High order semi lagrangian schemes

Michel Mehrenberger

Post-doctorant, Gamma project (INRIA Rocquencourt)

http://www-rocq.inria.fr/who/Michel.Mehrenberger/

PhD thesis advised by Eric Sonnendrücker

Université Louis Pasteur, Strasbourg and INRIA (CALVI project)

Outline

-Semi Lagrangian scheme with Lagrange and Spline interpolation (results obtained with Nicolas Besse)

-Von Neumann Stability Analysis for interpolets

-Description of a Semi-Lagrangian adaptive method with Hermite interpolation (current work with Eric Violard)

Characteristic form of an equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t, X(t), V(t)) = 0,$$

$X(t) = X(t; s, x, v)$ and $V(t) = V(t; s, x, v)$ are the characteristics.

In particular,

$$f(t, x, v) = f(t, X(t; t, x, v), V(t; t, x, v)) = f_0(X(0; t, x, v), V(0; t, x, v))$$

Vlasov equation:

$$\partial_t X = V \quad \partial_t V = E$$

General resolution scheme (semi-lagrangian method)

Succession of transports (T) and projections (Π) on a mesh $\mathcal{M}$

$$f_0 \longrightarrow (\Pi f_0, \mathcal{M}_0)$$

$$(f_n, \mathcal{M}_n) \longrightarrow (T f_n, T(\mathcal{M}_n)) \longrightarrow (\Pi T f_n, \Pi T(\mathcal{M}_n))$$

Choice of T?

Choice of $\mathcal{M}$ and Π? (interpolating operator)

Frequent choice of transport T: directional splitting

$T = T_x$ or $T = T_v^E$, with

$$T_x g = g(x - v\Delta t/2, v)$$

$$T_v^E g = g(x, v - \Delta t E_h),$$

where E_h is an approximation of electric field E on a fine mesh of size h.

Theorem 1 *(Nicolas Besse, M.M)*

Let $T > 0$, $0 \leq t^n \leq T$, $\Omega_R = [0, L] \times [-R, R]$ *(R big enough)*

$f_0 \in C^{m+1}(\Omega_R)$ *L-periodic in* x, *in compact support in* v.

Then

$$\|f(t^n) - f^n\|_{L^2(\Omega_R)} \leq C_T(\Delta t^2 + h^{m+1}/\Delta t),$$

for a semi-lagrangian method with splitting, with a reconstruction by B-splines or centered Lagrangian polynomials of degree m.

Von Neumann stability analysis:

Lemme 1 *(Reconstruction with polynoms of Lagrange)*

Let $n \in \mathbb{N}^*$, $\theta \in \mathbb{R}$ *and* P *the unique polynom of degree* $2n$ *interpolating the function* $\exp(i\theta\cdot)$ *on the points* $-n, \ldots, n$.

Then $|P| \leq 1$, *on* $[-1, 1]$.

(Reconstruction with splines)

Let $m \in \mathbb{N}$, $\theta \in \mathbb{R}$ *and* B_m *the B-spline of order* m *defined by convolution of the characteristic function of the interval* $[-\frac{1}{2}, \frac{1}{2}]$.

Then

$$\Phi_m(\alpha) := \Big|\sum_{k\in\mathbb{Z}} B_m(k+\alpha)\exp(ik\theta)\Big|^2$$

admit its maximum on the integers.

Von Neumann stability analysis for interpolets

We consider the symbol

$$m(\omega) = \frac{1}{2} \sum h_n e^{in\omega},$$

defining a refinable function φ via

$$\varphi(2x) = \sum h_n \varphi(2x - n).$$

We recall that φ is interpolatory (i.e. $\varphi(k) = \delta_k^0$) iff

$$m(\omega) + m(\omega + \pi) = 1.$$

One main example of refinable interpolatory functions are interpolets (of odd order $2d-1$). The corresponding symbol is then given by

$$m_d(\omega) = \cos^2(\omega/2)^N P_d(\sin^2(\omega/2)),$$

with

$$P_d(x) = \sum_{n=0}^{d-1} \binom{2d-1}{d+n} x^n (1-x)^{d-1-n},$$

In particular, we notice that $m_d(\omega) \geq 0$.

We consider now a dyadic discretization of a 1–periodic function, given by a sequence $(f_k)_{k\in\mathbb{Z}}$, with period $N = 2^J$. We can then reconstruct a function on the whole $\mathbb{R}$ coinciding with f_k at points $x_k = \frac{k}{N}$ by

$$\Pi_N f(x) = \sum f_k \varphi_{J,k}(x), \quad \varphi_{j,k}(x) = \varphi(2^{-j}x - k),$$

for an interpolatory function φ.
In particular, the value at a non dyadic point $(1-\alpha)x_k + \alpha x_{k+1}$ (for $0 < \alpha < 1$) is reconstructed by

$$f_{k+\alpha} := \sum_{\ell} \varphi_{J,\ell}(\alpha) f_{k+\ell}.$$

The interpolation is then stable in the context of the Von Neumann analysis, if

$$M(\alpha) := |\sum_{k\in\mathbb{Z}} \varphi_{J,k}(\alpha) e^{ik\omega}| \leq 1, \alpha, \omega \in \mathbb{R}.$$

We then have the following result

Proposition 1 *Suppose that*

$$m(\omega) + m(\omega + \pi) = 1 \quad \textit{and} \quad m(\omega) \geq 0, \quad \textit{for all } \omega \in \mathbb{R}.$$

Then, we have

$$M(\alpha) \leq 1,$$

for each dyadic point α *(i.e. of the form* $2^{-j}k, k, j \in \mathbb{N}$*).*

We define

$$\Phi(\alpha) = \sum_k e^{i\omega(k+\alpha)}\varphi(k+\alpha).$$

It suffices then to prove that $|\Phi(\alpha)| \leq 1$, for $\alpha = 0, 1/2, 1/4, 3/4, \ldots$. For $n, p \in \mathbb{N}$, we compute

$$\sum_{k=0}^{2^n-1} e^{i2\pi kp/2^n}\Phi(k/2^n) = 2^n \prod_{s=1}^{n} m((\omega + 2p\pi)/2^s)$$

The inverse Fourier transform then gives

$$\Phi(p/2^n) = \sum_{k=0}^{2^n-1} e^{-i2\pi kp/2^n} \prod_{s=1}^{n} m((\omega + 2k\pi)/2^s),$$

and thus

$$|\Phi(p/2^n)| \leq \sum_{k=0}^{2^n-1} \prod_{s=1}^{n} m((\omega + 2k\pi)/2^s) = \Phi(0) = 1,$$

Adaptive Numerical Resolution with Hermite reconstruction

Motivations:

-Adaptativity

-Locality

-High order

Description of the method (general framework)

-fix max resolution level J (fine grid)

-fix coarse grid level j_0

-fix tolerance number ε

-dyadic cell decomposition

-cell compression test

-advect mesh + solution backward following caracteristics

Specific interpolation operator:

on each cell, 16 degrees of freedom:

-$f, \partial_x f, \partial_v f$ on corners

-$\partial_v f$ on the middle of horizontal edges

-$\partial_x f$ on the middle of vertical edges

Hong D., Schumaker L. L., *Surface compression using a space of C^1 cubic splines with a hierarchical basis*, Geometric modelling Computing 72 (2004), no. 1-2, 79–92.

Numerical results:

$$f_0(x,v) = 20exp(-0.07((40(x-0.5)+4.8)^2 + (40(v-0.5)+4.8)^2))$$

on $[0,1]^2$

+ exact rotation with $\Delta t = 0.19635$ $(2\pi/32)$

Errors: (for $Q1$ after 100 iterations)

J	ϵ	Nx	$Nx * Nv$	L^1	L^2	L^∞
5	-	64	4096	0.479757	0.391226	0.4752230
6	-	128	16384	0.153018	0.138797	0.1854
7	-	256	65536	0.0410578	0.0387812	0.0546242
8	-	512	262144	0.0104429	0.0099829	0.014181
9	-	1024	1048576	0.00262001	0.00251314	0.00357201

Errors: (for $Q2$ after 100 iterations)

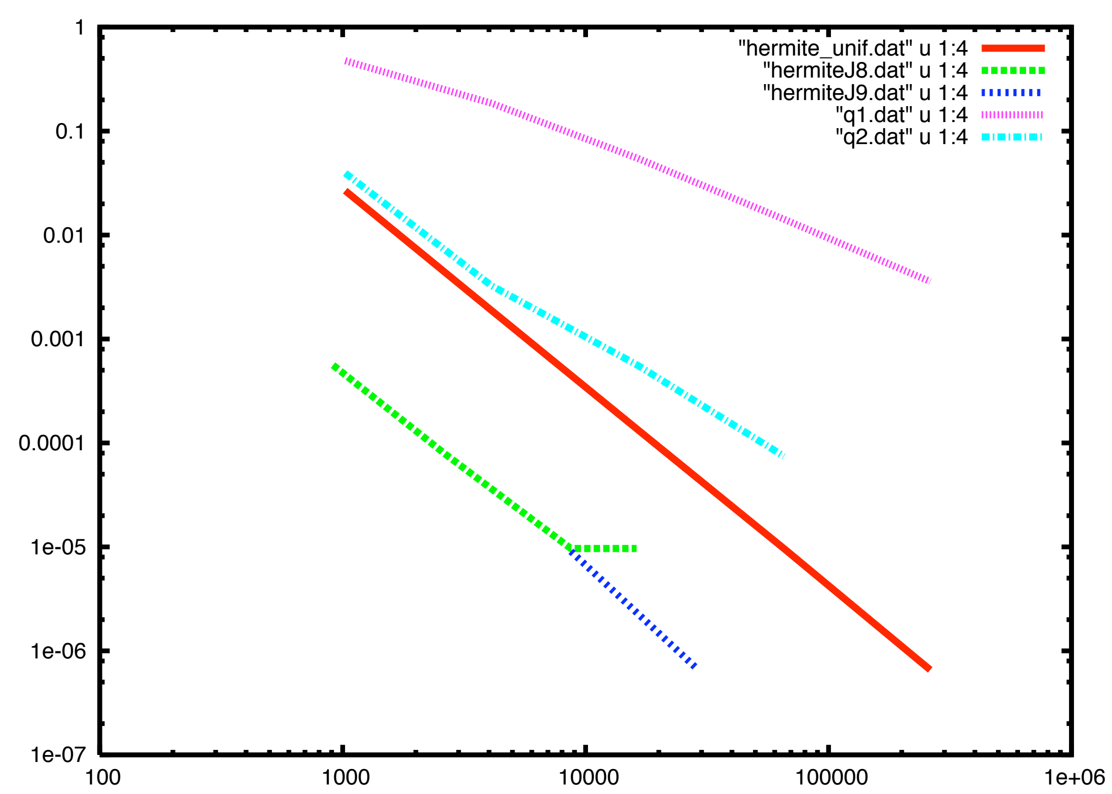
J	ϵ	Nx	$Nx * Nv$	L^1	L^2	L^∞
5	-	64	4096	0.0286834	0.0288427	0.0393988
6	-	128	16384	0.00268651	0.0024835	0.00326859
7	-	256	65536	0.000293845	0.000295532	0.000558552
8	-	512	262144	3.13587e-05	3.13749e-05	7.438e-05

Errors: (for Hermite uniform reconstruction after 100 iterations)

J	ϵ	Nx	$Nx * Nv$	L^1	L^2	L^∞
5	-	64	4096	0.0203536	0.0190892	0.0267576
6	-	128	16384	0.00128987	0.0012402	0.00188675
7	-	256	65536	8.11716e-05	7.9731e-05	0.000134138
8	-	512	262144	4.99702e-06	4.94857e-06	9.65621e-06
9	-	1024	1048576	3.21994e-07	3.20105e-07	6.56759e-07

Errors: (for Hermite adaptive reconstruction after 100 iterations)

J	ϵ	$ratio$	L^1	L^2	L^∞	nbcells
8	0.001	0.0139313	0.0034672	0.0012059	0.000561271	913
8	0.0001	0.04052732713	0.000317387	0.000116803	7.72862e-05	2656
8	0.00001	0.132629	3.84543e-05	1.28593e-05	9.65621e-06	8692
8	1.e-6	0.247574	7.91941e-06	5.08294e-06	9.65621e-06	16225
9	0.00001	0.0331802	3.8421e-05	1.28305e-05	9.07556e-06	8698
9	1.e-6	0.110897	4.01512e-06	1.31218e-06	6.56759e-07	29071



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"q2.dat" u 1:4
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0.01
0.001
0.0001
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1e-06
1e-07
100
1000
10000
100000
1e+06