LOCALIZED AND EXPANDING ENTIRE SOLUTIONS OF REACTION-DIFFUSION EQUATIONS

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Abstract. This paper is concerned with the spatio-temporal dynamics of nonnegative bounded entire solutions of some reaction-diffusion equations in $\mathbb{R}^N$ in any space dimension N. The solutions are assumed to be localized in the past. Under certain conditions on the reaction term, the solutions are then proved to be time-independent or heteroclinic connections between different steady states. Furthermore, either they are localized uniformly in time, or they converge to a constant steady state and spread at large time. This result is then applied to some specific bistable-type reactions.

In memory of Geneviève Raugel, with admiration and respect

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1. Introduction and the main result

In this paper we are concerned with nonnegative bounded entire solutions of the following reaction-diffusion equation:

$$u_t = \Delta u + f(u), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $f : [0, +\infty) \to \mathbb{R}$ is a C^1 function such that

$$f(0) = 0 \text{ and } f'(0) < 0. \tag{1.2}$$

The solutions are always understood in the classical sense $C^{1,2}_{t,x}(\mathbb{R}\times\mathbb{R}^N)$, from the parabolic regularity theory. Notice immediately that, for a nonnegative bounded solution u of (1.1), either $u(t, x) = 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, or $u(t, x) > 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, from the strong parabolic maximum principle and the uniqueness of the bounded solutions for the associated Cauchy problem.

The solutions u are called *entire* as they are defined for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. The solution u of (1.1) is called *bounded* if u is an entire solution of (1.1) and

$$\sup_{(t,x)\in\mathbb{R}^{N+1}} |u(t, x)| < \infty.$$

We are especially interested in the description of their limit profiles as $t \to \pm\infty$. If a solution u converges, in some sense to be made precise, to some limit states $\phi_\pm$ as $t \to \pm\infty$, then u is a heteroclinic connection between ϕ_- and ϕ_+ if $\phi_- \neq \phi_+$, while it is homoclinic to $\phi_\pm$ if $\phi_- = \phi_+$ (we will actually prove that the homoclinic connections reduce to time-independent solutions under the assumptions in this paper). The description and the properties of the entire solutions of (1.1) are of particular importance such as, for any element φ of the ω-limit set of any nonnegative initial condition of the associated Cauchy problem giving rise to a bounded global solution, and for any $t_0 \in \mathbb{R}$, there is a bounded entire solution u of (1.1) such that $u(t_0, \cdot) = \varphi$.

1.1. **Localized solutions in the past and localized steady states.** We are interested in solutions that are localized in the past, in the sense that

$$u(t,x) \to 0 \ \text{ as } |x| \to +\infty, \quad \text{uniformly in } t \le 0. \tag{1.3}$$

Throughout the paper, $x \mapsto |x|$ denotes the Euclidean norm in $\mathbb{R}^N$, $(x,y) \mapsto x \cdot y$ denotes the Euclidean inner product, $B(x,R)$ denotes the open Euclidean ball of center $x \in \mathbb{R}^N$ and radius $R > 0$, and $B_R = B(0,R)$. Notice that the condition $f(0) = 0$ is then forced by (1.3). Furthermore, from standard parabolic estimates and the boundedness of u, condition (1.3) is equivalent to $\lim_{|x|\to+\infty} u(t,x) = 0$ uniformly in $t \le t_0$ for some (or equivalently for all) $t_0 \in \mathbb{R}$. This, however, does not necessarily mean that $\lim_{|x|\to+\infty} u(t,x) = 0$ uniformly in $t \in \mathbb{R}$ (such solutions are called *uniformly localized*), and one of the main features of the paper is to show a dichotomy between the solutions that are uniformly localized and those that spread as $t \to +\infty$.

The description of the positive bounded solutions of (1.1) satisfying (1.3) is closely related to the study of the positive bounded localized steady states $\phi \in C^2(\mathbb{R}^N)$, solving

$$\begin{cases} \Delta\phi + f(\phi) = 0 \text{ and } \phi > 0 \text{ in } \mathbb{R}^N, \\ \phi(x) \to 0 \text{ as } |x| \to +\infty. \end{cases} \tag{1.4}$$

Under the condition $f'(0) < 0$, it is known [27, 38, 39] that any solution ϕ of (1.4) is radially symmetric and decreasing with respect to its center, namely there exist a point $x_0 \in \mathbb{R}^N$ and a $C^2([0,+\infty))$ function Φ such that $\Phi' < 0$ in $(0,+\infty)$ and

$$\phi(x) = \Phi(|x - x_0|) \ \text{ for all } x \in \mathbb{R}^N. \tag{1.5}$$

It then follows from the strong maximum principle applied to ϕ that

$$f(\Phi(0)) = f\Big(\max_{\mathbb{R}^N} \phi\Big) > 0, \tag{1.6}$$

hence, together with (1.2), there is a unique real number m_ϕ such that

$$0 < m_\phi < \max_{\mathbb{R}^N} \phi, \quad f(m_\phi) = 0 \ \text{ and } \ f > 0 \text{ in } \Big(m_\phi, \max_{\mathbb{R}^N} \phi\Big]. \tag{1.7}$$

Lastly, since

$$\Phi''(r) + \frac{N-1}{r}\,\Phi'(r) + f(\Phi(r)) = 0 \ \text{ for all } r \in (0,+\infty) \tag{1.8}$$

and $\Phi'(0) = \Phi'(+\infty) = 0$ (the limit $\Phi'(+\infty) = 0$ coming from (1.4)-(1.5) and standard elliptic estimates), integrating the above equation against Φ' over $(0,+\infty)$ yields

$$\begin{cases} F\Big(\max_{\mathbb{R}^N} \phi\Big) = F(\Phi(0)) = 0 \quad \text{if } N = 1, \\ F\Big(\max_{\mathbb{R}^N} \phi\Big) = F(\Phi(0)) > 0 \quad \text{if } N \ge 2, \end{cases} \tag{1.9}$$

where

$$F(s) = \int_0^s f(\sigma)\,d\sigma \quad \text{for } s \geq 0. \tag{1.10}$$

Since $f(\max_{\mathbb{R}^N}\phi) > 0$, there is then $\eta > 0$ such that $f > 0$ in $[\max_{\mathbb{R}^N}\phi - \eta, \max_{\mathbb{R}^N}\phi + \eta]$ and $F > 0$ in $(\max_{\mathbb{R}^N}\phi, \max_{\mathbb{R}^N}\phi + \eta]$ if $N = 1$ (resp. in $[\max_{\mathbb{R}^N}\phi, \max_{\mathbb{R}^N}\phi + \eta]$ if $N \geq 2$). In the sequel, we also set

$$M_\phi = \inf\Big\{s \geq \max_{\mathbb{R}^N}\phi : f(s) = 0\Big\} \in \Big(\max_{\mathbb{R}^N}\phi, +\infty\Big]. \tag{1.11}$$

Notice that

$$m_\phi < \max_{\mathbb{R}^N}\phi < M_\phi \quad \text{and} \quad f > 0 \text{ in } (m_\phi, M_\phi), \tag{1.12}$$

and that M_ϕ may be equal to $+\infty$ (we refer to some specific examples in Section 2).

Furthermore, not only the steady states of (1.4) are radially symmetric and decreasing with respect to some center, but so are the bounded entire solutions of (1.1) which are localized in the past. Namely, it follows from [54] that, for any positive bounded solution u of (1.1)-(1.3), there is a point $x_0 \in \mathbb{R}^N$ such that

$$\begin{cases} u(t,x) = u(t,y) & \text{for all } (t,x,y) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \text{ with } |x-x_0| = |y-x_0|, \\ \nabla u(t,x)\cdot(x-x_0) < 0 & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^N \text{ with } x \neq x_0. \end{cases}$$

1.2. **The main result.** In the following theorem, which is the main result of the paper, we call $\mathcal{E}$ the set of $C^2(\mathbb{R}^N)$ solutions of (1.4) and, for any continuous bounded function $\varphi : \mathbb{R}^N \to \mathbb{R}$ and any set $\mathcal{A}$ of continuous bounded functions, we denote

$$\mathrm{dist}(\varphi, \mathcal{A}) = \inf_{\psi\in\mathcal{A}} \|\varphi - \psi\|_{L^\infty(\mathbb{R}^N)}.$$

Theorem 1.1. *Assume that f satisfies* (1.2) *and*

$$F < 0 \text{ in } (0, m_\phi] \quad \text{for all } \phi \in \mathcal{E}. \tag{1.13}$$

If there exists a positive bounded solution u of (1.1) *satisfying* (1.3)*, then $\mathcal{E} \neq \emptyset$ and*

$$\mathrm{dist}(u(t,\cdot), \mathcal{E}) \to 0 \text{ as } t \to -\infty. \tag{1.14}$$

Furthermore,

(i) *either $u(t,\cdot) \to 0$ uniformly in $\mathbb{R}^N$ as $t \to +\infty$,*
(ii) *or there is $\phi \in \mathcal{E}$ such that $u(t,\cdot) \to \phi$ uniformly in $\mathbb{R}^N$ as $t \to +\infty$,*
(iii) *or else there is a continuous function $\xi : \mathbb{R} \to \mathbb{R}$ depending on u and some positive constants M and c only depending on f such that*

$$\begin{cases} \limsup_{t\to+\infty}\Big(\max_{|x|\leq\xi(t)-A}|u(t,x) - M|\Big) \to 0 \\ \limsup_{t\to+\infty}\Big(\max_{|x|\geq\xi(t)+A}u(t,x)\Big) \to 0 \end{cases} \quad \text{as } A \to +\infty \tag{1.15}$$

and

$$\lim_{t\to+\infty}\frac{\xi(t)}{t} = c, \tag{1.16}$$

with $f(M) = 0$, $f'(M) \le 0$ *and* c *characterized by the existence of a function* $\varphi \in C^2(\mathbb{R})$ *solving*

$$\varphi'' + c\varphi' + f(\varphi) = 0 \text{ in } \mathbb{R}, \quad \varphi' < 0 \text{ in } \mathbb{R}, \quad \varphi(-\infty) = M, \quad \varphi(+\infty) = 0. \tag{1.17}$$

Property (1.14) means that the α-limit sets, with respect to the uniform convergence in $\mathbb{R}^N$, of the positive bounded solutions u of (1.1) consist of steady states solving (1.4). As far as the behavior of a solution u as $t \to +\infty$ is concerned, it turns out that, in both cases (i) and (ii), $u(t,x) \to 0$ as $|x| \to +\infty$ uniformly in $t \in \mathbb{R}$, namely u is then called *uniformly localized*. As a consequence, the conclusion of Theorem 1.1 means there is a dichotomy between the uniformly localized solutions and the ones which converge locally uniformly to a positive constant, with a positive spreading rate. Notice that in all cases (i), (ii) and (iii), the solution u converges locally uniformly in $\mathbb{R}^N$ as $t \to +\infty$ to a steady state (either a necessarily non-constant solution of (1.4), or the constants 0 or M) and its ω-limit set (with respect to the uniform convergence in cases (i) and (ii) and to the locally uniform convergence in case (iii)) is a singleton.

This situation is in contrast with some non-convergence and even non-quasiconvergence results of some positive bounded solutions of the Cauchy problems of the Fujita equation

$$u_t = \Delta u + u^p, \tag{1.18}$$

for which the ω-limit set (with respect to the locally uniform convergence) of some initial conditions may not be reduced to a single steady state (non-convergence) or may even contain other elements than steady states (non-quasiconvergence). Such results have been proved in [60, 61] for (1.18) in high dimensions N for some ranges of values of p, even for solutions which are localized at large time (see also [18, 55, 56] for further non-quasiconvergence results with non-localized oscillating initial conditions and bistable nonlinearities of the type (1.19) below). On the other hand, convergence or quasiconvergence hold for all functions f in dimension $N = 1$ with compactly supported initial conditions [14] or for generic functions f in any dimension $N \ge 1$ with initial conditions converging to 0 at infinity [42, 43], while the existence of at least one steady state in bounded trajectories has been shown in dimensions $N \le 2$ [25]. We refer to [57] for a general overview on convergence and quasiconvergence properties in dimension $N = 1$, to [47, 48] for further convergence or quasiconvergence results for some bistable, ignition or monostable nonlinearities f in any dimension $N \ge 1$ with radially decreasing initial conditions, and to [32] for a general overview on convergence results for gradient-like parabolic or hyperbolic equations.

Remark 1.2. It actually follows from the proof of Theorem 1.1, in particular from Steps 4 and 5 in Section 4.2, that a similar result as (1.15)-(1.16) holds for the spreading solutions of the associated Cauchy problem with localized initial conditions. More precisely, let $0 < m < M$ be given, and let $f : [0, M] \to \mathbb{R}$ be a given $C^1([0, M])$ function such that $f(0) = f(m) = f(M) = 0$, $f'(0) < 0$, $f > 0$ in (m, M), $F < 0$ in $(0, m]$ and $F(M) > 0$. Let $u_0 : \mathbb{R}^N \to [0, M]$ be a continuous function such that $\lim_{|x|\to+\infty} u_0(x) = 0$ and $u_0 \not\equiv 0$ in $\mathbb{R}^N$. Now, if the bounded solution u of the Cauchy problem associated with (1.1) with initial condition u_0 is assumed to be such that

$$u(t, \cdot) \to M \text{ as } t \to +\infty \text{ locally uniformly in } \mathbb{R}^N,$$

then properties (1.15)-(1.16) still hold for some continuous function $\xi : [0,+\infty) \to \mathbb{R}$, where c is characterized by the existence of a solution $\varphi \in C^2(\mathbb{R})$ of (1.17). Notice that property (1.15) implies in particular that, for each $0 < \varepsilon \le M/2$ and each unit vector e, the diameter of the set $\{r \ge 0 : \varepsilon \le u(t, re) \le M - \varepsilon\}$ is bounded as $t \to +\infty$ (see also the second paragraph after Remark 1.3). Furthermore, the same conclusion holds if, instead of $\lim_{|x|\to+\infty} u_0(x) = 0$, one assumes that $\limsup_{|x|\to+\infty} u_0(x) \le \eta$, with $\eta > 0$ such that $f < 0$ in $(0, \eta]$ (in that case, one has $\limsup_{|x|\to+\infty} u(t, x) \to 0$ as $t \to +\infty$).

1.3. **Comments about the assumptions** (1.2) **and** (1.13) **on** f**, and** (1.3) **on** u**.** Let us make in the following paragraphs some comments about the role and necessity of the various assumptions on f and u used in Theorem 1.1.

Let us first discuss the linear stability assumption (1.2) on f. Firstly, as already emphasized, the equality $f(0) = 0$ is necessary for (1.3) to hold. Secondly, if $f'(0) = 0$, then Theorem 1.1 does not hold in general. For instance, in dimensions $N \ge 3$ and for $(N+2)/(N-2) < p < p_L$ with $p_L = (N-4)/(N-10)$ if $N \ge 11$ and $p_L = \infty$ if $N \le 10$, the Fujita equation (1.18) admits positive bounded entire solutions u which are uniformly localized and are homoclinic to 0, in the sense that $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to \pm\infty$, see [21]. Thirdly, if $f'(0) > 0$, then Theorem 1.1 does not make sense in general. Consider for instance a C^2 concave function $f : [0,+\infty) \to \mathbb{R}$ such that $f(0) = 0$, $f'(0) > 0$ and $f(b) = 0$ for some $b > 0$ (then, $f(s) < 0$ for all $s > b$). Any nonnegative bounded entire solution u of (1.1) and (1.3) necessarily satisfies $0 \le u < b$ in $\mathbb{R} \times \mathbb{R}^N$ from the maximum principle. Furthermore, $\max_{\mathbb{R}^N} u(t,\cdot) < b$ for all t negative enough (and then for all $t \in \mathbb{R}$), and it then follows from [33] that $u(t,x)$ is a function of t alone and (1.3) then yields $u \equiv 0$ in $\mathbb{R} \times \mathbb{R}^N$.

Let us now focus on condition (1.13) on f (assuming also the other condition (1.2)). First of all, (1.13) is automatically satisfied in dimension $N = 1$ (see Section 2.1). From (1.6)-(1.7) and (1.9), condition (1.13) is also fulfilled in any dimension $N \ge 1$ for bistable-type functions $f \in C^1([0,+\infty))$ for which there exist $0 < a < b$ such that

$$\begin{cases} f(0) = f(a) = f(b) = 0, \quad f'(0) < 0, \quad f'(b) < 0, \quad f'(a) > 0, \\ f < 0 \text{ in } (0,a), \quad f > 0 \text{ in } (a,b), \end{cases} \tag{1.19}$$

and

$$\int_0^b f(s)ds > 0, \tag{1.20}$$

together with $f < 0$ in $(b,+\infty)$.[1] In that case, one necessarily has $m_\phi = a < \max_{\mathbb{R}^N} \phi < b$ for any $\phi \in \mathcal{E}$, and $F < 0$ in $(0,a]$, with F defined as in (1.10). In any dimension $N \ge 1$, for such a function f and for any $0 < a' < a < b' < b$, there is then $\varepsilon > 0$ such that, if a $C^1([0,+\infty))$ function g satisfies $g = f$ in $[0,a'] \cup [b',+\infty)$ and $\|f - g\|_{L^\infty([a',b'])} \le \varepsilon$, then g fulfills (1.13) in the sense that $G(s) := \int_0^s g(\sigma)\,d\sigma < 0$ for all $s \in (0, m_\phi]$ and for every solution ϕ of (1.4) with g instead of f (indeed, after fixing c in (a, b') such that $F < 0$ in $(0,c]$, one then has $G < 0$ in $(0,c]$ and $g > 0$ in $[c,b)$ for $\varepsilon > 0$ small enough by continuity,

[1] The equation (1.1) with f satisfying (1.19) was originally proposed in [2, 49] and is accordingly often called the Allen-Cahn equation or the Nagumo equation. It arises in a wide variety of contexts such as phase transition, combustion, ecology and many models of biology.

hence $0 < m_\phi < c < \max_{\mathbb{R}^N}\phi < b$ by (1.6)-(1.7) and (1.9) applied to any solution ϕ of (1.4) with such a g, and finally $G < 0$ in $(0, m_\phi])$. Notice that such a function g, unlike f, may have more than one zero between 0 and b.

In the case where $N \geq 2$, (1.13) is not always fulfilled for all functions f satisfying (1.2). For instance, consider $0 < a < b < c$ and a $C^1([0,+\infty))$ function f satisfying (1.2) together with $f < 0$ in $(0,a)\cup(b,c)$, $f > 0$ in $(a,b)\cup(c,+\infty)$, $F(b) = 0$ (hence, $F < 0$ in $(0,b)\cup(b,c]$), and $F(\xi) > 0$ for some $\xi > c$ (and then for all large ξ). Under some additional growth conditions on $f(s)$ as $s \to +\infty$, according to the dimension $N \geq 2$ (see the references in the last paragraph of Section 2.2), the set $\mathcal{E}$ is not empty. It follows from (1.7) and (1.9) that, for such functions f, $c = m_\phi < \max_{\mathbb{R}^N}\phi$ for all $\phi \in \mathcal{E}$, whereas $F(b) = 0$ with $0 < b < c$, hence (1.13) is not satisfied.

Remark 1.3. Conditions (1.2) and (1.13) imply that the map $\phi \mapsto m_\phi$ is constant in $\mathcal{E}$. Indeed, for any $\phi, \widetilde{\phi} \in \mathcal{E}$, one has $F < 0$ in $(0, m_{\widetilde{\phi}}]$ by (1.13), while $F(\max_{\mathbb{R}^N}\phi) \geq 0$ and $f(\max_{\mathbb{R}^N}\phi) > 0$. Hence, owing to the definition (1.7) of m_ϕ and the fact that $f(m_{\widetilde{\phi}}) = 0$, one infers that $m_{\widetilde{\phi}} \leq m_\phi$ (otherwise, $m_\phi < \max_{\mathbb{R}^N}\phi < m_{\widetilde{\phi}}$ and thus $F(\max_{\mathbb{R}^N}\phi) < 0$, which is a contradiction). Finally $m_\phi = m_{\widetilde{\phi}}$ since ϕ and $\widetilde{\phi}$ are arbitrary in $\mathcal{E}$. Throughout the proof of Theorem 1.1, one will then use the notation

$$m = m_\phi > 0 \tag{1.21}$$

for all $\phi \in \mathcal{E}$.

From the proof of Theorem 1.1, it also turns out that, in case (iii) of Theorem 1.1, one necessarily has $M_\phi < +\infty$ for any (and all) $\phi \in \mathcal{E}$ and that the value M appearing in case (iii) is nothing but the constant quantity M_ϕ:

$$M = M_\phi \tag{1.22}$$

for all $\phi \in \mathcal{E}$. In particular, if the solution u spreads, it can not converge to an intermediate state smaller than M and the limit state M does not depend on the solution u itself. As a matter of fact, since $F < 0$ in $(0, m]$, $m < \max_{\mathbb{R}^N}\phi < M$, $F(\max_{\mathbb{R}^N}\phi) \geq 0$, $f > 0$ in (m, M) and $f(M) = 0$ (for all $\phi \in \mathcal{E}$), one then infers that M is the smallest zero of f for which $F(M) > 0$, that is,

$$M = \min\big\{s \geq 0 : f(s) = 0 \text{ and } F(s) > 0\big\}. \tag{1.23}$$

In short, the function $\phi \mapsto M_\phi$ is also constant in $\mathcal{E}$, and this actually holds whether M_ϕ be finite or not. Notice that (1.23) also implies that case (iii) is ruled out if $f > 0$ in $(m, +\infty)$ for any (and all) $\phi \in \mathcal{E}$, hence only cases (i) or (ii) may occur in this case.

Together with (1.2), assumption (1.13) plays a key-role in the dichotomy results between the uniformly localized solutions and the solutions converging as $t \to +\infty$ to a positive constant with a positive spreading rate. Without the assumption (1.13), the solutions u of (1.1) may well converge locally uniformly in $\mathbb{R}^N$ as $t \to +\infty$ to a steady state ϕ such that $\lim_{|x|\to+\infty}\phi(x) > 0$ (such behaviors are known for the solutions of the associated Cauchy problem with localized initial conditions in dimension $N = 1$ [42, 43] or with compactly supported initial conditions in dimensions $N \geq 1$ [16]).

The assumption (1.13) is also essential in the proof of formula (1.15) saying that, for spreading solutions, the transition between $M-\varepsilon$ and ε (for any $0<\varepsilon\le M/2$) has bounded width in any radial direction as $t\to+\infty$. This property refers to the notion of transition fronts (here, as $t\to+\infty$) introduced in [5]. In other words, the assumption (1.13) prevents the existence of terraces made of stacked propagating fronts between the top value M and the zero state. The existence and attractivity of radial terraces with radial positions $(0<)\ \xi_1(t)<\cdots<\xi_m(t)$ has been proved in [15] under an additional non-degeneracy assumption of all zeroes of f in $[0,M]$ (see also [17, 20, 28, 43] for further results on terraces for homogeneous or spatially periodic equations in dimension $N=1$, [62, 63] for the existence of one-dimensional and radially symmetric terraces for gradient multistable systems, [58, 59] for the existence of planar terraces for solutions with front-like initial data in $\mathbb{R}^N$, and [29] for the existence of terraces in spatially periodic equations in $\mathbb{R}^N$). The limit values of the ratios of $\xi_i(t)/t$ are also explicit, see [15]. Here, especially thanks to (1.13), only a single radial layer can exist, and the asymptotic position $\xi(t)$ of that layer at large times is given in terms of the unique speed of a traveling front φ connecting 0 and M for problem (1.17). We point out that M is asymptotically stable from below since f is positive in a left neighborhood of M, but M may not be linearly stable, in the sense that $f'(M)$ may vanish. Actually, formula (1.16) is proved even if $f'(M)=0$ (notice in particular that f may not be monotone in a left neighborhood of M)[2] and, as such, up to our knowledge, the spreading properties (1.15)-(1.16) are new even in dimension $N=1$. The exact position of the layer $\xi(t)$ and a quantitative estimate on the attractivity of the radial front with speed c are not clear without the assumption $f'(M)<0$ (see Remark 1.4 below for the case $f'(M)<0$). However, if x_0 denotes the point with respect to which the considered solution u is radially symmetric and decreasing, and if a is any fixed real number in $(0,M)$, then one knows from (1.16) that, for all t large enough, there is a unique $\xi_a(t)\in\mathbb{R}$ such that $u(t,x_0+\xi_a(t)e)=a$ for all unit vectors e, and

$$\limsup_{t\to+\infty}|\xi_a(t)-\xi(t)|<+\infty.$$

It is reasonable to conjecture that $u(t,x_0+\xi_a(t)e+x)\to\varphi(x\cdot e+\varphi^{-1}(a))$ as $t\to+\infty$ locally uniformly in $x\in\mathbb{R}^N$ for any unit vector e, albeit the proof of this property would require different arguments from the ones used here.

Remark 1.4. In alternative (iii) of Theorem 1.1, if M is further assumed to be nondegenerate, meaning here that $f'(M)<0$, then there are $x_0\in\mathbb{R}^N$ and $\tau\in\mathbb{R}$ depending on u, such that

$$\sup_{x\in\mathbb{R}^N}\Big|u(t,x)-\varphi\Big(|x-x_0|-ct+\frac{N-1}{c}\ln t+\tau\Big)\Big|\to 0\ \text{ as } t\to+\infty. \tag{1.24}$$

In particular, property (1.15) then holds with $\xi(t)=ct-((N-1)/c)\ln t$ (say for $t\ge 1$) if $f'(M)<0$. Property (1.24) makes the position $\xi(t)$ and the limit profile of u in all directions exactly known up to an $o(1)$ term as $t\to+\infty$. When 0 and M are nondegenerate

[2]On the other hand, if f were assumed to be monotone, namely nonincreasing, in a left neighborhood of M, then it would follow from [3, 64] that $\max_{|x|\le c't}|u(t,x)-M|\to 0$ and $\max_{|x|\ge c''t}u(t,x)\to 0$ as $t\to+\infty$ for every $0\le c'<c<c''$. This in particular yields (1.16), namely $\lim_{t\to+\infty}\xi(t)/t=c$, but this does not show property (1.15) on the boundedness of the radial width of the transition between 0 and M at large times.

and f has a single zero in the interval $(0,M)$, formula (1.24) for the solutions converging locally to M follows from [68, Corollary 2] (see also [65] for more precise estimates on the position of the front at large times, and [35] for earlier but less precise estimates). It is easily seen from [68] that the proof extends to the case when 0 and M are still nondegenerate and f has more than one zero in $(0,M)$, since the unique profile φ given in (1.17), with unique speed $c>0$, still satisfies $\varphi'<0$ in $\mathbb{R}$ and converges exponentially to 0 and M at $\pm\infty$.

Finally, let us comment the assumption (1.3) on u. It is essential in the derivation of (1.14) saying that the α-limit set of the considered solutions is included in $\mathcal{E}$, leading in particular to the quasiconvergence as $t\to-\infty$ in $L^\infty(\mathbb{R}^N)$. If, instead of (1.3), one only assumes that $u(t,x)\to 0$ as $|x|\to+\infty$ for each $t\le 0$ (and then, equivalently, for each $t\in\mathbb{R}$), but without any uniformity with respect to $t\le 0$, then the conclusion does not hold in general. For instance, consider any $b>0$ and a $C^1([0,b])$ bistable-type function f satisfying (1.19). If f satisfies (1.20) too, then there are entire solutions $u:\mathbb{R}\times\mathbb{R}\to(0,b)$ of (1.1) in dimension $N=1$ that satisfy $\lim_{x\to\pm\infty}u(t,x)=0$ for each $t\in\mathbb{R}$ and behave as two further and further pulses as $t\to-\infty$, see [41]. Namely, there is a solution $\phi:\mathbb{R}\to(0,a']$ of (1.4), where $a'\in(a,b)$ is such that $F(a')=0$ and ϕ is the unique, up to shifts, solution of (1.4) ranging in $[0,b]$. The solutions u constructed in [41] are such that

$$u(t,x)-\phi(-\xi(t))-\phi(\xi(t))\to 0\ \text{ as } t\to-\infty\ \text{ uniformly in } x\in\mathbb{R},$$

with $\lim_{t\to-\infty}\xi(t)=+\infty$. Therefore, these solutions u do not satisfy (1.14). However, these solutions are quasiconvergent, and even convergent to 0, as $t\to-\infty$ for the $L^\infty_{loc}(\mathbb{R})$ topology and, since $\phi(\pm\infty)=0$ and all the shifts of ϕ belong to $\mathcal{E}$, these solutions satisfy a property similar to (1.14) by replacing the $L^\infty(\mathbb{R})$ topology by the $L^\infty_{loc}(\mathbb{R})$ topology. On the other hand, if f satisfies (1.19) and

$$\int_0^b f(s)ds<0, \tag{1.25}$$

then there are entire solutions $u:\mathbb{R}\times\mathbb{R}\to(0,b)$ of (1.1) that satisfy $\lim_{x\to\pm\infty}u(t,x)=0$ for each $t\in\mathbb{R}$ and behave as two far fronts as $t\to-\infty$, see [30]. Namely, under (1.19) and (1.25), equation (1.1) admits a traveling front $\varphi(x-ct)$ with $c<0$ and such that $\varphi:\mathbb{R}\to(0,b)$ is decreasing with $\varphi(-\infty)=b$ and $\varphi(+\infty)=0$, that is, φ obeys (1.17) with $M=b$. The solutions u constructed in [30] satisfy

$$u(t,x)-\varphi(x-ct)-\varphi(-x-ct)+b\to 0\ \text{ as } t\to-\infty \text{ uniformly in } x\in\mathbb{R},$$

and also $\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}\to 0$ as $t\to+\infty$. Since in this case there is $\widetilde{b}\in(b,+\infty)$ such that $F<0$ in $(0,\widetilde{b}]$, equation (1.4) does not admit any solution $\phi:\mathbb{R}\to[0,\widetilde{b}]$, and the solutions u then do not satisfy (1.14). If one also assumes that $f<0$ in $(b,+\infty)$, one has $F<0$ in $(0,+\infty)$ and $\mathcal{E}=\emptyset$. Thus, even if the constructed solutions u are quasiconvergent, and even convergent to b, as $t\to-\infty$ for the $L^\infty_{loc}(\mathbb{R})$ topology, they do not satisfy (1.14) by replacing the $L^\infty(\mathbb{R})$ topology by the $L^\infty_{loc}(\mathbb{R})$ topology either.

Remark 1.5. In Theorem 1.1, the assumption (1.3) can nevertheless be relaxed, still keeping the uniformity with respect to $t\le 0$. More precisely, let any $\eta>0$ be such that $f<0$ in $(0,\eta]$.

Notice that such a real number η exists by (1.2). It then turns out that Theorem 1.1 holds if (1.3) is replaced by the assumption

$$\limsup_{|x|\to+\infty}\Big(\sup_{t\le 0}u(t,x)\Big)\le\eta, \tag{1.26}$$

or $\limsup_{|x|\to+\infty}\big(\sup_{t\le t_0}u(t,x)\big)\le\eta$ for some $t_0\in\mathbb{R}$. As a matter of fact, (1.26) implies (and is then equivalent to) (1.3). Indeed, assume by way of contradiction that (1.26) holds, but not (1.3). Then there is a sequence $(t_n,x_n)_{n\in\mathbb{N}}$ in $(-\infty,0]\times\mathbb{R}^N$ such that

$$\lim_{n\to+\infty}|x_n|=+\infty\quad\text{and}\quad 0<\liminf_{n\to+\infty}u(t_n,x_n)\le\limsup_{n\to+\infty}u(t_n,x_n)\le\eta.$$

Furthermore, it follows from standard parabolic estimates that, up to extraction of a subsequence, the functions $u_n:(t,x)\mapsto u_n(t,x)=u(t+t_n,x+x_n)$ converge in $C^{1,2}_{t,x}$ locally in $\mathbb{R}\times\mathbb{R}^N$ to a nonnegative bounded solution u_∞ of (1.1) such that $u_\infty(0,0)>0$ and $0\le u_\infty\le\eta$ in $(-\infty,0]\times\mathbb{R}^N$. Therefore, the maximum principle yields $u_\infty(t,x)\le\zeta(t+t_0)$ for all $t_0>0$ and $(t,x)\in[-t_0,+\infty)\times\mathbb{R}^N$, where ζ obeys $\zeta(0)=\eta$ and $\zeta'(t)=f(\zeta(t))$ for all $t\ge0$. In particular, $0<u_\infty(0,0)\le\zeta(t_0)$ for all $t_0>0$. But $\zeta(+\infty)=0$ since $f<0$ in $(0,\eta]$ and $f(0)=0$, leading to a contradiction. Therefore, (1.3) could equivalently be replaced by (1.26) in Theorem 1.1, but we preferred to state Theorem 1.1 with (1.3) since this assumption is simpler to write and does not involve the additional introduction of a quantity η.

1.4. **Existence of monotone heteroclinic connections.** Let us now discuss the existence of heteroclinic connections between a steady state ϕ of (1.4) and the constant states 0 or M. The following results are quite standard and inspired by similar ones in [24, 40, 41, 55, 56], so we just sketch the proof here for the sake of completeness.

Proposition 1.6. *Assume that $f'(0)<0$ and let ϕ be a solution of (1.4) with $M_\phi<+\infty$. Then, there are positive bounded solutions u_1 and u_2 of (1.1) such that*

$$\lim_{t\to-\infty}\|u_1(t,\cdot)-\phi\|_{L^\infty(\mathbb{R}^N)}=0,\qquad\lim_{t\to+\infty}\|u_1(t,\cdot)\|_{L^\infty(\mathbb{R}^N)}=0,$$
$$\lim_{t\to-\infty}\|u_2(t,\cdot)-\phi\|_{L^\infty(\mathbb{R}^N)}=0,\qquad\lim_{t\to+\infty}u_2(t,x)=M_\phi\ \textit{locally uniformly in } x\in\mathbb{R}^N.$$

Proof. Since ϕ decays exponentially to 0 at infinity (see also Section 4.1), so do its first- and second-order partial derivatives hence $\phi\in H^1(\mathbb{R}^N)$ and $\phi_{x_i}\in H^1(\mathbb{R}^N)$ for $1\le i\le N$. Since each first-order partial derivative ϕ_{x_i} changes sign, ϕ is then a strictly unstable solution of (1.4), in the sense that, for all $R>0$ large enough, the principal eigenvalue λ_R of the operator $-\Delta-f'(\phi)$ in the open Euclidean ball B_R with center 0 and radius R, with Dirichlet boundary condition, is negative. Let $\varphi_R\in C^2(\overline{B_R})$ be a principal eigenfunction associated to this operator, namely

$$-\Delta\varphi_R-f'(\phi)\varphi_R=\lambda_R\varphi_R\text{ in }B_R,\quad\varphi_R>0\text{ in }B_R,\quad\text{and}\quad\varphi_R=0\text{ on }\partial B_R. \tag{1.27}$$

Fix any $R>0$ large enough such that $\lambda_R<0$. There exists then $\varepsilon^*>0$ such that, for all $\varepsilon\in(0,\varepsilon^*)$, the $C^2(\overline{B_R})$ function $\phi_{R,\varepsilon}:=\phi-\varepsilon\varphi_R$ satisfies $0<\phi_{R,\varepsilon}\le\phi$ in $\overline{B_R}$, $\phi_{R,\varepsilon}=\phi$ on ∂B_R and $\Delta\phi_{R,\varepsilon}+f(\phi_{R,\varepsilon})<0$ in B_R. Denote $\widetilde{\phi}_{R,\varepsilon}(x)=\phi_{R,\varepsilon}(x)$ if $x\in\overline{B_R}$ and

$\widetilde{\phi}_{R,\varepsilon}(x) = \phi(x)$ if $x \in \mathbb{R}^N \backslash \overline{B_R}$. Thus, for any $\varepsilon \in (0, \varepsilon^*)$, the function $\widetilde{\phi}_{R,\varepsilon}$ is a generalized strict supersolution of (1.4), and the solution u^ε of the Cauchy problem

$$\begin{cases} u^\varepsilon_t = \Delta u^\varepsilon + f(u^\varepsilon) & \text{in } (0,+\infty) \times \mathbb{R}^N, \\ u^\varepsilon(0,\cdot) = \widetilde{\phi}_{R,\varepsilon} & \text{in } \mathbb{R}^N, \end{cases} \tag{1.28}$$

is strictly decreasing in t and satisfies $0 < u^\varepsilon < \widetilde{\phi}_{R,\varepsilon} \le \phi$ in $(0,+\infty) \times \mathbb{R}^N$. Hence, there exists a $C^2(\mathbb{R}^N)$ solution $\widetilde{\phi}$ of $\Delta\widetilde{\phi} + f(\widetilde{\phi}) = 0$ in $\mathbb{R}^N$ such that $0 \le \widetilde{\phi} < \phi$ in $\mathbb{R}^N$. Since by [9] any two solutions of (1.4) can not be ordered, it follows from the strong elliptic maximum principle that $\widetilde{\phi} \equiv 0$ in $\mathbb{R}^N$. Without loss of generality, one can assume that $\varepsilon^*\varphi_R(0) < \phi(0)/2$. Hence, for any $\varepsilon \in (0, \varepsilon^*)$, one has $u^\varepsilon(0,0) > \phi(0)/2$, and there is a unique time $t^\varepsilon > 0$ such that $u^\varepsilon(t^\varepsilon, 0) = \phi(0)/2$. Furthermore, since ϕ is a steady state, there holds $t^\varepsilon \to +\infty$ as $\varepsilon \to 0$. Consider now a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0, \varepsilon^*)$ and converging to 0. The functions

$$v_n(t,x) = u^{\varepsilon_n}(t + t^{\varepsilon_n}, x), \quad \text{for } (t,x) \in [-t^{\varepsilon_n}, +\infty) \times \mathbb{R}^N,$$

converge up to extraction of a subsequence in $C^{1,2}_{t,x}$ locally in $\mathbb{R} \times \mathbb{R}^N$ to a solution u_1 of (1.1), such that $0 \le u_1 \le \phi$ and $(u_1)_t \le 0$ in $\mathbb{R} \times \mathbb{R}^N$, together with $u_1(0,0) = \phi(0)/2$. Hence, $0 < u_1 < \phi$ in $\mathbb{R} \times \mathbb{R}^N$ from the strong maximum principle, and there are two steady states $\phi_\pm \in C^2(\mathbb{R}^N)$ such that $u_1(t,\cdot) \to \phi_\pm$ as $t \to \pm\infty$ locally uniformly in $\mathbb{R}^N$ (and then uniformly since $0 < u_1 < \phi$ in $\mathbb{R} \times \mathbb{R}^N$ and $\phi(x) \to 0$ as $|x| \to +\infty$), with $0 \le \phi_+ \le \phi_- \le \phi$ and $\phi_+(0) \le \phi(0)/2 \le \phi_-(0)$. The strong maximum principle and the non-existence of ordered solutions of (1.4) imply that $\phi_- \equiv \phi$ and $\phi_+ \equiv 0$ in $\mathbb{R}^N$. Lastly, $(u_1)_t < 0$ from the strong parabolic maximum principle applied to this function. In other words, the solution u_1 of (1.1) is a time-decreasing heteroclinic connection from ϕ to 0.

Similarly, the function $\phi + \varepsilon\varphi_R \in C^2(\overline{B_R})$ satisfies $\Delta(\phi + \varepsilon\varphi_R) + f(\phi + \varepsilon\varphi_R) > 0$ in B_R for all $\varepsilon > 0$ small enough, with $R > 0$ fixed large enough as above. Recall M_ϕ is defined in (1.11). Notice that the arguments of [9] based on the maximum principle and the sliding method imply that, for any classical positive solution $\widetilde{\phi}$ of $\Delta\widetilde{\phi} + f(\widetilde{\phi}) = 0$ in $\mathbb{R}^N$ such that $\widetilde{\phi} \ge \phi$, one has either $\widetilde{\phi} \equiv \phi$ in $\mathbb{R}^N$, or $\widetilde{\phi} \ge \max_{\mathbb{R}^N} \phi$ and then $\widetilde{\phi} \ge M_\phi$ in $\mathbb{R}^N$. Therefore, it follows that, for all $\varepsilon > 0$ small enough, the solutions u^ε of (1.28), with this time

$$\widetilde{\phi}_{R,\varepsilon}(x) = \phi(x) + \varepsilon\varphi_R(x) \text{ if } x \in \overline{B_R}, \quad \widetilde{\phi}_{R,\varepsilon}(x) = \phi(x) \text{ if } x \in \mathbb{R}^N \backslash \overline{B_R}, \tag{1.29}$$

are increasing in time in $(0,+\infty) \times \mathbb{R}^N$, and satisfy $u^\varepsilon(t,\cdot) \to M_\phi$ as $t \to +\infty$ locally uniformly in $\mathbb{R}^N$. Furthermore, since ϕ is radially symmetric (with respect to, say, the origin) and decreasing in $|x|$, with $\Delta\phi(0) = -f(\phi(0)) = -f(\max_{\mathbb{R}^N} \phi) < 0$, and since the principal eigenfunction φ_R of (1.27) is itself radially symmetric by uniqueness, the functions $\widetilde{\phi}_{R,\varepsilon}$ given in (1.29) are then also radially symmetric and decreasing in $|x|$, for all $\varepsilon > 0$ small enough. So are the functions $u^\varepsilon(t,\cdot)$ for all $t > 0$. With the same arguments as in the previous paragraph, by defining a time $t^\varepsilon > 0$ such that $u^\varepsilon(t^\varepsilon, 0) = (\phi(0) + M_\phi)/2$, one infers the existence of a time-increasing heteroclinic connection u_2 between ϕ and M_ϕ (the convergence to M_ϕ being this time only locally uniform in $\mathbb{R}^N$ as $t \to +\infty$). □

For bistable functions f of the type (1.19)-(1.20), we also refer to [34, 46] for the existence of other time-increasing heteroclinic connections $u(t, x_1, x_2) = U(x_1, x_2 - \gamma t)$ of (1.1), for any $\gamma > 0$ large enough, between the extended one-dimensional solution $\phi(x_1, x_2) = \phi(x_1)$ of (1.4) and the constant $b = M_\phi$, in the sense that $u(t, x_1, x_2) \to \phi(x_1)$ as $t \to -\infty$ uniformly in x_1 and locally uniformly in x_2, and $u(t, x_1, x_2) \to b$ as $t \to +\infty$ locally uniformly in (x_1, x_2). These connections are however not localized, that is, they do not satisfy (1.3): as a matter of fact, one has $\limsup_{|(x_1,x_2)|\to+\infty} u(t, x_1, x_2) = b > 0$ for each $t \in \mathbb{R}$.

Assume now that $M_\phi = +\infty$. In that case, the solutions u^ε of the previous paragraphs can not stay bounded and therefore blow up in finite or infinite time, according to the behavior of $f(s)$ as $s \to +\infty$. As above, without loss of generality, for all $\varepsilon > 0$ small enough, the functions $u^\varepsilon(t, \cdot)$ are radially symmetric and decreasing in $|x|$ for all t in their interval $(0, T^\varepsilon)$ of existence, hence $u^\varepsilon(t, 0) \to +\infty$ as $t \to T^\varepsilon$. For all $\varepsilon > 0$ small enough, there is then a time $t^\varepsilon \in (0, T^\varepsilon)$ such that

$$u^\varepsilon(t^\varepsilon, 0) = \phi(0) + 1,$$

and $t^\varepsilon \to +\infty$ as $\varepsilon \to 0$. Therefore, using again the strong parabolic maximum principle, there is a time-increasing solution u_∞ of (1.1), defined now in $(-\infty, T) \times \mathbb{R}^N$ with $T \in (0, +\infty]$, such that $u_\infty(0, 0) = \phi(0) + 1$, u_∞ is radially symmetric and decreasing with respect to $|x|$, $u_\infty(t, \cdot) \to \phi$ uniformly in $\mathbb{R}^N$ as $t \to -\infty$, and $u_\infty(t, 0) \to +\infty$ as $t \to T$. In other words, u_∞ blows up at time T (which may be finite or infinite, according to the function f).

Lastly, we point out that, since any two solutions of (1.4) can not be ordered [9], it follows that (1.1) can not have any time-monotone heteroclinic connection between two different solutions $\phi_\pm$ of (1.4). However, the existence of non-time-monotone heteroclinic connections is not a priori ruled out.

Remark 1.7. For any positive bounded solution u of (1.1) satisfying (1.3), the action

$$E[u(t, \cdot)] = \int_{\mathbb{R}^N} \Big(\frac{|\nabla u(t, x)|^2}{2} - F(u(t, x)) \Big)\, dx$$

is well defined and it is a Lyapunov functional, that is, $t \mapsto E[u(t, \cdot)]$ is non-increasing in $\mathbb{R}$ and even decreasing unless u does not depend on t. We refer to Section 4.1 for more details. Notice that $E[\phi] > 0 = E[0]$ for any solution ϕ of (1.4) (this can be viewed as a consequence of the aforementioned existence of heteroclinic connections between ϕ and 0). If M_ϕ is a real number and u is a heteroclinic connection between ϕ and M_ϕ, or more generally speaking in case (iii) of Theorem 1.1, then $E[u(t, \cdot)] \to -\infty$ as $t \to +\infty$. If u is a heteroclinic connection between two different solutions $\phi_\pm$ of (1.4) in the sense that $\|u(t, \cdot) - \phi_\pm\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to \pm\infty$ (this is a particular case of alternative (ii) of Theorem 1.1), then

$$E[\phi_-] > E[\phi_+].$$

Lastly, if a positive bounded solution u of (1.1) satisfying (1.3) does not converge to a single solution of (1.4) as $t \to -\infty$, in the sense that there are at least two different solutions ϕ and $\widetilde{\phi}$ of (1.4) such that

$$\|u(t_n, \cdot) - \phi\|_{L^\infty(\mathbb{R}^N)} \to 0 \ \text{ and } \ \|u(\widetilde{t}_n, \cdot) - \widetilde{\phi}\|_{L^\infty(\mathbb{R}^N)} \to 0 \ \text{ as } \ n \to +\infty$$

with $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} \widetilde{t}_n = -\infty$, then $E[\phi] = E[\widetilde{\phi}]$. Notice that in that situation, ϕ and $\widetilde{\phi}$ are necessarily radially symmetric with respect to the same origin, since so is u, and by connectedness of the α-limit set of u there is then a continuum of such limit steady states as $t \to -\infty$ in the α-limit set of u, all having the same Lagrangian (we also refer to the discussion before Corollary 2.3 below). As a consequence, if the Lagrangian E is one-to-one of the set the solutions of (1.4) which are symmetric with respect to the same point, then there is a single $\phi \in \mathcal{E}$ such that $\|u(t,\cdot) - \phi\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to -\infty$.

2. Some corollaries and particular cases

In this section, we list some corollaries of Theorem 1.1 which correspond to further assumptions or to some special cases. In particular, the conclusion (1.14) will be made more precise under further assumptions.

2.1. **Dimension $N = 1$.** The first corollary is concerned with the dimension $N = 1$. In this case, the solutions of (1.4) are unique, up to shifts. Indeed, for any such solution ϕ, it follows from (1.5) and (1.8) that $F < 0$ in $(0, \max_{\mathbb{R}} \phi) = (0, \Phi(0))$ and $\Phi'(r) = -\sqrt{-2F(\Phi(r))}$ for all $r \geq 0$, hence the radial profile Φ is unique from the Cauchy-Lipshitz theorem. Furthermore, $F(\Phi(0)) = 0$ and $f(\Phi(0)) > 0$. One can also infer from (1.8) that, if there is $\beta \in (0,+\infty)$ such that $F < 0$ in $(0, \beta)$ with $F(\beta) = 0$ and $f(\beta) > 0$ (notice that the hypotheses $F < 0$ in $(0, \beta)$ and $F(\beta) = 0$ imply that $f(\beta) \geq 0$), then (1.4) has a (unique up to shifts) solution.

Corollary 2.1. *Assume that $N = 1$, that f satisfies* (1.2) *and that there is $\beta \in (0, +\infty)$ such that $F < 0$ in $(0, \beta)$ with $F(\beta) = 0$ and $f(\beta) > 0$. Then $\mathcal{E} \neq \emptyset$ and, for any positive bounded solution u of* (1.1) *satisfying* (1.3)*, there is $\phi \in \mathcal{E}$ such that $\|u(t,\cdot) - \phi\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to -\infty$. Furthermore, either $\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to +\infty$, or $u(t,x) \equiv \phi(x)$ in $\mathbb{R} \times \mathbb{R}$, or else the alternative* (iii) *of Theorem* 1.1 *holds.*

Corollary 2.1 easily follows from Theorem 1.1, the previous observations and the fact that the solutions u are necessarily even in x with respect to some real number. The fact that case (ii) reduces to $u \equiv \phi$ is a consequence of the existence of a Lyapunov functional and the uniqueness of the solutions of (1.4) up to shifts. We refer to Section 5 for more details.

2.2. **Non-existence of positive bounded solutions.** In dimension $N = 1$, with (1.2), the existence of a smallest positive root β of F with $f(\beta) > 0$ is a necessary and sufficient condition for the existence of a solution of (1.4). In dimension $N \geq 2$, any solution ϕ of (1.4) satisfies $F(\max_{\mathbb{R}^N} \phi) > 0$. Therefore, the next result immediately follows from Theorem 1.1 and the strong maximum principle.

Corollary 2.2. *Assume that f satisfies* (1.2)*. In dimension $N = 1$, if $F < 0$ in $(0, +\infty)$ or if $F < 0$ in $(0, \beta)$ with $F(\beta) = 0$ and $f(\beta) = 0$, then the only nonnegative bounded solution u of* (1.1) *satisfying* (1.3) *is the trivial solution $u \equiv 0$ in $\mathbb{R} \times \mathbb{R}$. In dimension $N \geq 2$, if $F \leq 0$ in $[0, +\infty)$, then the same conclusion holds.*

The assumptions of Corollary 2.2 are simple conditions ruling out the existence of solutions to (1.4). These assumptions are however not optimal in dimensions $N \geq 2$. For instance, if $N \geq 3$, for any positive real numbers γ and δ and for any $p \geq (N+2)/(N-2)$, the equation (1.4) with

$$f(s) = -\gamma s + \delta s^p \tag{2.1}$$

does not admit any solution [36, 52]. The same property holds with

$$f(s) = -\gamma s - \delta s^p + \eta s^q \tag{2.2}$$

with $N \geq 3$, $\gamma > 0$, $\delta > 0$, $\eta > 0$, and $1 < p \leq (N+2)/(N-2) \leq q$ or $(N+2)/(N-2) < p < q$, see [36, 50, 52]. In these two examples, Theorem 1.1 implies that the only nonnegative bounded solution u of (1.1) satisfying (1.3) is then the trivial solution $u \equiv 0$ in $\mathbb{R} \times \mathbb{R}^N$.

On the other hand, much work has been devoted to the existence of solutions to (1.4) for some classes of functions f satisfying (1.2), see the book [36]. For instance, if $N \geq 3$ and $\sup_{[0,+\infty)} F > 0$ together with $\max(f(s), 0) = o(s^{(N+2)/(N-2)})$ as $s \to +\infty$, then (1.4) admits solutions, see [6, 7, 67]. The existence holds in dimension $N = 2$ if for instance f satisfies (1.19)-(1.20), see [8], or if $\sup_{[0,+\infty)} F > 0$ and $\max(f(s), 0) = o(e^{\alpha s^2})$ as $s \to +\infty$ for all $\alpha > 0$, see [4].

2.3. **Discreteness or uniqueness up to shifts of the localized steady states.** In Theorem 1.1, property (1.14) says that the solution u is close to the family of steady states of (1.4) as $t \to -\infty$, that is, the α-limit set of u (with respect to the uniform convergence in $\mathbb{R}^N$) consists of solutions of (1.4), which turn out to be all symmetric with respect to a same point in $\mathbb{R}^N$, since so is u. Any two different solutions of (1.4) can not be ordered by [9], but it is not clear in general to know whether u emanates from a single steady state or from a continuum of them (the possible existence of continua of solutions of (1.4) which are symmetric with respect to the same point is a difficult issue in general dimensions $N \geq 2$). However, under some further assumptions on the set of steady states, combined with the connectedness of the α-limit sets of the solutions, one can be sure that a single state is selected.

Corollary 2.3. *Assume that f satisfies* (1.2) *and* (1.13)*, and that the set of solutions of* (1.4) *which are radially symmetric with respect to the origin is discrete.*[3] *Let u be a positive bounded solution of* (1.1) *satisfying* (1.3). *Then there is $\phi \in \mathcal{E}$ such that $\|u(t,\cdot) - \phi\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to -\infty$. Furthermore,*

(i) *either $u(t,\cdot) \to 0$ uniformly in $\mathbb{R}^N$ as $t \to +\infty$,*
(ii) *or there is $\widetilde{\phi} \in \mathcal{E}$ such that $u(t,\cdot) \to \widetilde{\phi}$ uniformly in $\mathbb{R}^N$ as $t \to +\infty$, and, if $\phi = \widetilde{\phi}$, then $u(t,x) \equiv \phi(x) \equiv \widetilde{\phi}(x)$ in $\mathbb{R} \times \mathbb{R}^N$,*
(iii) *or else the alternative* (iii) *of Theorem* 1.1 *holds.*

In the particular case when the solutions ϕ of (1.4) are unique up to shifts, then $F \leq 0$ in $[0, m_\phi]$ since otherwise, by applying the results of [7, 8] to the function f extended by 0 in $(m_\phi, +\infty)$, there would exist other solutions $\widetilde{\phi}$ of (1.4) such that

$$\max_{\mathbb{R}^N} \widetilde{\phi} < m_\phi < \max_{\mathbb{R}^N} \phi.$$

Hence, condition (1.13) is necessarily fulfilled if the solutions of (1.4) are unique up to shifts. Therefore, since the bounded positive solutions u of (1.1) satisfying (1.3) are necessarily

[3]This means that, for any radially symmetric solution ϕ of (1.4), there is a positive constant ε such that $\|\psi - \phi\|_{L^\infty(\mathbb{R}^N)} \geq \varepsilon$ for every radially symmetric solution ψ of (1.4) except for ϕ.

radially symmetric and decreasing with respect to a single point in $\mathbb{R}^N$, the following corollary immediately holds.

Corollary 2.4. *Assume that f satisfies* (1.2) *and that the solutions of* (1.4) *exist and are unique up to shifts. Let u be a positive bounded solution of* (1.1) *satisfying* (1.3)*. Then there is $\phi \in \mathcal{E}$ such that $\|u(t,\cdot)-\phi\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to -\infty$. Moreover, either $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to +\infty$, or $u(t,x) \equiv \phi(x)$ in $\mathbb{R}\times\mathbb{R}^N$, or else the alternative* (iii) *of Theorem* 1.1 *holds.*

The existence and uniqueness up to shifts of the solutions of (1.4) is known for some classes of functions f. For instance, if f satisfies (1.2) and if there are $0 < a < a' < b \le +\infty$ such that $f < 0$ in $(0,a)$, $f > 0$ in (a,b), $F(a') = 0$, $f \le 0$ in $[b,+\infty)$ and

$$s \mapsto \frac{f(s)}{s-a'} \text{ is nonincreasing in } (a',b), \tag{2.3}$$

then there exists a unique up to shifts solution ϕ of (1.4), see [23, 51] (in this case, $m_\phi = a$ and $M_\phi = b$). Notice that the monotonicity condition (2.3) is especially fulfilled if f is nonincreasing in $[a',b)$. The condition (2.3) is not optimal for the uniqueness, since there are bistable functions f satisfying the above conditions but (2.3) for which the uniqueness up to shifts holds for (1.4) in any dimension $N \ge 1$ (see especially the cubic functions f of the type (2.9) below used in Corollary 2.7). The uniqueness up to shifts of the solutions of (1.4) also holds if f satisfies (1.2) and if there is $a \in (0,+\infty)$ such that $f \le 0$ in $[0,a]$, $f > 0$ in $(a,+\infty)$ and $s \mapsto sf'(s)/f(s)$ is nonincreasing in $(a,+\infty)$, see [1, 66]. An example is the function f given in (2.1), namely

$$f(s) = -\gamma s + \delta s^p,$$

with $\gamma > 0$, $\delta > 0$ and $p > 1$. As a matter of fact, for that function, the existence and uniqueness up to shifts of a solution ϕ of (1.4) holds if and only if $N \le 2$, or $N \ge 3$ and $1 < p < (N+2)/(N-2)$, see also [6, 7, 8, 10, 11, 12, 36, 37, 44, 45, 52, 69]. In this case, one has $m_\phi = (\gamma/\delta)^{1/(p-1)}$ and $M_\phi = +\infty$, and it follows from Corollary 2.4 that any positive bounded solution of (1.1) satisfying (1.3) is either independent of t or converges to 0 uniformly in $\mathbb{R}^N$ as $t \to +\infty$. Another important example is that of functions f of the type (2.2), namely

$$f(s) = -\gamma s - \delta s^p + \eta s^q,$$

with $\gamma > 0$, $\delta > 0$, $\eta > 0$, $p \ne q$, and $\min(p,q) > 1$. The uniqueness up to shifts of the solutions ϕ of (1.4) holds in that case, and the existence holds if and only if $N \le 2$, or $p < q < (N+2)/(N-2)$ with $N \ge 3$ (in this case, $M_\phi = +\infty$ and the bounded solutions of (1.1) satisfying (1.3) are either independent of t or converge to 0 uniformly in $\mathbb{R}^N$ as $t \to +\infty$), or $p > q$ with $N \ge 3$ and β is small enough (in this case, $M_\phi < +\infty$), see [66].

On the other hand, without (2.3) or the aforementioned conditions listed in the previous paragraph, some examples of non-uniqueness up to shifts in $\mathbb{R}^N$ with $N \ge 2$ are known, for functions f of the bistable type (1.19) with $f < 0$ in $(b,+\infty)$, see [51], or for some functions f having one single positive zero, see [53] (notice that conditions (1.2) and (1.13) are automatically fulfilled for the functions considered in [51, 53]). In [13], for functions f of the type $f(s) = -s + s^p + \lambda s^q$ with $\lambda > 0$ large, $1 < q < 3$ and $p < 5$ close to 5 (conditions (1.2) and (1.13) then hold), it was shown that (1.4) in dimension $N = 3$ admits at least three solutions which are radially symmetric with respect to the origin. Furthermore, it is reasonable to conjecture from the proof given in [13] that the set of all such solutions is discrete, in which case Corollary 2.3 can be applied.

2.4. **Bistable and cubic functions f.** We complete this section by considering the class of bistable functions f satisfying (1.19) for some $0 < a < b$, namely

$$\begin{cases} f(0) = f(a) = f(b) = 0, \quad f'(0) < 0, \quad f'(b) < 0, \quad f'(a) > 0, \\ f < 0 \text{ in } (0, a), \quad f > 0 \text{ in } (a, b), \end{cases} \tag{2.4}$$

together with

$$f < 0 \ \text{ in } (b, +\infty). \tag{2.5}$$

On the one hand, if

$$\int_0^b f(s)\, ds \le 0, \tag{2.6}$$

then $F < 0$ in $(0, b) \cup (b, +\infty)$, hence Corollary 2.2 immediately yields the following result.

Corollary 2.5. *If f satisfies* (2.4)-(2.6)*, then, in any dimension $N \ge 1$, the only nonnegative bounded solution u of* (1.1) *satisfying* (1.3) *is the trivial solution $u \equiv 0$ in $\mathbb{R} \times \mathbb{R}^N$.*

On the other hand, if

$$\int_0^b f(s)\, ds > 0, \tag{2.7}$$

then there are solutions ϕ of (1.4) [7, 8], and

$$a = m_\phi < \max_{\mathbb{R}^N} \phi < M_\phi = b.$$

It is also well known [3, 20] that there is a unique $c \in \mathbb{R}$, which is positive, and a unique up to shift function $\varphi \in C^2(\mathbb{R})$ such that

$$\varphi'' + c\varphi' + f(\varphi) = 0 \text{ in } \mathbb{R}, \quad \varphi' < 0 \text{ in } \mathbb{R}, \quad \varphi(-\infty) = b, \quad \varphi(+\infty) = 0. \tag{2.8}$$

An immediate corollary of Corollary 2.4, Theorem 1.1 and property (1.24) in Remark 1.4 is the following result.

Corollary 2.6. *Assume that f satisfies* (2.4) *and* (2.7) *and that the solutions of* (1.4) *are unique up to shifts. Let u be a positive bounded solution of* (1.1) *satisfying* (1.3)*. Then there is $\phi \in \mathcal{E}$ such that $\|u(t, \cdot) - \phi\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to -\infty$. Moreover, either $\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to +\infty$, or $u(t, x) \equiv \phi(x)$ in $\mathbb{R} \times \mathbb{R}^N$, or else there are $x_0 \in \mathbb{R}^N$ and $\tau \in \mathbb{R}$ such that*

$$\sup_{x \in \mathbb{R}^N} \Big| u(t, x) - \varphi\Big(|x - x_0| - ct + \frac{N-1}{c} \ln t + \tau\Big) \Big| \to 0 \quad \text{as } t \to +\infty,$$

where $c > 0$ and $\varphi \in C^2(\mathbb{R}^N)$ are given in (2.8)*.*

Consider finally an important example of functions f satisfying (1.19), namely the cubic nonlinearities

$$f(s) = s(b-s)(s-a) \tag{2.9}$$

with $0 < a < b$. Notice that (2.7) is fulfilled if and only if $a < b/2$. Furthermore, in that case, the solutions of (1.4) exist and are unique up to shifts, by [7, 8, 66]. As a consequence, the following corollary holds.

Corollary 2.7. *Let $0 < a < b$ and f be of the type (2.9). If $a \geq b/2$, then, in any dimension $N \geq 1$, the only nonnegative bounded solution u of (1.1) satisfying (1.3) is the trivial solution $u \equiv 0$ in $\mathbb{R}\times\mathbb{R}^N$. If $a < b/2$, then (1.4) admits solutions and, for any positive bounded solution u of (1.1) satisfying (1.3), there is $\phi \in \mathcal{E}$ such that $\|u(t,\cdot)-\phi\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to -\infty$, and either $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to +\infty$, or $u(t,x) \equiv \phi(x)$ in $\mathbb{R}\times\mathbb{R}^N$, or else there are $x_0 \in \mathbb{R}^N$ and $\tau \in \mathbb{R}$ such that*

$$\sup_{x\in\mathbb{R}^N}\Big|u(t,x) - \varphi\Big(|x-x_0| - ct + \frac{N-1}{c}\ln t + \tau\Big)\Big| \to 0 \quad \text{as } t \to +\infty,$$

where $c > 0$ and $\varphi \in C^2(\mathbb{R}^N)$ are given in (2.8).

Outline of the paper. Section 3 is concerned with some preliminary results on the existence of planar traveling fronts connecting 0 and M (these fronts are used in alternative (iii) of Theorem 1.1) and on the existence of steady states in large balls under some assumptions on the nonlinearity. Section 4 is devoted to the proof of Theorem 1.1 and Section 5 to the proof of Corollaries 2.1 and 2.3 (the other corollaries follow from the other results, as already emphasized).

3. Some preliminary facts

We start with the existence and uniqueness of traveling fronts (c,φ) solving (1.17) and arising in alternative (iii) of Theorem 1.1.

Lemma 3.1. *Let $0 < m < M$ and f be a $C^1([0,M])$ function such that $f(0) = f(M) = 0$, $f'(0) < 0$, $f > 0$ in (m,M), $F < 0$ in $(0,m]$ and $F(M) > 0$. Then there are a unique $c \in \mathbb{R}$ and $\varphi : \mathbb{R} \to (0,M)$ of class $C^2(\mathbb{R})$ solving*

$$\varphi'' + c\varphi' + f(\varphi) = 0 \text{ in } \mathbb{R}, \ \varphi(-\infty) = M, \text{ and } \varphi(+\infty) = 0, \tag{3.1}$$

where the uniqueness of φ is understood up to shifts. Furthermore, $c > 0$ and $\varphi' < 0$ in $\mathbb{R}$.

The result is expected since it has been well known under some additional assumptions on f. However we are not aware of a suitable reference for its proof, which is therefore sketched here for the sake of completeness.

Proof. First of all, the uniqueness of a pair (c,φ) is a direct consequence of [20, Corollary 2.3] and the property $\varphi' < 0$ in $\mathbb{R}$ follows from [20, Lemma 2.1]. Furthermore, $c > 0$ by integrating (3.1) against φ' over $\mathbb{R}$ and using the assumption $F(M) > 0$.

Let us now show the existence of a pair (c, φ) solving (3.1). From the assumptions made on f, it is easy to check that there are a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0, m)$ converging to 0, and a sequence $(\overline{f}_n)_{n\in\mathbb{N}}$ such that each function

$$\overline{f}_n : [\varepsilon_n, M + \varepsilon_n] \to \mathbb{R}$$

is of class $C^1([\varepsilon_n, M + \varepsilon_n])$, the sequence $(\|\overline{f}_n\|_{C^1([\varepsilon_n,M+\varepsilon_n])})_{n\in\mathbb{N}}$ is bounded, and for each $n \in \mathbb{N}$, there holds: $\overline{f}_n(\varepsilon_n) = \overline{f}_n(M + \varepsilon_n) = 0$, $\overline{f}'_n(\varepsilon_n) < 0$, $\overline{f}'_n(M+\varepsilon_n) < 0$, the zeroes of $\overline{f}_n$ are all non-degenerate (that is, $\{s \in [\varepsilon_n, M + \varepsilon_n] : \overline{f}_n(s) = \overline{f}'_n(s) = 0\} = \emptyset$), together with $\overline{f}_n \geq \overline{f}_{n'}$ in $[\varepsilon_n, M + \varepsilon_{n'}]$ if $n \leq n'$, $\overline{f}_n \geq f$ in $[\varepsilon_n, M]$, and $\max_{[\varepsilon_n,M]} |\overline{f}_n - f| \to 0$ as $n \to +\infty$. Furthermore, even if it means considering a subsequence, one can always assume that, for each $n \in \mathbb{N}$,

$$\overline{f}_n > 0 \text{ in } (m, M+\varepsilon_n), \ \int_{\varepsilon_n}^{s} \overline{f}_n(\sigma)\, d\sigma < 0 \text{ for all } s \in (\varepsilon_n, m], \text{ and} \int_{\varepsilon_n}^{M+\varepsilon_n} \overline{f}_n(\sigma)\, d\sigma > 0. \tag{3.2}$$

For each $n \in \mathbb{N}$, it then follows from [20, Theorem 2.8] that there are $p \in \mathbb{N}$, some real numbers $\varepsilon_n = a_0 < a_1 < \cdots < a_p = M + \varepsilon_n$ and

$$c_1 \geq \cdots \geq c_p \tag{3.3}$$

such that, for each $j \in \{1, \cdots, p\}$, there is a $C^2(\mathbb{R})$ function $\varphi_j : \mathbb{R} \to (a_{j-1}, a_j)$ satisfying

$$\varphi''_j + c_j\varphi'_j + \overline{f}_n(\varphi_j) = 0 \text{ in } \mathbb{R}, \ \varphi'_j < 0 \text{ in } \mathbb{R}, \ \varphi_j(-\infty) = a_j, \text{ and } \varphi_j(+\infty) = a_{j-1}. \tag{3.4}$$

Notice that the quantities p, a_j, c_j and φ_j actually depend on n, that $\overline{f}_n(a_j) = 0$ for all $0 \leq j \leq p$, and that the family $(c_j, \varphi_j)_{1\leq j\leq p}$ is then a stacked combination of traveling fronts with non-increasing speeds for the reaction term $\overline{f}_n$. Since $a_{p-1} \leq m$ and $\int_{a_{p-1}}^{M+\varepsilon_n} \overline{f}_n(\sigma)d\sigma > 0$ by (3.2), integrating (3.4) with $j = p$ against φ'_p over $\mathbb{R}$ implies that $c_p > 0$. Furthermore, if $p \geq 2$, one would have $a_1 \leq m$ and then $c_1 < 0$ by using again (3.2) and (3.4) with $j = 1$, contradicting (3.3). Therefore, $p = 1$ and there is then a solution $(\overline{c}_n, \overline{\varphi}_n)$ of

$$\overline{\varphi}''_n + \overline{c}_n\overline{\varphi}'_n + \overline{f}_n(\overline{\varphi}_n) = 0 \text{ in } \mathbb{R}, \ \overline{\varphi}'_n < 0 \text{ in } \mathbb{R}, \ \overline{\varphi}_n(-\infty) = M + \varepsilon_n, \text{ and } \overline{\varphi}_n(+\infty) = \varepsilon_n,$$

with $\overline{c}_n > 0$.

Now, if $n < n'$, then $M + \varepsilon_n > M + \varepsilon_{n'} > \varepsilon_n > \varepsilon_{n'}$ and, up to shifts, one has $\overline{\varphi}_n \geq \overline{\varphi}_{n'}$ in $\mathbb{R}$ with equality at a point ξ such that $\overline{\varphi}_n(\xi) = \overline{\varphi}_{n'}(\xi) \in (\varepsilon_n, M+\varepsilon_{n'})$. On the other hand, if $\overline{c}_n \leq \overline{c}_{n'}$, then one would have

$$\overline{\varphi}''_n + \overline{c}_{n'}\overline{\varphi}'_n + \overline{f}_{n'}(\overline{\varphi}_n) \leq \overline{\varphi}''_n + \overline{c}_n\overline{\varphi}'_n + \overline{f}_n(\overline{\varphi}_n) = 0 = \overline{\varphi}''_{n'} + \overline{c}_{n'}\overline{\varphi}'_{n'} + \overline{f}_{n'}(\overline{\varphi}_{n'})$$

in the open interval $I = \{x \in \mathbb{R} : \varepsilon_n < \overline{\varphi}_n(x) < M + \varepsilon_{n'}\}$. In other words, the functions $\overline{\varphi}_n$ and $\overline{\varphi}_{n'}$ are respectively a super-solution and a solution of the same elliptic equation in I, with $\overline{\varphi}_n \geq \overline{\varphi}_{n'}$ in $\mathbb{R} \supset I$. Since $\xi \in I$, it then follows from the strong maximum principle that $\overline{\varphi}_n(x) = \overline{\varphi}_{n'}(x)$ for all $x \in I$. Since I is of the type $I = (\zeta, +\infty)$ for some $\zeta \in \mathbb{R}$, one gets a contradiction by passing to the limit in $\overline{\varphi}_n(x) = \overline{\varphi}_{n'}(x)$ as $x \to +\infty$. As a consequence, $\overline{c}_n > \overline{c}_{n'}$ and the sequence $(\overline{c}_n)_{n\in\mathbb{N}}$ is decreasing, with $\overline{c}_n > 0$ for every $n \in \mathbb{N}$.

Finally, there is $c \in \mathbb{R}$ such that $\overline{c}_n \to c \geq 0$ as $n \to +\infty$. Let η be any real number in $(0, m)$ such that $f < 0$ in $(0, \eta]$. Up to shifts, one can assume without loss of generality that $\overline{\varphi}_n(0) = \eta$ for all n large enough (such that $\varepsilon_n < \eta$). From standard elliptic estimates, the functions $\overline{\varphi}_n$ converge in $C^2_{loc}(\mathbb{R})$ to a $C^2(\mathbb{R})$ function φ such that $\varphi(0)=\eta$, $0\le\varphi\le M$ in $\mathbb{R}$, $\varphi'\le 0$ in $\mathbb{R}$, and

$$\varphi''+c\varphi'+f(\varphi)=0 \text{ in } \mathbb{R}. \tag{3.5}$$

From standard elliptic estimates, $\varphi'(\pm\infty)=\varphi''(\pm\infty)=f(\varphi(\pm\infty))=0$ and the choice of η yields $\varphi(+\infty)=0$, hence $\varphi'<0$ in $\mathbb{R}$ from the strong maximum principle and $0<\varphi(-\infty)\le M$. Integrating (3.5) against φ' over $\mathbb{R}$ implies that $F(\varphi(-\infty))=c\int_{\mathbb{R}}(\varphi')^2\ge 0$. Therefore, the assumptions on f imply that $\varphi(-\infty)=M$. In other words, the pair (c,φ) solves (3.1), and the proof of Lemma 3.1 is thereby complete. □

Remark 3.2. The arguments used in the proof of Lemma 3.1 also lead to the approximation of the unique speed c from below. Namely, as above, there are a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,m)$ converging to 0, and a sequence $(\underline{f}_n)_{n\in\mathbb{N}}$ such that each function

$$\underline{f}_n:[-\varepsilon_n,M-\varepsilon_n]\to\mathbb{R}$$

is of class $C^1([-\varepsilon_n,M-\varepsilon_n])$, the sequence $(\|\underline{f}_n\|_{C^1([-\varepsilon_n,M-\varepsilon_n])})_{n\in\mathbb{N}}$ is bounded, and for each $n\in\mathbb{N}$, there holds: $\underline{f}_n(-\varepsilon_n)=\underline{f}_n(M-\varepsilon_n)=0$, $\underline{f}'_n(-\varepsilon_n)<0$, $\underline{f}'_n(M-\varepsilon_n)<0$, the zeroes of $\underline{f}_n$ are all non-degenerate, together with $\underline{f}_n\le\underline{f}_{n'}$ in $[-\varepsilon_{n'},M-\varepsilon_n]$ if $n\le n'$, $\underline{f}_n\le f$ in $[0,M-\varepsilon_n]$, and $\max_{[0,M-\varepsilon_n]}|\underline{f}_n-f|\to 0$ as $n\to+\infty$. Lastly, after fixing a real number $m'\in(m,M)$ such that $F<0$ in $(0,m']$, one can assume without loss of generality that, for each $n\in\mathbb{N}$,

$$\underline{f}_n>0 \text{ in } [m',M-\varepsilon_n),\ \int_{-\varepsilon_n}^{s}\underline{f}_n(\sigma)\,d\sigma<0 \text{ for all } s\in(-\varepsilon_n,m'],\ \text{and } \int_{-\varepsilon_n}^{M-\varepsilon_n}\underline{f}_n(\sigma)\,d\sigma>0.$$

As in the proof of Lemma 3.1, one can then show that, for each $n\in\mathbb{N}$, there is a solution $(\underline{c}_n,\underline{\varphi}_n)$ of $\underline{\varphi}''_n+\underline{c}_n\underline{\varphi}'_n+\underline{f}_n(\underline{\varphi}_n)=0$ in $\mathbb{R}$, $\underline{\varphi}'_n<0$ in $\mathbb{R}$, $\underline{\varphi}_n(-\infty)=M-\varepsilon_n$, $\underline{\varphi}_n(+\infty)=-\varepsilon_n$, and $\underline{c}_n>0$. Furthermore, the sequence $(\underline{c}_n)_{n\in\mathbb{N}}$ is increasing, and $\underline{c}_n<c$ for all $n\in\mathbb{N}$, with the same arguments as in the proof of Lemma 3.1. Finally, there is $\underline{c}\le c$ such that $\underline{c}_n\to\underline{c}$ as $n\to+\infty$, and there is a $C^2(\mathbb{R})$ solution $\underline{\varphi}$ of (3.1) with speed $\underline{c}$ instead of c. The uniqueness of (c,φ) then yields $\underline{c}=c$, hence $\underline{c}_n\to c$ as $n\to+\infty$.

The second preliminary result is concerned with the existence of solutions of some semi-linear elliptic equations in large balls.

Lemma 3.3. *Let $\alpha<\beta$ be two real numbers and $g:[\alpha,\beta]\to\mathbb{R}$ be a $C^1([\alpha,\beta])$ function such that $g(\alpha)=g(\beta)=0$ and*

$$G(\beta):=\int_\alpha^\beta g(\sigma)d\sigma>\int_\alpha^s g(\sigma)d\sigma=:G(s) \text{ for all } s\in[\alpha,\beta). \tag{3.6}$$

Then, for each $\nu\in(\alpha,\beta)$, there are $R>0$ and a $C^2(\overline{B_R})$ function ψ such that

$$\begin{cases} \Delta\psi+g(\psi) = 0 & \text{in } \overline{B_R},\\ \alpha\le\psi<\beta & \text{in } \overline{B_R},\\ \psi=\alpha & \text{on } \partial B_R,\\ \max_{\overline{B_R}}\psi=\psi(0)>\nu. \end{cases} \tag{3.7}$$

Proof. The proof is standard, based on [6], so we just sketch it. Let $\overline{g}:\mathbb{R}\to\mathbb{R}$ be the function defined by $\overline{g}(s)=g(s)$ for $s\in[\alpha,\beta]$, and $\overline{g}(s)=0$ for $s\in\mathbb{R}\backslash[\alpha,\beta]$, and let us still call G the primitive of $\overline{g}$ vanishing at α. For each $r>0$, there is a minimizer $\psi_r\in\alpha+H^1_0(B_r)$ of the Lagrangian J_r defined in $\alpha+H^1_0(B_r)$ by

$$J_r(\varphi)=\frac{1}{2}\int_{B_r}|\nabla\varphi(x)|^2\,dx-\int_{B_r}G(\varphi(x))\,dx,\quad J_r(\psi_r)=\min_{\varphi\in\alpha+H^1_0(B_r)}J_r(\varphi),$$

Owing to the definitions of $\overline{g}$ and G, one can assume without loss of generality that ψ_r ranges in $[\alpha,\beta]$, hence ψ_r is of class $C^2(\overline{B_r})$ from elliptic estimates and it solves $\Delta\psi_r+g(\psi_r)=0$ in $\overline{B_r}$ with $\psi_r=\alpha$ on ∂B_r. The strong elliptic maximum principle also yields $\psi_r<\beta$ in $\overline{B_r}$ and, either $\psi_r\equiv\alpha$ in $\overline{B_r}$, or $\psi_r>\alpha$ in B_r. In both cases, ψ_r is a radially symmetric and nonincreasing function of $|x|$ (from [26] in the latter). In particular, $\max_{\overline{B_r}}\psi_r=\psi_r(0)\in[\alpha,\beta)$.

Let us now show that

$$\max_{\overline{B_r}}\psi_r=\psi_r(0)\to\beta\ \text{ as } r\to+\infty, \tag{3.8}$$

which will then provide $R>0$ and a solution ψ of (3.7), given a fixed real number $\nu\in(\alpha,\beta)$. Assume by way of contradiction that there are $\theta\in(\alpha,\beta)$, a sequence $(r_k)_{k\in\mathbb{N}}\to+\infty$ and a sequence $(\psi_{r_k})_{k\in\mathbb{N}}$ of $C^2(\overline{B_{r_k}})$ functions such that each $\psi_{r_k}:\overline{B_{r_k}}\to[\alpha,\beta)$ minimizes J_{r_k} in $\alpha+H^1_0(B_{r_k})$ and $\max_{\overline{B_{r_k}}}\psi_{r_k}=\psi_{r_k}(0)\le\theta<\beta$. From the assumptions made on g, there is $\delta>0$ such that $G(s)\le G(\beta)-\delta$ for all $s\in[\alpha,\theta]$. Hence, $J_{r_k}(\psi_{r_k})\ge(\delta-G(\beta))\,\alpha_N r_k^N$ for all $k\in\mathbb{N}$, where $\alpha_N>0$ denotes the Lebesgue measure of the N-dimensional unit ball B_1. On the other hand, after assuming without loss of generality that $r_k>1$ for every $k\in\mathbb{N}$, consider the function $\varphi_k\in\alpha+H^1_0(B_{r_k})$ defined by $\varphi_k(x)=\beta$ for $x\in B_{r_k-1}$ and $\varphi_k(x)=\alpha+(\beta-\alpha)(r_k-|x|)$ for $x\in\overline{B_{r_k}}\backslash B_{r_k-1}$. For each $k\in\mathbb{N}$, one has

$$\begin{aligned} J_{r_k}(\psi_{r_k}) &\le J_{r_k}(\varphi_k)=\frac{\alpha_N}{2}\,(r_k^N-(r_k-1)^N)-G(\beta)\,\alpha_N\,(r_k-1)^N-\int_{B_{r_k}\backslash B_{r_k-1}}G(\varphi_k(x))\,dx\\ &\le \alpha_N\Big(\frac{1}{2}+\max_{[\alpha,\beta]}|G|\Big)\,(r_k^N-(r_k-1)^N)-G(\beta)\alpha_N(r_k-1)^N. \end{aligned}$$

This implies that

$$\delta\,r_k^N\le\Big(\frac{1}{2}+\max_{[\alpha,\beta]}|G|+G(\beta)\Big)\,(r_k^N-(r_k-1)^N).$$

It contradicts $\lim_{k\to+\infty}r_k=+\infty$, since $\delta>0$. As a consequence, (3.8) holds and the proof of Lemma 3.3 is thereby complete. □

4. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Throughout it, one assumes that f satisfies (1.2) and (1.13), and u denotes a positive bounded solution of (1.1) satisfying (1.3). Sections 4.1 and 4.2 are concerned with the behaviors of u as $t\to-\infty$ and $t\to+\infty$, respectively. Throughout the proof, from Remark 1.3, we denote $m=m_\phi>0$ the constant value of the map $\mathcal{E}\ni\phi\mapsto m_\phi$.

4.1. **The behavior of u as $t \to -\infty$.** We here establish (1.14) and further integral properties of the solution u. First of all, it follows from [54, Theorem 1.1] that there is a point $x_0 \in \mathbb{R}^N$ such that

$$(4.1)\quad \begin{cases} u(t,x) = u(t,y) & \text{for all } (t,x,y) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \text{ with } |x-x_0| = |y-x_0|,\\ \nabla u(t,x)\cdot(x-x_0) < 0 & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^N \text{ with } x \neq x_0. \end{cases}$$

Furthermore, [54, Corollary 2.5] implies that there are some positive constants C and ν such that

$$(4.2)\qquad 0 < u(t,x) \le C\,e^{-\nu|x|} \ \text{ for all } (t,x) \in (-\infty,0]\times\mathbb{R}^N.$$

Denote

$$M_0 = \|u\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)} > 0$$

and $L = \nu^2 + \max_{[0,M_0]}|f'|$. For any unit vector e of $\mathbb{R}^N$, the function $\overline{u}(t,x) = C\,e^{-\nu x\cdot e + Lt}$ satisfies $\overline{u}_t(t,x) - \Delta\overline{u}(t,x) - f(\overline{u}(t,x)) \ge 0$ for any $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ such that $\overline{u}(t,x) \le M_0$. Therefore, the maximum principle implies that $u(t,x) \le \overline{u}(t,x)$ for all $(t,x) \in [0,+\infty)\times\mathbb{R}^N$ and for any unit vector e, hence $u(t,x) \le C\,e^{-\nu|x|+Lt}$ for all $(t,x) \in [0,+\infty)\times\mathbb{R}^N$. By combining this inequality with (4.2), for every $T \in \mathbb{R}$, there is a real number $C_T > 0$ such that

$$(4.3)\qquad 0 < u(t,x) \le C_T\,e^{-\nu|x|} \ \text{ for all } (t,x) \in (-\infty,T]\times\mathbb{R}^N.$$

From standard parabolic estimates, one also infers that, for every $T \in \mathbb{R}$, there is a real number $C'_T > 0$ such that

$$(4.4)\quad u(t,x) + |u_t(t,x)| + |\nabla u(t,x)| + \sum_{1\le i,j\le N} |u_{x_ix_j}(t,x)| \le C'_T e^{-\nu|x|} \text{ for all } (t,x) \in (-\infty,T]\times\mathbb{R}^N.$$

In particular, the action

$$(4.5)\qquad E[u(t,\cdot)] = \int_{\mathbb{R}^N} \Big(\frac{|\nabla u(t,x)|^2}{2} - F(u(t,x))\Big)\,dx$$

is well defined at each time $t \in \mathbb{R}$, and Lebesgue's dominated convergence theorem implies that the function $t \mapsto E[u(t,\cdot)]$ is of class $C^1(\mathbb{R})$ with

$$(4.6)\qquad \frac{d}{dt}E[u(t,\cdot)] = -\int_{\mathbb{R}^N} (u_t(t,x))^2\,dx \le 0$$

for every $t \in \mathbb{R}$. Furthermore, (4.4) yields $\sup_{t\le T}|E[u(t,\cdot)]| < +\infty$ for every $T \in \mathbb{R}$, and there is then $\ell \in \mathbb{R}$ such that

$$(4.7)\qquad E[u(t,\cdot)] \to \ell \ \text{ as } t \to -\infty.$$

Consider now any sequence $(t_n)_{n\in\mathbb{N}}$ converging to $-\infty$, and denote

$$u_n(t,x) = u(t+t_n,x)$$

for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$. From (4.4) and further standard parabolic estimates, there is a classical nonnegative bounded solution u_∞ of (1.1) such that, up to extraction of a subsequence, $u_n \to u_\infty$ in $C^{1,2}_{t,x}$ locally in $\mathbb{R}\times\mathbb{R}^N$, together with

$$\|u_n(t,\cdot) - u_\infty(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ and } E[u_n(t,\cdot)] \to E[u_\infty(t,\cdot)] \ \text{ as } n \to +\infty,$$

for every $t \in \mathbb{R}$. Notice also that u_∞ satisfies (4.4) with the constant C_0' in the whole set $\mathbb{R}\times\mathbb{R}^N$. Since $E[u_n(t,\cdot)] = E[u(t+t_n,\cdot)] \to \ell$ as $n \to +\infty$, for every $t \in \mathbb{R}$, one infers that $E[u_\infty(t,\cdot)] = \ell$ for every $t \in \mathbb{R}$, hence $(u_\infty)_t \equiv 0$ in $\mathbb{R}\times\mathbb{R}^N$ from (4.6) applied to u_∞. As a consequence, u_∞ is a bounded nonnegative steady state solving

$$\begin{cases} \Delta u_\infty + f(u_\infty) = 0 \text{ and } u_\infty \geq 0 \text{ in } \mathbb{R}^N, \\ u_\infty(x) \to 0 \text{ as } |x| \to +\infty. \end{cases}$$

From the elliptic maximum principle, it follows that either $u_\infty \equiv 0$ in $\mathbb{R}^N$, or $u_\infty > 0$ in $\mathbb{R}^N$. In the former case, one has $u(t_n,\cdot) = u_n(0,\cdot) \to 0$ as $n \to +\infty$ uniformly in $\mathbb{R}^N$, hence $0 < u(t_n,\cdot) \leq \eta$ in $\mathbb{R}^N$ for all n large enough, where η is a positive real number such that

$$f < 0 \text{ in } (0,\eta]. \tag{4.8}$$

Thus, for all $t \in \mathbb{R}$, one gets that $0 < u(t,\cdot) = u(t-t_n+t_n,\cdot) \leq \zeta(t-t_n)$ in $\mathbb{R}^N$ for all n large enough, where ζ obeys $\zeta(0) = \eta$ and $\zeta'(t) = f(\zeta(t))$ for all $t \geq 0$. Since $\zeta(+\infty) = 0$ and $\lim_{n\to+\infty} t_n = -\infty$, it follows that $u(t,\cdot) \leq 0$ in $\mathbb{R}^N$ for all $t \in \mathbb{R}$, a contradiction. Therefore, $u_\infty > 0$ is a steady state solving (1.4), namely $u_\infty \in \mathcal{E}$. In particular, $\mathcal{E}$ is not empty.

Since in the previous paragraph the sequence $(t_n)_{n\in\mathbb{N}}$ converging to $-\infty$ was arbitrary, one concludes that

$$\inf_{\phi\in\mathcal{E}} \|u(t,\cdot) - \phi\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{as } t \to -\infty,$$

namely (1.14) has been proven. The observations of the previous paragraph also imply that

$$E[\phi] = \ell \tag{4.9}$$

for every $\phi \in \mathcal{E}$ belonging to the α-limit set of u.

Remark 4.1. Remember that, from Remark 1.3, the map $(\emptyset \neq)\,\mathcal{E} \ni \phi \mapsto m_\phi$ takes a constant value $m > 0$. The quantity $M \in (m,+\infty]$ defined in (1.11) and (1.22)-(1.23) is such that $f > 0$ in (m,M) from (1.12). In the present remark, we claim that

$$0 < u(t,x) < M \quad \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^N. \tag{4.10}$$

By assumption, u is positive. So there is nothing to show if $M = +\infty$. Assume now that $M < +\infty$. From the above proof of (1.14), there is $\phi \in \mathcal{E}$ and a sequence $(t_n)_{n\in\mathbb{N}}$ converging to $-\infty$ such that $\|u(t_n,\cdot) - \phi\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $n \to +\infty$. Since $\max_{\mathbb{R}^N}\phi < M$ by (1.11) and (1.22), one has $u(t_n,\cdot) < M$ in $\mathbb{R}^N$ for all n large enough. Since $f(M) = 0$, it then follows from the maximum principle (applied for n large enough) that, for each $t \in \mathbb{R}$, $u(t,\cdot) = u(t-t_n+t_n,\cdot) < M$ in $\mathbb{R}^N$, that is, (4.10) holds.

4.2. **The behavior of u as $t \to +\infty$.** In the section, we consider the behavior of the entire solution u as $t \to +\infty$. The proof is divided into five main steps.

Step 1: two key-lemmas. The proof of the dichotomy as $t \to +\infty$ between the uniformly localized solutions and the spreading solutions is based on two key-lemmas. The first one gives a sufficient condition for the finiteness and attractiveness of the quantity $M \in (m,+\infty]$ defined in (1.11) and (1.22)-(1.23).

Lemma 4.2. *For every $\varepsilon > 0$, there is a real number $\rho_\varepsilon > 0$ such that, if*

$$u(t_0, \cdot) \ge m + \varepsilon \text{ in } \overline{B(y_0, \rho_\varepsilon)} \tag{4.11}$$

for some $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^N$, then

$$M < +\infty$$

and

$$\max_{|x| \le \gamma t} |u(t, x) - M| \to 0 \text{ as } t \to +\infty \tag{4.12}$$

for some $\gamma > 0$.[4]

Remark 4.3. For a function f such that M is a priori assumed to be finite, the conclusion (4.12) can also be viewed as a consequence of [16, Lemma 2.4], which is based on the existence of approximated planar fronts defined in bounded intervals (see also [20, Theorem 3.2] and [3, Theorem 6.2] for related results with more specific nonlinearities f in the one- and multi-dimensional cases). We here both show (4.12) and the finiteness of M under assumption (4.11). Moreover, the proof of (4.12) given below differs from that of [16, Lemma 2.4] as it is based on Lemma 3.3 and on the existence of compactly supported steady states, together with the sliding method.

Proof of Lemma 4.2. Let $\varepsilon > 0$ be fixed throughout the proof. Assume first, by way of contradiction, that $M = +\infty$, and that there exists a sequence $(t_n, y_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \times \mathbb{R}^N$ such that $u(t_n, \cdot) \ge m + \varepsilon$ in $\overline{B(y_n, n)}$. From (1.12), it then follows that $f > 0$ in $(m, +\infty)$. From standard parabolic estimates, the functions $u_n : (t, x) \mapsto u_n(t, x) = u(t+t_n, x+y_n)$ converge in $C^{1,2}_{t,x}$ locally in $\mathbb{R} \times \mathbb{R}^N$, up to extraction of a subsequence, to a nonnegative bounded solution u_∞ of (1.1) such that $u_\infty(0, \cdot) \ge m + \varepsilon$ in $\mathbb{R}^N$. Hence, $u_\infty(t, \cdot) \ge \varsigma(t)$ in $\mathbb{R}^N$ for all $t \ge 0$, where ς obeys

$$\begin{cases} \varsigma'(t) = f(\varsigma(t)) & \text{for } t \ge 0, \\ \varsigma(0) = m + \varepsilon. \end{cases} \tag{4.13}$$

This is impossible since u_∞ is bounded while $f > 0$ in $[m+\varepsilon, +\infty)$. As a consequence, there is $\widetilde{\rho}_\varepsilon > 0$ such that if

$$u(t_0, \cdot) \ge m + \varepsilon \text{ in } \overline{B(y_0, \widetilde{\rho}_\varepsilon)}$$

for some $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^N$, then

$$M < +\infty.$$

We now claim that there is $\rho_\varepsilon \in [\widetilde{\rho}_\varepsilon, +\infty)$ such that, if condition (4.11) is fulfilled for some $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^N$, then $M < +\infty$ (from the previous paragraph) and

$$u(t, \cdot) \to M \text{ as } t \to +\infty \text{ locally uniformly in } \mathbb{R}^N. \tag{4.14}$$

Assume not. Then there is a sequence $(\tau_n, z_n)_{n\in\mathbb{N},\, n\ge\widetilde{\rho}_\varepsilon}$ in $\mathbb{R} \times \mathbb{R}^N$ such that $u(\tau_n, \cdot) \ge m + \varepsilon$ in $\overline{B(z_n, n)}$ (hence, $M < +\infty$) and

$$u(t, \cdot) \not\to M \text{ as } t \to +\infty \text{ locally uniformly in } \mathbb{R}^N. \tag{4.15}$$

[4]We point out that, when $\varepsilon > M_0 - m$, ρ_ε can be arbitrary because (4.11) is not fulfilled in that case.

Notice that (4.10) then implies that $m < m + \varepsilon < M$. On the other hand, since $F < 0$ in $(0, m]$ by (1.13) and (1.21), since $F(M) > 0$ by (1.22)-(1.23) and since $f > 0$ in (m, M) by (1.12), one infers that $F(s) < F(M)$ for all $s \in [0, M)$. Lemma 3.3 applied with $\alpha = 0$, $\beta = M$, $\gamma = m$ and $g = f$ yields the existence of $R > 0$ and a $C^2(\overline{B_R})$ function ψ such that

$$\left\{\begin{array}{rcll} \Delta\psi + f(\psi) & = & 0 & \text{in } \overline{B_R}, \\ 0 \ \leq \ \psi & < & M & \text{in } \overline{B_R}, \\ \psi & = & 0 & \text{on } \partial B_R, \\ M > \max\limits_{\overline{B_R}} \psi \ = \ \psi(0) & > & m > 0. & \end{array}\right. \tag{4.16}$$

Let ς be the solution of (4.13). Since $m + \varepsilon < M$ and $f > 0$ in (m, M) with $f(M) = 0$, one has $\varsigma(t) \to M$ as $t \to +\infty$. Hence, there is a positive real number $T > 0$ such that

$$\varsigma(T) > \psi(0). \tag{4.17}$$

Up to extraction of a subsequence, the functions

$$v_n : (t, x) \mapsto v_n(t, x) = u(t + \tau_n, x + z_n)$$

converge in $C^{1,2}_{t,x}$ locally in $\mathbb{R} \times \mathbb{R}^N$ to a nonnegative bounded solution v_∞ of (1.1) such that $v_\infty(0, \cdot) \geq m + \varepsilon$ in $\mathbb{R}^N$. Hence, $v_\infty(T, \cdot) \geq \varsigma(T) > \psi(0)$. It then follows from the last line in (4.16) that there is $n_0 \in \mathbb{N}$ (with $n_0 \geq \widetilde{\rho}_\varepsilon$) such that

$$u(T + \tau_{n_0}, \cdot + z_{n_0}) > \psi \ \text{ in } \overline{B_R}. \tag{4.18}$$

Let then w be the solution of the equation $w_t = \Delta w + f(w)$ in $(0, +\infty) \times \mathbb{R}^N$ with initial condition given by

$$w(0, x) = \left\{\begin{array}{ll} \psi(x) & \text{if } x \in \overline{B_R}, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \overline{B_R}. \end{array}\right. \tag{4.19}$$

Since ψ satisfies (4.16) and $f(M) = f(0) = 0$, the parabolic maximum principle implies that $0 < w < M$ in $(0, +\infty) \times \mathbb{R}^N$ and w is increasing with respect to t in $[0, +\infty) \times \mathbb{R}^N$. From standard parabolic estimates and uniqueness of the limit, there is a $C^2(\mathbb{R}^N)$ solution w_∞ of $\Delta w_\infty + f(w_\infty) = 0$ in $\mathbb{R}^N$ with $0 < w_\infty \leq M$ in $\mathbb{R}^N$ and $w_\infty > \psi$ in $\overline{B_R}$. It is then standard to show, by sliding ψ is all directions and using the strong elliptic maximum principle, that $w_\infty > \psi(0)$ in $\mathbb{R}^N$, hence $M \geq w_\infty > \psi(0) > m$ in $\mathbb{R}^N$. The positivity of f in (m, M) then implies that $w_\infty \equiv M$ in $\mathbb{R}^N$, hence

$$w(t, \cdot) \to M \text{ as } t \to +\infty \text{ locally uniformly in } \mathbb{R}^N. \tag{4.20}$$

Together with (4.18)-(4.19) and the maximum principle, one infers that

$$\liminf_{t \to +\infty} \Big(\min_K u(t, \cdot) \Big) \geq M$$

for any compact set $K \subset \mathbb{R}^N$, and finally $u(t, \cdot) \to M$ as $t \to +\infty$ locally uniformly in $\mathbb{R}^N$ from (4.10). This contradicts (4.15).

As a conclusion of the previous paragraph, there is a positive real number $\rho_\varepsilon (\geq \widetilde{\rho}_\varepsilon)$ such that if condition (4.11) is fulfilled for some $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^N$, then $M < +\infty$ and (4.14) holds. Let us finally show that this implies the stronger property (4.12): $\max_{|x| \leq \gamma t} |u(t, x) - M| \to 0$

as $t \to +\infty$, for some $\gamma > 0$. To do so, observe on the one hand that, since $w(t,\cdot) \to M$ as $t \to +\infty$ locally uniformly in $\mathbb{R}^N$ by (4.20) and since $\max_{\overline{B_R}} \psi < M$, there is a time $\tau > 0$ such that

$$w(\tau,\cdot) \ge \psi(\cdot - se) \ \text{ in } \overline{B(se,R)} \ \text{ for every unit vector } e \text{ and every } s \in [0,1].$$

In other words, $w(\tau,\cdot) \ge w(0,\cdot - se)$ in $\mathbb{R}^N$ for every unit vector e and every $s \in [0,1]$. From the maximum principle, one gets that $w(2\tau,\cdot) \ge w(\tau,\cdot - se) \ge w(0,\cdot - 2se)$ in $\mathbb{R}^N$ for every unit vector e and every $s \in [0,1]$. Hence, by an immediate induction,

$$w(k\tau,\cdot) \ge w(0,\cdot - se) \text{ in } \mathbb{R}^N \text{ for every } k \in \mathbb{N}, \text{ every unit vector } e, \text{ and every } s \in [0,k].$$

On the other hand, (4.14) and (4.16) yield the existence of a time $\tau_* > 0$ such that $u(\tau_*,\cdot) \ge \psi$ in $\overline{B_R}$, that is, $u(\tau_*,\cdot) \ge w(0,\cdot)$ in $\mathbb{R}^N$. Therefore, $u(\tau_* + k\tau,\cdot) \ge w(0,\cdot - se)$ in $\mathbb{R}^N$ for every $k \in \mathbb{N}$, every unit vector e, and every $s \in [0,k]$. In particular,

$$\min_{\overline{B_k}} u(\tau_* + k\tau,\cdot) \ge w(0,0) = \psi(0) \ \text{ for all } k \in \mathbb{N}. \tag{4.21}$$

We finally claim that

$$\max_{|x| \le t/(2\tau)} |u(t,x) - M| \to 0 \ \text{ as } t \to +\infty, \tag{4.22}$$

which will give the desired conclusion (4.12) with $\gamma = 1/(2\tau)$. Assume by way of contradiction that (4.22) does not hold. Since $0 < u < M$ in $\mathbb{R} \times \mathbb{R}^N$ by (4.10), there are then a real number $\theta \in [0,M)$ and a sequence $(s_n,\xi_n)_{n\in\mathbb{N}}$ in $(0,+\infty) \times \mathbb{R}^N$ such that

$$\lim_{n\to+\infty} s_n = +\infty, \quad \lim_{n\to+\infty} u(s_n,\xi_n) = \theta, \quad \text{and} \quad |\xi_n| \le \frac{s_n}{2\tau} \text{ for all } n \in \mathbb{N}. \tag{4.23}$$

Consider any integer $j \in \mathbb{N}$. For all n large enough, write

$$s_n = \tau_* + k_n\tau + s'_n, \quad \text{with } k_n \in \mathbb{N} \text{ and } s'_n \in [j\tau,(j+1)\tau) \tag{4.24}$$

(the quantities k_n and s'_n depend on j as well, but this does not matter). Up to extraction of a subsequence, there is $s'_\infty \in [j\tau,(j+1)\tau]$ such that $s'_n \to s'_\infty$ as $n \to +\infty$. Up to extraction of another subsequence, the functions

$$U_n : (t,x) \mapsto U_n(t,x) = u(t + \tau_* + k_n\tau, x + \xi_n)$$

converge in $C^{1,2}_{t,x}$ locally in $\mathbb{R}\times\mathbb{R}^N$ to a solution U_∞ of (1.1) such that $0 \le U_\infty \le M$ in $\mathbb{R}\times\mathbb{R}^N$. For each $x \in \mathbb{R}^N$, one has $|x + \xi_n| \le k_n$ for all n large enough, since $|\xi_n| \le s_n/(2\tau)$ for all n and $s_n \sim k_n\tau$ as $n \to +\infty$ from (4.24) and $\lim_{n\to+\infty} s_n = +\infty$. It then follows from (4.21) that $U_\infty(0,\cdot) \ge \psi(0)$ in $\mathbb{R}^N$, hence

$$U_\infty(t,\cdot) \ge \omega(t) \ \text{ in } \mathbb{R}^N \text{ for all } t \ge 0,$$

where ω obeys $\omega'(t) = f(\omega(t))$ and $\omega(0) = \psi(0)$. Since $m < \psi(0) < M$ and $f > 0$ in (m,M) with $f(M) = 0$, one has $\omega(t) \to M$ as $t \to +\infty$ (notice that U_∞ and $s'_\infty \in [j\tau,(j+1)\tau]$ depend on $j \in \mathbb{N}$, but $\psi(0)$ and ω do not). It also follows from (4.23)-(4.24) that $U_\infty(s'_\infty,0) = \theta < M$, hence $M > \theta \ge \omega(s'_\infty)$. Since $s'_\infty \in [j\tau,(j+1)\tau]$ and $\omega(+\infty) = M$, the passage to the limit as $j \to +\infty$ in the inequality $M > \theta \ge \omega(s'_\infty)$ leads to a contradiction. As a conclusion, (4.22) has been shown and the proof of Lemma 4.2 is thereby complete. □

The second key-lemma gives a quantitative estimate of the time the solution takes to go from $m+\varepsilon$ to any value λ less than M in large balls.

Lemma 4.4. *Under the notations of Lemma 4.2, for every $\varepsilon>0$, $\lambda<M$ and $r\ge 0$, there are some real numbers $\rho_{\varepsilon,\lambda,r}\ge\rho_\varepsilon>0$ and $T_{\varepsilon,\lambda,r}>0$ such that, if*

$$u(t_0,\cdot)\ge m+\varepsilon \text{ in } \overline{B(y_0,R)}$$

for some $(t_0,y_0)\in\mathbb{R}\times\mathbb{R}^N$ and $R\ge\rho_{\varepsilon,\lambda,r}$, then

$$u(t,\cdot)\ge\lambda \text{ in } \overline{B(y_0,R+r)} \text{ for all } t\ge t_0+T_{\varepsilon,\lambda,r}.$$

Proof. Let us fix $\varepsilon>0$, $\lambda<M$ and $r\ge 0$, and let $\rho_\varepsilon>0$ be given by Lemma 4.2. Assume by way of contradiction that the conclusion of Lemma 4.4 does not hold. Then there exist two sequences $(R_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ of positive real numbers converging to $+\infty$, and a sequence $(t_n,y_n,z_n)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$ such that

$$u(t_n,\cdot)\ge m+\varepsilon \text{ in } \overline{B(y_n,R_n)},\ z_n\in\overline{B(y_n,R_n+r)}, \text{ and } u(t_n+T_n,z_n)<\lambda, \tag{4.25}$$

for all $n\in\mathbb{N}$. Notice that Lemma 4.2 then implies that $M<+\infty$, and that $m+\varepsilon<M$ by (4.10) and (4.25).

Let now $R>0$ and $\psi\in C^2(\overline{B_R})$ be as in (4.16), let w be the solution of the Cauchy problem $w_t=\Delta w+f(w)$ in $(0,+\infty)\times\mathbb{R}^N$ with initial condition $w(0,\cdot)$ given by (4.19), and let $\varsigma\in C^1([0,+\infty))$ and $T>0$ be defined as in (4.13) and (4.17). For any $\varrho>0$, call v_ϱ the solution of $(v_\varrho)_t=\Delta v_\varrho+f(v_\varrho)$ in $(0,+\infty)\times\mathbb{R}^N$ with initial condition $v_\varrho(0,\cdot)$ defined by:

$$v_\varrho(0,x)=\begin{cases} m+\varepsilon & \text{if } x\in B_\varrho,\\ 0 & \text{if } x\in\mathbb{R}^N\setminus B_\varrho.\end{cases}$$

From standard parabolic estimates, there holds $v_\varrho(T,\cdot)\to\varsigma(T)$ $(>\psi(0))$ as $\varrho\to+\infty$ locally uniformly in $\mathbb{R}^N$ (e.g. see [31, Theorem 4.1]). Hence, there is $\varrho_0>0$ such that $v_{\varrho_0}(T,\cdot)>\psi(0)$ in $\overline{B_R}$, and then

$$v_{\varrho_0}(T,\cdot)>w(0,\cdot) \text{ in } \mathbb{R}^N.$$

Since $\lambda<M$ by assumption and since $w(t,\cdot)\to M$ as $t\to+\infty$ locally uniformly in $\mathbb{R}^N$ by (4.20), there is $T'>0$ such that

$$w(t,\cdot)\ge\lambda \text{ in } \overline{B_{r+\varrho_0}} \text{ for all } t\ge T'. \tag{4.26}$$

Notice that the parameters and functions introduced in the previous paragraph do not depend on n. Coming back to (4.25), one can assume without loss of generality that $R_n\ge\varrho_0$ for all $n\in\mathbb{N}$. Hence, by (4.25), for each $n\in\mathbb{N}$, there is a point y'_n such that

$$|z_n-y'_n|\le r+\varrho_0 \text{ and } B(y'_n,\varrho_0)\subset B(y_n,R_n), \tag{4.27}$$

and thus $u(t_n,\cdot)\ge v_{\varrho_0}(0,\cdot-y'_n)$ in $\mathbb{R}^N$. The maximum principle then yields

$$u(t_n+T,\cdot)\ge v_{\varrho_0}(T,\cdot-y'_n)>w(0,\cdot-y'_n) \text{ in } \mathbb{R}^N$$

and $u(t_n+t,\cdot)>w(t-T,\cdot-y'_n)$ in $\mathbb{R}^N$ for all $t\ge T$. For all n large enough so that $T_n\ge T+T'$, it then follows that $u(t_n+T_n,z_n)>w(T_n-T,z_n-y'_n)\ge\lambda$ by (4.26)-(4.27), contradicting the last property of (4.25).

To sum up, the existence of the sequences $(R_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$ and $(t_n, y_n, z_n)_{n\in\mathbb{N}}$ is ruled out and the proof of Lemma 4.4 is thereby complete. □

Step 2: a dichotomy. The conclusions (i)-(ii) of Theorem 1.1 on the one hand, and the conclusion (iii) on the other hand, are consequences of a fundamental dichotomy, see (4.33)-(4.34) below, obtained by comparing u to a shift of a sort of barrier function φ defined in (4.28) below. To introduce φ and this dichotomy, let us first define a few auxiliary parameters. Fix a real number $\eta > 0$ such that (4.8) holds. Since $f(m) = 0$ with $m > 0$, one has

$$0 < \eta < m.$$

Define $g(s) = -f(-s)$ for all $s \in [-m, 0]$. From assumption (1.13), the $C^1([-m, 0])$ function g satisfies (3.6) with $\alpha = -m$ and $\beta = 0$. Lemma 3.3 applied with $\nu = -\eta \in (-m, 0)$ then provides the existence of $R_1 > 0$ and of a function $\psi \in C^2(\overline{B_{R_1}})$ solving $\Delta\psi + g(\psi) = 0$ and $-m \le \psi < 0$ in $\overline{B_{R_1}}$ with $\psi = -m$ on ∂B_{R_1} and $\max_{\overline{B_{R_1}}} \psi = \psi(0) > -\eta$. In other words, the function $\varphi = -\psi \in C^2(\overline{B_{R_1}})$ solves

$$\left\{\begin{array}{rcll} \Delta\varphi + f(\varphi) & = & 0 & \text{in } \overline{B_{R_1}},\\ 0 \ <\ \varphi & \le & m & \text{in } \overline{B_{R_1}},\\ \varphi & = & m & \text{on } \partial B_{R_1},\\ \min\limits_{\overline{B_{R_1}}} \varphi \ =\ \varphi(0) & < & \eta. & \end{array}\right. \tag{4.28}$$

Furthermore, it follows from [26] that φ is radially symmetric, namely there is a $C^2([0, R_1])$ function $\widetilde{\varphi}$ such that

$$\varphi(x) = \widetilde{\varphi}(|x|) \text{ for all } x \in \overline{B_{R_1}} \tag{4.29}$$

and the Hopf lemma (or, here, the Cauchy-Lipschitz theorem) implies that

$$\delta := \widetilde{\varphi}'(R_1) > 0. \tag{4.30}$$

Since u is bounded by assumption, it follows from standard parabolic estimates that there is a positive constant M_2 such that

$$|u_{x_i,x_j}(t,x)| \le M_2 \ \text{ for all } (t,x) \in \mathbb{R}\times\mathbb{R}^N \text{ and } 1 \le i,j \le N. \tag{4.31}$$

From Lemma 4.2 applied with $\varepsilon = \delta^2/(4M_2) > 0$, there is a real number

$$R_2 = \rho_\varepsilon = \rho_{\delta^2/(4M_2)} > 0 \tag{4.32}$$

such that, if $u(t_0,\cdot) \ge m + \delta^2/(4M_2)$ in $\overline{B(y_0, R_2)}$ for some $(t_0, y_0) \in \mathbb{R}\times\mathbb{R}^N$, then $M < +\infty$ and $\max_{|x|\le\gamma t} |u(t,x) - M| \to 0$ as $t \to +\infty$ for some $\gamma > 0$. Here, by choosing M_2 large if necessary, we may assume that $\varepsilon = \delta^2/(4M_2) < M - m$.

Remember now that $x_0 \in \mathbb{R}^N$ is a center of symmetry given by (4.1) and that u is localized for $t \le 0$, in the sense of (1.3). There is then a point $x_1 \in \mathbb{R}^N$ such that

$$|x_1 - x_0| \ge R_2 + \frac{\delta}{2M_2} + R_1 \ \text{ and } \ u(t,\cdot) < \varphi(0) \text{ in } \overline{B(x_1, R_1)} \text{ for all } t \le 0.$$

We shall then compare u with $\varphi(\cdot - x_1)$ in $\overline{B(x_1, R_1)}$. First of all, owing to (4.28), one has $u(t,\cdot) < \varphi(\cdot - x_1)$ in $\overline{B(x_1, R_1)}$ for all $t \le 0$.

The aforementioned dichotomy can then be stated as follows, namely, two cases may then occur:

$$\text{either } u(t,\cdot) < \varphi(\cdot - x_1) \text{ in } \overline{B(x_1,R_1)} \text{ for all } t\in\mathbb{R}, \tag{4.33}$$

$$\begin{array}{c}\text{or there is } t_0\in\mathbb{R} \text{ such that } u(t,\cdot) < \varphi(\cdot - x_1) \text{ in } \overline{B(x_1,R_1)} \text{ for all } t<t_0\\ \text{and } u(t_0,\cdot) \le \varphi(\cdot - x_1) \text{ in } \overline{B(x_1,R_1)} \text{ with equality somewhere in } \overline{B(x_1,R_1)}.\end{array} \tag{4.34}$$

It will turn out that (4.33) will lead to the conclusions (i) or (ii) of Theorem 1.1, whereas (4.34) will lead to the spreading case (iii). We consider in Step 3 the alternative (4.33), while (4.34) will be dealt with in Steps 4 and 5.

Step 3: convergence at large times if u is uniformly localized. We assume here that (4.33) holds. Thus, $u(t,x_1) < \varphi(0) < \eta$ for all $t\in\mathbb{R}$ and property (4.1) implies that $u(t,x) < \varphi(0) < \eta$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ with $|x-x_0|\ge|x_1-x_0|$. The arguments used in Remark 1.5 then yield

$$u(t,x)\to 0 \text{ as } |x|\to+\infty \text{ uniformly in } t\in\mathbb{R},$$

that is, u is uniformly localized. From [9, Theorem 1.1] (see also [22]) and standard parabolic estimates, it follows that either $u(t,\cdot)\to 0$ as $t\to+\infty$ in $H^1(\mathbb{R}^N)\cap C^2(\mathbb{R}^N)$ (that is, the alternative (i) holds in Theorem 1.1), or there is positive steady state $\phi\in\mathcal{E}$ solving (1.4) such that $u(t,\cdot)\to\phi$ as $t\to+\infty$ in $H^1(\mathbb{R}^N)\cap C^2(\mathbb{R}^N)$ (that is, the alternative (ii) holds in Theorem 1.1). Notice that, in the former case, the action $E[u(t,\cdot)]$ defined by (4.5) satisfies $E[u(t,\cdot)]\to E[0]=0$ as $t\to+\infty$, while in the latter case,

$$E[u(t,\cdot)]\to E[\phi] \text{ as } t\to+\infty. \tag{4.35}$$

In all cases, the function $t\mapsto E[u(t,\cdot)]$ is then bounded in $\mathbb{R}$.

Step 4: the transition is radially bounded if u spreads, proof of (1.15). We assume in the sequel that (4.34) holds. We shall see that this case leads to the alternative (iii) of the conclusion of Theorem 1.1. We prove the property (1.15) in the present Step 4, and property (1.16) in Step 5. The proof of (1.15) is based on the maximum principle and on suitable estimates on the oscillations of the radial positions of the level sets of u at large time, as well as on the key-lemmas of Step 1.

First of all, since φ solves (4.28), the alternative (4.34) and the parabolic strong maximum principle imply that there is a point $x_2\in\partial B(x_1,R_1)$ such that

$$u(t_0,x_2)=\varphi(x_2-x_1)=m.$$

Because of (4.1), (4.34) and of the inequality $|x_1-x_0|\ge R_2+\delta/(2M_2)+R_1>R_1$, together with the fact that $\varphi<m$ in B_{R_1} (from the elliptic strong maximum principle), it turns out that x_2 is the unique point lying at the intersection of the sphere $\partial B(x_1,R_1)$ and the segment $[x_0,x_1]$. In particular,

$$|x_2-x_0|=|x_1-x_0|-R_1\ge R_2+\frac{\delta}{2M_2}.$$

Furthermore, from (4.34) and the definitions of $\widetilde{\varphi}$ and δ satisfying (4.29)-(4.30), it follows that

$$-|\nabla u(t_0,x_2)| = \nabla u(t_0,x_2)\cdot\frac{x_2-x_0}{|x_2-x_0|} \le -\delta < 0$$

(we also recall that u is radially symmetric and decreasing with respect to the point x_0). Together with (4.31) and (4.1) again, one infers that

$$\nabla u(t_0,x)\cdot\frac{x-x_0}{|x-x_0|} \le -\frac{\delta}{2} < 0 \ \text{ for all } x \text{ such that } |x_2-x_0| - \frac{\delta}{2M_2} \le |x-x_0| \le |x_2-x_0|.$$

Since $u(t_0,\cdot)=m$ at the point x_2 and then on $\partial B(x_0,|x_2-x_0|)$, it follows that

$$u(t_0,\cdot) \ge m + \frac{\delta^2}{4M_2} \text{ on } \partial B\big(x_0,|x_2-x_0|-\frac{\delta}{2M_2}\big), \text{ with } |x_2-x_0|-\frac{\delta}{2M_2} \ge R_2,$$

hence

$$u(t_0,\cdot) \ge m + \frac{\delta^2}{4M_2} \text{ in } \overline{B(x_0,R_2)} \tag{4.36}$$

by (4.1) again. Lemma 4.2 and the definition (4.32) of R_2 then imply that

$$M < +\infty$$

and

$$\max_{|x|\le\gamma t} |u(t,x)-M| \to 0 \text{ as } t\to+\infty \tag{4.37}$$

for some $\gamma > 0$.

Secondly, (4.1) together with (4.3) and (4.37) yield the existence of a real number τ_1 such that

$$\text{for each } t \ge \tau_1, \begin{cases} \max_{\mathbb{R}^N} u(t,\cdot) = u(t,x_0) > m \\ \text{there is a unique } \xi(t) > 0 \text{ such that } u(t,\cdot) = m \text{ on } \partial B(x_0,\xi(t)), \end{cases} \tag{4.38}$$

and $\liminf_{t\to+\infty}\xi(t)/t \ge \gamma > 0$. In particular,

$$\lim_{t\to+\infty}\xi(t) = +\infty. \tag{4.39}$$

The implicit function theorem with (4.1) implies that the function $t\mapsto\xi(t)$ is of class $C^1([\tau_1,+\infty))$. Define also $\xi(t)=\xi(\tau_1)$ for all $t<\tau_1$. The function ξ is then continuous in $\mathbb{R}$. We shall show in this Step 4 that (1.15) holds with this function ξ. To do so, we first prove in the following two lemmas some key-properties on the local oscillations of the function ξ.

Lemma 4.5. *There is a positive constant τ_2 such that $\xi(t+s) > \xi(t)$ for all $t \ge \tau_1$ and $s \ge \tau_2$.*

Proof. By (1.3) and (4.3), there is a real number $R_3 > 0$ such that

$$u(t,x) < \varphi(0) \text{ for all } t \le \tau_1 \text{ and } x \text{ such that } |x-x_0| \ge R_3. \tag{4.40}$$

With $\varepsilon=\delta^2/(4M_2)>0$, $\lambda=(m+M)/2<M$ and $r=\delta/(2M_2)\geq 0$, denote, using the notations of Lemma 4.4 and the definition (4.32) of R_2,

$$R_4=\max\Big(\rho_{\varepsilon,\lambda,r}+\frac{\delta}{2M_2},R_3,\xi(\tau_1)+1\Big)>0 \tag{4.41}$$

and

$$\tau_2=T_{\varepsilon,\lambda,r}>0.$$

Let also $\tau_3\in\mathbb{R}$ be such that

$$\xi(t)\geq R_4\ \text{ for all } t\geq\tau_3$$

(hence, $\tau_3>\tau_1$, since $R_4>\xi(\tau_1)$).

Consider now any $t\geq\tau_3$ and $s\geq\tau_2$ and let us show that $\xi(t+s)>\xi(t)$. Let $x_3\in\mathbb{R}^N$ be such that

$$|x_3-x_0|=\xi(t)+R_1,$$

where $R_1>0$ is given in (4.28). Thus, $|x_3-x_0|\geq R_4+R_1\geq R_3+R_1$, hence $u(t',\cdot)<\varphi(0)\leq\varphi(\cdot-x_3)$ in $\overline{B(x_3,R_1)}$ for all $t'\leq\tau_1$ by (4.40). Observe that $\xi(t')=\xi(\tau_1)<R_4\leq\xi(t)$ for all $t'\leq\tau_1$, and, by continuity of ξ, denote

$$t^*=\min\big\{t'\in(-\infty,t]:\xi(t')=\xi(t)\big\}\in(\tau_1,t].$$

Let x_4 be the intersection point of the segment $[x_0,x_3]$ with $\partial B(x_3,R_1)$. One has

$$|x_4-x_0|=|x_3-x_0|-R_1=\xi(t)=\xi(t^*),$$

hence $u(t^*,x_4)=m=\varphi(x_4-x_3)$. Furthermore, $u(t',\cdot)<\varphi(0)\leq\varphi(\cdot-x_3)$ in $\overline{B(x_3,R_1)}$ for all $t'\leq\tau_1$ by (4.28) and (4.40), while

$$u(t',\cdot)\leq m=\varphi(\cdot-x_3)\text{ on }\partial B(x_3,R_1)\text{ for all }t'\in[\tau_1,t^*]$$

by (4.1) and the definition of t^* (and even $u(t',\cdot)<m$ on $\partial B(x_3,R_1)$ for all $t'\in[\tau_1,t^*)$). It then follows from the maximum principle that

$$u(t',\cdot)\leq\varphi(\cdot-x_3)\text{ in }\overline{B(x_3,R_1)}\text{ for all }t'\in[\tau_1,t^*]$$

(actually with strict inequality for $t'\in[\tau_1,t^*)$ and even for $t'\in(-\infty,t^*)$). In particular, $u(t^*,\cdot)\leq\varphi(\cdot-x_3)$ in $\overline{B(x_3,R_1)}$ and since $x_4\in\partial B(x_3,R_1)$ with $|x_4-x_0|=\xi(t^*)$ and $t^*\geq\tau_1$, one has $u(t^*,x_4)=m=\varphi(x_4-x_3)$. Therefore,

$$-|\nabla u(t^*,x_4)|=\nabla u(t^*,x_4)\cdot\frac{x_4-x_0}{|x_4-x_0|}\leq-\delta$$

owing to the definition of δ in (4.29)-(4.30). Hence, as in the proof of (4.36), one infers that

$$u(t^*,\cdot)\geq m+\frac{\delta^2}{4M_2}=m+\varepsilon\text{ in }\overline{B\big(x_0,|x_4-x_0|-\delta/(2M_2)\big)}, \tag{4.42}$$

with $|x_4-x_0|-\delta/(2M_2)=\xi(t)-\delta/(2M_2)\geq R_4-\delta/(2M_2)\geq\rho_{\varepsilon,\lambda,r}$ by (4.41). Lemma 4.4 then yields

$$u(t',\cdot)\geq\lambda=\frac{m+M}{2}\text{ in }\overline{B\big(x_0,|x_4-x_0|-\delta/(2M_2)+r\big)}=\overline{B(x_0,|x_4-x_0|)}=\overline{B(x_0,\xi(t))}$$

for all $t' \geq t^* + T_{\varepsilon,\lambda,r} = t^* + \tau_2$. Since $t + s \geq t^* + \tau_2$, one has $u(t+s,\cdot) \geq (m+M)/2 > m$ in $\overline{B(x_0,\xi(t))}$ and the definition of $\xi(t+s)$ together with (4.1) and $t + s > t^* > \tau_1$ finally yields $\xi(t+s) > \xi(t)$.

As a consequence, $\xi(t+s) > \xi(t)$ for all $t \geq \tau_3$ and $s \geq \tau_2$. Since ξ is continuous in $\mathbb{R}$ and $\xi(t) \to +\infty$ as $t \to +\infty$, the conclusion of Lemma 4.5 follows, even if it means increasing τ_2 if necessary. □

Lemma 4.6. *For each $\tau > 0$, there is a positive constant A_τ such that $\xi(t+s) \leq \xi(t) + A_\tau$ for all $t \in \mathbb{R}$ and $s \in [0,\tau]$.*

Proof. Assume that the conclusion does not hold. Then there are $\tau > 0$ and some sequences $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $(s_n)_{n\in\mathbb{N}}$ in $[0,\tau]$ such that $\xi(t_n+s_n) > \xi(t_n)+n$. Since ξ is continuous in $\mathbb{R}$ and constant in $(-\infty,\tau_1]$, it follows that $t_n \to +\infty$ as $n \to +\infty$, hence $\xi(t_n) \to +\infty$ as $n \to +\infty$ by (4.39). Without loss of generality, one can assume that, for every $n \in \mathbb{N}$,

$$t_n \geq \tau_1 + \tau_2 \quad \text{and} \quad \xi(t_n) \geq \max(R_3, \xi(\tau_1) + 1),$$

where $\tau_1 \in \mathbb{R}$ and $\tau_2 > 0$ are given in (4.38) and in Lemma 4.5, and $R_3 > 0$ is given in (4.40).

Now, for every $n \in \mathbb{N}$, Lemma 4.5 yields the existence of $t_n^* \in (t_n - \tau_2, t_n]$ ($\subset (\tau_1, t_n]$) such that $\xi(t_n^*) = \xi(t_n)$ and $\xi(t) < \xi(t_n^*) = \xi(t_n)$ for all $t < t_n^*$. Let $y_n \in \mathbb{R}^N$ be such that

$$|y_n - x_0| = \xi(t_n^*) + R_1 = \xi(t_n) + R_1 \; (\geq R_3 + R_1).$$

Since $u(t,\cdot) < \varphi(0) \leq \varphi(\cdot - y_n)$ in $\overline{B(y_n,R_1)}$ for all $t \leq \tau_1$ by (4.40), and since $u(t,\cdot) < m = \varphi(\cdot - y_n)$ on $\partial B(y_n,R_1)$ for all $t \in [\tau_1, t_n^*)$ by (4.1) and definition of t_n^*, the maximum principle implies that

$$u(t_n^*,\cdot) \leq \varphi(\cdot - y_n) \text{ in } \overline{B(y_n,R_1)}. \tag{4.43}$$

In particular, $u(t_n^*, y_n) \leq \varphi(0)$ and

$$u(t_n^*, x) \leq \varphi(0) \text{ for all } x \text{ such that } |x - x_0| \geq |y_n - x_0| = \xi(t_n) + R_1, \tag{4.44}$$

by (4.1).

On the other hand, for every $n \in \mathbb{N}$, one has $t_n + s_n \geq t_n \geq \tau_1 + \tau_2 > \tau_1$, and there is a point z_n such that $|z_n - x_0| = \xi(t_n + s_n)$, hence $u(t_n + s_n, z_n) = m$. Notice also that $t_n - t_n^* + s_n \in [0, \tau_2 + \tau)$ for each $n \in \mathbb{N}$. Up to extraction of a subsequence, one can assume without loss of generality that $t_n - t_n^* + s_n \to s_\infty \in [0,\tau_2+\tau]$ as $n \to +\infty$ and that the functions

$$u_n : (t,x) \mapsto u_n(t,x) = u(t + t_n^*, x + z_n)$$

converge in $C^{1,2}_{t,x}$ locally in $\mathbb{R}\times\mathbb{R}^N$ to a bounded nonnegative solution u_∞ of (1.1) such that $u_\infty(s_\infty, 0) = m$. Furthermore, for each $x \in \mathbb{R}^N$, there holds

$$|x + z_n - x_0| \geq |z_n - x_0| - |x| = \xi(t_n + s_n) - |x| > \xi(t_n) + n - |x|,$$

hence $|x + z_n - x_0| \geq \xi(t_n) + R_1$ for all n large enough and $u_n(0,x) = u(t_n^*, x + z_n) \leq \varphi(0)$ by (4.44). As a consequence, $u_\infty(0,x) \leq \varphi(0)$ for all $x \in \mathbb{R}^N$. Since $0 < \varphi(0) < \eta$ by (4.28) and $f < 0$ in $(0,\eta]$ by (4.8), it follows from the maximum principle that $u_\infty \leq \varphi(0) < \eta$ in $[0,+\infty)\times\mathbb{R}^N$. In particular, $u_\infty(s_\infty,0) < \eta$, which is impossible since $u_\infty(s_\infty,0) = m$ and $m > \eta$ (remember that $f(m) = 0$ and $m > 0$). One has then reached a contradiction, and the proof of Lemma 4.6 is thereby complete. □

With Lemmas 4.5 and 4.6 in hand, we can now complete the proof of (1.15). Let us begin with the first statement in (1.15). Assume by way of contradiction that it does not hold. Then, thanks to (4.10), there are $M' \in (0, M)$ and some sequences $(t_n)_{n\in\mathbb{N}}$ converging to $+\infty$ and $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$0 < u(t_n, x_n) \le M' < M \text{ for all } n \in \mathbb{N}, \quad \text{and } |x_n| - \xi(t_n) \to -\infty \text{ as } n \to +\infty. \tag{4.45}$$

Let $\tau > 0$ be an arbitrary positive real number, and let $R_1 > 0$ be given as in (4.28). Consider in this paragraph the indices n large enough so that $t_n - \tau \ge \tau_1 + \tau_2$ for every $n \in \mathbb{N}$, where $\tau_1 \in \mathbb{R}$ and $\tau_2 > 0$ are given in (4.38) and Lemma 4.5, and

$$\xi(t_n - \tau) \ge \max\big(R_3, \xi(\tau_1) + 1, \frac{\delta}{2M_2} + 1\big),$$

where $R_3 > 0$ is given in (4.40), $\delta > 0$ in (4.29)-(4.30) and $M_2 > 0$ in (4.31). Notice that the quantities τ, τ_1, τ_2, R_1, R_3, δ and M_2 are independent of n. Now, for each n large enough, Lemma 4.5 yields the existence of $t_n^* \in (t_n - \tau - \tau_2, t_n - \tau]\,(\subset (\tau_1, t_n - \tau])$ such that $\xi(t_n^*) = \xi(t_n - \tau)$ and $\xi(t) < \xi(t_n^*) = \xi(t_n - \tau)$ for all $t < t_n^*$. Let $y_n \in \mathbb{R}^N$ be such that

$$|y_n - x_0| = \xi(t_n^*) + R_1 = \xi(t_n - \tau) + R_1\,(\ge R_3 + R_1)$$

and z_n be the intersection point of $[x_0, y_n]$ with $\partial B(y_n, R_1)$ such that

$$|z_n - x_0| = |y_n - x_0| - R_1 = \xi(t_n - \tau) = \xi(t_n^*), \quad u(t_n^*, z_n) = m$$

by (4.38). As in the proof of (4.43), there holds $u(t_n^*, \cdot) \le \varphi(\cdot - y_n)$ in $\overline{B(y_n, R_1)}$ with $u(t_n^*, z_n) = m = \varphi(z_n - y_n)$. Therefore,

$$-|\nabla u(t_n^*, z_n)| = \nabla u(t_n^*, z_n) \cdot \frac{z_n - x_0}{|z_n - x_0|} \le -\delta,$$

with $\delta > 0$ given by (4.29)-(4.30). Hence, as in the proof of (4.42), one infers that

$$u(t_n^*, \cdot) \ge m + \frac{\delta^2}{4M_2} \quad \text{in } \overline{B\big(x_0, |z_n - x_0| - \delta/(2M_2)\big)} = \overline{B\big(x_0, \xi(t_n - \tau) - \delta/(2M_2)\big)}, \tag{4.46}$$

with $|z_n - x_0| - \delta/(2M_2) = \xi(t_n - \tau) - \delta/(2M_2) > 0$. Together with (4.10), this implies in particular that $m < m + \delta^2/(4M_2) < M$. Notice also that $\tau \le t_n - t_n^* < \tau + \tau_2$ for each n (large enough). Up to extraction of a subsequence, one has $t_n - t_n^* \to s_\infty \in [\tau, \tau + \tau_2]$ as $n \to +\infty$ and the functions

$$u_n : (t, x) \mapsto u_n(t, x) = u(t + t_n^*, x + x_n)$$

converge in $C^{1,2}_{t,x}$ locally in $\mathbb{R} \times \mathbb{R}^N$ to a bounded nonnegative solution u_∞ of (1.1) such that

$$u_\infty(s_\infty, 0) \le M' < M$$

by (4.45). Furthermore, for each $x \in \mathbb{R}^N$, one has

$$|x + x_n - x_0| \le \xi(t_n) + |x_n| - \xi(t_n) + |x - x_0| \le \xi(t_n - \tau) + A_\tau + |x_n| - \xi(t_n) + |x - x_0|$$

from Lemma 4.6, where $A_\tau > 0$ is given in Lemma 4.6, hence $|x+x_n-x_0| \le \xi(t_n-\tau)-\delta/(2M_2)$ for all n large enough, from the second statement of (4.45). As a consequence, $u_n(0, x) =$

$u(t_n^*, x+x_n) \ge m+\delta^2/(4M_2)$ for all n large enough, by (4.46). Thus, $u_\infty(0,\cdot) \ge m+\delta^2/(4M_2)$ in $\mathbb{R}^N$ and $u_\infty(t,\cdot) \ge \varpi(t)$ in $\mathbb{R}^N$ for all $t \ge 0$, where ϖ obeys

$$\begin{cases} \varpi'(t) = f(\varpi(t)) & \text{for } t \ge 0,\\ \varpi(0) = m + \dfrac{\delta^2}{4M_2} > m. \end{cases}$$

Since $m < m + \delta^2/(4M_2) < M$ and $f > 0$ in (m, M) with $f(M) = 0$, the function ϖ is increasing in $[0, +\infty)$ and $\varpi(+\infty) = M$. The inequality $u_\infty(t,\cdot) \ge \varpi(t)$ applied at $t = s_\infty \ge \tau > 0$ and $x = 0$ yields $u_\infty(s_\infty, 0) \ge \varpi(s_\infty) \ge \varpi(\tau)$, hence $M > M' \ge u_\infty(s_\infty, 0) \ge \varpi(\tau)$. Since M' is given in (4.45) independently of $\tau > 0$ and $\tau > 0$ can be arbitrarily large, one infers that $M > M' \ge \varpi(+\infty) = M$, a contradiction. As a consequence, the first line in (1.15) has been proved.

Let us now show the second statement in (1.15). Assume by way of contradiction that it does not hold. Then, thanks to (4.10), there are $\kappa \in (0, M)$ and some sequences $(t_n)_{n\in\mathbb{N}}$ converging to $+\infty$ and $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$0 < \kappa \le u(t_n, x_n) < M \text{ for all } n \in \mathbb{N}, \quad \text{and } |x_n| - \xi(t_n) \to +\infty \text{ as } n \to +\infty. \tag{4.47}$$

Let $\sigma > 0$ be an arbitrary positive real number such that

$$\sigma \ge \tau_2,$$

where $\tau_2 > 0$ is given in Lemma 4.5, and let $R_1 > 0$ be given as in (4.28). Consider in this paragraph the indices n large enough so that $t_n - \sigma \ge \tau_1 + \tau_2$ for every $n \in \mathbb{N}$, where $\tau_1 \in \mathbb{R}$ is given in (4.38), and

$$\xi(t_n - \sigma) \ge \max(R_3, \xi(\tau_1) + 1),$$

where $R_3 > 0$ is given in (4.40). For each n large enough, Lemma 4.5 yields the existence of $t_n^* \in (t_n-\sigma-\tau_2, t_n-\sigma]$ ($\subset (\tau_1, t_n-\sigma]$) such that $\xi(t_n^*) = \xi(t_n-\sigma)$ and $\xi(t) < \xi(t_n^*) = \xi(t_n-\sigma)$ for all $t < t_n^*$. Let $y_n \in \mathbb{R}^N$ satisfy

$$|y_n - x_0| = \xi(t_n^*) + R_1 = \xi(t_n - \sigma) + R_1 \,(\ge R_3 + R_1).$$

As for (4.43), one then has $u(t_n^*, \cdot) \le \varphi(\cdot - y_n)$ in $\overline{B(y_n, R_1)}$. In particular, $u(t_n^*, y_n) \le \varphi(0)$ and, from (4.1),

$$u(t_n^*, x) \le \varphi(0) \text{ for all } x \text{ such that } |x - x_0| \ge |y_n - x_0| = \xi(t_n - \sigma) + R_1. \tag{4.48}$$

Notice also that $\sigma \le t_n - t_n^* < \sigma + \tau_2$ for each n (large enough). Up to extraction of a subsequence, one has $t_n - t_n^* \to t_\infty \in [\sigma, \sigma + \tau_2]$ as $n \to +\infty$ and the functions

$$u_n : (t, x) \mapsto u_n(t, x) = u(t + t_n^*, x + x_n)$$

converge in $C^{1,2}_{t,x}$ locally in $\mathbb{R} \times \mathbb{R}^N$ to a bounded nonnegative solution u_∞ of (1.1) such that

$$0 < \kappa \le u_\infty(t_\infty, 0)$$

by (4.47). Furthermore, for each $x \in \mathbb{R}^N$, one has

$$|x + x_n - x_0| \ge |x_n| - \xi(t_n) + \xi(t_n) - |x - x_0| > |x_n| - \xi(t_n) + \xi(t_n - \sigma) - |x - x_0|$$

from Lemma 4.5, since $t_n - \sigma \ge \tau_1$ and $\sigma \ge \tau_2$. Hence $|x + x_n - x_0| \ge \xi(t_n - \sigma) + R_1$ for all n large enough, from (4.47). As a consequence, $u_n(0, x) = u(t_n^*, x + x_n) \le \varphi(0)$ for all n

large enough, by (4.48). Thus, $u_\infty(0,\cdot) \le \varphi(0)$ in $\mathbb{R}^N$ and $u_\infty(t,\cdot) \le \vartheta(t)$ in $\mathbb{R}^N$ for all $t \ge 0$, where ϑ obeys

$$\begin{cases} \vartheta'(t) = f(\vartheta(t)) & \text{for } t \ge 0, \\ \vartheta(0) = \varphi(0) \in (0,\eta). \end{cases}$$

Since $f < 0$ in $(0,\eta)$ with $f(0) = 0$, the function ϑ is decreasing in $[0,+\infty)$ and $\vartheta(+\infty) = 0$. The inequality $u_\infty(t,\cdot) \le \vartheta(t)$ applied at $t = t_\infty \ge \sigma > 0$ and $x = 0$ yields $u_\infty(t_\infty, 0) \le \vartheta(t_\infty) \le \vartheta(\sigma)$, hence $0 < \kappa \le \vartheta(\sigma)$. Since κ is given in (4.47) independently of σ and since $\sigma \ge \tau_2$ can be arbitrarily large, one infers that $0 < \kappa \le \vartheta(+\infty) = 0$, a contradiction. As a conclusion, the proof of (1.15) is thereby complete.

Remark 4.7. The quantities $\xi(t)$ given in (4.38) (for $t \ge \tau_1$) are the radial positions (with respect to the point x_0) of the level sets with level m. Now, for any level λ in $(0, M)$, there is a unique real number $\xi_\lambda(t) > 0$ such that $u(t,x) = \lambda$ if and only if $|x - x_0| = \xi_\lambda(t)$, for all t large enough. One then infers from (1.15) that

$$\limsup_{t\to+\infty} |\xi_\lambda(t) - \xi(t)| < +\infty.$$

Furthermore, it also follows from (4.1) that, for any unit vector e and any sequence $(t_n)_{n\in\mathbb{N}}$ converging to $+\infty$, the functions

$$u_n : (t,x) \mapsto u_n(t,x) = u(t + t_n, x + \xi_\lambda(t_n)e)$$

converge in $C^{1,2}_{t,x}$ locally in $\mathbb{R} \times \mathbb{R}^N$, up to extraction of a subsequence, to a bounded nonnegative solution u_∞ of (1.1) which only depends on t and the variable $x \cdot e$ and is nonincreasing in the direction e. Moreover, (1.15) implies that $u_\infty(0,x) \to M$ as $x \cdot e \to -\infty$ and $u_\infty(0,x) \to 0$ as $x \cdot e \to +\infty$. In particular, the nonpositive function $e \cdot \nabla u_\infty$ can not be identically 0 in $(-\infty, 0] \times \mathbb{R}^N$ and the strong parabolic maximum principle then yields $e \cdot \nabla u_\infty(0,0) < 0$. From the arbitrariness of the sequence $(t_n)_{n\in\mathbb{N}}$ converging to $+\infty$ and from (4.1), we conclude that

$$\liminf_{t\to+\infty,\, u(t,x)=\lambda} |\nabla u(t,x)| > 0, \ \text{that is,} \ \limsup_{t\to+\infty,\, u(t,x)=\lambda} \nabla u(t,x) \cdot \frac{x - x_0}{|x - x_0|} < 0,$$

for any level $\lambda \in (0, M)$. In other words, the radial derivatives of the function u do not degenerate at large times along any level set of u.

Step 5: asymptotic position of the level sets if u spreads, proof of (1.16). We still assume that (4.34) holds, hence $M < +\infty$ and (1.15) holds, together with (4.37). We shall show here that $\xi(t)/t$ has a well determined limit as $t \to +\infty$. Such a property has been well known since the seminal paper [3], under some additional assumptions on the function f. The proof given in [3] was based on some comparison arguments and on the existence of approximated fronts defined in bounded intervals or in half-lines. The proof of (1.16) given here is still based on comparison arguments with suitable sub- and supersolutions, but the approximated fronts moving at speeds arbitrarily close to c that are here used are defined in the whole real line and are given in Lemma 3.1.

First of all, as already emphasized, one has $0 < m < M$, $f(0) = f(M) = 0$, $f'(0) < 0$, $f > 0$ in (m, M), $F < 0$ in $(0, m]$ and $F(M) > 0$. Lemma 3.1 then yields the existence and uniqueness of a pair (c, φ) solving (1.17), with $c > 0$.

Consider now any $c' \in (0,c)$, and let us show that $\liminf_{t\to+\infty}\xi(t)/t \ge c'$. From Remark 3.2, there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,m)$ converging to 0 such that, for each $n\in\mathbb{N}$, there is a $C^1([-\varepsilon_n, M-\varepsilon_n])$ function $\underline{f}_n$ such that $\underline{f}_n(-\varepsilon_n)=\underline{f}_n(M-\varepsilon_n)=0$, $\underline{f}_n\le f$ in $[0,M-\varepsilon_n]$, and there is a unique pair $(c_n,\varphi_n)\in\mathbb{R}\times C^2(\mathbb{R})$ solving

$$\varphi_n''+c_n\varphi_n'+\underline{f}_n(\varphi_n)=0 \text{ in } \mathbb{R},\ \varphi_n'<0 \text{ in } \mathbb{R},\ \varphi_n(-\infty)=M-\varepsilon_n,\ \varphi_n(+\infty)=-\varepsilon_n. \tag{4.49}$$

Furthermore, $c_n<c$ and $c_n\to c$ as $n\to+\infty$. Fix $\varepsilon>0$ arbitrary, and then n large enough such that

$$0<\varepsilon_n\le\varepsilon \text{ and } c'<c_n<c.$$

Let then $\rho>0$ large enough such that

$$\frac{N-1}{\rho}<\frac{c_n-c'}{2}$$

and denote $c_n'=(c'+c_n)/2\in(c',c_n)\subset(c',c)$. Since $u(t,\cdot)\to M$ as $t\to+\infty$ locally uniformly in $\mathbb{R}^N$ by (4.37), there is a time $T>0$ such that

$$u(t,x)\ge M-\varepsilon_n \ \text{ for all } t\ge T \text{ and } |x-x_0|\le\rho. \tag{4.50}$$

Let then $A>0$ be such that $\varphi_n(r-c_n'T+A)<0$ for all $r\ge\rho$ (that is possible since $\varphi_n(+\infty)=-\varepsilon_n<0$). Let us finally define

$$\underline{u}(t,x):=\max\big(\varphi_n(|x-x_0|-c_n't+A),0\big)$$

and show that this function is a generalized subsolution of (1.1) for $t\ge T$ and $|x-x_0|\ge\rho$. First of all, at time $t=T$, for all $|x-x_0|\ge\rho$, one has $\varphi_n(|x-x_0|-c_n'T+A)<0$, hence $\underline{u}(T,x)=0<u(T,x)$. Furthermore, for all $t\ge T$ and $|x-x_0|=\rho$, one has $\underline{u}(t,x)<M-\varepsilon_n\le u(t,x)$. Since $f(0)=0$, it just remains to show that, for any (t,x) such that $t>T$ and $|x-x_0|>\rho$ with $\underline{u}(t,x)>0$, then $\underline{u}_t(t,x)\le\Delta\underline{u}(t,x)+f(\underline{u}(t,x))$. Pick any such (t,x) and notice that $\underline{u}(t,x)=\varphi_n(|x-x_0|-c_n't+A)\in(0,M-\varepsilon_n)$ in a neighborhood of (t,x). Hence, having (4.49) in mind, it follows that

$$\begin{aligned}
&\underline{u}_t(t,x)-\Delta\underline{u}(t,x)-f(\underline{u}(t,x))\\
&\qquad=-c_n'\varphi_n'(|x-x_0|-c_n't+A)-\varphi_n''(|x-x_0|-c_n't+A)\\
&\qquad\quad-\frac{N-1}{|x-x_0|}\varphi_n'(|x-x_0|-c_n't+A)-f(\varphi_n(|x-x_0|-c_n't+A))\\
&\qquad\le\Big(c_n-c_n'-\frac{N-1}{|x-x_0|}\Big)\varphi_n'(|x-x_0|-c_n't+A)\\
&\qquad<0
\end{aligned}$$

since $\underline{f}_n\le f$ in $[0,M-\varepsilon_n]$, $(N-1)/|x-x_0|\le(N-1)/\rho<(c_n-c')/2=c_n-c_n'$, and $\varphi_n'<0$ in $\mathbb{R}$. The maximum principle then implies that

$$u(t,x)\ge\underline{u}(t,x)\ge\varphi_n(|x-x_0|-c_n't+A) \ \text{ for all } t\ge T \text{ and } |x-x_0|\ge\rho.$$

Therefore, together with (4.50), one gets that, for all $t\ge T$,

$$\min_{|x-x_0|\le c't}u(t,x)\ge\varphi_n(c't-c_n't+A)\to M-\varepsilon_n \text{ as } t\to+\infty,$$

since $c' < c'_n$ and $\varphi_n(-\infty) = M - \varepsilon_n$. Together with the inequality $0 < u < M$ in $\mathbb{R} \times \mathbb{R}^N$, one infers that $\limsup_{t\to+\infty} \max_{|x-x_0|\le c't} |u(t,x) - M| \le \varepsilon_n \le \varepsilon$. Since $\varepsilon > 0$ and $c' \in (0,c)$ were arbitrary, one concludes from the definition (4.38) of $\xi(t)$ that

$$\liminf_{t\to+\infty} \frac{\xi(t)}{t} \ge c. \tag{4.51}$$

For the converse inequality, consider any $c'' > c$ and let us show that $\limsup_{t\to+\infty} \xi(t)/t \le c''$. From the proof of Lemma 3.1, there is a sequence $(\eta_k)_{k\in\mathbb{N}}$ in $(0,m)$ converging to 0 such that, for each $k \in \mathbb{N}$, there is a $C^1([\eta_k, M+\eta_k])$ function $\overline{f}_k$ such that $\overline{f}_k(\eta_k) = \overline{f}_k(M+\eta_k) = 0$, $\overline{f}_k \ge f$ in $[\eta_k, M]$, and there is a unique pair $(\gamma_k, \phi_k) \in \mathbb{R} \times C^2(\mathbb{R})$ solving

$$\phi''_k + \gamma_k\phi'_k + \overline{f}_k(\phi_k) = 0 \text{ in } \mathbb{R},\ \phi'_k < 0 \text{ in } \mathbb{R},\ \phi_k(-\infty) = M + \eta_k,\ \phi_k(+\infty) = \eta_k. \tag{4.52}$$

Furthermore, $\gamma_k > c$ and $\gamma_k \to c$ as $k \to +\infty$. Fix $\varepsilon > 0$ arbitrary, and then k large enough such that

$$0 < \eta_k \le \varepsilon \text{ and } c < \gamma_k < c''.$$

Since $0 < u < M$ in $\mathbb{R} \times \mathbb{R}^N$ and $u(0,x) \to 0$ as $|x| \to +\infty$ by (1.3), and since $\phi_k(-\infty) = M + \eta_k > M$ and $\phi_k > \phi_k(+\infty) = \eta_k > 0$, there is $A' > 0$ such that $\phi_k(1 - A') > M$ and $u(0,x) < \phi_k(|x-x_0| - A')$ for all $x \in \mathbb{R}^N$. Let us finally define

$$\overline{u}(t,x) := \min\big(\phi_k(|x-x_0| - \gamma_k t - A'), M\big)$$

and show that this function is a generalized supersolution of (1.1) for $t \ge 0$ and $|x-x_0| \ge 1$. First of all, at time $t = 0$, one has $u(0,x) < \overline{u}(0,x)$ for all $|x-x_0| \ge 1$ (and even for all $x \in \mathbb{R}^N$, by construction). Furthermore, for all $t \ge 0$ and $|x-x_0| = 1$, one has $\gamma_k t \ge 0$ and $\phi_k(|x-x_0| - \gamma_k t - A') \ge \phi_k(1-A') > M$, hence $\overline{u}(t,x) = M > u(t,x)$. Since $f(M) = 0$, it just remains to show that, for any (t,x) such that $t > 0$ and $|x-x_0| > 1$ with $\overline{u}(t,x) < M$, then $\overline{u}_t(t,x) \ge \Delta\overline{u}(t,x) + f(\overline{u}(t,x))$. Pick any such (t,x) and notice that $\overline{u}(t,x) = \phi_k(|x-x_0| - \gamma_k t - A') \in (\eta_k, M)$ in a neighborhood of (t,x). Hence, having (4.52) in mind, it follows that

$$\begin{aligned}
\overline{u}_t(t,x) &- \Delta\overline{u}(t,x) - f(\overline{u}(t,x)) \\
&= -\gamma_k\phi'_k(|x-x_0| - \gamma_k t - A') - \phi''_k(|x-x_0| - \gamma_k t - A') \\
&\quad - \frac{N-1}{|x-x_0|}\phi'_k(|x-x_0| - \gamma_k t - A') - f(\phi_k(|x-x_0| - \gamma_k t - A')) \\
&\ge -\frac{N-1}{|x-x_0|}\phi'_k(|x-x_0| - \gamma_k t - A') \\
&\ge 0
\end{aligned}$$

since $\overline{f}_k \ge f$ in $[\eta_k, M]$ and $\phi'_k < 0$ in $\mathbb{R}$. The maximum principle then implies that

$$u(t,x) \le \overline{u}(t,x) \le \phi_k(|x-x_0| - \gamma_k t - A') \ \text{ for all } t \ge 0 \text{ and } |x-x_0| \ge 1.$$

Therefore, for all $t \ge 1/c''$, one has $\max_{|x-x_0|\ge c''t} u(t,x) \le \phi_k(c''t - \gamma_k t - A') \to \eta_k$ as $t \to +\infty$, since $c'' > \gamma_k$ and $\phi_k(+\infty) = \eta_k$. One then infers that

$$\limsup_{t\to+\infty} \max_{|x-x_0|\ge c''t} u(t,x) \le \eta_k \le \varepsilon.$$

Since $\varepsilon > 0$ and $c'' > c$ were arbitrary, one concludes from the definition (4.38) of $\xi(t)$ that $\limsup_{t\to+\infty} \xi(t)/t \le c$. Together with (4.51), the inequality (1.16) follows. The proof of Theorem 1.1 is thereby complete. □

5. Proof of Corollaries 2.1 and 2.3

As already emphasized in Section 2, Corollary 2.2 follows directly from Theorem 1.1, while Corollaries 2.4-2.7 follow from Corollaries 2.2 and 2.3 and the results of Section 1. It just remains to complete the proof of Corollaries 2.1 and 2.3.

Proof of Corollary 2.1. From the observations in the paragraph before Corollary 2.1 and from the assumptions made in Corollary 2.1, there is a unique solution ϕ_0 of (1.4) such that $\max_{\mathbb{R}} \phi_0 = \phi_0(0) = \beta$ (and ϕ_0 is then even and decreasing in $|x|$). Furthermore, $0 < m_{\phi_0} < \beta$ and $F < 0$ in $(0, m_{\phi_0}]$. All conditions of Theorem 1.1 are then fulfilled.

Let now u be a positive bounded solution of (1.1) satisfying (1.3). As in (4.1), there is $x_0 \in \mathbb{R}$ such that, for every $t \in \mathbb{R}$, the function $x \mapsto u(t, x+x_0)$ is even in x and decreasing in $|x|$. Since $\phi := \phi_0(\cdot - x_0)$ is the only solution of (1.4) which is symmetric with respect to x_0, it follows from property (1.14) of Theorem 1.1 that

$$\|u(t,\cdot) - \phi\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } t \to -\infty.$$

Lastly, if alternative (ii) holds in the conclusion of Theorem 1.1, then again by the uniqueness of the symmetric (with respect to x_0) solution ϕ of (1.4), one has $\|u(t,\cdot) - \phi\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to +\infty$. It then follows from (4.7), (4.9) and (4.35) that the action $E[u(t,\cdot)]$ defined by (4.5) has the same limit $E[\phi]$ as $t \to \pm\infty$. From (4.6), one concludes that $u_t \equiv 0$ in $\mathbb{R}\times\mathbb{R}$, that is, $u(t,x) \equiv \phi(x)$ in $\mathbb{R} \times \mathbb{R}$. The proof of Corollary 2.1 is thereby complete. □

Proof of Corollary 2.3. As in (4.1), there is a point $x_0 \in \mathbb{R}^N$ such that, for every $t \in \mathbb{R}$, the function $x \mapsto u(t, x+x_0)$ is radially symmetric, and decreasing in $|x|$. From the assumptions made in Corollary 2.3, the set of solutions of (1.4) which are radially symmetric with respect to the point x_0 is discrete. Since the α-limit set of u is non-empty, connected and made of solutions of (1.4) which are radially symmetric with respect to x_0 (from the proof of (1.14) in Section 4.1), it follows that there is $\phi \in \mathcal{E}$ such that

$$\|u(t,\cdot) - \phi\|_{L^\infty(\mathbb{R}^N)} \to 0 \text{ as } t \to -\infty.$$

In case alternative (ii) of the conclusion of Theorem 1.1 occurs, then there is $\widetilde{\phi} \in \mathcal{E}$ such that $\|u(t,\cdot) - \widetilde{\phi}\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to +\infty$. Furthermore, if $\phi = \widetilde{\phi}$, then, as in the above proof of Corollary 2.1, one has $E[u(t,\cdot)] \to E[\phi] = E[\widetilde{\phi}]$ as $t \to \pm\infty$, hence $u_t \equiv 0$ and $u(t,x) \equiv \phi(x) \equiv \widetilde{\phi}(x)$ in $\mathbb{R} \times \mathbb{R}^N$. □

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