Miscellaneous Remarks about Orthogonality

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rencontre de réalisabilité, Marseille, June 2018
Warning!

Contrary to popular (?) belief, I am no expert on realizability.
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I only organized 6 realizability workshops because...
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... it was fun!
Warning!

Contrary to popular (?) belief, I am no expert on realizability. I only organized 6 realizability workshops because... ... it was fun! Today, I won’t directly talk about realizability,
Warning!

Contrary to popular (?) belief, I am no expert on realizability.
I only organized 6 realizability workshops because...
... it was fun!

Today, I won’t directly talk about realizability,
... but about one thing I like in (classical) realizability:
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I only organized 6 realizability workshops because...

... it was fun!

Today, I won’t directly talk about realizability,

... but about one thing I like in (classical) realizability:

orthogonality.
(non) definition

The second line of Wikipedia’s entry for orthogonality is

*In mathematics, orthogonality is the generalization of the notion of perpendicularity to the linear algebra of bilinear forms.*

This is related to what we have in mind...
(non) definition

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Definition

Orthogonality is a tool used to define (sometimes) interesting models.
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*In mathematics, orthogonality is the generalization of the notion of perpendicularity to the linear algebra of bilinear forms.*

This is related to what we have in mind...

**Definition**

Orthogonality is a tool used to define (sometimes) interesting models.

---

\(\dagger\): the first line is “Orthogonal’ redirects here. For the trilogy of novels by Greg Egan, see Orthogonal (novel).”

but this isn’t really relevant.
Why “Perpendicularity”? 

For a finitely dimensional complex vector spaces $E$, we have

- $u \perp v$ is defined by $u \cdot v = 0$, ("perpendicularity")

- If $A$ is a subvector space, then $A^\perp$ is defined by $A^\perp = \{v \mid \forall u \in A, u \perp v\}$.

- Every subvector space satisfies $A = A^{\perp\perp}$. 
Why “Perpendicularity”? 

For a finitely dimensional complex vector spaces $E$, we have

- $u \perp v$ is defined by $u \cdot v = 0$, (“perpendicularity”)
- If $A$ is a subvector space, then $A^\perp$ is defined by $A^\perp = \{ v \mid \forall u \in A, u \perp v \}$.
- Every subvector space satisfies $A = A^{\perp\perp}$.

We have

**Proposition**

An arbitrary set of vectors $V$ is a subvector space if and only if $V = V^{\perp\perp}$.
Why “Perpendicularly”?

For a finitely dimensional complex vector spaces $E$, we have

- $\mathbf{u} \perp \mathbf{v}$ is defined by $\mathbf{u} \cdot \mathbf{v} = 0$, ("perpendicularity")
- If $A$ is a subvector space, then $A^\perp$ is defined by $A^\perp = \{ \mathbf{v} \mid \forall \mathbf{u} \in A, \mathbf{u} \perp \mathbf{v} \}$.
- Every subvector space satisfies $A = A^{\perp\perp}$.

We have

**Proposition**

An arbitrary set of vectors $V$ is a subvector space if and only if $V = V^{\perp\perp}$.

**Idea**

Orthogonality: defining interesting “spaces” as sets of “things” $T$ satisfying $T = T^{\perp\perp}$, for an appropriate relation “$\perp$” between “things".
Relations and Orthogonality

**Definition**

*Given a relation $\perp$ between sets $X$ and $Y$, we define the following operator from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$:

$$x^\perp = \{ b \in Y \mid \forall a \in x, a \perp b \}$$*
Relations and Orthogonality

**Definition**

Given a relation \( \perp \) between sets \( X \) and \( Y \), we define the following operator from \( \mathcal{P}(X) \) to \( \mathcal{P}(Y) \):

\[
x \perp = \{ b \in Y \mid \forall a \in X, a \in x \Rightarrow a \perp b \}\]

---

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Miscellaneous Remarks about Orthogonality
Relations and Orthogonality

Definition

Given a relation \(\bot\) between sets \(X\) and \(Y\), we define the following operator from \(\mathcal{P}(X)\) to \(\mathcal{P}(Y)\):

\[
x \downarrow = \{ b \in Y \mid \forall a \in X, a \in x \Rightarrow a \bot b \}\n\]

Lemma

\(\phi\) is of the form \(x \mapsto x\downarrow\) iff \(\phi\) transforms arbitrary unions into intersections.

Note that any such \(\phi\) is antitonic...
Relations and Orthogonality

Definition

Given a relation $\perp$ between sets $X$ and $Y$, we define the following operator from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$:

$$ x^\perp = \{ b \in Y \mid \forall a \in X, a \in x \Rightarrow a \perp b \} $$

Lemma

$\phi$ is of the form $x \mapsto x^\perp$ iff $\phi$ transforms arbitrary unions into intersections.

Note that any such $\phi$ is antitonic...

Proof idea: define $a \perp b$ iff $b \in \phi(\{a\})$. We have

$$ b \in \phi(x) \iff b \in \phi \left( \bigcup_{a \in x} \{a\} \right) \iff b \in \bigcap_{a \in x} \phi(\{a\}) \iff b \in x^\perp. $$
Relations and Orthogonality

**Definition**

*Given a relation \( \perp \) between sets \( X \) and \( Y \), we define the following operator from \( \mathcal{P}(X) \) to \( \mathcal{P}(Y) \):*

\[
x \perp = \{ b \in Y \mid \forall a \in X, a \in x \Rightarrow a \perp b \}\]

This can be generalized with the notion of “double-glueing”.
Relations and Orthogonality

Definition

Given a relation $\bot$ between sets $X$ and $Y$, we define the following operator from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$:

$$x^\bot = \{ b \in Y \mid \forall a \in X, a \in x \Rightarrow a \bot b \}$$

This can be generalized with the notion of “double-glueing”.

...but I am not going to say anything about that...
Two Orthogonalities

Lemma

Any monotonic \( \phi : \mathcal{P}(X) \to \mathcal{P}(Y) \) can be factorized as \( x \mapsto x \downarrow_1 \downarrow_2 \) for some set \( Z \) and relations \( \downarrow_1 \subset X \times Z \) and \( \downarrow_2 \subset Z \times Y \).
**Two Orthogonalities**

**Lemma**

*Any monotonic* \( \phi : \mathcal{P}(X) \rightarrow \mathcal{P}(Y) \) *can be factorized as* \( x \leftrightarrow x_{\downarrow 1 \downarrow 2} \) *for some set* \( Z \) *and relations* \( \downarrow_1 \subseteq X \times Z \) *and* \( \downarrow_2 \subseteq Z \times Y \).*

**Proof:** define \( Z = \mathcal{P}(X) \) and

\[
\begin{align*}
(a, x) &\in \downarrow_1 \iff a \in x, \\
(x, b) &\in \downarrow_2 \iff b \in \phi(x).
\end{align*}
\]
Two Orthogonality

Lemma

Any monotonic $\phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ can be factorized as $x \leftrightarrow x_{\perp 1 \perp 2}$ for some set $Z$ and relations $\perp_1 \subseteq X \times Z$ and $\perp_2 \subseteq Z \times Y$.

Proof: define $Z = \mathcal{P}(X)$ and

1. $(a, x) \in \perp_1 \iff a \in x$,
2. $(x, b) \in \perp_2 \iff b \in \phi(x)$.

\[ b \in x_{\perp 1 \perp 2} \iff \forall x', x' \in x_{\perp 1} \Rightarrow x' \perp_2 b \]

definition of $\perp_2$
Two Orthogonalitys

Lemma

Any monotonic $\phi : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ can be factorized as $x \leftrightarrow x^{\perp_1 \perp_2}$ for some set $Z$ and relations $\perp_1 \subset X \times Z$ and $\perp_2 \subset Z \times Y$.

Proof: define $Z = \mathcal{P}(X)$ and

- $(a, x) \in \perp_1 \iff a \in x$,
- $(x, b) \in \perp_2 \iff b \in \phi(x)$.

$$
\begin{align*}
  b \in x^{\perp_1 \perp_2} & \iff \forall x', x' \in x^{\perp_1} \Rightarrow x' \perp_2 b \\
  & \iff \forall x', x' \in x^{\perp_1} \Rightarrow b \in \phi(x')
\end{align*}
$$

- definition of $\perp_2$
- definition of $\perp_2$
**Two Orthogonalitys**

**Lemma**

Any monotonic $\phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ can be factorized as $x \mapsto x_{\perp_1 \perp_2}$ for some set $Z$ and relations $\perp_1 \subseteq X \times Z$ and $\perp_2 \subseteq Z \times Y$.

**Proof:** define $Z = \mathcal{P}(X)$ and

- $(a, x) \in \perp_1 \iff a \in x$,
- $(x, b) \in \perp_2 \iff b \in \phi(x)$.

\[
b \in x_{\perp_1 \perp_2} \iff \forall x', x' \in x_{\perp_1} \Rightarrow x' \perp_2 b \quad \text{definition of } \perp_1 \perp_2
\]
\[
\iff \forall x', x' \in x_{\perp_1} \Rightarrow b \in \phi(x') \quad \text{definition of } \perp_2
\]
\[
\iff \forall x', \left( \forall a \in x, a \perp_1 x' \right) \Rightarrow b \in \phi(x') \quad \text{definition of } \perp_1
\]
Two Orthogonalities

Lemma

Any monotonic $\phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ can be factorized as $x \mapsto x \perp_1 \perp_2$ for some set $Z$ and relations $\perp_1 \subseteq X \times Z$ and $\perp_2 \subseteq Z \times Y$.

Proof: define $Z = \mathcal{P}(X)$ and

- $(a, x) \in \perp_1$ $\iff$ $a \in x$,
- $(x, b) \in \perp_2$ $\iff$ $b \in \phi(x)$.

\[
\begin{align*}
b \in x \perp_1 \perp_2 & \iff \forall x', x' \in x \perp_1 \Rightarrow x' \perp_2 b \\
& \iff \forall x', x' \in x \perp_1 \Rightarrow b \in \phi(x') \\
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& \iff \forall x', (\forall a \in x, a \in x') \Rightarrow b \in \phi(x')
\end{align*}
\]

\[\text{definition of } \perp_1 \]

\[\text{definition of } \perp_2 \]

\[\text{definition of } \phi \]

\[\text{definition of } \phi \]

\[\text{definition of } \phi \]
Two Orthogonalities

Lemma

Any monotonic \( \phi : \mathcal{P}(X) \to \mathcal{P}(Y) \) can be factorized as \( x \mapsto x \bot_1 \bot_2 \) for some set \( Z \) and relations \( \bot_1 \subset X \times Z \) and \( \bot_2 \subset Z \times Y \).

Proof: define \( Z = \mathcal{P}(X) \) and

\[
\begin{align*}
(a, x) &\in \bot_1 \iff a \in x, \\
(x, b) &\in \bot_2 \iff b \in \phi(x).
\end{align*}
\]

\[
\begin{align*}
b &\in x \bot_1 \bot_2 \iff \forall x', x' \in x \bot_1 \Rightarrow x' \bot_2 b \\
&\iff \forall x', x' \in x \bot_1 \Rightarrow b \in \phi(x') & \text{definition of } \bot_2 \\
&\iff \forall x', \left( \forall a \in x, a \bot_1 x' \right) \Rightarrow b \in \phi(x') & \text{definition of } \bot_1 \\
&\iff \forall x', \left( \forall a \in x, a \in x' \right) \Rightarrow b \in \phi(x') & \text{simplification}
\end{align*}
\]
Two Orthogonalities

Lemma

Any monotonic $\phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ can be factorized as $x \mapsto x_{\perp_1 \perp_2}$ for some set $Z$ and relations $\perp_1 \subset X \times Z$ and $\perp_2 \subset Z \times Y$.

Proof: define $Z = \mathcal{P}(X)$ and

\begin{align*}
(a, x) &\in \perp_1 \iff a \in x, \\
(x, b) &\in \perp_2 \iff b \in \phi(x).
\end{align*}

\begin{align*}
b \in x_{\perp_1 \perp_2} &\iff \forall x', x' \in x_{\perp_1} \Rightarrow x' \perp_2 b \\
&\iff \forall x', x' \in x_{\perp_1} \Rightarrow b \in \phi(x') \\
&\iff \forall x', (\forall a \in x, a \perp_1 x') \Rightarrow b \in \phi(x') \\
&\iff \forall x', (\forall a \in x, a \in x') \Rightarrow b \in \phi(x') \\
&\iff \forall x', x \subset x' \Rightarrow b \in \phi(x') \\
&\iff b \in \phi(x)
\end{align*}

definition of $\perp_2$
definition of $\perp_2$
definition of $\perp_1$
definition of $\perp_1$
simplification
monotonicity of $\phi$
Two Orthogonalities

Lemma

Any monotonic $\phi: \mathcal{P}(X) \to \mathcal{P}(Y)$ can be factorized as $x \mapsto x \downarrow_1 \downarrow_2$ for some set $Z$ and relations $\downarrow_1 \subseteq X \times Z$ and $\downarrow_2 \subseteq Z \times Y$.

Proof: define $Z = \mathcal{P}(X)$ and

- $(a, x) \in \downarrow_1 \iff a \in x$, 
- $(x, b) \in \downarrow_2 \iff b \in \phi(x)$.

(comment for L. R.: this is impredicative...)
## Closure Operators

### Definition (Closure operator)

A closure operator on $\mathcal{P}(X)$ is an operator $\phi$ satisfying

1. $\phi$ is monotonic,
2. $\phi$ is expansive: $\forall x, x \subseteq \phi(x)$,
3. $\phi$ is idempotent, or equivalently: $\forall x, \phi(\phi(x)) \subseteq \phi(x)$. 

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Closure Operators

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The following is well known

Lemma

For any relation $\perp \subset X \times Y$, $x \mapsto x\perp\perp$ is a closure operator on $\mathcal{P}(X)$. 

(I implicitly reverse the relation where appropriate)
**Closure Operators**

**Definition (Closure operator)**

A closure operator on \( \mathcal{P}(X) \) is an operator \( \phi \) satisfying

1. \( \phi \) is monotonic,
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3. \( \phi \) is idempotent, or equivalently: \( \forall x, \phi(\phi(x)) \subseteq \phi(x) \).

The following is well known

**Lemma**

For any relation \( \bot \subseteq X \times Y \), \( x \mapsto x_{\bot} \) is a closure operator on \( \mathcal{P}(X) \).

(I implicitly reverse the relation where appropriate)

The following is less well known

**Proposition**

Any closure operator \( \phi : \mathcal{P}(X) \rightarrow \mathcal{P}(Y) \) can be factorized as \( \phi(x) = x_{\bot} \) for some relations \( \bot \subseteq X \times Z \).
Partial Proof

Write $\text{Fix}(\phi)$ for the set of fixpoint of $\phi$.

1. Because $\phi$ is a closure operator, $\text{Fix}(\phi)$ is the set of pre-fixpoints of $\phi$: $\text{Fix}(\phi) = \{x \mid \phi(x) \subseteq x\}$.
2. By the Knaster-Tarski theorem, $(\text{Fix}(\phi), \sqcap)$ is complete inf-lattice.
Partial Proof

Write $\text{Fix}(\phi)$ for the set of fixpoint of $\phi$.

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2. By the Knaster-Tarski theorem, $(\text{Fix}(\phi), \subseteq)$ is complete inf-lattice.

Lemma

If $\phi$ is a closure operator, we have $\phi(x) = \bigcap \{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \}$. 
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Lemma

If $\phi$ is a closure operator, we have $\phi(x) = \bigcap \{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \}$.

Define $Z = \text{Fix}(\phi)$ and $\bot \subseteq X \times Z$ by $a \bot x \iff a \in x$.  

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Partial Proof

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1. Because $\phi$ is a closure operator, $\text{Fix}(\phi)$ is the set of pre-fixpoints of $\phi$: $\text{Fix}(\phi) = \{ x \mid \phi(x) \subseteq x \}$.

2. By the Knaster-Tarski theorem, $(\text{Fix}(\phi), \bigcap)$ is complete inf-lattice.

Lemma

If $\phi$ is a closure operator, we have $\phi(x) = \bigcap\{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \}$.

Define $Z = \text{Fix}(\phi)$ and $\bot \subseteq X \times Z$ by $a \bot x \iff a \in x$.

\[ a \in x^{\bot} \iff \forall x' \in \text{Fix}(\phi), x' \in x^{\bot} \Rightarrow a \bot x' \]  

\[ \text{definition of } \bot \]
Partial Proof

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2. By the Knaster-Tarski theorem, $(\text{Fix}(\phi), \cap)$ is complete inf-lattice.

Lemma

*If $\phi$ is a closure operator, we have $\phi(x) = \bigcap \{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \}$.*

Define $Z = \text{Fix}(\phi)$ and $\bot \subseteq X \times Z$ by $a \perp x \iff a \in x$.

\[
\begin{align*}
a \in x^{\bot} & \iff \forall x' \in \text{Fix}(\phi), \ x' \in x^{\bot} \Rightarrow a \perp x' & \text{definition of } x^{\bot} \\
& \iff \forall x' \in \text{Fix}(\phi), \ x' \in x^{\bot} \Rightarrow a \in x' & \text{definition of } \perp
\end{align*}
\]
Partial Proof

Write $\text{Fix}(\phi)$ for the set of fixpoint of $\phi$.

1. Because $\phi$ is a closure operator, $\text{Fix}(\phi)$ is the set of pre-fixpoints of $\phi$: $\text{Fix}(\phi) = \{x \mid \phi(x) \subseteq x\}$.

2. By the Knaster-Tarski theorem, $\left(\text{Fix}(\phi), \cap\right)$ is complete inf-lattice.

Lemma

If $\phi$ is a closure operator, we have $\phi(x) = \bigcap\{x' \in \text{Fix}(\phi) \mid x \subseteq x'\}$.

Define $Z = \text{Fix}(\phi)$ and $\bot \subseteq X \times Z$ by $a \bot x \Leftrightarrow a \in x$.

\[
\begin{align*}
a \in x \bot \Leftrightarrow & \forall x' \in \text{Fix}(\phi), x' \in x \bot \Rightarrow a \bot x' \\
\Leftrightarrow & \forall x' \in \text{Fix}(\phi), x' \in x \bot \Rightarrow a \in x' \\
\Leftrightarrow & \forall x' \in \text{Fix}(\phi), (\forall a \in x, a \bot x') \Rightarrow a \in x' \\
& \text{definition of } \bot \\
& \text{definition of } \bot \\
& \text{definition of } \bot
\end{align*}
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Partial Proof

Write $\text{Fix}(\phi)$ for the set of fixpoint of $\phi$.

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2. By the Knaster-Tarski theorem, $(\text{Fix}(\phi), \sqcap)$ is complete inf-lattice.

Lemma

If $\phi$ is a closure operator, we have $\phi(x) = \bigcap \{x' \in \text{Fix}(\phi) \mid x \subseteq x'\}$.

Define $Z = \text{Fix}(\phi)$ and $\bot \subset X \times Z$ by $a \bot x \iff a \in x$.

\[
\begin{align*}
a \in x^{\bot} & \iff \forall x' \in \text{Fix}(\phi), x' \in x^{\bot} \Rightarrow a \bot x' \quad \text{definition of } \bot \\
& \iff \forall x' \in \text{Fix}(\phi), x' \in x^{\bot} \Rightarrow a \in x' \quad \text{definition of } \bot \\
& \iff \forall x' \in \text{Fix}(\phi), \left( \forall a \in x, a \bot x' \right) \Rightarrow a \in x' \quad \text{definition of } \bot \\
& \iff \forall x' \in \text{Fix}(\phi), \left( \forall a \in x, a \in x' \right) \Rightarrow a \in x' \quad \text{definition of } \bot
\end{align*}
\]
Partial Proof

Write \( \text{Fix}(\phi) \) for the set of fixpoint of \( \phi \).

1. Because \( \phi \) is a closure operator, \( \text{Fix}(\phi) \) is the set of pre-fixpoints of \( \phi \): \( \text{Fix}(\phi) = \{ x \mid \phi(x) \subseteq x \} \).
2. By the Knaster-Tarski theorem, \( \left( \text{Fix}(\phi), \bigcap \right) \) is complete inf-lattice.

Lemma

If \( \phi \) is a closure operator, we have \( \phi(x) = \bigcap \{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \} \).

Define \( Z = \text{Fix}(\phi) \) and \( \bot \subseteq X \times Z \) by \( a \bot x \iff a \in x \).

\[
\begin{align*}
a \in x \bot \ &\iff \forall x' \in \text{Fix}(\phi), x' \in x \bot \Rightarrow a \bot x' \\
&\iff \forall x' \in \text{Fix}(\phi), x' \in x \bot \Rightarrow a \in x' \\
&\iff \forall x' \in \text{Fix}(\phi), \left( \forall a \in x, a \bot x' \right) \Rightarrow a \in x' \\
&\iff \forall x' \in \text{Fix}(\phi), \left( \forall a \in x, a \in x' \right) \Rightarrow a \in x' \\
&\iff \forall x' \in \text{Fix}(\phi), x \subseteq x' \Rightarrow a \in x' \\
&\quad \text{definition of } \bot \\
&\quad \text{definition of } \bot \\
&\quad \text{definition of } \bot \\
&\quad \text{definition of } \bot \\
&\quad \text{simplification}
\end{align*}
\]
Partial Proof

Write \( \text{Fix}(\phi) \) for the set of fixpoint of \( \phi \).

1. Because \( \phi \) is a closure operator, \( \text{Fix}(\phi) \) is the set of pre-fixpoints of \( \phi \): \( \text{Fix}(\phi) = \{ x \mid \phi(x) \subseteq x \} \).
2. By the Knaster-Tarski theorem, \( (\text{Fix}(\phi), \cap) \) is complete inf-lattice.

Lemma

If \( \phi \) is a closure operator, we have \( \phi(x) = \bigcap \{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \} \).

Define \( Z = \text{Fix}(\phi) \) and \( \bot \subseteq X \times Z \) by \( a \bot x \iff a \in x \).

\[
\begin{align*}
a \in x^\bot &\iff \forall x' \in \text{Fix}(\phi), x' \in x^\bot \Rightarrow a \bot x' & \text{definition of } x^\bot \\
&\iff \forall x' \in \text{Fix}(\phi), x' \in x^\bot \Rightarrow a \in x' & \text{definition of } \bot \\
&\iff x' \in \text{Fix}(\phi), (\forall a \in x, a \bot x') \Rightarrow a \in x' & \text{definition of } x^\bot \\
&\iff \forall x' \in \text{Fix}(\phi), (\forall a \in x, a \in x') \Rightarrow a \in x' & \text{definition of } \bot \\
&\iff a \in \bigcap \{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \} & \text{simplification} \\
&\iff a \in \bigcap \{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \} & \text{definition}
\end{align*}
\]
Partial Proof

Write \( \text{Fix}(\phi) \) for the set of fixpoint of \( \phi \).

1. Because \( \phi \) is a closure operator, \( \text{Fix}(\phi) \) is the set of pre-fixpoints of \( \phi \):
   \[ \text{Fix}(\phi) = \{ x \mid \phi(x) \subseteq x \} . \]

2. By the Knaster-Tarski theorem, \( (\text{Fix}(\phi), \subseteq) \) is complete inf-lattice.

Lemma

If \( \phi \) is a closure operator, we have
\[ \phi(x) = \bigcap \{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \} . \]

Define \( Z = \text{Fix}(\phi) \) and \( \bot \subseteq X \times Z \) by \( a \bot x \iff a \in x \).

\[
\begin{align*}
a \in x^{\bot} &\iff \forall x' \in \text{Fix}(\phi), x' \in x^{\bot} \Rightarrow a \bot x' & \text{definition of } \bot \\
&\iff \forall x' \in \text{Fix}(\phi), x' \in x^{\bot} \Rightarrow a \in x' & \text{definition of } \bot \\
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&\iff \forall x' \in \text{Fix}(\phi), \left( \forall a \in x, a \in x' \right) \Rightarrow a \in x' & \text{simplification} \\
&\iff a \in \bigcap \{ x' \in \text{Fix}(\phi) \mid x \subseteq x' \} & \text{definition} \\
&\iff a \in \phi(x) & \text{lemma}
\end{align*}
\]
Partial Proof

Write $\text{Fix}(\phi)$ for the set of fixpoint of $\phi$.

1. Because $\phi$ is a closure operator, $\text{Fix}(\phi)$ is the set of pre-fixpoints of $\phi$: $\text{Fix}(\phi) = \{ x \mid \phi(x) \subseteq x \}$.
2. By the Knaster-Tarski theorem, $(\text{Fix}(\phi), \subseteq)$ is complete inf-lattice.

Lemma

If $\phi$ is a closure operator, we have $\phi(x) = \bigcap\{x' \in \text{Fix}(\phi) \mid x \subseteq x'\}$.

Define $Z = \text{Fix}(\phi)$ and $\bot \subseteq X \times Z$ by $a \bot x \Leftrightarrow a \in x$.

(comment for L. R.: this is impredicative...)

P. Hyvernat

Miscellaneous Remarks about Orthogonality

9/26
So...

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-Closure operators are equivalent to "bi-orthogonals".

Note however that not all closure operators can be obtained from a "homogeneous" relation \( K \subseteq X \times X \).

Counterexample: \( x \in X \) if \( X \) is cofinite, \( x \) otherwise.
So...

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  - "Kuratowski closure axioms": \( \phi(\emptyset) = \emptyset \)
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Counter example: $x \cup X \cap \# x$ if $X$ is cofinite $x$ otherwise

P. Hyvernat
Miscellaneous Remarks about Orthogonality
10/26
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\text{counter example: } x \in X \mapsto \begin{cases} 
  x & \text{if } X \text{ is cofinite} \\
  x & \text{otherwise}
\end{cases}
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Plan

1. Framework
2. Realizability
3. Linear Logic

P. Hyvernat

Miscellaneous Remarks about Orthogonality
Interpreting Types

Constant atomic types are easy: take all terms with that type.
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Interpreting types with parameters (system F) is more difficult.
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"C ⊆ SN", "C is → closed", "t neutral with its one step reducts in C ⇒ t ∈ C"
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- reducibility candidates (Girard),

  \( C \in S\mathcal{N} \), \( C \rightarrow_\beta \) closed, \( t \) neutral with its one step reducts in \( C \Rightarrow t \in C \)

- saturated sets (Tait),

  \( S \in \mathcal{S\mathcal{N}} \), \( (x)\bar{u} \in S \) if \( \bar{u} \in \mathcal{S\mathcal{N}} \), \( (t[x = v])\bar{u} \in S \Rightarrow (\lambda x.t)\bar{u} \in S \)
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- orthogonality between terms and contexts (Krivine, Miquel, ...)

  \[ O \in \mathcal{SN} \], \[ O = O \perp \perp \]
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  \[ O \subseteq SN \], \[ O = O^\perp \perp \]

The relation \( t \perp C \) depends on the model.

Remark: reducibility candidates and saturated sets are “closed” for some operations. In theory, they can be obtained using an orthogonality relation.
In practice, it is important that $\perp$ is closed under **backward** reduction:

“if $t \perp C$ and $t' \rightarrow t$ then $t' \perp C$.”
Strong Normalization

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This is not the case when $t \perp C$ is defined as $C[t] \in SN$!

Some care is needed to prove strong normalization using this technique...
**Strong Normalization**

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This is **not** the case when $t \perp C$ is defined as $C[t] \in SN$!

Some care is needed to prove strong normalization using this technique...

In many models, strong normalization isn’t important!

- “$t \perp \pi$” when $\langle t, \pi \rangle$ loops or reduces to 0 or 1. (Mellies & Vouillon, 2005)
- “$t \perp \pi$” if $\langle t, \pi \rangle \rightarrow^* \langle \text{stop}, n \cdot \pi' \rangle$ for some $n$ s.t. $f(n) = 0$. (Miquel, 2009)

Here $f$ is an arbitrary primitive recursive function.
Not Every Set is Closed!

Sets of terms that are not closed can be important...
Not Every Set is Closed!

Sets of terms that are not closed can be important...

... Lepigre uses the following for interpreting the type \( A \rightarrow B \)

\[ \forall \text{ values: } \| A \rightarrow B \| = \left\{ \lambda x.t \mid \forall v \in \| A \|, t[x = v] \in \| B \|^{\perp} \right\}, \]
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P. Hyvernat
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14/26
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- terms: $||A \to B||^{\perp \perp}$.
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This is important because Rodolphe works in a call-by value setting.
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This is important because Rodolphe works in a call-by-value setting.

More about that in Rodolphe’s talk...
Personal Remarks

In many examples, (Krivine’s) orthogonal based realizability is used to prove results on terms and reduction.
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**Question:** are there interesting “realizability” models without computational content?

(except forcing models)
Plan

1. Framework
2. Realizability
3. Linear Logic
Interpreting Formulas

This is easy:

- boolean algebras (classical logic)
- Heyting algebras (intuitionistic logic)
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For linear logic, more care is needed...
Interpreting Formulas and Proofs

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For linear logic, more care is needed...

Interpreting proofs for classical logic also requires more care...
**Phase Semantics**

**Definition (Girard, “Linear Logic”, 1987)**

A phase space is given by:

- a commutative monoid (whose elements are called “phases”),
- a set $\bot$ of phases.

Two phases are orthogonal, written $p \perp q$ when $pq \in \bot$, and fixpoints for $\bot \bot$ are called **facts**.
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This gives a (complete) provability semantics for linear logic, where the connectives are given by

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The proof of completeness uses the free commutative monoid on formulas (finite multisets) with \( \Gamma \perp \Delta \) iff \( \vdash \Gamma, \Delta \) is provable.
Coherent Spaces

They give a denotational semantics for linear proofs (Girard, 1987)

Definition

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$Z^i = \{x \mid \exists z \in Z, x \subseteq z\}$. 
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Other connectives are defined by de Morgan duality...

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Other connectives are defined by de Morgan duality...

This is more “abstract” than realizability models.

$Z^\perp = \{x \mid \exists z \in Z, x \subseteq z\}.$
Finiteness Spaces

They give a denotational semantics for differential proofs (Ehrhard, 2003)

**Definition**

A *finiteness space* over \( X \) is a \( C \subset \mathcal{P}(X) \) such that \( C = C^{\perp \perp} \), where \( x \perp y \) iff “\( x \cap y \) is finite”.

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Other connectives are defined by de Morgan duality...
Finiteness Spaces

They give a denotational semantics for differential proofs (Ehrhard, 2003)

**Definition**

A *finiteness space* over $X$ is a $C \subseteq \mathcal{P}(X)$ such that $C = C^\perp\perp$, where $x \perp y$ iff “$x \cap y$ is finite”.

The connectives are given by

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Finiteness spaces give a model of differential $\lambda$-calculus...

“The operations of identification could be seen as formal derivation or formal primitive. The interest of this approach was to propose, at the theoretical level, to replace brutal beta-conversion by iterated linear conversions.”

Girard, “Linear Logic”, 1987
Other Notable Models

- totality spaces (Loader, 1994) $x \perp y$ iff $x \cap y$ contains exactly one element
- Köthe spaces (Ehrhard, 2002)
- probabilistic coherent spaces (Girard, 2004)
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where \( u \) and \( v \) are self-adjoint operators on a finite dimensional Hilbert space

"One of the wild hopes that this suggests is the possibility of a direct connection with quantum mechanics... but let’s not dream too much!", (Girard, "Linear Logic", 1987)
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Old Fashioned Coherent Spaces

The original presentation of coherent spaces uses (reflexive) graphs.

**Definition**

- A coherent space over $X$ is a reflexive graph, $a \bowtie b$ means that $a$ and $b$ are related.
- A coherent set, or clique is a complete subgraph,
- The dual $G^\perp$ of a coherent space is the reflexive closure of its complement.
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Idea of proof:

- Given $G$ over $X$, define $C = \{x \mid x$ is a clique of $G\}$,
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- Don’t forget to check the transformations are inverse to each other.
Comparing Coherence and Finiteness

Even though “$\perp_c \subseteq \perp_f$”, the resulting models are very different.
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In particular

1. finite sets are always finitary, they usually are not cliques
2. they are closed under finite unions, they interpret algebraic \( \lambda \)-calculus, ask, L. Vaux
3. “&” and “⊕” coincide for finitary sets.
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In particular

1. finite sets are always finitary,
2. they are closed under finite unions,
3. “&” and “⊕” coincide for finitary sets.

With that in mind, the following is surprising

**Theorem**

*There is a canonical “inclusion” of Coh into Fin that preserves the linear structure.*
Finitely Incoherent Sets

Considering “finite unions of cliques” would make points 1 and 2 true.
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But that’s not really well behaved.
Finitely Incoherent Sets

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**Definition**

If $C$ is a coherent space over $X$, $x \subseteq X$ is **finitely incoherent** when $x$ doesn’t contain any infinite anticliques,
Finitely Incoherent Sets

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But that’s not really well behaved.

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*If C is a coherent space over X, x ⊆ X is finitely incoherent when x doesn’t contain any infinite anticliques, i.e. when*

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*We write \( \mathcal{F}(C) = C_{\perp_c \perp_f} \) for the set of finitely incoherent sets.*
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Quite surprisingly, we have

**Lemma**

$$\mathcal{F}(C^\perp) = \mathcal{F}(C)^\perp$$
Magic Happens

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Proof: by definition, we need to show that \( C^{\perp \perp} = C = C^{\perp \perp} \).
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**Lemma**

$$\mathcal{F}(\mathcal{C}^\perp c) = \mathcal{F}(\mathcal{C})^\perp f$$

**Proof:** by definition, we need to show that $\mathcal{C}^\perp c^\perp f = \mathcal{C}^\perp f = \mathcal{C}^\perp c^\perp f^\perp f$.

We have $\mathcal{C} \subset \mathcal{C}^\perp f = \mathcal{F}(\mathcal{C})$ and thus $\mathcal{C}^\perp f \supset \mathcal{C}^\perp c^\perp f^\perp f$. 
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\[ \text{if } x \in C^\perp \text{ and } y \in C^\perp C^\perp C^\perp C^\perp: \]

\[ x \cap y \subset y \in \mathcal{F}(C): \text{ doesn’t contain infinite anticliques,} \]

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\[ \text{if } x \cap y \subseteq x \in C^\perp: \text{ doesn’t contain infinite cliques,} \]

by Ramsey’s theorem, \( x \cap y \) is finite and thus \( x \in C^\perp \perp \perp \). We have

\[ C^\perp \subseteq C^\perp \subseteq \mathcal{F}(C)^\perp. \]

P. Hyvernats

Miscellaneous Remarks about Orthogonality
Magic Happens

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...

**Corollary**

\[ C^\perp f \perp f = C^\perp c \perp f \]
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We also have

**Lemma**

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Suppose \( \pi_1(r) \) contains an infinite anticlique \( A \). For each \( a \in A \), take \( b \) such that \((a, b) \in A\).
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Unfortunately

Lemma

*This doesn’t extend to the exponentials.*