The minimal cone of an algebraic Laurent series

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Abstract

For a given Laurent series that is algebraic over the field of power series in several indeterminates over a characteristic zero field, we show that the convex hull of its support is essentially a polyhedral rational cone. One of the main tools for proving this is the Abhyankar-Jung Theorem. Then we prove a positive characteristic analogue of this result by replacing the use of the Abhyankar-Jung Theorem with a result of Ewald and Ishida asserting that the set of orders on $\mathbf{Z}^n$ is compact.
Finally we apply this to obtain a bound on the gaps in the expansions of Laurent series algebraic over the field of power series of any characteristic.

Contents

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1. Introduction

When $\mathbf{K}$ is an algebraically closed field of characteristic zero and $x = (x_1, \ldots, x_n)$ is a vector of n indeterminates, we denote by $\mathbf{K}(\!(x)\!)$ the field of formal power series in n indeterminates. The problem we investigate here is the determination of the algebraic closure of $\mathbf{K}(\!(x)\!)$. When $n = 1$ it is well known that the elements that are algebraic over $\mathbf{K}(\!(x)\!)$ can be expressed as Puiseux series, i.e. as formal sums of the form $\sum_{k=k_0}^{\infty} a_k x^{k/q}$ for some positive integer q.

When $n \geq 2$ the question remains open in general. A classical result of McDonald [MD95] asserts that the elements that are algebraic over $\mathbf{K}(\!(x)\!)$ can be expressed as Puiseux series with support in the translation of a strongly convex rational cone. However the converse is wrong: a Puiseux series with support in the translation of a strongly convex rational cone is not algebraic over $\mathbf{K}(\!(x)\!)$ in general.

So a natural problem is characterizing the elements that are algebraic over $\mathbf{K}(\!(x)\!)$ among the Puiseux series with support in the translation of a strongly convex rational cone. Without loss of generality, when $\mathbf{K}$ is a characteristic zero field, we can restrict to the Laurent series (with integer exponents) with such a support (see for instance the introduction of [AR19]).

Here we investigate characterizations in terms of the support of the series. Indeed, such characterizations have already been studied for series in one indeterminate that are algebraic over the ring of polynomials. For these algebraic series in one indeterminate this problem is important and is related to several fields such as tropical geometry, number theory, and combinatorics (see [AB12], [HM17], and [AM-K31] for example). For such a series $f(x_1)$ algebraic over $\mathbf{K}[x_1]$, one can express all the coefficients of $f(x_1)$ in terms of a finite number of data: the coefficients of the minimal polynomial of $f(x_1)$ and the first coefficients of $f(x_1)$ up to an order N (see [FS98] and [HM17]). This order N is determined by the discriminant of the minimal polynomial of $f(x_1)$. However, these expressions, which are explicit, are usually difficult to handle (let us mention the work [HM] that recently extends such expressions in the multivariable case). Another approach is based on the fact that an algebraic series is D-finite, that is, a solution of a linear differential equation with polynomial coefficients. From this point of view the coefficients of $f(x_1)$ satisfy a linear recurrence with polynomial coefficients (see [St80] for instance). But once again it is still a difficult problem to handle such recurrences (see [AB12] for a presentation of the problem).

In fact our problem is much more subtle than the case of a series algebraic over $\mathbf{K}[x]$. Indeed a Laurent series ξ algebraic over $\mathbf{K}(\!(x)\!)$ will be determined by its minimal polynomial. But the coefficients of this minimal polynomial are formal power series in several indeterminates. Therefore the support of ξ depends on infinitely many coefficients in $\mathbf{K}$.

A first natural question is to find obstructions for the algebraicity of a Laurent series with support in the translation of a strongly convex rational cone in terms of the shape of the support.

For instance could there be an algebraic series in 3 indeterminates the convex hull of iwhose support is a right circular cone? A natural problem is to find how far from a strongly convex rational cone is the support of an algebraic series.

In [AR19] we began to investigate this kind of question. One of our results is that an algebraic series cannot have too many gaps in its expansion (see [AR19, Theorem 6.4] for a precise statement). In order to prove this result we proved a technical result asserting that, for a given algebraic series ξ with support in the translation of a strongly convex rational cone, there exists a hyperplane $H \subset \mathbf{R}^n$ such that $\text{Supp}(\xi) \cap H$ is infinite and one of the half-spaces delimited by H contains only a finite number of elements of $\text{Supp}(\xi)$.
In fact, the set $\tau(\xi)$ of normal vectors to such hyperplanes H has been defined as being the boundary of some strongly convex (open) cone. But nothing more has been proved about this cone $\tau(\xi)$. In particular we do not know if it is rational or even polyhedral.
The first aim of this paper is to prove that this cone is a strongly convex rational cone. Then we relate the support of such an algebraic series ξ to the dual of the cone $\tau(\xi)$. The same questions can be asked for a positive characteristic field. We will discuss the positive characteristic case, showing the differences with the zero characteristic case.

Let us present our main results in more detail. We begin by defining the cone $\tau(\xi)$:

Definition 1.1. *Let ξ be a series with support in $\mathbf{Q}^n$ and coefficients in a field $\mathbf{K}$. We set*

$$\tau(\xi) := \{\omega \in {\mathbf{R}_{\geq 0}}^n \mid \exists k \in \mathbf{R},\ \mathit{Supp}(\xi) \cap \{u \in \mathbf{R}^n \mid u \cdot \omega \leq k\} = \emptyset\}.$$

Here the support of the series $\xi = \sum_{\alpha \in \mathbf{Q}^n} \xi_\alpha x^\alpha$ is the set

$$\text{Supp}(\xi) := \{\alpha \in \mathbf{Q}^n \mid \xi_\alpha \neq 0\}.$$

Our first main result is the following:

Theorem 1.2. *Let ξ be a Laurent series whose support is included in a translation of a strongly convex cone containing ${\mathbf{R}_{\geq 0}}^n$ and with coefficients in a field $\mathbf{K}$ of any characteristic. Assume that ξ is algebraic over $\mathbf{K}(\!(x)\!)$. Then the set $\tau(\xi)$ is a strongly convex rational cone.*

Our second main result relies on the support of a Laurent series that is algebraic over $\mathbf{K}(\!(x)\!)$ to the cone $\tau(\xi)$:

Theorem 1.3. *Let ξ be a Laurent series whose support is included in a translation of a strongly convex cone containing ${\mathbf{R}_{\geq 0}}^n$ and with coefficients in a field $\mathbf{K}$ of any characteristic. Assume that ξ is algebraic over $\mathbf{K}(\!(x)\!)$. We have the following properties:*

i) There exist a finite set $C \subset \mathbf{Z}^n$, a Laurent polynomial $p(x)$, and a power series $f(x) \in \mathbf{K}[[x]]$ such that

a) *Supp*$(\xi + p(x) + f(x)) \subset C + \tau(\xi)^\vee$,

b) *for every* $(n-1)$*-dimensional (unbounded) face* F *of* $\mathrm{Conv}(C + \tau(\xi)^\vee)$*, the cardinal of*

$$Supp(\xi + p(x) + f(x)) \cap F$$

is infinite.

ii) If $\sigma \subset \tau(\xi)^\vee$ *is a cone containing* ${\mathbf{R}_{\geq 0}}^n$ *for which there exist a Laurent polynomial* $p'(x)$*, a power series* $f'(x) \in \mathbf{K}[[x]]$*, and a finite set* C' *such that*

$$Supp(\xi + p'(x) + f'(x)) \subset C' + \sigma,$$

then $\sigma = \tau(\xi)^\vee$*.*

We will see in Example 4.17 that, in general, the set C cannot be chosen to be one single point.

We will begin to treat the characteristic zero case because this case is simpler than the positive characteristic case, and because we feel that in this way the paper is easier to read. In this case The proof of Theorem 1.2 is essentially based on two tools: a version of Abhyankar-Jung Theorem for series with support in a strongly convex cone (see Theorem 3.3), and the construction, for every order $\preceq$ on $\mathbf{Q}^n$, of an algebraically closed field $\mathcal{S}_{\preceq}^{\mathbf{K}}$ containing $\mathbf{K}(\!(x)\!)$. This general version of Abhyankar-Jung Theorem has been proved in [GP00],[Ar04] and [PR12]. It will allow us to have a fan (defined by the Newton polyhedron of the discriminant) of ${\mathbf{R}_{\geq 0}}^n$, such that each full dimensional cone of this fan contains the support of all the roots of the minimal polynomial of ξ.

The construction of the algebraically closed fields $\mathcal{S}_{\preceq}^{\mathbf{K}}$ has been proved in [AR19] (see Theorem 2.10) and is based on systematic constructions of algebraically closed valued fields due to Rayner [Ra68]. In particular our proof does not involve our previous results in [AR19].

The proof of Theorem 1.3 is more involved and requires the introduction of new cones. These are denoted by $\tau_0(\xi)$, $\tau_0'(\xi)$, $\tau_1(\xi)$ and $\tau_1'(\xi)$. The definitions of $\tau_0(\xi)$ and $\tau_1(\xi)$ are purely algebraic (they are defined in terms of the fields $\mathcal{S}_{\preceq}^{\mathbf{K}}$), and the definitions of $\tau_0'(\xi)$ and $\tau_1'(\xi)$ are "geometric" (that is, they are defined in terms of the support of ξ).

We prove that the set $\tau(\xi)$ is the closure of $\tau_0'(\xi)$. Then, we prove that $\tau_1(\xi)$ and $\tau_1'(\xi)$ are equal and that $\tau_0'(\xi)$ is almost equal to $\tau_0(\xi)$ (see Proposition 4.6 - in particular they have the same closure). The main important property of $\tau_0(\xi)$ and $\tau_1(\xi)$ is that these two sets are open sets (see Proposition 4.7). Then, we will prove that the boundary of $\tau_0'(\xi)$ does not intersect $\tau_1'(\xi)$ (this comes from the openness of $\tau_0(\xi)$ and $\tau_1(\xi)$, see Corollary 4.8). In particular, the vectors in the boundary of $\tau(\xi)$ will correspond to "faces" of the support of ξ. The main tool used to prove the existence of the finite set C of Theorem 1.3, is a generalization of Dickson's Lemma that we prove here (see Corollary 4.13).

In the following part we investigate the positive characteristic case, which is quite different

from the zero characteristic case. Before giving the proofs of Theorems 1.2 and 1.3 in the positive characteristic case, we will investigate how the elements of an algebraic closure of $\mathbf{K}(\!(x)\!)$ can be described in this case. Indeed, in positive characteristic, the roots of polynomials with coefficients in the field of power series are not series with support in a lattice in general. But these roots can be expressed as series with support in a strongly convex cone with rational exponents whose denominators are not necessarily bounded (see the work [Sa17] where this analogue of MacDonald's Theorem is proved). First we show that Theorem 1.3 is no longer true for such series (see Example 5.2). This example shows that the problem is that the support of a root can have accumulation points, and therefore we need to take into account that its support is well ordered for the considered order. The first main difference with the characteristic zero case, is the fact that Abhyankar-Jung Theorem is no longer true in positive characteristic. This tool will be replaced by the compacity of the space of orders on ${\mathbf{R}_{\geq 0}}^n$, and will allow for the definition a fan of ${\mathbf{R}_{\geq 0}}^n$ as in the zero characteristic case. This result of compacity is due to Ewald and Ishida [EI06] (see also [Te18]) and is a purely topological result.
Subsequently we extend the result of Saavedra by constructing algebraically closed fields, each of them depending on a given order on ${\mathbf{R}_{\geq 0}}^n$, that contain $\mathbf{K}(\!(x)\!)$ (see Theorem 5.10). Then we introduce a new cone analogous to $\tau(\xi)$, but whose definition is more natural in that the support of such ξ is not a lattice in general. This allows us to prove that this cone is rational (see Theorem 5.16). Then we give an analogue of Theorem 1.3 in the positive characteristic case (see Theorem 5.18). This version is weaker than Theorem 1.3, but we show that there is no possibility for a stronger version for algebraic series with accumulation points in their support (see Example 5.20).

Then we will give the proof of Theorems 1.2 and 1.3 in the positive characteristic case (but only for power series with integer coefficients) by explaining the differences with the zero characteristic case. Then we use this to give a bound on the gaps in the expansion of a Laurent series algebraic over $\mathbf{K}(\!(x)\!)$ (see Theorem 7.1).

Let us mention that Theorem 1.2 has been announced in [ADR].

2. Orders and algebraically closed fields containing $\mathbf{K}(\!(x)\!)$

In this section we introduce the tools needed for the proof of Theorem 1.2.

2.1 The space of orders on ${\mathbf{R}_{\geq 0}}^n$

Definition 2.1. *Let us recall that a* cone $\tau \subset \mathbf{R}^n$ *is a subset of* $\mathbf{R}^n$ *such that for every* $t \in \tau$ *and* $\lambda \geq 0$*,* $\lambda t \in \tau$*. A cone* $\tau \subset \mathbf{R}^n$ *is* polyhedral *if it has the form*

$$\tau = \{\lambda_1 u_1 + \cdots + \lambda_s u_s \mid \lambda_1, \ldots, \lambda_s \geq 0\}$$

for some given vectors $u_1, \ldots, u_s \in \mathbf{R}^n$. *A cone is said to be a* rational cone *if it is polyhedral, and the* u_i *can be chosen in* $\mathbf{Z}^n$.
A cone is strongly convex *if it does not contain any non trivial linear subspace.*

Definition 2.2. *A* preorder *on an abelian group* G *is a binary relation* $\preceq$ *such that*

i) $\forall u, v \in G$, $u \preceq v$ *or* $v \preceq u$,

ii) $\forall u, v, w \in G$, $u \preceq v$ *and* $v \preceq w$ *implies* $u \preceq w$,

iii) $\forall u, v, w \in G$, $u \preceq v$ *implies* $u + w \preceq v + w$,

The set of preorders on G *is denoted by* $\mathrm{ZR}(G)$. *The set of orders on* G *is a subset of* $\mathrm{ZR}(G)$ *denoted by* $\mathrm{Ord}(G)$.

Definition 2.3. *By [Ro86, Theorem 2.5] for every* $\preceq \in \mathrm{ZR}(\mathbf{Q}^n)$ *there exist an integer* $s \geq 0$ *and orthogonal vectors* $u_1, \ldots, u_s \in \mathbf{R}^n$ *such that*

$$\forall u, v \in \mathbf{Q}^n,\ u \preceq v \Longleftrightarrow (u \cdot u_1, \ldots, u \cdot u_s) \leq_{lex} (v \cdot u_1, \ldots, v \cdot u_s).$$

For such a preorder we set $\preceq := \leq_{(u_1, \ldots, u_s)}$. *Such a preorder extends in an obvious way to a preorder on* $\mathbf{R}^n$ *and the preorders of this form are called* continuous preorders.

Definition 2.4. *Let* $A \subset \mathbf{R}^n$ *and* $\preceq$ *be a continuous preorder on* $\mathbf{R}^n$. *We say that* A *is* $\preceq$*-positive if*

$$\forall a \in A, \quad \underline{0} \preceq a.$$

Definition 2.5. *The set of continuous orders* $\preceq$ *such that* ${\mathbf{R}_{\geq 0}}^n$ *is* $\preceq$*-positive is denoted by* Ord_n, *and will be simply called orders on* ${\mathbf{R}_{\geq 0}}^n$.
In the rest of the paper all the orders that we consider will be exclusively orders on ${\mathbf{R}_{\geq 0}}^n$. *For simplicity we shall call them simply orders.*

Definition 2.6. *Given two preorders* $\preceq_1$ *and* $\preceq_2$, *one says that* $\preceq_2$ refines $\preceq_1$ *if*

$$\forall u, v \in \mathbf{R}^n,\ u \preceq_2 v \Longrightarrow u \preceq_1 v.$$

The next easy lemma will be used several times:

Lemma 2.7. *[AR19, Lemma 2.4] Let* σ_1 *and* σ_2 *be two cones and* γ_1 *and* γ_2 *be vectors of* $\mathbf{R}^n$. *Let us assume that* $\sigma_1 \cap \sigma_2$ *is full dimensional. Then there exists a vector* $\gamma \in \mathbf{Z}^n$ *such that*

$$(\gamma_1 + \sigma_1) \cap (\gamma_2 + \sigma_2) \subset \gamma + \sigma_1 \cap \sigma_2.$$

Finally we give the following result, which will be used in the proof of Theorem 1.3 (this is a generalization of [AR19, Corollary 3.10]):

Lemma 2.8. *Let* $\sigma_1, \ldots, \sigma_N$ *be strongly convex cones and let* $\omega \in \mathbf{R}^n$. *The following properties are equivalent:*

i) We have $\omega\in\operatorname{Int}\left(\bigcup_{i=1}^N\sigma_i^\vee\right)$.

ii) For every order $\preceq\in\operatorname{Ord}(\mathbf{Q}^n)$ *refining* $\leq_\omega$*, there is an index* i *such that* σ_i *is* $\preceq$*-positive.*

Proof. Let us prove that i) implies ii). We are going to show that for every non zero vector $v_1,\ldots,$ $v_{n-1}\in\langle\omega\rangle^\perp$ such that $v_j\in\langle\omega,v_1,\ldots,v_{j-1}\rangle^\perp$, there is an integer i such that σ_i is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive. This proves the implication by [Ro86, Theorem 2.5] (see Definition 2.3).
By assumption there is an integer i such that $\omega\in\sigma_i^\vee$. If $\omega\in\operatorname{Int}(\sigma_i^\vee)$ for some i, then σ_i is $\preceq$-positive for every order $\preceq$ refining $\leq_\omega$. Otherwise let E_1 be the set of indices i such that $\omega\in\sigma_i^\vee$. We have $\omega\in\operatorname{Int}(\cup_{i\in E_1}\sigma_i^\vee)$, since the $\sigma_i^\vee$ are full dimensional.
Now let $v_1\in\langle\omega\rangle^\perp\setminus\{\underline{0}\}$. Since $\omega\in\operatorname{Int}(\cup_{i\in E_1}\sigma_i^\vee)$, there is $\lambda_1>0$ such that $\omega+\lambda_1v_1\in\operatorname{Int}(\cup_{i\in E_1}\sigma_i^\vee)$. Hence there is an integer $j\in E_1$ such that $\omega+\lambda_1v_1\in\sigma_j^\vee$. Then two cases may occur:
If $\omega+\lambda_1v_1\in\operatorname{Int}(\sigma_j^\vee)$, then σ_j is $\preceq$-positive for every order $\preceq$ refining $\leq_{(\omega,v_1)}$. Indeed for $s\in\sigma_j$, either $\omega\cdot s>0$, or $\omega\cdot s=0$. In this last case we have $v_1\cdot s>0$ since $(\omega+\lambda_1v_1)\cdot s>0$ and $\lambda_1>0$.
If $\omega+\lambda_1v_1\notin\operatorname{Int}(\sigma_j^\vee)$, we denote by E_2 the set of $i\in E_1$ such that $\omega+\lambda_1v_1\in\sigma_i^\vee$. Now let $v_2\in\langle\omega,v_1\rangle^\perp\setminus\{\underline{0}\}$. There is $\lambda_2>0$ such that $\omega+\lambda_1v_1+\lambda_2v_2\in\operatorname{Int}(\cup_{i\in E_2}\sigma_i^\vee)$. Once again, if $\omega+\lambda_1v_1+\lambda_2v_2\in\operatorname{Int}(\sigma_i^\vee)$ for some $i\in E_2$, σ_i is $\preceq$-positive for every order $\preceq$ refining $\leq_{(\omega,v_1,v_2)}$. Otherwise we repeat the same process.

Now assume that for every order $\preceq\in\operatorname{Ord}(\mathbf{Q}^n)$ refining $\leq_\omega$, there is an index j such that σ_j is $\preceq$-positive. Let J be the set of such cones. In particular $\omega\in\bigcap_{j\in J}\sigma_j^\vee\subset\bigcup_{i=1}^n\sigma_i^\vee$. If $n=1$, $\leq_\omega=\leq_{\omega'}$ for every ω' in a small neighborhood of ω, and there is nothing to show. Therefore we assume that $n\geq 2$.
Let v be a vector with $\|v\|=1$. By assumption there is an index i such that σ_i is $\leq_{(\omega,v)}$-positive. Let $s_1,\ldots,s_l$ be generators of σ_i that we assume to be of norm equal to 1. Reordering the s_i, there is an integer k such that $s_j\cdot\omega>0$ for every $j\leq k$, and $s_j\cdot\omega=0$ for every $j>k$. Take $\lambda:=\dfrac{\min_{j\leq k}\{s_j\cdot\omega\}}{2}>0$. Then we claim that $\omega+\lambda v\in\sigma_i^\vee$. Indeed, if $j\leq k$ we have

$$(\omega+\lambda v)\cdot s_j=\omega\cdot s_j+\lambda v\cdot s_j\geq\omega\cdot s_j-\lambda\|v\|\|s_j\|\geq\frac{\min_{j\leq k}\{s_j\cdot\omega\}}{2}>0.$$

If $j>k$ we have

$$(\omega+\lambda v)\cdot s_j=\lambda v\cdot s_j\geq 0$$

since σ_i is $\leq_{(\omega,v)}$-positive. This implies that $\omega+\lambda v\in\sigma_i^\vee$. Since this is true for every v, we have $\omega\in\operatorname{Int}\left(\bigcup_{i=1}^N\sigma_i^\vee\right)$.

Corollary 2.9. *Let* $\omega\in{\mathbf{R}_{\geq0}}^n$ *and let* $\sigma_1,\ldots,\sigma_N$ *be strongly convex full dimensional cones which are* $\leq_\omega$*-positive. Assume that for every order* $\preceq\in\operatorname{Ord}_n$ *refining* $\leq_\omega$*, there is an index* i *such that* σ_i *is* $\preceq$*-positive. Then there is a neighborhood* V *of* ω *such that, for every* $\omega'\in V$ *and every* $\preceq'\in\operatorname{Ord}_n$ *refining* $\leq_{\omega'}$*, there is an index* i *such that* σ_i *is* $\preceq'$*-positive.*

Proof. We have $\omega \in \mathrm{Int}\left(\bigcup_{i=1}^{N} \sigma_i^\vee\right)$ by the previous lemma. Therefore, the previous lemma shows that we can choose $V = \mathrm{Int}\left(\bigcup_{i=1}^{N} \sigma_i^\vee\right)$.

2.2 Algebraically closed fields containing $\mathbf{K}(\!(x)\!)$

Let n be a positive integer and $\preceq \in \mathrm{Ord}_n$.
For a field $\mathbf{K}$ of characteristic zero we set

$$\mathcal{S}_{\preceq}^{\mathbf{K}} = \left\{ \xi \mid \exists k \in \mathbf{N}^*, \gamma \in \mathbf{Z}^n, \sigma \preceq\text{-positive rational cone, } \mathrm{Supp}(\xi) \subset (\gamma + \sigma) \cap \frac{1}{k}\mathbf{Z}^n \right\}.$$

We have the following result:

Theorem 2.10. *[AR19, Theorem 4.5] Assume that $\mathbf{K}$ is an algebraically closed field of characteristic zero. The set $\mathcal{S}_{\preceq}^{\mathbf{K}}$ is an algebraically closed field containing $\mathbf{K}(\!(x)\!)$.*

3. Proof of Theorem 1.2

Lemma 3.1. *Let ξ be a Laurent series whose support is included in a translation of a strongly convex cone σ containing $\mathbf{R}_{\geq 0}{}^n$ and with coefficients in a characteristic zero field $\mathbf{K}$, and let $P \in \mathbf{K}[[x]][T]$ be a monic polynomial of degree d with $P(\xi) = 0$. Let $\sigma_0 \subset \mathbf{R}_{\geq 0}{}^n$ be a strongly convex rational cone such that there are d distinct series $\xi_1, \ldots, \xi_d$, belonging to $\mathcal{S}_{\preceq}^{\mathbf{K}}$ for some $\preceq \in \mathrm{Ord}_n$, with support in $\gamma + \sigma_0$ for some $\gamma \in \mathbf{Z}^n$, with $P(\xi_i) = 0$ for $i = 1, \ldots, d$.*
Then

$$\mathrm{Int}(\sigma_0^\vee) \cap \tau(\xi) \neq \emptyset \Longrightarrow \sigma_0^\vee \subset \tau(\xi).$$

Proof. Consider a non zero vector $\omega \in \mathrm{Int}(\sigma_0^\vee) \cap \tau(\xi)$. Since $\omega \in \tau(\xi)$ we have that

$$\mathrm{Supp}(\xi) \subset \gamma + \langle \omega \rangle^\vee$$

for some $\gamma \in \mathbf{Z}^n$. By Lemma 2.7 we have

$$\mathrm{Supp}(\xi) \subset \gamma' + \sigma \cap \langle \omega \rangle^\vee$$

for some $\gamma' \in \mathbf{Z}^n$. Since $\sigma \cap \langle \omega \rangle^\vee$ is $\leq_\omega$-positive, there exists an order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$ such that $\sigma \cap \langle \omega \rangle^\vee$ is $\preceq$-positive (see for example [AR19, Lemma 3.8]). Thus ξ is a root of P in $\mathcal{S}_{\preceq}^{\mathbf{K}}$. On the other hand, ω is in the interior of $\sigma_0^\vee$, thus $\sigma_0 \cap \langle \omega \rangle^\perp = \{\underline{0}\}$. This implies that for every $u \in \sigma_0$, $\underline{0} \preceq u$, since $\preceq$ is refining $\leq_\omega$. That is σ_0 is $\preceq$-positive. In particular the ξ_i are the roots of P in $\mathcal{S}_{\preceq}^{\mathbf{K}}$ and $\xi = \xi_i$ for some i. Hence there is some $\gamma'' \in \mathbf{Z}^n$ such that

$$\mathrm{Supp}(\xi) \subset \gamma'' + \sigma_0.$$

Therefore for every $\omega' \in \sigma_0^\vee$ we have

$$\mathrm{Supp}(\xi) \cap \left\{ u \in \mathbf{R}^n \mid u \cdot \omega' \leq \gamma'' \cdot \omega' - 1 \right\} = \emptyset.$$

Hence $\sigma_0^\vee \subset \tau(\xi)$.

Corollary 3.2. *Let ξ be a Laurent series with support in the translation of a strongly convex cone σ containing ${\mathbf{R}_{\geq 0}}^n$ and with coefficients in a characteristic zero field $\mathbf{K}$, and let $P \in \mathbf{K}[[x]][T]$ be a monic polynomial of degree d with $P(\xi) = 0$. Let $\sigma_k \subset {\mathbf{R}_{\geq 0}}^n$, $k = 1, \ldots, N$, be finitely generated strongly convex rational cones satisfying the following properties:*

i) $\displaystyle\bigcup_{k=1}^{N} \sigma_k^\vee = {\mathbf{R}_{\geq 0}}^n$,

ii) for every k there are d series $\xi_1^{(k)}, \ldots, \xi_d^{(k)}$, belonging to $\mathcal{S}_{\preceq}^{\mathbf{K}}$ for some $\preceq \in \mathrm{Ord}_n$, with support in $\gamma_k + \sigma_k$ for some $\gamma_k \in \mathbf{Z}^n$, with $P(\xi_i^{(k)}) = 0$ for $i = 1, \ldots, d$.

Then, after renumbering the σ_k, there is an integer $l \leq N$ such that

$$\tau(\xi) = \bigcup_{k=1}^{l} \sigma_k^\vee.$$

Proof. Since the σ_k are strongly convex, the $\sigma_k^\vee$ are full dimensional and $\mathrm{Int}(\sigma_k^\vee) \neq \emptyset$ for every k.

By Lemma 3.1 we can renumber the σ_k such that $\sigma_k^\vee \subset \tau(\xi)$ for $k \leq l$ and $\mathrm{Int}(\sigma_k^\vee) \cap \tau(\xi) = \emptyset$ for every $k > l$. So we have $\displaystyle\bigcup_{k=1}^{l} \sigma_k^\vee \subset \tau(\xi)$.

Now, suppose that this inclusion is strict, there is an element $\omega \in \tau(\xi)$ such that $\omega \notin \displaystyle\bigcup_{k=1}^{l} \sigma_k^\vee$. By Hahn-Banach Theorem there is a hyperplane H separating ω and the convex closed set $\displaystyle\bigcup_{k=1}^{l} \sigma_k^\vee$ in the following sense: one open half space delimited by H, denoted by O, contains ω and $\displaystyle\bigcup_{k=1}^{l} \sigma_k^\vee \subset \mathbf{R}^n \backslash \overline{O}$. Since $\displaystyle\bigcup_{k=1}^{l} \sigma_k^\vee$ is full dimensional, the convex envelop $\mathcal{C}$ of ω and $\displaystyle\bigcup_{k=1}^{l} \sigma_k^\vee$ is full dimensional:

$$\mathcal{C} := \left\{ \lambda\omega + (1-\lambda)v \mid v \in \bigcup_{k=1}^{l} \sigma_k^\vee, 1 \geq \lambda \geq 0 \right\}.$$

Thus $\mathcal{C} \cap O$ contains an open ball B.

But $\tau(\xi)$ is convex because for every ω, $\omega' \in \mathbf{R}^n$, k, $l \in \mathbf{R}$:

$$\{u \in \mathbf{R}^n \mid u \cdot (\omega + \omega') \leq k + l\} \subset \{u \in \mathbf{R}^n \mid u \cdot \omega \leq k\} \cup \{u \in \mathbf{R}^n \mid u \cdot \omega' \leq l\}.$$

Thus $\mathcal{C} \subset \tau(\xi)$ and $B \subset \tau(\xi)$. Then B intersects one $\sigma_i^\vee$ for $i > l$ because $B \subset O$ and we have assumed $\displaystyle\bigcup_{k=1}^{N} \sigma_k^\vee = {\mathbf{R}_{\geq 0}}^n$. Since B is open, $B \cap \mathrm{Int}(\sigma_i^\vee) \neq \emptyset$, but this contradicts the fact that $i > l$.

For a formal power series $f \in \mathbf{K}[[x]]$ we denote by $\mathrm{NP}(f)$ its Newton polyhedron. Let p be a vertex of $\mathrm{NP}(f)$. The set of vectors $v \in \mathbf{R}^n$ such that $p + \lambda v \in \mathrm{NP}(f)$ for some $\lambda \in \mathbf{R}_{\geq 0}$ is a rational strongly convex cone. Such a cone is called the *cone of the Newton polyhedron of f associated with the vertex p*. We have the following generalization of Abhyankar-Jung Theorem:

Theorem 3.3 (Abhyankar-Jung Theorem)**.** *[GP00, Théorème 3][Ar04, Theorem 7.1][PR12, Theorem 6.2]*

Let $\mathbf{K}$ be a characteristic zero field. Let $P(Z) \in \mathbf{K}[[x]][Z]$ be a monic polynomial and let Δ be its discriminant. Let $\mathrm{NP}(\Delta)$ denote the Newton polyhedron of Δ. Then the set of cones of $\mathrm{NP}(\Delta)$ satisfies the properties of Corollary 3.2.

Proof of Theorem 1.2. By Theorem 2.10 for every order $\preceq \in \mathrm{Ord}_n$ there are an element $\gamma_{\preceq} \in \mathbf{Z}^n$ and a $\preceq$-positive rational strongly convex cone $\sigma_{\preceq}$ such that the roots of P can be expanded as Puiseux Laurent series with support in $\gamma_{\preceq} + \sigma_{\preceq}$. Thus $\tau(\xi)$ is a strongly convex rational cone by Theorem 3.3 and Corollary 3.2. This proves Theorem 1.2.

4. Proof of Theorem 1.3

4.1 Preliminary results

Definition 4.1. *For a Laurent series ξ we set*

$$\tau_0'(\xi) = \{\omega \in \mathbf{R}_{\geq 0}{}^n \backslash \{\underline{0}\} \mid \#\left(Supp(\xi) \cap \{u \in \mathbf{R}^n \mid u \cdot \omega \leq k\}\right) < \infty, \forall k \in \mathbf{R}\},$$

$$\tau_1'(\xi) = \{\omega \in \mathbf{R}_{\geq 0}{}^n \backslash \{\underline{0}\} \mid \#\left(Supp(\xi) \cap \{u \in \mathbf{R}^n \mid u \cdot \omega \leq k\}\right) = \infty, \forall k \in \mathbf{R}\}.$$

We have the following lemma:

Lemma 4.2. *Let ξ be a Laurent series with support in the translation of a strongly convex cone containing $\mathbf{R}_{\geq 0}{}^n$. We have $\tau_0'(\xi) \subset \tau(\xi) \subset \overline{\tau_0'(\xi)}$.*

Proof. We have $\tau_0'(\xi) \subset \tau(\xi)$ by definition.
Let $\omega \in \tau(\xi)$. Then, for some $k \in \mathbf{R}$:

$$\mathrm{Supp}(\xi) \cap \{u \in \mathbf{R}^n \mid u \cdot \omega \leq k\} = \emptyset$$

that is $\mathrm{Supp}(\xi) \subset \gamma + \langle\omega\rangle^\vee$ for some $\gamma \in \mathbf{Z}^n$. On the other hand, by hypothesis, $\mathrm{Supp}(\xi)$ is included in $\gamma' + \sigma$ for some $\gamma' \in \mathbf{Z}^n$ and σ a strongly convex cone such that $\mathbf{R}_{\geq 0}{}^n \subset \sigma$. Thus, by Lemma 2.7, $\mathrm{Supp}(\xi)$ is included in a translation of the strongly convex cone $\sigma \cap \langle\omega\rangle^\vee$. Moreover $\omega \in (\sigma \cap \langle\omega\rangle^\vee)^\vee$, and $(\sigma \cap \langle\omega\rangle^\vee)^\vee$ is full dimensional. Thus there exists a sequence $(\omega_k)_k \in \mathrm{Int}\,(\sigma \cap \langle\omega\rangle^\vee)^\vee$ such that $(\omega_k)_k$ converges to ω. Moreover, for all index k, we have $\omega_k \in \mathbf{R}_{\geq 0}{}^n$ since $\mathbf{R}_{\geq 0}{}^n \subset \sigma \cap \langle\omega\rangle^\vee$. Thus

$$\sigma \cap \langle\omega\rangle^\vee \cap \langle\omega_k\rangle^\perp = \{0\}.$$

Thus $\omega_k \in \tau_0'(\xi)$ for all k, therefore $\omega \in \overline{\tau_0'(\xi)}$.

Corollary 4.3. *Under the hypothesis of Theorem 1.3, we have*

$$\tau(\xi)=\overline{\tau_0'(\xi)}.$$

Proof. By Lemma 4.2 we have $\tau_0'(\xi)\subset\tau(\xi)\subset\overline{\tau_0'(\xi)}$. Since $\tau(\xi)$ is closed (it is a rational cone, thus a polyhedral cone, by Theorem 1.2) we have $\tau(\xi)=\overline{\tau_0'(\xi)}$.

Definition 4.4. *In the rest of this section we consider the following setting: ξ is a Laurent series with support in a translated strongly convex rational cone and is algebraic over $\mathbf{K}[[x]]$ where $\mathbf{K}$ is a characteristic zero field. We denote by $P\in\mathbf{K}[[x]][T]$ the minimal polynomial of ξ and, for any order $\preceq\in\mathrm{Ord}_n$, $\xi_1^{\preceq},\ldots,\xi_d^{\preceq}$ denote the roots of $P(T)$ in $\mathcal{S}_{\preceq}^{\mathbf{K}}$. We set*

$$\tau_0(\xi):=\left\{\omega\in{\mathbf{R}_{\geq0}}^n\backslash\{\underline{0}\}\mid \text{ for all }\preceq\text{ that refines }\leq_\omega,\ \exists i\text{ such that }\xi=\xi_i^{\preceq}\right\},$$

$$\tau_1(\xi):=\left\{\omega\in{\mathbf{R}_{\geq0}}^n\backslash\{\underline{0}\}\mid \xi\neq\xi_i^{\preceq},\text{ for all }\preceq\text{ that refines }\leq_\omega,\ \forall i=1,\ldots,d\right\},$$

Remark 4.5. *These sets were introduced in [AR19], but only for $\omega\in{\mathbf{R}_{>0}}^n$. In this case it was proved that $\tau_0(\xi)\cap{\mathbf{R}_{>0}}^n=\tau_0'(\xi)\cap{\mathbf{R}_{>0}}^n$ and $\tau_1(\xi)\cap{\mathbf{R}_{>0}}^n=\tau_1'(\xi)\cap{\mathbf{R}_{>0}}^n$ (see [AR19, Lemmas 5.8, 5.11]). Taking into account all the $\omega\in{\mathbf{R}_{\geq0}}^n$ changes the situation. In particular we do not have $\tau_0(\xi)=\tau_0'(\xi)$ in general (see Example 4.11).*

Proposition 4.6. *We have $\tau_1(\xi)=\tau_1'(\xi)$ and $\tau_0'(\xi)\subset\tau_0(\xi)$.*

Proof. The proof of the equality $\tau_1(\xi)=\tau_1'(\xi)$ is exactly the proof of [AR19, Lemma 5.11]. Let us prove $\tau_0'(\xi)\subset\tau_0(\xi)$. Let $\omega\in\tau_0'(\xi)$, in particular:

$$\#\left(\mathrm{Supp}(\xi)\cap\{u\in\mathbf{R}^n\mid u\cdot\omega\leq k\}\right)<\infty,\ \forall k\in\mathbf{R},\tag{1}$$

and let us consider an order $\preceq$ that refines $\leq_\omega$.
By (1) we have that $\mathrm{Supp}(\xi)$ is $\preceq$-well-ordered. Thus by [AR19, Corollary 4.6] ξ is an element of $\mathcal{S}_{\preceq}^{\mathbf{K}}$. This shows that $\omega\in\tau_0(\xi)$.

Proposition 4.7. *The sets $\tau_0(\xi)$ and $\tau_1(\xi)$ are open subsets of ${\mathbf{R}_{\geq0}}^n$.*

Proof. We apply Lemma 2.8 and Theorem 3.3 to see the following:
For every $\omega\in{\mathbf{R}_{\geq0}}^n$, there exists a finite set of strongly convex cones $\mathcal{T}_\omega$, such that, for any order $\preceq\in\mathrm{Ord}_n$ refining $\leq_\omega$, there is $\sigma\in\mathcal{T}_\omega$ such that the roots of P in $\mathcal{S}_{\preceq}^{\mathbf{K}}$ have support in a translation of σ.
Moreover, let us choose $\mathcal{T}$ to be minimal among the sets of cones having this property. Then Corollary 2.9 implies that, for every $\omega'\in{\mathbf{R}_{\geq0}}^n$ close enough to ω, and for any order $\preceq'\in\mathrm{Ord}_n$ refining $\leq_{\omega'}$, there is $\sigma\in\mathcal{T}_\omega$ such that the roots of P in $\mathcal{S}_{\preceq'}^{\mathbf{K}}$ have support in a translation of σ. Since $\mathcal{T}_\omega$ is minimal with this property, for every ω' close enough to ω, for every order $\preceq'\in\mathrm{Ord}_n$ refining $\leq_{\omega'}$ and for every $i=1,\ldots,d$, there is an order $\preceq\in\mathrm{Ord}_n$ refining $\leq_\omega$ such that $\xi_i^{\preceq'}=\xi_{j_i}^{\preceq}$

for some j_i.

If $\omega \in \tau_0(\xi)$ then ξ is equal to some $\xi_i^{\preceq}$ for every order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$. Thus, for every $\omega' \in \mathbf{R}_{\geq 0}{}^n$ close enough to ω and every order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$, $\xi = \xi_j^{\preceq'}$ for some j. Thus $\omega' \in \tau_0(\xi)$. This proves that $\tau_0(\xi)$ is open in $\mathbf{R}_{\geq 0}{}^n$.

If $\omega \in \tau_1(\xi)$ then $\xi \neq \xi_i^{\preceq}$ for every i and for every order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$. Thus, for $\omega' \in \mathbf{R}_{\geq 0}{}^n$ close enough to ω and every order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$, $\xi \neq \xi_j^{\preceq'}$ for every j. Hence $\omega' \in \tau_1(\xi)$ and $\tau_1(\xi)$ is open.

Corollary 4.8. *We have*

$$\overline{\tau_0'(\xi)} \cap \tau_1'(\xi) = \emptyset.$$

Proof. The sets $\tau_0(\xi)$ and $\tau_1(\xi)$ are disjoint and open in $\mathbf{R}_{\geq 0}{}^n$. Thus $\overline{\tau_0(\xi)} \cap \tau_1(\xi) = \emptyset$. This proves the corollary because $\tau_0'(\xi) \subset \tau_0(\xi)$ and $\tau_1'(\xi) = \tau_1(\xi)$ by Proposition 4.6.

Lemma 4.9. *We have*

$$\overline{\tau_0'(\xi)} = \overline{\tau_0(\xi) \cap \mathbf{R}_{>0}{}^n} = \overline{\tau_0(\xi)}.$$

Proof. The set $\tau_0(\xi)$ is open. Therefore every $w \in \tau_0(\xi) \cap (\mathbf{R}_{\geq 0}{}^n \backslash \mathbf{R}_{>0}{}^n)$ can be approximated by elements of $\tau_0(\xi) \cap \mathbf{R}_{>0}{}^n$. Hence

$$\overline{\tau_0(\xi) \cap \mathbf{R}_{>0}{}^n} = \overline{\tau_0(\xi)}.$$

By [AR19, Lemma 5.8] $\tau_0'(\xi) \cap \mathbf{R}_{>0}{}^n = \tau_0(\xi) \cap \mathbf{R}_{>0}{}^n$. We have that $\tau_0'(\xi)$ is convex (the proof is exactly the same as the proof of [AR19, Lemma 5.9]). Thus we have

$$\overline{\tau_0'(\xi) \cap \mathbf{R}_{>0}{}^n} = \overline{\tau_0'(\xi)}$$

by [Bo53, Prop. 16 - Cor. 1; II.2.6]. Hence

$$\overline{\tau_0'(\xi)} = \overline{\tau_0'(\xi) \cap \mathbf{R}_{>0}{}^n} = \overline{\tau_0(\xi) \cap \mathbf{R}_{>0}{}^n} = \overline{\tau_0(\xi)}.$$

Corollary 4.10. *For every $f \in \mathbf{K}(\!(x)\!)$ we have*

$$\tau_0(\xi + f) = \tau_0(\xi),\ \tau_1(\xi + f) = \tau_1(\xi),\ \overline{\tau_0'(\xi + f)} = \overline{\tau_0'(\xi)}.$$

Proof. The minimal polynomial of $\xi + f$ is $Q(T) := P(T - f)$. Thus, for a given $\preceq \in \mathrm{Ord}_n$, the roots of $Q(T)$ in $\mathcal{S}_{\preceq}^{\mathbf{K}}$ are $\xi_1^{\preceq} + f, \ldots, \xi_d^{\preceq} + f$. This shows that

$$\tau_0(\xi + f) = \tau_0(\xi),\ \tau_1(\xi + f) = \tau_1(\xi).$$

Lemma 4.9 implies that

$$\overline{\tau_0'(\xi + f)} = \overline{\tau_0'(\xi)}.$$

Example 4.11. *We can see on a basic example that $\tau_0'(\xi + f) \neq \tau_0'(\xi)$ in general: let $n = 2$ and fix $\xi = \sum_{k \in \mathbf{N}} x_1^k$ and $f = 1 - \xi$. Then $\tau_0'(\xi) = \mathbf{R}_{>0} \times \mathbf{R}_{\geq 0}$ but $\tau_0'(\xi + f) = \mathbf{R}_{\geq 0}{}^2$. This also shows that $\tau_0(\xi) \neq \tau_0'(\xi)$ in general.*

4.2 A generalization of Dickson's Lemma

We will prove here a strengthened version of Lemma 2.7 that we will need in the proof of Theorem 1.2. For this we need the following lemma:

Lemma 4.12. *Let U and V be two vectors of indeterminates, and I and J be ideals of $\mathbf{K}[U,V]$ such that I is generated by binomials and J by monomials. Then there is an ideal J' generated by monomials such that*

$$(J+I)\cap\mathbf{K}[U]=J'+I\cap\mathbf{K}[U].$$

Proof. We will use the idea of the proof of [ES96, Corollary 1.3]. We consider the right-lexicographic order on the set of monomials in U and V and fix a Gröbner basis B of I with respect to this order. To compute such a basis we begin with binomials generating I and follow Buchberger's Algorithm. The reader may consult [CLO07, Definition 4, p. 83 and Theorem 2 p. 90] for details about this algorithm and the notion of S-polynomial. It is straightforward to see that the elements produced step by step in this algorithm are still binomials (this is in fact the content of [ES96, Proposition 1.1]). In particular $I\cap\mathbf{K}[U]$ is generated by binomials.

Now we wish to determine a Gröbner basis of $J+I$. We begin with the Gröbner basis B of I formed of binomials and we add the monomials generating J. Following Buchberger's Algorithm we may produce new elements which are not in B in the following cases:

- We consider the S-polynomials of two binomials in B and divide it by a monomial: in this case the remainder is either the S-polynomial that is in B or a monomial.
- We consider the S-polynomials of two monomials and it is always 0.
- We consider the S-polynomial of one binomial of B and one monomial. It is a monomial and its remainder under the division by a binomial is always a monomial.

Therefore we see that the Gröbner basis of $I+J$ obtained by Buchberger's Algorithm consists of B along with a finite number of monomials. Thus $(J+I)\cap\mathbf{K}[U]$ is generated by the elements of B that do not depend on V (i.e. the generators of $I\cap\mathbf{K}[U]$) and a finite number of monomials (defining a monomial ideal J').

Corollary 4.13 (Dickson's Lemma)**.** *Let $\sigma_1,\ldots,\ \sigma_k$ be convex rational cones such that $\sigma:=\bigcap_{j=1}^k\sigma_j$ is a full dimensional convex rational cone. Let $\gamma_1,\ldots,\ \gamma_k\in\mathbf{Z}^n$. Then there exists a finite set $C\subset\mathbf{Z}^n$ such that*

$$\bigcap_{j=1}^k(\gamma_j+\sigma_j)\cap\mathbf{Z}^n=C+\sigma\cap\mathbf{Z}^n.$$

Proof. Up to a translation we may assume that $\gamma_j\in\sigma\cap\mathbf{Z}^n$ for every j because σ is full dimensional. Let $u_1,\ \ldots,\ u_s$ be integer coordinate vectors generating $\sigma\cap\mathbf{Z}^n$. Then the ring R_σ of polynomials in $x_1,\ \ldots,\ x_n$ with support in $\sigma\cap\mathbf{Z}^n$ is isomorphic to $\mathbf{K}[U_1,\ldots,U_s]/I$ for some

binomial ideal I. This can be described as follows:
for any linear relation $L := \{\sum_{i=1}^{s}\lambda_i u_i = 0\}$ with $\lambda_i \in \mathbf{Z}$ we consider the binomial

$$B_L := \prod_{i|\lambda_i\geq 0} U_i^{\lambda_i} - \prod_{i|\lambda_i<0} U_i^{-\lambda_i}.$$

Then I is the ideal generated by the B_L for L running over the $\mathbf{Z}$-linear relations between the u_i. Moreover, for $\gamma \in \sigma\cap\mathbf{Z}^n$, the isomorphism sends x^γ onto U^{α_γ} where $\alpha_\gamma \in \mathbf{Z}^s_{\geq 0}$ is defined by $\gamma = \sum_{i=1}^{s}\alpha_{\gamma,i}u_i$.
By assumption we have $R_\sigma \subset R_{\sigma_j}$ for every j and $R_\sigma = \bigcap_{j=1}^{k} R_{\sigma_j}$. For every j we consider the ideal $x^{\gamma_j}R_{\sigma_j}$ of R_{σ_j} generated by x^{γ_j}. Since $R_\sigma = \bigcap_{j=1}^{k} R_{\sigma_j}$ we have

$$\bigcap_{j=1}^{k} x^{\gamma_j}R_{\sigma_j} = \bigcap_{j=1}^{k}(x^{\gamma_j}R_{\sigma_j}\cap R_\sigma).$$

Let us fix an index j. As for R_σ, the ring R_{σ_j} of polynomials in $x_1, \ldots, x_n$ with support in $\sigma_j\cap\mathbf{Z}^n$ is isomorphic to a ring of polynomials modulo a binomial ideal. Moreover we can consider the generators $u_1, \ldots, u_s$ of σ and add vectors $v_1, \ldots, v_r$ such that σ_j is generated by the u_i and v_l. Then R_{σ_j} is isomorphic to $\mathbf{K}[U,V]/I_j$ where $U = (U_1,\ldots,U_s)$ and $V = (V_1,\ldots,V_r)$ are vectors of indeterminates, and I_j is a binomial ideal such that $I_j\cap\mathbf{K}[U] = I$. The ideal $x^{\gamma_j}R_{\sigma_j}$ is isomorphic to the image of a monomial ideal J_j in $\mathbf{K}[U,V]/I_j$. By Lemma 4.12 we have

$$(J_j + I_j)\cap\mathbf{K}[U] = J'_j + I$$

for some monomial ideal J'_j of $\mathbf{K}[U]$. Thus $x^{\gamma_j}R_{\sigma_j}\cap R_\sigma$ is isomorphic to $J'_j\mathbf{K}[U]/I$. Therefore we have

$$\bigcap_{j=1}^{k} x^{\gamma_j}R_{\sigma_j} \simeq \bigcap_{j=1}^{k} J'_j\mathbf{K}[U]/I.$$

This is a monomial ideal in the indeterminates U_l by [ES96, Corollary 1.6]. By Noetherianity this monomial ideal is generated by finitely many monomials:

$$U^{\beta_1}, \ldots, U^{\beta_r}.$$

For every i we have $U^{\beta_i} = x^{\gamma'_i}$ for some $\gamma'_i \in \sigma\cap\mathbf{Z}^n$. Set $C = \{\gamma'_1,\ldots,\gamma'_r\}$. Then we have

$$\bigcap_{j=1}^{k}(\gamma_j+\sigma_j)\cap\mathbf{Z}^n = C + \sigma\cap\mathbf{Z}^n.$$

Lemma 4.14. *Let C be a finite subset of $\mathbf{R}^n$ and let σ be a convex cone. Then*

$$\mathrm{Conv}(C) + \sigma = \mathrm{Conv}(C+\sigma).$$

Proof. We may make a translation and assume that $\underline{0} \in C$. Let $u \in \mathrm{Conv}(C+\sigma)$. This means that

$$u = \sum_{i=1}^{k}\lambda_i c_i + \sum_{j=1}^{l}\mu_j s_j$$

where the c_i are in C, the s_j in σ, $\lambda_i, \mu_j \geq 0$ and $\sum_i \lambda_i + \sum_j \mu_j = 1$. Since σ is a convex cone then $s := \sum_{j=1}^{l} \mu_j s_i \in \sigma$. Moreover $c := \sum_{i=1}^{k} \lambda_i c_i \in \mathrm{Conv}(C)$ because $\sum_i \lambda_i \leq 1$ and $\underline{0} \in C$. Hence $u \in \mathrm{Conv}(C) + \sigma$.

4.3 Proof of Theorem 1.3

• Because $\tau(\xi)$ is a rational cone it is generated by finitely many integer coordinate vectors $u_1, \ldots, u_s \in \mathbf{R}_{\geq 0}{}^n$. We assume that the set $\{u_1, \ldots, u_s\}$ is minimal, i.e. the rays $\mathbf{R}_{\geq 0} u_i$ are the extremal rays of $\tau(\xi)$. For every $i = 1, \ldots, s$ and $t \in \mathbf{R}$ we set

$$H_i(t) = \{x \in \mathbf{R}^n \mid u_i \cdot x = t\},\ H_i(t)^+ = \{x \in \mathbf{R}^n \mid u_i \cdot x \geq t\}.$$

We have

$$\tau(\xi)^\vee = \bigcap_{i=1}^{s} H_i(0)^+.$$

Since the $\mathbf{R}_{\geq 0} u_i$ are the extremal rays of $\tau(\xi)$, for every $(n-1)$-dimensional face σ of $\tau(\xi)^\vee$, there is an index i such that $H_i(0) \cap \tau(\xi)^\vee = \sigma$.

Moreover the vectors $u_1, \ldots, u_s$ are in the boundary of $\tau_0'(\xi)$ (indeed $\tau(\xi) = \overline{\tau_0'(\xi)}$ by Corollary 4.3). Hence by Corollary 4.8 we have $u_i \notin \tau_1'(\xi)$ for any i. Thus for every i we have $u_i \in \tau_0'(\xi)$ or $u_i \in \mathbf{R}_{\geq 0}{}^n \backslash (\tau_0'(\xi) \cup \tau_1'(\xi))$.

Because $\tau_0'(\xi) \cap \mathbf{R}_{>0}{}^n$ is open by Proposition 4.7 and Remark 4.5, if $u_i \in \tau_0'(\xi)$ then $u_i \in \mathbf{R}_{\geq 0}{}^n \backslash \mathbf{R}_{>0}{}^n$ because u_i is in the boundary of $\tau_0'(\xi)$. In particular $\langle u_i \rangle^\perp \cap \mathbf{R}_{\geq 0}{}^n \neq \{\underline{0}\}$. Because u_i has integer coordinates, $\langle u_i \rangle^\perp$ is generated by vectors with integer coordinates. Take $f_i(x) \in \mathbf{K}[[x]]$ with support in $\langle u_i \rangle^\perp \cap \mathbf{R}_{\geq 0}{}^n$ and such that

$$\#\left\{\mathrm{Supp}(\xi + f_i(x)) \cap \langle u_i \rangle^\perp \cap \mathbf{R}_{\geq 0}{}^n\right\} = +\infty. \tag{2}$$

By Corollary 4.10 $\tau(\xi) = \tau(\xi + f_i(x))$. So we can replace ξ with $\xi + f_i(x)$. Therefore by (2) we have $u_i \in \mathbf{R}_{\geq 0}{}^n \backslash (\tau_0'(\xi) \cup \tau_1'(\xi))$. By doing the same for every i such that $u_i \in \tau_0'(\xi)$, we may replace ξ with $\xi + f(x)$ for some formal power series $f(x) \in \mathbf{K}[[x]]$ and assume that none of the u_i is in $\tau_0'(\xi)$.

Hence we can repeat the proof of [AR19, Theorem 5.13] and see that for every i there exist a Laurent polynomial $p_i(x)$ and a real number t_i such that

$$\mathrm{Supp}(\xi + p_i(x)) \subset H_i(t_i)^+ \text{ and } \#\left(\mathrm{Supp}(\xi + p_i(x)) \cap H_i(t_i)\right) = +\infty.$$

Then, modulo a finite number of monomials and a formal power series $f(x) \in \mathbf{K}[[x]]$, the support of ξ is included in $\bigcap_{i=1}^{s} H_i(t_i)^+ \cap \mathbf{Z}^n$. Moreover each $H_i(t_i)$ contains infinitely many monomials of ξ, i.e there is a Laurent polynomial $p(x)$ such that

$$\mathrm{Supp}(\xi + p(x)) \subset \bigcap_{i=1}^{s} H_i(t_i)^+ \cap \mathbf{Z}^n \text{ and } \#\left(\mathrm{Supp}(\xi + p(x)) \cap H_i(t_i)\right) = +\infty \quad \forall i.$$

For every i we have $H_i(t_i)^+ = \gamma_i + H_i(0)^+$ for some $\gamma_i \in \mathbf{Z}^n$. By Corollary 4.13 there is a finite set $C \subset \mathbf{Z}^n$ such that

$$\bigcap_{i=1}^{s} H_i(t_i)^+ \cap \mathbf{Z}^n = C + \bigcap_{i=1}^{s} H_i(0)^+ \cap \mathbf{Z}^n = C + \tau(\xi)^\vee \cap \mathbf{Z}^n.$$

By Lemma 4.14 we have

$$\mathrm{Conv}(C + \tau(\xi)^\vee) = \mathrm{Conv}(C) + \tau(\xi)^\vee$$

is an unbounded convex polytope. Moreover its faces of highest dimension are all of the form $H_i(t_i) \cap \mathrm{Conv}(C + \tau(\xi)^\vee)$. Indeed the convex hull of $\bigcap_{i=1}^{s} H_i(t_i)^+ \cap \mathbf{Z}^n$ is $\bigcap_{i=1}^{s} H_i(t_i)^+$ because the $H_i(t_i)$ are affine hyperplanes defined over $\mathbf{Z}$. This proves i)

- Assume now that there are $C' \in \mathbf{R}^n$ and a convex cone $\sigma \subset \tau(\xi)^\vee$ such that

$$\mathrm{Supp}(\xi + p'(x) + f'(x)) \subset \gamma' + \sigma$$

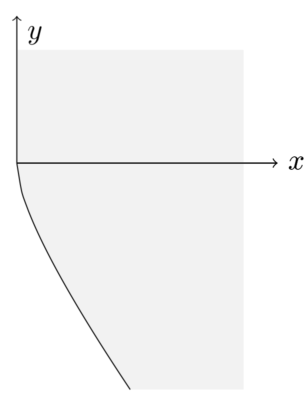
for some Laurent polynomial $p'(x)$ and some formal power series $f'(x) \in \mathbf{K}[[x]]$. Then by definition of $\tau(\xi)$ we have

$$\sigma^\vee \subset \tau(\xi).$$

Therefore $\sigma = \tau(\xi)^\vee$. This proves ii).

4.4 Three examples

Example 4.15. *Let* $E := \{(x,y) \in \mathbf{R}_{\geq 0} \times \mathbf{R} \mid y \geq -x - \sqrt{x}\}$ *and let* ξ *be a Laurent series whose support is* $\mathbf{Z}^2 \cap E$.



Then $\tau(\xi)^\vee$ *is the set*

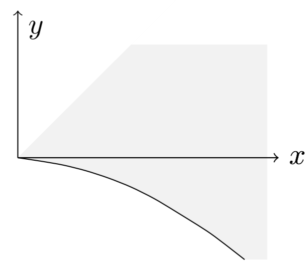
$$\{(x,y) \in \mathbf{R}_{\geq 0} \times \mathbf{R} \mid y > -x\}.$$

Thus, $\tau(\xi)$ *is a not a polyhedral cone. Therefore* ξ *is not algebraic over* $\mathbf{K}(\!(x,y)\!)$.
Moreover $\tau_1'(\xi)$ *is the closed cone generated by* $(1,0)$ *and* $(1,1)$. *So* $\tau_1'(\xi)$ *is not open. In this case* ${\mathbf{R}_{\geq 0}}^n = \tau_0'(\xi) \cup \tau_1'(\xi)$.

Example 4.16. *We consider the set*

$$E := \{(x, y) \in \mathbf{R}_{\geq 0} \times \mathbf{R} \mid y \geq \ln(x+1)\}.$$

We rotate it by an angle of $-\pi/4$ and denote this set by Γ. We denote a Laurent series whose support is $\Gamma \cap \mathbf{Z}^2$ by ξ.

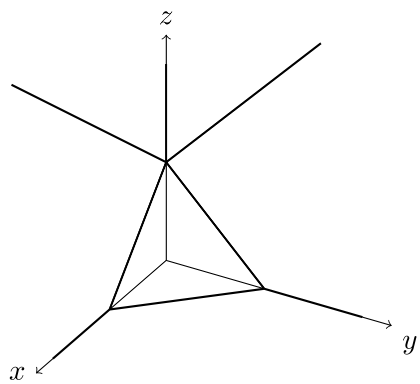


Then $\tau(\xi)^\vee$ is the cone generated by $(1, -1)$ and $(0, 1)$, so it is rational, but ξ is not algebraic since Theorem 1.3 ii) is not satisfied.
Moreover $\tau(\xi)$ is generated by $(0, 1)$ and $(1, 1)$. Thus the vector $(1, 1)$ is in the boundary of $\tau(\xi)$ but here $(1, 1) \in \tau_0'(\xi)$. Thus $\tau_0'(\xi)$ is closed and by Proposition 4.6 and Lemma 4.9 $\tau_0(\xi)$ is closed.

Example 4.17. *Let C be the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, and let σ be the cone generated by the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(1, -1, 1)$, $(-1, 1, 1)$ and $(0, 0, 1)$. We can construct a Laurent series ξ, algebraic over $\mathbf{K}[[x, y, z]]$, with support in $\mathrm{Conv}(C) + \sigma$, such that all the unbounded faces of $\mathrm{Conv}(C) + \sigma$ contain infinitely many monomials of ξ as follows:*
We fix an algebraic series $G(T)$ not in $\mathbf{K}(T)$. Then we set

$$\xi = xG(x) + yG(y) + zG(z) + zG\left(\frac{xz}{y}\right) + zG\left(\frac{yz}{x}\right).$$

Then ξ is algebraic over $\mathbf{K}((x, y, z))$, its support is $\mathrm{Conv}(C) + \sigma$ and all the unbounded faces of $\mathrm{Conv}(C) + \sigma$ contain infinitely many monomials of ξ. Therefore $\tau(\xi)^\vee = \sigma$. Moreover we can see that there is no $\gamma \in \mathbf{R}^n$ such that Supp$(\xi) \subset \gamma + \sigma$ and every face of $\gamma + \sigma$ contains infinitely many monomials of ξ. Indeed, if it were the case, the five unbounded 1-dimensional faces of $\mathrm{Conv}(C) + \sigma$ would intersect at one point and this is clearly not the case. Thus we cannot assume that the finite set C of Theorem 1.3 is a single point.



5. The positive characteristic case

In the positive characteristic case, the roots of polynomials with coefficients in $\mathbf{K}(\!(x)\!)$, with $x=(x_1,\ldots,x_n)$, are not Laurent Puiseux series in general. This was first noticed by Chevalley in [Ch51] for the case $n=1$: he showed that the solutions of the equation

$$T^p - x_1^{p-1}T - x_1^{p-1} = 0$$

cannot been expressed as Puiseux series. Then Abhyankar noticed that for such a polynomial, the roots can be expressed as series with support in $\mathbf{Q}$ with the additional property that their support is well ordered. Here such a root can be written as

$$\sum_{k=1}^{\infty} x_1^{1-\frac{1}{p^k}}.$$

The determination of the algebraic closure of $\mathbf{K}(\!(x_1)\!)$ for $n=1$, when $\mathbf{K}$ is a positive characteristic field, was finally achieved recently (see [Ke01], [Ke17]).

For $n \geq 2$, this problem has recently been investigated by Saavedra [Sa17]. He generalized Macdonald's Theorem to the positive characteristic case as follows:

Theorem 5.1. *[Sa17, Theorem 5.3] Let $\mathbf{K}$ be an algebraically closed field of characteristic $p>0$. Let $\omega \in {\mathbf{R}_{>0}}^n$ be a vector whose coordinates are $\mathbf{Q}$-linearly independent. The set*

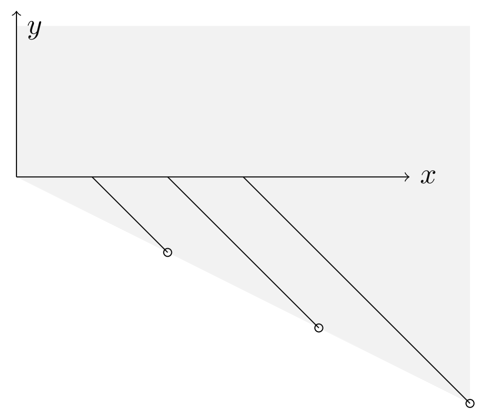
$$\mathcal{S}_\omega^{\mathbf{K}} = \Big\{ \xi \mid \exists k \in \mathbf{N}^*, \gamma \in \mathbf{Z}^n, \sigma \text{ a } \leq_\omega\text{-positive rational cone,}$$
$$\mathit{Supp}(\xi) \subset (\gamma+\sigma) \cap \cup_{l\in\mathbf{N}} \frac{1}{kp^l}\mathbf{Z}^n \text{ and } \mathit{Supp}(\xi) \text{ is } \leq_\omega \text{-well ordered} \Big\}$$

is an algebraically closed field.

It is a natural question to extend the problem of the shape of the support of an element of $\mathcal{S}_\omega^{\mathbf{K}}$ that is algebraic over $\mathbf{K}(\!(x)\!)$. Firstly we can remark that Theorem 1.3 is no longer true in this situation:

Example 5.2. *Let $\mathbf{K}$ be a field of characteristic $p>0$. Set $f=\sum_{k=1}^{\infty} t^{1-\frac{1}{p^k}}$. The series f is*

algebraic over $\mathbf{K}[t]$ *since* $f^p - t^{p-1}f - t^{p-1} = 0$*. Thus* $g := \sum_{k=1}^{\infty}\left(\frac{x}{y}\right)^{1-\frac{1}{p^k}}$ *is algebraic over* $\mathbf{K}[x,y]$*. We set* $\xi = \sum_{k=1}^{\infty}(xg)^k$*. Because* $\xi = \frac{xg}{1-xg}$*,* ξ *is rational over the field extension of* $\mathbf{K}(x,y)$ *by* g*. Hence* ξ *is algebraic over* $\mathbf{K}[x,y]$*. The support of* ξ *is included in the cone* σ *generated by* $(2,-1)$ *and* $(0,1)$*. Moreover the support of* $(xg)^k$ *contains a sequence of points converging to* $(2k,-k)$*. But* $(2k,-k)$ *does not belong to the support of* ξ *since* $(1,-1)$ *does not belong to the support of* g*. Hence* $\tau(\xi) = \sigma^\vee$ *is generated by* $(0,1)$ *and* $(1,2)$*.*
But Theorem 1.2 ii) does not hold in this case: there is no hyperplane $H_\lambda = \{(x,y) \in \mathbf{R}^2 \mid x+2y = \lambda\}$ *containing infinitely many elements of Supp*(ξ) *such that* $H_\lambda^- := \{(x,y) \in \mathbf{R}^2 \mid x+2y < \lambda\}$ *contains only finitely many elements of Supp*(ξ)*.*



Here $\tau'_0(\xi) = \{0\}$*. This shows that Lemma 4.2 is not valid for series with exponents in* $\mathbf{Q}^n$ *that are algebraic over* $\mathbf{K}((x))$*, for* $\mathbf{K}$ *a positive characteristic field.*

We can also remark that a positive characteristic version of Theorem 1.2 could not be proved in the same way as in characteristic zero since Lemma 3.1 is no longer true in positive characteristic. The following example is given in [Sa17]:

Example 5.3. *[Sa17, Example 3] Set* $P(T) = T^p - x^{p-1}T - x^{p-1}y^3$ *over a field* $\mathbf{K}$ *of characteristic* $p > 0$*. Set*

$$\omega_1 = \left(1, \sqrt{2}\right),\ \omega_2 = \left(1, \frac{\sqrt{2}}{6}\right).$$

The roots of P *in* $\mathcal{S}^{\mathbf{K}}_{\omega_2}$ *have support in a translation of* ${\mathbf{R}_{\geq 0}}^2$ *since these roots are*

$$\sum_{k=1}^{\infty} x^{1-\frac{1}{p^k}} y^{\frac{3}{p^k}} + cx, \quad c \in \mathbf{F}_p.$$

But the roots of P *in* $\mathcal{S}^{\mathbf{K}}_{\omega_1}$ *have support in the cone* σ *generated by* $(1,0)$ *and* $(-1,3)$*, and the face generated by* $(-1,3)$ *contains infinitely many exponents of each of these roots. Indeed these roots are*

$$-\sum_{k=1}^{\infty} x^{1-p^k} y^{3p^k} + cx, \quad c \in \mathbf{F}_p.$$

Let ξ be one root of P in $\mathcal{S}_{\omega_1}^{\mathbf{K}}$. So $\tau(\xi)=\sigma^\vee$. Set $\sigma_0 := {\mathbf{R}_{\geq 0}}^2$. Then

$$\omega := (2,1) \in \tau(\xi)\cap \operatorname{Int}(\sigma_0).$$

But σ_0 is not included in $\tau(\xi)$ since $(4,1)$ is not in $\tau(\xi)$. Thus Lemma 3.1 is not valid in positive characteristic.

Nevertheless we can extend some of the previous results, proved in characteristic zero, to the positive characteristic case. The main problems are as follows:
first we must find an alternative to Abhyankar-Jung Theorem (Theorem 3.3) that is only valid in characteristic zero. For this we will use the fact that the set of orders Ord_n is a topological compact space for a well chosen topology. This topology has been introduced by Ewald and Ishida [EI06].
Then we prove an extension of Theorem 5.1 analogous to Theorem 2.10. This is based on the notion of field-family introduced by Rayner [Ra68] that gives aa method for the construction of Henselian valued fields which are close to be algebraically closed.
Finally we introduce a natural analogue of the cone $\tau(\xi)$ in the positive characteristic case. We prove that this cone is rational and we relate it to the support of ξ (see Theorem 5.16 and 5.18).

5.1 The space Ord_n as a compact topological space

Definition 5.4. *[EI06][Te18] The set $\operatorname{ZR}(\mathbf{Q}^n)$ is endowed with a topology for which the sets*

$$\mathcal{U}_\sigma := \{\preceq \in \operatorname{ZR}(\mathbf{Q}^n) \text{ such that } \sigma \text{ is } \preceq\text{-positive}\}$$

form a basis of open sets where σ runs over the full dimensional strongly convex rational cones.

Remark 5.5. *We this definition we have $\operatorname{Ord}_n = \mathcal{U}_{{\mathbf{R}_{\geq 0}}^n}\cap \operatorname{Ord}(\mathbf{Q}^n)$.*

We have the following result:

Theorem 5.6. *[EI06] The space $\operatorname{ZR}(\mathbf{Q}^n)$ is compact and $\operatorname{Ord}(\mathbf{Q}^n)$ is closed in $\operatorname{ZR}(\mathbf{Q}^n)$. Moreover every $\mathcal{U}_\sigma$ is compact. Therefore Ord_n is compact.*

This allows us to prove the following result that will replace the use of Theorem 3.3 in the positive characteristic case:

Corollary 5.7. *Let σ be a full dimensional strongly convex rational cone. For every $\preceq\in\mathcal{U}_\sigma$ let $\sigma_\preceq$ be a full dimensional $\preceq$-positive cone. Then there is a finite number of orders $\preceq_1, \ldots, \preceq_N$ such that*

$$\sigma^\vee \subset \bigcup_{i=1}^N \sigma_{\preceq_i}^\vee.$$

Proof. We have that $\mathcal{U}_\sigma \subset \bigcup_{\preceq\in\mathcal{U}_\sigma} \mathcal{U}_{\sigma_\preceq}$. Thus, by Theorem 5.6, there exist $\sigma_{\preceq_1}, \ldots, \sigma_{\preceq_N}$ such that $\mathcal{U}_\sigma \subset \bigcup_{i=1}^{N} \mathcal{U}_{\sigma_{\preceq_i}}$. Let ω be a non zero element of $\sigma^\vee$. Let $u_2, \ldots, u_n \in \mathbf{R}^n$ such that $\leq_{(\omega,u_2,\ldots,u_n)}\in \mathcal{U}_\sigma$. Then, there exists $i \in \{1, \ldots, N\}$ such that $\leq_{(\omega,u_2,\ldots,u_n)}\in \mathcal{U}_{\sigma_{\preceq_i}}$. In particular $\omega \in \sigma^\vee_{\preceq_i}$. This proves the corollary.

5.2 Algebraically closed fields in positive characteristic

Definition 5.8. *Let $\preceq\in \mathrm{Ord}_n$ and $A \subset \mathbf{R}^n$. We say that A is $\preceq$-well ordered if A is well ordered with respect to $\preceq$.*

Definition 5.9. *We fix an order $\preceq\in \mathrm{Ord}_n$ and a field $\mathbf{K}$ of positive characteristic $p > 0$. We set*

$$\mathcal{S}^{\mathbf{K}}_{\preceq} := \Big\{ \xi \in \mathbf{K}(\!(x^{\mathbf{Q}^n})\!) \mid \exists k \in \mathbf{N}^*, \gamma \in \mathbf{Z}^n, \sigma \text{ a } \preceq\text{-positive rational cone, such that}$$
$$\mathit{Supp}(\xi) \subset (\gamma+\sigma) \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbf{Z}^n, \text{ and } \forall \preceq'\in \mathrm{Ord}_n \cap \mathcal{U}_\sigma,\ \mathit{Supp}(\xi) \text{ is } \preceq'\text{-well ordered} \Big\}.$$

We have the following theorem:

Theorem 5.10. *Let $\preceq\in \mathrm{Ord}_n$. If $\mathbf{K}$ is an algebraically closed field of positive characteristic $p > 0$, the set $\mathcal{S}^{\mathbf{K}}_{\preceq}$ is an algebraically closed field containing $\mathbf{K}(\!(x)\!)$.*

In order to prove this theorem we will use the notion of field-family introduced by Rayner:

Definition 5.11. *[Ra68] A family $\mathcal{F}$ of subsets of an ordered abelian group $(G, \preceq)$ is said to be a field-family with respect to G if we have the following.*

- (i) *Every element of $\mathcal{F}$ is a well ordered subset of G.*
- (ii) *The elements of the members of $\mathcal{F}$ generate Γ as an abelian group.*
- (iii) $\forall (A, B) \in \mathcal{F}^2, A \cup B \in \mathcal{F}$.
- (iv) $\forall A \in \mathcal{F}$ *and* $B \subset A, B \in \mathcal{F}$.
- (v) $\forall (A, \gamma) \in \mathcal{F} \times G,\ \gamma + A \in \mathcal{F}$.
- (vi) $\forall A \in \mathcal{F} \cap \{\delta \in \Gamma \mid \delta \succeq 0\}$, *the semigroup generated by A belongs to $\mathcal{F}$.*

Theorem 5.12. *[Ra68, Theorem 2] If $\mathcal{F}$ is a field-family with respect to G then the set*

$$\left\{ \sum_{g\in\Gamma} a_g x^g \mid \{g \mid a_g \neq 0\} \in \mathcal{F} \right\}$$

is a Henselian valued field.

For $\preceq\in \mathrm{Ord}_n$ we set

$$\mathcal{F}_{\preceq} := \left\{ A \subset \mathbf{Q}^n \mid \exists k \in \mathbf{N}^*, \gamma \in \mathbf{Z}^n, \sigma \text{ a } \preceq\text{-positive rational cone, such that} \right.$$
$$\left. A \subset (\gamma+\sigma) \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbf{Z}^n, \text{ and } \forall \preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma,\ A \text{ is } \preceq'\text{-well ordered} \right\}.$$

Proposition 5.13. *The set $\mathcal{F}_{\preceq}$ is a field-family with respect to $(\mathbf{Q}^n, \preceq)$.*

Proof. It is straightforward to verify that $\mathcal{F}_{\preceq}$ satisfies the five first items of Definition 5.11. Therefore we only prove (6) here.
Let us consider an element $A \in \mathcal{F}_{\preceq} \cap \{\delta \in \mathbf{Q}^n \mid \delta \succeq \underline{0}\}$. So

$$A \subset (\gamma+\sigma) \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbf{Z}^n$$

for some $k \in \mathbf{N}^*, \gamma \in \mathbf{Z}^n$ and σ a $\preceq$-positive rational cone. We know that there exist $(u_1, \ldots, u_s) \in (\mathbf{R}^n)^s$ and $(q_1, \ldots, q_r) \in (\mathbf{Q}^n)^r$ such that $\leq = \leq_{(u_1,\ldots,u_s)}$ and $\sigma = \langle q_1, \ldots, q_r\rangle$.
Assume first that $\gamma \succeq \underline{0}$. Then $A \subset \sigma' = \langle \gamma, q_1, \ldots, q_r\rangle$ and σ' is a $\preceq$-positive rational cone. Hence the semigroup generated by A is included in $\sigma' \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbf{Z}^n$. Moreover, for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma'}$, A is $\preceq'$-well ordered. Indeed this is true for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma$ and $\sigma'^\vee \subset \sigma^\vee$. Therefore, since for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma'}$ the set A is $\preceq'$-positive, by [Ne49, Theorem 3.4, p. 206] the semigroup generated by A is $\preceq'$-well ordered.

Now assume that $\gamma \prec \underline{0}$. By [Sa17, Lemma 3.11] we may assume that $\underline{0} \in \gamma + \sigma$. We consider $a := \min(A \setminus \{\underline{0}\})$ and we set

$$H_i := \{x \in \mathbf{R}^n \text{ such that } x \cdot u_i = a \cdot u_i\}$$

and

$$H_i^+ := \{x \in \mathbf{R}^n \text{ such that } x \cdot u_i \geq a \cdot u_i\}.$$

Since $A \subset \{\delta \in \mathbf{Q}^n \mid \delta \succeq \underline{0}\}$, we know that $a \succeq \underline{0}$. Hence $a \cdot u_1 \geq 0$.
If $a \cdot u_1 > 0$ we set $\sigma' := \mathrm{cone}(H_1 \cap \sigma)$. It is a $\preceq$-positive cone such that $(\gamma+\sigma) \cap H_1^+ \subset \sigma'$ and $\sigma' \cap u_1^\perp = \{\underline{0}\}$ (see [Sa17, Lemma 3.8]). Therefore $A \subset (\gamma+\sigma) \cap H_1^+ \subset \sigma'$. Then the semigroup generated by A is included in $\sigma' \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbf{Z}^n$. Since $\underline{0} \in \gamma+\sigma$, we have that $\sigma \subset \gamma+\sigma \subset \sigma'$. Hence $\sigma'^\vee \subset \sigma^\vee$, and therefore, for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma'}$, A is $\preceq'$-well ordered and $\preceq'$-positive. This implies (by [Ne49, Theorem 3.4, p. 206]) that the semigroup generated by A is $\preceq'$-well ordered for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma'}$.
Now assume that $a \cdot u_1 = 0$. We denote the set $A \cap H_1$ by B, and we set $a_1 := \min(A \setminus B)$. Since $A \subset \{\delta \in \mathbf{Q}^n \mid \delta \succeq \underline{0}\}$ and $a_1 \notin H_1$, we have $a_1 \cdot u_1 > 0$. Using the same argument given above there exists a rational $\preceq$-positive cone σ_1 containing σ such that $(A \setminus B) \subset (\gamma+\sigma) \cap H_1^+ \subset \sigma_1$. By definition of a, we have $a = \min(B \setminus \{\underline{0}\})$. If $a \cdot u_2 > 0$ then with the same argument there exists a rational $\preceq$-positive cone σ_2 such that $B \subset (\gamma+\sigma) \cap H_2^+ \subset \sigma_2$. Then the cone $\sigma' = \sigma_1 \cup \sigma_2$

is $\preceq$-positive, $A \subset \sigma'$, and $\sigma \subset \sigma'$. If $a \cdot u_2 = 0$ we repeat the same process and consider $B \setminus H_2$. This process ends because, since $a \succ 0$, there exists an index i such that $a \cdot u_j = 0$ forall $j < i$ and $a \cdot u_i > 0$. Thus, by induction, we have a $\preceq$-positive rational cone σ'' such that $A \subset \sigma''$ and $\sigma \subset \sigma''$. Hence the semigroup generated by A is included in $\sigma'' \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbf{Z}^n$. Since $\sigma \subset \sigma''$, for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma''}$, A is $\preceq'$-well ordered and $\preceq'$-positive. This implies (by [Ne49, Theorem 3.4, p. 206]) that the semigroup generated by A is $\preceq'$-well ordered for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma''}$. This concludes the proof.

Proof of Theorem 5.10. By Proposition 5.13 and Theorem 5.12, the set $\mathcal{S}_{\preceq}^{\mathbf{K}}$ is a Henselian valued field.

Assume that $\mathcal{S}_{\preceq}^{\mathbf{K}}$ is not algebraically closed. By [Ra68, Lemma 4] there exists $a \in \mathcal{S}_{\preceq}^{\mathbf{K}}$ such that $T^p - T - a$ is irreducible in $\mathcal{S}_{\preceq}^{\mathbf{K}}[T]$. Let us write

$$a = a^+ + a^-$$

where $\mathrm{Supp}(a^-) \subset \{b \in \mathbf{Q}^n \mid b \prec \underline{0}\}$ and $\mathrm{Supp}(a^+) \subset \{b \in \mathbf{Q}^n \mid b \succeq \underline{0}\}$. Because the map $b \longmapsto b^p$ is an additive map, if ξ^+ is a root of $T^p - T - a^+$ and ξ^- is root of $T^p - T - a^-$, then $\xi^+ + \xi^-$ is a root of $T^p - T - a$. We will prove that $T^p - T - a^+$ and $T^p - T - a^-$ admit a root in $\mathcal{S}_{\preceq}^{\mathbf{K}}$ contradicting the fact that $T^p - T - a$ is irreducible.

Since $\mathcal{S}_{\preceq}^{\mathbf{K}}$ is a Henselian valued field,

$$\mathfrak{O} := \left\{\xi \in \mathcal{S}_{\preceq}^{\mathbf{K}} \mid \forall b \in \mathrm{Supp}(\xi), b \succeq \underline{0}\right\}$$

is a Henselian local ring with maximal ideal

$$\mathfrak{m} := \left\{\xi \in \mathcal{S}_{\preceq}^{\mathbf{K}} \mid \forall b \in \mathrm{Supp}(\xi), b \succ \underline{0}\right\}.$$

The polynomial $T^p - T - a^+ \in \mathfrak{O}[T]$ has a root modulo $\mathfrak{m}$ since $\mathbf{K}$ is algebraically closed (here $\mathfrak{O}/\mathfrak{m} = \mathbf{K}$). Moreover the derivative of this polynomial is -1. Thus this polynomial satisfies Hensel's Lemma and admits a root ξ^+ in $\mathcal{S}_{\preceq}^{\mathbf{K}}$.

In order to prove that $T^p - T - a^-$ has a root in $\mathcal{S}_{\preceq}^{\mathbf{K}}$, we follow the proofs of [Ra68, Theorem 3], and [Sa17, Theorem 5.3]. We write $a^- = \sum_{q \in \mathbf{Q}^n} a_q^- x^q$ and we define

$$\xi^- := \sum_{q \in \mathbf{Q}^n} \left(\sum_{i=1}^{\infty} \left(a_{p^i q}^- \right)^{\frac{1}{p^i}} \right) x^q.$$

We can verify that ξ^- is well defined: for a given $q \in \mathrm{Supp}(a^-)$, the sequence $(p^i q)_i$ is strongly decreasing for the order $\preceq$ since $q \prec \underline{0}$. Therefore $a_{p^i q}^- = 0$ for i large enough because $\mathrm{Supp}(a^-)$ is $\preceq$-well ordered. Hence the sum $\sum_{i=1}^{\infty} \left(a_{p^i q}^-\right)^{\frac{1}{p^i}}$ is in fact a finite sum.

Exactly as done in the proof of [Sa17, Theorem 5.3], there exists a $\preceq$-positive cone σ and $\gamma \in \mathbf{Z}^n$

such that

$$\operatorname{Supp}(\xi^-) \subset (\gamma+\sigma) \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbf{Z}^n,$$

and for every order $\preceq' \in \operatorname{Ord}_n \cap \mathcal{U}_\sigma$, $\operatorname{Supp}(\xi^-)$ is $\preceq'$-well ordered. Thus $\xi^- \in \mathcal{S}_{\preceq}^{\mathbf{K}}$.
Moreover an easy computation shows that ξ^- is a root of $T^p - T - a^-$. This proves the theorem.

5.3 Positive analogue of $\tau(\xi)$ in positive characteristic

By Theorem 2.10, for a Laurent series ξ algebraic over $\mathbf{K}(\!(x)\!)$ where $\mathbf{K}$ is a field of characteristic zero, the cone $\tau(\xi)$ is the set of vectors $\omega \in {\mathbf{R}_{\geq 0}}^n$ such that $\operatorname{Supp}(\xi)$ is included in a translation of $\leq_\omega$-positive cone. But, in positive characteristic, Examples 5.2 and 5.3 show that the condition for the support of the series to be well ordered for a given order is a crucial condition. Therefore we define the following cone, which agrees with $\tau(\xi)$ for a Laurent series ξ, and we will prove its rationality:

Definition 5.14. *Let ξ be a series with support in $\mathbf{Q}^n$. We set*

$$\widetilde{\tau}(\xi) = \big\{\omega \in {\mathbf{R}_{\geq 0}}^n \mid \exists \sigma \subset \langle\omega\rangle^\vee, \gamma \in \mathbf{Z}^n,\ \mathit{Supp}(\xi) \subset \gamma+\sigma \text{ and } \forall \preceq \in \operatorname{Ord}_n \cap \mathcal{U}_\sigma, \mathit{Supp}(\xi) \text{ is } \preceq \text{-well ordered}\big\}.$$

Remark 5.15. *If ξ is a Laurent series, then $\tau(\xi) = \widetilde{\tau}(\xi)$.*

Then we have the following analogue of Theorem 1.2 in positive characteristic:

Theorem 5.16. *Let $\xi \in \mathcal{S}_{\preceq}^{\mathbf{K}}$ where $\mathbf{K}$ is a positive characteristic field and $\preceq \in \operatorname{Ord}_n$. Then $\widetilde{\tau}(\xi)$ is a strongly convex rational cone.*

Proof. By Theorem 5.10 for every order $\preceq' \in \operatorname{Ord}_n$ there are an element $\gamma_{\preceq'} \in \mathbf{Z}^n$, and a $\preceq'$-positive rational strongly convex cone $\sigma_{\preceq'}$ such that the roots of P, the minimal polynomial of ξ, can be expanded as series in $\mathcal{S}_{\preceq'}^{\mathbf{K}}$ with support in $\gamma_{\preceq'} + \sigma_{\preceq'}$. We may replace $\sigma_{\preceq'}$ by $\sigma_{\preceq'} + {\mathbf{R}_{\geq 0}}^n$ and assume that $\sigma_{\preceq'}$ contains the first orthant for every $\preceq'$. Moreover for every $\preceq'' \in \operatorname{Ord}_n \cap \mathcal{U}_{\sigma_{\preceq'}}$, the supports of these roots are $\preceq''$-well ordered.
Therefore, by Corollary 5.7, there exist σ_k containing ${\mathbf{R}_{\geq 0}}^n$, $k = 1, \ldots, N$, finitely generated strongly convex rational cones satisfying the following properties:

i) $\displaystyle\bigcup_{k=1}^{N} \sigma_k^\vee = {\mathbf{R}_{\geq 0}}^n$,

ii) for every k there are d Laurent Puiseux series with support in $\gamma_k + \sigma_k$ for some $\gamma_k \in \mathbf{Z}^n$, denoted by $\xi_1^{(k)}, \ldots, \xi_d^{(k)}$ with $P(\xi_i^{(k)}) = 0$ for $i = 1, \ldots, d$,

iii) for every k, every $\preceq' \in \operatorname{Ord}_n \cap \mathcal{U}_{\sigma_k}$ and every $i = 1, \ldots, d$, $\operatorname{Supp}(\xi_i^{(k)})$ is $\preceq'$-well ordered.

Thus Lemma 5.17 given below implies (exactly as for Corollary 3.2) that, after renumbering the

σ_k, there is an integer $l \leq N$ such that

$$\widetilde{\tau}(\xi) = \bigcup_{k=1}^{l} \sigma_k^{\vee}.$$

Therefore $\widetilde{\tau}(\xi)$ is a strongly convex rational cone.

Lemma 5.17. *Let ξ be a series belonging to $\mathcal{S}_{\preceq'}^{\mathbf{K}}$ for some $\preceq' \in \mathrm{Ord}_n$ and whose support is included in a translation of a strongly convex cone σ containing ${\mathbf{R}_{\geq 0}}^n$. Let $P \in \mathbf{K}[[x]][T]$ be a monic polynomial of degree d with $P(\xi) = 0$. Let $\sigma_0 \subset {\mathbf{R}_{\geq 0}}^n$ be a strongly convex rational cone such that*

i) there are d distinct series with rational exponents whose support is in $\gamma + \sigma_0$ for some $\gamma \in \mathbf{Z}^n$, denoted by $\xi_1, \ldots, \xi_d$ with $P(\xi_i) = 0$ for $i = 1, \ldots, d$,

ii) $Supp(\xi_j)$ is $\preceq'$-well ordered for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma_0}$.

Then

$$\mathrm{Int}(\sigma_0^{\vee}) \cap \widetilde{\tau}(\xi) \neq \emptyset \Longrightarrow \sigma_0^{\vee} \subset \widetilde{\tau}(\xi).$$

Proof. Consider a non zero vector $\omega \in \mathrm{Int}(\sigma_0^{\vee}) \cap \widetilde{\tau}(\xi)$. Since $\omega \in \widetilde{\tau}(\xi)$, there are $k \in \mathbf{N}$, $\gamma_0 \in \mathbf{Z}^n$, and σ a $\leq_\omega$-positive rational cone, such that

$$\mathrm{Supp}(\xi) \subset (\gamma_0 + \sigma) \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l} \mathbf{Z}^n,$$

and $\forall \preceq \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma$, $\mathrm{Supp}(\xi)$ is $\preceq$-well ordered. Since σ is $\leq_\omega$-positive and strongly convex, there exists an order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$ such that σ is $\preceq$-positive (see [AR19, Lemma 3.8]). Therefore $\mathrm{Supp}(\xi)$ is $\preceq$-well ordered. Thus ξ is a root of P in $\mathcal{S}_{\preceq}^{\mathbf{K}}$ since $\mathrm{Supp}(\xi)$ is $\preceq$-well ordered. On the other hand, ω is in the interior of $\sigma_0^{\vee}$, thus $\sigma_0 \cap \langle \omega \rangle^{\perp} = \{\underline{0}\}$. This implies that for every $u \in \sigma_0$, $\underline{0} \preceq u$, since $\preceq$ is refining $\leq_\omega$. That is, σ_0 is $\preceq$-positive. In particular the ξ_i are the roots of P in $\mathcal{S}_{\preceq}^{\mathbf{K}}$ by ii) and because they are $\preceq$-well ordered. Thus $\xi = \xi_i$ for some i. Hence there is some $\gamma'' \in \mathbf{Z}^n$ such that

$$\mathrm{Supp}(\xi) \subset \gamma'' + \sigma_0.$$

Thus, for $\omega' \in \sigma_0^{\vee}$, we have that

$$\sigma_0 \subset \langle \omega' \rangle^{\vee}.$$

Moreover $\mathrm{Supp}(\xi)$ is $\preceq'$-well ordered for every order $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma_0}$ since $\xi = \xi_i$ for some i. Hence $\omega' \in \widetilde{\tau}(\xi)$. This proves the lemma.

Now we are able to prove the following analogue of Theorem 1.3 i) a) and ii):

Theorem 5.18. *Let $\xi \in \mathcal{S}_{\preceq'}^{\mathbf{K}}$ for some $\preceq' \in \mathrm{Ord}_n$ that is algebraic over $\mathbf{K}(\!(x)\!)$.*

i) There exists $\gamma \in \mathbf{Z}^n$ such that

$$Supp(\xi) \subset \gamma + \widetilde{\tau}(\xi)^{\vee}$$

and for every $\preceq \in \mathrm{Ord}_n \cap \mathcal{U}_{\widetilde{\tau}(\xi)}$, *Supp*$(\xi)$ *is* $\preceq$*-well ordered.*

ii) Let σ *be a cone such that*

$$\mathit{Supp}(\xi) \subset \gamma + \sigma$$

for some γ*, and such that for every* $\preceq \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma^\vee}$, *Supp*$(\xi)$ *is* $\preceq$*-well ordered. Then* $\widetilde{\tau}(\xi)^\vee \subset \sigma$.

Proof. By assumption, for every $\omega \in \widetilde{\tau}(\xi)$, there are $\gamma_\omega \in \mathbf{Z}^n$ and a $\leq_\omega$-positive rational cone σ_ω such that $\mathrm{Supp}(\xi) \subset \gamma_\omega + \sigma_\omega$ and $\mathrm{Supp}(\xi)$ is $\preceq$-well ordered for every $\preceq \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma_\omega}$.
Therefore for $\omega \in \widetilde{\tau}(\xi)$ we have $\omega \in \sigma_\omega^\vee$, thus, $\widetilde{\tau}(\xi) \subset \bigcup_{\omega \in \widetilde{\tau}(\xi)} \sigma_\omega^\vee$.
On the other hand, let us fix $\omega \in \widetilde{\tau}(\xi)$. Let $\omega' \in \sigma_\omega^\vee$. Since σ_ω is $\leq_{\omega'}$-positive, by definition of $\widetilde{\tau}(\xi)$, $\omega' \in \widetilde{\tau}(\xi)$. This proves that

$$\widetilde{\tau}(\xi) = \bigcup_{\omega \in \widetilde{\tau}(\xi)} \sigma_\omega^\vee$$

or, equivalently,

$$\mathcal{U}_{\widetilde{\tau}(\xi)^\vee} = \bigcup_{\omega \in \widetilde{\tau}(\xi)} \mathcal{U}_{\sigma_\omega}.$$

By Theorem 5.6, there are $\omega_1, \ldots, \omega_N \in \widetilde{\tau}(\xi)$ such that

$$\widetilde{\tau}(\xi) = \bigcup_{i=1}^N \sigma_{\omega_i}^\vee.$$

In particular

$$\mathrm{Supp}(\xi) \subset \bigcap_{i=1}^N \left(\gamma_{\omega_i} + \sigma_{\omega_i}\right).$$

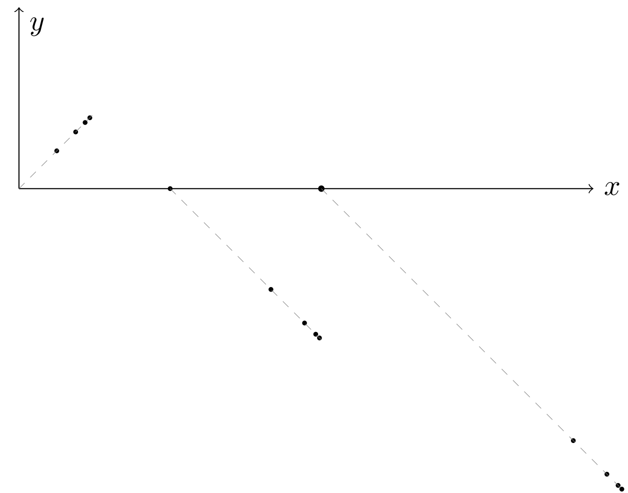
Therefore, by Lemma 2.7, there is $\gamma \in \mathbf{Z}^n$ such that

$$\mathrm{Supp}(\xi) \subset \gamma + \widetilde{\tau}(\xi)^\vee.$$

Moreover, for $\preceq \in \mathcal{U}_{\widetilde{\tau}(\xi)^\vee}$, there is $\omega \in \widetilde{\tau}(\xi)$ such that $\preceq$ refines $\leq_\omega$. Hence, by definition of $\widetilde{\tau}(\xi)$, $\mathrm{Supp}(\xi)$ is $\preceq$-well ordered. This proves i).
Now let σ as in ii). Let $\omega \in \sigma^\vee$ and let $\preceq \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma$ refining ω. By the assumption on σ, $\omega \in \widetilde{\tau}(\xi)$. Therefore $\sigma^\vee \subset \widetilde{\tau}(\xi)$ and $\widetilde{\tau}(\xi)^\vee \subset \sigma$. This proves ii).

Remark 5.19. *Let us consider the algebraic series* ξ *of Example 5.2. In this case* $\widetilde{\tau}(\xi)$ *is the dual of the cone generated by* $(0,1)$ *and* $(1,-1)$*. Therefore we have* $\widetilde{\tau}(\xi) \subsetneq \tau(\xi)$.

Example 5.20. *Still in Example 5.2 the series* ξ *satisfies Theorem 1.3 i) b) by replacing* $\tau(\xi)$ *with* $\widetilde{\tau}(\xi)$*: we only need to remove the constant term of* ξ*, and add a series in* y*, in order to obtain a series whose support is* $(0,1) + \widetilde{\tau}(\xi)^\vee$*, and both faces of* $(0,1) + \widetilde{\tau}(\xi)^\vee$ *contain infinitely many exponents of this series.*
Now consider the series $\xi' = \xi + f\left(x^{\frac{1}{2}} y^{\frac{1}{2}}\right)$ *where* $f(t) = \sum_{k=1}^\infty t^{1-\frac{1}{p^k}}$ *as in Example 5.2.*



Then ξ' is algebraic over $\mathbf{K}(\!(x)\!)$. We remark that Supp(ξ') contains the sequence

$$\left(\left(\frac{1}{2},\frac{1}{2}\right)-\left(\frac{1}{2p^k},\frac{1}{2p^k}\right)\right)_{k\in\mathbf{N}}.$$

Here $\widetilde{\tau}(\xi')=\widetilde{\tau}(\xi)$. For $\omega=(1,-1)$ (which is in the boundary of $\widetilde{\tau}(\xi')$) and $s\in\mathbf{R}$, we define

$$H(s):=\{x\in\mathbf{R}^n \mid x\cdot\omega=s\},\ H^-(s)=\{x\in\mathbf{R}^n\mid x\cdot\omega<s\}.$$

Then we see that, for $s<1$, the sets Supp$(\xi)\cap H^-(s)$ and Supp$(\xi')\cap H(s)$ are finite. But Supp$(\xi')\cap H^-(1)$ is infinite.
Therefore the series ξ' does not satisfy Theorem 1.3 i) b), even by replacing $\tau(\xi)$ with $\widetilde{\tau}(\xi)$ in this statement.

6. Proofs of Theorems 1.2 and 1.3 in positive characteristic

In this section we explain why Theorem 1.2 and 1.3 remain valid in positive characteristic (for ξ a power series with integer exponents).

Proposition 6.1. *Let $\omega\in{\mathbf{R}_{\geq 0}}^n$ and $P\in\mathbf{K}[[x]][T]$. There exists a finite set $\mathcal{T}_\omega$ of strongly convex rational cones such that:*

i) for any order $\preceq\in\mathrm{Ord}_n$ refining $\leq_\omega$, there is $\sigma\in\mathcal{T}_\omega$, σ being $\preceq$-positive, such that the roots of P in $\mathcal{S}_{\preceq}^{\mathbf{K}}$ have support in a translation of σ,

ii) for every $\sigma\in\mathcal{T}_\omega$ and $\omega'\in\sigma^\vee$, the supports of the roots of P in $\mathcal{S}_{\preceq}^{\mathbf{K}}$ are $\preceq'$-well ordered for every $\preceq'\in\mathrm{Ord}_n$ refining $\leq_{\omega'}$.

Moreover for a given $\omega\in{\mathbf{R}_{\geq 0}}^n$ and a given finite set of cones $\mathcal{T}_\omega$ satisfying the former property, for every ω'' close enough to ω, we can choose $\mathcal{T}_{\omega''}=\mathcal{T}_\omega$.

Proof. By Theorem 5.10, for every $\preceq\in\mathrm{Ord}_n$ there is a cone $\sigma_\preceq$ such that the roots of P in $\mathcal{S}_{\preceq}^{\mathbf{K}}$ have support in a translation of $\sigma_\preceq$, and for every $\omega'\in\sigma_\preceq^\vee$, the supports of the roots of P in $\mathcal{S}_{\preceq}^{\mathbf{K}}$

are $\preceq'$-well ordered, for every $\preceq'\in \mathrm{Ord}_n$ refining $\leq_{\omega'}$.
Then $\mathrm{Ord}_n=\bigcup_{\preceq\in\mathrm{Ord}_n}\mathcal{U}_{\sigma_\preceq}$. Thus, by Theorem 5.6, there exists a finite set of orders $\preceq_1,\ldots,\preceq_N$ such that $\mathrm{Ord}_n=\bigcup_{i=1}^N\mathcal{U}_{\sigma_{\preceq_i}}$. Therefore for every $\omega\in{\mathbf{R}_{\geq 0}}^n$, we can choose $\mathcal{T}_\omega=\{\sigma_{\preceq_1},\ldots,\sigma_{\preceq_N}\}$.

Then the last claim follows from Corollary 2.9.

Now let ξ be a Laurent series (that is, with integer exponents) whose support is included in a translation of a strongly convex cone containing ${\mathbf{R}_{\geq 0}}^n$ and with coefficients in a positive characteristic field $\mathbf{K}$. Assume that ξ is algebraic over $\mathbf{K}(\!(x)\!)$. Then Theorem 1.2 remains valid. Indeed the proof is still valid by using the definition of $\mathcal{S}_\preceq^{\mathbf{K}}$ given in Definition 5.9, and by using Corollary 5.7 instead of Theorem 3.3.
Moreover Theorem 1.3 also remains valid. Indeed, we can also define $\tau_0(\xi)$, $\tau_1(\xi)$, $\tau_0'(\xi)$ and $\tau_1'(\xi)$ for such a ξ. Proposition 4.6 is still valid. Moreover we can prove that $\tau_0(\xi)$ and $\tau_1(\xi)$ are open, exactly as in the zero characteristic case, by using Proposition 6.1. Therefore Corollary 4.8 and Lemma 4.9 remain valid in positive characteristic.

7. Gaps in the expansion of algebraic Laurent series

Here we apply Theorem 1.3 to give a result concerning the gaps of a Laurent series algebraic over the field of power series. This is the following:

Theorem 7.1. *Let ξ be a Laurent series whose support is included in a translation of a strongly convex cone containing ${\mathbf{R}_{\geq 0}}^n$ and with coefficients in a field $\mathbf{K}$ of any characteristic. Assume that ξ is algebraic over $\mathbf{K}(\!(x)\!)$ and $\xi\notin\mathbf{K}[[x]]_{(x)}$. Let $\omega=(\omega_1,\ldots,\omega_n)\in\mathrm{Int}(\tau(\xi))$. We expand ξ as*

$$\xi=\sum_{i\in\mathbf{N}}\xi_{k(i)}$$

where

- *i) for every $l\in\Gamma=\mathbf{Z}\omega_1+\cdots+\mathbf{Z}\omega_n$, ξ_l is a (finite) sum of monomials of the form cx^α with $\omega\cdot\alpha=l$,*
- *ii) the sequence $k(i)$ is a strictly increasing sequence of elements of Γ,*
- *iii) for every integer i, $\xi_{k(i)}\neq 0$.*

Then there exists a constant $C>0$ such that

$$\frac{k(i+1)}{k(i)}\leq C\qquad\forall i\in\mathbf{N}.$$

Proof. First we can multiply ξ by a monomial and assume that $k(0)\geq 0$.
By Theorem 1.3 there exist a finite set $C\subset\mathbf{Z}^n$, a Laurent polynomial $p(x)$, and a power series

$f(x) \in \mathbf{K}[[x]]$ such that $\mathrm{Supp}(\xi+p(x)+f(x)) \subset C+\tau(\xi)^\vee$. Moreover for every $(n-1)$-dimensional (unbounded) face F of $\mathrm{Conv}(C+\tau(\xi)^\vee)$, the cardinal of

$$\mathrm{Supp}(\xi+p(x)+f(x)) \cap F$$

is infinite.

If $\tau(\xi)^\vee = {\mathbf{R}_{\geq 0}}^n$, then $\xi \in \mathbf{K}[[x]]_{(x)}$. Therefore, by assumption, ${\mathbf{R}_{\geq 0}}^n \subsetneq \tau(\xi)^\vee$ and $\tau(\xi) \subsetneq {\mathbf{R}_{\geq 0}}^n$. Hence there is $\omega' \in {\mathbf{R}_{>0}}^n$ in the boundary of $\tau(\xi)$, such that one $(n-1)$-dimensional (unbounded) face F of $\mathrm{Conv}(C+\tau(\xi)^\vee)$ is parallel to $\langle\omega'\rangle^\perp$.

Let $\mathrm{in}_{\omega'}(f)$ denote the initial term of the series f for the monomial valuation $\nu_{\omega'}$ defined by the weights $\omega'_1,\ldots,\ \omega'_n$. Since ξ is algebraic over $\mathbf{K}[[x]]$ there exist an integer d and formal power series $a_0,\ldots,a_d \in \mathbf{K}[[x]]$ such that

$$a_d\xi^d+\cdots+a_1\xi+a_0=0.$$

Thus

$$\sum_{i\in E} \mathrm{in}_{\omega'}(a_i)\,\mathrm{in}_{\omega'}(\xi)^i = 0 \tag{3}$$

where

$$E=\{i\in\{0,\ldots,n\}\ /\ \nu_{\omega'}(a_i\xi^i)=\min_j \nu_{\omega'}(a_j\xi^j)\}.$$

Since the a_i are in $\mathbf{K}[[x]]$, and $\omega' \in \mathbf{R}_{>0}$, the $\mathrm{in}_{\omega'}(a_i)$ are polynomials.

We set $b_i := \mathrm{in}_{\omega'}(a_i)$ for every i and $\xi' := \mathrm{in}_{\omega'}(\xi)$. We have

$$\sum_{i\in E} b_i{\xi'}^i = 0.$$

Now let us write $\xi' = \sum_{i\in\mathbf{N}} \xi'_{k(i)}$ such that

i) for every $l \in \Gamma = \mathbf{Z}\omega_1+\cdots+\mathbf{Z}\omega_n$, ξ'_l is a (finite) sum of monomials of the form cx^α with $\omega\cdot\alpha = l$,

ii) the sequence $k(i)$ is a strictly increasing sequence of elements of Γ,

iii) for every integer i, $\xi'_{k(i)} \neq 0$.

Let $N \in \mathbf{N}$ and set ${\xi'}^{(N)} := \sum_{i\leq N}\xi'_{k(i)}$. We set $P(T) = \sum_{i\in E} b_iT^i$, $d := \deg_T(P(T))$ and let ν be the maximum of the $\nu_\omega(x^\alpha)$ where α runs over the exponents of the b_i. Then we have $P({\xi'}^{(N)}) \neq 0$ for N large enough. We have

$$\frac{P({\xi'}^{(N)})}{{\xi'}^{(N)}-\xi'} = \frac{P({\xi'}^{(N)})-P(\xi')}{{\xi'}^{(N)}-\xi'} = \sum_{i\in E} b_i\left({{\xi'}^{(N)}}^i + {{\xi'}^{(N)}}^{i-1}\xi' + \cdots + {\xi'}^i\right).$$

Because the valuation of the right side term is positive, the valuation of $P({\xi'}^{(N)})$ is greater than the valuation of $\xi'-{\xi'}^{(N)}$. However the maximal valuation of the monomials of $P({\xi'}^{(N)})$ is $\nu+dk(N)$. Since the valuation of $\xi'-{\xi'}^{(N)}$ is $k(N+1)$ we have that

$$k(N+1) \leq \nu+dk(N) \leq (\nu+d)k(N).$$

Since the non zero monomials of ξ' are non zero monomials of ξ, this proves the result.

Remark 7.2. *Let us mention that this result was already known for power series of* $\mathbf{K}[[x]]$ *algebraic over* $\mathbf{K}[x]$ *(see [Sc33, Hilfssatz 5]).*

Remark 7.3. *This result can be strengthened in characteristic zero in the sense that the differences* $k(i+1)-k(i)$ *are uniformly bounded (see [AR19, Theorem 6.4]). However this statement is sharp in positive characteristic. For instance the series*

$$\xi = \sum_{i\in\mathbf{N}} \left(\frac{x}{y}\right)^{p^i}$$

is a root of the polynomial

$$T^p - T + \frac{x}{y}$$

over a field of characteristic $p > 0$*.*

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