Finite group actions on 3-manifolds and cyclic branched covers of knots

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Abstract

As a consequence of a general result about finite group actions on 3-manifolds, we show that a hyperbolic 3-manifold can be the cyclic branched cover of at most fifteen inequivalent knots in $S^3$ (in fact, a main motivation of the present paper is to establish the existence of such a universal bound). A similar, though weaker, result holds for arbitrary irreducible 3-manifolds: an irreducible 3-manifold can be a cyclic branched cover of odd prime order of at most six knots in $S^3$. We note that in most other cases such a universal bound does not exist.

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1 Introduction

A classical way to construct closed, connected, orientable 3-manifolds is to consider cyclic covers of the 3-sphere branched along knots. A natural question in this setting is to understand in how many different ways a closed, connected, orientable 3-manifold can be presented as the (total space of a) cyclic branched cover of a knot. There is an extensive literature on this problem, mainly focussing on the case where the branching order is fixed. For instance, it is known that a closed hyperbolic 3-manifold is the $n$-fold cyclic branched cover of at most two knots in $S^3$, provided $n > 2$ [Z2], and at most nine if $n = 2$ [Re]. For arbitrary closed, connected, orientable, irreducible 3-manifolds some results are known but only for prime orders of ramification. More precisely, such manifolds can be 2-fold cyclic branched covers of arbitrary many knots [Mon1, Mon2] but can cover at most two knots if the order is an odd prime [BoPa]. For branching order equal to 2, several analyses of how the different quotient knots are related can also be found (see, for instance, [MR] for hyperbolic manifolds, [Mon1, Mon2, V, MonW] for toroidal ones, [P] for Conway reducible hyperbolic knots, and [Gr] for alternating ones).

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Possibly due to the fact that the knots that are covered by the same manifold but with different orders of branching are harder to relate, not much was known so far on the general problem, even for hyperbolic manifolds (see [RZ]). The prime motivation for this work was to understand whether it is possible to establish bounds on the number of ways a 3-manifold can be presented as a cyclic branched cover of a knot, without fixing the order of the cover. By the above discussion, no universal bound can be given in general, however the first main result of the present paper assures the existence of a universal bound for the class of closed hyperbolic 3-manifolds.

**Theorem 1.** A closed hyperbolic 3-manifold is a cyclic branched cover of at most fifteen inequivalent knots in $S^3$.

We call two knots equivalent if one is mapped to the other by an orientation-preserving diffeomorphism of $S^3$. In the present paper, all manifolds are closed, connected and orientable, and all maps are smooth and orientation-preserving.

The orientation-preserving isometry group of a closed hyperbolic 3-manifold $M$ is finite, and every finite group occurs for some hyperbolic $M$. Suppose that $M$ is a cyclic branched cover of a knot in $S^3$; then the group of covering transformations acting on $M$ is generated by a hyperelliptic rotation: we call a periodic diffeomorphism of a closed 3-manifold a hyperelliptic rotation if all of its non-trivial powers have connected, non-empty fixed-point set (a simple closed curve), and its quotient (orbit) space is $S^3$. By the geometrization of 3-orbifolds, or of finite group actions on 3-manifolds ([BLP], [BoP], [DL]), the group of covering transformations is conjugate to a subgroup of the isometry group of $M$. Hence establishing a universal upper bound for hyperbolic 3-manifolds as in Theorem 1 is equivalent to bounding the number of conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of the isometry group of a hyperbolic 3-manifold. Now Theorem 1 is a consequence of the following more general result on finite group actions on closed 3-manifolds.

**Theorem 2.** Let $M$ be a closed 3-manifold not homeomorphic to $S^3$. Let $G$ be a finite group of orientation-preserving diffeomorphisms of $M$. Then $G$ contains at most fifteen conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation (at most six for cyclic subgroups whose order is not a power of two).

Note that the 3-sphere is the $n$-fold cyclic branched cover of the trivial knot for any integer $n \geq 2$ (and, by the solution to the Smith conjecture, only of the trivial knot). It is well-known that, for any branching order $n$, a 3-manifold can be the $n$-fold cyclic branched cover of an arbitrary number of non-prime knots (such a manifold is not irreducible), and that an irreducible 3-manifold can be the 2-fold branched cover of arbitrarily many prime knots (see [Mon1, Mon2, V]). Moreover, Proposition 7 in Section 6.1 shows that Seifert fibred manifolds can cover an arbitrary number of knots, all with non-prime orders. It is thus natural to restrict our attention to covers of odd prime order.

For irreducible 3-manifolds, the following holds:

**Theorem 3.** Let $M$ be a closed, irreducible 3-manifold. Then there are at most six inequivalent knots in $S^3$ having $M$ as a cyclic branched cover of odd prime order.
The proof of Theorem 3 uses Theorem 2 in connection with the equivariant torus-decomposition of irreducible 3-manifolds into geometric pieces, see [BoP], [BLP], [CHK], and [KL]. For arbitrary 3-manifolds, as a direct consequence of Theorem 3 as well as the equivariant prime decomposition for 3-manifolds [MSY], the following remains true.

**Corollary 1.** Let $M$ be a closed 3-manifold not homeomorphic to $S^3$. Then $M$ is a $p$-fold cyclic branched cover of a knot in $S^3$ for at most six distinct odd prime numbers $p$.

Another consequence of Theorem 2 is the following characterisation of the 3-sphere which generalises the main result of [BPZ] from the case of $Z$-homology 3-spheres to arbitrary closed 3-manifolds.

**Corollary 2.** A closed 3-manifold $M$ is homeomorphic to $S^3$ if and only if there is a finite group $G$ of orientation-preserving diffeomorphisms of $M$ such that $G$ contains sixteen conjugacy classes of subgroups generated by hyperelliptic rotations.

A noteworthy aspect of the proof of Theorem 2 is the substantial use of finite group theory, in particular of the classification of finite simple groups. For a prime $p$, the $p$-fold cyclic branched cover of a knot in $S^3$ is a rational homology 3-sphere, and we will prove in Section 7 that every finite group acts non-freely on some rational homology 3-sphere, so the use of the classification seems to be intrinsic to the proofs of our results. We note that the class of finite groups acting on a $Z/2$-homology 3-sphere instead is quite restricted (see [MZ]), and in this case the much shorter Gorenstein-Harada classification of finite simple groups of sectional 2-rank at most four is sufficient for our proofs. The bounds that can be derived for $Z/2$-homology 3-spheres are, however, precisely the same as those we get for arbitrary manifolds.

We have tried to separate the algebraic, purely group theoretical parts of the proof (Section 4) from the topological parts (Sections 3 and 5), so they can be read independently.

We note that, although the upper bounds in our results are quite small, at this point we do not know if they are really optimal. In the hyperbolic case, one can easily construct manifolds that are covers of orders $> 2$ of three distinct knots [RZ]. For hyperbolic double covers, the bound (nine) is sharp according to a result of Kawauchi [Ka], but no explicit examples are known so far. For general irreducible manifolds, Brieskorn spheres of type $\Sigma(p,q,r)$, where $p$, $q$, and $r$ are three pairwise different odd primes, provide examples of manifolds that cover four knots: $\Sigma(p,q,r)$ is the $p$-fold (resp. $q$-fold and $r$-fold) cyclic cover of $S^3$ branched along the torus knot $T(q,r)$ (resp. $T(p,r)$ and $T(p,q)$) as well as the double branched cover of a Montesinos knot.

The paper is organised as follows. In Section 2 we present a brief sketch of the proof of Theorem 2. Hyperelliptic rotations and their properties are considered in Section 3. Section 4 contains the main group-theoretical part of the paper, Section 5 the proof of Theorem 2, and Section 6 the proof of Theorem 3 for the irreducible case. Finally, in an Appendix we prove that every finite group acts non-freely on some rational homology sphere (adapting the result of [CL] that deals with free actions).
2 Sketch of the proof of Theorem 2

The proof of Theorem 2 is based on a series of preliminary results which are presented in Sections 3 and 4. Our choice to present the group-theoretical part of the proof in a separate section (Section 4) allows for the group-theoretical results to be read independently of the other parts of the paper. In the following, in order to make the paper more accessible, we explain the main steps of the proof.

We begin with a more detailed definition of hyperelliptic rotation. Note that throughout the paper, unless otherwise stated, 3-manifold will mean orientable, connected, closed 3-manifold. Also, all finite group actions by diffeomorphisms will be faithful and orientation-preserving.

Definition 1. Let \( \psi : M \rightarrow M \) be a finite order diffeomorphism of a 3-manifold \( M \). We shall say that \( \psi \) is a rotation if it preserves the orientation of \( M \), \( \text{Fix}(\psi) \) is non-empty and connected, and \( \text{Fix}(\psi) = \text{Fix}(\psi^k) \) for all non-trivial powers \( \psi^k \) of \( \psi \). \( \text{Fix}(\psi) \) will be referred to as the axis of the rotation. Note that if \( \psi \) is a periodic diffeomorphism of prime order, then \( \psi \) is a rotation if and only if \( \text{Fix}(\psi) = S^1 \). We shall say that a rotation \( \psi \) is hyperelliptic if the space of orbits \( M/\psi \) of its action is \( S^3 \), and a hyperelliptic group is a cyclic group generated by a hyperelliptic rotation.

We start by observing that the case of hyperelliptic rotations whose order is a power of two is already well-understood by work of Reni and Mecchia (see [Re] and [Mec1]). In particular, there are at most nine conjugacy classes of cyclic groups generated by such hyperelliptic rotations. As a consequence, from now on, we exclude this case and consider only hyperelliptic groups whose order is not a power of two.

Section 3 collects various simple facts on the geometry of hyperelliptic rotations. In particular, we prove that there are at most three hyperelliptic groups commuting pairwise. This result implies the existence of a universal bound in the solvable case. In fact, in the solvable case, the presence of Hall subgroups assures that all hyperelliptic subgroups commute, up to conjugacy, implying that there at most three conjugacy classes of such groups.

The case of non-solvable groups, where local approaches on the basis of \( p \)-groups fail, is more involved. We need a global description of the groups that may arise, which is provided in Section 4. In that section, we introduce the notion of an algebraically hyperelliptic collection of cyclic subgroups which have the same algebraic properties as the hyperelliptic subgroups; this allows a purely algebraic approach in Section 4. The main result there (Theorem 4) is that a non-solvable finite group generated by an algebraically hyperelliptic collection is of a very special type, in particular \( G \) has a quotient by a normal solvable subgroup which is isomorphic to the direct product of at most two simple groups.

The next step is to cover the hyperelliptic subgroups by a bounded number of conjugacy classes of solvable subgroups (i.e., to find a collection of solvable subgroups such that each element of odd prime order in a hyperelliptic subgroup has a conjugate in one of these solvable subgroups); this concept of a solvable cover of a finite group is central for the proofs in the present paper since a bound on the number of elements of such a cover implies that there is a bound on the number of conjugacy classes of hyperelliptic subgroups. Using the classification of finite simple groups, we prove, in Proposition 4, that in any
finite simple group the hyperelliptic subgroups can be covered, up to conjugacy, by at most four solvable subgroups. This result, together with the characterisation of groups generated by algebraically hyperelliptic collections, gives directly the existence of a universal bound of fifty-seven, much larger than the bound of fifteen obtained in Theorem 2 by exploiting extra topological considerations. Note that the existence of a universal (although non explicit) bound is ensured by the existence of only a finite number of simple sporadic groups.

In Section 5 we conclude the proof of Theorem 2. We can suppose that $G$ is generated by an algebraically hyperelliptic collection and that $G$ is not solvable. Under these hypotheses, the proof is divided into two cases.

In the first case, we suppose that $G$ contains no rotation of order two. By geometric motivations, an involution acting dihedrally by conjugation on a hyperelliptic subgroup is a rotation. The absence of this type of involutions induces further restrictions on the structure of $G$; in particular, up to a quotient by a solvable subgroup, $G$ is a single simple group. By using these properties of $G$ and the solvable covers we prove that four is the upper bound in this case.

In the second case, $G$ contains a rotation of order two. The groups that may act on 3-manifolds containing such an involution are listed in [Mec2]. We combine this with our result about groups generated by an algebraically hyperelliptic collection and we obtain that a quotient of $G$ by a solvable subgroup must be isomorphic to one of $A_8$, $PSL_2(q)$ or $PSL_2(q) \times PSL_2(q')$. For these groups we explicitly find a solvable cover with a bounded number of elements and hence a universal bound as in Theorem 2.

### 3 Rotations and their properties

In this section we shall establish some properties of rotations in general and hyperelliptic ones in particular.

**Remark 1.** Assume that $\psi$ is a hyperelliptic rotation acting on a 3-manifold $M$. The natural projection from $M$ to the space of orbits $M/\psi$ of $\psi$ is a cyclic cover of $S^3$ branched along a knot $K = \text{Fix}(\psi)/\psi$. The converse is also true, that is any deck transformation generating the automorphism group of a cyclic covering of $S^3$ branched along a knot is a hyperelliptic rotation.

We observe that cyclic branched covers of prime order are closely related to $\mathbb{Q}$-homology 3-spheres.

**Remark 2.**

1. If the order of $\psi$ is a prime $p$, then $M$ is a $\mathbb{Z}/p$-homology sphere [Go].

2. By Smith theory, if $f$ is a periodic diffeomorphism of order $p$, a prime number, acting on a $\mathbb{Z}/p$-homology sphere, then $f$ either acts freely or is a rotation.

We start with a somehow elementary remark which is however central to determine constraints on finite groups acting on 3-manifolds.

**Remark 3.** Let $G \subset \text{Diff}^+(M)$ be a finite group of diffeomorphisms acting on a 3-manifold $M$. One can choose a Riemannian metric on $M$ which is invariant by $G$ and with respect to which $G$ acts by isometries. Let now $\psi \in G$ be
a rotation. Since the normaliser $N_G(\psi)$ of $\psi$ in $G$ consists precisely of those diffeomorphisms that leave the circle $Fix(\psi)$ invariant, we deduce that $N_G(\psi)$ is a finite subgroup of $\mathbb{Z}/2 \times (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$, where the nontrivial element in $\mathbb{Z}/2$ acts by conjugation sending each element of $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ to its inverse. Note that the elements of $N_G(\psi)$ are precisely those that rotate about $Fix(\psi)$, translate along $Fix(\psi)$, or invert the orientation of $Fix(\psi)$; in the last case the elements have order 2 and non-empty fixed-point set meeting $Fix(\psi)$ in two points.

Remark 4. Let us consider the 3-sphere $S^3$. According to Smith’s theory, an orientation-preserving finite-order diffeomorphism of $S^3$ is a rotation if and only if its fixed-point set is non-empty. Because of the positive solution to the Smith conjecture the fixed-point set of a rotation of $S^3$ is the trivial knot. Moreover, any group of symmetries of a non-trivial knot $K$ (that is, any finite group of orientation-preserving diffeomorphisms of $S^3$ acting on the pair $(S^3, K)$) is either cyclic or dihedral.

Lemma 1. Let $\varphi$ and $\psi$ be two rotations contained in a finite group of orientation-preserving diffeomorphisms of a 3-manifold $M$.

1. A non-trivial power of $\psi$ of order different from 2 commutes with a non-trivial power of $\varphi$, if and only if $\varphi$ and $\psi$ commute.

2. Assume $M \neq S^3$. If $\varphi$ and $\psi$ are hyperelliptic and $Fix(\varphi) = Fix(\psi)$, then $\langle \varphi \rangle = \langle \psi \rangle$ (in particular they have the same order).

3. Assume $M \neq S^3$. If $\varphi$ and $\psi$ are hyperelliptic, then $\langle \varphi \rangle$ and $\langle \psi \rangle$ are conjugate if and only if some non-trivial power of $\varphi$ is conjugate to some non-trivial power of $\psi$.

Proof. Part 1 The sufficiency of the condition being obvious, we only need to prove the necessity. Remark that we can assume that both rotations act as isometries for some fixed Riemannian metric on the manifold. Denote by $f$ and $g$ the non trivial powers of $\varphi$ and $\psi$, respectively. Note that, by definition, $Fix(\psi) = Fix(g)$ and $Fix(\varphi) = Fix(f)$. Since $g$ and $f$ commute, $g$ leaves invariant $Fix(\varphi) = Fix(f)$ and thus normalises every rotation about $Fix(\varphi)$. Moreover $g$ and $\varphi$ commute, for the order of $g$ is not 2 (see Remark 3). In particular, $\varphi$ leaves $Fix(\psi) = Fix(g)$ invariant and normalises every rotation about $Fix(\psi)$. The conclusion follows.

Part 2 Reasoning as in Part 1, one sees that the two rotations commute. Assume, by contradiction, that the subgroups they generate are different. Under this assumption, at least one of the two subgroups is not contained in the other. Without loss of generality we can assume that $\langle \varphi \rangle \not\subseteq \langle \psi \rangle$. Take the quotient of $M$ by the action of $\psi$. The second rotation $\varphi$ induces a non-trivial rotation of $S^3$ which leaves the quotient knot $K = Fix(\psi)/\psi \subset S^3$ invariant. Moreover, this induced rotation fixes pointwise the knot $K$. The positive solution to the Smith conjecture implies now that $K$ is the trivial knot and thus $M = S^3$, against the hypothesis.

Part 3 follows from 2 since the conjugate of a hyperelliptic rotation is again a hyperelliptic rotation.

Corollary 3. Let $G$ be a finite group of orientation-preserving diffeomorphisms acting on a 3-manifold $M \neq S^3$. Let $\psi$ be a rotation and let $f \in G$ be an element
of odd prime order which is a power of \( \psi \). Then we have \( C_G(\langle f \rangle) = C_G(\langle \psi \rangle) \) and \( N_G(\langle f \rangle) = N_G(\langle \psi \rangle) \).

There is a natural bound on the number of hyperelliptic subgroups of order not a power of two which commute pairwise; we begin analysing the situation of the symmetry group of a knot.

**Definition 2.** A rotation of a knot \( K \) in \( S^3 \) is a rotation \( \psi \) of \( S^3 \) such that \( \psi(K) = K \) and \( K \cap \text{Fix}(\psi) = \emptyset \). We shall say that \( \psi \) is a full rotation if \( K/\psi \) in \( S^3 = S^3/\psi \) is the trivial knot.

**Remark 5.** Let \( \psi \) and \( \varphi \) be two commuting rotations acting on some manifold \( M \) and with orders not both equal to 2. Assume that \( \psi \) is hyperelliptic and \( \varphi \) is not a power of \( \psi \). According to Remark 3 we have two situations: Either \( \text{Fix}(\psi) \cap \text{Fix}(\varphi) = \emptyset \) and \( \varphi \) induces a rotation \( \phi = \text{Fix}(\psi)/\psi \), or \( \text{Fix}(\psi) = \text{Fix}(\varphi) \). In the former situation we have that \( \varphi \) is hyperelliptic if and only if \( \phi \) is a full rotation. This can be shown by considering the quotient of \( M \) by the action of the group generated by \( \psi \) and \( \varphi \). This quotient is \( S^3 \) and the projection onto it factors through \( M/\varphi \), which can be seen as a cyclic cover of \( S^3 = S^3/\psi \) branched along \( K/\phi \). By the positive solution to the Smith conjecture, \( M/\varphi \) is \( S^3 \) if and only if \( K/\phi \) is the trivial knot. In the latter situation, again because of the positive solution to Smith's conjecture, we have \( M = S^3 \) (compare also Part 2 of Lemma 1).

The following finiteness result about commuting rotations of a non-trivial knot in \( S^3 \) is one of the main ingredients in the proof of Theorem 2 (see [BoPa, Proposition 2], and [BoPa, Theorem 2] for a stronger result where commutativity is not required).

**Proposition 1.** Let \( K \) be a non-trivial knot in \( S^3 \). Let us consider a set of pairwise commuting full rotations in \( \text{Diff}^+(S^3, K) \). The elements of the set generate at most two pairwise distinct cyclic subgroups.

**Proof.** Note, first of all, that, according to Remark 4 the finite subgroup of \( \text{Diff}^+(S^3, K) \) generated by a finite set of pairwise commuting full rotations is cyclic.

Assume now, by contradiction, that there are three pairwise distinct cyclic subgroups generated by commuting full rotations of \( K, \varphi, \psi \) and \( \rho \) respectively. Note that such cyclic subgroups have distinct orders. Assume that two of them -say \( \varphi, \psi\)- have the same axis. Fix the one with smaller order -say \( \psi\)- since \( \psi \) is a full rotation, the quotient \( K/\psi \) is the trivial knot, and \( \varphi \) induces a rotation of \( K/\psi \) which is non-trivial since \( \psi \) commutes with \( \psi \) and its order is larger than that of \( \psi \). The axis of this induced symmetry is the image of \( \text{Fix}(\psi) \) in the quotient \( S^3/\psi \) by the action of \( \psi \). In particular \( K/\psi \) and \( A \) form a Hopf link and \( K \) is the trivial knot: this follows from the equivariant Dehn lemma, see [Hil].

We can thus assume that the axes are pairwise disjoint. Indeed, this follows from Remark 3 taking into account that the rotations commute pairwise and at most one of them has order 2. Therefore we would have that the axis of \( \rho \), which is a trivial knot, admits two commuting rotations, \( \varphi \) and \( \psi \), with distinct axes, which is impossible: this follows, for instance, from the fact (see [EL, Thm 5.2]) that one can find a fibration of the complement of the trivial knot which is equivariant with respect to the two symmetries.
Observe that the proof of the proposition shows that two commuting full rotations of a non-trivial knot either generate the same cyclic subgroup or have disjoint axes.

We stress that if a knot $K \subset S^3$ admits a full rotation, then it is a prime knot, see [BoPa, Lemma 2].

**Lemma 2.** Let $G$ be a finite group of diffeomorphisms acting on a 3-manifold $M \neq S^3$ and $\{H_1, \ldots, H_m\}$ be a set of hyperelliptic subgroups of $G$ of order not a power of two. Suppose that there exists an abelian subgroup of $G$, containing at least an element of odd prime order of each $H_i$, then $m \leq 3$. Moreover either the orders of the $H_i$ are pairwise coprime or $m \leq 2$.

**Proof.**

By Lemma 1 we obtain that the subgroups $H_i$ commute and have trivial intersection pairwise. Consider the cyclic branched covering $M \rightarrow M/H_1 \cong S^3$ over the knot $\text{Fix}(H_1)/H_1$. By projecting $H_i$ with $i \geq 2$ to $M/H_1$ we obtain full rotations of $\text{Fix}(H_i)/H_1$. Note that if two subgroups $H_i$ have the same order that is moreover a divisor of the order of $H_1$, a priori they might map to the same subgroup in $\langle H_1, \ldots, H_m\rangle/H_1$. We claim, however, that the induced full rotations are distinct so there are $m - 1$ of them. This follows from the fact that, for different indices $i$ and $j$, $\text{Fix}(H_i)$ and $\text{Fix}(H_j)$ are disjoint according to Part 2 of Lemma 1. Now, since the subgroups commute, for all $i = 1, \ldots, m$ we have $H_1(\text{Fix}(H_i)) = \text{Fix}(H_i)$, so that the fixed-point sets of the induced full rotations are disjoint, too, and the full rotations are pairwise distinct. By Proposition 1 we obtain $m - 1 \leq 2$. Note that, by Remark 4 a non-trivial knot cannot admit two distinct and commuting cyclic groups of symmetries of the same order. This proves the latter part of the lemma.

The above lemma implies directly the following corollary.

**Corollary 4.** Let $p$ be an odd prime and assume that $H \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$ acts on a 3-manifold $M \neq S^3$, the group $H$ contains at most two distinct cyclic subgroups generated by non-trivial powers of hyperelliptic rotations.

We now collect several facts relative to hyperelliptic rotations that can be deduced from the discussion in this section.

**Remark 6.** Let $G$ be a finite group of orientation-preserving diffeomorphisms of a 3-manifold $M \neq S^3$. Let $\psi \in G$ be a hyperelliptic rotation of order not a power of 2. Recall that the structures of the centraliser and of the normaliser of $\langle \psi \rangle$ are described in Remark 3. Let $N$ denote the normaliser $N_G(\langle \psi \rangle)$.

1. The centraliser $C_G(\langle \psi \rangle)$ of $\psi$ in $G$ satisfies $1 \rightarrow \langle \psi \rangle \rightarrow C_G(\langle \psi \rangle) \rightarrow H \rightarrow 1$, where $H$ is cyclic, possibly trivial. This follows from Remark 4.

2. Since the symmetries of a knot not acting freely have connected fixed-point set (see again Remark 4), the possible elements of $N \setminus C_G(\langle \psi \rangle)$ are rotations of order two.

3. According to last part of Lemma 2, the centraliser $C_G(\langle \psi \rangle)$ contains at most one more cyclic subgroup of the same order as $\langle \psi \rangle$ and generated by a hyperelliptic rotation.

4. Observe that we have $N_G(N) = N_G(C_G(\langle \psi \rangle))$. 

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5. Of course, the conjugate of a hyperelliptic rotation is again a hyperelliptic rotation. Consider an element \( g \in \mathcal{N}_G(N) \). Then either \( g \) conjugates \( \langle \psi \rangle \) to itself and so \( g \) belongs to \( N \), or \( g \) does not normalise \( \langle \psi \rangle \). In this case, \( C_G(\langle \psi \rangle) \) contains precisely two cyclic subgroups generated by hyperelliptic rotations of the same order as \( \psi \) (and another one) and \( g \) exchanges them. It follows that \( C_G(\langle \psi \rangle) \) has index a divisor of 4 in \( \mathcal{N}_G(N) \).

6. Let \( f \in G \) be an element of odd prime order which is a power of \( \psi \). Then we have \( C_G(\langle f \rangle) = C_G(\langle \psi \rangle) \) and \( \mathcal{N}_G(\langle f \rangle) = N \). This is just Corollary 3.

The following general observation will be useful in the sequel.

Remark 7. With the notation used in the previous remark, assume that there are elements in \( \mathcal{N}_G(N) \) which do not act in the same way on \( C_G(\langle \psi \rangle) \), then necessarily the index of \( N \) in \( \mathcal{N}_G(N) \) is two and \( C_G(\langle \psi \rangle) \) contains two distinct subgroups conjugate to \( \langle \psi \rangle \).

We end this section by providing a dictionary translating between algebraic properties of the structure of \( \mathcal{N}_G(N) \) and the symmetries of the knot \( K = \text{Fix}(\psi)/\psi \). This will not be needed in the proofs of our results but will provide a geometric interpretation of the different situations that occur in the proof of Theorem 2 (see Remark 9). We start with the following definition.

Definition 3. A 2-component link is called exchangeable if there exists an orientation-preserving diffeomorphism of \( S^3 \) which exchanges the two components of the link.

Let \( K \) be a knot and \( \rho \) a rotation of \( K \) of order \( n \) and with axis \( A \). Consider the 2-component link \( K \cup A \) consisting of the images of the knot \( K \) and of the axis \( A \) in the quotient \( S^3/\rho \) of the 3-sphere by the action of \( \rho \). Note that at least one component of this link (i.e. \( A \)) is trivial. We call \( K \) \( n \)-self-symmetric if \( K \cup A \) is exchangeable. In this case \( \rho \) is a full rotation of \( K \).

Figure 1: A 5-self-symmetric knot on the left, and its exchangeable quotient link on the right.

Since the structure of the normaliser of \( \langle \psi \rangle \) and of its centraliser, only depend on the symmetries of \( K \) that lift to \( G \), we introduce the following definitions:

Definition 4. Let \( G \) be a finite group of orientation preserving diffeomorphisms of a closed connected 3-manifold \( M \). Let \( \psi \) be a hyperelliptic rotation contained in \( G \) with quotient knot \( K \). We say that \( K \) is strongly invertible with respect
to $G$ if $K$ admits a strong inversion that lifts to $G$. Similarly we say that $K$ is self-symmetric with respect to $G$ if $G$ contains an element $\psi'$ conjugate to $\psi$ such that the subgroup $\langle \psi, \psi' \rangle$ is abelian of rank 2, i.e. not cyclic. Remark that in the latter case $K$ is $n$-self-symmetric, where $n$ is the order of $\psi$.

The proof of the following facts is elementary and left to the reader.

**Proposition 2.** Let $K$, $\psi$, $N$, and $G$ be as above.

- The centraliser of $\langle \psi \rangle$ in $G$ is contained with index 2 in $N$ if and only if $K$ is strongly invertible with respect to $G$.
- $N$ is contained with index 2 in $N_G(N)$ if and only if $K$ is self-symmetric with respect to $G$.

Moreover, if $M$ is hyperbolic and $G = \text{Iso}^+(M)$, $K$ is strongly invertible if and only if it is strongly invertible with respect to $G$, and it is self-symmetric with respect to $G$ if and only if it is $n$-self-symmetric, where $n$ is the order of the hyperelliptic rotation $\psi$.

### 4 Group theoretical results

This section is devoted to the proofs of the group-theoretical results that are used to obtain the bound provided by Theorem 2.

The first result, Theorem 4, is probably the more interesting of the two from a group-theoretical point of view. It describes the finite groups $G$ that may be generated by hyperelliptic rotations. In this settings, a hyperelliptic rotation is simply an element of $G$ of order not a power of two, satisfying a purely algebraic condition (see Definition 5, below) on the structure of the normaliser of (a power of odd prime order of) the element. More precisely, Theorem 4 states that either $G$ is solvable, or it admits a quotient by a normal solvable subgroup so that the quotient is either the product of a simple group times a cyclic group of odd order, or the product of two simple groups. We see that the fact of being generated by hyperelliptic rotations puts very strong constraints on the structure of $G$.

The second result, Proposition 4, although possibly not as striking as the previous one, is however key to be able to carry out the strategy of bounding the number of hyperelliptic subgroups by covering them with solvable subgroups (see Definition 6). Indeed, in Proposition 4 we establish the fact that four conjugacy classes of solvable subgroups are sufficient to contain, up to conjugacy, all hyperelliptic subgroups of orders not a power of two that may sit inside a finite simple group.

Note that these two main results imply that $4 \times 4$ solvable subgroups suffice to cover all hyperelliptic groups in $G$, and together with Lemma 3, and Lemma 2 show that a finite group $G$ can contain at most forty-eight conjugacy classes of hyperelliptic subgroups of orders that are not powers of 2.

In the following $G$ will denote a finite group. We start with some preliminary observations and definitions.
Definition 5. A collection \( \{C_i\} \) of subgroups of \( G \), each of odd prime order \( p_i \), is said to be *algebraically hyperelliptic* if, for each \( i \), the following conditions are satisfied:

1. the centraliser of \( C_i \) in \( G \) is abelian of rank at most two and has index at most two in the normaliser of \( C_i \) in \( G \);
2. each element belonging to the normaliser of \( C_i \) but not to the centraliser inverts by conjugation each element in the centraliser;
3. if \( C_i \) is contained in a Sylow \( p_i \)-subgroup \( S_i \) of \( G \), then \( S_i \) contains at most two distinct conjugates of \( C_i \).

We remark that in this definition the primes \( p_i \) are not necessarily pairwise distinct.

Proposition 3. Let \( S \) be a Sylow \( p_i \)-subgroup where \( p_i \) is the order of a group \( C_i \) belonging to an algebraically hyperelliptic collection.

1. \( S \) is either cyclic or the product of two cyclic groups, and
2. \( N_G(S) \) contains with index at most 2 the normaliser of a conjugate of \( C_i \), and contains an abelian subgroup of rank at most 2 with index a divisor of 4. In particular \( N_G(S) \) is solvable.

Proof. Up to conjugation we can suppose that \( S \) contains \( C_i \). By Properties 1 and 2 in Definition 5 and the fact that \( p_i \) is odd, the normaliser \( N_S(C_i) \) is abelian of rank at most two. Property 3 in Definition 5 implies that \( N_S(N_S(C_i)) = N_S(C_i) \). Since \( S \) is a \( p_i \)-subgroup we obtain that \( S = N_S(C_i) \) and we get the first part of the thesis.

Since the subgroup \( N_G(S) \) normalises the maximal elementary abelian subgroup of \( S \), we obtain also the second part of the thesis.

Lemma 3. Let \( G \) be a solvable group containing an algebraically hyperelliptic collection \( \{C_1, \ldots, C_m\} \) of subgroups of odd prime order. Then there exists an abelian subgroup of \( G \) containing a conjugate of \( C_i \), for each \( i = 1, \ldots, m \). In particular the subgroups \( C_i \) commute pairwise up to conjugacy.

Proof. Let \( \pi \) be the set of the orders of the \( C_i \) and let \( B \) be a Hall \( \pi \)-subgroup of \( G \). Each \( C_i \) is conjugate to a subgroup of \( B \). Since \( \pi \) contains only odd primes, Definition 5 yields that centraliser and normaliser of every Sylow \( p \)-subgroup of \( B \) coincide. By Burnside’s \( p \)-complement theorem (see [Su, Theorem 2.10 page 144]), every Sylow \( p \)-subgroup of \( B \) has a normal complement, and hence \( B \) is abelian.

Remark 8. Suppose that \( N \) is a normal subgroup of \( G \) and \( H \) is a \( p \)-subgroup of \( G \). If the order of \( N \) is coprime with \( p \), then the normaliser of the projection of \( H \) to \( G/N \) is the projection of the normaliser of \( H \) in \( G \), that is

\[
N_{G/N}(HN/N) = N_G(H)N/N.
\]

The inclusion \( \supseteq \) holds trivially. We prove briefly the other inclusion. Let \( fN \) be an element of \( N_{G/N}(HN/N) \), then \( Hf \subseteq HN \). Both \( Hf \) and \( H \) are Sylow
$p$-subgroups of $HN$ and by the second Sylow theorem they are conjugate by an element $hn \in HN$. We obtain that $H^{hn} = H$, and hence $f \in N_G(H)N$.

Analogously if $f$ is an element of prime order coprime with the order of $N$, then $C_{G/N}(fN) = C_G(f)N/N$.

Recall that a finite group $Q$ is quasisimple if it is perfect (the abelianised group is trivial) and the factor group $Q/Z$ of $Q$ by its centre $Z$ is a nonabelian simple group (see [Su, chapter 6.6]). A group $E$ is semisimple if it is perfect and the factor group $E/Z(E)$ is a direct product of nonabelian simple groups. A semisimple group $E$ is a central product of quasisimple groups which are uniquely determined. Any finite group $G$ has a unique maximal semisimple normal subgroup $E(G)$ (maybe trivial), which is characteristic in $G$. The subgroup $E(G)$ is called the layer of $G$ and the quasisimple factors of $E(G)$ are called the components of $G$.

The maximal normal nilpotent subgroup of a finite group $G$ is called the Fitting subgroup and is usually denoted by $F(G)$. The Fitting subgroup commutes elementwise with the layer of $G$. The normal subgroup generated by $E(G)$ and by $F(G)$ is called the generalised Fitting subgroup and is usually denoted by $F^*(G)$. The generalised Fitting subgroup has the important property to contain its centraliser in $G$, which thus coincides with the centre of $F^*(G)$. For further properties of the generalised Fitting subgroup see [Su, Section 6.6.].

**Theorem 4.** Suppose that $G$ is generated by the algebraically hyperelliptic collection $\mathcal{H} := \{C_1, \ldots, C_m\}$. Denote by $p_i$ the order of $C_i$ and by $A$ the maximal normal solvable subgroup of order coprime with every $p_i$.

If $G$ is non-solvable, then the following properties hold:

1. every $p_i$ divides the order of any component of $G/A$;
2. either $G/A$ is the direct product of a cyclic group of odd order and a simple group or it is the direct product of two simple groups;
3. if in addition $G$ does not contain any involution acting dihedraly on any $C_i$, then $E(G/A)$ is simple, every $p_i$ divides the order of $F(G/A)$ and a Sylow $p_i$-subgroup of $G$ contains exactly two conjugates of $C_i$.

**Proof.** Let $\pi$ be the set of the primes $p_i$.

By Remark 8 we can suppose that $A$ is trivial and $F(G)$ is a $\pi$-group.

**Claim 1.** $F(G)$ is cyclic and $E(G)$ is not trivial.

Suppose first that $F(G)$ contains an abelian $p_i$-subgroup $S$ of rank two. Then $S$, being the maximal elementary abelian $p_i$-subgroup contained in $F(G)$, is normal in $G$ and contains $C_i$. This implies that $G$ is solvable and we get a contradiction. Hence $F(G)$ is cyclic.

If $E(G)$ is trivial, then $F(G) = F^*(G)$ and $G/F(G)$ is isomorphic to a subgroup of $\text{Aut}F(G)$. Since $F(G)$ is cyclic $\text{Aut}F(G)$ is solvable, and we get again a contradiction.

**Claim 2.** Each $p_i$ divides the order of any component of $G$. Moreover the components of $G$ are simple groups and are at most two.
Since the Sylow $p_i$-subgroups are abelian and $A$ is trivial, by [Su, Exercise 1, page 161] the components of $G$ have trivial centre.

Now we prove that each component of $G$ is normalised by any $C_i$. Let $f_i$ be a generator of $C_i$ and $Q$ a component of $G$. Suppose by contradiction that $Q$ is not normalised by $f_i$. We define the following subgroup:

$$Q_c = \{ x f_1 f_1^{-1} \cdots f_i f_i^{-1} x f_i^{p_i-1} | x \in Q \}.$$ 

Since the components of $G$ commute elementwise, $Q_c$ is a subgroup of $G$ isomorphic to $Q$. Moreover, each element of $Q_c$ commutes with $f_i$ and this gives a contradiction.

We have that $C_i$ normalises $Q$ but cannot centralise it, so the action by conjugation of $f_i$ on $Q$ is not trivial.

Assume that $Q$ is either sporadic or alternating. Since the order of the outer automorphism group of any such simple group is a (possibly trivial) power of 2 (see [GLS, Section 5.2 and 5.3]), we conclude that $f$ must induce an inner automorphism of $Q$. In particular $p_i$ divides the order of $Q$.

We can thus assume that $Q$ is a simple group of Lie type.

Recall that, by [GLS, Theorem 2.5.12], $Aut(Q)$ is the semidirect product of a normal subgroup $Inndiag(Q)$, containing the subgroup $Inn(Q)$ of inner automorphisms, and a group $\Phi \Gamma$, where, roughly speaking, $\Phi$ is the group of automorphisms of $Q$ induced by the automorphisms of the defining field and $\Gamma$ is the group of automorphisms of $Q$ induced by the symmetries of the Dynkin diagram associated to $Q$ (see [GLS] for the exact definition). By [GLS, Theorem 2.5.12.(c)], every prime divisor of $|Inndiag(Q)|$ divides $|Q|$. Thus we can assume that the automorphism induced by $f_i$ on $Q$ is not contained in $Inndiag(Q)$ and its projection $\theta$ on $Aut(Q)/Inndiag(Q) \cong \Phi \Gamma$ has order $p_i$.

We will find a contradiction showing that in this case the centraliser of $f_i$ in $Q$ is not abelian.

Write $\theta = \phi \gamma$, with $\phi \in \Phi$ and $\gamma \in \Gamma$. If $\phi = 1$, then $\gamma$ is nontrivial and $f_i$ induces a graph automorphism according to [GLS, Definition 2.5.13]. Since $p_i$ is odd, the only possibility is that $Q$ is $D_4(q)$ and $p_i = 3$ (see [GLS, Theorem 2.5.12 (e)]). The centraliser of $f_i$ in $Q$ is nonabelian by [GLS, Table 4.7.3 and Proposition 4.9.2.]. If $\phi \neq 1$ and $Q$ is not isomorphic to the group $^3D_4(q)$, then the structure of the centraliser of $f_i$ in $Q$ is described by [GLS, Theorem 4.9.1], and it is nonabelian. Finally, if $\phi \neq 1$ and $Q \cong ^3D_4(q)$, the structure of the nonabelian centraliser of $f_i$ in $Q$ follows from [GLS, Proposition 4.2.4]. We proved that the automorphism induced by $f_i$ is contained in $Inndiag(Q)$ and $p_i$ divides $|Q|$.

Since $p_i$ divides the order of any component, $G$ has at most two components.

**Claim 3.** $G = E(G)F(G)$.

We prove first that $G = E(G)C_G(E(G))$. Let us assume by contradiction that there exists $C_i$ with trivial intersection with $E(G)C_G(E(G))$ and denote by $f$ a generator of $C_i$. Since $p_i$ divides the order of every component of $G$ and the Sylow $p_i$-subgroup has rank at most 2, we get that $E(G)$ has only one component which we denote by $Q$. The Sylow $p_i$-subgroups of $Q$ are cyclic. Moreover, by the first part of the proof, the automorphism induced by $f$ on $Q$ is inner-diagonal. If it is inner, we obtain $f$ as a product of an element that centralises $Q$ and an element in $Q$, a contradiction to our assumption; otherwise, we get again...
contradiction, since, by [GLS, Theorem 2.5.12] and [A, (33.14)], a group of Lie type with cyclic Sylow $p_i$-subgroup cannot have a diagonal automorphism of order $p_i$.

Hence, all the subgroups $C_i$ are contained in $E(G)C_G(E(G))$ and, since they generate $G$, we obtain that $G = E(G)C_G(E(G))$.

Now if $F(G) = 1$, then $F^*(G) = E(G)$ and hence $C_G(E(G)) = C_G(F^*(G)) = Z(E(G)) = 1$ and the claim is proved. So suppose that $F(G) \neq 1$. Then, since $F(G)$ is a $\pi$-group, there is at least one subgroup $C_i$ that is contained in $E(G)F(G)$. Hence there is a subgroup $T_1$ of $E(G)$ with order $p_i$ and a subgroup $T_2$ of $F(G)$ with order $p_i$ such that $C_i \leq T_1T_2$. Since $F(G)$ is cyclic, $T_2$ is normal in $G$ and so $C_G(E(G)) \leq N_G(T_1T_2)$. But $N_G(T_1T_2)$ has an abelian normal subgroup of index a divisor of 4, so $C_G(E(G))$ is solvable. This shows that every Sylow $p$-subgroup of $C_G(E(G))$ for $p$ odd is contained in $F(G)$, whence it follows that $G = E(G)F(G)$.

Claim 4. If no involution of $G$ acts dihedraly on any $C_i$, then $E(G)$ is simple, and, for every $i$, $p_i$ divides the order of $F(G)$ and a Sylow $p_i$-subgroup of $G$ contains exactly two conjugates of $C_i$.

Suppose no involution of $G$ acts dihedraly on any $C_i$. By Definition 5, it follows that $N_G(C_i) = C_G(C_i)$ for every $i = 1, \ldots, m$. Let $Q$ be a component of $E(G)$ and suppose by contradiction that $Q$ contains a Sylow $p_i$-subgroup $S$ of rank two, for some $i$. Up to conjugation we can suppose that $S$ contains $C_i$. By Definition 5, $N_Q(S)$ contains with index at most two the abelian group $C_Q(C_i)$. Since $Q$ is simple, Burnside’s $p$-complement theorem (see [Su, Theorem 2.10 page 144]) yields that $N_Q(S)$ is not abelian. Therefore, $N_Q(S)$ contains with index two $N_Q(C_i)$ and the elements of $N_Q(S) \setminus N_Q(C_i)$ conjugate $C_i$ to a cyclic subgroup distinct from $C_i$. By using [Su, Exercise 1, page 161] and the fact that $Q$ is perfect we get again a contradiction. Hence, for every $i \in \{1, \ldots, m\}$, the Sylow $p_i$-subgroups of $Q$ are cyclic. Now, as above for every Sylow subgroup $S$ of $Q$ we have $N_Q(S) \neq C_Q(S)$. Since $N_G(C_i) = C_G(C_i)$ for every $i = 1, \ldots, m$, that implies that the Sylow $p_i$-subgroups of $G$ are not cyclic and hence $p_i$ divides the order of $F(G)$ for every $i$.

To bound above the number of conjugacy classes of hyperelliptic rotations, our strategy will consist of conjugating hyperelliptic rotations in solvable subgroups of $G$, where they are forced to commute, hence, in analogy with the standard definition of normal covers, we introduce the notion of solvable normal $\pi$-cover and we prove the following lemma.

Definition 6. Let $G$ be a finite group. Let $\pi$ be a set of primes dividing $|G|$. We will call a collection $\mathcal{C}$ of subgroups of $G$ a solvable normal $\pi$-$\text{cover}$ of $G$ if every element of $G$ of prime order $p$ belonging to $\pi$ is contained in an element of $\mathcal{C}$ and for every $g \in G$, $H \in \mathcal{C}$ we have that $H^g \in \mathcal{C}$. We denote by $\gamma^\pi_\mathcal{C}(G)$ the smallest number of conjugacy classes of subgroups in a solvable normal $\pi$-$\text{cover}$ of $G$. Here, the letter “$s$” stands for “solvable”, and is used to distinguish this number from $\gamma(G)$, that is the standard notation in the case of covers by subgroups that are not requested to be solvable. Note that, since Sylow subgroups are clearly solvable, $\gamma^\pi_\mathcal{C}(G) \leq |\pi|$.

Proposition 4. Let $G$ be a finite nonabelian simple group. If $\pi$ is the set of odd primes $p$ such that $G$ has cyclic Sylow $p$-subgroup, the centraliser of $C_G(g)$ is
abelian for every element \( q \in G \) of order \( p \) and the normaliser of any subgroup of order \( p \) contains with index at most two its centraliser, then \( \gamma_2^G(G) \leq 4 \).

**Proof.**

If \( G \) is a sporadic group, the primes dividing the order of the group do not satisfy the condition on the normaliser.

If \( G \) is isomorphic to the alternating group \( A_n \) and \( p \in \pi \), then the condition on the centraliser of the elements of order \( p \) implies that \( p > n - 4 \) and \( \gamma_2^G(G) \leq |\pi| \leq 2 \).

The only remaining case is that of groups of Lie type.

Let \( G \cong \Sigma_n(q) \) or \( G \cong d\Sigma_n(q) \), where \( q \) is a power of a prime \( t \). Here we use the same notation as in [GLS]: the symbol \( \Sigma(q) \) (resp. \( d\Sigma(q) \)) may refer to finite groups in different isomorphism classes, each of them is an untwisted (resp. twisted) finite group of Lie type with root system \( \Sigma \) (see [GLS, Remark 2.2.5]). Any finite group of Lie type is quasisimple with the exception of the following groups: \( A_1(2) \), \( A_1(3) \), \( 2A_2(2) \), \( 2B_2(2) \), \( B_2(2) \), \( G_2(2) \), \( 2F_4(2) \) and \( 2G_2(3) \) (see [GLS, Theorem 2.2.7]).

If \( t \in \pi \), then by [GLS, Theorem 3.3.3], either \( t = 3 \) and \( G \cong (2G_2(3))' \) or \( G \cong A_1(s) \). In the former case the order of \( (2G_2(3))' \) is divided only by two odd primes, thus \( \gamma_2^G(G) \leq 2 \); in the latter case we have \( \gamma_2^G(G) \leq 2 \) (see for example [H]).

Assume now that \( t \notin \pi \). By [GLS, Paragraph 4.10], since the Sylow subgroups are cyclic, every element of order \( p \in \pi \) is contained in a maximal torus of \( G \), and clearly a maximal torus is abelian.

Therefore, we need only to bound the number of conjugacy classes of cyclic maximal tori in \( G \) with abelian centraliser. Note that the number of conjugacy classes of maximal tori in \( G \) is bounded by the number of different cyclotomic polynomials evaluated in \( q \) appearing as factors of \( |G| \). Moreover the power of a cyclotomic polynomial in the order of \( G \) gives the rank of the corresponding maximal torus (except possibly when the prime divides the order of the centre but in this case the Sylow subgroup is not cyclic, see [A, (33.14)])

Recall \( \Sigma \) is the root system associated to \( G \) as in [GLS, 2.3.1]; let \( \Pi = \{ \alpha_1, \ldots, \alpha_n \} \) be a fundamental system for \( \Sigma \) as in Table 1.8 in [GLS], \( \alpha_i \) be the lowest root relative to \( \Pi \) as defined in [GLS, Paragraph 1.8] and set \( \Pi^* = \Pi \cup \{ \alpha_i \} \). We recall that \( |G| \) can be deduced from [GLS, Table 2.2] and the Dynkin diagrams can be found in [GLS, Table 1.8]. Observe that, by [GLS, Proposition 2.6.2], if \( \Sigma_0 \) is a root subsystem of \( \Sigma \), then \( G \) contains a subsystem subgroup \( H \), which is a central product of groups of Lie type corresponding to the irreducible constituents of \( \Sigma_0 \). In order to prove the lemma, we shall show that for every group \( G \) and for every element \( g \) of order \( p \) a prime \( r \) lying in a maximal torus belonging to any but four conjugacy classes of maximal tori, either the Sylow \( r \)-subgroup is not cyclic or we find a subsystem subgroup \( H \) that is a central product of two groups \( H_1 \) and \( H_2 \) such that \( H_1 \) contains \( q \) and \( H_2 \) is not abelian. Note that for every prime power \( q \), \( A_1(q) \) is a non-abelian group (see [GLS, Theorem 2.2.7]).

We treat the case \( G \cong A_n(q) \) in detail as an example. All other cases can be dealt similarly. Assume \( G \cong A_n(q) \). Let \( m \) be the minimum index \( i \) such that \( r \) divides \( q^{i+1} - 1 \) and let \( \Sigma_0 \) be generated by \( \Pi^* \setminus \{ \alpha_1, \alpha_n \} \). Then the corresponding subsystem subgroup is \( H = H_1 \cdot H_2 \), where \( H_1 \cong A_{n-2}(q) \) and \( H_2 \cong A_1(q) \).

Thus if \( m \leq n - 1 \), then \( H_1 \) contains an element \( g \) of order \( r \) and \( C_G(g) \) contains...
which is not abelian. Therefore, since \( g \) has an abelian centraliser, \( r \) may divide only \((q^n - 1)(q^{n+1} - 1)\), that is \( r \) divides \( \Phi_n(q)\Phi_{n+1}(q) \). Hence we have at most two conjugacy classes of maximal tori with abelian centraliser.

5 Proof of Theorem 2

Let \( G \) be a finite group of orientation preserving diffeomorphisms of an orientable connected 3-manifold which is not homeomorphic to \( S^3 \).

As noted in Section 2, there at most nine conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation of order \( 2^n \), so we concentrate on hyperelliptic subgroups whose order is not a power of two and we will prove that there are at most six conjugacy classes of such subgroups.

Let \( S = \{C_1, \ldots, C_m\} \) be the set of cyclic subgroups of odd prime orders that are generated by powers of the hyperelliptic rotations of \( G \). We recall that the conjugate of a power of a hyperelliptic rotation is the power of a hyperelliptic rotation too. By Remark 3 and Corollary 4, \( S \) is an algebraically hyperelliptic collection, and actually the definition of an algebraically hyperelliptic collection was chosen precisely to capture the behaviour of cyclic subgroups of odd prime orders generated by powers of the hyperelliptic rotations. For \( i \in \{1, \ldots, m\} \), let \( p_i \) be the prime order of \( C_i \) and \( \pi \) be the set including every \( p_i \). We denote by \( G_0 \) the subgroup generated by the subgroups \( C_i \). Let \( A \) be the maximal normal solvable subgroup of \( G_0 \) of order coprime with all \( p_i \). We denote by \( G_0/A \) and by \( \overline{C_i} \) the projection of \( C_i \) to \( G_0 \).

**Case 1.** If \( G_0 \) is solvable, we are in the hypotheses of Lemma 3 which affirms that up to conjugacy the hyperelliptic rotations of \( G \) commute. Since they commute we can apply Lemma 2 and conclude that there are at most three conjugacy classes of subgroups generated by hyperelliptic rotations of order not a power of two.

Even if its proof is easier than that of the remaining situations, the solvable case is interesting in its own right, in particular for it plays an important role also in the proof of Theorem 3. We then summarise the conclusions in the following proposition.

**Proposition 5.** Let \( M \) be a closed 3-manifold not homeomorphic to \( S^3 \). Let \( G \) be a finite group of orientation preserving diffeomorphisms of \( M \). If \( G \) is solvable, it contains at most three conjugacy classes of hyperelliptic subgroups of order not a power of two, and any two such subgroups commute up to conjugacy.

Moreover, if there are three conjugacy classes, then their orders must be pairwise coprime.

**Case 2.** Suppose \( G_0 \) is not solvable and it has no rotation of order 2 outside \( A \). Then, because of the structure of the normaliser of a hyperelliptic rotation as described in Remark 3, \( G_0 \) has no involution acting dihedrally on any \( C_i \) and by Part 3 of Theorem 4 describing the structure of \( G_0 \), \( E(G_0) \) is simple and, for each \( p_i \), any Sylow \( p_i \)-subgroup contains exactly two distinct conjugates of \( \overline{C_i} \). By Remark 3 every hyperelliptic rotation commuting with one of these two subgroups of order \( p_i \) commutes also with the other one. From this fact, Lemma 3, and Lemma 2, it follows that \( \gamma_2^*(G_0) \) bounds from above the number of conjugacy classes of hyperelliptic subgroups of order not a power of two (see
also Proposition 5). It is easy to see that \( \gamma_s^\pm(G_0) \leq \gamma_s^\pm(E(\overline{G_0})) \). By Proposition 4 we have \( \gamma_s^\pm(E(\overline{G_0})) \leq 4 \) and so we get the thesis in this case.

**Case 3.** Suppose \( G_0 \) is not solvable and it has a rotation of order 2 not contained in \( A \). The groups containing a rotation of order two are studied in [Mec2] where the following result was proved.

**Theorem 5.** [Mec2] Let \( D \) be a finite group of orientation-preserving diffeomorphisms of a closed orientable 3-manifold. Let \( O \) be the maximal normal subgroup of odd order and \( E(D) \) be the layer of \( D = D/O \). Suppose that \( D \) contains an involution which is a rotation.

1. If \( E(D) \) is trivial, there exists a normal subgroup \( H \) of \( D \) such that \( H \) is solvable and \( D/H \) is isomorphic to a subgroup of \( \mathfrak{s}_8 \), the alternating group on 8 letters.

2. If the semisimple group \( E(D) \) is not trivial, it has at most two components and the factor group of \( D/E(D) \) is solvable.

Moreover if \( D \) contains a rotation of order 2 such that its projection is contained in \( E(D) \), then \( E(D) \) is isomorphic to one of the following groups:

\[
PSL_2(q), \ SL_2(q) \times_{Z/2} SL_2(q')
\]

where \( q \) and \( q' \) are odd prime powers greater than 4.

Applying Theorem 5 and Theorem 4 to \( G_0 \), we get that \( E(\overline{G_0}) \) is isomorphic either to a subgroup of \( \mathfrak{s}_8 \), or to \( PSL_2(q) \), or to \( PSL_2(q) \times PSL_2(q') \). In the first case, there are at most three odd primes dividing the order of \( E(\overline{G_0}) \) and the thesis follows again from Lemma 2 and Lemma 3.

In the remaining cases, we will use a solvable normal \( \pi \)-cover to bound the number of conjugacy classes. We have that \( \gamma_s^\pm(PSL_2(q)) \leq 2 \). In fact the upper triangular matrices form a solvable subgroup of \( SL_2(q) \) of order \( (q - 1)q \), moreover \( SL_2(q) \) contains a cyclic subgroup of order \( q + 1 \) (see [H]). The conjugates of the projections of these two subgroups to \( PSL_2(q) \) give a solvable normal \( \pi \)-cover of \( PSL_2(q) \). It is again easy to see that, if \( E(\overline{G_0}) \) is isomorphic to \( PSL_2(q) \), then \( \gamma_s^\pm(G) \leq 2 \). As above, by Lemma 3 and Lemma 2, we get a bound of six in this case.

Finally, if \( E(\overline{G}) \) is isomorphic to \( PSL_2(q) \times PSL_2(q') \), then it follows from the discussion above that \( \gamma_s^\pm(E(\overline{G_0})) \leq 4 \), and hence \( \gamma_s^\pm(G_0) \leq 4 \). Now, by Part 1 of Theorem 4, for each \( p_i \in \pi \), \( p_i \) divides the order of both \( PSL_2(q) \) and \( PSL_2(q') \), so that the Sylow \( p_i \)-subgroup of \( \overline{G_0} \) has rank 2. For each Sylow \( p_i \)-subgroup \( S_i \), one can find in \( E(\overline{G_0}) \) elements of the group which normalise \( S_i \) but do not act in the same way on all of its elements. Indeed, let \( H \) be a cyclic subgroup of order \( p_i \) in \( PSL_2(q) \). If \( p_i \) does not divide \( q \), then there is an element of order 2 in \( PSL_2(q) \) which acts dihedrally on \( H \) but commutes with all elements of order \( p \) in \( PSL_2(q') \). If \( p_i \) divides \( q \), then \( q = p_i \) since the Sylow \( p_i \)-subgroup of the component must be cyclic. Since \( q > 3 \), we have again elements in \( PSL_2(q) \) that normalise the Sylow \( p \)-subgroup but do not centralise it (note that in this case the structure of the normaliser is compatible with the description given in Remark 3 only if \( p = q = 5 \)). We now deduce from Remark 7 and Corollary 3 that \( S_i \) contains two subgroups conjugate to \( C_i \).
Reasoning as in Case 2 we get that $\gamma^+_s(G_0)$ bounds from above the number of conjugacy classes of hyperelliptic subgroups of order not a power of two. This concludes the proof.

Remark 9. We wish to stress that the three cases that appear in the proof of Theorem 2 correspond to different types of symmetries possessed by the knots that are branched covered by the manifold. Let $M$ be a closed, connected, oriented 3-manifold and let $G$ a finite group of diffeomorphisms of $M$ generated by a set $\{\psi_1, \ldots, \psi_k\}$ of hyperelliptic rotations. Denote by $K_i$ the quotient knot $\text{Fix}(\psi_i)/\psi_i$, $i = 1, \ldots, k$. If there exists a $K_i$ which is neither strongly invertible nor self-symmetric with respect to $G$, then we are in Case 1, that is, the hyperelliptic rotations commute up to conjugacy (see [Su, Thm 2.10, page 144]). If there is a $K_i$ which is strongly invertible with respect to $G$, then $G$ contains a rotation of order 2 and we are in Case 3. Otherwise, every $K_i$ is self-symmetric with respect to $G$ but not strongly invertible with respect to $G$, and we are in Case 2.

The proof of Theorem 2 shows that topology and geometry can impose extra constraints on the conditions that can be derived by the algebra alone. In this spirit, note that if $M$ is a (closed, connected, oriented) reducible 3-manifold and $G$ a finite group of diffeomorphisms of $M$ generated by hyperelliptic rotations, then, by the equivariant sphere theorem, $G$ is isomorphic to a finite subgroup of $SO(3)$, that is cyclic, dihedral or a spherical triangular group. It follows readily, that, up to conjugacy, at most three cyclic hyperelliptic subgroups can be contained in any such $G$. We know, however, that the group of diffeomorphisms of a reducible 3-manifolds can admit arbitrarily many conjugacy classes of cyclic hyperelliptic subgroups.

The main point here is that, generically, one expects that two hyperelliptic rotations in the group $\text{Diff}^+(M)$ generate an infinite subgroup. As a consequence, Theorem 2 cannot be directly exploited to obtain bounds for non-hyperbolic manifolds and new strategies must be developed, as we will see in the next section.

6 Proof of Theorem 3

The statement of Theorem 3 is equivalent to the following:

Theorem 6. Let $M$ be a closed, orientable, connected, irreducible 3-manifold which is not homeomorphic to $S^3$, then the group $\text{Diff}^+(M)$ of orientation preserving diffeomorphisms of $M$ contains at most six conjugacy classes of hyperelliptic subgroups of odd prime order.

6.1 Proof of Theorem 6 for Seifert manifolds

In this section we prove Proposition 6 which implies Theorem 6 for closed Seifert fibred 3-manifolds. We also show that the assumption that the hyperelliptic rotations have odd prime orders cannot be avoided in general by exhibiting examples of closed Seifert fibred 3-manifolds $M$ such that $\text{Diff}^+(M)$ contains an arbitrarily large number of conjugacy classes of hyperelliptic subgroups of odd, but not prime, orders.
Proposition 6. Let $M$ be a closed Seifert fibred 3-manifold which is not homeomorphic to $S^3$. Then the group $\text{Diff}^+(M)$ of orientation preserving diffeomorphisms of $M$ contains at most one conjugacy class of hyperelliptic subgroups of odd prime order except if $M$ is a Brieskorn integral homology sphere with 3 exceptional fibres. In this latter case $\text{Diff}^+(M)$ contains at most three non conjugate hyperelliptic subgroups of odd prime orders.

Proof.

By hypothesis $M$ is a cyclic cover of $S^3$ branched over a knot, so it is orientable and a rational homology sphere by Remark 2. Notably, $M$ cannot be $S^1 \times S^2$ nor a Euclidean manifold, except for the Hantzsche-Wendt manifold, see [Or, Chap. 8.2]. In particular, since $M$ is prime it is also irreducible.

Consider a hyperelliptic rotation $\psi$ on $M$ of odd prime order $p$ and let $K$ be the image of $\text{Fix}(\psi)$ in the quotient $S^3 = M/\psi$ by the action of $\psi$. The knot $K$ must be hyperbolic or a torus knot, otherwise its exterior would be toroidal and have a non-trivial JSJ-collection of essential tori which would lift to a non-trivial JSJ-collection of tori for $M$, since the order of $\psi$ is $p > 2$ (see [JS, J] and [BS]). By the orbifold theorem (see [BoP], [CHK]), the cyclic branched cover with order $p \geq 3$ of a hyperbolic knot is hyperbolic, with a single exception for $p = 3$ when $K$ is the figure-eight knot and $M$ is the Hantzsche-Wendt Euclidean manifold. But then, by the orbifold theorem and the classification of 3-dimensional crystallographic groups, $\psi$ generates the unique, up to conjugacy, Euclidean hyperelliptic cyclic subgroup of $\text{Diff}^+(M)$, see for example [Dun], [Zi].

So we can assume that $M$ is the $p$-fold cyclic cover of $S^3$ branched along a non-trivial torus knot $K$ of type $(a,b)$, where $a > 1$ and $b > 1$ are coprime integers. Then $M$ is a Brieskorn-Pham manifold $M = V(p,a,b) = \{z^p + x^a + y^b = 0 \}$ with $(z, x, y) \in \mathbb{C}^3$ and $|z|^2 + |x|^2 + |y|^2 = 1$. A simple computation shows that $M$ admits a Seifert fibration with 3, $p$ or $p + 1$ exceptional fibres and base space $S^2$, see [Ko, Lem. 2], or [BoPa, Lemma 6 and proof of Lemma 7]. In particular $M$ has a unique Seifert fibration, up to isotopy: by [Wa], [Sc] and [BOt] the only possible exception with base $S^2$ and at least 3 exceptional fibres is the double of a twisted $I$-bundle, which is not a rational homology sphere, since it fibers over the circle. We distinguish now two cases:

Case 1: The integers $a$ and $b$ are coprime with $p$, and there are three singular fibres of pairwise relatively prime orders $a$, $b$ and $p$. By the orbifold theorem any hyperelliptic rotation of $M$ of order $> 2$ is conjugate into the circle action $S^1 \subset \text{Diff}^+(M)$ inducing the Seifert fibration, hence the uniqueness of the Seifert fibration, up to isotopy, implies that $M$ admits at most 3 non conjugate hyperelliptic groups of odd prime orders belonging to \{a, b, p\}. Indeed $M$ is a Brieskorn integral homology sphere, see [BPZ].

Case 2: Either $a = p$ and $M$ has $p$ singular fibres of order $b$, or $a = a'p$ with $a' > 1$, and $M$ has $p$ singular fibres of order $b$ and one extra singular fibre of order $a'$. In both situations, there are $p \geq 3$ exceptional fibres of order $b$ which are cyclically permuted by the hyperelliptic rotation $\psi$. As before, $M$ has a unique Seifert fibration, up to isotopy. Therefore, up to conjugacy, $\psi$ is the only hyperelliptic rotation of order $p$ on $M$, and by the discussion above $M$ cannot admit a hyperelliptic rotation of odd prime order $q \neq p$. □
Remark 10. The requirement that the rotations are hyperelliptic is essential in the proof of Proposition 6. The Brieskorn homology sphere \( \Sigma(p_1, \ldots, p_n) \), \( n \geq 4 \), with \( n \geq 4 \) exceptional fibres admits \( n \) rotations of pairwise distinct prime orders but which are not hyperelliptic.

The hypothesis that the orders of the hyperelliptic rotations are \( \neq 2 \) cannot be avoided either.

Indeed, Montesinos’s construction of fibre preserving hyperelliptic involutions on Seifert fibered rational homology spheres [Mon1], [Mon2], (see also [BS, Appendix A], [BZH, Chapter 12]), shows that for any given integer \( n \) there are infinitely many closed orientable Seifert fibred 3-manifolds with at least \( n \) conjugacy classes of hyperelliptic rotations of order 2.

On the other hand, the hypothesis that the orders are odd primes is sufficient but not necessary: A careful analysis of the Seifert invariants shows that if \( M \neq S^3 \) is a Seifert rational homology sphere, then \( M \) can be the cyclic branched cover of a knot in \( S^3 \) of order \( > 2 \) in at most three ways.

The hypotheses of Proposition 6 cannot be relaxed further, though: Proposition 7 below shows that there exist closed 3-dimensional circle bundles with arbitrarily many conjugacy classes of hyperelliptic rotations of odd, but not prime, orders.

**Proposition 7.** Let \( N \) be an odd prime integer. For any integer \( 1 \leq q < \frac{N}{2} \) the Brieskorn-Pham manifold \( M = V((2^q + 1)(2^{(N-q)} - 1), 2^q + 1, 2^{(N-q)} + 1) \) is a circle bundle over a closed surface of genus \( g = \frac{N-1}{2} \) with Euler class \pm 1. Hence, up to homeomorphism (possibly reversing the orientation), \( M \) depends only on the integer \( N \) and admits at least \( \frac{N-1}{2} \) conjugacy classes of hyperelliptic groups of odd orders.

**Proof.** We remark first that the integers \( q \) and \( N-q \) are relatively prime, because \( N \) is prime. If \( k \) is a common prime divisor of \( 2^q + 1 \) and \( 2^{(N-q)} + 1 \), by the Bezout identity we have \( 2^1 = 2^{a+b(N-q)} \equiv (-1)^{a+b} \mod k \) which implies that \( k = 3 \). But then \((-1)^q \equiv (-1)^{(N-q)} \equiv -1 \mod 3 \) and thus \((-1)^N \equiv 1 \mod 3 \) which is impossible since \( N \) is odd. Hence \( 2^q + 1 \) and \( 2^{(N-q)} + 1 \) are relatively prime.

So the Brieskorn-Pham manifold \( M \) is the \((2^q + 1)(2^{(N-q)} + 1)\)-fold cyclic cover of \( S^3 \) branched over the torus knot \( K_q = T(2^q + 1, 2^{(N-q)} + 1) \). It is obtained by Dehn filling the \((2^q + 1)(2^{(N-q)} + 1)\)-fold cyclic cover of the exterior of the torus knot \( K_q \) along the lift of its meridian. The \((2^q + 1)(2^{(N-q)} + 1)\)-fold cyclic cover of the exterior of \( K_q \) is a trivial circle bundle over a once punctured surface of genus \( g = 2^N - 1 \). On the boundary of the torus-knot exterior the algebraic intersection between a meridian and a fibre of the Seifert fibration of the exterior is \( \pm 1 \) (the sign depends on a choice of orientation, see for example [BZH, Chapter 3]). So on the torus boundary of the \((2^q + 1)(2^{(N-q)} + 1)\)-fold cyclic cover the algebraic intersection between the lift of a meridian of the torus knot and an \( S^1 \)-fiber is again \( \pm 1 \). Hence the circle bundle structure of the \((2^q + 1)(2^{(N-q)} + 1)\)-fold cyclic cover of the exterior of the torus knot \( K_q \) can be extended with Euler class \( \pm 1 \) to the Dehn filling along the lift of the meridian. So \( M \) is a circle bundle over a closed surface of genus \( g = 2^{N-1} + 1 \) with Euler class \( \pm 1 \).

Since the torus knots \( K_q = T(2^q + 1, 2^{(N-q)} + 1) \) are pairwise inequivalent for \( 1 \leq q \leq \frac{N-1}{2} \), the hyperelliptic subgroups corresponding to the \((2^q + 1)(2^{(N-q)} + 1)\)-fold cyclic cover of the exterior of the torus knot \( K_q \) can be extended with Euler class \( \pm 1 \) to the Dehn filling along the lift of the meridian. So \( M \) is a circle bundle over a closed surface of genus \( g = 2^{N-1} + 1 \) with Euler class \( \pm 1 \).
1)-fold cyclic branched covers of the knots $K_q$ are pairwise not conjugate in $\text{Diff}^+(M)$.

**Remark 11.** Note that the Seifert manifolds $M$ and their hyperelliptic rotations constructed in Proposition 7 enjoy the following properties: If $N > 8$, then no hyperelliptic rotation can commute up to conjugacy with all the remaining ones (see Proposition 5 and [BoPa, Theorem 2]). If $N > 14$ no finite subgroup of $\text{Diff}^+(M)$ can contain up to conjugacy all hyperelliptic rotations of $M$, according to Theorem 2.

### 6.2 Reduction to the finite group action case

The fact that Theorem 6 implies Corollary 1 follows from the existence of a decomposition of a closed, orientable 3-manifold $M$ as a connected sum of prime manifolds and the observation that a hyperelliptic rotation on $M$ induces a hyperelliptic rotation on each of its prime summands. A 3-manifold admitting a hyperelliptic rotation must be a rational homology sphere, and so $M$ cannot have $S^2 \times S^1$ summands. Hence all prime summands are irreducible and at least one is not homeomorphic to $S^3$, since $M$ itself is not homeomorphic to $S^3$. This is enough to conclude.

The remaining of this section is devoted to the proof that Theorem 2 implies Theorem 6.

We prove the following proposition:

**Proposition 8.** If $M$ is a closed, orientable, irreducible 3-manifold such that there are $k \geq 7$ conjugacy classes of hyperelliptic subgroups of $\text{Diff}^+(M)$ whose order is an odd prime, then $M$ is homeomorphic to $S^3$.

**Proof.**

Let $M$ be a closed, orientable, irreducible 3-manifold such that $\text{Diff}^+(M)$ contains $k \geq 7$ conjugacy classes of hyperelliptic subgroups of odd prime orders.

According to the orbifold theorem (see [BoP], [BMP], [CHK]), a closed orientable irreducible manifold $M$ admitting a rotation has geometric decomposition. This means that $M$ can be split along a (possibly empty) finite collection of $\pi_1$-injective embedded tori into submanifolds carrying either a hyperbolic or a Seifert fibered structure. This splitting along tori is unique up to isotopy and is called the JSJ-decomposition of $M$, see for example [NS], [BMP, chapter 3]. In particular, if its JSJ-decomposition is trivial, $M$ admits either a hyperbolic or a Seifert fibered structure.

First we see that $M$ cannot be hyperbolic. Indeed, if the manifold $M$ is hyperbolic then, by the orbifold theorem, any hyperelliptic rotation is conjugate into the finite group $\text{Isom}^+(M)$ of orientation preserving isometries of $M$. Hence, applying Theorem 2 to $G = \text{Isom}^+(M)$, we see that $k \leq 6$ against the hypothesis.

If the manifold $M$ is Seifert fibered, it follows readily from Proposition 6 of the previous section that $M = S^3$. So we are left to exclude the case where the JSJ-decomposition of $M$ is not empty.

Consider the JSJ-decomposition of $M$: each geometric piece admits either a complete hyperbolic structure with finite volume or a Seifert fibered product.
structure with orientable base. Moreover, the geometry of each piece is unique, up to isotopy.

Let \( \Psi = \{ \psi_1, \ldots, \psi_k, k \geq 7 \} \) be the set of hyperelliptic rotations which generate non conjugate cyclic subgroups in \( \text{Diff}^+(M) \). By the orbifold theorem [BoP], [BMP], [CHK], after conjugacy, one can assume that each hyperelliptic rotation preserves the JSJ-decomposition, acts isometrically on the hyperbolic pieces, and respects the product structure on the Seifert pieces. We say that they are geometric.

Let \( \Gamma \) be the dual graph of the JSJ-decomposition: it is a tree, for \( M \) is a rational homology sphere (in fact, the dual graph of the JSJ-decomposition for a manifold which is the cyclic branched cover of a knot is always a tree, regardless of the order of the covering). Let \( H \subset \text{Diff}^+(M) \) be the group of diffeomorphisms of \( M \) generated by the set \( \Psi \) of geometric hyperelliptic rotations. By [BoPa, Thm 1], there is a subset \( \Psi_0 \subset \Psi \) of \( k_0 \geq 4 \) hyperelliptic rotations with pairwise distinct odd prime orders, say \( \Psi_0 = \{ \psi_i, i = 1, \ldots, k_0 \} \).

Let \( H_\Gamma \) denote the image of the induced representation of \( H \) in \( \text{Aut}(\Gamma) \). Since rotations of finite odd order cannot induce an inversion on any edge of \( \Gamma \), the finite group \( H_\Gamma \) must fix pointwise a non-empty subtree \( \Gamma_f \) of \( \Gamma \).

The idea of the proof is now analogous to the ones in [BoPa] and [BPZ]: we start by showing that, up to conjugacy, the \( k_0 \geq 4 \) hyperelliptic rotations with pairwise distinct odd prime orders can be chosen to commute on the submanifold \( M_f \) of \( M \) corresponding to the subtree \( \Gamma_f \). We consider then the maximal subtree corresponding to a submanifold of \( M \) on which these hyperelliptic rotations commute up to conjugacy and prove that such subtree is in fact \( \Gamma \). Then the conclusion follows by applying Lemma 2.

The first step of the proof is achieved by the following proposition:

**Proposition 9.** The hyperelliptic rotations in \( \Psi_0 \) commute, up to conjugacy in \( \text{Diff}^+(M) \), on the submanifold \( M_f \) of \( M \) corresponding to the subtree \( \Gamma_f \).

**Proof.**

Since the hyperelliptic rotations in \( \Phi \) have odd orders, either \( \Gamma_f \) contains an edge, or it consists of a single vertex. We shall analyse these two cases.

**Case 1:** \( M_f \) contains an edge.

**Claim 5.** Assume that \( \Gamma_f \) contains an edge and let \( T \) denote the corresponding torus. The hyperelliptic rotations in \( \Psi_0 \) commute, up to conjugacy in \( \text{Diff}^+(M) \), on the geometric pieces of \( M \) adjacent to \( T \).

**Proof.**

The geometric pieces adjacent to \( T \) are left invariant by the hyperelliptic rotations in \( \Psi_0 \), since their orders are odd. Let \( V \) denote one of the two adjacent geometric pieces: each hyperelliptic rotation acts non-trivially on \( V \) with odd prime order. We distinguish two cases according to the geometry of \( V \).

\( V \) is hyperbolic. In this case all rotations act as isometries and leave a cusp invariant. Since their order is odd, the rotations must act as translations along horospheres, and thus commute.

Note that, even in the case where a rotation has order 3, its axis cannot meet a torus of the JSJ-decomposition of \( M \) for each such torus is separating and cannot meet the axis in an odd number of points.
V is Seifert fibred. In this case we can assume that the hyperelliptic rotations in Ψ preserve the Seifert fibration with orientable base. Since their orders are odd and prime, each one preserves the orientation of the fibres and of the base. The conjugacy class of a fiber-preserving rotation of V with odd prime order depends only on its combinatorial behaviour, i.e. its translation action along the fibre and the induced permutation on cone points and boundary components of the base. In particular, two geometric rotations with odd prime order having the same combinatorial data are conjugate via a diffeomorphism isotopic to the identity.

Since the hyperelliptic rotations in Ψ₀ have pairwise distinct odd prime orders, an analysis of the different cases described in Lemma 4 below shows that at most one among these hyperelliptic rotations can induce a non-trivial action on the base of the fibration, and thus the remaining ones act by translation along the fibres and induce the identity on the base. Since the translation along the fibres commutes with every fiber-preserving diffeomorphism of V, the hyperelliptic rotations in Ψ₀ commute on V.

Lemma 4 describes the Seifert fibred pieces of a manifold admitting a hyperelliptic rotation of odd prime order, as well as the action of the rotation on the pieces. Its proof can be found in [BoPa, Lemma 6 and proof of Lemma 7], see also [Ko, Lem. 2].

**Lemma 4.** Let M be an irreducible 3-manifold admitting a non-trivial JSJ-decomposition. Assume that M admits a hyperelliptic rotation of prime odd order p. Let V be a Seifert piece of the JSJ-decomposition for M. Then the base B of V can be:

1. A disc with 2 cone points. In this case either the rotation freely permutes p copies of V or leaves V invariant and acts by translating along the fibres.

2. A disc with p cone points. In this case the rotation leaves V invariant and cyclically permutes the singular fibres, while leaving a regular one invariant.

3. A disc with p + 1 cone points. In this case the rotation leaves V and a singular fibre invariant, while cyclically permuting the remaining p singular fibres.

4. An annulus with 1 cone point. In this case either the rotation freely permutes p copies of V or leaves V invariant and acts by translating along the fibres.

5. An annulus with p cone points. In this case the rotation leaves V invariant and cyclically permutes the p singular fibres.

6. A disc with p − 1 holes and 1 cone point. In this case the rotation leaves V invariant and cyclically permutes all p boundary components, while leaving invariant the only singular fibre and a regular one.

7. A disc with k holes, k ≥ 2. In this case either the rotation freely permutes p copies of V or leaves V invariant. In this latter case either the rotation acts by translating along the fibres, or k = p − 1 and the rotation permutes all the boundary components (while leaving invariant two fibres), or k = p
and the rotation permutes p boundary components, while leaving invariant the remaining one and a regular fibre.

We conclude that the rotations in \( \Psi_0 \) can be chosen to commute on the submanifold \( M_f \) of \( M \) corresponding to \( \Gamma_f \) by using inductively at each edge of \( \Gamma_f \) the gluing lemma below (see [Lemma 6][BPZ]). We give the proof for the sake of completeness.

**Lemma 5.** If the rotations preserve a JSJ-torus \( T \) then they commute on the union of the two geometric pieces adjacent to \( T \).

**Proof.**
Let \( V \) and \( W \) be the two geometric pieces adjacent to \( T \). By Claim 5, after conjugacy in \( Diff^+(M) \), the rotations in \( \Psi_0 \) commute on \( V \) and \( W \). Since they have pairwise distinct odd prime orders, their restrictions on \( V \) and \( W \) generate two cyclic groups of the same finite odd order. Let \( g_V \) and \( g_W \) be generators of these two cyclic groups. Since they have odd order, they both act by translation on \( T \). We need the following result about the slope of translation for such periodic transformation of the torus:

**Claim 6.** Let \( \psi \) be a periodic diffeomorphism of the product \( T^2 \times [0,1] \) which is isotopic to the identity and whose restriction to each boundary torus \( T \times \{i\} \), \( i = 0,1 \), is a translation with rational slopes \( \alpha_0 \) and \( \alpha_1 \) in \( H_1(T^2;\mathbb{Z}) \). Then \( \alpha_0 = \alpha_1 \).

**Proof.**
By Meeks and Scott [MS, Thm 8.1], see also [BS, Prop. 12], there is a Euclidean product structure on \( T^2 \times [0,1] \) preserved by \( \psi \) such that \( \psi \) acts by translation on each fiber \( T \times \{t\} \) with rational slope \( \alpha_t \). By continuity, the rational slopes \( \alpha_t \) are constant.

Now the following claim shows that the actions of \( g_V \) and \( g_W \) can be glued on \( T \).

**Claim 7.** The translations \( g_V|_T \) and \( g_W|_T \) have the same slope in \( H_1(T^2;\mathbb{Z}) \).

**Proof.**
Let \( \Psi_0 = \{ \psi_i, i = 1, \ldots, k_0 \} \). Let \( p_i \) the order of \( \psi_i \) and \( q_i = \Pi_j p_j \). Then the slopes \( \alpha_V \) and \( \alpha_W \) of \( g_V|_T \) and \( g_W|_T \) verify: \( q_i \alpha_V = q_i \alpha_W \) for \( i = 1,\ldots,k_0 \), by applying Claim 6 to each \( \psi_i \). Since the \( GCD \) of the \( q_i \) is 1, it follows that \( \alpha_V = \alpha_W \).

This finishes the proof of Lemma 5 and of Proposition 9 when \( M_f \) contains an edge.

To complete the proof of Proposition 9 it remains to consider the case where \( \Gamma_f \) is a single vertex.

**Case 2:** \( M_f \) is a vertex.

**Claim 8.** Assume that \( \Gamma_f \) consists of a single vertex and let \( V \) denote the corresponding geometric piece. Then the hyperelliptic rotations in \( \Psi_0 \) commute on \( V \), up to conjugacy in \( Diff^+(M) \).
Proof.

We consider again two cases according to the geometry of $V$.

The case where $V$ is Seifert fibred follows once more from Lemma 4.
We consider now the case where $V$ is hyperbolic.

In this case, the hyperelliptic rotations in $\Psi$ act non-trivially on $V$ by isometries of odd prime orders. The restriction $H_V \subset Isom^+(V)$ of the action of the subgroup $H$ that they generate in $Diff^+(M)$ is finite.

If the action on $V$ of the cyclic subgroups generated by two of the hyperelliptic rotations in $\Psi$ are conjugate in $H_V$, one can conjugate the actions in $Diff^+(M)$ to coincide on $V$, since any diffeomorphism in $H_V$ extends to $M$. Then by [BoPa, Lemma 10] these actions must coincide on $M$, contradicting the hypothesis that the conjugacy classes of cyclic subgroups generated by the hyperelliptic rotations in $\Psi$ are pairwise distinct in $Diff^+(M)$.

Hence, the cyclic subgroup generated by the $k \geq 7$ hyperelliptic rotations in $\Psi$ are pairwise not conjugate in the finite group $H_V \subset Isom^+(V)$.

Since the dual graph of the JSJ-decomposition of $M$ is a tree, a boundary torus $T \subset \partial V$ is separating and bounds a component $U_T$ of $M \setminus \text{int}(V)$. Since, by hypothesis, $\Gamma_f$ consists of a single vertex, no boundary component $T$ is setwise fixed by the finite group $H_V$. This means that there is a hyperelliptic rotation $\psi_i \in \Psi$ of odd prime order $p_i$ such that the orbit of $U_T$ under $\psi_i$ is the disjoint union of $p_i$ copies of $U_T$. In particular $U_T$ projects homeomorphically onto a knot exterior in the quotient $S^3 = M/\psi_i$. Therefore on each boundary torus $T = \partial U_T \subset \partial V$, there is a simple closed curve $\lambda_T$, unique up to isotopy, that bounds a properly embedded incompressible and $\partial$-incompressible Seifert surface $S_T$ in the knot exterior $U_T$.

By pinching the surface $S_T$ onto a disc $D^2$, in each component $U_T$ of $M \setminus \text{int}(V)$, one defines a degree-one map $\pi : M \longrightarrow M'$, where $M'$ is the rational homology sphere obtained by Dehn filling each boundary torus $T \subset V$ along the curve $\lambda_T$.

For each hyperelliptic rotation $\psi_i$ in $\Psi$, of odd prime order $p_i$, the $\psi_i$-orbit of each component $U_T$ of $M \setminus \text{int}(V)$ consists of either one or $p_i$ elements. As a consequence, by [Sa], $\psi_i$ acts equivariantly on the set of isotopy classes of curves $\lambda_T \subset \partial V$. Hence, each $\psi_i$ extends to periodic diffeomorphism $\psi_i'$ of order $p_i$ on $M'$. Moreover, $M'$ is a $\mathbb{Z}/p_i$-homology sphere, since so is $M$ and $\pi : M \longrightarrow M'$ is a degree-one map. According to Smith theory, if $\text{Fix}(\psi_i')$ is non-empty on $M'$, then $\psi_i'$ is a rotation on $M'$. To see that $\text{Fix}(\psi_i') \neq \emptyset$ on $M'$ it suffices to observe that either $\text{Fix}(\psi) \subset V$ or $\psi_i$ is a rotation of some $U_T$; in this latter case, $\psi_i'$ must have a fixed point on the disc $D^2$ onto which the surface $S_T$ is pinched. To show that $\psi_i'$ is hyperelliptic it remains to show that the quotient $M'/\psi_i'$ is homeomorphic to $S^3$.

Since $\psi_i$ acts equivariantly on the components $U_T$ of $M \setminus \text{int}(V)$ and on the set of isotopy classes of curves $\lambda_T \subset \partial V$, the quotient $S^3 = M/\psi_i$ is obtained from the compact 3-manifold $V/\psi_i$ by gluing knot exteriors (maybe solid tori) to its boundary components, in such a way that the boundaries of the Seifert surfaces of the knot exteriors are glued to the curves $\lambda_T/\psi_i \subset \partial V/\psi_i$.

In the same way, the rotation $\psi_i'$ acts equivariantly on the components $M' \setminus \text{int}(V)$ and on the set of isotopy classes of curves $\lambda_T \subset \partial V$. By construction, these components are solid tori, and either the axis of the rotation is contained in $V$ or there exists a unique torus $T \subset \partial V$ such that the solid torus glued to $T$ to obtain $M'$ contains the axis. In the latter case, by [EL, Cor. 2.2], the
rotation \( \psi'_i \) preserves a meridian disc of this solid torus and its axis is a core of \( W_T \). It follows that the images in the quotient \( M'/\psi'_i \) of the the solid tori glued to \( \partial V \) are again solid tori. Hence \( M'/\psi'_i \) is obtained from \( S^3 \) by replacing each components of \( S^3 \setminus V/\psi_i \) by a solid torus, in such way that boundaries of meridian discs of the solid tori are glued to the curves \( \lambda_T/\psi'_i \subset \partial V/\psi'_i \). It follows that \( M'/\psi'_i \) is again \( S^3 \).

So far we have constructed a closed orientable 3-manifold \( M' \) with a finite subgroup of orientation preserving diffeomorphisms \( H_V \) that contains at least seven conjugacy classes of hyperelliptic subgroups of odd prime orders. Theorem 2 implies that \( M' \) must be \( S^3 \), and thus by the orbifold theorem [BLP] \( H_V \) is conjugate to a finite subgroup of \( SO(4) \). In particular the subgroup \( H_0 \subset H_V \) generated by the subset \( \Psi_0 \) of at least 4 hyperelliptic rotations with pairwise distinct odd prime orders must be solvable. Therefore, by Proposition 5 the induced rotations commute on \( M' \) and, by restriction, the hyperelliptic rotations in \( \Psi_0 \) commute on \( V \).

In the final step of the proof we extend the commutativity on \( M_f \) to the whole manifold \( M \). The proof of this step is analogous to the one given in [BPZ], since the proof there was not using the homology assumption. We give the argument for the sake of completeness.

**Proposition 10.** The \( k_0 \geq 4 \) hyperelliptic rotations in \( \Psi_0 \) commute, up to conjugacy in \( Diff^+(M) \), on \( M \).

**Proof.**

Let \( \Gamma_c \) be the largest subtree of \( \Gamma \) containing \( \Gamma_f \), such that, up to conjugacy in \( Diff^+(M) \), the rotations in \( \Psi_0 \) commute on the corresponding invariant submanifold \( M_c \) of \( M \). We shall show that \( \Gamma_c = \Gamma \). If this is not the case, we can choose an edge contained in \( \Gamma \) corresponding to a boundary torus \( T \) of \( M_c \). Denote by \( U_T \) the submanifold of \( M \) adjacent to \( T \) but not containing \( M_c \) and by \( V_T \subset U_T \) the geometric piece adjacent to \( T \).

Let \( H_0 \subset Diff^+(M) \) be the group of diffeomorphisms of \( M \) generated by the set of geometric hyperelliptic rotations \( \Psi_0 = \{ \psi_i, i = 1, \ldots, k_0 \} \). Since the rotations in \( \Psi_0 \) commute on \( M_c \) and have pairwise distinct odd prime orders, the restriction of \( H_0 \) on on \( M_c \) is a cyclic group with order the product of the orders of the rotations. Since \( \Gamma_f \subset \Gamma_c \), the \( H_0 \)-orbit of \( T \) cannot be reduced to only one element. Moreover each rotation \( \psi \in \Psi_0 \) either fixes \( T \) or acts freely on the orbit of \( T \) since its order is prime.

If no rotation in \( \Psi_0 \) leaves \( T \) invariant, the \( H_0 \)-orbit of \( T \) contains as many elements as the product of the orders of the rotations, for they commute on \( M_c \). In particular, only the identity (which extends to \( U \)) stabilises a torus in the \( H_0 \)-orbit of \( T \). Note that all components of \( \partial M_c \) that are in the \( H_0 \)-orbit of \( T \) bound a manifold homeomorphic to \( U_T \).

Since the rotation \( \psi_i \) acts freely on the \( H_0 \)-orbit of \( U_T \), \( U_T \) is a knot exterior in the quotient \( M/\psi_i = S^3 \). Hence there is a well defined meridian-longitude system on \( T = \partial U_T \) and also on each torus of the \( H_0 \)-orbit of \( T \). This set of meridian-longitude systems is cyclically permuted by each \( \psi_i \) and thus equivariant under the action of \( H_0 \).

Let \( M_c/H_0 \) be the quotient of \( M_c \) by the induced cyclic action of \( H_0 \) on \( M_c \). Then there is a unique boundary component \( T' \subset \partial( M_c/H_0 ) \) which is the image of the \( H_0 \)-orbit of \( T \). We can glue a copy of the knot exterior \( U_T \) to \( M_c/H_0 \) along
components extend to the whole manifold $M$.

Dehn twist act trivially on the homology of the boundary. The manifolds adjacent to these components are all homeomorphic and that there is a natural and well-defined way to identify each element of the orbit of $\pi\cdot M$ with the total space of such covering. By construction it follows that $N$, is the quotient $(M_c \cup H_0 \cdot U_T)/\psi_i$. This implies that the $\psi_i$'s commute on $M_c \cup H_0 \cdot U_T$ contradicting the maximality of $\Gamma_c$.

We can thus assume that some rotations fix $T$ and some do not. Since all rotations commute on $M_c$ and have pairwise distinct odd prime orders, we see that the orbit of $T$ consists of as many elements as the product of the orders of the rotations which do not fix $T$ and each element of the orbit is fixed by the rotations which leave $T$ invariant. The rotations which fix $T$ commute on the orbit of $V_T$ according to Claim 5 and Lemma 5, and form a cyclic group generated by, say, $\gamma$. The argument for the previous case shows that the rotations acting freely on the orbit of $T$ commute on the orbit of $U_T$ and thus on the orbit of $V_T$, and form again a cyclic group generated by, say, $\eta$. To reach a contradiction to the maximality of $M_c$, we shall show that $\gamma$, after perhaps some conjugacy, commutes with $\eta$ on the $H_0$-orbit of $V_T$, in other words that $\gamma$ and $\eta \gamma \eta^{-1}$ coincide on $H_0 \cdot V_T$. Since $\eta$ acts freely and transitively on the $H_0$-orbit of $V_T$ there is a natural and well-defined way to identify each element of the orbit $H_0 \cdot V_T$ to $V_T$ itself. Note that this is easily seen to be the case if $V_T$ is hyperbolic: this follows from Claim 5 and Claim 6. We now consider the case when $V_T$ is Seifert fibred.

Claim 9. Assume that $V_T$ is Seifert fibred and that the restriction of $\gamma$ induces a non-trivial action on the base of $V_T$. Then $\gamma$ induces a non-trivial action on the base of each component of the $H_0$-orbit of $V_T$. Moreover, up to conjugacy on $H_0 \cdot V_T \setminus V_T$ by diffeomorphisms which are the identity on $H_0 \cdot T$ and extend to $M$, we can assume that the restrictions of $\gamma$ to these components induce the same permutation of their boundary components and the same action on their bases.

Proof.

By hypothesis $\gamma$ and $\eta \gamma \eta^{-1}$ coincide on $\partial M_c$. The action of $\gamma$ on the base of $V_T$ is non-trivial if and only if its restriction to the boundary circle of the base corresponding to the fibres of the torus $T$ is non-trivial. Therefore the action of $\gamma$ is non-trivial on the base of each component of $H_0 \cdot V_T$.

By Lemma 4 and taking into account the fact that $V_T$ is a geometric piece in the JSJ-decomposition of the knot exterior $U_T$, the only situation in which the action of $\gamma$ on the base of $V_T$ is non-trivial is when the base of $V_T$ consists of a disc with $p$ holes, where $p$ is the order of one of the rotations that generate $\gamma$. Moreover, the restriction of $\gamma$ to the elements of $H_0 \cdot V_T$ cyclically permutes their boundary components which are not adjacent to $M_c$. Up to performing Dehn twists, along vertical tori inside the components of $H_0 \cdot V_T \setminus V_T$, which permute the boundary components, we can assume that the restriction of $\gamma$ induces the same cyclic permutations on the boundary components of each element of the orbit $H_0 \cdot V_T$. We only need to check that Dehn twists permuting two boundary components extend to the whole manifold $M$. This follows from the fact that the manifolds adjacent to these components are all homeomorphic and that Dehn twist act trivially on the homology of the boundary.
Since the actions of the restrictions of $\gamma$ on the bases of the elements of $H_0 \cdot V_T$ are combinatorially equivalent, after perhaps a further conjugacy by an isotopy, the different restrictions can be chosen to coincide on the bases.

By Claim 5 and Claim 9 we can now deduce that the restrictions of $\gamma$ and $\eta \gamma \eta^{-1}$ to the $H_0$-orbit of $V_T$ commute, up to conjugacy of $\gamma$ which is the identity on the $H_0$-orbit of $T$. Since $\gamma$ and $\eta \gamma \eta^{-1}$ coincide on this $H_0$-orbit of $T$, we can conclude that they coincide on the $H_0$-orbit of $V_T$. This finishes the proof of Proposition 10.

Since there are at least four hyperelliptic rotations with pairwise distinct odd prime orders in $\Psi_0$, Proposition 8 is consequence of Proposition 10 and Proposition 1, like in the solvable case.

Remark 12. As we have seen, the strategy to prove that an irreducible manifold $M$ with non-trivial JSJ-decomposition cannot admit more than six conjugacy classes of hyperelliptic subgroups of odd prime order inside $Diff^+(M)$ consists in modifying by conjugacy any given set of hyperelliptic rotations so that the new hyperelliptic rotations commute pairwise. Note that this strategy cannot be carried out in general if the orders are not pairwise coprime (see, for instance, [BoPa, Section 4.1] where the case of two hyperelliptic rotations of the same odd prime order, generating non conjugate subgroups, is considered). Similarly, for hyperelliptic rotations of arbitrary orders $> 2$ lack of commutativity might arise on the Seifert fibred pieces of the decomposition, as it does for the Seifert fibred manifolds constructed in Proposition 7.

7 Appendix: non-free finite group actions on rational homology spheres

In this section we will show that every finite group $G$ admits a faithful action by orientation preserving diffeomorphisms on some rational homology sphere so that some elements of $G$ have non-empty fixed-point sets.

Cooper and Long’s construction of $G$-actions on rational homology spheres in [CL] consists in starting with a $G$-action on some 3-manifold and then modifying the original manifold, notably by Dehn surgery, so that the new manifold inherits a $G$-action but has “smaller” rational homology. Since their construction does not require that the $G$-action is free, it can be used to prove the existence of non-free $G$-actions. We will thus start by exhibiting non-free $G$-actions on some 3-manifold before pointing out what need to be taken into account in this setting in order for Cooper and Long’s construction to work.

Since every (non-trivial) cyclic group acts as a group of rotations of $S^3$, for simplicity we will assume that $G$ is a finite non-cyclic group.

Claim 10. Let $G$ be a finite non-cyclic group. There is a closed, connected, orientable 3-manifold $M$ on which $G$ acts faithfully, by orientation preserving diffeomorphisms so that there are $g \in G \setminus \{1\}$ with the property that $Fix(g) \neq \emptyset$.

Proof.
Let $k \geq 2$ and let $\{g_i\}_{1 \leq i \leq k+1}$ be a system of generators for $G$ satisfying the following requirements:

- for all $1 \leq i \leq k + 1$, the order of $g_i$ is $n_i \geq 2$;
\[ \cdot g_{k+1} = g_1g_2 \cdots g_k. \]

Since \( G \) is not cyclic these conditions are not difficult to fulfill, and actually they can be fulfilled even in the case when \( G \) is cyclic for an appropriate choice of the set \( \{g_i\}_{1 \leq i \leq k+1} \).

Consider the free group of rank \( k \) that we wish to see as the fundamental group of a \((k+1)\)-punctured 2-sphere: each generator \( x_i \) corresponds to a loop around a puncture of the sphere so that a loop around the \( k+1 \)st puncture corresponds to the element \( x_1x_2 \cdots x_k \). Build an orbifold \( \mathcal{O} \) by compactifying the punctured-sphere with cone points so that the \( i \)th puncture becomes a cone point of order \( n_i \). The resulting orbifold has (orbifold) fundamental group with the following presentation:

\[
\langle \ x_1, x_2, \ldots, x_k, x_{k+1} \ | \ \{x_i^{n_i}\}_{1 \leq i \leq k+1}, x_1 \cdots x_kx_{k+1}^{-1} \rangle.
\]

Clearly this group surjects onto \( G \). Such surjection gives rise to an orbifold covering \( \Sigma \rightarrow \mathcal{O} \), where \( \Sigma \) is an orientable surface on which \( G \) acts in such a way that each element \( g_i \) has non-empty fixed-point set. One can consider the 3-manifold \( \Sigma \times S^1 \): the action of \( G \) on \( \Sigma \) extends to a product action of \( G \) on \( \Sigma \times S^1 \) which is trivial on the \( S^1 \) factor.

To be able to repeat Cooper and Long’s construction it is now easy to observe that it is always possible to choose \( G \)-equivariant families of simple closed curves in \( M \) so that they miss the fixed-point sets of elements of \( G \) and either their homology classes generate \( H_1(M; \mathbb{Q}) \) (so that the hypothesis of [CL, Lemma 2.3] are fulfilled when we choose \( X \) to be the exterior of such families) or the family is the \( G \)-orbit of a representative of some prescribed homology class (as in the proof of [CL, Proposition 2.5]).

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