A short proof of the logarithmic Bramson correction in Fisher-KPP equations

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Abstract

In this paper, we explain in simple PDE terms a famous result of Bramson about the logarithmic delay of the position of the solutions $u(t,x)$ of Fisher-KPP reaction-diffusion equations in $\mathbb{R}$, with respect to the position of the travelling front with minimal speed. Our proof is based on the comparison of u to the solutions of linearized equations with Dirichlet boundary conditions at the position of the minimal front, with and without the logarithmic delay. Our analysis also yields the large-time convergence of the solutions u along their level sets to the profile of the minimal travelling front.

1 Introduction

This paper is concerned with the large time behavior of the solutions of the Cauchy problem for the reaction-diffusion equation

$$\begin{aligned} &u_t = u_{xx} + f(u), \quad t>0,\ x\in\mathbb{R}, \\ &u(0,x) = u_0(x), \quad x\in\mathbb{R}. \end{aligned} \tag{1}$$

We assume that u_0 is localized (in a sense to be made more precise below), and we are interested in how the "positions" of the level sets of u compare to those of travelling fronts. The reaction function $f \in C^2$ is assumed to be of the Fisher-KPP (for Kolmogorov-Petrovskii-Piskunov) type [11, 14], that is

$$f(0) = f(1) = 0,\quad f'(0) > 0,\quad f'(1) < 0 \ \text{ and } \ 0 < f(s) \le f'(0)s \text{ for all } s \in (0,1). \tag{2}$$

A typical example is a concave positive function f on $(0,1)$ that vanishes at 0 and 1, such as $f(u) = u(1-u)$. The initial datum $u_0 \in L^\infty(\mathbb{R})$ is such that

$$0 \le u_0 \le 1,\quad u_0 \not\equiv 0,\ \text{ and } \ u_0 = 0 \text{ in } (A,+\infty)$$

for some real number A. The solution $u(t,x)$ is classical for $t>0$, and $0 < u(t,x) < 1$ for all $t>0$ and $x\in\mathbb{R}$ from the strong maximum principle. Such equations arise in many mathematical models in biology, ecology or genetics, see e.g. [10, 11, 15, 19], and u typically stands for the density of a population.

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The assumptions on f imply that 0 and 1 are, respectively, unstable and stable equilibria for the ODE $\dot{\zeta} = f(\zeta)$. For the PDE (1) the state $u \equiv 1$ invades the state 0. Specifically, a celebrated result of Aronson and Weinberger [1] states that the solution u spreads at the speed $c^* = 2\sqrt{f'(0)}$, in the sense that

$$\min_{|x|\leq ct} u(t,x) \to 1 \text{ as } t \to +\infty, \text{ for all } 0 \leq c < 2\sqrt{f'(0)}, \tag{3}$$

and

$$\sup_{x\geq ct} u(t,x) \to 0 \text{ as } t \to +\infty, \text{ for all } c > 2\sqrt{f'(0)}. \tag{4}$$

Sharp asymptotics of the location of the level sets of $u(t,x)$ was given by Bramson in the celebrated papers [3, 4]. Given $m \in (0,1)$, let $E_m(t)$ be the set of points in $(0,+\infty)$ where $u(t,\cdot)$ equals m, that is $E_m(t) = \big\{x > 0, \ u(t,x) = m\big\}$. Bramson [3, 4] has shown with probabilistic arguments that there exist a shift x_m depending on m and the initial data u_0, and some constants $\gamma > 0$ and $C_m > 0$ such that

$$E_m(t) \subset \Big[c^*t - \frac{3}{2\lambda^*}\ln t - x_m - \frac{\gamma}{\sqrt{t}} - \frac{C_m}{t}, c^*t - \frac{3}{2\lambda^*}\ln t - x_m - \frac{\gamma}{\sqrt{t}} + \frac{C_m}{t}\Big] \text{ for } t \text{ large enough}, \tag{5}$$

with $\lambda^* = c^*/2$.[1] The goal of the present paper is to explain the logarithmic shift in (5) in simple PDE terms, at the expense of losing precision in the $O(1)$ terms. More precisely, we will show the following result.

Theorem 1.1 *For every $m \in (0,1)$, there is $C \geq 0$ such that*

$$E_m(t) \subset \Big[c^*t - \frac{3}{2\lambda^*}\ln t - C, c^*t - \frac{3}{2\lambda^*}\ln t + C\Big] \quad \textit{for } t \textit{ large enough.}$$

To observe that the deviation of $E_m(t)$ from c^*t grows in time, notice that since $f(s) \leq f'(0)s$, the maximum principle implies that $u(t,x) \leq \widetilde{u}(t,x)$, where $\widetilde{u}$ solves the linear heat equation

$$\widetilde{u}_t = \widetilde{u}_{xx} + f'(0)\widetilde{u}, \quad t > 0, \quad x \in \mathbb{R}, \tag{6}$$

with the initial condition $\widetilde{u}(0,x) = u_0(x)$. Therefore,

$$u(t,x) \leq \frac{e^{f'(0)t}}{\sqrt{4\pi t}} \int_{-\infty}^{A} e^{-\frac{(x-y)^2}{4t}} dy = \frac{e^{f'(0)t}}{\sqrt{\pi}} \int_{-\infty}^{(A-x)/\sqrt{4t}} e^{-z^2} dz \text{ for all } t > 0 \text{ and } x \in \mathbb{R}. \tag{7}$$

Remember that $c^* = 2\sqrt{f'(0)} = 2\lambda^*$. As a consequence,

$$\limsup_{t\to+\infty} \Big(\max_{x\geq C} u\big(t, c^*t - \frac{\ln t}{2\lambda^*} + x\big)\Big) \to 0 \text{ as } C \to +\infty. \tag{8}$$

For every $m \in (0,1)$, there is then $C \in \mathbb{R}$ such that

$$\max E_m(t) \leq c^*t - \frac{\ln t}{2\lambda^*} + C \quad \text{for all } t \text{ large enough.} \tag{9}$$

In other words, the positions of the levels sets $E_m(t)$ are corrected by a term which is at least of the order $(\ln t)/(2\lambda^*)$ to the left of the position c^*t at large times. However, this calculation turns out

[1] We keep a separate notation for λ^* since it is a mere coincidence that in a homogeneous medium the relation between c^* and λ^* is so explicit. The $O(t^{-1/2})$ and $O(t^{-1})$ terms in (5) are actually due to Ebert and Van Saarloos [8].

to underestimate the gap $c^*t - \max E_m(t)$ (or $c^*t - \min E_m(t)$). The fact that the function u cannot exceed the value 1 shall force level sets of u to lag even further behind those of $\widetilde{u}$.

As (3) implies that $u(t,x) \to 1$ as $t \to +\infty$ locally uniformly in x, it follows from Theorem 1.1 that

$$\liminf_{t\to+\infty} \Big(\min_{0\le x\le c^*t-(3/(2\lambda^*))\ln t - C} u(t,x) \Big) \to 1 \text{ as } C \to +\infty. \tag{10}$$

Furthermore, since $u(t,+\infty) = 0$ for all $t \ge 0$, there also holds

$$\limsup_{t\to+\infty} \Big(\max_{x\ge c^*t-(3/(2\lambda^*))\ln t + C} u(t,x) \Big) \to 0 \text{ as } C \to +\infty. \tag{11}$$

In other words, the region of points $x \ge 0$ where $u(t,x)$ is bounded away from 0 and 1 has a bounded width as $t \to +\infty$, and is located around the position $c^*t - (3/(2\lambda^*))\ln t$.

Theorem 1.1, together with more precise estimates on the behavior of u to the right of $c^*t - (3/(2\lambda^*))\ln t$, implies that the solution u approaches the family of shifted travelling fronts $U_{c^*}(x - c^*t + (3/(2\lambda^*))\ln t + \xi)$ uniformly in $\{x \ge 0\}$. Indeed, it is well-known [1, 14] that problem (1) admits travelling fronts of the type $U_c(x - ct)$ with $U_c(+\infty) = 0 < U_c < 1 = U_c(-\infty)$ if and only if $c \ge c^* = 2\sqrt{f'(0)}$. Furthermore, each profile U_c satisfies

$$U_c'' + cU_c' + f(U_c) = 0 \quad \text{in } \mathbb{R}, \tag{12}$$

is unique up to a shift, and is decreasing. Here is the result.

Theorem 1.2 *There exist a constant $C \ge 0$ and a function $\xi : (0,+\infty) \to \mathbb{R}$ such that $|\xi(t)| \le C$ for all $t > 0$ and*

$$\lim_{t\to+\infty} \Big\| u(t,\cdot) - U_{c^*}\Big(\cdot - c^*t + \frac{3}{2\lambda^*}\ln t + \xi(t)\Big) \Big\|_{L^\infty(0,+\infty)} = 0. \tag{13}$$

Furthermore, for every $m \in (0,1)$ and every sequence (t_n, x_n) such that $t_n \to +\infty$ as $n \to +\infty$ and $x_n \in E_m(t_n)$ for all $n \in \mathbb{N}$, there holds

$$u(t+t_n, x+x_n) \underset{n\to+\infty}{\longrightarrow} U_{c^*}\big(x - c^*t + U_{c^*}^{-1}(m)\big) \text{ locally uniformly in } (t,x) \in \mathbb{R}^2, \tag{14}$$

where $U_{c^*}^{-1}$ denotes the inverse of the function U_{c^*}.

Note that Bramson's result is more precise, in the sense that it identifies a unique travelling wave in the possible limiting trajectories of $u(t,x)$. This is of course due to the precision of the front location asymptotics (5).

A brief bibliographical survey is now in order. The first result on the large time behavior for the parabolic Cauchy problem (1) is the work of Kolmogorov, Petrovskii and Piskunov [14]: here, u_0 is a Heaviside type function and the authors prove, the existence of a function $s(t)$ such that $\lim_{t\to+\infty} s(t)/t = 0$, and such that

$$\lim_{t\to+\infty} |u(t,x) - U_{c^*}(x - c^*t - s(t))| = 0, \qquad \text{uniformly in } x \in \mathbb{R}.$$

The key point in [14] is the beautiful observation that $t \mapsto |u_x(t,.)|$ decreases on each E_m. The KPP theorem is generalized in [16], for a large class of monotone initial data. The observation that $s(t)$ might have a nontrivial behavior was proved by Uchiyama [18]: $s(t) = -3/(2\lambda^*)\ln t + O(\ln\ln t)$ for a large class of Heaviside-like initial data. This was obtained by a refinement of the KPP argument. The sharpest asymptotics is, as we have mentioned, due to Bramson [3, 4]. Those results were also proved by Lau [13], using the decrease of the number of intersection points between any two solutions of the parabolic Cauchy problem (1). Both Bramson and Lau used the additional assumption $f'(s) \le f'(0)$. Let us also mention Eckmann and Gallay [7]: they prove the stability of the critical wave under (essentially) compact perturbations, thus a context quite different from ours (the solution converges to the wave with no shift). However, their work makes it quite clear that subtle diffusion phenomena are at work in the tail of the solution.

The logarithmic shift in KPP type equations has been much revisited in the recent years: in [5], the link is made with the behavior of (1) when f has a tiny cut-off. In [8, 9] the question is cast in the more general problem of the dynamics of pulled fronts. In these works the authors show, from the point of view of formal asymptotics, the universal character of this shift, retrieving it not only in KPP or Ginzburg-Landau type equations, but also in systems and 4th order parabolic equations (e.g. Swift-Hohenberg). We finally mention [6]: here the problem is studied in an heterogeneous medium, and the solutions are shown (in a formal fashion) to have a behavior different from that of the homogeneous case.

Let us now describe our approach. It is based on the following observation: in the region $\{x \ge c^*t\}$, u is small and should then be close to (or at least bounded from below by) the solution $\underline{u}$ of the linearized equation (6) with the Dirichlet boundary condition $\underline{u}(t, c^*t) = 0$ at the position $x = c^*t$ for $t \ge 1$, and with initial condition $u(1, x)$ having a Gaussian decay as $x \to +\infty$ at time $t = 1$ (we here use the notation $\underline{u}$ instead of $\widetilde{u}$ in (6) because the Dirichlet boundary condition $\underline{u}(t, c^*t) = 0$ will make $\underline{u}$ a subsolution of the linearized equation and even a subsolution of the nonlinear equation up to some time-dependent prefactor). The function $p(t, y) = e^{\lambda^* y}\underline{u}(t, y + c^*t)$ solves the heat equation

$$p_t = p_{yy}$$

for $y > 0$ with the Dirichlet boundary condition $p(t, 0) = 0$. The explicit expression for p implies that $p(t, y)$ is of the order $y(t)/t^{3/2}$ (up to some bounded positive prefactors) at the position $1 \ll y(t) \le O(\sqrt{t})$ as $t \to +\infty$, meaning that

$$\underline{u}(t, c^*t + y(t)) \sim C_t \frac{y(t)}{t^{3/2}} e^{-\lambda^* y(t)} \tag{15}$$

for some C_t which is bounded from above and below by two positive constants as $t \to +\infty$. On the other hand, recall that

$$U_{c^*}(s) \sim B\, s\, e^{-\lambda^* s} \text{ as } s \to +\infty, \tag{16}$$

with some constant $B > 0$ [1]. When $\ln t \ll y(t) \le O(\sqrt{t})$, due to the s prefactor in (16), the value (15) corresponds to that of the front shifted to the position $c^*t - 3/(2\lambda^*) \ln t$:

$$U_{c^*}(x - c^*t + (3/(2\lambda^*)) \ln t)\big|_{x=c^*t+y(t)} \sim B\,(y(t) + \frac{3}{2\lambda^*} \ln t)\, e^{-\lambda^*(y(t)+3/(2\lambda^*)\ln t)} \sim \frac{B\,y(t)}{t^{3/2}} e^{-\lambda^* y(t)}.$$

This gives the exact estimate $c^*t - (3/(2\lambda^*)) \ln t$ as a lower bound of the location of u. Notice that the above heuristic arguments work for $\ln t \ll y(t) \le O(\sqrt{t})$ as $t \to +\infty$, and that these bounds are sharp.

The above ideas can be adapted to periodic equations of the type

$$u_t = u_{xx} + g(x) f(u),$$

where g is a periodic function which is bounded between two positive constants, and with weaker regularity assumptions on the function f. The calculations in the periodic case are more involved and are combined with further more sophisticated estimates, see [12]. This approach can also be used in higher dimensions. Our aim in the present paper is to present the key ideas in the simple homogeneous setting (1), where the analysis is very explicit. The paper is organized as follows: in Section 2, we present the proof of Theorem 1.1 with the additional assumption that f is linear in a vicinity of 0. The estimates of this section, together with a refinement, are used in Section 3 to prove the theorem for a general C^2 nonlinearity. Theorem 1.2 is proved in Section 4.

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2 The Dirichlet problem in the logarithmically shifted frame

The linearized Dirichlet problem

As we have mentioned, one of the key observations is that the solution of the nonlinear KPP equation behaves as the solution of the linearized equation with the Dirichlet boundary condition imposed at an appropriate point $X(t)$:

$$\underline{u}_t = \underline{u}_{xx} + f'(0)\underline{u}, \quad t > 0, \quad x > X(t), \tag{17}$$

with $\underline{u}(t, X(t)) = 0$. Our goal is to devise a reference frame in which the Dirichlet problem will have solutions that remain bounded both from above and below in a certain region, and this is exactly what the $(3/(2\lambda^*)) \ln t$ shift achieves, see Lemma 2.1 below.

Motivated by (9), we choose $X(t) = c^*t - (r/\lambda^*) \ln(t + t_0)$, with $t_0 > 0$ to be chosen later, and change variables in (17) as

$$x' = x - \left(c^*t - \frac{r}{\lambda^*}\ln(t + t_0)\right).$$

For the moment we keep $r > 0$ general. After dropping the primes the problem becomes

$$z_t - z_{xx} - \left(c^* - \frac{r}{\lambda^*(t + t_0)}\right) z_x - f'(0)z = 0, \quad t > 0,\ x > 0, \tag{18}$$

with the boundary condition $z(t, 0) = 0$, and with compactly supported initial data $z_0 \geq 0$ in $(0, +\infty)$ and $z_0 \not\equiv 0$ (that is fixed and does not depends on t_0). The following lemma explains why $r = 3/2$ is a good choice, and contains the essence of the PDE reason for the Bramson correction.

Lemma 2.1 *If $r = 3/2$, then there is a constant $t_0 > 0$ that depends on z_0 such that, for all $0 < a \leq b < +\infty$, we have*

$$0 < \inf_{t\geq 1,\, a\leq x\leq b} z(t, x) \leq \sup_{t\geq 1,\, a\leq x\leq b} z(t, x) < +\infty.$$

Proof. Let us introduce the function $v(t, x)$ by

$$z(t, x) = e^{-\lambda^* x} v(t, x).$$

This transforms (18) into

$$v_t - v_{xx} + \frac{r}{\lambda^*(t + t_0)}(v_x - \lambda^* v) = 0, \quad t > 0,\ x > 0,$$

with $v(t,0)=0$. In the self-similar variables $\tau=\ln(t+t_0)-\ln t_0$, $y=\dfrac{x}{\sqrt{t+t_0}}$ this becomes

$$v_\tau - v_{yy} - \frac{y}{2}v_y + \frac{re^{-\tau/2}}{t_0^{1/2}\lambda^*}v_y - rv = 0,$$

with $v(\tau,0)=0$. We rewrite this as

$$v_\tau + Lv = (r-1)v - \varepsilon e^{-\tau/2}v_y, \quad \tau>0,\ y>0, \tag{19}$$

with

$$Lv = -v_{yy} - \frac{y}{2}v_y - v,$$

and $\varepsilon = r/(t_0^{1/2}\lambda^*)$. From Lemma 2.2 below we have

$$v(\tau,y) = e^{(r-1)\tau}y\Big(\frac{e^{-y^2/4}}{2\sqrt{\pi}}\int_0^{+\infty}\xi v_0(\xi)d\xi + O(\varepsilon) + O(e^{-\tau/2})\Big).$$

Here $O(\varepsilon)$ and $O(e^{-\tau/2})$ denote functions of t and y which are of that order for $\tau>0$, and for y in any fixed compact set.

Let us now come back to the solution $z(t,x)$ of (18) with boundary condition $z(t,0)=0$, and with compactly supported initial data $z_0\geq 0$ in $(0,+\infty)$ and $z_0\not\equiv 0$. It follows from the previous paragraph that there exist a constant $C>0$ depending on z_0 and a constant $t_0>0$ depending on C and r such that

$$z(t,x) = \frac{(t+t_0)^{r-3/2}}{t_0^{r-1}}xe^{-\lambda^* x}\left[Ce^{-x^2/4(t+t_0)} + h(t,x)\right], \tag{20}$$

where, for each $\sigma>0$,

$$\limsup_{t\to+\infty}\ \sup_{0\leq x\leq\sigma\sqrt{t+1}}|h(t,x)| < \frac{C}{2}. \tag{21}$$

Therefore, if we choose $r=3/2$ (and only with that value of r), and t_0 sufficiently large, the function $z(t,x)$ will remain bounded from above and below away from zero on any fixed interval $a\leq x\leq b$ for all $t\geq 1$ with $0<a\leq b<+\infty$. For times $t\geq t_0$, this follows from (20-21) and, for times $1\leq t\leq t_0$, this follows from the continuity and positivity of z in $(0,+\infty)\times(0,+\infty)$, due to the strong maximum principle. Therefore,

$$0 < \inf_{t\geq 1}\Big(\min_{[a,b]} z(t,\cdot)\Big) \leq \sup_{t\geq 1}\Big(\max_{[a,b]} z(t,\cdot)\Big) < +\infty$$

for all $0<a\leq b$. Note finally that, in order to get these lower and upper bounds by positive constants, the logarithmic shift is needed – without the shift we get a solution that decays as $t^{-3/2}$. The proof of Lemma 2.1 is thereby complete. $\square$

We now prove the perturbation result used in the proof of Lemma 2.1. Although it is quite standard, we present its proof for the sake of completeness.

Lemma 2.2 *Let $v(t,y)$ solve*

$$v_\tau + Lv = -\varepsilon e^{-\tau/2}v_y, \quad \tau>0,\ y>0; \qquad v(\tau,0)=0. \tag{22}$$

There exists $\varepsilon_0>0$ so that, for all compact sets K of $\mathbb{R}_+$ there is $C_K>0$ such that, for $0<\varepsilon<\varepsilon_0$:

$$v(\tau,y) = y\Big(\frac{e^{-y^2/4}}{2\sqrt{\pi}}\big(\int_0^{+\infty}\xi v_0(\xi)d\xi + O(\varepsilon)\big) + e^{-\tau/2}\widetilde{v}(\tau,y)\Big) \quad y>0,\ \ \tau>0,$$

where $|\widetilde{v}(\tau,y)|\leq C_K$ for all $y\in K$, $\tau>0$.

Proof. Let us introduce the function $w(\tau,y)=e^{y^2/8}v(\tau,y)$. This new function solves

$$w_\tau+Mw=-\varepsilon e^{-\tau/2}(w_y-\frac{y}{4}w),\quad \tau>0,\ y>0, \tag{23}$$

with

$$Mw=-w_{yy}+(\frac{y^2}{16}-\frac{3}{4})w.$$

For later purposes, we introduce the quadratic form

$$Q(w)=\langle Mw,w\rangle_{L^2((0,+\infty))}=\int_0^{+\infty}\big(w_y^2+(\frac{y^2}{16}-\frac{3}{4})w^2\big)dy$$

in the space $H_0^1((0,+\infty))$ with $yw\in L^2((0,+\infty))$. The operator M is symmetric, its null space is generated by the unit eigenfunction $e_0(y)=ye^{-y^2/8}/(2\sqrt{\pi})^{1/2}$ and the form Q is nonnegative. Moreover, we have

$$Q(w)\geq \|w\|^2_{L^2((0,+\infty))}\ \text{ in } e_0^\perp, \tag{24}$$

as the second eigenfunction of M is $e_1(y)=(ye^{-y^2/4})''e^{y^2/8}$. Higher order eigenfunctions of M can be easily expressed in terms of the Hermite polynomials. Let us first notice that $\|w(\tau,\cdot)\|_{L^2}=\|w(\tau,\cdot)\|_{L^2(0,+\infty)}$ is uniformly controlled from above. Indeed, multiplying (23) by w and integrating by parts over $(0,+\infty)$ gives immediately

$$\frac{d}{d\tau}\|w(\tau,\cdot)\|^2_{L^2}+2Q(w(\tau,\cdot))=2\varepsilon e^{-\tau/2}\int_0^{+\infty}\frac{y}{4}w(\tau,y)^2dy.$$

Note that

$$\begin{aligned}\int_0^{+\infty}\frac{y}{4}w(\tau,y)^2dy\leq\int_0^{+\infty}(\frac{y^2}{16}+\frac{1}{4})w(\tau,y)^2dy &\leq \int_0^{+\infty}\Big(w_y(\tau,y)^2+(\frac{y^2}{16}+\frac{1}{4})w(\tau,y)^2\Big)dy\\ &= Q(w(\tau,\cdot))+\|w(\tau,\cdot)\|^2_{L^2},\end{aligned}$$

whence

$$\frac{d}{d\tau}\|w(\tau,\cdot)\|^2_{L^2}+2(1-\varepsilon e^{-\tau/2})Q(w(\tau,\cdot))\leq 2\varepsilon e^{-\tau/2}\|w(\tau,\cdot)\|^2_{L^2}.$$

If $\varepsilon<1$, this implies

$$\|w(\tau,\cdot)\|_{L^2}\leq C, \tag{25}$$

for all times. Throughout this proof we denote by C various constants that depend only on the initial data.

Set now, for all $w\in L^2((0,+\infty))$:

$$w=\langle e_0,w\rangle e_0+\widetilde{w}.$$

Thus, $\widetilde{w}$ is orthogonal to e_0. First, $w_1(\tau)=\langle e_0,w(\tau,\cdot)\rangle$ satisfies

$$w_1'(\tau)=-\varepsilon e^{-\tau/2}\langle e_0,w_y(\tau,\cdot)-\frac{y}{4}w(\tau,\cdot)\rangle=\varepsilon e^{-\tau/2}\langle (e_0)_y+\frac{y}{4}e_0,w(\tau,\cdot)\rangle.$$

It follows from (25) that

$$|w_1(\tau)-w_1(0)|\leq C\varepsilon. \tag{26}$$

Next, the equation for $\widetilde{w}$ is

$$\widetilde{w}_\tau+M\widetilde{w}=-\varepsilon e^{-\tau/2}\bigg(\langle (e_0)_y+\frac{y}{4}e_0,w(\tau,\cdot)\rangle e_0+\langle e_0,w(\tau,\cdot)\rangle((e_0)_y-\frac{y}{4}e_0)+\widetilde{w}_y(\tau,\cdot)-\frac{y}{4}\widetilde{w}(\tau,\cdot)\bigg). \tag{27}$$

Multiplication by $\widetilde{w}$, integration by parts and use of (25) yields the inequality

$$\frac{d}{d\tau}\|\widetilde{w}(\tau,\cdot)\|_{L^2}^2 + 2Q(\widetilde{w}(\tau,\cdot)) \le C\varepsilon e^{-\tau/2}(Q(\widetilde{w}(\tau,\cdot)) + \|\widetilde{w}(\tau,\cdot)\|_{L^2} + \|\widetilde{w}(\tau,\cdot)\|_{L^2}^2).$$

Using (24) for $\widetilde{w}$ gives the following inequality for the function $\phi(\tau) := \|\widetilde{w}(\tau,\cdot)\|_{L^2}^2$, for ε sufficiently small:

$$\phi' + 2(1 - C\varepsilon - C\varepsilon e^{-\tau/2})\phi \le C\varepsilon e^{-\tau},$$

which yields $\|\widetilde{w}(\tau,\cdot)\|_{L^2} \le Ce^{-\tau/2}$. By parabolic regularity, we have, for any compact set K,

$$\|\widetilde{w}_y(\tau,\cdot)\|_{L^\infty(K)} \le C_K e^{-\tau/2},$$

and so $|\widetilde{w}(\tau,y)| \le C_K y e^{-\tau/2}$. This implies the lemma, with

$$\widetilde{v}(\tau,y) = \frac{\widetilde{w}(t,y)e^{-y^2/8+\tau/2}}{y},$$

and $|\widetilde{v}(\tau,y)| \le C_K e^{-y^2/8}$. This completes the proof. □

An upper bound for u

Let us now come back to the solution $u(t,x)$ of the nonlinear problem (1) on the whole line, with the initial data $u_0(x)$ supported in $(-\infty, A)$ and such that $0 \le u_0(x) \le 1$. Without loss of generality, one can assume that $A > 0$. Lemma 2.1 leads to the following upper bounds on u.

Proposition 2.3 *There holds*

$$\limsup_{t\to+\infty} u\Big(t, c^*t - \frac{3}{2\lambda^*}\ln t + y\Big) < 1 \quad \textit{for all } y \in \mathbb{R}. \tag{28}$$

Moreover,

$$\limsup_{t\to+\infty}\Big(\max_{x\ge c^*t-(3/(2\lambda^*))\ln t+y} u(t,x)\Big) \to 0 \quad \textit{as } y \to +\infty. \tag{29}$$

Lastly, for every $\sigma > 0$, there is a positive constant $\rho > 0$ such that

$$u\Big(t, c^*t - \frac{3}{2\lambda^*}\ln t + y\Big) \le \rho\,(y+1)\,e^{-\lambda^* y} \quad \textit{for all } t \ge 1 \textit{ and } 0 \le y \le \sigma\sqrt{t}. \tag{30}$$

Proof. In the moving frame with the logarithmic correction, the function

$$U(t,x) = u(t, c^*t - (3/(2\lambda^*))(\ln(t+t_0) - \ln(t_0)) + x) \tag{31}$$

satisfies

$$U_t - \left(c^* - \frac{3}{2\lambda^*(t+t_0)}\right)U_x = U_{xx} + f(U),\ t > 0,\ x \in \mathbb{R}. \tag{32}$$

Let z_A be the solution of (18), with initial condition equal to the indicator function of the interval $[0, 2A]$. The function $z_A(t,x)$ is a super-solution to (32) for $x > 0$, as is any multiple $Bz_A(t,x)$ with $B > 0$. However, we can not immediately compare $U(t,x)$ and $z_A(t,x)$ since $z_A(t,0) = 0$ while $U(t,x) > 0$. In order to overcome this difficulty, fix $B > 0$ large enough so that, thanks to Lemma 2.1 (where $t_0 > 0$ in (31) only depends on A), we have

$$Bz_A(t,A) \ge 1 \quad \text{for all } t > 0.$$

Define $\bar{U}(t,x)$ for all $x \in \mathbb{R}$ as

$$\bar{U}(t,x) = \begin{cases} 1, & \text{if } x \le A \\ \min(1, Bz_A(t,x)), & \text{if } x \ge A. \end{cases}$$

The role of the $3/(2\lambda^*) \ln t$ logarithmic shift is to ensure that the two supersolutions $U \equiv 1$ and Bz_A intersect at a point $x(t)$ whose location is uniformly bounded in time. The function $\bar{U}$ being a super-solution to (32), and sitting above $U_0 = u_0$ at $t = 0$, we have

$$u\Big(t, c^*t - \frac{3}{2\lambda^*} \ln(1 + t/t_0) + x\Big) = U(t,x) \le \bar{U}(t,x) \text{ for all } t > 0 \text{ and } x \in \mathbb{R}.$$

The above inequality, together with (20-21) and the boundedness of u lead to (30) and (29).

Finally, let us prove (28). Assume it does not hold. There exist then $y_0 \in \mathbb{R}$ and a sequence of positive times $t_n \to +\infty$ such that $u(t_n, c^*t_n - (3/(2\lambda^*)) \ln t_n + y_0) \to 1$ as $n \to +\infty$. Up to extraction of a subsequence, the functions $u_n(t,x) = u(t + t_n, x + c^*t_n - (3/(2\lambda^*)) \ln t_n)$ converge locally uniformly in $\mathbb{R}^2$ to a classical solution u_∞ of

$$(u_\infty)_t = (u_\infty)_{xx} + f(u_\infty), \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{33}$$

such that $0 \le u_\infty \le 1$ in $\mathbb{R}^2$ and $u_\infty(0,0) = 1$. The strong maximum principle implies that $u_\infty \equiv 1$, whereas $u_\infty(0,y) \le 1/2$ for y large enough, from (29). One has reached a contradiction and the proof of Proposition 2.3 is thereby complete. □

Property (29) can be rewritten as an upper bound for the level sets $E_m(t)$:

Corollary 2.4 *For any $m \in (0,1)$, there are some constants $t_0 > 0$ and $C \in \mathbb{R}$ such that*

$$\max E_m(t) \le c^*t - \frac{3}{2\lambda^*} \ln t + C \quad \textit{for all } t \ge t_0.$$

A lower bound for u when f is linear at zero

It is immediate to see that in the special case when $f(s) = f'(0)s$ for $s \in [0, s_0)$ with some $s_0 > 0$, the same argument can be used to obtain a lower bound on $u(t,x)$. Indeed, in that case the function $\widetilde{z} = \delta z_A$ is a subsolution for u provided that $\delta > 0$ is chosen sufficiently small so as to ensure that $0 \le \widetilde{z} \le s_0$. This will ensure that

$$u\Big(t, c^*t - \frac{3}{2\lambda^*} \ln(t+1) + x\Big) \ge \widetilde{z}(t,x) \text{ for all } t > 0 \text{ and } x \in \mathbb{R}. \tag{34}$$

It is quite straightforward to get the full lower bound on u as in Theorem 1.1 from (34). We will not do that here since the argument is similar to that for the general f considered in the next section.

3 Lower bound on the location of u

In this section we prove lower bounds on the function u at the right of the position $c^*t - (3/(2\lambda^*)) \ln t$. The strategy is to construct a subsolution on the interval $[c^*t, +\infty)$, with the Dirichlet boundary condition at a point close to $x = c^*t$, that behaves as $y\, e^{-\lambda^* y}\, t^{-3/2}$ at $x = c^*t + y$ as $t \to +\infty$, see (42) and (43) below.

Proposition 3.1 *There holds*

$$\liminf_{t\to+\infty} u\Big(t, c^*t - \frac{3}{2\lambda^*}\ln t + y\Big) > 0 \quad \textit{for all } y \in \mathbb{R}, \tag{35}$$

uniformly in y in any compact set. Moreover, for every $\sigma > 0$, there is a positive constant $\kappa > 0$ such that

$$u\Big(t, c^*t - \frac{3}{2\lambda^*}\ln t + y\Big) \geq \kappa\, y\, e^{-\lambda^* y} \quad \textit{for all } t \geq 1 \textit{ and } 0 \leq y \leq \sigma\sqrt{t}. \tag{36}$$

Proof. We will follow the strategy described above for the proof of (35), the inequality (36) being a by-product of the proof.

Step 1: The linearized problem with Dirichlet boundary condition at c^*t. In the frame moving with speed c^*, the function $v(t,y) = u(t, c^*t + y)$ solves

$$v_t - c^* v_y = v_{yy} + f(v). \tag{37}$$

Consider the linearized equation

$$w_t - c^* w_y = w_{yy} + f'(0)w \tag{38}$$

with Dirichlet boundary condition at $y = 0$:

$$w(t,0) = 0, \text{for all } t > 0, \tag{39}$$

and the same initial condition $w(0,\cdot) = v(0,\cdot)$ on $(0,+\infty)$. Since $c^* = 2\sqrt{f'(0)} = 2\lambda^*$, the function $p(t,y) = e^{\lambda^* y} w(t,y)$ solves

$$\begin{aligned} &p_t = p_{yy}, \quad t > 0, \quad y > 0, \\ &p(t,0) = 0, \quad t > 0, \end{aligned} \tag{40}$$

whence

$$w(t,y) = \frac{e^{-\lambda^* y}}{\sqrt{4\pi t}} \int_0^{+\infty} \Big(e^{-\frac{(y-y')^2}{4t}} - e^{-\frac{(y+y')^2}{4t}}\Big)\, p(0,y')\, dy' \quad \text{for all } t > 0 \text{ and } y \geq 0, \tag{41}$$

which implies that

$$w(t,y) \sim C\, y\, e^{-\lambda^* y - y^2/(4t)}\, t^{-3/2} \text{ as } t \to +\infty, \tag{42}$$

in the interval $y \in [0, \sqrt{t}]$, where $C > 0$ only depends on $p(0,\cdot)$ (without loss of generality, $p(0,\cdot)$ is nonnegative and not identically zero on $(0,+\infty)$).

Step 2: Lower bound at $x = c^*t + O(\sqrt{t})$. For simplicity, let us first suppose that f is actually linear in a small neighborhood of 0: $f(s) = f'(0)s$ for $s \in [0, s_0)$. In this case, $\underline{v}(t,y) := \delta w(t,y)$ is a subsolution of the nonlinear problem problem (37) for all $t > 0$ and $y \geq 0$, if δ is sufficiently small. It follows then from the maximum principle that

$$u(t, c^*t + y) = v(t,y) \geq \underline{v}(t,y) \text{ for all } t \geq 0 \text{ and } y \geq 0.$$

In particular, for any $\sigma > 0$, there is $\widetilde{\delta} > 0$ such that

$$u(t, c^*t + y) \geq \underline{v}(t,y) \geq \widetilde{\delta}\, y\, e^{-\lambda^* y}\, t^{-3/2} \text{ for all } t \geq 1 \text{ and } y \in [0, \sigma\sqrt{t}]. \tag{43}$$

This inequality holds first for large t from (42). Even if it means decreasing $\widetilde{\delta}$, the inequality (43) then holds for all $t \geq 1$ from the continuity and positivity of $\underline{v}$ in $(0,+\infty) \times (0,+\infty)$ and from the

Hopf lemma at $y = 0$, implying that the function $t \mapsto w_y(t,0)$ is positive and continuous for all $t > 0$.

If $f(u)$ is not linear in a neighborhood of $u = 0$ (such as $f(u) = u(1-u)$), then (43) can be proved via a slight modification of the subsolution $\underline{v}(t,y)$. As $f \in C^2([0,1])$, there exists $M > 0$ so that $f(s) - f'(0)s \geq -Ms^2$ for $s \in [0, s_0)$. The function $\underline{v}(t,y) := a(t)w(t,y)$ satisfies

$$\underline{v}_t - c^* \underline{v}_y - \underline{v}_{yy} - f(\underline{v}) = a'(t)w + f'(0)aw - f(aw) \leq a'(t)w + M(aw)^2.$$

Then, $\underline{v}$ is a subsolution to (37) provided that $a'(t)w + M(aw)^2 \leq 0$. Since $w(t,y) \leq C(t+1)^{-3/2}$ for all $t > 0$ and $y \geq 0$ (since $p(0,\cdot)$ has compact support), it suffices to choose $a(t)$ to solve

$$a'(t) = -CM(t+1)^{-3/2}a^2, \quad t > 0.$$

Hence, $a(t)$ can be chosen uniformly bounded from above and below: $0 < a_0 \leq a(t) \leq a_1 < +\infty$, hence (43) still holds.

Step 3: The approximate travelling fronts are subsolutions for $0 \leq x \leq c^*t + O(\sqrt{t})$. Let us now prove property (35). Fix $\sigma > 0$ and let $\xi(t) = \sigma\sqrt{t}$. Using the estimate (43), we will construct an explicit subsolution to (1) on the interval $0 \leq x \leq c^*t + \xi(t)$ (in the original frame) for t large enough. This subsolution will be an approximate travelling front, moving at a speed close to c^*.

By (43) there exist $\widetilde{\delta} > 0$ and $T_1 \geq 1$ such that

$$u(t, c^*t + \xi(t)) \geq \underline{v}(t, \xi(t)) \geq \widetilde{\delta}\, \xi(t)\, e^{-\lambda^* \xi(t)}\, t^{-3/2} \text{ for all } t \geq T_1. \tag{44}$$

It is known from [1] that $u(t,x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}$. Therefore, there exists $T_2 \geq T_1$ such that $u(t,0) \geq 1/2$ for all $t \geq T_2$. Let now f_1 be a C^1 function such that $f_1 \leq f$ in $[0,1/2]$, $f_1(0) = f_1(1/2) = 0$, $f_1'(0) = f'(0)$ and $f_1 > 0$ on $(0,1/2)$. The function f_1 then satisfies $f_1(s) \leq f(s) \leq f'(0)s = f_1'(0)s$ for all $s \in [0,1/2]$. Thus, there exists a travelling front $\widetilde{U}_{c^*}(x - c^*t)$ of (1) with nonlinearity f_1 instead of f, such that $0 < \widetilde{U}_{c^*} < 1/2$ in $\mathbb{R}$, $\widetilde{U}_{c^*}(-\infty) = 1/2$, $\widetilde{U}_{c^*}(+\infty) = 0$, with minimal speed $c^* = 2\sqrt{f_1'(0)} = 2\sqrt{f'(0)}$. The profile $\widetilde{U}_{c^*}$ is decreasing in $\mathbb{R}$ and is such that

$$\widetilde{U}_{c^*}(s) \sim \widetilde{B}\, s\, e^{-\lambda^* s} \text{ as } s \to +\infty, \tag{45}$$

for some constant $\widetilde{B} > 0$. Let now $\gamma > 0$ and fix $x_1 \in \mathbb{R}$ large enough so that $\widetilde{B}\,(\gamma+1)\, e^{-\lambda^* x_1} < \widetilde{\delta}$. Since there exists $T_3 \geq T_2$ such that $(\frac{3}{2\lambda^*}\ln(t) + x_1) < \gamma\xi(t)$ for $t \geq T_3$, it follows that

$$\widetilde{U}_{c^*}\Big(\frac{3}{2\lambda^*}\ln t + \xi(t) + x_1\Big) \leq \widetilde{\delta}\, \xi(t)\, e^{-\lambda^* \xi(t)}\, t^{-3/2} \quad \text{for all } t \geq T_3. \tag{46}$$

On the other hand, since $\min_{x \in [0, c^*T_3 + \xi(T_3)]} u(T_3, x) > 0$ and $\widetilde{U}_{c^*}(+\infty) = 0$, there exists $x_2 \geq x_1$ such that

$$\widetilde{U}_{c^*}\Big(x - c^*T_3 + \frac{3}{2\lambda^*}\ln T_3 + x_2\Big) \leq u(T_3, x) \quad \text{for all } x \in [0, c^*T_3 + \xi(T_3)]. \tag{47}$$

Define the subsolution $\underline{u}$ as follows:

$$\underline{u}(t,x) = \widetilde{U}_{c^*}\Big(x - c^*t + \frac{3}{2\lambda^*}\ln t + x_2\Big) \quad \text{for all } t \geq T_3 \text{ and } x \in [0, c^*t + \xi(t)].$$

It follows from (47) that $\underline{u}(T_3, x) \leq u(T_3, x)$ for all $x \in [0, c^*T_3 + \xi(T_3)]$. Using (44) and (46), and since $x_2 \geq x_1$, and $\widetilde{U}_{c^*}$ is decreasing, we have

$$\underline{u}(t, c^*t + \xi(t)) = \widetilde{U}_{c^*}\Big(\frac{3}{2\lambda^*}\ln t + \xi(t) + x_2\Big) \leq \widetilde{\delta}\, \xi(t)\, e^{-\lambda^* \xi(t)}\, t^{-3/2} \leq u(t, c^*t + \xi(t)).$$

for all $t \geq T_3\,(\geq T_1)$. Furthermore, $\underline{u}(t,0) \leq 1/2 \leq u(t,0)$ for all $t \geq T_3\,(\geq T_2)$, Lastly, since $f_1 \leq f$ in $[0,1/2]$ and since $\widetilde{U}_{c^*}$ is decreasing and

$$\widetilde{U}''_{c^*} + c^* \widetilde{U}'_{c^*} + f_1(\widetilde{U}_{c^*}) = 0,$$

for all $t \geq T_3$ and $x \in [0, c^*t + \xi(t)]$, we get

$$\underline{u}_t(t,x) - \underline{u}_{xx}(t,x) - f(\underline{u}(t,x)) \leq \Big(-c^* + \frac{3}{2\lambda^* t}\Big)\widetilde{U}'_{c^*}(z) - \widetilde{U}''_{c^*}(z) - f_1(\widetilde{U}_{c^*}(z)) = \frac{3}{2\lambda^* t}\,\widetilde{U}'_{c^*}(z) \;\leq\; 0,$$

where $z = x - c^*\,t + (3/(2\lambda^*))\ln t + x_2$. To sum up, the function $\underline{u}$ is a subsolution of (1) for all $t \geq T_3$ and $x \in [0, c^*t + \xi(t)]$. The maximum principle implies that

$$u(t,x) \geq \underline{u}(t,x) = \widetilde{U}_{c^*}\Big(x - c^*t + \frac{3}{2\lambda^*}\ln t + x_2\Big) \text{ for all } t \geq T_3 \text{ and } 0 \leq x \leq c^*t + \xi(t). \tag{48}$$

Step 4: Conclusion and proof of (35-36). From the inequality (48), it follows in particular that, for any given $y \in \mathbb{R}$,

$$u\Big(t, c^*t - \frac{3}{2\lambda^*}\ln t + y\Big) \geq \underline{u}\Big(t, c^*t - \frac{3}{2\lambda^*}\ln t + y\Big) = \widetilde{U}_{c^*}(y + x_2) > 0$$

for t large enough. This provides (35), together with uniformity in y in compact sets. On the other hand, fixing $\sigma > 0$ and $\xi(t) = \sigma\sqrt{t}$, (45) and (48) lead to (36). This completes the proof of Proposition 3.1. □

Corollary 3.2 *For any $m \in (0,1)$, there are some constants $t_0 > 0$ and $C \in \mathbb{R}$ such that*

$$\min E_m(t) \geq c^*t - \frac{3}{2\lambda^*}\ln t + C \quad \textit{for all } t \geq t_0.$$

Proof. The proof of Proposition 3.1 has already established this result for $m < 1/2$. In order to show it for $1/2 \leq m < 1$, we can simply repeat the argument in Steps 3 and 4 of that proof but replacing f_1 by a nonlinearity that vanishes at $s = 0$ and $s = (1+m)/2$ rather than at $s = 1/2$. □

4 Convergence to the family of shifted approximated minimal fronts

This section contains the sketch of the proof of Theorem 1.2. It is based on the fact that the set where u is bounded away from 0 or 1 is located around the position $c^*t - (3/(2\lambda^*))\ln t$ for t large enough. To the right of this position, the solution u has the same type of decay as the critical front U_{c^*} in (16), because of the upper and lower bounds on u obtained in the previous sections. Therefore, u is almost trapped between two finite shifts of the profile of the front U_{c^*}. From a Liouville-type result, similar to that in [2] and based on the sliding method, the convergence to the shifted approximated minimal fronts follows.

First, let $0 \leq \kappa \leq \rho$ be given as in the statements of Proposition 3.1 and 2.3 with, say, $\sigma = 1$. Let $B > 0$ be given as in (16) and let $C \geq 0$ be such that

$$B\,e^{-\lambda^* C} \leq \kappa \leq \rho \leq B\,e^{\lambda^* C}. \tag{49}$$

Let us prove that (13) holds with this choice of C. Assume not. There are then $\varepsilon > 0$ and a sequence of positive times $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to +\infty$ as $n \to +\infty$ and

$$\min_{|\xi|\leq C} \Big\| u(t_n,\cdot) - U_{c^*}\big(\,\cdot - c^*t_n + \frac{3}{2\lambda^*}\ln t_n + \xi\big) \Big\|_{L^\infty(0,\infty)} \geq \varepsilon$$

for all $n \in \mathbb{N}$. Since $U_{c^*}(-\infty) = 1$, $U_{c^*}(+\infty) = 0$, properties (10) and (11) then give the existence of a constant $\theta \geq 0$ such that

$$\min_{|\xi|\leq C}\Big(\max_{|y|\leq\theta}\big|u\big(t_n, y + c^* t_n - \frac{3}{2\lambda^*}\ln t_n\big) - U_{c^*}(y+\xi)\big|\Big) \geq \varepsilon \tag{50}$$

for all $n \in \mathbb{N}$. Up to extraction of a subsequence, the functions u_n defined by

$$u_n(t,x) = u\big(t + t_n, x + c^* t_n - \frac{3}{2\lambda^*}\ln t_n\big)$$

converge locally uniformly in $\mathbb{R}^2$ to a solution u_∞ of (33) such that $0 \leq u_\infty \leq 1$ in $\mathbb{R}^2$. Furthermore, the limits (10) and (11) imply that

$$\lim_{y\to+\infty}\Big(\sup_{(t,x)\in\mathbb{R}^2,\, x\geq c^*t+y} u_\infty(t,x)\Big) = 0$$

and

$$\lim_{y\to-\infty}\Big(\inf_{(t,x)\in\mathbb{R}^2,\, x\leq c^*t+y} u_\infty(t,x)\Big) = 1. \tag{51}$$

On the other hand, for each fixed $t \in \mathbb{R}$ and $y > 0$, there holds $y_n = y + (3/(2\lambda^*))\ln((t+t_n)/t_n) \geq 0$ for n large enough, together with $t + t_n \geq 1$ and $0 \leq y_n \leq \sqrt{t+t_n}$, whence

$$\kappa\, y_n\, e^{-\lambda^* y_n} \leq u_n(t, c^*t + y) \leq \rho\,(y_n+1)\, e^{-\lambda^* y_n}$$

for n large enough. Therefore,

$$\kappa\, y\, e^{-\lambda^* y} \leq u_\infty(t, c^*t + y) \leq \rho\,(y+1)\, e^{-\lambda^* y} \ \text{ for all } t \in \mathbb{R} \text{ and } y \geq 0. \tag{52}$$

The following Liouville-type result gives a classification of the time-global solutions u_∞ satisfying the above properties.

Lemma 4.1 *For any solution $0 \leq u_\infty \leq 1$ of (33) in $\mathbb{R}^2$ satisfying (51) and (52) for some positive constants κ and ρ, there is $\xi_0 \in \mathbb{R}$ such that*

$$u_\infty(t,x) = U_{c^*}(x - c^*t + \xi_0) \ \text{ for all } (t,x) \in \mathbb{R}^2. \tag{53}$$

The conclusion of this lemma follows directly from Theorem 3.5 of [2]. The proof is based on the exponential estimates (52) and on the sliding method. In particular, it can be proved that the set $\big\{\xi \in \mathbb{R},\ u_\infty(t, x+\xi) \leq u_\infty(t,x) \text{ for all } (t,x) \in \mathbb{R}^2\big\}$ is not empty, is bounded from below and that its minimum ξ_0 satisfies (53). An interested reader may consult [12] for more details, where the proof is done in the more general case of x-periodic equations.

We first complete the proof of Theorem 1.2. It follows from Lemma 4.1, from (52) and from the exponential decay (16) of U_{c^*}, that $\kappa \leq B\, e^{-\lambda^*\xi_0} \leq \rho$, whence $|\xi_0| \leq C$ from (49). But since (at least for a subsequence) $u_n \to u_\infty$ locally uniformly in $\mathbb{R}^2$, it follows in particular that $u_n(0,\cdot) - U_{c^*}(\cdot + \xi_0) \to 0$ uniformly in $[-\theta, \theta]$, that is

$$\max_{|y|\leq\theta}\Big|u\big(t_n, y + c^* t_n - \frac{3}{2\lambda^*}\ln t_n\big) - U_{c^*}(y+\xi_0)\Big| \to 0 \ \text{ as } n \to +\infty.$$

Since $|\xi_0| \leq C$, one gets a contradiction with (50). Therefore, (13) is proved.

Let us now turn to the proof of (14). Let $m \in (0,1)$ be fixed and let $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that $t_n \to +\infty$ as $n \to +\infty$ and $u(t_n, x_n) = m$ for all $n \in \mathbb{N}$. Set

$$\xi_n = x_n - c^* t_n + \frac{3}{2\lambda^*} \ln t_n.$$

Theorem 1.1 implies that the sequence $(\xi_n)_{n\in\mathbb{N}}$ is bounded, and then converges to a real number ξ_∞, up to extraction of a subsequence. From the previous paragraph, the functions

$$v_n(t,x) = u(t+t_n, x+x_n) = u\Big(t+t_n, x+\xi_n + c^* t_n - \frac{3}{2\lambda^*}\ln t_n\Big)$$

converge, up to extraction of another subsequence, locally uniformly in $\mathbb{R}^2$ to $v_\infty(t,x) = U_{c^*}(x - c^*t + \xi_\infty + \xi)$ for some $\xi \in [-C, C]$, where $C \geq 0$ is given in (13). Since $v_n(0,0) = m$ for all $n \in \mathbb{N}$, one gets that $U_{c^*}(\xi_\infty + \xi) = m$, that is $\xi_\infty + \xi = U_{c^*}^{-1}(m)$. Finally, the limit v_∞ is uniquely determined and the whole sequence $(v_n)_{n\in\mathbb{N}}$ therefore converges to the travelling front $U_{c^*}(x - c^*t + U_{c^*}^{-1}(m))$. The proof of Theorem 1.2 is thereby complete. □

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