20] O. Savin, *Regularity of flat level sets in phase transitions*, *Ann. Math.* **169** (2009), 41-78.