

Heteroclinic connections for multidimensional bistable reaction-diffusion equations

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À Claude-Michel Brauner, avec toute notre amitié

Abstract

In this paper, non-planar two-dimensional travelling fronts connecting an unstable one-dimensional periodic limiting state to a constant stable state are constructed for some reaction-diffusion equations with bistable nonlinearities. The minimal speeds are characterized in terms of the spatial period of the unstable limiting state. The limits of the minimal speeds and of the travelling fronts as the period converges to a critical minimal value or to infinity are analyzed. The fronts converge to flat fronts or to some curved fronts connecting an unstable ground state to a constant stable state.

1 Introduction and main results

This paper is concerned with special time-global solutions $v(t, x, y)$ of the reaction-diffusion equation

$$\frac{\partial v}{\partial t} - \Delta v = f(v) \tag{1.1}$$

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set in the two-dimensional space $\mathbb{R}^2 = \{(x, y), x \in \mathbb{R}, y \in \mathbb{R}\}$. The function f is assumed to be of class $C^1([0, 1])$ and bistable, that is there exists $\theta \in (0, 1)$ such that

$$\begin{cases} f(0) = f(\theta) = f(1), & f'(0) < 0, & f'(\theta) > 0, & f'(1) < 0, \\ f < 0 \text{ on } (0, \theta), & \text{and } f > 0 \text{ on } (\theta, 1). \end{cases}$$

The special solutions $v : \mathbb{R} \times \mathbb{R}^2 \rightarrow [0, 1]$ of (1.1) which are interested in are travelling fronts

$$v(t, x, y) = u(x, y + ct)$$

which propagate with speed c in the direction $-y$ and which connect the stable state 1 to another limiting state $\varphi = \varphi(x)$, that is

$$\begin{cases} -\Delta u + cu_y = f(u) & \text{in } \mathbb{R}^2, \\ u(x, y) \rightarrow 1 & \text{as } y \rightarrow +\infty, \text{ uniformly in } x \in \mathbb{R}, \\ u(x, y) \rightarrow \varphi(x) & \text{as } y \rightarrow -\infty, \text{ uniformly in } x \in \mathbb{R}, \\ \varphi(x) \leq u(x, y) \leq 1 & \text{for all } (x, y) \in \mathbb{R}^2, \end{cases} \quad (1.2)$$

where $\varphi : \mathbb{R} \rightarrow [0, 1]$ is a solution of the limiting equation

$$-\varphi''(x) = f(\varphi(x)) \text{ for all } x \in \mathbb{R}. \quad (1.3)$$

Notice that any solution of (1.2) satisfies $\varphi(x) < u(x, y) < 1$, from the strong maximum principle.

The state 0 is a stable state of (1.3) and it is well-known that there exists a unique speed $c_0 \in \mathbb{R}$ for which there is a solution $U(x, y)$ of (1.2) with $\varphi = 0$, and, furthermore, this solution depends only on y , it is increasing in y and unique up to shifts in y (see e.g. [4, 11]). Furthermore, the speed c_0 has the sign of the integral of the function f over the interval $[0, 1]$.

It is also well-known that there exists a minimal speed c^* such that

$$c^* > c_0$$

for which there exist solutions $U_c(y)$ of (1.2) with $\varphi = \theta$ if and only if $c \geq c^*$. Furthermore, if f is concave on $[\theta, 1]$, the only solutions $U(x, y)$ of (1.2) with $\varphi = \theta$ are of the type $U(x, y) = U_c(y)$ ([17]).

However, when φ is constant and the limits as $y \rightarrow \pm\infty$ are only assumed to be pointwise in $x \in \mathbb{R}$, then there may be many more solutions. For instance, there exist finite-dimensional (resp. infinite-dimensional, if f is concave in $[\theta, 1]$) manifolds of non-planar solutions $0 < u < 1$ of

$$-\Delta u + cu_y = f(u) \text{ in } \mathbb{R}^2$$

such that $u(x, +\infty) = 1$ uniformly in $x \in \mathbb{R}$, and $u(x, -\infty) = 0$ (resp. θ , with $u(x, y) > \theta$) pointwise in $x \in \mathbb{R}$, see [8, 10, 13, 15, 16, 17, 18, 22, 28] for the existence of such travelling fronts, and [22, 23, 26, 29] for stability results).

In this paper, we are interested in the connections $u(x, y)$, in the sense of (1.2), between 1 and a non-constant solution φ of (1.3). As far as problem (1.3) is concerned, there is a one-dimensional family

$$(\varphi_L)_{L \in (L_{\min}, +\infty)}$$

of non-constant periodic solutions, where

$$L_{\min} = 2\pi\sqrt{f'(\theta)} > 0$$

and, for each $L > L_{\min}$, $\varphi_L : \mathbb{R} \rightarrow (0, 1)$ has period L . Actually, φ_L is unique up to shifts in x , and one can assume, up to shifts, that $\varphi_L(0) = \max_{x \in \mathbb{R}} \varphi_L(x)$. Each function φ_L is then even in x and decreasing in $[0, L/2]$. Notice also that

$$\min_{x \in \mathbb{R}} \varphi_L(x) = \varphi(L/2) < \theta < \varphi_L(0) = \max_{x \in \mathbb{R}} \varphi_L(x) \quad \text{and} \quad \int_{\varphi_L(L/2)}^{\varphi_L(0)} f(s) ds = 0$$

for each $L > L_{\min}$. Furthermore,

$$\varphi_L \rightarrow \theta \quad \text{as } L \xrightarrow{\geq} L_{\min} \quad (\text{that is as } L \rightarrow L_{\min}^+) \quad \text{uniformly in } \mathbb{R}.$$

Lastly, all functions φ_L are unstable solutions of (1.3) in the sense that the principal periodic eigenvalue of the linearized operator $\phi \mapsto -\phi'' - f'(\varphi_L(x))\phi$ acting on the set of L -periodic functions is negative.

The first result states the existence of non-planar and L -periodic in x solutions of (1.2) connecting $\varphi = \varphi_L$ to 1 as $y \rightarrow \pm\infty$.

Theorem 1.1 *For each $L \in (L_{\min}, +\infty)$, there exists a minimal speed $c_L^* \in \mathbb{R}$ such that there exist solutions $u_{c,L}(x, y)$ of (1.2) with $\varphi = \varphi_L$, that is*

$$\begin{cases} -\Delta u + cu_y - f(u) = 0 & \text{in } \mathbb{R}^2 = \{(x, y), x \in \mathbb{R}, y \in \mathbb{R}\}, \\ u(x, y) \rightarrow 1 & \text{as } y \rightarrow +\infty, \quad \text{uniformly in } x \in \mathbb{R}, \\ u(x, y) \rightarrow \varphi_L(x) & \text{as } y \rightarrow -\infty, \quad \text{uniformly in } x \in \mathbb{R}, \\ \varphi_L(x) < u(x, y) < 1 & \text{for all } (x, y) \in \mathbb{R}^2. \end{cases} \quad (1.4)$$

satisfying

$$u(x + L, y) = u(x, y) \quad \text{in } \mathbb{R}^2, \quad (1.5)$$

if and only if $c \geq c_L^*$. Furthermore,

$$c_L^* > c_0.$$

Lastly, for any solution $u(x, y)$ of (1.4) and (1.5), with $c \geq c_L^*$, there holds

$$u_y(x, y) > 0, \quad u(x, y) = u(-x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2,$$

and $u(x, y) = u_{c,L}(x, y + b)$ for some $b \in \mathbb{R}$.

Actually, we shall prove in Section 3 that, for any solution of (1.2) with $\varphi = \varphi_L$ for some $L > L_{\min}$, the speed c is positive. Consequently, it is always true that

$$c_L^* > 0$$

for all $L > L_{\min}$, even if the planar speed c_0 is negative.

It is important to notice that, even if the function f is globally bistable over the interval $[0, 1]$, problem (1.2) with $\varphi = \varphi_L$ is really monostable due to the unstability of φ_L . Therefore, as for $\varphi = \theta$, there exists a continuum of possible speeds and, as soon as a speed is admissible, all larger speeds are also admissible (this situation is very different from the case of connections between two stable limiting states, see [7, 8, 15, 16, 22, 23, 28, 29] for various examples of bistable curved connections). Furthermore, if the function f is of the Kolmogorov-Petrovsky-Piskunov [19] type with respect to the function φ_L , that is if

$$f(\varphi_L(x) + s) \leq f(\varphi_L(x)) + f'(\varphi_L(x))s \quad \text{for all } x \in \mathbb{R} \text{ and } s \in [0, 1 - \varphi_L(x)], \quad (1.6)$$

then it can be proved as in [5] or [6] that the minimal speed c_L^* is given by a variational formula which only involves linearized operators around the limiting state φ_L , that is

$$c_L^* = \min_{\lambda > 0} \frac{-k(\lambda)}{\lambda},$$

where $k(\lambda)$ denotes the periodic principal eigenvalue of the operator

$$\phi \mapsto -\phi'' - (\lambda^2 + f'(\varphi_L(x)))\phi.$$

In the general non-KPP case, a variational min-max type formula for the minimal speed c_L^* always holds, see Section 4 for further details.

To derive the symmetry, monotonicity and uniqueness (up to shifts in y) properties stated in Theorem 1.1, we use the fact that the solutions are assumed to satisfy the periodicity condition (1.5). However, as already said, in the case when φ is equal to the constant 0 (resp. θ if f is concave over $[\theta, 1]$), then the independence with respect to x at the limit as $y \rightarrow \pm\infty$ is inherited at all y , that is the solutions only depend on y , see [4, 17]. The same type of question can be addressed here, namely if u solves (1.4) and is thus L -periodic in x as $y \rightarrow \pm\infty$, then does u necessarily fulfill the periodicity condition (1.5)? The answer is positive under the assumption that the relative x -variations of the difference between u and φ_L are controlled as $y \rightarrow -\infty$.

Theorem 1.2 *For each $L > L_{\min}$, if u solves (1.4) and if*

$$\limsup_{y \rightarrow -\infty} \frac{\sup_{x \in \mathbb{R}} (u(x, y) - \varphi_L(x))}{\inf_{x \in \mathbb{R}} (u(x, y) - \varphi_L(x))} < +\infty, \quad (1.7)$$

then the function u is L -periodic with respect to x , that is u satisfies (1.5).

It then follows that, under assumption (1.7), the function u satisfies the monotonicity and symmetry properties stated in Theorem 1.1 and it is then unique up to shifts in y . Condition (1.7) is fulfilled in particular if there exist a solution $u_{c,L}$ of (1.4) satisfying (1.5), and $a < b \in \mathbb{R}$ such that

$$u_{c,L}(x, y + a) \leq u(x, y) \leq u_{c,L}(x, y + b) \quad \text{for all } (x, y) \in \mathbb{R}^2, \quad (1.8)$$

We refer to Section 3, after the proof of Theorem 1.2, for the details. As a consequence, from Theorem 1.1, any solution u satisfying (1.8) can be written as $u(x, y) = u_{c,L}(x, y + \gamma)$, for some $\gamma \in [a, b]$. Let us mention that conditions similar to (1.8) have been used in [2, 3] to get uniqueness results for a class of generalized almost-planar fronts in monostable homogeneous or periodic equations.

The last two results of this paper are concerned with the dependence of the minimal speeds c_L^* with respect to L and the analysis of the limits as $L \rightarrow L_{\min}^+$ and $L \rightarrow +\infty$. Remember that $\varphi_L(x) \rightarrow \theta$ as $L \rightarrow L_{\min}^+$, uniformly in $x \in \mathbb{R}$.

Theorem 1.3 *The limit as $L \rightarrow L_{\min}^+$ is given by*

$$c_L^* \rightarrow c^* \quad \text{as } L \rightarrow L_{\min}^+$$

and the map $L \mapsto c_L^*$ is locally Lipschitz-continuous in $(L_{\min}, +\infty)$. Furthermore, for any $c > c^*$, for any solution $U_c(y)$ of (1.2) with $\varphi = \theta$ and for any family of solutions $(u_{c,L})_L$ of (1.4) (for $L - L_{\min} > 0$ small enough so that $c \geq c_L^*$) satisfying (1.5) and, say, $u_{c,L}(0, 0) = U_c(0)$, one has

$$u_{c,L}(x, y) \rightarrow U_c(y) \quad \text{as } L \rightarrow L_{\min}^+ \quad \text{in } C^2(\mathbb{R}^2).$$

In other words, Theorem 1.3 says that the fronts $u_{c,L}$ become flat as $L \rightarrow L_{\min}^+$, as do the limiting values $\varphi_L(x)$ as $y \rightarrow -\infty$. On the other hand, the limit as $L \rightarrow +\infty$ will give rise to completely different and non-planar fronts. Let us assume here that the function f has, say, a positive mass over $[0, 1]$, that is

$$\int_0^1 f(s) ds > 0.$$

Then it is easy to see that

$$\varphi_L(x) \rightarrow \varphi_\infty(x) \quad \text{as } L \rightarrow +\infty \quad \text{locally uniformly in } x \in \mathbb{R},$$

where φ_∞ is the unique solution of (1.3) such that $0 < \varphi_\infty(x) < 1$ for all $x \in \mathbb{R}$, φ_∞ is even, decreasing in $[0, +\infty)$, $\varphi_\infty(0) = \max_{x \in \mathbb{R}} \varphi_\infty(x) > \theta$ and $\varphi_\infty(x) \rightarrow 0$ as $x \rightarrow \pm\infty$. Actually, $\varphi_\infty(0)$ is the unique real number in $(\theta, 1)$ such that

$$\int_0^{\varphi_\infty(0)} f(s) ds = 0.$$

The function φ_∞ can be viewed as an unstable nonlinear ground state for problem (1.3). When $L \rightarrow +\infty$, the limiting fronts will then connect the limiting function φ_∞ (uniformly in $x \in \mathbb{R}$ as $y \rightarrow -\infty$) to 1 (locally uniformly in $x \in \mathbb{R}$ as $y \rightarrow +\infty$):

Theorem 1.4 *There is $c_\infty^* \in (c_0, +\infty)$ such that*

$$c_L^* \rightarrow c_\infty^* \quad \text{as } L \rightarrow +\infty.$$

Furthermore, for each $c > c_\infty^$, if $u_{c,L}$ denotes a solution of (1.4) satisfying (1.5) and, say $u_{c,L}(0,0) = (1 + \varphi_\infty(0))/2$, then*

$$u_{c,L}(x,y) \rightarrow u_{c,\infty}(x,y) \quad \text{as } L \rightarrow +\infty, \quad \text{locally uniformly in } \mathbb{R}^2,$$

where $u_{c,\infty}(x,y)$ solves

$$\begin{cases} -\Delta u + cu_y = f(u) & \text{in } \mathbb{R}^2 = \{(x,y), x \in \mathbb{R}, y \in \mathbb{R}\}, \\ u(x,y) \rightarrow 1 & \text{as } y \rightarrow +\infty, \quad \text{locally uniformly in } x \in \mathbb{R}, \\ u(x,y) \rightarrow \varphi_\infty(x) & \text{as } y \rightarrow -\infty, \quad \text{uniformly in } x \in \mathbb{R}, \\ \varphi_\infty(x) < u(x,y) < 1 & \text{for all } (x,y) \in \mathbb{R}^2. \end{cases} \quad (1.9)$$

Moreover we have

$$(u_{c,\infty})_y(x,y) > 0, \quad u_{c,\infty}(x,y) = u_{c,\infty}(-x,y) \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Lastly, solutions $u_{c,\infty}(x,y)$ of (1.9) with $\partial_y u_{c,\infty} > 0$ in \mathbb{R}^2 exist if and only if $c \geq c_\infty^$.*

When the integral of the function f over the interval $[0, 1]$ is negative, then there exists an even solution $0 < \tilde{\varphi}_\infty < 1$ of (1.3) such that $\tilde{\varphi}_\infty(\pm\infty) = 1$. If the functions φ_L are shifted in such a way that $\varphi_L(0) = \min_{\mathbb{R}} \varphi_L$, then the limiting functions $u_{c,\infty}$ will solve (1.2) with limiting value $\varphi = \tilde{\varphi}_\infty$ as $y \rightarrow -\infty$.

Remark 1.5 Throughout the paper, the solutions of (1.2) or (1.9) given in Theorems 1.1 and 1.4 are two-dimensional y -connections between an unstable limiting profile φ and the stable constant state 1. Each of these solutions $u_{c,L}$ or $u_{c,\infty}$ is expected to be stable for the Cauchy problem (1.1), at least if the initial condition v_0 is above $\max \varphi_L$ or $\max \varphi_\infty$ as $y \rightarrow +\infty$ and if the initial perturbation $v_0 - u_{c,L}$ or $v_0 - u_{c,\infty}$ decays to 0 as $y \rightarrow -\infty$ exponentially faster than the difference $u_{c,L} - \varphi_L$ or $u_{c,\infty} - \varphi_\infty$ (see Sections 4 and 5 for further details about the exponential decay of $u_{c,L} - \varphi_L$ and $u_{c,\infty} - \varphi_\infty$ as $y \rightarrow -\infty$).

Notice that such monostable connections are also known to exist for the one-dimensional equation

$$u_t = u_{xx} + f(u). \quad (1.10)$$

Namely, for each $L > 0$, there exist pulsating travelling fronts $u(t,x) = \phi(x - ct, x)$ such that ϕ is L -periodic in its second variable, and $\phi(-\infty, \cdot) = 1$, $\phi(+\infty, \cdot) = \varphi_L$, for large enough speeds c , see [27]. Roughly speaking, even if the proofs are completely different, the y variable would play in our problems (1.2) or (1.9) a role analogue to the time variable for problem (1.10).

Remark 1.6 During the preparation of this work, we learnt about an alternative proof of the existence of solutions of (1.9), by Y. Morita and H. Ninomiya [21]. A similar existence result actually holds in any spatial dimension and for some heterogeneous equations as well. However, only the existence of solutions for large enough speeds c is proved in [21] and the minimal speeds is not characterized. The methods are also different: the paper [21] is based on the direct construction of suitable sub- and super-solutions for problem (1.9), while our construction is based on approximated problems which are L -periodic in x and whose properties are analyzed in this paper.

2 Existence and properties of periodic connections

This section is devoted to the proof of Theorem 1.1. We divide the proof into several steps. In this section, L denotes a fixed real number such that $L > L_{\min}$, and problem (1.2) with $\varphi = \varphi_L$ corresponds to (1.4), that is :

$$\begin{cases} -\Delta u + cu_y - f(u) = 0 & \text{in } \mathbb{R}^2, \\ u(x, y) \rightarrow 1 & \text{as } y \rightarrow +\infty, \text{ uniformly in } x \in \mathbb{R}, \\ u(x, y) \rightarrow \varphi_L(x) & \text{as } y \rightarrow -\infty, \text{ uniformly in } x \in \mathbb{R}, \\ \varphi_L(x) \leq u(x, y) \leq 1 & \text{for all } (x, y) \in \mathbb{R}^2. \end{cases}$$

Actually, from the strong maximum principle, since $\varphi_L(x) < 1$ in \mathbb{R} , the inequalities

$$\varphi_L(x) < u(x, y) < 1$$

are strict in \mathbb{R}^2 .

Step 1 : existence results. First, it is straightforward to check that the change of functions

$$u(x, y) = \varphi_L(x) + v(x, y) \times (1 - \varphi_L(x)),$$

makes problem (1.4) equivalent to

$$\begin{cases} -\Delta v + cu_y + \alpha_L(x)v_x - g_L(x, v) = 0 & \text{in } \mathbb{R}^2, \\ v(x, y) \rightarrow 1 & \text{as } y \rightarrow +\infty, \text{ uniformly in } x \in \mathbb{R}, \\ v(x, y) \rightarrow 0 & \text{as } y \rightarrow -\infty, \text{ uniformly in } x \in \mathbb{R}, \\ 0 < v(x, y) < 1 & \text{for all } (x, y) \in \mathbb{R}^2, \end{cases} \quad (2.1)$$

where

$$\alpha_L(x) = \frac{2\varphi_L'(x)}{1 - \varphi_L(x)} \text{ for all } x \in \mathbb{R}$$

and

$$g_L(x, s) = \frac{f(\varphi_L(x) + s \times (1 - \varphi_L(x))) - (1 - s) \times f(\varphi_L(x))}{1 - \varphi_L(x)} \text{ for all } (x, s) \in \mathbb{R} \times [0, 1].$$

The function α_L is of class $C^2(\mathbb{R})$ and it satisfies $\alpha_L(x+L) = \alpha_L(x)$ for all $x \in \mathbb{R}$. The function g_L is of class $C^1(\mathbb{R} \times [0, 1])$ and it satisfies

$$g_L(x, 0) = g_L(x, 1) = 0, \quad g_L(x+L, s) = g_L(x, s) \quad \text{for all } (x, s) \in \mathbb{R} \times [0, 1].$$

Furthermore, the condition $u(x+L, y) = u(x, y)$ in \mathbb{R}^2 is equivalent to $v(x+L, y) = v(x, y)$ in \mathbb{R}^2 .

The states 0 and 1 are constant solutions of the limiting equation

$$-V''(x) + \alpha_L(x)V'(x) - g_L(x, V(x)) = 0 \quad \text{in } \mathbb{R}, \quad (2.2)$$

which is obtained from (2.1) by passing to the limit as $y \rightarrow \pm\infty$. Let us study the linear stability of these two solutions of (2.2) with respect to L -periodic perturbations. Let ψ_0 be the (unique, up to multiplications by positive constants) principal eigenfunction of the linearized operator around 0, with principal eigenvalue λ_0 . That is, ψ_0 is positive in \mathbb{R} , it satisfies $\psi_0(x+L) = \psi_0(x)$ for all $x \in \mathbb{R}$, and

$$-\psi_0''(x) + \alpha_L(x)\psi_0'(x) - \frac{\partial g_L}{\partial s}(x, 0)\psi_0(x) = \lambda_0\psi_0(x) \quad \text{for all } x \in \mathbb{R},$$

where

$$\frac{\partial g_L}{\partial s}(x, 0) = f'(\varphi_L(x)) + \frac{f(\varphi_L(x))}{1 - \varphi_L(x)}.$$

The function

$$\phi_0(x) = \psi_0(x) \times (1 - \varphi_L(x))$$

is positive in \mathbb{R} and it satisfies $\phi_0(x+L) = \phi_0(x)$ in \mathbb{R} and

$$-\phi_0''(x) - f'(\varphi_L(x))\phi_0(x) = \lambda_0\phi_0(x) \quad \text{in } \mathbb{R}.$$

In other words, λ_0 is the principal eigenvalue of the symmetric operator

$$\phi \mapsto -\phi'' - f'(\varphi_L(x))\phi$$

acting on the set of L -periodic functions ϕ , that is

$$\lambda_0 = \min_{\phi \in H_{per}^1, \phi \neq 0} \frac{\int_0^L [(\phi'(x))^2 - f'(\varphi_L(x))\phi(x)^2] dx}{\int_0^L \phi(x)^2 dx}, \quad (2.3)$$

where $H_{per}^1 = \{\phi \in H_{loc}^1(\mathbb{R}), \phi(x+L) = \phi(x) \text{ a.e. in } \mathbb{R}\}$. Moreover, the minimum in (2.3) is reached only at the principal eigenfunctions $\gamma\phi_0$ with $\gamma \in \mathbb{R}^*$. Differentiating (1.3) with $\varphi = \varphi_L$ yields

$$-(\varphi_L')'' - f'(\varphi_L)\varphi_L' = 0 \quad \text{in } \mathbb{R}.$$

By multiplying this last equation by $\phi_L := \varphi_L'$, integrating by parts over $[0, L]$ and using the L -periodicity of φ_L' , it follows that

$$\int_0^L [(\phi_L'(x))^2 - f'(\varphi_L(x))\phi_L(x)^2] dx = 0.$$

But the function $\phi_L = \varphi'_L$ does not have a constant sign in \mathbb{R} , whence

$$\lambda_0 < 0,$$

that is 0 is a linearly unstable solution of (2.2).

Similarly, denote ψ_1 a principal eigenfunction of the operator which is obtained by linearizing (2.2) around 1, with principal eigenvalue λ_1 . That is, ψ_1 is positive, L -periodic in \mathbb{R} and it satisfies

$$\begin{aligned} -\psi_1''(x) + \alpha_L(x)\psi_1'(x) - \frac{\partial g_L}{\partial s}(x, 1)\psi_1(x) \\ = -\psi_1''(x) + \alpha_L(x)\psi_1'(x) - \left(f'(1) + \frac{f(\varphi_L(x))}{1 - \varphi_L(x)} \right) \psi_1(x) = \lambda_1 \psi_1(x) \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

The function $\phi_1 = \psi_1 \times (1 - \varphi_L)$ is positive, L -periodic in \mathbb{R} and it satisfies

$$-\phi_1''(x) - f'(1)\phi_1(x) = \lambda_1 \phi_1(x) \quad \text{in } \mathbb{R}.$$

As a consequence, by uniqueness, ϕ_1 is constant and $\lambda_1 = -f'(1) > 0$, that is 1 is a linearly stable solution of (2.2).

Now, if W denotes a solution of (2.2) such that $0 \leq W(x + L) = W(x) \leq 1$ in \mathbb{R} , then the function w defined by

$$w(x) = \varphi_L(x) + W(x) \times (1 - \varphi_L(x)) \quad \text{for all } x \in \mathbb{R}$$

solves

$$-w''(x) - f(w(x)) = 0, \quad \varphi_L(x) \leq w(x) \leq 1, \quad w(x + L) = w(x) \quad \text{for all } x \in \mathbb{R}.$$

But since the functions $(\varphi_L)_{L > L_{\min}}$ are the only periodic non-constant solutions of (1.3) ranging in $(0, 1)$ and since the map $L \mapsto \max_{x \in \mathbb{R}} \varphi_L(x)$ is increasing with respect to $L \in (L_{\min}, +\infty)$, one gets that w is either identically equal to φ_L or identically equal to 1. In other words, either $W = 0$ in \mathbb{R} or $W = 1$ in \mathbb{R} .

From the results of H. Berestycki and L. Nirenberg [6] (even if it means adapting them to the periodic case as in [3], see also [1, 5] and [31, 32] for similar problems with Dirichlet boundary conditions), it follows that there exists a real number c_L^* such that solutions $v_{c,L}$ of (2.1) and (1.5) exist if and only if

$$c \geq c_L^*.$$

In other words, solutions $u_{c,L}$ of (1.4) and (1.5) exist if and only if $c \geq c_L^*$. Furthermore, all solutions $v_{c,L}$ (and $u_{c,L}$) are increasing in y and, for each $c \geq c_L^*$, they are unique up to shifts in y .

Let $u_{c,L}$ denote any solution of (1.4) and (1.5), with a speed $c \geq c_L^*$. The function

$$\tilde{u}(x, y) = u_{c,L}(-x, y)$$

is also a solution of (1.4) and (1.5), since the function φ_L is even. From the uniqueness up to shifts in y , it follows that

$$\tilde{u}(x, y) = u_{c,L}(x, y + b) \quad \text{in } \mathbb{R}^2,$$

for some $b \in \mathbb{R}$. In particular, $u_{c,L}(0, 0) = u_{c,L}(0, b)$, whence $b = 0$ since $u_{c,L}$ is increasing in y . One concludes that

$$u_{c,L}(-x, y) = u_{c,L}(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Notice that this property, combined with (1.5), implies also that

$$u_{c,L}(L/2 - x, y) = u_{c,L}(L/2 + x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2,$$

that is $u_{c,L}$ is also symmetric with respect to the axis $\{x = L/2\}$. Actually, it is symmetric with respect to any axis of the type $\{x = kL/2\}$ with $k \in \mathbb{Z}$.

Step 2 : inequality $c_L^ > c_0$.* Let us now prove that $c_L^* > c_0$, where c_0 denotes the unique speed of the planar connections between 0 and 1. Namely, for the speed c_0 , there exists a unique (up to shifts) function U_0 such that

$$-U_0'' + c_0 U_0' - f(U_0) = 0, \quad U_0(-\infty) = 0 < U_0 < 1 = U_0(+\infty) \quad \text{in } \mathbb{R},$$

and $U_0' > 0$ in \mathbb{R} . Let $u := u_{c_L^*, L}$ denote a solution of (1.4) and (1.5) with the speed c_L^* . Let us assume that $c_L^* \leq c_0$. The function $U_0(x, y) = U_0(y)$ satisfies

$$-\Delta U_0 + c_L^* U_{0,y} - f(U_0) = (c_L^* - c_0) U_0'(y) \leq 0 \quad \text{in } \mathbb{R}^2.$$

In other words, U_0 is a subsolution for the equation which is satisfied by u . Let $\rho > 0$ be chosen so that

$$f'(s) < 0 \quad \text{for all } s \in [1 - \rho, 1], \quad (2.4)$$

and let $A \in \mathbb{R}$ so that

$$1 - \rho \leq u(x, y) \leq 1 \quad \text{for all } (x, y) \in \mathbb{R} \times [A, +\infty). \quad (2.5)$$

Such a real number A does exist since $u(x, y) \rightarrow 1$ as $y \rightarrow +\infty$, uniformly in $x \in \mathbb{R}$. Since

$$U_0(-\infty) = 0 < \min_{x \in \mathbb{R}} \varphi_L(x) = \inf_{(x,y) \in \mathbb{R}^2} u(x, y),$$

there exists $\tau_0 > 0$ such that

$$U_0(A - \tau) \leq \min_{x \in \mathbb{R}} \varphi_L(x) = \inf_{(x,y) \in \mathbb{R}^2} u(x, y) \quad (2.6)$$

for all $\tau \geq \tau_0$. Fix a real number $\tau \in [\tau_0, +\infty)$. We shall now use a comparison method similar to those used in [1, 15, 30] for instance. Namely, call

$$\varepsilon^* = \inf \{ \varepsilon > 0, U_0(y - \tau) \leq u(x, y) + \varepsilon \quad \text{in } \mathbb{R} \times [A, +\infty) \}.$$

The nonnegative real number ε^* is well-defined since both U_0 and u are bounded. Notice that $U_0(y - \tau) \leq u(x, y) + \varepsilon^*$ for all $(x, y) \in \mathbb{R} \times [A, +\infty)$. Assume that $\varepsilon^* > 0$. Then,

by periodicity of u with respect to x and since $U_0(+\infty) = u(x, +\infty) = 1$ (uniformly in x), there is a point $(x^*, y^*) \in \mathbb{R} \times [A, +\infty)$ such that

$$U_0(y^* - \tau) = u(x^*, y^*) + \varepsilon^*.$$

Furthermore, $y^* > A$ because of (2.6). Extend the function f by 0 outside the interval $[0, 1]$. Consequently, the function f is nonincreasing in the interval $[1 - \rho, +\infty)$. Because of (2.5), one gets that

$$-\Delta u + c_L^* u_y - f(u + \varepsilon^*) \geq 0 \quad \text{in } \mathbb{R} \times [A, +\infty).$$

The strong maximum principle then implies that $U_0(y - \tau) = u(x, y) + \varepsilon^*$ for all $(x, y) \in \mathbb{R} \times [A, +\infty)$, which is clearly impossible as $y \rightarrow +\infty$. Therefore, $\varepsilon^* = 0$, whence

$$U_0(y - \tau) \leq u(x, y) \quad \text{for all } (x, y) \in \mathbb{R} \times [A, +\infty) \text{ and for all } \tau \geq \tau_0.$$

On the other hand, for all $(x, y) \in \mathbb{R} \times (-\infty, A]$, there holds

$$U_0(y - \tau) \leq U_0(A - \tau) \leq u(x, y)$$

from (2.6) and since U_0 is increasing. Eventually,

$$U_0(y - \tau) \leq u(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2 \text{ and for all } \tau \geq \tau_0.$$

Define

$$\tau^* = \inf \{ \tau \in \mathbb{R}, U_0(y - \tau) \leq u(x, y) \text{ for all } (x, y) \in \mathbb{R}^2 \}.$$

One has $\tau^* \leq \tau_0$, and $\tau^* > -\infty$ since $U_0(+\infty) = 1 > u(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Notice also that $U_0(y - \tau^*) \leq u(x, y)$ in \mathbb{R}^2 . If there is equality somewhere in \mathbb{R}^2 , then $U_0(y - \tau^*) = u(x, y)$ in \mathbb{R}^2 from the strong maximum principle, which is impossible since

$$U_0(-\infty) = 0 < \inf_{(x,y) \in \mathbb{R}^2} u(x, y). \quad (2.7)$$

Thus,

$$U_0(y - \tau^*) < u(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2. \quad (2.8)$$

From (2.7), (2.8) and the continuity and periodicity of u with respect to x , there exists then $\eta > 0$ such that

$$U_0(y - \tau) < u(x, y) \quad \text{for all } (x, y) \in \mathbb{R} \times (-\infty, A] \text{ and for all } \tau^* - \eta \leq \tau \leq \tau^*.$$

But the same arguments as above imply that $U_0(y - \tau) \leq u(x, y)$ for all $(x, y) \in \mathbb{R} \times [A, +\infty)$ and for all $\tau \in [\tau^* - \eta, \tau^*]$. Thus, $U_0(y - \tau) \leq u(x, y)$ in \mathbb{R}^2 for all $\tau \in [\tau^* - \eta, \tau^*]$, which contradicts the minimality of τ^* .

As a conclusion, the assumption $c_L^* \leq c_0$ leads to a contradiction. Therefore, $c_L^* > c_0$ and the proof of Theorem 1.1 is complete. \square

3 A priori periodicity

This section is devoted to the proof of Theorem 1.2, namely that any solution of (1.4) satisfying the assumption (1.7) is periodic in the variable x . To do so, we first prove a useful proposition which says that any solution of (1.4), even without (1.7), has lower and upper exponential decay rates as $y \rightarrow -\infty$, which are controlled uniformly with respect to $x \in \mathbb{R}$. Throughout this section, L denotes a fixed real number such that $L > L_{\min}$.

Proposition 3.1 *Let $u(x, y)$ be a solution of (1.4), with a speed $c \in \mathbb{R}$. Then $c > 0$ and*

$$0 < \lambda_m := \liminf_{y \rightarrow -\infty} \left(\inf_{x \in \mathbb{R}} \frac{u_y(x, y)}{u(x, y) - \varphi_L(x)} \right) \leq \limsup_{y \rightarrow -\infty} \left(\sup_{x \in \mathbb{R}} \frac{u_y(x, y)}{u(x, y) - \varphi_L(x)} \right) =: \lambda_M < +\infty.$$

Proof. Remember first that any solution of (1.4) satisfies the strict inequalities

$$\varphi_L(x) < u(x, y) < 1 \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

In this section, let v be the function defined in \mathbb{R}^2 by

$$v(x, y) = u(x, y) - \varphi_L(x). \quad (3.1)$$

It satisfies $v(x, y) \in (0, 1 - \varphi_L(x))$ and

$$-\Delta v + cv_y - f(v + \varphi_L) + f(\varphi_L) = 0 \quad \text{in } \mathbb{R}^2.$$

Since f is of class $C^1([0, 1])$, it follows from Schauder interior elliptic estimates and Harnack inequality that the function

$$(x, y) \mapsto \frac{|\nabla v(x, y)|}{v(x, y)}$$

is bounded in \mathbb{R}^2 . In particular, the quantities λ_m and λ_M defined in Proposition 3.1 are real numbers.

Let now $(x_n, y_n)_{n \in \mathbb{N}}$ be a sequence of points such that $y_n \rightarrow -\infty$ and

$$\frac{u_y(x_n, y_n)}{u(x_n, y_n) - \varphi_L(x_n)} = \frac{v_y(x_n, y_n)}{v(x_n, y_n)} \rightarrow \lambda_m \quad \text{as } n \rightarrow +\infty.$$

The family of positive functions $(v_n)_{n \in \mathbb{N}}$ defined in \mathbb{R}^2 by

$$v_n(x, y) = \frac{v(x + x_n, y + y_n)}{v(x_n, y_n)}$$

are then locally bounded. They satisfy the equations

$$-\Delta v_n + c(v_n)_y - \frac{f(v(x + x_n, y + y_n) + \varphi_L(x + x_n)) - f(\varphi_L(x + x_n))}{v(x + x_n, y + y_n)} \times v_n(x, y) = 0$$

in \mathbb{R}^2 . On the other hand, since $v(x_n, y_n) \rightarrow 0$ as $n \rightarrow +\infty$ (because of the uniformity of the limit as $y \rightarrow -\infty$ in (1.4)), one has that $v(x + x_n, y + y_n) \rightarrow 0$ as $n \rightarrow +\infty$, locally uniformly in $(x, y) \in \mathbb{R}^2$. Write

$$x_n = x'_n + x''_n,$$

where $x'_n \in LZ$ and $x''_n \in [0, L]$. Up to extraction of a subsequence, one can assume without loss of generality that $x''_n \rightarrow x_\infty \in [0, L]$ as $n \rightarrow +\infty$. From standard elliptic estimates, the functions v_n converge in $W_{loc}^{2,p}(\mathbb{R}^2)$ spaces (for all $1 \leq p < +\infty$), up to extraction of a subsequence, to a nonnegative solution v_∞ of

$$-\Delta v_\infty + c(v_\infty)_y - f'(\varphi_L(x + x_\infty))v_\infty = 0 \quad \text{in } \mathbb{R}^2$$

such that $v_\infty(0, 0) = 1$. The function v_∞ is then positive in \mathbb{R}^2 from the strong maximum principle. Moreover,

$$\frac{v_y(x + x_n, y + y_n)}{v(x + x_n, y + y_n)} = \frac{(v_n)_y(x, y)}{v_n(x, y)} \rightarrow \frac{(v_\infty)_y(x, y)}{v_\infty(x, y)} =: V(x, y) \quad \text{as } n \rightarrow +\infty,$$

locally uniformly in $(x, y) \in \mathbb{R}^2$. Therefore, $V(x, y) \geq \lambda_m$ for all $(x, y) \in \mathbb{R}^2$, whereas $(v_\infty)_y(0, 0) = \lambda_m = \lambda_m v_\infty(0, 0)$ by definition of the sequence $(x_n, y_n)_{n \in \mathbb{N}}$. In other words, the function V , which is bounded, reaches its minimum at the point $(0, 0)$. But it satisfies the equation

$$-\Delta V - 2 \frac{\nabla v_\infty}{v_\infty} \cdot \nabla V + cV_y = 0 \quad \text{in } \mathbb{R}^2.$$

The strong maximum principle yields $V(x, y) = \lambda_m$ for all $(x, y) \in \mathbb{R}^2$.

As a consequence, the function v_∞ can be written as

$$v_\infty(x, y) = e^{\lambda_m y} w(x),$$

where w is a positive solution of the ordinary differential equation

$$-w'' - \lambda_m^2 w + c\lambda_m w - f'(\varphi_L(x + x_\infty))w = 0 \quad \text{in } \mathbb{R}. \quad (3.2)$$

Furthermore, the function

$$\xi \mapsto \frac{w(\xi + L)}{w(\xi)}$$

is globally bounded. Call $\mu = \sup_{\xi \in \mathbb{R}} (w(\xi + L)/w(\xi)) > 0$ and let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that

$$\frac{w(\xi_n + L)}{w(\xi_n)} \rightarrow \mu \quad \text{as } n \rightarrow +\infty.$$

Write $\xi_n = \xi'_n + \xi''_n$, where $\xi'_n \in LZ$ and $\xi''_n \in [0, L]$. Up to extraction of a subsequence, one can assume that $\xi''_n \rightarrow \xi_\infty \in [0, L]$ as $n \rightarrow +\infty$. The sequence of positive functions $(w_n)_{n \in \mathbb{N}}$ defined in \mathbb{R} by

$$w_n(\xi) = \frac{w(\xi + \xi'_n)}{w(\xi_n)}$$

is locally bounded, and each function w_n satisfies the same equation (3.2) as w , since φ_L is L -periodic and $\xi'_n \in LZ$. Up to extraction of a subsequence, the functions w_n converge,

in $W_{loc}^{2,p}(\mathbb{R})$ spaces (for all $1 \leq p < +\infty$), to a nonnegative function w_∞ solving (3.2) and such that $w_\infty(\xi_\infty) = 1$. Consequently, w_∞ is positive in \mathbb{R} . On the other hand,

$$\mu \geq \frac{w(\xi + \xi'_n + L)}{w(\xi + \xi'_n)} = \frac{w_n(\xi + L)}{w_n(\xi)} \xrightarrow{n \rightarrow +\infty} \frac{w_\infty(\xi + L)}{w_\infty(\xi)} =: W(\xi) \quad \text{for all } \xi \in \mathbb{R},$$

whence $W(\xi) \leq \mu$ for all $\xi \in \mathbb{R}$. But $W(\xi_\infty) = \mu$ from the choice of the sequence $(\xi_n)_{n \in \mathbb{N}}$. Since the function W satisfies

$$-W'' - 2\frac{w'_\infty}{w_\infty} \times W' = 0 \quad \text{in } \mathbb{R},$$

the strong maximum principle implies that $W(\xi) = \mu$ for all $\xi \in \mathbb{R}$. That is, $w_\infty(\xi + L) = \mu w_\infty(\xi)$ for all $\xi \in \mathbb{R}$. Call $\alpha = (\ln \mu)/L$. Hence, the positive function ψ defined in \mathbb{R} by

$$\psi(x) = e^{-\alpha x} w_\infty(x)$$

satisfies $\psi(x + L) = \psi(x)$ for all $x \in \mathbb{R}$, and

$$-\psi'' - 2\alpha\psi' - [\alpha^2 + \lambda_m^2 - c\lambda_m + f'(\varphi_L(x + x_\infty))] \psi = 0 \quad \text{in } \mathbb{R}. \quad (3.3)$$

For each $\beta \in \mathbb{R}$, call $k(\beta)$ the principal eigenvalue of the operator

$$\phi \mapsto -\phi'' - 2\beta\phi' - [\beta^2 + f'(\varphi_L(x + x_\infty))] \phi$$

with L -periodicity condition. It is known (see for instance [24], or the proof of Lemma 3.1 of [5]) that the function k is concave and that $k'(0) = 0$. Therefore, $k(\beta) \leq k(0)$ for all $\beta \in \mathbb{R}$. But $k(0) = \lambda_0 < 0$, under the notations of Section 2. But (3.3) means that $k(\alpha) = \lambda_m^2 - c\lambda_m$, by uniqueness of the principal eigenvalue. One concludes that

$$\lambda_m^2 - c\lambda_m < 0.$$

Similarly, one has that $\lambda_M^2 - c\lambda_M < 0$, where λ_M is defined in Proposition 3.1. As a conclusion, $c\lambda_m$ and $c\lambda_M$ are both positive, and λ_m and λ_M have the same sign. Since they cannot be both negative (because $u(x, y) > \varphi_L(x)$ for all $(x, y) \in \mathbb{R}^2$), one concludes that c , λ_m and λ_M are positive. That completes the proof of Proposition 3.1. \square

Lemma 3.2 *Let $u(x, y)$ be a solution of (1.4), with a speed $c \in \mathbb{R}$. Then, for any real numbers $a \leq b$,*

$$\inf_{(x,y) \in \mathbb{R} \times [a,b]} v(x, y) > 0,$$

where v is defined in (3.1).

Proof. Remember first that the function v is positive in \mathbb{R}^2 . Assume now that the conclusion does not hold, for some $a \leq b \in \mathbb{R}$. Then there exists a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ in $\mathbb{R} \times [a, b]$ such that

$$v(x_n, y_n) = u(x_n, y_n) - \varphi_L(x_n) \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (3.4)$$

Write $x_n = x'_n + x''_n$, where $x'_n \in LZ$ and $x''_n \in [0, L)$. One can assume without loss of generality that $x''_n \rightarrow x_\infty \in [0, L]$ and $y_n \rightarrow y_\infty \in [a, b]$ as $n \rightarrow +\infty$. Call

$$u_n(x, y) = u(x + x'_n, y).$$

Since φ_L is L -periodic in x , the functions u_n still satisfy (1.4). Furthermore, since the functions u_n are shifts in the direction x of the same function u , they satisfy the limiting conditions $u_n(x, y) \rightarrow \varphi_L(x)$ as $y \rightarrow -\infty$ and $u_n(x, y) \rightarrow 1$ as $y \rightarrow +\infty$, uniformly with respect to $x \in \mathbb{R}$ and $n \in \mathbb{N}$. From standard elliptic estimates, the functions u_n converge in $C^2_{loc}(\mathbb{R}^2)$, up to extraction of a subsequence, to a function u_∞ which satisfies (1.4) as well. On the other hand, $u_\infty(x_\infty, y_\infty) = \varphi(x_\infty)$, from (3.4). Since $u_\infty \geq \varphi_L$ in \mathbb{R}^2 , the strong maximum principle yields

$$u_\infty(x, y) = \varphi_L(x) \text{ for all } (x, y) \in \mathbb{R}^2.$$

This is clearly impossible since $u_\infty(x, y) \rightarrow 1$ as $y \rightarrow +\infty$ and $\varphi_L(x) < 1$ for every $x \in \mathbb{R}$. \square

Proof of Theorem 1.2. Let u be a solution of (1.4) satisfying (1.7). In other words, the function $v(x, y) = u(x, y) - \varphi_L(x)$ satisfies

$$\limsup_{y \rightarrow -\infty} \frac{\sup_{x \in \mathbb{R}} v(x, y)}{\inf_{x \in \mathbb{R}} v(x, y)} < +\infty. \quad (3.5)$$

Notice that, for each $y \in \mathbb{R}$, $\inf_{x \in \mathbb{R}} v(x, y) > 0$ from Lemma 3.2.

Fix an integer $N \in \mathbb{N}$. For any $\tau \in \mathbb{R}$, call u^τ the function defined in \mathbb{R}^2 by

$$u^\tau(x, y) = u(x + NL, y - \tau)$$

and observe that u^τ still satisfies (1.4). Let $\rho > 0$ and $A \in \mathbb{R}$ be chosen so that (2.4) and (2.5) hold. We now claim that there exists $\tau_0 > 0$ such that

$$\forall \tau \geq \tau_0, \forall (x, y) \in \mathbb{R} \times (-\infty, A], \quad u^\tau(x, y) \leq u(x, y). \quad (3.6)$$

Assume not. Then there exists two sequences $(\tau_n)_{n \in \mathbb{N}}$ in \mathbb{R} and $(x_n, y_n)_{n \in \mathbb{N}}$ in $\mathbb{R} \times (-\infty, A]$ such that $\tau_n \rightarrow +\infty$ as $n \rightarrow +\infty$, and

$$\forall n \in \mathbb{N}, \quad u^{\tau_n}(x_n, y_n) = u(x_n + NL, y_n - \tau_n) > u(x_n, y_n). \quad (3.7)$$

Two cases may occur : either $y_n \rightarrow y_\infty \in (-\infty, A]$, or $y_n \rightarrow -\infty$ as $n \rightarrow +\infty$, up to extraction of a subsequence. In the former case,

$$u(x_n + NL, y_n - \tau_n) - \varphi_L(x_n) = u(x_n + NL, y_n - \tau_n) - \varphi_L(x_n + NL) \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

from the uniformity of the limit of $u(x, y)$ as $y \rightarrow -\infty$. However,

$$\liminf_{n \rightarrow +\infty} (u(x_n, y_n) - \varphi_L(x_n)) > 0$$

from Lemma 3.2. This is in contradiction with (3.7). Therefore, $y_n \rightarrow -\infty$ as $n \rightarrow +\infty$. From Proposition 3.1, there exists $B \in \mathbb{R}$ such that

$$v_y(x, y) \geq \frac{\lambda_m v(x, y)}{2} \quad \text{for all } (x, y) \in \mathbb{R} \times (-\infty, B], \quad (3.8)$$

where $\lambda_m > 0$. For n large enough, there holds $y_n - \tau_n \leq y_n \leq B$, whence

$$v(x_n + NL, y_n - \tau_n) \leq e^{-\lambda_m \tau_n / 2} v(x_n + NL, y_n).$$

On the other hand, as already underlined in the proof of Proposition 3.1, the Harnack inequality applied to the positive function v provides the existence of a positive constant C_1 such that

$$v(x + NL, y) \leq C_1 v(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Thus, for n large enough,

$$v(x_n + NL, y_n - \tau_n) \leq C_1 e^{-\lambda_m \tau_n / 2} v(x_n, y_n),$$

whence $v(x_n + NL, y_n - \tau_n) \leq v(x_n, y_n)$ for n large enough, since $\tau_n \rightarrow +\infty$. As a consequence, $u(x_n + NL, y_n - \tau_n) \leq u(x_n, y_n)$ for n large, which contradicts (3.7). The claim (3.6) is proved.

Let now $\tau_0 > 0$ be as in (3.6). For each $\tau \geq \tau_0$, the function u^τ satisfies in particular $u^\tau(x, A) \leq u(x, A)$ for all $x \in \mathbb{R}$, while $u(x, y) \geq 1 - \rho$ for all $(x, y) \in \mathbb{R} \times [A, +\infty)$. Since both u and u^τ solve (1.4), the arguments of Step 2 of the proof of Theorem 1.1 imply that $u^\tau \leq u$ in $\mathbb{R} \times [A, +\infty)$ for all $\tau \geq \tau_0$. Together with (3.6), one gets that

$$\forall \tau \geq \tau_0, \quad u^\tau \leq u \quad \text{in } \mathbb{R}^2.$$

Define

$$\tau^* = \inf \{ \tau > 0, \quad u^\tau \leq u \text{ in } \mathbb{R}^2 \}. \quad (3.9)$$

The real number τ^* is well-defined and satisfies $0 \leq \tau^* \leq \tau_0$. The function w defined in \mathbb{R}^2 by

$$w(x, y) = u(x, y) - u^{\tau^*}(x, y) = u(x, y) - u(x + NL, y - \tau^*)$$

is nonnegative in \mathbb{R}^2 . Assume that $\tau^* > 0$. Two cases may occur : either there is $\varepsilon > 0$ such that

$$w(x, y) \geq \varepsilon v(x, y) \quad \text{for all } (x, y) \in \mathbb{R} \times (-\infty, A], \quad (3.10)$$

or not.

Assume first that such a positive ε exists. Lemma 3.2 then implies that $\inf_{x \in \mathbb{R}} w(x, A) > 0$. Since u is (at least) uniformly continuous in \mathbb{R}^2 , it follows that there exists $\eta > 0$ such that

$$u(x, A) \geq u(x + NL, A - \tau) \quad \text{for all } x \in \mathbb{R} \text{ and } \tau^* - \eta \leq \tau \leq \tau^*.$$

Once again, as in Step 2 of the proof of Theorem 1.1, this yields

$$u(x, y) \geq u(x + NL, y - \tau) \quad \text{for all } (x, y) \in \mathbb{R} \times [A, +\infty) \text{ and } \tau^* - \eta \leq \tau \leq \tau^*. \quad (3.11)$$

On the other hand, the Harnack inequality applied to the positive function v provides the existence of another positive constant C_2 such that $v(x, y) \geq C_2 v(x + NL, y - \tau^*)$ for all $(x, y) \in \mathbb{R}^2$. Assumption (3.10) implies that

$$u(x, y) - u(x + NL, y - \tau^*) = w(x, y) \geq \varepsilon C_2 \times [u(x + NL, y - \tau^*) - \varphi_L(x + NL)]$$

for all $(x, y) \in \mathbb{R} \times (-\infty, A]$. Thus,

$$u(x, y) - \varphi_L(x + NL) \geq (1 + \varepsilon C_2) \times [u(x + NL, y - \tau^*) - \varphi_L(x + NL)] \quad (3.12)$$

for all $(x, y) \in \mathbb{R} \times (-\infty, A]$. But since the function $|\nabla(u - \varphi_L)|/(u - \varphi_L)$ is globally bounded (as in the proof of Proposition 3.1), there exists a real number $\eta' \in (0, \eta)$ such that

$$(1 + \varepsilon C_2) \times ([u(x + NL, y - \tau^*) - \varphi_L(x + NL)]) \geq u(x + NL, y - \tau) - \varphi_L(x + NL) \quad (3.13)$$

for all $(x, y) \in \mathbb{R}^2$ and $\tau^* - \eta' \leq \tau \leq \tau^*$. It follows from (3.12) and (3.13) that

$$u(x, y) \geq u(x + NL, y - \tau) \quad \text{for all } (x, y) \in \mathbb{R} \times (-\infty, A] \text{ and } \tau^* - \eta' \leq \tau \leq \tau^*.$$

Together with (3.11), and since $\eta' < \eta$, one concludes that $u \geq u^\tau$ in \mathbb{R}^2 for all $\tau^* - \eta' \leq \tau \leq \tau^*$. This contradicts the minimality of τ^* in (3.9).

Therefore, (3.10) cannot occur and there exists a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ in $\mathbb{R} \times (-\infty, A]$ such that

$$\forall n \in \mathbb{N}, \quad 0 \leq w(x_n, y_n) < \frac{v(x_n, y_n)}{n + 1}.$$

Assume first that the sequence $(y_n)_{n \in \mathbb{N}}$ is bounded, and write $x_n = x'_n + x''_n$, where $x'_n \in LZ$ and $x''_n \in [0, L]$. Up to extraction of a subsequence, there holds $(x''_n, y_n) \rightarrow (x_\infty, y_\infty) \in [0, L] \times (-\infty, A]$ as $n \rightarrow +\infty$ and the functions $u_n(x, y) = u(x + x'_n, y)$ converge in $C_{loc}^2(\mathbb{R}^2)$ to a solution u_∞ of (1.4) such that

$$u_\infty(x, y) \geq u_\infty(x + NL, y - \tau^*) \quad \text{in } \mathbb{R}^2$$

with equality at the point (x_∞, y_∞) . It follows from the strong maximum principle that

$$u_\infty(x, y) = u_\infty(x + NL, y - \tau^*) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

As a consequence, $u_\infty(0, 0) = u_\infty(kNL, -k\tau^*)$ for all $k \in \mathbb{Z}$. Since u_∞ still satisfies the same uniform limiting conditions as u as $y \rightarrow \pm\infty$ and since $\tau^* > 0$, one concludes, by letting $k \rightarrow \pm\infty$, that $1 = \varphi_L(0)$, which is a contradiction.

Thus, the sequence $(y_n)_{n \in \mathbb{N}}$ is not bounded, and up to extraction of a subsequence, one can then assume without loss of generality that $y_n \rightarrow -\infty$ as $n \rightarrow +\infty$. Since the nonnegative function w satisfies

$$-\Delta w + cw_y - f(u(x, y)) + f(u(x + NL, y - \tau^*)) = 0 \quad \text{in } \mathbb{R}^2$$

and since f is of class $C^1([0, 1])$, the Harnack inequality applied to w provides the existence of a positive constant C_3 such that

$$w(x + NL, y - \tau^*) \leq C_3 w(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Consequently, for all $n \in \mathbb{N}$ and $0 \leq j \leq k \in \mathbb{N}$,

$$\begin{aligned} & u(x_n + jNL, y_n - j\tau^*) - u(x_n + (j+1)NL, y_n - (j+1)\tau^*) \\ &= w(x_n + jNL, y_n - j\tau^*) \leq C_3^j \times w(x_n, y_n) < C_3^j \times \frac{v(x_n, y_n)}{n+1}, \end{aligned}$$

whence, by summing from $j = 0$ to k ,

$$u(x_n, y_n) - u(x_n + (k+1)NL, y_n - (k+1)\tau^*) < \frac{1 + C_3 + \cdots + C_3^k}{n+1} \times v(x_n, y_n).$$

Since φ_L is L -periodic, the previous inequality can be rewritten as

$$v(x_n, y_n) - v(x_n + (k+1)NL, y_n - (k+1)\tau^*) < \frac{1 + C_3 + \cdots + C_3^k}{n+1} \times v(x_n, y_n)$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$. On the other hand, remember that v is bounded in \mathbb{R}^2 and that $\inf_{x \in \mathbb{R}} v(x, y) \rightarrow 1 - \sup_{x \in \mathbb{R}} \varphi_L(x) > 0$ as $y \rightarrow +\infty$. Together with assumption (3.5) and Lemma 3.2, it follows that there exists a constant $C_4 > 0$ such that

$$\forall y \in \mathbb{R}, \quad 0 < \frac{\sup_{x \in \mathbb{R}} v(x, y)}{\inf_{x \in \mathbb{R}} v(x, y)} \leq C_4.$$

In particular,

$$\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, \quad v(x_n + (k+1)NL, y_n - (k+1)\tau^*) \leq C_4 v(x_n, y_n - (k+1)\tau^*),$$

whence

$$v(x_n, y_n) < C_4 v(x_n, y_n - (k+1)\tau^*) + \frac{1 + C_3 + \cdots + C_3^k}{n+1} \times v(x_n, y_n). \quad (3.14)$$

Because of (3.8) and the positivity of τ^* , there holds

$$\forall (x, y) \in \mathbb{R} \times (-\infty, B], \forall k \in \mathbb{N}, \quad v(x, y) \geq e^{\lambda_m(k+1)\tau^*/2} v(x, y - (k+1)\tau^*). \quad (3.15)$$

Choose now $k_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that

$$e^{\lambda_m(k_0+1)\tau^*/2} \geq 2C_4, \quad \frac{1 + C_3 + \cdots + C_3^{k_0}}{n_0 + 1} \leq \frac{1}{2} \quad \text{and} \quad \sup_{n \geq n_0} y_n \leq B.$$

It follows from (3.14), applied with $n \geq n_0$ and k_0 , that

$$\forall n \geq n_0, \quad v(x_n, y_n) < 2C_4 v(x_n, y_n - (k_0 + 1)\tau^*).$$

Together with (3.15) (applied with k_0 and (x_n, y_n) , $n \geq n_0$), it follows that

$$e^{\lambda_m(k_0+1)\tau^*/2} v(x_n, y_n - (k_0 + 1)\tau^*) < 2C_4 v(x_n, y_n - (k_0 + 1)\tau^*),$$

which contradicts the choice of k_0 , since $v > 0$ in \mathbb{R}^2 .

As a conclusion, τ^* cannot be positive, which implies that $\tau^* = 0$ and

$$u(x + NL, y) \leq u(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Since the previous inequality is true for all $N \in \mathbb{Z}$, one concludes that

$$u(x + L, y) = u(x, y) \quad \text{for all } (x, y) \in \mathbb{R}^2,$$

which completes the proof of Theorem 1.2. \square

Assume now that u satisfies (1.4) and that there exist a solution $u_{c,L}$ of (1.4) satisfying (1.5), and $a < b \in \mathbb{R}$ such that

$$u_{c,L}(x, y + a) \leq u(x, y) \leq u_{c,L}(x, y + b) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

Let us check that, in this case, assumption (1.7) is fulfilled. Call

$$v(x, y) = u(x, y) - \varphi_L(x) \quad \text{and} \quad \tilde{v}(x, y) = u_{c,L}(x, y) - \varphi_L(x).$$

There holds $\tilde{v}(x, y + a) \leq v(x, y) \leq \tilde{v}(x, y + b)$ for all $(x, y) \in \mathbb{R}^2$. Therefore, for every $y \in \mathbb{R}$,

$$\sup_{x \in \mathbb{R}} v(x, y) \leq \sup_{x \in \mathbb{R}} \tilde{v}(x, y + b) = \max_{0 \leq x \leq L} \tilde{w}(x, y + b),$$

since $x \mapsto \tilde{w}(x, y)$ is continuous and L -periodic. But, from the Harnack inequality applied to $\tilde{w} \geq 0$, there exists a constant $C_5 > 0$ such that

$$\forall y \in \mathbb{R}, \quad \max_{0 \leq x \leq L} \tilde{w}(x, y + b) \leq C_5 \times \min_{0 \leq x \leq L} \tilde{w}(x, y + a).$$

Thus, using again the L -periodicity of \tilde{w} with respect to x , one gets that, for every $y \in \mathbb{R}$,

$$\sup_{x \in \mathbb{R}} v(x, y) \leq C_5 \times \min_{0 \leq x \leq L} \tilde{w}(x, y + a) = C_5 \times \min_{x \in \mathbb{R}} \tilde{w}(x, y + a) \leq C_5 \times \inf_{x \in \mathbb{R}} v(x, y).$$

As a conclusion, assumption (1.7) is fulfilled. It then follows from Theorems 1.1 and 1.2 that $u(x, y) = u_{c,L}(x, y + \gamma)$ for all $(x, y) \in \mathbb{R}^2$ and for some $\gamma \in [a, b]$.

4 Continuity and convergence to flat fronts as $L \rightarrow L_{\min}^+$

The basic tool to prove Theorem 1.3 is the min-max formula for the minimal velocity c_L^* . It is essentially proved in [9, 12]; because the context is slightly different here (the nonlinearity does not have any sign) we will recall the main ideas. First, for $0 < L \leq L_{\min}$, we define $\varphi_L = \theta$ and $c_L^* = c^*$, that is the minimal speed of planar fronts connecting θ to 1 monotonically.

Let us consider the exponential solutions as $y \rightarrow -\infty$ of (2.1), i.e. the solutions $e^{\lambda_L y} \psi_L(x)$ of (2.1) linearised at $v = 0$. We have the relation

$$c_L^* \geq 2\sqrt{\mu_1(L)} := c_L^{KPP}, \quad (4.1)$$

where $-\mu_1(L)$ is the first periodic eigenvalue of

$$-\frac{d^2}{dx^2} + \alpha_L \frac{d}{dx} - \partial_v g_L(x, 0)$$

on $(-L, L)$. Set also, for $c \geq c_L^{KPP}$:

$$\lambda_L^\pm(c) = \frac{c \pm \sqrt{c^2 - (c_L^{KPP})^2}}{2}.$$

The following is true:

Proposition 4.1 ([6, 25]) *For any $L > 0$, if $c > c_L^*$, then, for a convenient translation of $u_{c,L}$ in y we have, as $y \rightarrow -\infty$:*

$$u_{c,L}(x, y) = \psi_L(x) e^{\lambda_L^-(c)y} + O(e^{(\lambda_L^-(c)+\delta)y}) \quad (4.2)$$

for some $\delta > 0$. If $c_L^* > c_L^{KPP}$, then, for a convenient translation of $u_{c,L}$ in y we have, as $y \rightarrow -\infty$:

$$u_{c_L^*,L}(x, y) = \psi_L(x) e^{\lambda_L^+(c_L^*)y} + O(e^{(\lambda_L^+(c_L^*)+\delta)y}) \quad (4.3)$$

for some $\delta > 0$.

The min-max formula for c_L^* is then the following

Theorem 4.2 ([12]) *Let $L > 0$ be given and \mathcal{E} be the set of all functions $w(x, y)$ which are C^2 and L -periodic in \mathbb{R}^2 , such that*

$$\lim_{y \rightarrow -\infty} u(x, y) = 0, \quad \lim_{y \rightarrow +\infty} u(x, y) = 1$$

uniformly with respect to x , and such that $\partial_y u > 0$. Then with the notations of Section 2 – and, in particular, (2.1) – we have

$$c_L^* = \min_{w \in \mathcal{E}} \sup_{(x,y) \in [-L,L] \times \mathbb{R}} \frac{\Delta w - \alpha_L(x) \partial_x w + g_L(x, w)}{\partial_y w}.$$

Proof. Obviously, c_L^* is above the right handside, it remains to see that it cannot be strictly above. Suppose this is not the case, there exists $\varepsilon > 0$ and an element w_ε of \mathcal{E} such that

$$-\Delta w_\varepsilon + (c_L^* - \varepsilon) \partial_y w_\varepsilon + \alpha_L(x) \partial_x w_\varepsilon \geq g_L(x, w_\varepsilon).$$

In particular, if H is the Heaviside function and $\delta > 0$ is small, there exists $y_\delta \in \mathbb{R}$ such that

$$w_\varepsilon(x, y + y_\delta) \geq (1 - \delta)H(y).$$

And, if $w(t, x, y)$ solves the Cauchy problem

$$\begin{aligned} w_t - \Delta w + \alpha_L(x)\partial_x w &= g_L(x, w) \\ w(0, x, y) &= (1 - \delta)H(y) \end{aligned} \quad (4.4)$$

we have

$$w_\varepsilon(x, y + y_\delta + (c_L^* - \varepsilon)t) \geq w(t, x, y). \quad (4.5)$$

Case 1. $c_L^* > c_L^{KPP}$. Proposition 4.1 implies – see [25] – that the family of waves $u_{c_L^*, L}$ attracts all the solutions of (4.4), provided that they initially decay faster than $e^{\lambda_L^-(c_L^*)y}$ as $y \rightarrow -\infty$, and have a positive liminf as $y \rightarrow +\infty$ – which is certainly the case here. As $t \rightarrow +\infty$ we infer from (4.5) that

$$w_\varepsilon(x, y + y_\delta + (c_L^* - \varepsilon)t) \geq u_{c_L^*, L}(x, y + c_L^*t) + o_{t \rightarrow +\infty}(1),$$

an impossibility.

Case 2. $c_L^* = c_L^{KPP}$. Let us choose $\varepsilon' > 0$ such that

$$c_L^* - \varepsilon < 2\sqrt{\mu_1(L) - \varepsilon'}. \quad (4.6)$$

There is $\delta > 0$ such that

$$g_L(x, v) \geq (\partial_v g_L(x, 0) - \varepsilon')v \quad \text{if } v \in [0, \delta].$$

From (4.6) and [20], there is a nontrivial compactly supported solution of

$$-\Delta\phi + \alpha_L(x)\partial_x\phi + (c_L^* - \frac{\varepsilon}{2})\partial_y\phi = (\partial_v g_L(x, 0) - \varepsilon')\phi, \quad \phi \geq 0$$

and, for $q > 0$ small enough, $\underline{v} := q\phi$ is a solves

$$-\Delta\underline{v} + \alpha_L(x)\partial_x\underline{v} + (c_L^* - \frac{\varepsilon}{2})\partial_y\underline{v} \leq g_L(x, \underline{v}).$$

And so we have, possibly by restricting q a little bit more:

$$\underline{v}(x, y) \leq w_\varepsilon(x, y).$$

Thus

$$\underline{v}(x, y + (c_L^* - \frac{\varepsilon}{2})t) \leq w_\varepsilon(x, y + (c_L^* - \varepsilon)t),$$

a contradiction. \square

Proof of Theorem 1.3. Consider $(L_1, L_2) \in (0, +\infty)^2$ and the minimal velocities $c_{L_i}^*$. Let also $u_i^* := u_{c_{L_i}^*, L_i}$ be the corresponding wave solutions. Let us also set, for $v \in \mathcal{E}$:

$$Q_L[v] = \frac{\Delta w - \alpha_L(x)\partial_x w + g_L(x, w)}{\partial_y w}.$$

Set $u_{1,2}(x, y) = u_1^*(\frac{L_1}{L_2}x, y)$ and compute

$$Q_{L_2}[u_{1,2}](x, y) = \left(\frac{L_1^2}{L_2^2} - 1\right) \frac{\partial_{xx} u_1^*(\frac{L_1}{L_2}x, y)}{\partial_y u_1^*(\frac{L_1}{L_2}x, y)} + (\alpha_{L_1}(\frac{L_1}{L_2}x) - \alpha_{L_2}(x)) \frac{\partial_x u_1^*(\frac{L_1}{L_2}x, y)}{\partial_y u_1^*(\frac{L_1}{L_2}x, y)} \\ + \frac{g_{L_2}(x, u_{1,2}) - g_{L_1}(\frac{L_1}{L_2}x, u_{1,2})}{\partial_y u_1^*(\frac{L_1}{L_2}x, y)} + Q_{L_1}[u_1^*]$$

Because $\partial_x u_1^*$ and $\partial_{xx} u_1^*$ decay at the same exponential rate as $\partial_y u_1^*$ and u_1^* as $y \rightarrow \pm\infty$ – see [6] – we have

$$Q_{L_2}[u_{1,2}](x, y) \leq C_{L_1, L_2} |L_1 - L_2| + Q_{L_1}[u_1^*].$$

The constant C_{L_1, L_2} is bounded on every compact set in (L_1, L_2) and this implies, passing to the inf, that $L \mapsto c_L$ is locally Lipschitz, whence $c_L^* \rightarrow c^*$ as $L \rightarrow L_{\min}^+$.

It especially implies that, for a given $c > c^*$, the functions $u_{c,L}$ (normalized by $u_{c,L}(0, 0) = U_c(0)$) have a limit $u_{c, L_{\min}}$ as $L \rightarrow L_{\min}^+$ (up to a subsequence). We claim that the limit can only be the flat front connecting θ to 1: indeed, as $L \rightarrow L_{\min}^+$, we have

$$u_{c,L}(x, y) \geq \varphi_L(x)$$

which yields, sending $L \rightarrow L_{\min}^+$:

$$u_{c, L_{\min}}(x, y) \geq \theta.$$

Because $u_{c, L_{\min}}$ is nontrivial and above θ , the nonexistence of a steady solution at $-\infty$ above θ implies the convergence of $u_{c, L_{\min}}$ to θ as $y \rightarrow -\infty$ and to 1 as $y \rightarrow +\infty$. By uniqueness and the L_{\min} -periodicity in x of $u_{c, L_{\min}}$, we conclude that $u_{c, L_{\min}}$ is the planar front U_c and $u_{c,L}(x, y) \rightarrow U_c(y)$ locally uniformly in $(x, y) \in \mathbb{R}^2$ as $L \rightarrow L_{\min}^+$. \square

5 Connections between 1 and the ground state

Let $\lambda_{\infty}^{\pm}(c)$ be given by

$$\lambda_{\infty}^{\pm} = \frac{c \pm \sqrt{c^2 - (c_{\infty}^{KPP})^2}}{2}$$

Notice that there is here no difficulty in defining $\mu_1(+\infty)$: the Rayleigh formula for $\mu_1(L)$ and the property $f'(0) < 0$ ensures that $\mu_1(L)$ has a limit as $L \rightarrow +\infty$.

Let us say a little more about the geometry of a possible solution $u_{c, \infty}$ to (1.9), where the limits as $y \rightarrow \pm\infty$ are a priori assumed to be only pointwise, which is even in x and which satisfies $\partial_y u_{c, \infty} > 0$ in \mathbb{R}^2 with $c > 0$. The arguments involved are all borrowed to the recent papers [14, 15, 16], and will therefore not be given in full detail.

Recall that c_0 is the velocity of the planar front connecting 0 to 1. The first lemma that comes up is the

Lemma 5.1 *We have $c > c_0$.*

Proof. Consider $\lambda \in (\sup \varphi_\infty, 1)$. By [15], the function $\partial_y u_{c,\infty}$ is uniformly bounded away from 0 on the level set $\{u_{c,\infty} = \lambda\}$, hence the set $\{u_{c,\infty} = \lambda\}$ is a globally Lipschitz curve $\{y = h_\lambda(x)\}$. Notice then that, by Theorem 1.6 of [15], if $(x_n)_n$ is a sequence converging to $+\infty$ such that

$$\lim_{n \rightarrow +\infty} \frac{\partial_x u_{c,\infty}(x_n, h_\lambda(x_n))}{\partial_y u_{c,\infty}(x_n, h_\lambda(x_n))} = - \limsup_{x \rightarrow +\infty} h'_\lambda(x),$$

then the sequence $(u_{c,\infty}(x_n + x, h_\lambda(x_n) + y))_n$ converges to a planar solution of (1.2). This planar solution is nontrivial, and connects either 0 to 1 or θ to 1. Let \tilde{c} be its normal velocity, there holds $\tilde{c} \geq c_0$. We have, still arguing as in Theorem 1.6 of [15]:

$$\lim_{x \rightarrow +\infty} h'_\lambda(x) = \pm \cot \alpha, \quad \sin \alpha = \frac{\tilde{c}}{c} \in (0, 1], \quad \alpha \in \left(0, \frac{\pi}{2}\right]. \quad (5.1)$$

This implies in particular that $c \geq c_0$, because $\tilde{c} \geq c_0$. Thus it remains to see what happens if $c = c_0$; in this case we have

$$\lim_{x \rightarrow +\infty} h'_\lambda(x) = 0.$$

In fact - see Theorem 1 of [16] - we not only have the above limit, we also have that $h_\lambda(x)$ is bounded. Thus, the function $u_{c,\infty}(x, y)$ is trapped, as $x \rightarrow +\infty$, between two translates of the 1D front $\phi_0(y)$ connecting 0 to 1. As a consequence, $u_{c,\infty}(x, y) \rightarrow 1$ as $y \rightarrow +\infty$ uniformly in $x \in \mathbb{R}$ and there is $y_0 > 0$ such that

$$u_{c,\infty}(x, y + y_0) \geq \phi_0(y)$$

and we may apply the standard sliding method to conclude to a contradiction. \square

To decide between the plus and minus sign in (5.1), suppose that the minus sign holds. Then (5.1) holds - [15] once again - for *all* level sets, and comparison with the classical conical wave with the same velocity, in the same spirit as in Lemma 5.1, provides a contradiction. Therefore, $h'_\lambda(x) \rightarrow \cot \alpha$ as $x \rightarrow +\infty$. This in turn implies the convergence of $u_{c,\infty}$ to 0 in every strict subcone of

$$\tilde{\mathcal{C}}_\alpha = \{\arg X \in (-\frac{\pi}{2}, \frac{\pi}{2} - \alpha)\}, \quad X = (x, y) \in \mathbb{R}^2,$$

and the limit $\lim_{y \rightarrow -\infty} u_{c,\infty}(x, y) = \varphi_\infty(x)$ holds uniformly. In the following figure, the level sets of $u_{c,\infty}$ are depicted.

The proof of Theorem 1.4 will result from the following

Lemma 5.2 *Assume the existence of a solution u_{c_0} to (1.2), with $u_{c_0}(x, -\infty) = \varphi_\infty(x)$, with the monotonicity and symmetry properties stated in Theorem 1.4. Assume that $\lambda_\infty^-(c_0) < \lambda_\infty^+(c_0)$ and that, for some translation in y of u_{c_0} , we have*

$$u_{c_0}(x, y) = \varphi_\infty(x) + e^{\lambda_\infty^-(c_0)y} \psi_\infty(x) + O(e^{(\lambda_\infty^-(c_0)+\delta)y}) \quad \text{as } y \rightarrow -\infty,$$

where $\delta > 0$ and ψ_∞ is a local uniform limit of the functions ψ_L suitably normalized. There exists $\varepsilon_0 > 0$ and $L_0 > 0$ large such that, for any $c \in [c_0 - \varepsilon_0, c_0 + \varepsilon_0]$ and for any $L \geq L_0$, Problem (1.2) has a solution $u_{c,L}$ for $\varphi = \varphi_L$.

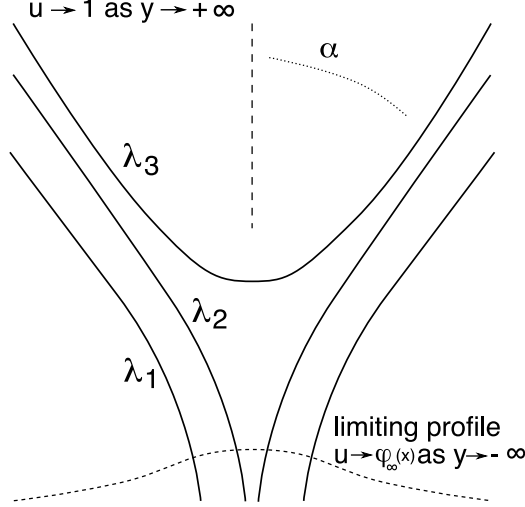


Figure 1: Three level sets $\{(x, y) \in \mathbb{R}^2, u(x, y) = \lambda_i\}$ of an even in x solution u of (1.9) such that $\partial_y u > 0$: $0 < \lambda_1 < \max_{\mathbb{R}} \varphi_\infty$, $\lambda_2 = \max_{\mathbb{R}} \varphi_\infty$, $\max_{\mathbb{R}} \varphi_\infty < \lambda_3 < 1$.

This lemma itself follows from the following theorem:

Theorem 5.3 *Under the assumptions of Lemma 5.2, let*

$$\mathcal{L}_\infty^{c_0} = -\Delta + c\partial_y - f'(u_{c_0})$$

in the space X_r of even functions $v \in BUC(\mathbb{R}^2)$ such that $(1 + e^{-ry})v(x, y) \in BUC(\mathbb{R}^2)$, with

$$r = \frac{1}{2}(\lambda_\infty^-(c_0) + \lambda_\infty^+(c_0)).$$

Then $\mathcal{L}_\infty^{c_0}$ is an isomorphism of X_r . Moreover, assume the conclusion of Lemma 5.2 to hold true, and let $u_{c,L}$ be the solution of Problem (1.2) for $\varphi = \varphi_L$. Set

$$\mathcal{L}_L^c = -\Delta + c\partial_y - f'(u_{c,L})$$

in the subspace X_r^L of all L -periodic functions of X_r . Then the family $(\mathcal{L}_L^c)^{-1}$ is uniformly bounded for c in $[c_0 - \varepsilon_0, c_0 + \varepsilon_0]$ and $L \geq L_0$.

Proof. (sketch) The property of $\mathcal{L}_\infty^{c_0}$ is proved exactly in the same fashion as Theorem 4.1 in [14]. To prove the property on $\mathcal{L}_\infty^{c_0}$, it is sufficient to prove that there cannot be a sequence $(U_{c,L})_{L \geq L_0}$ such that

$$\|U_{c,L}\|_{D(\mathcal{L}_L^c)} = 1, \quad \lim_{L \rightarrow +\infty, c \rightarrow c_0} \|\mathcal{L}_L^c U_{c,L}\|_{X_r} = 0.$$

If $X_{c,L} := (x_{c,L}, y_{c,L})$ is a point where $U_{c,L}$ reaches half its supremum, set

$$\tilde{U}_{c,L} = U_{c,L}(\cdot + X_{c,L}).$$

By compactness, it converges to some function $U_{c_0,\infty}$ locally. Because $U_{c,L}$ is uniformly bounded in X_r^L , there are only two cases to work out.

Case 1. The family $(X_{c,L})_{c,L}$ is bounded. Thus it can be assumed to converge, and thus there is a bounded solution U of $\mathcal{L}_\infty^c U = 0$. Impossible.

Case 2. We have $\lim_{L \rightarrow +\infty, c \rightarrow c_0} y_{c,L} = +\infty$ and the family $\frac{x_{c,L}}{y_{c,L}}$ is bounded. Then the last sequence may be assumed to converge to some real number γ , that we may assume to be 0 by the evenness of $u_{c,L}$. If $\gamma < \cot\alpha$, we have found a bounded, nonzero solution of

$$-\Delta U + c\partial_y U - f'(0)U = 0,$$

an impossibility. If, on the contrary, we have $\gamma > \cot\alpha$, we have found a solution of

$$-\Delta U + c\partial_y U - f'(1)U = 0,$$

once again an impossibility. Finally, if $\gamma = \cot\alpha$, we have (depending on how the sequence $(y_{c,L} - \cot\alpha x_{c,L})_{c,L}$ behaves, i.e. if it diverges or remains bounded): one of the above cases or, on the contrary, a solution $U(x, y)$ of

$$-\Delta U + c\partial_y U - f'(\phi_0(X.e_\alpha))U = 0,$$

where e_α is the (counter-clockwise) rotation of e_1 by the angle α , and ϕ_0 the basic connection between 0 and 1. This (see the proof of Theorem 4.1 in [14]) is impossible. \square

Proof of Lemma 5.2. If $\gamma(y)$ is the usual cut-off function which is zero for $y \leq -M - 1$ ($M > 0$ large) and equal to 1 for $y \geq -M$, we look for a solution $u_{c,L}$ under the form

$$\varphi_L(x)(1 - \gamma(y)) + u_{c,\infty}(x, y)\gamma(y)$$

and argue as in the proof of Theorem 2.1 in [25], having in mind the uniform convergence of φ_L to φ_∞ in $[-\frac{L}{2}, \frac{L}{2}]$ and using Theorem 5.3. \square

Lemma 5.2 shows that (i) the set of possible velocities for Problem (1.2) with $\varphi = \varphi_\infty$ is, just as in the bounded case, an interval; (ii) either the bottom speed is c_∞^{KPP} , or it is strictly larger than c_∞^{KPP} and, in this case, the corresponding wave chooses the fastest decay - in contrast with the waves of higher velocities.

All this leads to the

Proof of Theorem 1.4. It is enough to prove that the set of possible velocities is nonzero, the rest of the properties of Theorem 1.4 follows easily from Lemma 5.2. To prove the existence of a solution $u_{c,\infty}$ to Problem (1.2) with $\varphi = \varphi_\infty$ and c large, we look for $u_{c,L}$ under the form

$$u_{c,L}(x, y) = \gamma(y)(\varphi_L(x) + e^{\lambda_\infty^-(c)y}\psi_\infty(x)) + e^{r(c)y}v(x, y)$$

where

$$r(c) = \frac{1}{2}(\lambda_\infty^-(c) + \lambda_\infty^+(c)).$$

Notice that $\lim_{c \rightarrow +\infty} r(c) = +\infty$. The equation for v has the form

$$-\Delta v + c\partial_y v = h(x, y, v)$$

and the important features of h are that $h(x, y, 0)$ is supported on $[-M - 1, M + 1]$ and

$$|h(x, y, v_1) - h(x, y, v_2)| \leq K|v_1 - v_2|, \quad K \text{ independent of } c.$$

The change of unknowns

$$v(x, y) = e^{r(c)y} w(x, y)$$

transforms the equation for v into (we drop the c -dependence of r):

$$-\Delta w + (c - 2r)\partial_y w + (cr - r^2)w = e^{-ry} h(x, y, e^{ry} w) := k(x, y, w). \quad (5.2)$$

We have

$$cr - r^2 \sim_{c \rightarrow +\infty} \frac{c^2}{4},$$

and k is K -Lipschitz, K independent of c . Therefore there is a unique w satisfying (5.2), which ends the proof of Theorem 1.4. \square

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