

The analogue of Izumi's Theorem for Abhyankar valuations

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Dedicated to the memory of Shreeram Abhyankar and David Rees.

ABSTRACT

A well known theorem of Shuzo Izumi, strengthened by David Rees, asserts that all the divisorial valuations centered in an analytically irreducible local noetherian ring (R, \mathfrak{m}) are linearly comparable to each other. This is equivalent to saying that any divisorial valuation ν centered in R is linearly comparable to the \mathfrak{m} -adic order. In the present paper we generalize this theorem to the case of Abhyankar valuations ν with archimedean value semigroup Φ . Indeed, we prove that in a certain sense linear equivalence of topologies characterizes Abhyankar valuations with archimedean semigroups, centered in analytically irreducible local noetherian rings. In other words, saying that R is analytically irreducible, ν is Abhyankar and Φ is archimedean is equivalent to linear equivalence of topologies plus another condition called weak noetherianity of the graded algebra $\text{gr}_\nu R$.

We give some applications of Izumi's theorem and of Lemma 2.5, which is a crucial step in our proof of the main theorem. We show that some of the classical results on equivalence of topologies in noetherian rings can be strengthened to include linear equivalence of topologies. We also prove a new comparison result between the \mathfrak{m} -adic topology and the topology defined by the symbolic powers of an arbitrary ideal.

1. Introduction

Let $(R, \mathfrak{m}, \mathbb{k})$ be a local noetherian domain with the maximal ideal \mathfrak{m} and residue field \mathbb{k} . Let \mathbb{K} denote the field of fractions of R . Consider a valuation $\nu : \mathbb{K}^* \rightarrow \Gamma$ of \mathbb{K} with value group Γ . We denote by R_ν its valuation ring and by \mathfrak{m}_ν its maximal ideal.

DEFINITION 1. We say that ν is centered in R if ν is non-negative on R and strictly positive on \mathfrak{m} .

Consider a valuation $\nu : R \rightarrow \Gamma$, centered in R . Then $\mathfrak{m}_\nu \cap R = \mathfrak{m}$; thus \mathbb{k} is a subfield of $\mathbb{k}_\nu := \frac{R_\nu}{\mathfrak{m}_\nu}$.

DEFINITION 2. We say that ν is a *divisorial valuation* if its value group $\Gamma = \mathbb{Z}$ and $\text{tr.deg}_{\mathbb{k}} \mathbb{k}_\nu = \dim R - 1$.

The purpose of this paper is to generalize the following theorem of Shuzo Izumi and David Rees (often called Izumi's Theorem for short) to a larger class of valuations than just the divisorial ones.

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THEOREM 1.1. [7], [14] *Let R be an analytically irreducible local domain. Then for any two divisorial valuations ν and ν' , centered in R , there exists a constant $k > 0$ such that*

$$\nu(f) \leq k\nu'(f) \quad \forall f \in R.$$

This result has played a central role in the study of ideal-adic topologies and other questions about commutative rings during the last decades.

To highlight its applications, we start with some basic definitions and then recall two related theorems due to David Rees.

Let R be a commutative noetherian ring and I an ideal of R . For an element $f \in R$ the I -order of f is defined to be

$$I(f) := \max\{n \in \mathbb{N} \mid f \in I^n\}.$$

This function takes its values in $\mathbb{N} \cup \{\infty\}$ and $I(f) = \infty$ if and only if $f \in I^n$ for all $n \in \mathbb{N}$. Moreover it is easy to see that $I(fg) \geq I(f) + I(g)$ for all $f, g \in R$ since I is an ideal.

Next we introduce a more invariant notion, defined by David Rees and Pierre Samuel, namely the *reduced order*:

$$\bar{I}(f) := \lim_{n \rightarrow \infty} \frac{I(f^n)}{n}.$$

A priori, it is not obvious that $\bar{I}(f)$ is a rational number, or even finite. The fact that $\bar{I}(f)$ is always rational in a noetherian domain R is a consequence of the following theorem of D. Rees:

THEOREM 1.2. [11]/[12] *For any ideal I in a noetherian domain R there exists a unique finite set of valuations $\{\nu_i\}_{1 \leq i \leq r}$ of R (with values in \mathbb{Z}) such that*

$$\bar{I}(f) = \min_{1 \leq i \leq r} \frac{\nu_i(f)}{\nu_i(I)}$$

and this representation is irredundant. These valuations ν_i are called the *Rees valuations* of the ideal I .

REMARK 1. This Theorem has been stated and proved by Rees assuming only that R is noetherian (not necessarily a domain); here we restrict ourselves to the case of domains in order to simplify the exposition.

REMARK 2. In view of Theorem 1.2, Theorem 1.1 can be reformulated as follows: every divisorial valuation ν is linearly equivalent to the \mathfrak{m} -adic order. This means that there exists a positive integer k such that $\nu(f) < k\mathfrak{m}(f)$ for all $f \in R$.

In the case R is a local domain and $I = \mathfrak{m}$, the valuations ν_i appearing in the statement of Theorem 1.2 are divisorial valuations centered in R .

From the definitions we see easily that $\bar{I}(f) \geq I(f)$ for all f . We will need the following result of D. Rees in order to derive some corollaries of Theorem 1.1 about ideal-adic topologies in noetherian rings:

THEOREM 1.3. [13] *Let R be an analytically unramified local ring. Then there exists a constant $C > 0$ such that*

$$\bar{I}(f) \leq I(f) + C \quad \forall f \in R.$$

The goal of this paper is to generalize Theorem 1.1 to a larger class of valuations: the Abhyankar valuations whose semigroup is archimedean.

Let us begin with some definitions. Let \mathbb{K} denote the field of fractions of a local domain R . Consider a valuation $\nu : \mathbb{K}^* \rightarrow \Gamma$ of \mathbb{K} with value group Γ , centered in R . Let R_ν denote the valuation ring of ν and \mathfrak{m}_ν the maximal ideal of R_ν . Since ν is centered in R , we have a natural injection $\mathbb{k} \subset \frac{R_\nu}{\mathfrak{m}_\nu}$. The three basic invariants associated with ν are

$$\text{tr.deg}_{\mathbb{k}} \nu := \text{tr.deg} \left(\frac{R_\nu}{\mathfrak{m}_\nu} / \mathbb{k} \right) \tag{1.1}$$

$$\text{rat.rk } \nu := \dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}) \tag{1.2}$$

$$\text{rk } \nu := \dim R_\nu, \tag{1.3}$$

where “dimension” means the Krull dimension. In 1956 S. Abhyankar proved that

$$\text{rat.rk } \nu + \text{tr.deg}_{\mathbb{k}} \nu \leq \dim R \tag{1.4}$$

(cf. [1], Theorem 1, p. 330). This inequality is called the *Abhyankar Inequality*. An *Abhyankar valuation* is a valuation centered in R such that the above inequality is an equality. Any divisorial valuation centered in R satisfies $\text{rat.rk } \nu = 1$ and $\text{tr.deg}_{\mathbb{k}} \nu = \dim R - 1$; hence it is an Abhyankar valuation.

Let

$$\Phi := \nu(R \setminus \{0\}) \subset \Gamma.$$

Then Φ is an ordered semigroup contained in Γ . Since R is noetherian, Φ is a well-ordered set. For $\alpha \in \Phi$, let

$$p_\alpha := \{x \in R \mid \nu(x) \geq \alpha\}$$

and

$$p_{\alpha+} := \{x \in R \mid \nu(x) > \alpha\}$$

(here we adopt the convention that $\nu(0) > \alpha$ for any $\alpha \in \Gamma$). Ideals of R which are contractions to R of ideals in R_ν are called ν -ideals. All of the p_α and $p_{\alpha+}$ are ν -ideals and $\{p_\alpha\}_{\alpha \in \Phi}$ is the complete list of ν -ideals in R . Specifying all the ν -ideals in R is equivalent to specifying ν (see [20], Appendix 3). The following is a characterization of ν -ideals: an ideal $I \subset R$ is a ν -ideal if and only if, for any elements $a \in R$, $b \in I$ such that $\nu(a) \geq \nu(b)$ we have $a \in I$.

We associate to ν the following graded algebra:

$$\text{gr}_\nu R := \bigoplus_{\alpha \in \Phi} \frac{p_\alpha}{p_{\alpha+}}.$$

DEFINITION 3. We say that Φ is *archimedean* if for any $\alpha, \beta \in \Phi$, $\alpha \neq 0$, there exists $r \in \mathbb{N}$ such that $r\alpha > \beta$.

This is equivalent to saying that every ν -ideal in R is \mathfrak{m} -primary and weaker than saying that $\text{rk } \nu = 1$. For instance the monomial valuation ν with values in $\mathbb{Z}_{\leq x}^2$ defined on $R = \mathbb{C}[[x, y]]$

by

$$\nu(x) = (1, 0) \text{ and } \nu(y) = (1, 1)$$

satisfies $\text{rk } \nu = 2$ but its semi-group $\nu(R \setminus \{0\}) = \{(n, m) \in \mathbb{Z}^2 / n \geq m \geq 0\}$ is archimedean.

For $l \in \mathbb{N}$, let Q_l denote the ν -ideal

$$Q_l := \{x \in R \mid \nu(x) \geq l\nu(\mathfrak{m})\}. \tag{1.5}$$

Of course, $\mathfrak{m}^l \subset Q_l$ for all $l \in \mathbb{N}$.

DEFINITION 4. We say that the ν -adic and the \mathfrak{m} -adic topologies are linearly equivalent if there exists $r \in \mathbb{N}$ such that

$$Q_{rl} \subset \mathfrak{m}^l$$

for all $l \in \mathbb{N}$.

Thus Theorem 1.1 is equivalent to saying that for any divisorial valuation ν of R the ν -adic topology is linearly equivalent to the \mathfrak{m} -adic topology.

Let $A = \bigoplus_{\alpha \in \Phi} A_\alpha$ be a Φ -graded \mathbb{k} -algebra. Assume that $A_0 = \mathbb{k}$. By abuse of notation, let us denote

$$1 := \nu(\mathfrak{m})$$

and for $l \in \mathbb{N}$

$$l := l \cdot 1.$$

DEFINITION 5. We say that A is weakly noetherian of dimension d if A contains d algebraically independent elements and the function

$$F(l) = \sum_{0 \leq \alpha \leq l} \dim_{\mathbb{k}} A_\alpha$$

is bounded above by a polynomial in l of degree d .

If A is weakly noetherian then $\dim_{\mathbb{k}} A_\alpha < \infty$ for all $\alpha \in \Phi$. We now state the main theorem of this paper.

THEOREM 1.4. *Let $(R, \mathfrak{m}, \mathbb{k})$ be a local noetherian domain with field of fractions \mathbb{K} . Let ν be a valuation of \mathbb{K} centered in R with value semigroup Φ . Then the following two conditions are equivalent:*

(i) *R is analytically irreducible, Φ is archimedean and*

$$\text{rat.rk } \nu + \text{tr.deg}_{\mathbb{k}} \nu = \dim R \tag{1.6}$$

(ii) *the ν -adic and the \mathfrak{m} -adic topologies are linearly equivalent and $gr_\nu R$ is weakly noetherian.*

This theorem is proved in Part 2. Part 3 is devoted to applications of Izumi's Theorem and of Lemma 2.5.

The most difficult part in the proof of Theorem 1.4 is the implication (1) \implies "linear equivalence between the ν -adic and the \mathfrak{m} -adic topologies". Let us mention that this implication

has been proved by D. Cutkosky [2] in the case R contains a characteristic zero field. More precisely he proved in this case that an Abhyankar valuation is quasi-monomial after a sequence of blow-ups. B. Teissier proved a similar statement for a complete equicharacteristic local domain R when the residue field of the Abhyankar valuation is equal to the residue field of R and is algebraically closed (see Proposition 6.24 [18]). Thus in these cases, by Theorem 1.1, the ν -adic and the \mathfrak{m} -adic topologies are linearly equivalent. This implication has also been proved in the case when (R, \mathfrak{m}) is a complete local ring containing a field of positive characteristic and ν is the composition of a morphism $R \rightarrow S$, where (S, \mathfrak{m}_0) is a regular complete local ring with $\frac{S}{\mathfrak{m}_0} \simeq \frac{R}{\mathfrak{m}}$, and of the \mathfrak{m}_0 -adic valuation of S (see [15]).

Theorem 1.4 has been announced by the second author and a proof was sketched in [16] without details but the entire proof was never published. The proof presented here follows the sketched proof announced in [16].

1.1. Conventions

Let R be an integral domain with field of fractions \mathbb{K} . Let ν be a valuation of \mathbb{K} with value group Γ .

The notation R_ν will stand for the valuation ring of ν (the ring of all the elements of \mathbb{K} whose values are non-negative) and \mathfrak{m}_ν the maximal ideal of R_ν (the elements with strictly positive values). If $p \subset R$ is an ideal, we put

$$\nu(p) := \min \{ \nu(x) \mid x \in p \}.$$

If R is noetherian, this minimum is always achieved.

If x is an element of R , \bar{x} will always mean the natural image of x in $\text{gr}_\nu R$.

We freely use the multi-index notation: if $x = (x_1, \dots, x_n)$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, then x^α stands for $\prod_{i=1}^n x_i^{\alpha_i}$. The symbol $|\alpha|$ will stand for $\sum_{i=1}^n \alpha_i$.

Let R be a regular local ring with regular system of parameters $x = (x_1, \dots, x_n)$. A valuation ν , centered at R , is said to be *monomial* with respect to x if all the ν -ideals of R are generated by monomials in x .

2. Proof of Theorem 1.4

The purpose of this section is to prove Theorem 1.4. We start with a few remarks.

REMARK 3. Suppose the \mathfrak{m} -adic and the ν -adic topologies in R are equivalent. Then Φ is archimedean. Indeed, to say that Φ is not archimedean is equivalent to saying that there exists a ν -ideal p in R which is not \mathfrak{m} -primary. But then p is open in the ν -adic topology, but not in the \mathfrak{m} -adic topology, which is a contradiction.

Moreover, the equivalence of topologies implies that R is analytically irreducible. Indeed, let $\{a_n\}, \{b_n\}$ be two Cauchy sequences for the \mathfrak{m} -adic topology in R such that

$$\lim_{n \rightarrow \infty} a_n \neq 0 \tag{2.1}$$

$$\lim_{n \rightarrow \infty} b_n \neq 0, \tag{2.2}$$

but

$$\lim_{n \rightarrow \infty} a_n b_n = 0.$$

By the equivalence of topologies, (2.1) and (2.2), $\nu(a_n)$ and $\nu(b_n)$ are independent of n for $n \gg 0$, hence so is $\nu(a_n b_n)$, which contradicts the fact that

$$\lim_{n \rightarrow \infty} a_n b_n = 0$$

in the ν -adic topology.

REMARK 4. The fact that Φ is archimedean together with the equality (1.6) implies that the \mathfrak{m} -adic and the ν -adic topologies are equivalent (see [19], pp. 63–64).

LEMMA 2.1. *Let R be any local domain whatsoever and ν any valuation of the field of fractions, centered in R . Let x_1, \dots, x_r be elements of R such that $\{\nu(x_i)\}_{1 \leq i \leq r}$ are linearly independent over \mathbb{Z} . Let y_1, \dots, y_t be elements of R_ν such that the natural images \bar{y}_i of the y_i in $\frac{R_\nu}{\mathfrak{m}_\nu}$ are algebraically independent over \mathbb{k} . Assume that there exists a monomial $b = x^\omega$ such that $by_i \in R$ for all i , $1 \leq i \leq t$. Then the natural images of $x_1, \dots, x_r, by_1, \dots, by_t$ in $\text{gr}_\nu R$ are algebraically independent over \mathbb{k} .*

Proof. The algebra $\text{gr}_\nu R$ is an integral domain, on which ν induces a natural valuation, which we shall also denote by ν . Let

$$z_i := by_i, \quad 1 \leq i \leq s.$$

Consider an algebraic relation

$$\sum_{\alpha, \beta} \bar{c}_{\alpha, \beta} \bar{x}^\alpha \bar{z}^\beta = 0, \quad \alpha \in \mathbb{N}_0^r, \beta \in \mathbb{N}_0^s, \bar{c}_{\alpha, \beta} \in \mathbb{k}, \quad (2.3)$$

where the $\bar{c}_{\alpha, \beta}$ are not all zero. Here \bar{x}_i, \bar{z}_i denote the natural images in $\text{gr}_\nu R$ of x_i and z_i , respectively. We may assume that (2.3) is homogeneous with respect to ν . At least two of the $\bar{c}_{\alpha, \beta}$ must be non-zero. Take a pair $(\alpha, \beta), (\gamma, \delta) \in \mathbb{N}_0^{s+r}$ such that

$$\bar{c}_{\alpha, \beta} \neq 0 \quad (2.4)$$

$$\bar{c}_{\gamma, \delta} \neq 0. \quad (2.5)$$

Then

$$\bar{x}^\alpha \bar{b}^{|\beta|} = \bar{x}^\gamma \bar{b}^{|\delta|}$$

otherwise the equality

$$\nu(\bar{c}_{\alpha, \beta} \bar{x}^\alpha \bar{z}^\beta) = \nu(\bar{c}_{\gamma, \delta} \bar{x}^\gamma \bar{z}^\delta)$$

would give a rational dependence between the $\nu(x_i)$. This implies that (2.3) can be rewritten in the form

$$\bar{x}^\lambda \sum_{\alpha, \beta} \bar{c}_{\alpha, \beta} \bar{y}^\beta = 0$$

and hence

$$\sum_{\alpha, \beta} \bar{c}_{\alpha, \beta} \bar{y}^\beta = 0, \quad (2.6)$$

since $\text{gr}_\nu R$ is an integral domain. But this contradicts the choice of the y_i and the Lemma is proved. \square

COROLLARY 2.2. *Under the assumptions of Lemma 2.1, let*

$$r := \text{rat.rk } \nu \tag{2.7}$$

$$t := \text{tr.deg}_{\mathbb{k}} \nu. \tag{2.8}$$

Then $\text{tr.deg}_{\mathbb{k}} \text{gr}_{\nu} R = r + t$. Here we allow the possibility for both sides to be infinite. In particular, if $\text{gr}_{\nu} R$ is weakly noetherian, its dimension must be $r + t$.

Proof. We work under the assumption that r and t are finite and leave the general case as an easy exercise. Let y_1, \dots, y_t be a maximal set of elements of R_{ν} such that the \bar{y}_i are algebraically independent over \mathbb{k} . Let $x_1 \in R$ be any element of strictly positive value such that $x_1 y_i \in R$ for all $i, 1 \leq i \leq t$. Choose x_2, \dots, x_r in such a way that $\{\nu(x_i)\}_{1 \leq i \leq r}$ form a basis for $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ over \mathbb{Q} . By Lemma 2.1, $\bar{x}_1, \dots, \bar{x}_r, \bar{x}_1 \bar{y}_1, \dots, \bar{x}_1 \bar{y}_t$ are algebraically independent over \mathbb{k} in $\text{gr}_{\nu} R$. Hence,

$$\text{tr.deg}_{\mathbb{k}} \text{gr}_{\nu} R \geq r + t.$$

On the other hand, take any $z \in R$. By the choice of the x_i , there exist $l \in \mathbb{N}, c_i \in \mathbb{Z}$ such that

$$\nu(z^l) = \nu\left(\prod_{i=1}^r x_i^{c_i}\right).$$

By the choice of the y_i , $\frac{\bar{z}^l}{\prod_{i=1}^r \bar{x}_i^{c_i}}$ is algebraic over $\mathbb{k}(\bar{y}_1, \dots, \bar{y}_t)$. Writing down the algebraic dependence relation and clearing denominators, we get that \bar{z} is algebraic over $\mathbb{k}[\bar{x}_1, \dots, \bar{x}_r, \bar{x}_1 \bar{y}_1, \dots, \bar{x}_1 \bar{y}_t]$ and the proof is complete. \square

Proof of Theorem 1.4 (2) \implies (1). By Remark 3 we only have to prove that

$$\text{rat.rk } \nu + \text{tr.deg}_{\mathbb{k}} \nu = \dim R.$$

Let

$$d := \text{rat.rk } \nu + \text{tr.deg}_{\mathbb{k}} \nu$$

and for $l \in \mathbb{N}$

$$N(l) := \text{length} \frac{R}{Q_l}$$

where Q_l is as in (1.5). By Corollary 2.2, $\text{gr}_{\nu} R$ is weakly noetherian of dimension d . Hence $N(l)$ is bounded above by a polynomial in l of degree d . By the linear equivalence of topologies, there exists $r \in \mathbb{N}$ such that

$$Q_{rl} \subset \mathfrak{m}^l \quad \text{for all } l \in \mathbb{N}.$$

Hence

$$\text{length} \frac{R}{\mathfrak{m}^l} \leq \text{length} \frac{R}{Q_{rl}},$$

and so $\text{length} \frac{R}{\mathfrak{m}^l}$ is bounded above by a polynomial of degree d in l . Therefore $\dim R \leq d$, hence $\dim R = d$ by Abhyankar's Inequality (1.4).

(1) \implies (2). Let d be as above. By Corollary 2.2, $\text{gr}_{\nu} R$ contains d algebraically independent elements over \mathbb{k} . For $l \in \mathbb{N}, \mathfrak{m}^l \subset Q_l$, so that

$$\text{length} \frac{R}{\mathfrak{m}^l} \geq \text{length} \frac{R}{Q_l}.$$

Therefore $\text{length} \frac{R}{Q_l}$ is bounded above by a polynomial in l of degree d and $\text{gr}_{\nu} R$ is weakly noetherian.

We have now come to the hard part of the Theorem: proving the linear equivalence of topologies.

Let \widehat{R} be the \mathfrak{m} -adic completion of R . Since R is analytically irreducible and Φ archimedean, there exists a unique extension $\widehat{\nu}$ of ν to \widehat{R} (see [19], pp. 63–64). Moreover, $\widehat{\nu}$ has the same value group as ν and $\frac{R_{\widehat{\nu}}}{\mathfrak{m}_{\widehat{\nu}}} = \frac{R_{\nu}}{\mathfrak{m}_{\nu}}$. Since

$$\mathfrak{m}^n \widehat{R} = (\mathfrak{m}\widehat{R})^n,$$

it is sufficient to prove that the $\widehat{\nu}$ -adic and the $\mathfrak{m}\widehat{R}$ -adic topologies are linearly equivalent in \widehat{R} . Thus we may assume that R is complete.

Claim. There exists a system of parameters (x_1, \dots, x_d) of R such that $\bar{x}_1, \dots, \bar{x}_d$ are algebraically independent in $\text{gr}_{\nu}R$ over \mathbb{k} , and if $\text{char}(R) = 0$ and $\text{char}(\mathbb{k}) = p > 0$ we have $x_1 = p$.

Proof of Claim. We construct the x_i recursively. First of all we choose any non-zero element x_1 in \mathfrak{m} except in the case $\text{char}(R) = 0$ and $\text{char}(\mathbb{k}) = p > 0$ where we choose $x_1 := p$. Assume that we already constructed elements

$$x_1, \dots, x_i \in \mathfrak{m}$$

such that $\bar{x}_1, \dots, \bar{x}_i$ are algebraically independent over \mathbb{k} and $\text{ht}(x_1, \dots, x_i) = i < d$. By Corollary 2.2 there exists $y \in R$ such that \bar{y} is transcendental over $\mathbb{k}[\bar{x}_1, \dots, \bar{x}_i]$ in $\text{gr}_{\nu}R$. Let P_1, \dots, P_s denote the minimal prime ideals of $(x_1, \dots, x_i)R$. Renumbering the P_j , if necessary, we may assume that there exists $j \in \{0, \dots, s\}$ such that $y \in P_l$, $1 \leq l \leq j$ and $y \notin P_l$, $j < l \leq s$.

Take any $z \in \bigcap_{l=j+1}^s P_l \setminus \bigcup_{l=1}^j P_l$ (where we take $\bigcup_{l=1}^j P_l = \emptyset$ if $j = 0$ and $\bigcap_{l=j+1}^s P_l = R$ if $j = s$), and let

$$x_{i+1} := y + z^N,$$

where N is an integer such that $N\nu(z) > \nu(y)$. We have constructed elements

$$x_1, \dots, x_i, x_{i+1} \in \mathfrak{m}$$

such that $\bar{x}_1, \dots, \bar{x}_{i+1}$ are algebraically independent over \mathbb{k} and

$$\text{ht}(x_1, \dots, x_{i+1}) = i + 1.$$

For $i = d$ we obtain the desired system of parameters (x_1, \dots, x_d) . The Claim is proved. \square

If R is not equicharacteristic, it contains a complete non-equicharacteristic Dedekind domain W (cf. [8], Theorem 84) whose maximal ideal is generated by $p = x_1$. Since R is an integral domain, R is finite over $\mathbb{k}[[x_1, \dots, x_d]]$ or $W[[x_2, \dots, x_d]]$, depending on whether R is equicharacteristic or not (cf. [8], Theorem 84). Let

$$S := \mathbb{k}[[x_1, \dots, x_d]] \quad \text{or} \quad W[[x_2, \dots, x_d]],$$

depending on which of the two cases we are in. Let \mathbb{L} denote the field of fractions of S and $\mathfrak{m}_0 := \mathfrak{m} \cap S$. Let t_1, \dots, t_n be a system of generators of R as an S -algebra. Let T_1, \dots, T_n be independent variables and write

$$R = \frac{S[T_1, \dots, T_n]}{p},$$

where p is the kernel of the natural map $S[T] \rightarrow R$, given by $T_i \rightarrow t_i$, $1 \leq i \leq n$. Let

$$\tilde{S} = S \left[\frac{x_2}{x_1}, \dots, \frac{x_d}{x_1} \right]$$

and let \bar{S} be the localization of \tilde{S} at the prime ideal $(x_1)\tilde{S}$:

$$\bar{S} = \tilde{S}_{(x_1)}.$$

The ring \bar{S} is the local ring of the generic point of the exceptional divisor of the blowing-up of σS at \mathfrak{m}_0 . Let $\bar{R} := R \left[\frac{x_2}{x_1}, \dots, \frac{x_d}{x_1} \right]$. Then \bar{R} is a semi-local integral domain, finite over \bar{S} and birational to R . Let $q_0 := \mathfrak{m}_0\bar{S} = (x_1)\bar{S}$; $q := \mathfrak{m}\bar{R}$.

Let ν_0 denote the restriction of ν to \mathbb{L} .

LEMMA 2.3. *Let $(S, \mathfrak{m}_0, \mathbb{k})$ be a regular local ring with regular system of parameters*

$$x = (x_1, \dots, x_d)$$

and field of fractions \mathbb{L} . Let ν_0 be a valuation of \mathbb{L} , centered at S , such that $\bar{x}_1, \dots, \bar{x}_d$ are algebraically independent in $\text{gr}_{\nu_0} S$ over \mathbb{k} . Then the valuation ν_0 is monomial with respect to x .

Proof. In what follows, "monomial" will mean a monomial in x and "monomial ideal" - an ideal, generated by monomials in x . For an element $f \in S$, let $M(f)$ denote the smallest (in the sense of inclusion) monomial ideal of S , containing f . The ideal $M(f)$ is well defined: it is nothing but the intersection of all the monomial ideals containing f . Let $Mon(f)$ denote the minimal set of monomials generating $I(f)$. In other words, $Mon(f)$ is the smallest set of monomials in x such that f belongs to the ideal of S generated by $Mon(f)$. The set $Mon(f)$ can also be characterized as follows. It is the unique set $\{\omega_1, \dots, \omega_s\}$ of monomials, none of which divide each other and such that f can be written as

$$f = \sum_{i=1}^s c_i \omega_i \quad \text{with } c_i \text{ units of } S. \tag{2.9}$$

A key point is the following: since $\bar{x}_1, \dots, \bar{x}_n$ are algebraically independent in $\text{gr}_{\nu_0} S$ by assumption, (2.9) implies that

$$\nu_0(f) = \min_{1 \leq i \leq s} \{\nu(\omega_i)\}. \tag{2.10}$$

If I is an ideal of S , let $M(I)$ denote the smallest monomial ideal, containing I . The ideal $M(I)$ is generated by the set $\{Mon(f) \mid f \in I\}$.

We want to prove that all the ν_0 -ideals of S are monomial. Let I be a ν_0 -ideal of S . It is sufficient to prove that $I = M(I)$. Obviously, $I \subset M(I)$. To prove the opposite inclusion, take a monomial $\omega \in M(I)$. By the above, there exists $f \in I$ and $\omega' \in Mon(f)$ such that

$$\omega' \mid \omega. \tag{2.11}$$

By (2.10), we have $\nu_0(\omega') \geq \nu_0(f)$. Since I is a ν_0 -ideal, this implies that $\omega' \in I$. Hence $\omega \in I$ by (2.11). This completes the proof. \square

LEMMA 2.4. *Let (S, \mathfrak{m}_0, k) be a regular local ring with regular system of parameters*

$$x = (x_1, \dots, x_d)$$

and field of fractions \mathbb{L} . Let ν_0 be a monomial valuation of \mathbb{L} , centered at S , such that the semigroup $\nu_0(S \setminus \{0\})$ is archimedean. Then the ν_0 -adic topology on S is linearly equivalent to the \mathfrak{m}_0 -adic topology.

Proof. Renumbering the x_i , if necessary, we may assume that $\nu_0(x_1) \leq \dots \leq \nu_0(x_d)$. Since $\nu_0(S \setminus \{0\})$ is archimedean, there exists a natural number N such that $\nu_0(x_d) \leq N\nu_0(x_1)$. For $l \in \mathbb{N}$, let Q_l denote the ν_0 -ideal

$$Q_l := \{x \in S \mid \nu_0(x) \geq l\nu_0(\mathfrak{m}_0)\}. \quad (2.12)$$

Then for all $l \in \mathbb{N}$ we have $Q_{Nl} \subset \mathfrak{m}_0^l \subset Q_l$. This completes the proof of the Lemma. \square

Next, R' be the normalization of R . Since R is complete, R' is a local ring [10], (37.9). Let \mathfrak{m}' be the maximal ideal of R' . If the \mathfrak{m}' -adic topology in R' is linearly equivalent to the ν -adic one then the same is true for the \mathfrak{m} -adic topology in R . Hence we may assume that R is normal.

Let \mathbb{K}_1 be a finite extension of \mathbb{K} which is normal over \mathbb{L} (in the sense of field theory). Then there exists a valuation ν_1 of \mathbb{K}_1 whose restriction to \mathbb{K} is ν [20], Chapter VI, 4,6,11. Let R_1 be the integral closure of R in \mathbb{K}_1 . Then R_1 is a product of complete local rings and since it is an integral domain (it is a subring of \mathbb{K}_1) it is a complete local ring. Then ν_1 is centered in the maximal ideal \mathfrak{m}_1 of R_1 . We have $\dim(R_1) = \dim R$, $\text{rat.rk } \nu_1 = \text{rat.rk } \nu$ and $\left[\frac{R_{\nu_1}}{\mathfrak{m}_{\nu_1}} : \frac{R_\nu}{\mathfrak{m}_\nu}\right] < \infty$. The ring R_1 is analytically irreducible by [10], (37.8). Finally, since R_1 is algebraic over R , for any $x \in R_1$ there exist $n \in \mathbb{N}$ and $y \in R$ such that $n\nu_1(x) \geq \nu_1(y) = \nu(y)$. Hence ν_1 is archimedean on R_1 . Therefore ν_1 satisfies (1) of Theorem 1.4. To prove that the \mathfrak{m} -adic topology on R is linearly equivalent to the ν -adic one, it is sufficient to prove that the same is true of the \mathfrak{m}_1 -adic topology on R_1 . Thus we may assume that the field extension $\mathbb{L} \hookrightarrow \mathbb{K}$ is normal. Let $p = \text{char } \mathbb{K}$ if $\text{char } \mathbb{K} > 0$ and $p = 1$ otherwise. Let p^n denote the inseparability degree of \mathbb{K} over \mathbb{L} .

By Lemmas 2.3–2.4 the ν_0 -adic topology on S is linearly equivalent to the \mathfrak{m}_0 -adic topology. Now, let $f \in \mathfrak{m}$. Since R is assumed to be integrally closed in \mathbb{K} , it equals the integral closure of S in \mathbb{K} . Therefore R is mapped to itself by all the automorphisms of \mathbb{K} over \mathbb{L} . In particular, for every $\sigma \in \text{Aut}(\mathbb{K}/\mathbb{L})$ we have $\nu(\sigma f) > 0$. Then

$$\nu(f) < \nu \left(\prod_{\sigma \in \text{Aut}(\mathbb{K}/\mathbb{L})} \sigma f^{p^n} \right) = \nu(N_{\mathbb{K}/\mathbb{L}}(f)) = \nu_0(N_{\mathbb{K}/\mathbb{L}}(f)).$$

By the linear equivalence of topologies on S , there exists $r \in \mathbb{N}$, such that

$$\nu_0(N_{\mathbb{K}/\mathbb{L}}(f)) \leq r\mathfrak{m}_0(N_{\mathbb{K}/\mathbb{L}}(f)).$$

Now Theorem 1.4 follows from the next Lemma (we state Lemma 2.5 in greater generality than is necessary for Theorem 1.4 for future reference).

LEMMA 2.5. *Let $S \subset R$ be two noetherian domains with fields of fractions \mathbb{L} and \mathbb{K} , respectively. Let \mathfrak{m} be a maximal ideal of R and $\mathfrak{m}_0 := \mathfrak{m} \cap S$. Assume that $S_{\mathfrak{m}_0}$ and $R_{\mathfrak{m}}$ are analytically irreducible and that $R_{\mathfrak{m}}$ is finite over $S_{\mathfrak{m}_0}$. Assume that the following conditions hold:*

- (i) *for any $f \in R$, $N_{\mathbb{K}/\mathbb{L}}(f) \in S$.*
- (ii) *The \mathfrak{m}_0 -adic order on S is a valuation*
(*so that the expression $\mathfrak{m}_0(N_{\mathbb{K}/\mathbb{L}}(f))$ makes sense*). *Then there exists $r \in \mathbb{N}$ such that for any $f \in R$*

$$\mathfrak{m}_0(N_{\mathbb{K}/\mathbb{L}}(f)) \leq r\mathfrak{m}(f).$$

Proof. Arguing as above, we reduce the problem to the case when R and S are complete local rings, R is integrally closed in \mathbb{K} and the field extension $\mathbb{L} \hookrightarrow \mathbb{K}$ is normal. We will work under all these assumptions from now on.

First, we prove the equivalence of topologies under the additional assumptions that \mathbb{K} is separable (hence Galois) over \mathbb{L} and S is regular. We fix a regular system of parameters $x = (x_1, \dots, x_d)$ of S . Let $G := \text{Gal}(\mathbb{K}/\mathbb{L})$. Then G acts on R .

Let \bar{R} be as above and let \bar{R}' denote the integral closure of \bar{R} in \mathbb{K} . Then \bar{R}' is a 1-dimensional semi-local ring, finite over \bar{S} . Let $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ denote the maximal ideals of \bar{R}' . Write

$$q\bar{R}' = \bigcap_{i=1}^s \mathfrak{m}_i^{k_i} = \prod_{i=1}^s \mathfrak{m}_i^{k_i}.$$

Then

$$q^n \bar{R}' = \bigcap_{i=1}^s \mathfrak{m}_i^{nk_i} = \prod_{i=1}^s \mathfrak{m}_i^{nk_i}. \quad (2.13)$$

Each \mathfrak{m}_i defines a divisorial valuation of \mathbb{K} , centered in R . We denote it by ν_i . The group G acts on \bar{R}' and permutes the \mathfrak{m}_i . By Theorem 1.1, all the ν_i are linearly comparable in R to the \mathfrak{m} -adic pseudo-valuation of R . Hence there exists $r \in \mathbb{N}$ such that for any i , $1 \leq i \leq s$ and any $f \in R$,

$$\nu_i(f) \leq r\mathfrak{m}(f). \quad (2.14)$$

Since S is a UFD, $q_0(f) = \mathfrak{m}_0(f)$ for all $f \in S$. Now, take any $f \in R$. Without loss of generality, assume that

$$\nu_1(f) = \max_{1 \leq i \leq s} \nu_i(f).$$

Finally, let $l \in \mathbb{N}$ be such that $q^l \cap S \subset q_0$. Then for any $g \in S$, $q_0(g) \leq lq(g)$. We have

$$\mathfrak{m}_0(N_{\mathbb{K}/\mathbb{L}}(f)) = q_0(N_{\mathbb{K}/\mathbb{L}}(f)) \leq lq(N_{\mathbb{K}/\mathbb{L}}(f)) \leq \quad (2.15)$$

$$\leq l \max_{1 \leq i \leq s} k_i \max_{1 \leq i \leq s} \nu_i(N_{\mathbb{K}/\mathbb{L}}(f)) \quad (2.16)$$

$$\leq l \max_{1 \leq i \leq s} k_i \max_{1 \leq i \leq s} \prod_{\sigma \in G} \nu_i(\sigma f) \quad (2.17)$$

$$\leq l \max_{1 \leq i \leq s} k_i [\mathbb{K} : \mathbb{L}] \nu_1(f) \leq rl \max_{1 \leq i \leq s} k_i [\mathbb{K} : \mathbb{L}] \mathfrak{m}(f). \quad (2.18)$$

This proves Lemma 2.5 in the case when \mathbb{K} is Galois over \mathbb{L} and S is regular.

Continue to assume that S is regular, but drop the assumption of separability of \mathbb{K} over \mathbb{L} . Let \mathbb{K}_s denote the maximal separable extension of \mathbb{L} in \mathbb{K} . Let $R_s := R \cap \mathbb{K}_s$. Suppose $\text{char } \mathbb{K} = p > 0$. Then there exists $n \in \mathbb{N}$ such that $R^{p^n} \subset R_s$. Let \mathfrak{m}_s denote the maximal ideal of R_s . Since $\mathfrak{m}_s \subset \mathfrak{m}$, for any $g \in R_s$ we have $\mathfrak{m}_s(g) \leq \mathfrak{m}(g)$. By the separable case there exists $r \in \mathbb{N}$ such that for any $f \in R_s$

$$\nu(f) \leq r\mathfrak{m}_s(f).$$

Then for any $f \in R$,

$$\nu(f) = \frac{1}{p^n} \nu(f^{p^n}) \leq \frac{r}{p^n} \mathfrak{m}_s(f^{p^n}) \quad (2.19)$$

$$\leq \frac{r}{p^n} \mathfrak{m}(f^{p^n}) \leq r\mathfrak{m}(f). \quad (2.20)$$

Hence $\nu(f)$ and $\mathfrak{m}(f)$ are linearly comparable, as desired. This proves Lemma 2.5 assuming S is regular.

Finally, drop the assumption that S is regular. There exists a complete regular local ring $T \subset S$ such that S is finite over T . Since Lemma 2.5 is already known for T , it is also true for S by the multiplicativity of the norm. This proves Lemma 2.5 and with it Theorem 1.4. \square

3. Applications

The rest of the paper is devoted to the applications of Izumi's Theorem and of Lemma 2.5. The first application is to rewrite some of the classical theorems on comparison of topologies in noetherian rings (which were traditionally proved by Chevalley's Lemma) to include linear equivalence of topologies.

The following observation will be useful in the sequel.

LEMMA 3.1. *Let R be a noetherian ring, \mathfrak{m} a maximal ideal of R . Let $\varphi : R \rightarrow R_{\mathfrak{m}}$ denote the localization homomorphism. Then*

$$\varphi^{-1}(\mathfrak{m}^n R_{\mathfrak{m}}) = \mathfrak{m}^n + \text{Ker } \varphi = \mathfrak{m}^n.$$

In other words, the symbolic powers of \mathfrak{m} coincide with the usual powers. In particular, the \mathfrak{m} -adic topology on R coincides with the restriction of the $\mathfrak{m}^n R_{\mathfrak{m}}$ -adic topology to R .

Proof. Consider an element

$$x \in \varphi^{-1}(\mathfrak{m}^n R_{\mathfrak{m}}).$$

Then there exists $u \in R \setminus \mathfrak{m}$ such that $ux \in \mathfrak{m}^n$. For every natural number n , we have $(u) + \mathfrak{m}^n = R$. Then there exist $v_n \in R$, $m_n \in \mathfrak{m}^n$ such that $uv_n + m_n = 1$. We have $x = x \cdot 1 = x(uv_n + m_n) = xuv_n + xm_n \in \mathfrak{m}^n$. This proves that $\varphi^{-1}(\mathfrak{m}^n R_{\mathfrak{m}}) \subset \mathfrak{m}^n$ for all natural numbers n , and the Lemma follows. \square

The following Corollary is a partial generalization of Corollary 2, [20] p. 273:

COROLLARY 3.2. *Let R be a noetherian ring and \mathfrak{m} a maximal ideal of R , such that $R_{\mathfrak{m}}$ is an analytically irreducible local domain. Let \bar{R} be a finitely generated R -algebra, containing R . Let p be a prime ideal of \bar{R} , lying over \mathfrak{m} . Then the p -adic topology on R is linearly equivalent to the \mathfrak{m} -adic topology. In other words, there exists $r \in \mathbb{N}$ such that for any $n \in \mathbb{N}$*

$$p^{rn} \cap R \subset \mathfrak{m}^n.$$

Proof. By Lemma 3.1, we may assume that R is an analytically irreducible local noetherian domain with maximal ideal \mathfrak{m} . Let \mathbb{K} be the field of fractions of R . Let $\psi : R \rightarrow \hat{R}$ denote the \mathfrak{m} -adic completion of R . The homomorphism ψ is faithfully flat, hence so is the induced map $\bar{R} \rightarrow \bar{R} \otimes_R \hat{R}$. Then there exists a prime ideal \hat{p} in $\bar{R} \otimes_R \hat{R}$ which lies over p . The ring $\bar{R} \otimes_R \hat{R}$ is finitely generated over \hat{R} , so we may assume that R is \mathfrak{m} -adically complete.

If \bar{R} is a purely transcendental extension of R , then $p^n \cap R = \mathfrak{m}^n$ and the Corollary is trivial.

The normalization R' of R is a complete local domain, finite over R ([10], Corollary 37.9). Hence the Corollary is true when $\bar{R} = R'$. Replacing (R, \bar{R}) with $(R', R' \otimes_R \bar{R})$, we may assume that R is normal, hence analytically normal.

Since we know the Corollary for the case of purely transcendental extensions, we may replace R by the normalization of a maximal purely transcendental extension of R , contained in \bar{R} . Thus we may assume that the total ring of fractions of \bar{R} is finite over \mathbb{K} .

Let q be a minimal prime of \bar{R} such that $q \subset p$. Since R is a domain, $p \cap R = (0)$. Replacing \bar{R} with $\frac{\bar{R}}{q}$, we may assume that \bar{R} is a domain. Let $\bar{\mathbb{K}}$ denote its field of fractions. Let $\pi : X \rightarrow \text{Spec } \bar{R}$ be the normalized blowing-up along p . Since R is Nagata, so is \bar{R} and π is of finite type. Let E be any irreducible component of $\pi^{-1}(p)$. Let ν denote the divisorial valuation of $\bar{\mathbb{K}}$ associated to E , and let

$$p_l := \{x \in \bar{R} \mid \nu(x) \geq l\}, \quad l \in \mathbb{N}.$$

Then $p^l \subset p_l$ for all $l \in \mathbb{N}$. Hence it is sufficient to show that the ν -adic and the \mathfrak{m} -adic topologies in R are linearly equivalent. Since $[\bar{\mathbb{K}} : \mathbb{K}] < \infty$, ν induces a divisorial valuation of \mathbb{K} , centered in R . Now the Corollary follows from Theorem 1.4. \square

The above Corollary can be strengthened as follows.

COROLLARY 3.3. *Let R be a noetherian ring and \mathfrak{m} a maximal ideal of R such that $R_{\mathfrak{m}}$ is an analytically irreducible local domain. Let \bar{R} be a finitely generated R -algebra, containing R . Let p be a prime ideal of \bar{R} lying over \mathfrak{m} and I an ideal of \bar{R} such that $I \subset p$ and $I \cap R = (0)$. Then there exists $r \in \mathbb{N}$ such that for any $n \in \mathbb{N}$,*

$$(I + p^{rn}) \cap R \subset \mathfrak{m}^n.$$

Proof. Replacing \bar{R} by $\frac{\bar{R}}{I}$ does not change the problem. Now Corollary 3.3 follows from Corollary 3.2. \square

REMARK 5. In terms of the analogy with functional analysis, this Corollary says that if $I \cap R = (0)$, then I is “transversal” to R .

COROLLARY 3.4. *Let R be a noetherian ring and $T = (T_1, \dots, T_n)$ independent variables. Let \mathfrak{m} be a maximal ideal of R . Let \bar{R} be a finitely generated extension of R and $\bar{p} \subset \bar{\mathfrak{m}} \subset \bar{R}[[T]]$ a pair of prime ideals of $\bar{R}[[T]]$ such that $\bar{\mathfrak{m}} \cap R[[T]] = (\mathfrak{m}, T)$. Let*

$$p := \bar{p} \cap R[[T]] \subset (\mathfrak{m}, T). \quad (3.1)$$

Assume that $\frac{R[[T]]_{(\mathfrak{m}, T)}}{p}$ is analytically irreducible. Then there exists $r \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$(\bar{\mathfrak{m}}^{rn} + \bar{p}) \cap R[[T]] \subset (\mathfrak{m}, T)^n + p.$$

REMARK 6. In particular, we can apply this Corollary to the following situation. Let R , T , \mathfrak{m} , \bar{R} , $\bar{\mathfrak{m}}$ be as in Corollary 3.4. Assume, in addition, that R is a UFD. Let $p = (F)$ be a principal prime ideal generated by a single irreducible power series $F \in R[[T]]$. Let

$$p\bar{R}[[T]] = q_1 \cap \dots \cap q_s \quad (3.2)$$

be a primary decomposition of p in $\bar{R}[[T]]$. We must have $\sqrt{q_i} \subset \bar{\mathfrak{m}}$ for some i . Let $\bar{p} := \sqrt{q_i}$. Finally, assume that the equality (3.1) is satisfied. The equality (3.2) corresponds to a factorization F in $\bar{R}[[T]]$. Say, $F = F_1 F_2$ in $\bar{R}[[T]]$. Then Corollary 3.4 says that there exists $r \in \mathbb{N}$ such that if

$$\tilde{F} \equiv F_1 \tilde{F}_2 \pmod{\bar{\mathfrak{m}}^{rn}},$$

where $\tilde{F}_2 \in \bar{R}[[T]]$, $\tilde{F} \in R[[T]]$, then

$$\tilde{F} \cong Fg \pmod{(\mathfrak{m}, T)^n}$$

for some $g \in R[[T]]$. Thus $F_1(F_2g - \tilde{F}_2) \in (\mathfrak{m}, T)^n \subset \tilde{\mathfrak{m}}^n$ and by Artin-Rees Lemma there exists a constant c depending only on F_1 such that $\tilde{F}_2 \cong F_2g \pmod{\tilde{\mathfrak{m}}^{n-c}}$. In other words, approximate factorization of an element of $R[[T]]$ in $\bar{R}[[T]]$ is close to an actual factorization, and the estimate is linear in n .

Proof Proof of Corollary 3.4. Let $\tilde{R} := \bar{R}[[T]]_{\tilde{\mathfrak{m}}}$ and let

$$S = R[[T]]_{(\mathfrak{m}, T)} \otimes_R \bar{R}_{\tilde{\mathfrak{m}} \cap \bar{R}} \subset \tilde{R}.$$

Let $\mathfrak{m}_0 := \tilde{\mathfrak{m}}\tilde{R} \cap S$. Then \tilde{R} is the (T) -adic completion of S , followed by localization at $\tilde{\mathfrak{m}}$. We can decompose the injective homomorphism $S \rightarrow \tilde{R}$ in two steps: $S \rightarrow S_{\mathfrak{m}_0} \rightarrow \tilde{R}$, where the second arrow is a faithfully flat homomorphism and the first a localization with respect to a maximal ideal. By faithful flatness, we have $\tilde{\mathfrak{m}}^n \cap S_{\mathfrak{m}_0} = \mathfrak{m}_0^n \tilde{R} \cap S_{\mathfrak{m}_0} = \mathfrak{m}_0^n S_{\mathfrak{m}_0}$, so that the $\tilde{\mathfrak{m}}$ -adic topology on $S_{\mathfrak{m}_0}$ is linearly equivalent to the \mathfrak{m}_0 -adic topology. Combining this with Lemma 3.1, we see that the $\tilde{\mathfrak{m}}$ -adic topology on S is linearly equivalent to the \mathfrak{m}_0 -adic topology. The ring S is a localization of a finitely generated $R[[T]]$ -algebra, and we apply Corollary 3.2 to the ring extension $\frac{R[[T]]}{p} \hookrightarrow \frac{S}{\bar{p} \cap S}$. This completes the proof. \square

REMARK 7. Corollary 3.4 can be strengthened as follows. Let R be a noetherian ring, T_1, \dots, T_n independent variables and A a noetherian ring such that

$$R[T] \subset A \subset R[[T]].$$

Let \mathfrak{m} be a maximal ideal of R . Let \bar{R} be a finitely generated R -algebra and let B be a noetherian \bar{R} -algebra such that

$$A \otimes_R \bar{R} \subset B \subset \bar{R}[[T]].$$

Assume that (\mathfrak{m}, T) is a maximal ideal of A and that both $A \otimes_R \bar{R}$ and B have $\bar{R}[[T]]$ as their (T) -adic completion. Let $\tilde{\mathfrak{m}}$ be any prime ideal of \bar{R} lying over (\mathfrak{m}, T) and \bar{p} a prime ideal of B such that $\bar{p} \subset \tilde{\mathfrak{m}}$.

Let $p := \bar{p} \cap A$. Assume that $\frac{A_{\tilde{\mathfrak{m}} \cap A}}{pA_{\tilde{\mathfrak{m}} \cap A}}$ is analytically irreducible. Then there exists $r \in \mathbb{N}$ such that for any $n \in \mathbb{N}$

$$(\tilde{\mathfrak{m}}^{rn} + \bar{p}) \cap A \subset (\mathfrak{m}, T)^n + p.$$

The proof is exactly the same as for Corollary 3.4.

The following result is not a corollary of Theorem 1.4 but uses Theorem 1.2:

PROPOSITION 3.5. (cf. [10] Theorem 3.12, p. 11). *Let R be a noetherian domain and I an ideal of R . Let x be a non-zero element of R . Assume that R is analytically unramified. Then there exists $r \in \mathbb{N}$ such that for any $k, n \in \mathbb{N}$*

$$I^k : x^n \subset I^{k-rn}.$$

Here we adopt the convention that $I^n = R$ if $n \leq 0$.

Proof. By Theorem 1.2, there exist valuations ν_1, \dots, ν_s such that for any $f \in R$

$$\bar{I}(f) = \min_{1 \leq i \leq s} \frac{\nu_i(f)}{\nu_i(I)}.$$

By Theorem 1.3 there exists $r_1 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and any $f \in R$

$$\bar{I}(f) \leq r_1 + I(f). \tag{3.3}$$

Hence, for any $l \in \mathbb{N}$, if $\bar{I}(f) \geq r_1 + l$ then $f \in I^l$. For any $y \in I^k : x^n$ we must have

$$\nu_i(y) + n\nu_i(x) \geq k\nu_i(I) \quad \text{for all } i, 1 \leq i \leq s.$$

Now take a positive integer R such that

$$r \geq \max_{1 \leq i \leq s} \frac{\nu_i(x)}{\nu_i(I)} + r_1.$$

Then for any $y \in I^k : x^n$ and any $i \in \{1, \dots, s\}$ we have

$$\nu_i(y) \geq k\nu_i(I) - n\nu_i(x) \geq k\nu_i(I) - n(r - r_1)\nu_i(I) \geq (k - nr + r_1)\nu_i(I).$$

By (3.3) this implies that $y \in I^{k-nr}$, as desired. □

The following corollary is a generalization of the main result of [9]:

COROLLARY 3.6. *Let R be an analytically irreducible noetherian local ring. Then there exists $a \in \mathbb{N}$ such that for any proper ideal I of R we have:*

$$I^{(ac)} \subset \mathfrak{m}^c \quad \forall c \in \mathbb{N}.$$

Here, if W denotes the complement of the union of the associated primes of I , $I^{(n)}$ is the contraction of $I^n R_W$ to R where R_W denotes the localization of R with respect to the multiplicative system W . The ideal $I^{(n)}$ is called the n -th symbolic power of I .

Let us mention that it is known that if (R, \mathfrak{m}) is a regular local ring of dimension d then $I^{(dc)} \subset I^c$ for any ideal I of R and any integer c [4]. If (R, \mathfrak{m}) has an isolated singularity ring then $I^{(kc)} \subset I^c$ for any ideal I of R and any integer c [5] for some constant k independent on p . For a general local ring R and for any ideal I , there exists a constant k depending on I such that $I^{(kc)} \subset I^c$ for any c [17] but it is still an open question to know if such a k may be chosen independently of I in general.

Proof. First let us prove the result when R is a complete local domain and $I = p$ is a prime ideal. By Cohen's structure theorem R is finite over a ring of power series over a field or over a complete Dedekind domain S . We denote by \mathfrak{m}_0 the maximal ideal of S . Let \mathbb{K} (resp. \mathbb{L}) denote the field of fractions of R (resp. S).

First let us assume that \mathbb{K}/\mathbb{L} is Galois and R is normal. Let $q := p \cap S$. Since \mathbb{K}/\mathbb{L} is Galois and R is the integral closure of S in \mathbb{K} , there exists an integer l which is independent of p such that for any integer N , $x \in p^{(N)}$ implies $N_{\mathbb{K}/\mathbb{L}}(x)^l \in q^{(N)}$ (see Proposition 3.10 of [3]). Since S is a regular local ring we have $q^{(k)} \subset \mathfrak{m}_0^k$ for any k (see [3] p. 9). By Lemma 2.5 there exists an integer $r \in \mathbb{N}$ such that for any $f \in R$, $\mathfrak{m}_0(N_{\mathbb{K}/\mathbb{L}}(f)) \leq r\mathfrak{m}(f)$. Thus if $x \in p^{(rkl)}$ then $N_{\mathbb{K}/\mathbb{L}}(x)^l \in \mathfrak{m}_0^{rkl}$, hence $N_{\mathbb{K}/\mathbb{L}}(x) \in \mathfrak{m}_0^{rk}$ and we have $x \in \mathfrak{m}^k$. Finally we obtain

$$p^{(rlc)} \subset \mathfrak{m}^c \quad \forall p \subset R \text{ prime and } c \in \mathbb{N}.$$

Next, keep the assumptions that R is complete and p is prime, but drop the assumptions that

R is normal and that the extension $\mathbb{L} \rightarrow \mathbb{K}$ is Galois. Let $p = \text{char } \mathbb{K}$ if $\text{char } \mathbb{K} > 0$ and $p = 1$ otherwise. Let p^n be the inseparability degree of \mathbb{K} over \mathbb{L} . Let \mathbb{K}_s be the maximal separable extension of \mathbb{L} in \mathbb{K} and set $R_s := R \cap \mathbb{K}_s$. Then R_s is a complete local domain whose maximal ideal \mathfrak{m}_s equals $\mathfrak{m} \cap R_s$ and $R^{p^n} \subset R_s$. The ideal $p_s := p \cap R_s$ is a prime ideal of R_s and $p^{p^n} \subset p_s$. For any element $y \in p^{(c)}$ there exists $a \in R \setminus p$ such that $ay \in p^c$. Thus $a^{p^n} y^{p^n} \in p_s^c$ and $y^{p^n} \in p_s^{(c)}$.

If $p_s^{(ac)} \subset \mathfrak{m}_s^c$ for any integer c , then for any $x \in p^{(\alpha p^n c)}$ we have $x^{p^n} \in p_s^{(\alpha p^n c)} \subset \mathfrak{m}_s^{p^n c}$. Thus $x^{p^n} \in \mathfrak{m}^{p^n c}$ and by Rees theorem there exists a constant c_0 depending only on R such that $x \in \mathfrak{m}^{c-c_0}$. Thus we may assume that \mathbb{K}/\mathbb{L} is separable.

In this case let us denote by \mathbb{K}_1 a finite separable field extension of \mathbb{K} which is normal over \mathbb{L} and let R_1 be the integral closure of R in \mathbb{K}_1 . Then R_1 is a direct sum of complete local rings and since R_1 is a domain (it is a subring of a field) it is a complete local domain. Let \mathfrak{m}_1 be the maximal ideal of R_1 . By Lemma 2.4 [15] there exists $\alpha \in \mathbb{N}$ such that $\mathfrak{m}_1^{\alpha c} \cap R \subset \mathfrak{m}^c$ for any integer c . Since $R \rightarrow R_1$ is finite there exists a prime ideal p_1 of R_1 lying over p . Thus by replacing R and p by R_1 and p_1 , we may assume that \mathbb{K}/\mathbb{L} is Galois and R is normal and this case has been proved above.

Now let us assume that R is an analytically irreducible local ring and $I = p$ is a prime ideal of R . Let us consider an irredundant primary decomposition of $p\widehat{R}$:

$$p\widehat{R} = q_1 \cap \cdots \cap q_s$$

where \widehat{R} denotes the completion of R and the q_i are primary ideals of \widehat{R} . Let p_i be the radical of q_i for all i and set $W := \widehat{R} \setminus \cup_i p_i$. Since $p_i \cap R = p$ for any i , we have an inclusion of multiplicative systems: $R \setminus p \subset W$. Thus we have for any integer n :

$$p^{(n)} = p^n R_{R \setminus p} \cap R \subset p^n \widehat{R}_{R \setminus p} \cap R \subset p^n \widehat{R}_W \cap \widehat{R} \subset p_1^{(n)} \widehat{R}_{\widehat{R} \setminus p_1} \cap \widehat{R} = p_1^{(n)}.$$

By the previous case there exist a and b such that $p_1^{(ac+b)} \subset \widehat{\mathfrak{m}}^c$ for all integers c . Since $\widehat{\mathfrak{m}}^c \cap R = \mathfrak{m}^c$ the theorem is proved in this case.

Finally let us assume that R is an analytically irreducible local ring and I is any ideal of R , not necessarily prime. Let p_1, \dots, p_s be the associated primes of I and set $W = R \setminus \cup_i p_i$. Then we have for any integer n :

$$I^{(n)} = I^n R_W \cap R \subset p_1^n R_W \cap R \subset p_1^n R_{R \setminus p_1} \cap R = p_1^{(n)}.$$

Since the theorem is proved for the symbolic powers of p_1 , this proves the theorem for any ideal I . □

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