THE MINIMAL CONE OF AN ALGEBRAIC LAURENT SERIES

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Abstract. We study the algebraic closure of $\mathbb{K}(\!(x)\!)$, the field of power series in several indeterminates over an algebraically closed characteristic zero field. We show that an element algebraic over $\mathbb{K}(\!(x)\!)$ can be expressed as a Puiseux series such that the convex hull of its support is essentially a polyhedral rational cone. One of the main tools for proving this is the Abhyankar-Jung Theorem. Then we prove a positive characteristic analogue of this result by replacing the use of the Abhyankar-Jung Theorem by the compacity of the set of orders on $\mathbb{Z}^n$.
Finally we apply these results to obtain a bound on the gaps in the expansions of Laurent series algebraic over the field of power series of any characteristic.

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2010 *Mathematics Subject Classification.* 05E40, 06A05, 11J81, 12J99, 13F25, 14B05, 32B10.
Key words and phrases. power series rings, support of a Laurent series, algebraic closure, orders on a lattice, Dickson's Lemma, Henselian valued fields.
This work has been partially supported by ECOS Project M14M03, and by PAPIIT-IN108216. The third author is deeply grateful to the UMI LASOL of CNRS where this project has been carried out.

1. Introduction

When $\mathbb{K}$ is an algebraically closed field of characteristic zero and $x = (x_1, \ldots, x_n)$ is a vector of n indeterminates, we denote by $\mathbb{K}(\!(x)\!)$ the field of formal power series in n indeterminates. The problem we investigate here is the determination of the algebraic closure of $\mathbb{K}(\!(x)\!)$. When $n = 1$ it is well known that the elements that are algebraic over $\mathbb{K}(\!(x)\!)$ can be expressed as Puiseux series, i.e. as formal sums of the form $\sum_{k=k_0}^{\infty} a_k x^{k/q}$ for some positive integer q.
When $n \geq 2$ the question remains open in general, and this case is the first step in order to understand what is the algebraic closure of fields of generalized power series. A classical result of McDonald [MD95] asserts that the elements that are algebraic over $\mathbb{K}(\!(x)\!)$ can be expressed as Puiseux series with support in the translation of a strongly convex rational cone. However the converse is wrong: a Puiseux series with support in the translation of a strongly convex rational cone is not algebraic over $\mathbb{K}(\!(x)\!)$ in general.
So a natural problem is characterizing the elements that are algebraic over $\mathbb{K}(\!(x)\!)$ among the Puiseux series with support in the translation of a strongly convex rational cone. Without loss of generality, when $\mathbb{K}$ is a characteristic zero field, we can restrict to the Laurent series (with integer exponents) with such a support (see for instance the introduction of [AR19]).

Here we investigate characterizations in terms of the support of the series. Indeed, such characterizations have already been studied for series in one indeterminate that are algebraic over the ring of polynomials. For these algebraic series in one indeterminate this problem is important and is related to several fields such as tropical geometry, number theory, and combinatorics (see [AB12], [HM17], and [AM-K31] for example). For such a series $f(x_1)$ algebraic over $\mathbb{K}[x_1]$, one can express all the coefficients of $f(x_1)$ in terms of a finite number of data: the coefficients of the minimal polynomial of $f(x_1)$ and the first coefficients of $f(x_1)$ up to an order N (see [FS98] and [HM17]). This order N is determined by the discriminant of the minimal polynomial of $f(x_1)$. However, these expressions, which are explicit, are usually difficult to handle (let us mention the work [HM] that recently extends such expressions in the multivariable case). Another approach is based on the fact that an algebraic series is D-finite, that is, a solution of a linear differential equation with polynomial coefficients. From this point of view the coefficients of $f(x_1)$ satisfy a linear recurrence with polynomial coefficients (see [St80] for instance). But once again it is still a difficult problem to handle such recurrences (see [AB12] for a presentation of the problem).

In fact our problem is much more subtle than the case of a series algebraic over $\mathbb{K}[x]$. Indeed a Laurent series ξ algebraic over $\mathbb{K}(\!(x)\!)$ will be determined by its minimal polynomial. But the coefficients of this minimal polynomial are formal power

series in several indeterminates. Therefore the support of ξ depends on infinitely many coefficients in $\mathbb{K}$.
A first natural question is to find obstructions for the algebraicity of a Laurent series with support in the translation of a strongly convex rational cone in terms of the shape of the support. For instance could there be an algebraic series in 3 indeterminates for which the convex hull of its support is a right circular cone? A natural problem is to find how far from a strongly convex rational cone is the support of an algebraic series.

In [AR19] we began to investigate this kind of question. One of our results is that an algebraic series cannot have too many gaps in its expansion (see [AR19, Theorem 6.4] for a precise statement). In order to prove this result we proved a technical result asserting that, for a given algebraic series ξ with support in the translation of a strongly convex rational cone, there exists a hyperplane $H \subset \mathbb{R}^n$ such that $\mathrm{Supp}(\xi) \cap H$ is infinite and one of the half-spaces delimited by H contains only a finite number of elements of $\mathrm{Supp}(\xi)$.
In fact, the set $\tau(\xi)$ of normal vectors to such hyperplanes H has been defined as being the boundary of some strongly convex (open) cone. But nothing more has been proved about this cone $\tau(\xi)$. In particular we do not know if it is rational or even polyhedral.
The first aim of this paper is to prove that this cone is a strongly convex rational cone. Then we relate the support of such an algebraic series ξ to the dual of the cone $\tau(\xi)$. The same questions can be asked for a positive characteristic field. We will discuss the positive characteristic case, showing the differences with the characteristic zero case.

Let us present our main results in more details. We begin by two definitions:

Definition 1.1. A *(generalized) series* ξ (with support in $\mathbb{Q}^n$ and coefficients in a field $\mathbb{K}$) is a formal sum $\xi = \sum_{\alpha \in \mathbb{Q}^n} \xi_\alpha x^\alpha$, where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and the $\xi_\alpha \in \mathbb{K}$. Its support is the set

$$\mathrm{Supp}(\xi) := \{\alpha \in \mathbb{Q}^n \mid \xi_\alpha \neq 0\}.$$

Such a series is called a *Laurent series* (resp. *Laurent Puiseux series*) if $\mathrm{Supp}(\xi) \subset \mathbb{Z}^n$ (resp. $\mathrm{Supp}(\xi) \subset \frac{1}{k}\mathbb{Z}^n$ for some $k \in \mathbb{N}^*$).

Definition 1.2. Let ξ be a series with support in $\mathbb{Q}^n$ and coefficients in a field $\mathbb{K}$. We set

$$\tau(\xi) := \left\{\omega \in {\mathbb{R}_{\geq 0}}^n \mid \exists k \in \mathbb{R},\ \mathrm{Supp}(\xi) \cap \{u \in \mathbb{R}^n \mid u \cdot \omega \leq k\} = \emptyset\right\}.$$

Our first main result is the following:

Theorem 1.3. *Let ξ be a Laurent series whose support is included in a translation of a strongly convex cone containing ${\mathbb{R}_{\geq 0}}^n$ and with coefficients in a field $\mathbb{K}$ of any characteristic. Assume that ξ is algebraic over $\mathbb{K}(\!(x)\!)$. Then the set $\tau(\xi)$ is a strongly convex rational cone.*

Our second main result relies on the support of a Laurent series that is algebraic over $\mathbb{K}(\!(x)\!)$ to the cone $\tau(\xi)$:

Theorem 1.4. *Let ξ be a Laurent series whose support is included in a translation of a strongly convex cone containing ${\mathbb{R}_{\geq 0}}^n$ and with coefficients in a field $\mathbb{K}$ of any characteristic. Assume that ξ is algebraic over $\mathbb{K}(\!(x)\!)$. We have the following properties (here $\tau(\xi)^\vee$ denotes the dual of $\tau(\xi)$ - see Definition 2.1):*

i) *There exist a finite set $C \subset \mathbb{Z}^n$, a Laurent polynomial $p(x)$, and a power series $f(x) \in \mathbb{K}[[x]]$ such that*
$$\mathit{Supp}(\xi + p(x) + f(x)) \subset C + \tau(\xi)^\vee.$$

ii) *The triplet $(C, p(x), f(x))$ satisfying i) is not necessarily unique. But:*
   a) *There is a triplet $(C_1, p_1(x), f_1(x))$ satisfying i) such that for every $(n-1)$-dimensional (unbounded) face F of $\mathrm{Conv}(C_1 + \tau(\xi)^\vee)$, the cardinal of*
   $$\mathit{Supp}(\xi + p_1(x) + f_1(x)) \cap F$$
   is infinite,
   b) *There is a triplet $(C_2, p_2(x), f_2(x))$ satisfying i) such that every one dimensional face σ of $\tau(\xi)^\vee$, there is a one dimensional unbounded face of $\mathrm{Conv}(C_2 + \tau(\xi)^\vee)$ of the form $\gamma + \sigma$, for some $\gamma \in C$, such that the cardinal of*
   $$\mathit{Supp}(\xi + p_2(x) + f_2(x)) \cap (\gamma + \sigma)$$
   is infinite.

iii) *If $\sigma \subset \tau(\xi)^\vee$ is a convex cone (not necessarily polyhedral) containing ${\mathbb{R}_{\geq 0}}^n$ for which there exist a Laurent polynomial $p'(x)$, a power series $f'(x) \in \mathbb{K}[[x]]$, and a finite set C' such that*
$$\mathit{Supp}(\xi + p'(x) + f'(x)) \subset C' + \sigma,$$
then $\sigma = \tau(\xi)^\vee$.

We will see in Example 4.17 that, in general, the set C satisfying i) cannot be chosen to be one single point. We will also see in Example 4.16 that there is no minimal, maximal or canonical C satisfying i).

We will begin to treat the characteristic zero case because this case is simpler than the positive characteristic case, and because we feel that in this way the paper is easier to read. In this case the proof of Theorem 1.3 is essentially based on two tools: the compacity of the space of orders on ${\mathbb{R}_{\geq 0}}^n$, and the construction, for every order $\preceq$ on $\mathbb{Q}^n$, of an algebraically closed field $\mathcal{S}_{\preceq}^{\mathbb{K}}$ containing $\mathbb{K}(\!(x)\!)$. This result of compacity is due to Ewald and Ishida [EI06] (see also [Te18]) and is a purely topological result. It will allow us to have a decomposition of ${\mathbb{R}_{\geq 0}}^n$ into a union of finitely many rational strongly convex cones having the following property: for each order $\preceq$, the roots of the minimal polynomial of ξ in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ have support in one of these cones.

The construction of the algebraically closed fields $\mathcal{S}_{\preceq}^{\mathbb{K}}$ has been proved in [AR19] (see Theorem 2.18) and is based on systematic constructions of algebraically closed valued fields due to Rayner [Ra68].

The proof of Theorem 1.4 is more involved and requires the introduction of new cones. These are denoted by $\tau_0(\xi)$, $\tau_0'(\xi)$, $\tau_1(\xi)$ and $\tau_1'(\xi)$. The definitions of $\tau_0(\xi)$ and $\tau_1(\xi)$ are purely algebraic (they are defined in terms of the fields $\mathcal{S}_{\preceq}^{\mathbb{K}}$), and the definitions of $\tau_0'(\xi)$ and $\tau_1'(\xi)$ are "geometric" (that is, they are defined in terms of the support of ξ).

We prove that the set $\tau(\xi)$ is the closure of $\tau_0'(\xi)$. Then, we prove that $\tau_1(\xi)$ and $\tau_1'(\xi)$ are equal and that $\tau_0'(\xi)$ is almost equal to $\tau_0(\xi)$ (see Proposition 4.6 - in particular they have the same closure). The main important property of $\tau_0(\xi)$ and $\tau_1(\xi)$ is that these two sets are open sets (see Proposition 4.7). Then, we will prove that the boundary of $\tau_0'(\xi)$ does not intersect $\tau_1'(\xi)$ (this comes from the openness of $\tau_0(\xi)$ and $\tau_1(\xi)$, see Corollary 4.8). In particular, the vectors in the boundary of $\tau(\xi)$ will correspond to "faces" of the support of ξ. The main tool used to prove the existence of the finite set C of Theorem 1.4, is a generalization of Dickson's Lemma that we prove here (see Corollary 4.13).

In the following part we investigate the positive characteristic case, which is very different from the zero characteristic case. Before giving the proofs of Theorems 1.3 and 1.4 in the positive characteristic case, we will investigate how the elements of an algebraic closure of $\mathbb{K}(\!(x)\!)$ can be described in this case. Indeed, in positive characteristic, the roots of polynomials with coefficients in the field of power series are not series with support in a lattice in general. But these roots can be expressed as series with support in a strongly convex cone with rational exponents whose denominators are not necessarily bounded (see the work [Sa17] where this analogue of MacDonald's Theorem is proved). First we show that the analogue of Theorem 1.4 is no longer true for such series (see Example 5.2). This example shows that the problem is that the support of a root can have accumulation points, and therefore we need to take into account that its support is well-ordered for the considered order. This is the main difference with the characteristic zero case.

Therefore we begin by extending the result of Saavedra by constructing algebraically closed fields, each of them depending on a given order on ${\mathbb{R}_{\geq 0}}^n$, that contain $\mathbb{K}(\!(x)\!)$ (see Theorem 5.5). Then we introduce a new cone analogous to $\tau(\xi)$, but whose definition is more natural in that the support of such a ξ is not included in a lattice in general. This allows us to prove that this cone is rational (see Theorem 5.11). Then we give an analogue of Theorem 1.4 in the positive characteristic case for series whose support is not in a lattice (see Theorem 5.13). The conclusion is weaker than the one of Theorem 1.4, but we show that there is no possibility for a stronger version for algebraic series with accumulation points in their support (see Example 5.15).

Then we give the proofs of Theorems 1.3 and 1.4 in the positive characteristic case (that are satisfied only for power series with integer exponents) by explaining the differences with the zero characteristic case. Finally we use this to give a bound on the gaps in the expansion of a Laurent series algebraic over $\mathbb{K}(\!(x)\!)$:

Theorem 1.5. *Let ξ be a Laurent series whose support is included in a translation of a strongly convex cone containing ${\mathbb{R}_{\geq 0}}^n$ and with coefficients in a field $\mathbb{K}$ of any characteristic. Assume that ξ is algebraic over $\mathbb{K}(\!(x)\!)$ and $\xi \notin \mathbb{K}[[x]]_{(x)}$. Let $\omega = (\omega_1, \ldots, \omega_n) \in \operatorname{Int}(\tau(\xi))$. We expand ξ as*

$$\xi = \sum_{i \in \mathbb{N}} \xi_{k(i)}$$

where

i) *for every $l \in \Gamma := \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$, ξ_l is a (finite) sum of monomials of the form cx^α with $\omega \cdot \alpha = l$,*
ii) *the sequence $k(i)$ is a strictly increasing sequence of elements of Γ,*
iii) *for every integer i, $\xi_{k(i)} \neq 0$.*

Then there exists a constant $K > 0$ such that

$$\frac{k(i+1)}{k(i)} \leq K \qquad \forall i \in \mathbb{N}.$$

Let us mention that Theorem 1.3 has been announced in [ADR].

2. Orders and algebraically closed fields containing $\mathbb{K}(\!(x)\!)$

In this section we introduce the tools needed for the proof of Theorem 1.3.

2.1. The space of orders on ${\mathbb{R}_{\geq 0}}^n$.

Definition 2.1. Let us recall that a *cone* $\tau \subset \mathbb{R}^n$ is a subset of $\mathbb{R}^n$ such that for every $t \in \tau$ and $\lambda \geq 0$, $\lambda t \in \tau$. A cone $\tau \subset \mathbb{R}^n$ is *polyhedral* if it has the form

$$\tau = \{\lambda_1 u_1 + \cdots + \lambda_s u_s \mid \lambda_1, \ldots, \lambda_s \geq 0\}$$

for some given vectors $u_1, \ldots, u_s \in \mathbb{R}^n$. A cone is said to be a *rational cone* if it is polyhedral, and the u_i can be chosen in $\mathbb{Z}^n$.
A cone is *strongly convex* if it does not contain any non trivial linear subspace.
In practice, as almost all the cones that we consider in this paper are polyhedral

cones, the term cone will always refer to polyhedral cones (unless stated otherwise). The *dual* $\sigma^\vee$ of a cone σ is the cone given by

$$\sigma^\vee := \{v \in \mathbb{R}^n \mid v \cdot u \geq 0, \text{ for all } u \in \sigma\}$$

where $u \cdot v$ stands for the dot product $(u_1, \ldots, u_n) \cdot (v_1, \ldots, v_n) := u_1 v_1 + \cdots + u_n v_n$.

Remark 2.2. Let ξ be a series and $\omega \in \tau(\xi)$. Then $\mathrm{Supp}(\xi) \subset \gamma + \langle \omega \rangle^\vee$ for some $\gamma \in \mathbb{Z}^n$. Indeed it is enough to choose γ such that $\mathrm{Supp}(\xi) \cap \{u \in \mathbb{R}^n \mid u \cdot \omega \leq \gamma \cdot \omega\} = \emptyset$.

Definition 2.3. A *preorder* on an abelian group G is a binary relation $\preceq$ such that

i) $\forall u, v \in G$, $u \preceq v$ or $v \preceq u$,
ii) $\forall u, v, w \in G$, $u \preceq v$ and $v \preceq w$ implies $u \preceq w$,
iii) $\forall u, v, w \in G$, $u \preceq v$ implies $u + w \preceq v + w$,

The set of preorders on G is denoted by $\mathrm{ZR}(G)$. The set of orders on G is a subset of $\mathrm{ZR}(G)$ denoted by $\mathrm{Ord}(G)$.

Theorem-Definition 2.4. By [Ro86, Theorem 2.5] for every $\preceq \in \mathrm{ZR}(\mathbb{Q}^n)$ there exist an integer $s \geq 0$ and orthogonal vectors $u_1, \ldots, u_s \in \mathbb{R}^n$ such that

$$\forall u, v \in \mathbb{Q}^n, \; u \preceq v \Longleftrightarrow (u \cdot u_1, \ldots, u \cdot u_s) \leq_{\mathrm{lex}} (v \cdot u_1, \ldots, v \cdot u_s).$$

For such a preorder we set $\preceq := \leq_{(u_1,\ldots,u_s)}$. Such a preorder extends in an obvious way to a preorder on $\mathbb{R}^n$ and the preorders of this form are called *continuous preorders*.

Definition 2.5. Let $A \subset \mathbb{R}^n$ and $\preceq$ be a continuous preorder on $\mathbb{R}^n$. We say that A is $\preceq$-positive if

$$\forall a \in A, \quad \underline{0} \preceq a.$$

Definition 2.6. Let $\preceq \in \mathrm{Ord}_n$ and $A \subset \mathbb{R}^n$. We say that A is $\preceq$-well-ordered if A is well-ordered with respect to $\preceq$.

Definition 2.7. The set of continuous orders $\preceq$ such that ${\mathbb{R}_{\geq 0}}^n$ is $\preceq$-positive is denoted by Ord_n, and they will be simply called orders on ${\mathbb{R}_{\geq 0}}^n$.
In the rest of the paper all the orders that we consider will be exclusively orders on ${\mathbb{R}_{\geq 0}}^n$. For simplicity we shall call them simply orders.

Definition 2.8. Given two preorders $\preceq_1$ and $\preceq_2$, one says that $\preceq_2$ *refines* $\preceq_1$ if

$$\forall u, v \in \mathbb{R}^n, \; u \preceq_2 v \Longrightarrow u \preceq_1 v.$$

Remark 2.9. Let $(u_1, \ldots, u_s)$ be non zero vectors of $\mathbb{R}^n$. Using Theorem-Definition 2.4 it is easy to check that for a preorder $\preceq$, $\preceq$ refines $\leq_{(u_1,\ldots,u_s)}$ if and only if there exist vectors $u_{s+1}, \ldots, u_{s+k}$ such that $\preceq = \leq_{(u_1,\ldots,u_{s+k})}$.

Lemma 2.10. *Let $\omega \in \mathbb{R}^n$ and σ be a strongly convex cone with $\omega \in \mathrm{Int}(\sigma^\vee)$. Then σ is $\preceq$-positive for every order $\preceq$ refining $\leq_\omega$.*

Proof. If $\omega \in \mathrm{Int}(\sigma^\vee)$, we have that $s \cdot \omega > 0$ for every $s \in \sigma \setminus \{0\}$. By Theorem-Definition 2.9, every $\preceq$ refining $\leq_\omega$ is equal to $\leq_{(\omega,v_1,\ldots,v_s)}$ for some vectors v_i. Thus σ is $\preceq$-positive. □

The next easy lemma will be used several times:

Lemma 2.11. [AR19, Lemma 2.4] *Let σ_1 and σ_2 be two cones and γ_1 and γ_2 be vectors of $\mathbb{R}^n$. Let us assume that $\sigma_1 \cap \sigma_2$ is full dimensional. Then there exists a vector $\gamma \in \mathbb{Z}^n$ such that*

$$(\gamma_1 + \sigma_1) \cap (\gamma_2 + \sigma_2) \subset \gamma + \sigma_1 \cap \sigma_2.$$

Finally we give the following result, which will be used in the proof of Theorem 1.4 (this is a generalization of [AR19, Corollary 3.10]):

Lemma 2.12. *Let $\sigma_1, \ldots, \sigma_N$ be strongly convex cones and let $\omega \in \mathbb{R}^n \setminus \{\underline{0}\}$. The following properties are equivalent:*

i) *We have $\omega \in \mathrm{Int}\left(\bigcup_{i=1}^N \sigma_i^\vee\right)$.*
ii) *For every order $\preceq \in \mathrm{Ord}(\mathbb{Q}^n)$ refining $\leq_\omega$, there is an index i such that σ_i is $\preceq$-positive.*

Proof. Let us prove that i) implies ii). Let $\omega \in \mathrm{Int}\left(\bigcup_{i=1}^N \sigma_i^\vee\right)$. We are going to show that for every non zero vector $v_1, \ldots, v_{n-1} \in \langle\omega\rangle^\perp$ such that $v_j \in \langle \omega, v_1, \ldots, v_{j-1}\rangle^\perp$, there is an integer i such that σ_i is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive. Indeed, by Theorem-Definition 2.4 every preorder $\preceq$ refining $\leq_\omega$ is of the form $\leq_{(\omega,v_1,\ldots,v_{n-1})}$ for such vectors $v_1, \ldots, v_{n-1}$. Therefore ii) is satisfied. So from now on we fix such a set of vectors $v_1, \ldots, v_{n-1}$.
By Lemma 2.10, if $\omega \in \mathrm{Int}(\sigma_i^\vee)$ for some i, then σ_i is $\preceq$-positive for every $\preceq$-refining $\leq_\omega$. In particular it is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive. Otherwise let E_1 be the set of indices i such that $\omega \in \sigma_i^\vee$. If ω were in the boundary of $\bigcup_{i\in E_1} \sigma_i^\vee$, then ω would belong to some σ_i for $i \notin E_1$ because $\omega \in \mathrm{Int}\left(\bigcup_{i=1}^N \sigma_i^\vee\right)$. Thus $\omega \in \mathrm{Int}\left(\bigcup_{i\in E_1} \sigma_i^\vee\right)$.
Since $\omega \in \mathrm{Int}\left(\bigcup_{i\in E_1} \sigma_i^\vee\right)$, there is $\lambda_1 > 0$ such that $\omega + \lambda_1 v_1 \in \mathrm{Int}\left(\bigcup_{i\in E_1} \sigma_i^\vee\right)$. Then two cases may occur:
(1) Assume $\omega + \lambda_1 v_1 \in \mathrm{Int}(\sigma_i^\vee)$ for some $i \in E_1$. Because $i \in E_1$, for $s \in \sigma_i \setminus \{0\}$, either $\omega \cdot s > 0$, or $\omega \cdot s = 0$. In this last case we have $v_1 \cdot s > 0$ since $(\omega + \lambda_1 v_1) \cdot s > 0$ and $\lambda_1 > 0$. Therefore σ_i is $\preceq$-positive for every order $\preceq$ refining $\leq_{(\omega,v_1)}$ (In

particular it is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive).
(2) If $\omega+\lambda_1 v_1 \notin \mathrm{Int}(\sigma_i^\vee)$ for every $i\in E_1$, we denote by E_2 the set of $i\in E_1$ such that $\omega+\lambda_1 v_1\in\sigma_i^\vee$. As before we necessarily have $\omega+\lambda_1 v_1\in \mathrm{Int}\left(\bigcup_{i\in E_2}\sigma_i^\vee\right)$. Therefore there is $\lambda_2>0$ such that $\omega+\lambda_1 v_1+\lambda_2 v_2\in\mathrm{Int}\left(\bigcup_{i\in E_2}\sigma_i^\vee\right)$. Once again, if $\omega+\lambda_1 v_1+\lambda_2 v_2\in\mathrm{Int}(\sigma_i^\vee)$ for some $i\in E_2$, σ_i is $\preceq$-positive for every order $\preceq$ refining $\leq_{(\omega,v_1,v_2)}$. Otherwise we repeat the same process until one of the two situations occur:

a) there is $j<n-1$ such that $\omega+\lambda_1 v_1+\cdots+\lambda_j v_j\in\mathrm{Int}(\sigma_i^\vee)$ for some i. Then we can prove in the same way as (1) that σ_i is $\preceq$-positive for every $\preceq$ refining $\leq_{(\omega,v_1,\ldots,v_j)}$ (hence it is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive).
b) there is no such an index j. Thus we end with $\omega+\lambda_1 v_1+\cdots+\lambda_{n-1}v_{n-1}$ that belongs to (at least) one $\sigma_i^\vee$. Therefore the cone σ_i is $\leq_{(\omega,v_1,\ldots,v_{n-1})}$-positive, because $\omega\in\sigma_i^\vee$, $\mathbf{w}+\lambda_1 v_1\in\sigma_i^\vee$, ..., $\mathbf{w}+\lambda_1 v_1+\cdots+\lambda_{n-1}v_{n-1}\in\sigma_i^\vee$.

This proves that i) implies i)). Now we prove the converse. Assume that for every order $\preceq\in\mathrm{Ord}(\mathbb{Q}^n)$ refining $\leq_\omega$, there is an index i such that σ_i is $\preceq$-positive.
Let v be a vector with $\|v\|=1$. By assumption there is an index i such that σ_i is $\leq_{(\omega,v)}$-positive. Let s_1, ..., s_l be generators of σ_i that we assume to be of norm equal to 1. Reordering the s_j, there is an integer $k\geq 0$ such that $s_j\cdot\omega>0$ for every $j\leq k$, and $s_j\cdot\omega=0$ for every $j>k$, because σ_i is $\leq_{(\omega,v)}$-positive. Take $\lambda>0$. When $k>1$ assume moreover that $\dfrac{\min_{j\leq k}\{s_j\cdot\omega\}}{2}\geq\lambda$. Then we claim that $\omega+\lambda v\in\sigma_i^\vee$. Indeed, if $j\leq k$ we have

$$(\omega+\lambda v)\cdot s_j=\omega\cdot s_j+\lambda v\cdot s_j\geq\omega\cdot s_j-\lambda\|v\|\|s_j\|\geq\frac{\min_{j\leq k}\{s_j\cdot\omega\}}{2}>0.$$

If $j>k$ we have

$$(\omega+\lambda v)\cdot s_j=\lambda v\cdot s_j\geq 0$$

since σ_i is $\leq_{(\omega,v)}$-positive. This implies that $\omega+\lambda v\in\sigma_i^\vee$. Since this is true for every v, we have $\omega\in\mathrm{Int}\left(\bigcup_{i=1}^N\sigma_i^\vee\right)$. □

Corollary 2.13. *Let $\omega\in{\mathbb{R}_{\geq0}}^n$ and let σ_1, ..., σ_N be strongly convex cones which are $\leq_\omega$-positive. Assume that for every order $\preceq\in\mathrm{Ord}_n$ refining $\leq_\omega$, there is an index i such that σ_i is $\preceq$-positive. Then there is a neighborhood V of ω such that, for every $\omega'\in V$ and every $\preceq'\in\mathrm{Ord}_n$ refining $\leq_{\omega'}$, there is an index i such that σ_i is $\preceq'$-positive.*

Proof. We have $\omega\in\mathrm{Int}\left(\bigcup_{i=1}^N\sigma_i^\vee\right)$ by the previous lemma. Therefore, the previous lemma shows that we can choose $V=\mathrm{Int}\left(\bigcup_{i=1}^N\sigma_i^\vee\right)$. □

2.2. **The space Ord_n as a compact topological space.** One main tool for the proof of Theorem 1.3 is the fact that the set of orders Ord_n is a topological compact space for a well chosen topology. This topology has been introduced by Ewald and Ishida [EI06].

Definition 2.14. [EI06][Te18] The set $\mathrm{ZR}(\mathbb{Q}^n)$ is endowed with a topology for which the sets

$$\mathcal{U}_\sigma := \{\preceq\in \mathrm{ZR}(\mathbb{Q}^n) \text{ such that } \sigma \text{ is } \preceq\text{-positive}\}$$

form a basis of open sets where σ runs over the full dimensional strongly convex rational cones.

Remark 2.15. With this definition we have $\mathrm{Ord}_n = \mathcal{U}_{\mathbb{R}_{\geq 0}{}^n} \cap \mathrm{Ord}(\mathbb{Q}^n)$.

We have the following result:

Theorem 2.16. [EI06] *The space* $\mathrm{ZR}(\mathbb{Q}^n)$ *is compact and* $\mathrm{Ord}(\mathbb{Q}^n)$ *is closed in* $\mathrm{ZR}(\mathbb{Q}^n)$. *Moreover every* $\mathcal{U}_\sigma$ *is compact. Therefore* Ord_n *is compact.*

This allows us to prove the following result:

Lemma 2.17. *Let* $\sigma_1, \ldots, \sigma_N$ *be rational cones such that* $\mathrm{Ord}_n \subset \bigcup_{k=1}^N \mathcal{U}_{\sigma_k}$. *Then*

$$\mathbb{R}_{\geq 0}{}^n \subset \bigcup_{k=1}^N \sigma_k^\vee.$$

Proof. Let $\omega \in \mathbb{R}_{\geq 0}{}^n$. Let $\preceq\in \mathrm{Ord}_n$ refining $\leq_\omega$. Such a $\preceq$ exists by [AR19, Lemma 3.18]. Then $\preceq\in \mathcal{U}_{\sigma_k}$ for some k. Since $\preceq$ refines $\leq_\omega$, we have that σ_k is $\leq_\omega$-positive. This means that $\omega \in \sigma_k^\vee$. This proves that $\mathbb{R}_{\geq 0}{}^n \subset \bigcup_{k=1}^N \sigma_k^\vee$. □

2.3. **Algebraically closed fields containing** $\mathbb{K}(\!(x)\!)$**.** Let n be a positive integer and $\preceq\in \mathrm{Ord}_n$.
For a field $\mathbb{K}$ of characteristic zero we denote by $\mathcal{S}_\preceq^{\mathbb{K}}$ the following set

$$\left\{\xi \text{ series} \mid \exists k \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n, \sigma \preceq\text{-positive rational cone, } \mathrm{Supp}(\xi) \subset (\gamma+\sigma) \cap \frac{1}{k}\mathbb{Z}^n\right\}.$$

We have the following result:

Theorem 2.18. [AR19, Theorem 4.5] *Assume that* $\mathbb{K}$ *is an algebraically closed field of characteristic zero. The set* $\mathcal{S}_\preceq^{\mathbb{K}}$ *is an algebraically closed field containing* $\mathbb{K}(\!(x)\!)$.

The following lemma will be used several times:

Lemma 2.19. *Let* ξ *be a Laurent series with coefficients in a field* $\mathbb{K}$. *Assume that Supp*$(\xi) \subset \gamma + \sigma$ *where* $\gamma \in \mathbb{Z}^n$ *and* σ *is a rational cone. Let* $\omega \in \sigma^\vee$. *Then, for every* $t \in \mathbb{R}$, *the set*

$$\{u \cdot \omega \mid u \in \mathit{Supp}(\xi)\} \cap]-\infty, t]$$

is finite.

Proof. We can make a translation and assume that $\gamma = 0$. Since σ is a rational cone, by Gordan's Lemma, there exist vectors $v_1, \ldots, v_N \in \sigma \cap \mathbb{Z}^n$ such that $\sigma \cap \mathbb{Z}^n$ is generated as a semigroup by $v_1, \ldots, v_N$. Since $\omega \in \sigma^\vee$, we have $v_i \cdot \omega \geq 0$ for every i.
By assumption we have $\sigma = \left\{ \sum_{i=1}^N n_i v_i \mid n_i \in \mathbb{N} \right\}$. Therefore the set $\{u \cdot \omega \mid u \in \mathrm{Supp}(\xi)\}$ is included in the semigroup generated by $v_1 \cdot \omega, \ldots, v_N \cdot \omega$. Since this semigroup is finitely generated, the sets $\{u \cdot \omega \mid u \in \mathrm{Supp}(\xi)\} \cap]-\infty, t]$ are finite. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let ξ be a Laurent series whose support is included in a translation of a strongly convex cone σ containing ${\mathbb{R}_{\geq 0}}^n$ and with coefficients in a characteristic zero field $\mathbb{K}$, and let $P \in \mathbb{K}[[x]][T]$ be a monic polynomial of degree d with $P(\xi) = 0$. Let $\sigma_0 \subset {\mathbb{R}_{\geq 0}}^n$ be a strongly convex rational cone such that there are d distinct series $\xi_1, \ldots, \xi_d$, belonging to $\mathcal{S}_{\preceq}^{\mathbb{K}}$ for some $\preceq \in \mathrm{Ord}_n$, with support in $\gamma + \sigma_0$ for some $\gamma \in \mathbb{Z}^n$, with $P(\xi_i) = 0$ for $i = 1, \ldots, d$.*
Then
$$\mathrm{Int}(\sigma_0^\vee) \cap \tau(\xi) \neq \emptyset \Longrightarrow \sigma_0^\vee \subset \tau(\xi).$$

Proof. Consider a non zero vector $\omega \in \mathrm{Int}(\sigma_0^\vee) \cap \tau(\xi)$. By Remark 2.2
$$\mathrm{Supp}(\xi) \subset \gamma + \langle \omega \rangle^\vee$$
for some $\gamma \in \mathbb{Z}^n$. By Lemma 2.11 we have
$$\mathrm{Supp}(\xi) \subset \gamma' + \sigma \cap \langle \omega \rangle^\vee$$
for some $\gamma' \in \mathbb{Z}^n$. Since $\sigma \cap \langle \omega \rangle^\vee$ is $\leq_\omega$-positive, there exists an order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_\omega$ such that $\sigma \cap \langle \omega \rangle^\vee$ is $\preceq'$-positive (see for example [AR19, Lemma 3.8]). Thus ξ is a root of P in $\mathcal{S}_{\preceq'}^{\mathbb{K}}$.
On the other hand, ω is in the interior of $\sigma_0^\vee$, thus σ_0 is $\preceq'$-positive by Lemma 2.10. Hence the ξ_i belong to $S_{\preceq'}^{\mathbb{K}}$. In particular the ξ_i are the roots of P in $\mathcal{S}_{\preceq'}^{\mathbb{K}}$ because P has at most d roots in a given field. Therefore $\xi = \xi_i$ for some i. Hence there is some $\gamma'' \in \mathbb{Z}^n$ such that
$$\mathrm{Supp}(\xi) \subset \gamma'' + \sigma_0.$$
Therefore for every $\omega' \in \sigma_0^\vee$ we have
$$\mathrm{Supp}(\xi) \cap \{u \in \mathbb{R}^n \mid u \cdot \omega' \leq \gamma'' \cdot \omega' - 1\} = \emptyset.$$
Hence $\sigma_0^\vee \subset \tau(\xi)$. □

Corollary 3.2. *Let ξ be a Laurent series with support in a translation of a strongly convex cone σ containing ${\mathbb{R}_{\geq 0}}^n$ and with coefficients in a characteristic zero field $\mathbb{K}$, and let $P \in \mathbb{K}[[x]][T]$ be a monic polynomial of degree d with $P(\xi) = 0$. Let $\sigma_k \subset {\mathbb{R}_{\geq 0}}^n$, $k = 1, \ldots, N$, be strongly convex rational cones satisfying the following properties:*

i) $\bigcup_{k=1}^{N} \sigma_k^\vee = {\mathbb{R}_{\geq 0}}^n$,

ii) *for every k there are d series $\xi_1^{(k)}, \ldots, \xi_d^{(k)}$, belonging to $\mathcal{S}_\preceq^\mathbb{K}$ for some $\preceq \in$ Ord_n, with support in $\gamma_k + \sigma_k$ for some $\gamma_k \in \mathbb{Z}^n$, with $P(\xi_i^{(k)}) = 0$ for $i = 1, \ldots, d$.*

Then, after renumbering the σ_k, there is an integer $l \leq N$ such that

$$\tau(\xi) = \bigcup_{k=1}^{l} \sigma_k^\vee.$$

Proof. By Lemma 3.1 we can renumber the σ_k such that $\sigma_k^\vee \subset \tau(\xi)$ for $k \leq l$ and $\mathrm{Int}(\sigma_k^\vee) \cap \tau(\xi) = \emptyset$ for every $k > l$. So we have $\bigcup_{k=1}^{l} \sigma_k^\vee \subset \tau(\xi)$.
Now, suppose that this inclusion is strict: there is an element $\omega \in \tau(\xi)$ such that $\omega \notin \bigcup_{k=1}^{l} \sigma_k^\vee$. By Hahn-Banach Theorem there is a hyperplane H separating ω and the convex closed set $\bigcup_{k=1}^{l} \sigma_k^\vee$ in the following sense: one open half space delimited by H, denoted by O, contains ω and $\bigcup_{k=1}^{l} \sigma_k^\vee \subset \mathbb{R}^n \backslash \overline{O}$. Since $\bigcup_{k=1}^{l} \sigma_k^\vee$ is full dimensional, the convex envelop $\mathcal{C}$ of ω and $\bigcup_{k=1}^{l} \sigma_k^\vee$ is full dimensional:

$$\mathcal{C} := \left\{ \lambda\omega + (1-\lambda)v \mid v \in \bigcup_{k=1}^{l} \sigma_k^\vee, 1 \geq \lambda \geq 0 \right\}.$$

Thus $\mathcal{C} \cap O$ contains an open ball B.
But $\tau(\xi)$ is convex because for every ω, $\omega' \in \mathbb{R}^n$, k, $l \in \mathbb{R}$:

$$\{u \in \mathbb{R}^n \mid u \cdot (\omega + \omega') \leq k + l\} \subset \{u \in \mathbb{R}^n \mid u \cdot \omega \leq k\} \cup \{u \in \mathbb{R}^n \mid u \cdot \omega' \leq l\}.$$

Thus $\mathcal{C} \subset \tau(\xi)$ and $B \subset \tau(\xi)$. Then B intersects one $\sigma_i^\vee$ for $i > l$ because $B \subset O$ and we have assumed $\bigcup_{k=1}^{N} \sigma_k^\vee = {\mathbb{R}_{\geq 0}}^n$. But because B is open, $B \cap \mathrm{Int}(\sigma_i^\vee) \neq \emptyset$, and this is a contradiction because $B \subset \tau(\xi)$ and $\tau(\xi) \cap \mathrm{Int}(\sigma_i^\vee) = \emptyset$ for $i > l$. Therefore the inclusion is not strict and $\bigcup_{k=1}^{l} \sigma_k^\vee = \tau(\xi)$. $\square$

Proposition 3.3. *Let ξ be a Laurent series whose support is included in a translation of a strongly convex cone σ containing ${\mathbb{R}_{\geq 0}}^n$ and with coefficients in a characteristic zero field $\mathbb{K}$, and let $P \in \mathbb{K}[[x]][T]$ be a monic polynomial of degree d with $P(\xi) = 0$. Then there exist strongly convex rational cones σ_k containing ${\mathbb{R}_{\geq 0}}^n$, $k = 1, \ldots, N$, satisfying the following properties:*

i) $\mathrm{Ord}_n \subset \bigcup_{k=1}^N \mathcal{U}_{\sigma_{\preceq_k}}$ *and* $\displaystyle\bigcup_{k=1}^N \sigma_k^\vee = {\mathbb{R}_{\geq 0}}^n$,

ii) *for every k there are d series $\xi_1^{(k)}, \ldots, \xi_d^{(k)}$, belonging to $\mathcal{S}_{\preceq}^{\mathbb{K}}$ for some $\preceq \in$ Ord_n, with support in $\gamma_k + \sigma_k$ for some $\gamma_k \in \mathbb{Z}^n$, with $P(\xi_i^{(k)}) = 0$ for $i = 1, \ldots, d$.*

Proof. By Theorem 2.18, for every $\preceq \in \mathrm{Ord}_n$, there exist $\gamma_{\preceq} \in \mathbb{Z}^n$, a $\preceq$-positive rational strongly convex cone $\sigma_{\preceq}$, such that the roots of $P(T)$ in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ have support in $\gamma_{\preceq} + \sigma_{\preceq}$. By replacing $\sigma_{\preceq}$ by $\sigma_{\preceq} + {\mathbb{R}_{\geq 0}}^n$ we can assume that the $\sigma_{\preceq}$ contain the first orthant. Every cone $\sigma_{\preceq} + {\mathbb{R}_{\geq 0}}^n$ is strongly convex because $\sigma_{\preceq}$ and ${\mathbb{R}_{\geq 0}}^n$ are $\preceq$-positive.
In particular we have $\mathrm{Ord}_n \subset \bigcup_{\preceq} \mathcal{U}_{\sigma_{\preceq}}$. Hence, by Theorem 2.16, we can extract from this family of cones $\sigma_{\preceq}$, a finite number of cones, denoted by $\sigma_1, \ldots, \sigma_N$, such that $\mathrm{Ord}_n = \mathcal{U}_{{\mathbb{R}_{\geq 0}}^n} \subset \bigcup_{k=1}^N \mathcal{U}_{\sigma_{\preceq_k}}$. Therefore, by Lemma 2.17, we have that ${\mathbb{R}_{\geq 0}}^n \subset \displaystyle\bigcup_{k=1}^N \sigma_k^\vee$. But, since the $\sigma_{\preceq_k}$ contain ${\mathbb{R}_{\geq 0}}^n$, we have

$$ {\mathbb{R}_{\geq 0}}^n = \bigcup_{k=1}^N \sigma_k^\vee. $$

On the other hand this family satisfies the following property:

$$ \begin{aligned} &\forall \preceq \in \mathrm{Ord}_n, \exists \gamma_{\preceq} \in \mathbb{Z}^n,\ \exists k \in \{1, \ldots, N\}, \\ &\qquad\qquad \text{such that the roots of } P \text{ in } \mathcal{S}_{\preceq}^{\mathbb{K}} \text{ have support in } \gamma_{\preceq} + \sigma_k. \end{aligned} \tag{1} $$

Assume that the same integer $k \in \{1, \ldots, N\}$ satisfies the previous property for two orders $\preceq$ and $\preceq' \in \mathrm{Ord}_n$. That is, the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ (resp. in $\mathcal{S}_{\preceq'}^{\mathbb{K}}$) have support in $\gamma_{\preceq} + \sigma_k$ (resp. in $\gamma_{\preceq'} + \sigma_k$). Then the roots of P in $\mathcal{S}_{\preceq'}^{\mathbb{K}}$ are elements of $\mathcal{S}_{\preceq}^{\mathbb{K}}$, and, because P has only d roots in $\mathcal{S}_{\preceq}^{\mathbb{K}}$, the roots of P in $\mathcal{S}_{\preceq'}^{\mathbb{K}}$ coincide with its roots in $\mathcal{S}_{\preceq}^{\mathbb{K}}$. Therefore we may assume that the element $\gamma_{\preceq}$ of (1) does depend only on k. □

Proof of Theorem 1.3. By Proposition 3.3, there exist strongly convex rational cones $\sigma_1, \ldots, \sigma_N$ satisfying i) and ii) of Corollary 3.2. Therefore, by Corollary 3.2, we have that $\tau(\xi)$ is a strongly convex rational cone. This proves Theorem 1.3. □

Remark 3.4. For a formal power series $f \in \mathbb{K}[[x]]$ we denote by $\mathrm{NP}(f)$ its Newton polyhedron. Let p be a vertex of $\mathrm{NP}(f)$. The set of vectors $v \in \mathbb{R}^n$ such that $p + \lambda v \in \mathrm{NP}(f)$ for some $\lambda \in \mathbb{R}_{\geq 0}$ is a rational strongly convex cone. Such a cone is called the *cone of the Newton polyhedron of f associated with the vertex p.* We have the following generalization of Abhyankar-Jung Theorem that provides in an effective way some cones satisfying Corollary 3.2:

Theorem 3.5 (Abhyankar-Jung Theorem). [GP00, Théorème 3][Ar04, Theorem 7.1][PR12, Theorem 6.2]

Let $\mathbb{K}$ be a characteristic zero field. Let $P(Z) \in \mathbb{K}[[x]][Z]$ be a monic polynomial and let Δ be its discriminant. Let $\mathrm{NP}(\Delta)$ denote the Newton polyhedron of Δ. Then the set of cones of $\mathrm{NP}(\Delta)$ satisfies the properties of Corollary 3.2.

Therefore, if ξ is integral over $\mathbb{K}[[x]]$, that is $P(T)$ is a monic polynomial in T, we may replace the use of Corollary 3.2 (thus Proposition 3.3 and thus Theorem 2.16) by Theorem 3.5.

4. Proof of Theorem 1.4

4.1. Preliminary results.

Definition 4.1. For a Laurent series ξ we set

$$\tau_0'(\xi) = \left\{\omega \in {\mathbb{R}_{\geq 0}}^n \setminus \{\underline{0}\} \mid \#\left(\mathrm{Supp}(\xi) \cap \{u \in \mathbb{R}^n \mid u \cdot \omega \leq k\}\right) < \infty, \forall k \in \mathbb{R}\right\},$$

$$\tau_1'(\xi) = \left\{\omega \in {\mathbb{R}_{\geq 0}}^n \setminus \{\underline{0}\} \mid \#\left(\mathrm{Supp}(\xi) \cap \{u \in \mathbb{R}^n \mid u \cdot \omega \leq k\}\right) = \infty, \forall k \in \mathbb{R}\right\}.$$

We have the following lemma:

Lemma 4.2. *Let ξ be a Laurent series with support in a translation of a strongly convex cone containing ${\mathbb{R}_{\geq 0}}^n$. We have $\tau_0'(\xi) \subset \tau(\xi) \subset \overline{\tau_0'(\xi)}$.*

Proof. We have $\tau_0'(\xi) \subset \tau(\xi)$ by definition.
Let $\omega \in \tau(\xi)$. Then by Lemma 2.2, $\mathrm{Supp}(\xi) \subset \gamma + \langle\omega\rangle^\vee$ for some $\gamma \in \mathbb{Z}^n$.
On the other hand, by hypothesis, $\mathrm{Supp}(\xi)$ is included in $\gamma' + \sigma$ where $\gamma' \in \mathbb{Z}^n$ and σ is a strongly convex cone such that ${\mathbb{R}_{\geq 0}}^n \subset \sigma$. Thus, by Lemma 2.11, $\mathrm{Supp}(\xi)$ is included in a translation of the strongly convex cone $\sigma \cap \langle\omega\rangle^\vee$.
We have $\omega \in \langle\omega\rangle^{\vee\vee} \subset (\sigma \cap \langle\omega\rangle^\vee)^\vee$, and $(\sigma \cap \langle\omega\rangle^\vee)^\vee$ is full dimensional. Thus there exists a sequence $(\omega_k)_k$ of vectors in $\mathrm{Int}\left((\sigma \cap \langle\omega\rangle^\vee)^\vee\right)$ that converges to ω.
We have to prove that the ω_k belong to $\tau_0'(\xi)$. For $u \in (\sigma \cap \langle\omega\rangle^\vee) \setminus \{\underline{0}\}$, we have $u \cdot \omega_k \neq 0$ because $\omega_k \in \mathrm{Int}\left((\sigma \cap \langle\omega\rangle^\vee)^\vee\right)$. This shows that $\sigma \cap \langle\omega\rangle^\vee \cap \langle\omega_k\rangle^\perp = \{\underline{0}\}$. Therefore, because $\mathrm{Supp}(\xi)$ is included in a translation of $\sigma \cap \langle\omega\rangle^\vee$, for all k we have:

$$\omega_k \in \left\{\omega' \in \mathbb{R}^n \mid \#\left(\mathrm{Supp}(\xi) \cap \{u \in \mathbb{R}^n \mid u \cdot \omega' \leq k\}\right) < \infty, \forall k \in \mathbb{R}\right\}.$$

Moreover, because $\omega \in \tau(\xi) \subset {\mathbb{R}_{\geq 0}}^n$ and ${\mathbb{R}_{\geq 0}}^n \subset \sigma$, we have ${\mathbb{R}_{\geq 0}}^n = ({\mathbb{R}_{\geq 0}}^n)^\vee \subset \sigma \cap \langle\omega\rangle^\vee$. Therefore the ω_k are in ${\mathbb{R}_{\geq 0}}^n$, and they are non zero for k large enough because $(\omega_k)_k$ converges to ω which is non zero. This shows that $\omega_k \in \tau_0'(\xi)$ for k large enough, therefore $\omega \in \overline{\tau_0'(\xi)}$. □

Corollary 4.3. *Under the hypothesis of Theorem 1.4, we have*

$$\tau(\xi) = \overline{\tau_0'(\xi)}.$$

Proof. By Lemma 4.2 we have $\tau_0'(\xi) \subset \tau(\xi) \subset \overline{\tau_0'(\xi)}$. Since $\tau(\xi)$ is closed (it is a rational cone, thus a polyhedral cone, by Theorem 1.3) we have $\tau(\xi) = \overline{\tau_0'(\xi)}$. □

Definition 4.4. In the rest of this section we consider the following setting: ξ is a Laurent series with support included in the translation of a strongly convex rational cone, and ξ is algebraic over $\mathbb{K}[[x]]$ where $\mathbb{K}$ is a characteristic zero field. We denote by $P \in \mathbb{K}[[x]][T]$ the minimal polynomial of ξ and, for any order $\preceq \in \mathrm{Ord}_n$, $\xi_1^{\preceq}, \ldots, \xi_d^{\preceq}$ denote the roots of $P(T)$ in $\mathcal{S}_{\preceq}^{\mathbb{K}}$. We set

$$\tau_0(\xi) := \left\{\omega \in {\mathbb{R}_{\geq 0}}^n \backslash \{\underline{0}\} \mid \text{ for all } \preceq \text{ that refines } \leq_\omega, \; \exists i \text{ such that } \xi = \xi_i^{\preceq}\right\},$$

$$\tau_1(\xi) := \left\{\omega \in {\mathbb{R}_{\geq 0}}^n \backslash \{\underline{0}\} \mid \xi \neq \xi_i^{\preceq}, \text{ for all } \preceq \text{ that refines } \leq_\omega, \; \forall i = 1, \ldots, d\right\},$$

Remark 4.5. These sets were introduced in [AR19], but only for $\omega \in {\mathbb{R}_{>0}}^n$. In this case it was proved that $\tau_0(\xi) \cap {\mathbb{R}_{>0}}^n = \tau_0'(\xi) \cap {\mathbb{R}_{>0}}^n$ and $\tau_1(\xi) \cap {\mathbb{R}_{>0}}^n = \tau_1'(\xi) \cap {\mathbb{R}_{>0}}^n$ (see [AR19, Lemmas 5.8, 5.11]). Taking into account all the $\omega \in {\mathbb{R}_{\geq 0}}^n$ changes the situation. In particular we do not have $\tau_0(\xi) = \tau_0'(\xi)$ in general (see Example 4.11).

Proposition 4.6. *We have $\tau_1(\xi) = \tau_1'(\xi)$ and $\tau_0'(\xi) \subset \tau_0(\xi)$.*

Proof. The proof of the equality $\tau_1(\xi) = \tau_1'(\xi)$ is exactly the proof of [AR19, Lemma 5.11]. Let us prove $\tau_0'(\xi) \subset \tau_0(\xi)$. Let $\omega \in \tau_0'(\xi)$, in particular:

$$\# \left(\mathrm{Supp}(\xi) \cap \{u \in \mathbb{R}^n \mid u \cdot \omega \leq k\}\right) < \infty, \; \forall k \in \mathbb{R}, \tag{2}$$

and let us consider an order $\preceq$ that refines $\leq_\omega$.
Let $(u_l)_l$ be a sequence of elements of $\mathrm{Supp}(\xi)$ such that $u_l \succeq u_{l+1}$ for every $l \in \mathbb{N}$. Then $u_l \geq_\omega u_{l+1}$, that is $u_l \cdot \omega \geq u_{l+1} \cdot \omega$, for every $l \in \mathbb{N}$. Therefore by (2), this sequence contains only finitely many distinct terms. Therefore $u_{l+1} = u_l$ for l large enough because $\preceq$ is an order. This shows that $\mathrm{Supp}(\xi)$ is $\preceq$-well-ordered. Thus by [AR19, Corollary 4.6] ξ is an element of $\mathcal{S}_{\preceq}^{\mathbb{K}}$. This shows that $\omega \in \tau_0(\xi)$. $\square$

Proposition 4.7. *The sets $\tau_0(\xi)$ and $\tau_1(\xi)$ are open subsets of ${\mathbb{R}_{\geq 0}}^n$.*

Proof. Let us consider the cones σ_k given by Proposition 3.3. In particular, for every $\omega \in {\mathbb{R}_{\geq 0}}^n$, the set of orders $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$ is included in $\bigcup_{k=1}^N \mathcal{U}_{\sigma_k^\vee}$. For every $\omega \in {\mathbb{R}_{\geq 0}}^n$, we set $\mathcal{T}_\omega := \{\sigma_1, \ldots, \sigma_N\}$. Therefore we have proved that:
For every $\omega \in {\mathbb{R}_{\geq 0}}^n$, there exists a finite set $\mathcal{T}_\omega$ of strongly convex cones rational cones such that, for any order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$, there is $\sigma \in \mathcal{T}_\omega$ such that the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ have support in a translation of σ.
Moreover, let us choose $\mathcal{T}_\omega$ to be minimal among the sets of cones having this property. Then Corollary 2.13 implies that, for every $\omega' \in {\mathbb{R}_{\geq 0}}^n$ close enough to ω, and for any order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$, there is $\sigma \in \mathcal{T}_\omega$ such that the roots of P in $\mathcal{S}_{\preceq'}^{\mathbb{K}}$ have support in a translation of σ. Since $\mathcal{T}_\omega$ is minimal with this property, for every ω' close enough to ω, for every order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$ and for every $i = 1, \ldots, d$, there is an order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$ such that $\xi_i^{\preceq'} = \xi_{j_i}^{\preceq}$ for some j_i.
If $\omega \in \tau_0(\xi)$ then ξ is equal to some $\xi_i^{\preceq}$ for every order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$. Thus, for every $\omega' \in {\mathbb{R}_{\geq 0}}^n$ close enough to ω and every order $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$,

$\xi=\xi_j^{\preceq'}$ for some j. Thus $\omega'\in\tau_0(\xi)$. This proves that $\tau_0(\xi)$ is open in ${\mathbb{R}_{\geq 0}}^n$.
If $\omega\in\tau_1(\xi)$ then $\xi\neq\xi_i^{\preceq}$ for every i and for every order $\preceq\in\mathrm{Ord}_n$ refining $\leq_\omega$. Thus, for $\omega'\in{\mathbb{R}_{\geq 0}}^n$ close enough to ω and every order $\preceq'\in\mathrm{Ord}_n$ refining $\leq_{\omega'}$, $\xi\neq\xi_j^{\preceq'}$ for every j. Hence $\omega'\in\tau_1(\xi)$ and $\tau_1(\xi)$ is open. □

Corollary 4.8. *We have*

$$\overline{\tau_0'(\xi)}\cap\tau_1'(\xi)=\emptyset.$$

Proof. The sets $\tau_0(\xi)$ and $\tau_1(\xi)$ are disjoint and open in ${\mathbb{R}_{\geq 0}}^n$. Thus $\overline{\tau_0(\xi)}\cap\tau_1(\xi)=\emptyset$. This proves the corollary because $\tau_0'(\xi)\subset\tau_0(\xi)$ and $\tau_1'(\xi)=\tau_1(\xi)$ by Proposition 4.6. □

Lemma 4.9. *We have*

$$\overline{\tau_0'(\xi)}=\overline{\tau_0(\xi)\cap{\mathbb{R}_{>0}}^n}=\overline{\tau_0(\xi)}.$$

Proof. The set $\tau_0(\xi)$ is open. Therefore every $w\in\tau_0(\xi)\cap({\mathbb{R}_{\geq 0}}^n\backslash{\mathbb{R}_{>0}}^n)$ can be approximated by elements of $\tau_0(\xi)\cap{\mathbb{R}_{>0}}^n$. Hence

$$\overline{\tau_0(\xi)\cap{\mathbb{R}_{>0}}^n}=\overline{\tau_0(\xi)}.$$

By [AR19, Lemma 5.8] $\tau_0'(\xi)\cap{\mathbb{R}_{>0}}^n=\tau_0(\xi)\cap{\mathbb{R}_{>0}}^n$. We have that $\tau_0'(\xi)$ is convex (the proof is exactly the same as the proof of [AR19, Lemma 5.9]). Thus we have

$$\overline{\tau_0'(\xi)\cap{\mathbb{R}_{>0}}^n}=\overline{\tau_0'(\xi)}$$

by [Bo53, Prop. 16 - Cor. 1; II.2.6]. Hence

$$\overline{\tau_0'(\xi)}=\overline{\tau_0'(\xi)\cap{\mathbb{R}_{>0}}^n}=\overline{\tau_0(\xi)\cap{\mathbb{R}_{>0}}^n}=\overline{\tau_0(\xi)}.$$

□

Corollary 4.10. *For every $f\in\mathbb{K}(\!(x)\!)^*$ we have*

$$\tau_0(\xi+f)=\tau_0(\xi),\ \tau_1(\xi+f)=\tau_1(\xi),\ \tau(\xi+f)=\tau(\xi),$$

$$\tau_0(f\xi)=\tau_0(\xi),\ \tau_1(f\xi)=\tau_1(\xi),\ \tau(f\xi)=\tau(\xi).$$

Proof. We begin by proving these equalities for $f\in\mathbb{K}[[x]]^*$. The minimal polynomial of $\xi+f$ is $Q(T):=P(T-f)$. Thus, for a given $\preceq\in\mathrm{Ord}_n$, the roots of $Q(T)$ in $\mathcal{S}_\preceq^{\mathbb{K}}$ are $\xi_1^{\preceq}+f,\ldots,\xi_d^{\preceq}+f$. This shows that

$$\tau_0(\xi+f)=\tau_0(\xi),\ \tau_1(\xi+f)=\tau_1(\xi).$$

Lemma 4.9 and Corollary 4.3 imply that $\tau(\xi+f)=\tau(\xi)$.
Now, the polynomial $R(T):=f^dP(T/f)$ vanishes at $f\xi$. On the other hand if $\overline{R}(T)$ is a polynomial with $\overline{R}(f\xi)=0$, then $R(fT)$ is a polynomial vanishing at ξ. This shows that $P(T)$ divides $R(fT)$. Thus, the minimal polynomial of $f\xi$ has degree d and divides $R(T)$, thus it is of the form $\frac{1}{g}R(T)=\frac{f^d}{g}P(T/f)$ for some $g\in\mathbb{K}[[x]]^*$.

Therefore, for a given $\preceq \in \mathrm{Ord}_n$, the roots of $R(T)$ in $\mathcal{S}^{\mathbb{K}}_{\preceq}$ are $f\xi_1^{\preceq}, \ldots, f\xi_d^{\preceq}$. This shows that, as before, that

$$\tau_0(f\xi) = \tau_0(\xi),\ \tau_1(f\xi) = \tau_1(\xi),\ \overline{\tau_0'(f\xi)} = \overline{\tau_0'(\xi)}.$$

Now let $f = g/h$, where $g,\, h \in \mathbb{K}[[x]]^*$. Then we have

$$\tau_\bullet(\xi) = \tau_\bullet(g\xi) = \tau_\bullet(hg/h\xi) = \tau_\bullet(g/h\xi)$$

by the previous case (here $\bullet$ denotes indistinctively 0, 1 or $\emptyset$).
Moreover, again by the previous cases, we have

$$\tau_\bullet(\xi) = \tau_\bullet(h\xi) = \tau_\bullet(h\xi + g) = \tau_\bullet(\xi + g/h).$$

This proves the corollary. $\square$

Example 4.11. We can see on a basic example that $\tau_0'(\xi + f) \neq \tau_0'(\xi)$ in general: let $n = 2$ and fix $\xi = \sum_{k\in\mathbb{N}} x_1^k$ and $f = 1 - \xi$. Then $\tau_0'(\xi) = \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$ but $\tau_0'(\xi + f) = {\mathbb{R}_{\geq 0}}^2$. This also shows that $\tau_0(\xi) \neq \tau_0'(\xi)$ in general.

4.2. **A generalization of Dickson's Lemma.** We will prove here a strengthened version of Lemma 2.11 that we will need in the proof of Theorem 1.3. For this we need the following lemma:

Lemma 4.12. *Let U and V be two vectors of indeterminates, and I and J be ideals of $\mathbb{K}[U,V]$ such that I is generated by binomials and J by monomials. Then there exists a monomial ideal J' of $\mathbb{K}[U]$ such that*

$$(J + I) \cap \mathbb{K}[U] = J' + I \cap \mathbb{K}[U].$$

Proof. We will use the idea of the proof of [ES96, Corollary 1.3]. We consider the right-lexicographic order on the set of monomials in U and V and fix a Gröbner basis B of I with respect to this order. To compute such a basis we begin with binomials generating I and follow Buchberger's Algorithm. The reader may consult [CLO07, Definition 4, p. 83 and Theorem 2 p. 90] for details about this algorithm and the notion of S-polynomial. It is straightforward to see that the elements produced step by step in this algorithm are still binomials (this is in fact the content of [ES96, Proposition 1.1]). In particular $I \cap \mathbb{K}[U]$ is generated by binomials.
Now we wish to determine a Gröbner basis of $J + I$. As a set of generators of $I + J$, we take the Gröbner basis B of I formed of binomials and we add the monomials generating J. Following Buchberger's Algorithm we may produce new elements which are not in B in the following cases:

- We consider the S-polynomial of two binomials in B, and we take the remainder of the division of this S-polynomial by a monomial: in this case this remainder is either the S-polynomial that is in B, or a monomial.
- We consider the S-polynomial of two monomials. This S-polynomial it is always 0.

- We consider the S-polynomial of one binomial of B and one monomial. It is a monomial, and the remainder of its division by a binomial is always a monomial.

Therefore we see that the Gröbner basis of $I+J$ obtained by Buchberger's Algorithm consists of B along with a finite number of monomials. Thus $(J+I)\cap\mathbb{K}[U]$ is generated by the elements of B that do not depend on V (i.e. the generators of $I\cap\mathbb{K}[U]$) and a finite number of monomials (defining a monomial ideal J').

□

Corollary 4.13 (Dickson's Lemma)**.** *Let $\sigma_1,\ldots,\sigma_k$ be convex rational cones such that $\sigma:=\bigcap_{j=1}^k\sigma_j$ is a full dimensional convex rational cone. Let $\gamma_1,\ldots,\gamma_k\in\mathbb{Z}^n$. Then there exists a finite set $C\subset\mathbb{Z}^n$ such that*

$$\bigcap_{j=1}^k(\gamma_j+\sigma_j)\cap\mathbb{Z}^n=C+\sigma\cap\mathbb{Z}^n.$$

Proof. Up to a translation we may assume that $\gamma_j\in\sigma\cap\mathbb{Z}^n$ for every j because σ is full dimensional. Let $u_1,\ldots,u_s$ be integer coordinate vectors generating $\sigma\cap\mathbb{Z}^n$. Then the ring R_σ of polynomials in $x_1,\ldots,x_n$ with support in $\sigma\cap\mathbb{Z}^n$ is isomorphic to $\mathbb{K}[U_1,\ldots,U_s]/I$ for some binomial ideal I. This is well known and this can be described as follows (for instance see [CLS11, Proposition 1.1.9] for details):
for any linear relation $L:=\{\sum_{i=1}^s\lambda_iu_i=0\}$ with $\lambda_i\in\mathbb{Z}$ we consider the binomial

$$B_L:=\prod_{i\mid\lambda_i\geq 0}U_i^{\lambda_i}-\prod_{i\mid\lambda_i<0}U_i^{-\lambda_i}.$$

Then I is the ideal generated by the B_L for L running over the $\mathbb{Z}$-linear relations between the u_i. Moreover, for $\gamma\in\sigma\cap\mathbb{Z}^n$, the isomorphism $R_\sigma\longrightarrow\mathbb{K}[U]/I$ sends x^γ onto U^{α_γ} where $\alpha_\gamma\in\mathbb{Z}^s_{\geq 0}$ is defined by $\gamma=\sum_{i=1}^s\alpha_{\gamma,i}u_i$.
Because $\sigma=\bigcap_{j=1}^k\sigma_j$, we have $R_\sigma\subset R_{\sigma_j}$ for every j and $R_\sigma=\bigcap_{j=1}^kR_{\sigma_j}$. For every j we consider the ideal $x^{\gamma_j}R_{\sigma_j}$ of R_{σ_j} generated by x^{γ_j}. Since $R_\sigma=\bigcap_{j=1}^kR_{\sigma_j}$ we have

$$\bigcap_{j=1}^kx^{\gamma_j}R_{\sigma_j}=\bigcap_{j=1}^k(x^{\gamma_j}R_{\sigma_j}\cap R_\sigma).$$

Let us fix an index j. As for R_σ, the ring R_{σ_j} of polynomials in $x_1,\ldots,x_n$ with support in $\sigma_j\cap\mathbb{Z}^n$ is isomorphic to a ring of polynomials modulo a binomial ideal. Moreover we can consider the generators $u_1,\ldots,u_s$ of σ and add vectors $v_1,\ldots,v_r$ such that σ_j is generated by the u_i and v_l. Then R_{σ_j} is isomorphic to $\mathbb{K}[U,V]/I_j$ where $U=(U_1,\ldots,U_s)$ and $V=(V_1,\ldots,V_r)$ are vectors of indeterminates, and I_j is a binomial ideal such that $I_j\cap\mathbb{K}[U]=I$. This isomorphism sends the principal monomial ideal $x^{\gamma_j}R_{\sigma_j}$ onto a principal monomial ideal J_j in $\mathbb{K}[U,V]/I_j$. By Lemma 4.12 we have

$$(J_j+I_j)\cap\mathbb{K}[U]=J'_j+I$$

for some monomial ideal J'_j of $\mathbb{K}[U]$. Thus $x^{\gamma_j}R_{\sigma_j}\cap R_\sigma$ is isomorphic to $J'_j\mathbb{K}[U]/I$. Therefore we have

$$\bigcap_{j=1}^{k} x^{\gamma_j}R_{\sigma_j} \simeq \bigcap_{j=1}^{k} J'_j\mathbb{K}[U]/I.$$

This is a monomial ideal in the indeterminates U_l by [ES96, Corollary 1.6]. By Noetherianity this monomial ideal is generated by finitely many monomials:

$$U^{\beta_1},\dots,U^{\beta_r}.$$

For every i we have $U^{\beta_i}=x^{\gamma'_i}$ for some $\gamma'_i\in\sigma\cap\mathbb{Z}^n$. Set $C=\{\gamma'_1,\dots,\gamma'_r\}$. Then we have

$$\bigcap_{j=1}^{k}(\gamma_j+\sigma_j)\cap\mathbb{Z}^n = C+\sigma\cap\mathbb{Z}^n.$$

□

4.3. **Proof of Theorem 1.4.** By [Od88, Proposition 1.3], because $\tau(\xi)^\vee$ is a strongly convex rational cone, for each non zero face $\sigma\subset\tau(\xi)^\vee$, there is a vector u_σ in the boundary of $\tau(\xi)$ such that

$$\sigma=\langle u_\sigma\rangle^\perp\cap\tau(\xi)^\vee.$$

In fact, we can freely choose u_σ in the relative interior of $\sigma^\perp\cap\tau(\xi)$, where $\sigma^\perp\cap\tau(\xi)$ is a face of dimension $n-\dim(\sigma)$ of $\tau(\xi)$. Thus, when σ is a face of dimension one, $\sigma^\perp\cap\tau(\xi)$ is a cone of dimension $n-1$ in the hyperplane $\sigma^\perp$ which is defined by one equation with integer coordinates. Therefore we can choose $u_\sigma=(u_{\sigma,1},\dots,u_{\sigma,n})$ such that

$$\dim_{\mathbb{Q}}(\mathbb{Q}u_{\sigma,1}+\cdots+\mathbb{Q}u_{\sigma,n})=n-1. \tag{3}$$

For a non zero face σ of $\tau(\xi)^\vee$ and $t\in\mathbb{R}$, we set

$$H_\sigma(t):=\{u\in\mathbb{R}^n\mid u\cdot u_\sigma=t\},\ H_\sigma(t)^+=\{u\in\mathbb{R}^n\mid u\cdot u_\sigma\geq t\}.$$

We have

$$\tau(\xi)^\vee=\bigcap_{\sigma\text{ non zero face of }\tau(\xi)^\vee} H_\sigma(0)^+.$$

The vectors u_σ are in the boundary of $\tau'_0(\xi)$ because $\tau(\xi)=\overline{\tau'_0(\xi)}$ by Corollary 4.3. Hence by Corollary 4.8 we have $u_\sigma\notin\tau'_1(\xi)$ for any i. Thus for every non zero face σ of $\tau(\xi)^\vee$ we have $u_\sigma\in\tau'_0(\xi)$ or $u_\sigma\in{\mathbb{R}_{\geq0}}^n\backslash(\tau'_0(\xi)\cup\tau'_1(\xi))$. We will reduce to the situation where none of the u_σ are in $\tau'_0(\xi)$:

Let σ be a non zero face of $\tau(\xi)^\vee$ for which $u_\sigma\in\tau'_0(\xi)$. By Proposition 4.7, $\tau'_0(\xi)\cap{\mathbb{R}_{>0}}^n$ is open. Thus, because u_σ is in the boundary of $\tau'_0(\xi)$, we have $u_\sigma\in{\mathbb{R}_{\geq0}}^n\backslash{\mathbb{R}_{>0}}^n$. In particular at least one of the coordinates of u_σ is zero, hence $\langle u_\sigma\rangle^\perp$ contains at least one line generated by one vector with integer coordinates. Therefore there exists $f_\sigma(x)\in\mathbb{K}[[x]]$ with support in $\langle u_\sigma\rangle^\perp\cap{\mathbb{R}_{\geq0}}^n$ and such that

$$\#\left\{\mathrm{Supp}(\xi+f_\sigma(x))\cap\langle u_\sigma\rangle^\perp\cap{\mathbb{R}_{\geq0}}^n\right\}=+\infty.$$

Moreover we can do this simultaneously for every non zero face σ of $\tau(\xi)^\vee$ such that $u_\sigma \in \tau_0'(\xi)$, hence there exists $f(x) \in \mathbb{K}[[x]]$ such that for every such face σ:

$$\#\left\{\operatorname{Supp}(\xi + f(x)) \cap \langle u_\sigma \rangle^\perp \cap {\mathbb{R}_{\geq 0}}^n\right\} = +\infty. \tag{4}$$

By Corollary 4.10 $\tau(\xi) = \tau(\xi + f(x))$. But $u_\sigma \notin \tau_0'(\xi + f(x))$ by (4). Therefore, we replace ξ with $\xi + f(x)$. This does not change $\tau(\xi)$, but this allows to us to assume that $u_\sigma \in {\mathbb{R}_{\geq 0}}^n \backslash (\tau_0'(\xi) \cup \tau_1'(\xi))$. Therefore we may assume that none of the u_σ is in $\tau_0'(\xi)$.

The next step is to prove that for every non zero face σ of $\tau(\xi)^\vee$, there exist a Laurent polynomial $p_\sigma(x)$ and a real number t_σ such that

$$\operatorname{Supp}(\xi + p_\sigma(x)) \subset H_\sigma(t_\sigma)^+ \text{ and } \#\left(\operatorname{Supp}(\xi + p_\sigma(x)) \cap H_\sigma(t_\sigma)\right) = +\infty. \tag{5}$$

For this we do the following. First, because $u_\sigma \notin \tau_0'(\xi) \cup \tau_1'(\xi)$, the following set is non empty and bounded from above :

$$E_\sigma := \left\{t \in \mathbb{R} \mid \#\left(\operatorname{Supp}(\xi) \cap \{u \in \mathbb{R}^n \mid u \cdot u_\sigma < t\}\right) < \infty\right\}.$$

Let us set $t_\sigma := \sup E_\sigma$. By Lemma 2.19, $t_\sigma = \max E_\sigma$ and (5) is satisfied.

Then, modulo a finite number of monomials and a formal power series $f(x) \in \mathbb{K}[[x]]$, the support of ξ is included in $\displaystyle\bigcap_{\sigma \text{ non zero face of } \tau(\xi)^\vee} H_\sigma(t_\sigma)^+ \cap \mathbb{Z}^n$. Moreover each $H_\sigma(t_\sigma)$ contains infinitely many monomials of ξ, i.e there is a Laurent polynomial $p(x)$ such that

$$\operatorname{Supp}(\xi + p(x)) \subset \bigcap_{\sigma \text{ non zero face of } \tau(\xi)^\vee} H_\sigma(t_\sigma)^+ \cap \mathbb{Z}^n$$
$$\text{and } \#\left(\operatorname{Supp}(\xi + p(x)) \cap H_\sigma(t_\sigma)\right) = +\infty \quad \forall \sigma.$$

For every σ non zero face of we have $H_\sigma(t_\sigma)^+ = \gamma_\sigma + H_\sigma(0)^+$ for any $\gamma_\sigma \in H_\sigma(t_\sigma)$. But, since $H_\sigma(t_\sigma) \cap \mathbb{Z}^n \neq \emptyset$, we may fix $\gamma_\sigma \in \mathbb{Z}^n$. By Corollary 4.13 there is a finite set $C \subset \mathbb{Z}^n$ such that

$$\bigcap_{\sigma \text{ non zero face of } \tau(\xi)^\vee} H_\sigma(t_\sigma)^+ \cap \mathbb{Z}^n = C + \bigcap_{\sigma \text{ non zero face of } \tau(\xi)^\vee} H_\sigma(0)^+ \cap \mathbb{Z}^n = C + \tau(\xi)^\vee \cap \mathbb{Z}^n.$$

This proves i).

• Because the sum of two convex sets is a convex set, we have

$$\operatorname{Conv}(C + \tau(\xi)^\vee) = \operatorname{Conv}(C) + \tau(\xi)^\vee$$

is an unbounded convex polytope.

Because $\tau(\xi)^\vee$ is the convex hull of its one-dimensional faces, $\operatorname{Conv}(C + \tau(\xi)^\vee)$ is the convex hull of the union of all the sets of the form $\gamma + \sigma$, where $\gamma \in C$ and σ is one dimensional face of $\tau(\xi)^\vee$. Let σ be such a one dimensional face of $\tau(\xi)^\vee$. We have that $\operatorname{Supp}(\xi + p(x)) \subset \operatorname{Conv}(C + \tau(\xi)^\vee) \subset H_\sigma^+(t_\sigma)$. But $H_\sigma^+(t_\sigma) \cap \mathbb{Q}^n$ is a one dimensional $\mathbb{Q}$-vector space by (3). Moreover, by (4), $H_\sigma(t_\sigma) \cap \operatorname{Supp}(\xi + p(x))$ is infinite. Therefore $H_\sigma(t_\sigma) \cap \operatorname{Conv}(C + \tau(\xi)^\vee)$ is a one dimensional face of $\operatorname{Conv}(C) + \tau(\xi)^\vee$

of the form $\gamma+\sigma$ for some $\gamma\in C$, and this face contains infinitely many elements of $\mathrm{Supp}(\xi+p(x))$. This proves ii) b) with $(C_2,p_2(x),f_2(x))=(C,p(x),f(x))$.

• Now we remark that we also have

$$\mathrm{Supp}(\xi+p(x))\subset\bigcap_{\sigma\ (n-1)\text{-dim. face of }\tau(\xi)^\vee}H_\sigma(t_\sigma)^+\cap\mathbb{Z}^n.$$

Again by Lemma 4.13, there is a finite set $C_1\subset\mathbb{Z}^n$ such that

$$\bigcap_{\sigma\ (n-1)\text{-dim. face of }\tau(\xi)^\vee}H_\sigma(t_\sigma)^+\cap\mathbb{Z}^n=C_1+\tau(\xi)^\vee\cap\mathbb{Z}^n.$$

Moreover the $(n-1)$-dimensional faces of $\mathrm{Conv}(C_1+\tau(\xi)^\vee)$ are all of the form $H_\sigma(t_\sigma)\cap\mathrm{Conv}(C+\tau(\xi)^\vee)$. Indeed the convex hull of $\bigcap_\sigma H_\sigma(t_\sigma)^+\cap\mathbb{Z}^n$ is $\bigcap_\sigma H_\sigma(t_\sigma)^+$ because the $H_\sigma(t_\sigma)$ are affine hyperplanes defined over $\mathbb{Z}$. This proves ii) a).

• Assume now that there are $C'\in\mathbb{R}^n$ and a convex (non necessarily polyhedral) cone $\sigma\subset\tau(\xi)^\vee$ such that

$$\mathrm{Supp}(\xi+p'(x)+f'(x))\subset\gamma'+\sigma$$

for some Laurent polynomial $p'(x)$ and some formal power series $f'(x)\in\mathbb{K}[[x]]$. Then by definition of $\tau(\xi)$ we have

$$\sigma^\vee\subset\tau(\xi).$$

Therefore $\sigma=\tau(\xi)^\vee$. This proves ii).

4.4. **Some examples.**

Example 4.14. Let $E:=\{(x,y)\in\mathbb{R}_{\geq 0}\times\mathbb{R}\mid y\geq -x-\sqrt{x}\}$ and let ξ be a Laurent series whose support is $\mathbb{Z}^2\cap E$ as follows:

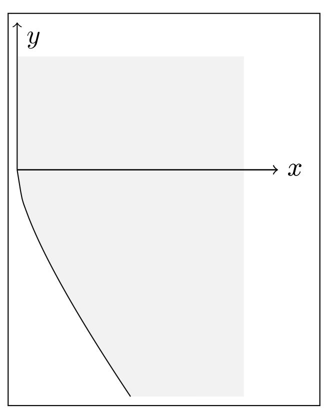

FIGURE 1.

Then $\tau(\xi)^\vee$ is the set

$$\{(x,y)\in\mathbb{R}_{\geq 0}\times\mathbb{R} \mid y>-x\}.$$

Thus, $\tau(\xi)$ is a not a polyhedral cone. Therefore ξ is not algebraic over $\mathbb{K}(\!(x,y)\!)$. Moreover $\tau_1'(\xi)$ is the rational cone generated by $(1,0)$ and $(1,1)$. So $\tau_1'(\xi)$ is not open. In this case ${\mathbb{R}_{\geq 0}}^n=\tau_0'(\xi)\cup\tau_1'(\xi)$.

Example 4.15. We consider the set

$$E:=\{(x,y)\in\mathbb{R}_{\geq 0}\times\mathbb{R} \mid y\geq \ln(x+1)\}.$$

We rotate it by an angle of $-\pi/4$ and denote this set by Γ. We denote a Laurent series whose support is $\Gamma\cap\mathbb{Z}^2$ by ξ (see Figure 2).

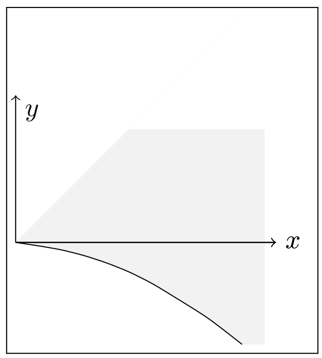

FIGURE 2.

Then $\tau(\xi)^\vee$ is the cone generated by $(1,-1)$ and $(0,1)$, so it is rational, but ξ is not algebraic as Theorem 1.4 ii) is not satisfied.
Moreover $\tau(\xi)$ is generated by $(0,1)$ and $(1,1)$. Thus the vector $(1,1)$ is in the boundary of $\tau(\xi)$ but here $(1,1)\in\tau_0'(\xi)$. Thus $\tau_0'(\xi)$ is closed.

Example 4.16. Let σ be the cone generated by the vectors $(1,0)$, $(0,1)$ and $(1,-1)$. Then the series $\xi:=\sum_{k=0}^{\infty}(xy^{-1})^k$ has support in σ and it is straightforward to see that $\sigma=\tau(\xi)^\vee$. Let $N\in\mathbb{Z}$ and set $p_N(x,y):=\sum_{k=0}^{N}$ (when $N\geq 0$) or $p_N(x,y)=\sum_{k=N}^{0}$ (when $N<0$). Let C_N denote the point $(N,-N)$. Then we have

$$\mathrm{Supp}(\xi-p_N(x,y))\subset C_N+\sigma.$$

This shows that there is no canonical choice for C_N in Theorem 1.4 i), neither a minimal or maximal C_N.

Example 4.17. Let C be the set $\{(1,0,0),(0,1,0),(0,0,1)\}$, and let σ be the cone generated by the vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,-1,1)$, $(-1,1,1)$, and $(1,1,-1)$. We can construct a Laurent series ξ, algebraic over $\mathbb{K}[[x,y,z]]$, with support in $\mathrm{Conv}(C)+\sigma$, such that all the unbounded faces of $\mathrm{Conv}(C)+\sigma$ contain infinitely many monomials of ξ as follows:
We fix an algebraic series $G(T)$ not in $\mathbb{K}(T)$. We remark that, for a, b, $c \in \mathbb{Z}$, the series $G(x^a y^b z^z)$ is algebraic over $\mathbb{K}(x,y,z)$, and it is a formal sum of monomials of the form $x^{ka}y^{kb}z^{kc}$ with $k \in \mathbb{N}$. Thus its support is included in the half line generated by the vector (a,b,c).
Then we set

$$\xi = G(x) + G(y) + zG(z) + zG\left(\frac{xz}{y}\right) + zG\left(\frac{yz}{x}\right) + (x+y)G\left(\frac{xy}{z}\right).$$

Then ξ is algebraic over $\mathbb{K}(\!(x,y,z)\!)$, its support is $\mathrm{Conv}(C)+\sigma$ and all the unbounded faces of $\mathrm{Conv}(C)+\sigma$ contain infinitely many monomials of ξ (see Figure 3). Therefore $\tau(\xi)^\vee = \sigma$. Moreover we can see that there is no $\gamma \in \mathbb{R}^n$ such that $\mathrm{Supp}(\xi) \subset \gamma + \sigma$ and every face of $\gamma+\sigma$ contains infinitely many monomials of ξ, even after removing monomials of ξ belonging to ${\mathbb{R}_{\geq 0}}^3$. Indeed, if it were the case, the four unbounded 1-dimensional faces of $\mathrm{Conv}(C)+\sigma$ that are not included in ${\mathbb{R}_{\geq 0}}^3$ would intersect at one point and this is clearly not the case. Thus we cannot assume that the finite set C of Theorem 1.4 i) is a single point.

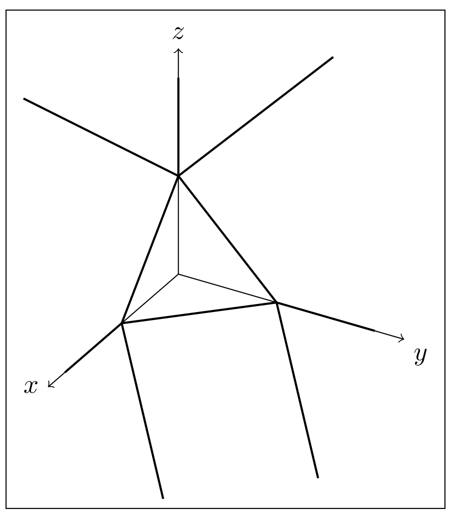

FIGURE 3.

5. The positive characteristic case

In the positive characteristic case, the roots of polynomials with coefficients in $\mathbb{K}(\!(x)\!)$, with $x = (x_1, \ldots, x_n)$, are not Laurent Puiseux series in general. This was first noticed by Chevalley in [Ch51] for the case $n = 1$: he showed that the solutions of the equation

$$T^p - x_1^{p-1}T - x_1^{p-1} = 0$$

cannot been expressed as Puiseux series. Then Abhyankar noticed that for such a polynomial, the roots can be expressed as series with support in $\mathbb{Q}$ with the additional property that their support is well-ordered. Here such a root can be written as

$$\sum_{k=1}^{\infty} x_1^{1-\frac{1}{p^k}}.$$

The determination of the algebraic closure of $\mathbb{K}(\!(x_1)\!)$ for $n = 1$, when $\mathbb{K}$ is a positive characteristic field, was finally achieved recently (see [Ke01], [Ke17]).
For $n \geq 2$, this problem has recently been investigated by Saavedra [Sa17]. He generalized Macdonald's Theorem to the positive characteristic case as follows:

Theorem 5.1. [Sa17, Theorem 5.3] *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. Let $\omega \in {\mathbb{R}_{>0}}^n$ be a vector whose coordinates are $\mathbb{Q}$-linearly independent. The set*

$$\mathcal{S}_\omega^{\mathbb{K}} = \Bigg\{\xi \textit{ series } \mid \exists k \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n, \sigma \textit{ a } \leq_\omega \textit{-positive rational cone,}$$
$$\textit{Supp}(\xi) \subset (\gamma + \sigma) \cap \cup_{l \in \mathbb{N}} \frac{1}{kp^l}\mathbb{Z}^n \textit{ and Supp}(\xi) \textit{ is } \leq_\omega \textit{-well-ordered}\Bigg\}$$

is an algebraically closed field.

It is a natural question to extend the problem of the shape of the support of an element of $\mathcal{S}_\omega^{\mathbb{K}}$ that is algebraic over $\mathbb{K}(\!(x)\!)$. Firstly we can remark that Theorem 1.4 is no longer true in this situation:

Example 5.2. Let $\mathbb{K}$ be a field of characteristic $p > 0$. Set $f = \displaystyle\sum_{k=1}^{\infty} t^{1-\frac{1}{p^k}}$. The series f is algebraic over $\mathbb{K}[t]$ because $f^p - t^{p-1}f - t^{p-1} = 0$. Thus $g := \displaystyle\sum_{k=1}^{\infty}\left(\frac{x}{y}\right)^{1-\frac{1}{p^k}}$ is algebraic over $\mathbb{K}[x, y]$. We set $\xi = \displaystyle\sum_{k=1}^{\infty}(xg)^k$. Because $\xi = \frac{xg}{1-xg}$, ξ is rational over the field extension of $\mathbb{K}(x, y)$ by g. Hence ξ is algebraic over $\mathbb{K}[x, y]$.
We see that all the monomials of $(xg)^k$ are of the form $x^{k-l}y^l$ for $l \in \mathbb{Q}_{\geq 0}$. Therefore the support of ξ is included in the cone σ generated by $(2, -1)$ and $(0, 1)$ (see Figure 4). Moreover the support of $(xg)^k$ contains a sequence of points converging to $(2k, -k)$. But $(2k, -k)$ does not belong to the support of ξ since $(1, -1)$ does not belong to the support of g. Hence $\tau(\xi) = \sigma^\vee$ is generated by $(0, 1)$ and $(1, 2)$.
But Theorem 1.3 ii) does not hold in this case: there is no hyperplane $H_\lambda = \{(x, y) \in \mathbb{R}^2 \mid x + 2y = \lambda\}$ containing infinitely many elements of $\mathrm{Supp}(\xi)$ such that $H_\lambda^- := \{(x, y) \in \mathbb{R}^2 \mid x + 2y < \lambda\}$ contains only finitely many elements of $\mathrm{Supp}(\xi)$.

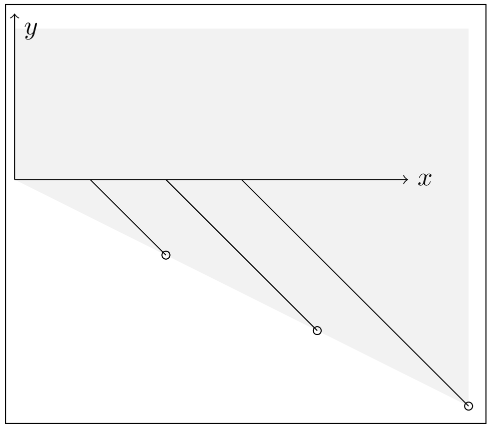

FIGURE 4.

Here $\tau_0'(\xi) = \{0\}$. This shows that Lemma 4.2 is not valid for series with exponents in $\mathbb{Q}^n$ that are algebraic over $\mathbb{K}(\!(x)\!)$, for a positive characteristic field $\mathbb{K}$.

We can also remark that a positive characteristic version of Theorem 1.3 could not be proved in the same way as in characteristic zero since Lemma 3.1 is no longer true in positive characteristic. The following example is given in [Sa17]:

Example 5.3. [Sa17, Example 3] Set $P(T) = T^p - x^{p-1}T - x^{p-1}y^3$ over a field $\mathbb{K}$ of characteristic $p > 0$. Set

$$\omega_1 = \left(1, \sqrt{2}\right),\ \omega_2 = \left(1, \frac{\sqrt{2}}{6}\right).$$

The roots of P in $\mathcal{S}^{\mathbb{K}}_{\omega_2}$ have support in a translation of ${\mathbb{R}_{\geq 0}}^2$ since these roots are

$$\sum_{k=1}^{\infty} x^{1-\frac{1}{p^k}} y^{\frac{3}{p^k}} + cx, \quad c \in \mathbb{F}_p.$$

But the roots of P in $\mathcal{S}^{\mathbb{K}}_{\omega_1}$ have support in the cone σ generated by $(1, 0)$ and $(-1, 3)$, and the face generated by $(-1, 3)$ contains infinitely many exponents of each of these roots. Indeed these roots are

$$-\sum_{k=1}^{\infty} x^{1-p^k} y^{3p^k} + cx, \quad c \in \mathbb{F}_p.$$

Let ξ be one root of P in $\mathcal{S}^{\mathbb{K}}_{\omega_1}$. So $\tau(\xi) = \sigma^\vee$. Set $\sigma_0 := {\mathbb{R}_{\geq 0}}^2$. Then

$$\omega := (2, 1) \in \tau(\xi) \cap \mathrm{Int}(\sigma_0).$$

But σ_0 is not included in $\tau(\xi)$ since $(4, 1)$ is not in $\tau(\xi)$. Thus Lemma 3.1 is not valid in positive characteristic, even if here ξ is a Laurent series.

Nevertheless we can extend some of the previous results, proved in characteristic zero, to the positive characteristic case. The main problems are as follows:
First, because Theorem 2.18 is not true in positive characteristic, we need an analogue of this theorem in positive characteristic. For this we prove an extension of Theorem 5.1 analogous to Theorem 2.18. This is based on the notion of field-family introduced by Rayner [Ra68] that gives a method of construction of Henselian valued fields which are close to be algebraically closed.
Then we introduce a natural analogue of the cone $\tau(\xi)$ in the positive characteristic case. We prove that this cone is rational and we relate it to the support of ξ (see Theorem 5.11 and 5.13).

5.1. **Algebraically closed fields in positive characteristic.** We give here a positive characteristic version of $\mathcal{S}_{\preceq}^{\mathbb{K}}$:

Definition 5.4. We fix an order $\preceq\in \mathrm{Ord}_n$ and a field $\mathbb{K}$ of positive characteristic $p>0$. We set

$$\mathcal{S}_{\preceq}^{\mathbb{K}} := \Bigg\{ \xi \text{ series } \mid \exists k\in\mathbb{N}^*, \gamma\in\mathbb{Z}^n, \sigma \text{ a } \preceq\text{-positive rational cone, such that}$$

$$\mathrm{Supp}(\xi)\subset(\gamma+\sigma)\cap\bigcup_{l=0}^{\infty}\frac{1}{kp^l}\mathbb{Z}^n, \text{ and } \forall \preceq'\in \mathrm{Ord}_n\cap\mathcal{U}_\sigma,\ \mathrm{Supp}(\xi) \text{ is } \preceq'\text{-well-ordered} \Bigg\}.$$

We have the following analogue of Theorem 2.18 in positive characteristic:

Theorem 5.5. *Let $\preceq\in \mathrm{Ord}_n$. If $\mathbb{K}$ is an algebraically closed field of positive characteristic $p>0$, the set $\mathcal{S}_{\preceq}^{\mathbb{K}}$ is an algebraically closed field containing $\mathbb{K}(\!(x)\!)$.*

In order to prove this theorem we will use the notion of field-family introduced by Rayner:

Definition 5.6. [Ra68] A family $\mathcal{F}$ of subsets of an ordered abelian group $(G,\preceq)$ is said to be a field-family with respect to G if we have the following.

(1) Every element of $\mathcal{F}$ is a well-ordered subset of G.
(2) The elements of the members of $\mathcal{F}$ generate G as an abelian group.
(3) $\forall (A,B)\in\mathcal{F}^2, A\cup B\in\mathcal{F}$.
(4) $\forall A\in\mathcal{F}$ and $B\subset A, B\in\mathcal{F}$.
(5) $\forall (A,\gamma)\in\mathcal{F}\times G$, $\gamma+A\in\mathcal{F}$.
(6) $\forall A\in\mathcal{F}\cap\{\delta\in G\mid\delta\succeq 0\}$, the semigroup generated by A belongs to $\mathcal{F}$.

Theorem 5.7. [Ra68, Theorem 2] *If $\mathcal{F}$ is a field-family with respect to G then the set*

$$\left\{\sum_{g\in G} a_g x^g \mid \{g\mid a_g\neq 0\}\in\mathcal{F}\right\}$$

is a Henselian valued field.

For $\preceq\in \mathrm{Ord}_n$ we set

$$\mathcal{F}_{\preceq} := \Big\{ A \subset \mathbb{Q}^n \mid \exists k \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n, \sigma \text{ a } \preceq\text{-positive rational cone, such that} \\ A \subset (\gamma+\sigma) \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbb{Z}^n, \text{ and } \forall \preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma,\ A \text{ is } \preceq'\text{-well-ordered} \Big\}.$$

Proposition 5.8. *The set $\mathcal{F}_{\preceq}$ is a field-family with respect to $(\mathbb{Q}^n, \preceq)$.*

Proof. It is straightforward to verify that $\mathcal{F}_{\preceq}$ satisfies the five first items of Definition 5.6. Therefore we only prove (6) here. The proof is done by induction on n. In fact we will prove a slightly stronger statement: we will prove by induction on n, that for $A \in \mathcal{F}_{\preceq} \cap \{\delta \in \mathbb{Q}^n \mid \delta \succeq \underline{0}\}$, there exists a $\preceq$-positive rational cone σ such that

$$\langle A \rangle \subset \sigma \cap \bigcup_{l=1}^{\infty} \frac{1}{kp^l}\mathbb{Z}^n$$

for some $k \in \mathbb{N}^*$ (here $\langle A \rangle$ denotes the semigroup generated by A) and A is $\preceq'$-well-ordered for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma$.
Let us consider an element $A \in \mathcal{F}_{\preceq} \cap \{\delta \in \mathbb{Q}^n \mid \delta \succeq \underline{0}\}$. So

$$A \subset (\gamma+\sigma) \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbb{Z}^n$$

for some $k \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n$ and σ a $\preceq$-positive rational cone.
If $n = 1$, we may assume that $\preceq$ is the usual order $\leq$ on $\mathbb{Q}$ and $\sigma = \mathbb{Q}_{\geq 0}$. Therefore we may assume that $\gamma = 0$ as $A \subset \mathbb{Q}_{\geq 0}$. In this case $\mathrm{Ord}_1 \cap \sigma = \{\leq\}$. Since A is $\leq$-positive and $\leq$-well-ordered, $\langle A \rangle \subset \mathbb{Q}_{\geq 0}$ is also $\leq$-well-ordered by [Ne49, Theorem 3.4, p. 206]. This settles the case $n = 1$.
So from now on, assume that $n > 1$ and that the result is satisfied for $n - 1$.
We know that there exist non zero vectors $(u_1, \ldots, u_s) \in (\mathbb{R}^n)^s$ and $(q_1, \ldots, q_r) \in (\mathbb{Q}^n)^r$ such that $\preceq = \leq_{(u_1,\ldots,u_s)}$ and $\sigma = \langle q_1, \ldots, q_r \rangle$.
Assume first that $\gamma \succeq \underline{0}$. Then $A \subset \sigma' = \langle \gamma, q_1, \ldots, q_r \rangle$ and σ' is a $\preceq$-positive rational cone. Hence the semigroup generated by A is included in $\sigma' \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbb{Z}^n$. Moreover, for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma'}$, A is $\preceq'$-well-ordered. Indeed this is true for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma$ and ${\sigma'}^\vee \subset \sigma^\vee$. Therefore, since for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma'}$ the set A is $\preceq'$-positive, by [Ne49, Theorem 3.4, p. 206] the semigroup generated by A is $\preceq'$-well-ordered.

Now assume that $\gamma \prec \underline{0}$. By replacing σ by the cone generated by σ and $-\gamma$, we may assume that $\underline{0} \in \gamma + \sigma$. We define $a := \min(A \setminus \{\underline{0}\})$ and we set

$$H := \{u \in \mathbb{R}^n \text{ such that } u \cdot u_i = a \cdot u_i\}$$

and

$$H^+ := \{u \in \mathbb{R}^n \text{ such that } u \cdot u_i \geq a \cdot u_i\}.$$

Since $A \subset \{\delta \in \mathbb{Q}^n \mid \delta \succeq \underline{0}\}$, we know that $a \succ \underline{0}$. Hence $a \cdot u_1 \geq 0$ because $\preceq = \leq_{(u_1,\ldots,u_s)}$.

Case 1: If $a \cdot u_1 > 0$ we set

$$\sigma' := \{\lambda u \mid \lambda \in \mathbb{R}_{\geq 0}, u \in H \cap \sigma\}.$$

It is a $\preceq$-positive cone such that $(\gamma+\sigma)\cap H^+ \subset \sigma'$ and $\sigma' \cap \langle u_1\rangle^\perp = \{\underline{0}\}$ (see [Sa17, Lemma 3.8]). Therefore $A \subset (\gamma+\sigma)\cap H^+ \subset \sigma'$. Then the semigroup generated by A is included in $\sigma' \cap \bigcup_{l=0}^\infty \frac{1}{kp^l}\mathbb{Z}^n$. Since $\underline{0} \in \gamma+\sigma$, we have that $\sigma \subset \gamma+\sigma \subset \sigma'$. Hence $\sigma'^\vee \subset \sigma^\vee$, and therefore, for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma'}$, A is $\preceq'$-well-ordered and $\preceq'$-positive. This implies (by [Ne49, Theorem 3.4, p. 206]) that the semigroup generated by A is $\preceq'$-well-ordered for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma'}$.

Case 2: Assume that $a \cdot u_1 = 0$. We denote the set $A \cap H$ by B, and we set $a_1 := \min(A \setminus B)$. Since $A \subset \{\delta \in \mathbb{Q}^n \mid \delta \succeq \underline{0}\}$ and $a_1 \notin H$, we have $a_1 \cdot u_1 > 0$. By Case 1, there exists a rational $\preceq$-positive cone σ_1 containing σ such that $\langle (A\setminus B)\rangle \subset \sigma_1$, and $\langle (A\setminus B)\rangle$ is $\preceq'$-well-ordered for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma_1}$.

Now we consider $B \subset H$. Here $H = \langle u_1\rangle^\perp$ is isomorphic to $\mathbb{R}^{n-1}$ under a $\mathbb{R}$-linear map φ. Since $u_1 \in \mathbb{Q}^n$ we may assume that φ is defined by a matrix with integral entries. In particular $\varphi(H \cap \mathbb{Z}^n) \subset \mathbb{Z}^{n-1}$.

For $\preceq' \in \mathrm{Ord}_n$ we define $\overline{\preceq}' \in \mathrm{Ord}_{n-1}$ by:

$$\forall u, v \in \mathbb{R}^{n-1}, u \,\overline{\preceq}'\, v \Longleftrightarrow \varphi^{-1}(u) \preceq' \varphi^{-1}(v).$$

On the other hand, for $\preceq' \in \mathrm{Ord}_{n-1}$, we define $\widetilde{\preceq}' \in \mathrm{Ord}_n$ as follows:

$$\forall \lambda_1, \lambda_2 \in \mathbb{R}, v_1, v_2 \in H,\ (\lambda_1 u_1 + v_1)\widetilde{\preceq}'(\lambda_2 u_1 + v_2) \Longleftrightarrow \left\{ \begin{array}{c} \lambda_1 < \lambda_2 \\ \text{or } \lambda_1 = \lambda_2 \text{ and } \varphi(v_1) \preceq' \varphi(v_2) \end{array}\right.$$

It is straightforward to check that for all $\preceq' \in \mathrm{Ord}_{n-1}$, $\overline{\widetilde{\preceq}'} = \preceq'$.

We denote by $\overline{\sigma} := \varphi(\sigma \cap H)$. Now let $\preceq' \in \mathrm{Ord}_{n-1}$ be such that $\overline{\sigma}$ is $\preceq'$-positive. Let $u \in \sigma$. We can write $u = \lambda u_1 + v$ where $v \in H$, that is $v \cdot u_1 = 0$. Because σ is $\preceq$-positive, we have $\lambda = \frac{1}{\|u_1\|^2} u \cdot u_1 \geq 0$. If $\lambda > 0$, we have $u \,\widetilde{\succ}'\, \underline{0}$. If $\lambda = 0$, then $u = v \in H \cap \sigma$. Therefore $\varphi(v) \succeq' \underline{0}$. This proves that $\widetilde{\preceq}' \in \mathcal{U}_\sigma$. Therefore the map $\preceq' \longmapsto \widetilde{\preceq}'$ sends $\mathcal{U}_{\overline{\sigma}} \cap \mathrm{Ord}_{n-1}$ on $\mathcal{U}_\sigma \cap \mathrm{Ord}_n$. On the other hand, it is straightforward to see that $\preceq' \longmapsto \overline{\preceq}'$ sends $\mathcal{U}_\sigma \cap \mathrm{Ord}_n$ on $\mathcal{U}_{\overline{\sigma}} \cap \mathrm{Ord}_{n-1}$.

By Lemma 2.11 there exists $\gamma_1 \in H$ such that $B \subset \gamma_1 + \sigma \cap H$. We set $\overline{\gamma} := \varphi(\gamma_1)$. Therefore we have

$$\left\{ \begin{array}{c} \varphi(B) \subset (\overline{\gamma} + \overline{\sigma}) \cap \bigcup_{l=0}^\infty \frac{1}{kp^l}\mathbb{Z}^{n-1} \\ \varphi(B) \subset \{\delta \in \mathbb{Q}^{n-1} \mid \delta \,\overline{\succeq}\, \underline{0}\} \\ \overline{\sigma} \text{ is } \overline{\preceq}\text{-positive} \end{array}\right.$$

Moreover, we have $\widetilde{\preceq}' \in \mathrm{Ord}_n \cap \sigma$ for every $\preceq' \in \mathrm{Ord}_{n-1} \cap \overline{\sigma}$. Therefore, because B is $\preceq'$-well-ordered for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma$, we have that $\varphi(B)$ is $\preceq'$-well-ordered for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\overline{\sigma}}$. Hence, by the inductive assumption, $\langle \varphi(B)\rangle \subset \sigma_0 \cap \bigcup_{l=0}^\infty \frac{1}{kp^l}\mathbb{Z}^{n-1}$, for some $\overline{\preceq}$-positive rational cone σ_0, and $k \in \mathbb{N}$. Moreover $\langle \varphi(B)\rangle$ is $\preceq'$-well-ordered for every $\preceq' \in \mathrm{Ord}_{n-1} \cap \overline{\sigma}$. Let q be a common denominator of the entries of the matrix of φ^{-1}. Then we have

$$\langle B\rangle=\varphi^{-1}(\langle\varphi(B)\rangle)\subset\varphi^{-1}(\sigma_0)\cap\bigcup_{l=0}^{\infty}\frac{1}{qkp^l}\mathbb{Z}^n$$

and $\varphi^{-1}(\sigma_0)$, is $\preceq$-positive.
Now let $\preceq'\in\mathrm{Ord}_n\cap\mathcal{U}_{\varphi^{-1}(\sigma_0)}$. Then $\widetilde{\preceq}'\in\mathrm{Ord}_n\cap\mathcal{U}_{\sigma_0}$. Thus $\langle\varphi(B)\rangle$ is $\widetilde{\preceq}'$-well-ordered, and $\langle B\rangle$ is $\widetilde{\widetilde{\preceq}}'$-well-ordered. But $\langle B\rangle\subset H$, therefore $\preceq'$ and $\widetilde{\widetilde{\preceq}}'$ coincide on $\langle B\rangle$. This shows that $\langle B\rangle$ is $\preceq'$-well-ordered.
We have $\langle A\rangle=\big\langle\langle(A\setminus B)\rangle\cup\langle B\rangle\big\rangle$. Thus we have

$$\langle A\rangle\subset(\sigma_1+\sigma_2)\cap\bigcup_{l=1}^{\infty}\frac{1}{kp^l}\mathbb{Z}^n$$

for some $k\in\mathbb{N}^*$, where $\sigma_2:=\varphi^{-1}(\sigma_0)$.
Because $\mathcal{U}_{\sigma_1+\sigma_2}=\mathcal{U}_{\sigma_1}\cap\mathcal{U}_{\sigma_2}$, $\langle(A\setminus B)\rangle\cup\langle B\rangle$ is $\preceq'$-well-ordered for every $\preceq'\in\mathrm{Ord}_n\cap\mathcal{U}_{\sigma_1+\sigma_2}$. Therefore $\langle A\rangle$ is $\preceq'$-well-ordered for every $\preceq'\in\mathrm{Ord}_n\cap\mathcal{U}_{\sigma_1+\sigma_2}$ by [Ne49, Theorem 3.4, p. 206]. This concludes the proof. □

Proof of Theorem 5.5. By Proposition 5.8 and Theorem 5.7, the set $\mathcal{S}^{\mathbb{K}}_{\preceq}$ is a Henselian valued field.
Assume that $\mathcal{S}^{\mathbb{K}}_{\preceq}$ is not algebraically closed. Then, by [Ra68, Lemma 4] there exists $a\in\mathcal{S}^{\mathbb{K}}_{\preceq}$ such that T^p-T-a is irreducible in $\mathcal{S}^{\mathbb{K}}_{\preceq}[T]$. Let us write

$$a=a^++a^-$$

where $\mathrm{Supp}(a^-)\subset\{b\in\mathbb{Q}^n\mid b\prec\underline{0}\}$ and $\mathrm{Supp}(a^+)\subset\{b\in\mathbb{Q}^n\mid b\succeq\underline{0}\}$. Because the map $b\longmapsto b^p$ is an additive map, if ξ^+ is a root of T^p-T-a^+ and ξ^- is root of T^p-T-a^-, then $\xi^++\xi^-$ is a root of T^p-T-a. We will prove that T^p-T-a^+ and T^p-T-a^- admit a root in $\mathcal{S}^{\mathbb{K}}_{\preceq}$ contradicting the fact that T^p-T-a is irreducible.

Since $\mathcal{S}^{\mathbb{K}}_{\preceq}$ is a Henselian valued field,

$$\mathfrak{O}:=\left\{\xi\in\mathcal{S}^{\mathbb{K}}_{\preceq}\mid\forall b\in\mathrm{Supp}(\xi),b\succeq\underline{0}\right\}$$

is a Henselian local ring with maximal ideal

$$\mathfrak{m}:=\left\{\xi\in\mathcal{S}^{\mathbb{K}}_{\preceq}\mid\forall b\in\mathrm{Supp}(\xi),b\succ\underline{0}\right\}.$$

The polynomial $T^p-T-a^+\in\mathfrak{O}[T]$ has a root modulo $\mathfrak{m}$ since $\mathbb{K}$ is algebraically closed (here $\mathfrak{O}/\mathfrak{m}=\mathbb{K}$). Moreover the derivative of this polynomial is -1. Thus this polynomial satisfies Hensel's Lemma and admits a root ξ^+ in $\mathcal{S}^{\mathbb{K}}_{\preceq}$.

In order to prove that T^p-T-a^- has a root in $\mathcal{S}^{\mathbb{K}}_{\preceq}$, we follow the proofs of [Ra68, Theorem 3], and [Sa17, Theorem 5.3]. We write $a^-=\sum_{q\in\mathbb{Q}^n}a^-_qx^q$ and we define

$$\xi^-:=\sum_{q\in\mathbb{Q}^n}\left(\sum_{i=1}^{\infty}\left(a^-_{p^iq}\right)^{\frac{1}{p^i}}\right)x^q.$$

We can verify that ξ^- is well defined: for a given $q \in \operatorname{Supp}(a^-)$, the sequence $(p^i q)_i$ is strongly decreasing for the order $\preceq$ since $q \prec \underline{0}$. Therefore $a^-_{p^i q} = 0$ for i large enough because $\operatorname{Supp}(a^-)$ is $\preceq$-well-ordered. Hence the sum $\sum_{i=1}^{\infty} \left(a^-_{p^i q}\right)^{\frac{1}{p^i}}$ is in fact a finite sum.
Exactly as done in the proof of [Sa17, Theorem 5.3], there exist a $\preceq$-positive cone σ and $\gamma \in \mathbb{Z}^n$ such that

$$\operatorname{Supp}(\xi^-) \subset (\gamma + \sigma) \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbb{Z}^n,$$

and for every order $\preceq' \in \operatorname{Ord}_n \cap \mathcal{U}_\sigma$, $\operatorname{Supp}(\xi^-)$ is $\preceq'$-well-ordered. Thus $\xi^- \in \mathcal{S}^{\mathbb{K}}_{\preceq}$. Moreover an easy computation shows that ξ^- is a root of $T^p - T - a^-$. This proves the theorem. $\square$

5.2. **Positive analogue of $\tau(\xi)$ in positive characteristic.** By Theorem 2.18, for a Laurent series ξ algebraic over $\mathbb{K}(\!(x)\!)$ where $\mathbb{K}$ is a field of characteristic zero, the cone $\tau(\xi)$ is the set of vectors $\omega \in \mathbb{R}_{\geq 0}{}^n$ such that $\operatorname{Supp}(\xi)$ is included in a translation of $\leq_\omega$-positive cone. But, in positive characteristic, Examples 5.2 and 5.3 show that the condition for the support of the series to be well-ordered for a given order is a crucial condition. Therefore we define the following cone, which agrees with $\tau(\xi)$ for a Laurent series ξ, and we will prove its rationality:

Definition 5.9. Let ξ be a series with support in $\mathbb{Q}^n$. We set

$$\begin{aligned}\widetilde{\tau}(\xi) = \{\omega \in \mathbb{R}_{\geq 0}{}^n \mid \exists \sigma \subset \langle\omega\rangle^\vee, \gamma \in \mathbb{Z}^n,\ \operatorname{Supp}(\xi) \subset \gamma + \sigma \text{ and}\\ \forall \preceq \in \operatorname{Ord}_n \cap \mathcal{U}_\sigma, \operatorname{Supp}(\xi) \text{ is } \preceq\text{-well-ordered}\}.\end{aligned}$$

Lemma 5.10. *For a Laurent series ξ whose support is included in a translation of a strongly convex rational cone σ, we have $\tau(\xi) = \widetilde{\tau}(\xi)$.*

Proof. Directly from the definitions we have $\widetilde{\tau}(\xi) \subset \tau(\xi)$.
Now let $\omega \in \tau(\xi)$. By Lemma 2.11, there is $\gamma \in \mathbb{Z}^n$ such that $\operatorname{Supp}(\xi) \subset \gamma + \sigma \cap \langle\omega\rangle^\vee$. Since ξ is a Laurent series, $\operatorname{Supp}(\xi)$ is $\preceq$-well-ordered for every $\preceq \in \operatorname{Ord}_n \cap \mathcal{U}_\sigma$. This means that $\omega \in \widetilde{\tau}(\xi)$, and the lemma is proved. $\square$

Then we have the following analogue of Theorem 1.3 in positive characteristic:

Theorem 5.11. *Let $\xi \in \mathcal{S}^{\mathbb{K}}_{\preceq}$ be algebraic over $\mathbb{K}(\!(x)\!)$, where $\mathbb{K}$ is a positive characteristic field and $\preceq \in \operatorname{Ord}_n$. Then $\widetilde{\tau}(\xi)$ is a strongly convex rational cone.*

Proof. Let P be the minimal polynomial of ξ, and let d denote its degree. By Theorem 5.5 for every order $\preceq' \in \operatorname{Ord}_n$ there are an element $\gamma_{\preceq'} \in \mathbb{Z}^n$, and a $\preceq'$-positive strongly convex rational cone $\sigma_{\preceq'}$ such that the roots of P can be expanded as series in $\mathcal{S}^{\mathbb{K}}_{\preceq'}$ with support in $\gamma_{\preceq'} + \sigma_{\preceq'}$. We may replace $\sigma_{\preceq'}$ by $\sigma_{\preceq'} + \mathbb{R}_{\geq 0}{}^n$ and assume that $\sigma_{\preceq'}$ contains the first orthant for every $\preceq'$. Moreover for every $\preceq'' \in \operatorname{Ord}_n \cap \mathcal{U}_{\sigma_{\preceq'}}$, the supports of these roots are $\preceq''$-well-ordered.

In particular we have $\mathrm{Ord}_n \subset \mathcal{U}_{\mathbb{R}_{\geq 0}{}^n} \subset \bigcup_{\preceq} \mathcal{U}_{\sigma_{\preceq}}$. Hence, by Theorem 2.16, we can extract from this family of cones $\sigma_{\preceq}$, a finite number of cones, denoted by σ_1, ..., σ_N, such that $\mathrm{Ord}_n \subset \bigcup_{k=1}^N \mathcal{U}_{\sigma_{\preceq_k}}$. Therefore, by Lemma 2.17, we have that ${\mathbb{R}_{\geq 0}}^n \subset \bigcup_{k=1}^N \sigma_k^\vee$. Because the $\sigma_{\preceq_k}$ contain ${\mathbb{R}_{\geq 0}}^n$, we have ${\mathbb{R}_{\geq 0}}^n = \bigcup_{k=1}^N \sigma_k^\vee$. Moreover these cones satisfy the following properties:

i) for every k there are d Laurent Puiseux series with support in $\gamma_k + \sigma_k$ for some $\gamma_k \in \mathbb{Z}^n$, denoted by $\xi_1^{(k)}, \ldots, \xi_d^{(k)}$ with $P(\xi_i^{(k)}) = 0$ for $i = 1, \ldots, d$,

ii) for every k, every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma_k}$ and every $i = 1, \ldots, d$, $\mathrm{Supp}(\xi_i^{(k)})$ is $\preceq'$-well-ordered.

Thus Lemma 5.12 given below implies (exactly as for Corollary 3.2) that, after renumbering the σ_k, there is an integer $l \leq N$ such that

$$\widetilde{\tau}(\xi) = \bigcup_{k=1}^{l} \sigma_k^\vee.$$

Therefore $\widetilde{\tau}(\xi)$ is a strongly convex rational cone. $\square$

Lemma 5.12. *Let ξ be a series belonging to $\mathcal{S}_{\preceq'}^{\mathbb{K}}$ for some $\preceq' \in \mathrm{Ord}_n$ and whose support is included in a translation of a strongly convex cone σ containing ${\mathbb{R}_{\geq 0}}^n$. Let $P \in \mathbb{K}[[x]][T]$ be a monic polynomial of degree d with $P(\xi) = 0$. Let $\sigma_0 \subset {\mathbb{R}_{\geq 0}}^n$ be a strongly convex rational cone such that*

i) *there are d distinct series with rational exponents whose support is in $\gamma + \sigma_0$ for some $\gamma \in \mathbb{Z}^n$, denoted by $\xi_1, \ldots, \xi_d$ with $P(\xi_i) = 0$ for $i = 1, \ldots, d$,*

ii) *Supp(ξ_j) is $\preceq'$-well-ordered for every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma_0}$.*

Then

$$\mathrm{Int}(\sigma_0^\vee) \cap \widetilde{\tau}(\xi) \neq \emptyset \Longrightarrow \sigma_0^\vee \subset \widetilde{\tau}(\xi).$$

Proof. Consider a non zero vector $\omega \in \mathrm{Int}(\sigma_0^\vee) \cap \widetilde{\tau}(\xi)$. Since $\omega \in \widetilde{\tau}(\xi)$, there are $k \in \mathbb{N}$, $\gamma_0 \in \mathbb{Z}^n$, and σ a $\leq_\omega$-positive rational cone, such that

$$\mathrm{Supp}(\xi) \subset (\gamma_0 + \sigma) \cap \bigcup_{l=0}^{\infty} \frac{1}{kp^l}\mathbb{Z}^n,$$

and $\forall \preceq \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma$, $\mathrm{Supp}(\xi)$ is $\preceq$-well-ordered. Since σ is $\leq_\omega$-positive and strongly convex, there exists an order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$ such that σ is $\preceq$-positive (see [AR19, Lemma 3.8]). Therefore $\mathrm{Supp}(\xi)$ is $\preceq$-well-ordered. Thus ξ is a root of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ since $\mathrm{Supp}(\xi)$ is $\preceq$-well-ordered.

On the other hand, ω is in the interior of $\sigma_0^\vee$, so σ_0 is $\preceq$-positive by Lemma 2.10. Therefore, because the supports of the ξ are $\preceq$-well-ordered, ii) implies that the ξ_i are the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$. Thus $\xi = \xi_i$ for some i. Hence there is some $\gamma'' \in \mathbb{Z}^n$ such that

$$\mathrm{Supp}(\xi) \subset \gamma'' + \sigma_0$$

and $\mathrm{Supp}(\xi)$ is $\preceq'$-well-ordered for every order $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma_0}$.

Now let $\omega' \in \sigma_0^\vee$. We have $\sigma_0 \subset \langle \omega' \rangle^\vee$. Hence $\omega' \in \widetilde{\tau}(\xi)$. This proves the lemma. $\square$

Now we are able to prove the following analogue of Theorem 1.4 i) and iii):

Theorem 5.13. *Let $\xi \in \mathcal{S}^{\mathbb{K}}_{\preceq'}$ for some $\preceq' \in \mathrm{Ord}_n$ that is algebraic over $\mathbb{K}((x))$.*

i) *There exists $\gamma \in \mathbb{Z}^n$ such that*

$$Supp(\xi) \subset \gamma + \widetilde{\tau}(\xi)^\vee$$

and for every $\preceq \in \mathrm{Ord}_n \cap \mathcal{U}_{\widetilde{\tau}(\xi)}$, $Supp(\xi)$ is $\preceq$-well-ordered.

ii) *Let σ be a cone such that*

$$Supp(\xi) \subset \gamma + \sigma$$

for some γ, and such that for every $\preceq \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma^\vee}$, $Supp(\xi)$ is $\preceq$-well-ordered. Then $\widetilde{\tau}(\xi)^\vee \subset \sigma$.

Proof. Let P be the minimal polynomial of ξ and let d be its degree. As shown in the proof of Theorem 5.11, there exist strongly convex rational cones σ_k containing ${\mathbb{R}_{\geq 0}}^n$, $k = 1, \ldots, l$, satisfying the following properties:

i) $\widetilde{\tau}(\xi) = \bigcup_{k=1}^l \sigma_k^\vee$,

ii) for every k there are d Laurent Puiseux series with support in $\gamma_k + \sigma_k$ for some $\gamma_k \in \mathbb{Z}^n$, denoted by $\xi_1^{(k)}, \ldots, \xi_d^{(k)}$ with $P(\xi_i^{(k)}) = 0$ for $i = 1, \ldots, d$,

iii) for every k, every $\preceq' \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma_k}$ and every $i = 1, \ldots, d$, $\mathrm{Supp}(\xi_i^{(k)})$ is $\preceq'$-well-ordered.

Let $k \in \{1, \ldots, l\}$ and let $\omega \in \mathrm{Int}(\sigma_k^\vee)$. Since $\sigma_k^\vee \subset \widetilde{\tau}(\xi)$, there is a rational strongly convex cone $\sigma \subset \langle \omega \rangle^\vee$ and $\gamma \in \mathbb{Z}^n$ such that

$$\mathrm{Supp}(\xi) \subset \gamma + \sigma$$

and $\mathrm{Supp}(\xi)$ is $\preceq$-well-ordered for every $\preceq \in \mathrm{Ord}_n \cap \mathcal{U}_\sigma$. Let $\preceq_0 \in \mathrm{Ord}_n$ such that σ is $\preceq_0$-positive and $\preceq_0$ refines $\leq_\omega$. Such a $\preceq_0$ exists by [AR19, Lemma 3.8]. By definition, $\xi \in \mathcal{S}^{\mathbb{K}}_{\preceq_0}$.

On the other hand, P has d distinct roots whose supports are in $\gamma_k + \sigma_k$ for some $\gamma_k \in \mathbb{Z}^n$ and are $\preceq$-well-ordered for every $\preceq \in \mathrm{Ord}_n \cap \mathcal{U}_{\sigma_k}$. By Lemma 2.10, σ_k is $\preceq_0$-positive. Therefore these d roots are in $\mathcal{S}^{\mathbb{K}}_{\preceq_0}$, hence one of them is equal to ξ.

This shows that $\mathrm{Supp}(\xi) \subset \gamma_k + \sigma_k$ for every k. Therefore, by Lemma 2.11, there is $\gamma \in \mathbb{Z}^n$ such that

$$\mathrm{Supp}(\xi) \subset \gamma + \widetilde{\tau}(\xi)^\vee$$

because $\widetilde{\tau}(\xi)^\vee = \bigcap_{k=1}^l \sigma_k$.

Moreover, for $\preceq \in \mathcal{U}_{\widetilde{\tau}(\xi)^\vee}$, there is $\omega \in \widetilde{\tau}(\xi)$ such that $\preceq$ refines $\leq_\omega$. Hence, by definition of $\widetilde{\tau}(\xi)$, $\mathrm{Supp}(\xi)$ is $\preceq$-well-ordered. This proves i).

Now let σ as in ii). Let $\omega \in \sigma^\vee$. By the assumption on σ, $\omega \in \widetilde{\tau}(\xi)$. Therefore $\sigma^\vee \subset \widetilde{\tau}(\xi)$ and $\widetilde{\tau}(\xi)^\vee \subset \sigma$. This proves ii). □

Remark 5.14. Let us consider the algebraic series ξ of Example 5.2. In this case $\widetilde{\tau}(\xi)$ is the dual of the cone generated by $(0, 1)$ and $(1, -1)$. Therefore we have $\widetilde{\tau}(\xi) \subsetneq \tau(\xi)$.

Example 5.15. Still in Example 5.2 the series ξ satisfies Theorem 1.4 ii) a) and b) by replacing $\tau(\xi)$ with $\widetilde{\tau}(\xi)$: we only need to remove the constant term of ξ, and add a series in y, in order to obtain a series whose support is $(0,1)+\widetilde{\tau}(\xi)^\vee$, and both faces of $(0,1)+\widetilde{\tau}(\xi)^\vee$ contain infinitely many exponents of this series.
Now consider the series $\xi' = \xi + f\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right)$ where $f(t) = \sum_{k=1}^\infty t^{1-\frac{1}{p^k}}$ as in Example 5.2 (each element of the support of ξ' is given by a black dot in Figure 5).

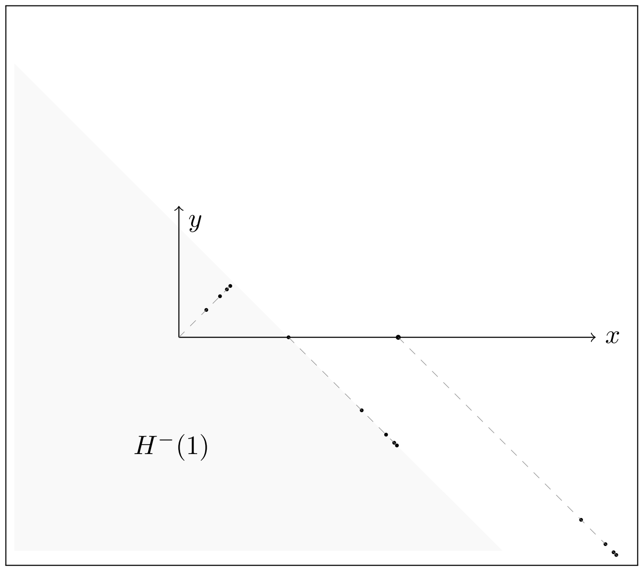

FIGURE 5.

Then ξ' is algebraic over $\mathbb{K}(\!(x)\!)$. We remark that $\operatorname{Supp}(\xi')$ contains the sequence

$$\left(\left(\frac{1}{2},\frac{1}{2}\right)-\left(\frac{1}{2p^k},\frac{1}{2p^k}\right)\right)_{k\in\mathbb{N}}.$$

Here $\widetilde{\tau}(\xi') = \widetilde{\tau}(\xi)$. For $\omega = (1,-1)$ (which is in the boundary of $\widetilde{\tau}(\xi')$) and $s\in\mathbb{R}$, we define

$$H(s) := \{x\in\mathbb{R}^n \mid x\cdot\omega = s\},\ H^-(s) = \{x\in\mathbb{R}^n \mid x\cdot\omega < s\}.$$

Then we see that, for $s<1$, the sets $\operatorname{Supp}(\xi)\cap H^-(s)$ and $\operatorname{Supp}(\xi')\cap H(s)$ are finite. But $\operatorname{Supp}(\xi')\cap H^-(1)$ is infinite ($H^-(1)$ is the grey area in Figure 5).
Therefore the series ξ' does not satisfy Theorem 1.4 i) b), even by replacing $\tau(\xi)$ with $\widetilde{\tau}(\xi)$ in this statement.

6. PROOFS OF THEOREMS 1.3 AND 1.4 IN POSITIVE CHARACTERISTIC

In this section we explain why Theorem 1.3 and 1.4 remain valid in positive characteristic (for ξ a power series with integer exponents).

Proposition 6.1. *Let $\omega \in {\mathbb{R}_{\geq 0}}^n$ and $P \in \mathbb{K}[[x]][T]$. There exists a finite set $\mathcal{T}_\omega$ of strongly convex rational cones such that:*

i) *for any order $\preceq \in \mathrm{Ord}_n$ refining $\leq_\omega$, there is $\sigma \in \mathcal{T}_\omega$, σ being $\preceq$-positive, such that the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ have support in a translation of σ,*

ii) *for every $\sigma \in \mathcal{T}_\omega$ and $\omega' \in \sigma^\vee$, the supports of the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ are $\preceq'$-well-ordered for every $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$.*

Moreover for a given $\omega \in {\mathbb{R}_{\geq 0}}^n$ and a given finite set of cones $\mathcal{T}_\omega$ satisfying the former property, for every ω'' close enough to ω, we can choose $\mathcal{T}_{\omega''} = \mathcal{T}_\omega$.

Proof. By Theorem 5.5, for every $\preceq \in \mathrm{Ord}_n$ there is a cone $\sigma_{\preceq}$ such that the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ have support in a translation of $\sigma_{\preceq}$, and for every $\omega' \in \sigma_{\preceq}^\vee$, the supports of the roots of P in $\mathcal{S}_{\preceq}^{\mathbb{K}}$ are $\preceq'$-well-ordered, for every $\preceq' \in \mathrm{Ord}_n$ refining $\leq_{\omega'}$.
Then $\mathrm{Ord}_n = \bigcup_{\preceq \in \mathrm{Ord}_n} \mathcal{U}_{\sigma_{\preceq}}$. Thus, by Theorem 2.16, there exists a finite set of orders $\preceq_1, \ldots, \preceq_N$ such that $\mathrm{Ord}_n = \bigcup_{i=1}^N \mathcal{U}_{\sigma_{\preceq_i}}$. Therefore (as shown in the proof of Proposition 4.7) the set $\mathcal{T}_\omega = \{\sigma_{\preceq_1}, \ldots, \sigma_{\preceq_N}\}$ satisfies the desired property.
And the last claim follows from Corollary 2.13. □

Now let ξ be a Laurent series (that is, with integer exponents) whose support is included in a translation of a strongly convex cone containing ${\mathbb{R}_{\geq 0}}^n$ and with coefficients in a positive characteristic field $\mathbb{K}$. Assume that ξ is algebraic over $\mathbb{K}(\!(x)\!)$. Then Theorem 1.3 remains valid. The proof given in characteristic zero is no longer valid by using the definition of $\mathcal{S}_{\preceq}^{\mathbb{K}}$ given in Definition 5.4, because Lemma 3.1 does not hold anymore in positive characteristic (see Example 5.3). But Theorem 1.3 comes form Theorem 5.11 and Lemma 5.10
Moreover Theorem 1.4 also remains valid. Indeed, we can also define $\tau_0(\xi)$, $\tau_1(\xi)$, $\tau_0'(\xi)$ and $\tau_1'(\xi)$ for such a ξ. Proposition 4.6 is still valid. Moreover we can prove that $\tau_0(\xi)$ and $\tau_1(\xi)$ are open, exactly as in the zero characteristic case, by using Proposition 6.1. Therefore Corollary 4.8 and Lemma 4.9 remain valid in positive characteristic.

7. Gaps in the expansion of algebraic Laurent series

The aim of this part is to prove Theorem 1.5.

Proof of Theorem 1.5. First we can multiply ξ by a monomial and assume that $k(0) \geq 0$. This does not affect neither the hypothesis, neither the conclusion of Theorem 1.5.
By Theorem 1.4 there exist a finite set $C \subset \mathbb{Z}^n$, a Laurent polynomial $p(x)$, and a power series $f(x) \in \mathbb{K}[[x]]$ such that $\mathrm{Supp}(\xi + p(x) + f(x)) \subset C + \tau(\xi)^\vee$. Moreover, for every one dimensional face σ of $\tau(\xi)^\vee$, there exists $\gamma_\sigma \in C$ such that the set $\gamma_\sigma + \sigma$ is a one dimensional face of $\mathrm{Conv}(C + \tau(\xi)^\vee)$, and

$$\#\big(\mathrm{Supp}(\xi + p(x) + f(x)) \cap (\gamma_\sigma + \sigma)\big) = \infty. \tag{6}$$

From now on we replace ξ by $\xi + p(x)$ allowing us to assume that $p(x) = 0$. This does not change the hypothesis, nor the conclusion of the theorem.
If $\tau(\xi)^\vee = {\mathbb{R}_{\geq 0}}^n$, then $\operatorname{Supp}(\xi) \subset \gamma + \mathbb{N}^n$ for some $\gamma \in \mathbb{Z}$ by Theorem 1.4. This means that $\xi = x^\gamma f(x)$ where $f(x) \in \mathbb{K}[[x]]$, hence $\xi \in \mathbb{K}[[x]]_{(x)}$, contradicting the hypothesis. Therefore, by assumption, ${\mathbb{R}_{\geq 0}}^n \subsetneq \tau(\xi)^\vee$ and $\tau(\xi) \subsetneq {\mathbb{R}_{\geq 0}}^n$. In particular $\tau(\xi)^\vee$ has a one dimensional face σ that is not included in one of the coordinate axis. Let $\omega' \in \sigma^\perp \cap \operatorname{Int}(\tau(\xi))$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\omega'_1 + \cdots + \mathbb{Q}\omega'_n) = n-1$. Since σ is not included in one of the coordinate axis, we have $\omega' \in {\mathbb{R}_{>0}}^n$.
Let g be a Laurent series with support in $C + \tau(\xi)^\vee$. Because $\omega' \in \tau(\xi)$, by Lemma 2.19, for every $t \in \mathbb{R}$, the set

$$\{u \cdot \omega' \mid u \in \operatorname{Supp}(g)\} \cap \,]-\infty, t]$$

is finite. Therefore, if we write $g = \sum_{\alpha \in \mathbb{Z}^n} g_\alpha x^\alpha$,

$$\nu_{\omega'}(g) = \min\{\alpha \cdot \omega' \mid g_\alpha \neq 0\}$$

is well defined. The function $\nu_{\omega'}$ is a monomial valuation. For such a Laurent series g we denote by $\operatorname{in}_{\omega'}(g)$ the initial term of g for the valuation $\nu_{\omega'}$, that is:

$$\operatorname{in}_{\omega'}(g) := \sum_{\alpha \in \mathbb{Z}^n,\, \alpha\cdot\omega' = \nu_{\omega'}(g)} g_\alpha x^\alpha.$$

Since ξ is algebraic over $\mathbb{K}[[x]]$, $\xi + f(x)$ also, and there exist an integer d and formal power series $a_0, \ldots, a_d \in \mathbb{K}[[x]]$ such that

$$a_d(\xi + f(x))^d + \cdots + a_1(\xi + f(x)) + a_0 = 0.$$

Thus

$$\sum_{i \in E} \operatorname{in}_{\omega'}(a_i) \operatorname{in}_{\omega'}(\xi + f(x))^i = 0 \tag{7}$$

where

$$E = \{i \in \{0, \ldots, n\} \;/\; \nu_{\omega'}(a_i(\xi + f(x))^i) = \min_j \nu_{\omega'}(a_j(\xi + f(x))^j)\}.$$

For a given $t \geq 0$, the set $\{\alpha \in \mathbb{N}^n \mid \alpha \cdot \omega' = t\}$ is finite because $\omega' \in \mathbb{R}_{>0}$. Therefore, since the a_i are in $\mathbb{K}[[x]]$, the $\operatorname{in}_{\omega'}(a_i)$ are polynomials.
We set $b_i := \operatorname{in}_{\omega'}(a_i) \in \mathbb{K}[x]$ for every i and $\xi' := \operatorname{in}_{\omega'}(\xi + f(x))$. We have

$$\sum_{i \in E} b_i \xi'^i = 0.$$

Now, for every Laurent series g with support in $C + \tau(\xi)^\vee$, because $\omega \in \operatorname{Int}(\tau(\xi))$, we can expand g as $g = \sum_{i \in \mathbb{N}} g_{k(i)}$ where

i) for every $l \in \Gamma = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$, g_l is a sum of finitely many monomials of the form cx^α with $\omega \cdot \alpha = l$,
ii) the sequence $k(i)$ is a strictly increasing sequence of elements of Γ,
iii) for every integer i, $g_{k(i)} \neq 0$.

Therefore we consider the following corresponding expansions:

$$\xi=\sum_{i\in\mathbb{N}}\xi_{k_1(i)},\ \mathrm{in}_{\omega'}(\xi)=\sum_{i\in\mathbb{N}}a_{k_2(i)},\ \xi'=\sum_{i\in\mathbb{N}}\xi'_{k_3(i)}.$$

We claim that it is enough to show the existence of a constant $K>0$ such that

$$\forall i\in\mathbb{N},\ k_2(i+1)\leq Kk_2(i).$$

Indeed, assume that such a constant exists. Because $\mathrm{Supp}(\mathrm{in}_{\omega'}(\xi))\subset\mathrm{Supp}(\xi)$, for every $i\in\mathbb{N}$, there is an integer $n(i)\in\mathbb{N}$ such that $k_2(i)=k_1(n(i))$. Let us fix $i\in\mathbb{N}$ large enough for insuring that $k_1(i)\geq k_2(0)$. For such a i, we denote by j the largest integer such that $k_2(j)\leq k_1(i)$. Therefore we have

$$k_2(j)\leq k_1(i)\leq k_1(i+1)\leq k_2(j+1),$$

Thus

$$\frac{k_1(i+1)}{k_1(i)}\leq\frac{k_2(j+1)}{k_2(j)}\leq K.$$

This proves the claim.

Now we remark that, for a given $t\in\mathbb{R}$, the set of monomials in the expansion of f of the form cx^α, with $\alpha\cdot\omega'=t$, is finite because $\omega'\in{\mathbb{R}_{>0}}^n$. Therefore $\mathrm{in}_{\omega'}(\xi)$ and $\xi'=\mathrm{in}_{\omega'}(\xi+f(x))$ differ only by a finite number of monomials. Therefore there is constant $K_2>0$ such that

$$\forall i\in\mathbb{N},\ k_2(i+1)\leq K_2k_2(i)$$

if and only if there is a constant $K_3>0$ such that

$$\forall i\in\mathbb{N},\ k_3(i+1)\leq K_3k_3(i).$$

Therefore we only need to prove that the theorem is valid for ξ'.

Let $N\in\mathbb{N}$ and set ${\xi'}^{(N)}:=\sum_{i\leq N}\xi'_{k_3(i)}$. We set $P(T)=\sum_{i\in E}b_iT^i$, $d:=\deg_T(P(T))$ and let ν be the maximum of the $\nu_\omega(x^\alpha)$ where α runs over the exponents of the b_i. Then we have $P({\xi'}^{(N)})\neq 0$ for N large enough. We have

$$\frac{P({\xi'}^{(N)})}{{\xi'}^{(N)}-\xi'}=\frac{P({\xi'}^{(N)})-P(\xi')}{{\xi'}^{(N)}-\xi'}=\sum_{i\in E}b_i\left({{\xi'}^{(N)}}^i+{{\xi'}^{(N)}}^{i-1}\xi'+\cdots+{\xi'}^i\right).$$

Because the valuation of the right side term is positive, the valuation of $P({\xi'}^{(N)})$ is greater than the valuation of ${\xi'}^{(N)}-\xi'$. However the maximal valuation of the monomials of $P({\xi'}^{(N)})$ is $\nu+dk_3(N)$. Since the valuation of $\xi'-{\xi'}^{(N)}$ is $k_3(N+1)$ we have that

$$k_3(N+1)\leq\nu+dk_3(N)\leq(\nu+d)k_3(N).$$

This proves the result. □

Remark 7.1. Let us mention that this result was already known for power series of $\mathbb{K}[[x]]$ algebraic over $\mathbb{K}[x]$ (see [Sc33, Hilfssatz 5]).

Remark 7.2. This result can be strengthened in characteristic zero in the sense that the differences $k(i+1) - k(i)$ are uniformly bounded (see [AR19, Theorem 6.4]). However this statement is sharp in positive characteristic. For instance the series

$$\xi = \sum_{i \in \mathbb{N}} \left(\frac{x}{y}\right)^{p^i}$$

is a root of the polynomial $T^p - T + \frac{x}{y}$ over a field of characteristic $p > 0$.

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