Almost periodic solutions for first order differential equations

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(7th September 2005)

Abstract

We study here the almost periodic solutions of first order differential equations. We give sufficient conditions for the existence and uniqueness. The method relies on penalization and a priori estimates. One of the main difficulties consists of verifying that the limit of the sequence of perturbed solutions remains almost periodic. We introduce the notions of minimal/maximal solutions.

Keywords: Almost periodic solutions, sub/supersolutions, minimal/maximal solutions.

AMS classification: 35Q99, 35B40.

1 Introduction

The theory of almost periodic functions has been developed in connection with problems of differential equations, stability theory, dynamical systems, and so on. The applications include not only ordinary differential equations, but also partial differential equations or equations in Banach spaces. There are several results concerning the existence and uniqueness of almost periodic solution for first order differential equations. Demidovitch [5] proved that if $f : \mathbb{R} \to \mathbb{R}$ is almost periodic function with bounded primitive $F(t) = \int_0^t f(s)\ ds$, $t \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is monotone C^1 function, then all bounded solution of

$$x'(t) + g(x(t)) = f(t), \quad t \in \mathbb{R}, \tag{1}$$

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is almost periodic. This result was generalized by Gheorghiu [7] for first order differential equations

$$x'(t) + g(t, x(t)) = 0, \quad t \in \mathbb{R}. \tag{2}$$

He proved that if g is C^1 almost periodic of t, uniformly wrt $x \in [-R, R]$, $\forall R > 0$ and

$$\forall R > 0, \quad \exists \gamma_R > 0 \quad : \frac{\partial g}{\partial x} \geq \gamma_R > 0 (\text{resp. } \frac{\partial g}{\partial x} \leq \gamma_R < 0), \forall (t, x) \in \mathbb{R} \times [-R, R],$$

then all bounded solution of (2) is almost periodic. The previous result was extended by Opial [8] for functions $g = g(t, x)$ almost periodic of t, uniformly wrt x on bounded sets and monotone wrt x. Other results have been obtained by Amerio [1], Corduneanu [4], Favard [6]. In all these works the authors suppose the existence of bounded solutions and prove that these solutions are almost periodic. The aim of this paper is to give sufficient conditions in terms of the functions f, g in (1), (2) which ensure the existence of almost periodic solution for first order differential equations. We indicate also a necessary condition. In order to present the ideas let us analyze the periodic solutions of (1) where f is T periodic continuous function and g is nondecreasing continuous function. Notice that if there is at least one T periodic solution for (1) then

$$\int_0^T g(x(t)) \, dt = \int_0^T f(t) \, dt,$$

and therefore, by mean value theorem, we deduce that there is $t_0 \in [0, T]$ such that

$$g(x(t_0)) = \langle f \rangle := \frac{1}{T} \int_0^T f(t) \, dt.$$

We obtained the following necessary condition

$$\langle f \rangle \in g(\mathbb{R}). \tag{3}$$

Conversely, we can prove that the above condition is sufficient for the existence of time periodic solution for (1). For this it is convenient to use the penalization method. For all $\alpha > 0$ it is easy to construct the unique T periodic solution for the perturbed equation

$$\alpha x_\alpha(t) + x'_\alpha(t) + g(x_\alpha(t)) = f(t), \quad t \in \mathbb{R}. \tag{4}$$

As usual we obtain a solution for (1) by passing $\alpha \searrow 0$ in the sequence $(x_\alpha)_{\alpha > 0}$. This can be done by using the Arzela-Ascoli theorem if we find uniform bounds wrt $\alpha > 0$ for $\|x_\alpha\|_{L^\infty(\mathbb{R})}$. Suppose that the condition (3) is satisfied and let us derive a priori estimates for $(x_\alpha)_{\alpha > 0}$. As before, by using mean value theorem we obtain

$$\exists x_\alpha = x_\alpha(t_\alpha), \quad \alpha x_\alpha + g(x_\alpha) = \langle f \rangle,$$

which can be written

$$\alpha x_\alpha + g(x_\alpha) - g(x_0) = 0, \tag{5}$$

where $g(x_0) = \langle f \rangle$. After multiplication of (5) by $x_\alpha - x_0$ and by using the monotony of g we deduce that the sequence $(x_\alpha(t_\alpha))_\alpha$ is bounded

$$|x_\alpha(t_\alpha)| \le |x_0|, \ \ \forall \alpha > 0,$$

and after standard computations we obtain

$$\|x_\alpha\|_{L^\infty(\mathbb{R})} \le |x_0| + \|f - \langle f \rangle\|_{L^1(]0,T[)}, \ \ \forall \alpha > 0.$$

The same method applies for evolution equations with periodic source term

$$x'(t) + Ax(t) = f(t), \ \ t \in \mathbb{R},$$

where $A : D(A) \subset H \to H$ is linear maximal monotone symmetric operator on a Hilbert space H and f is T periodic. We can prove that there is a periodic solution iff $\langle f \rangle \in \text{Range}(A)$, *i.e.*, $\exists x_0 \in D(A)$ such that $\langle f \rangle = Ax_0$. For details the reader can refer to [2], [3].

The main results of this paper are the following sufficient conditions which guarantee the existence of almost periodic solution for first order differential equations

$$x'(t) + g(t, x(t)) = 0, \ \ t \in \mathbb{R}. \tag{6}$$

Theorem 3.3 *Assume that $g = g(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is nondecreasing wrt x and almost periodic of t, uniformly wrt x on bounded sets. If there is $M > 0$ such that*

$$g(t, -M) \le 0 \le g(t, M), \ \ \forall t \in \mathbb{R},$$

then there is at least one almost periodic solution x for (6) satisfying

$$-M \le x(t) \le M, \ \ \forall t \in \mathbb{R}.$$

Theorem 4.1 *Assume that $g = g(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is nondecreasing wrt x and almost periodic of t, uniformly wrt x on bounded sets. If there is $X \in \mathbb{R}$ such that $\langle g(\cdot, X) \rangle := \lim_{T\to+\infty} \frac{1}{T}\int_0^T g(t,X)\,dt = 0$ and $\sup_{s,t\in\mathbb{R}} \left(-\int_s^t g(\sigma, X)\,d\sigma\right) < +\infty$ then there is at least one almost periodic solution x for (6) satisfying*

$$\|x(\cdot) - X\|_{L^\infty(\mathbb{R})} \le \sup_{s,t\in\mathbb{R}} \left(-\int_s^t g(\sigma, X)\,d\sigma\right).$$

We give also a uniqueness result for the almost periodic solution of

$$x'(t) + g(x(t)) = f(t), \ \ t \in \mathbb{R}. \tag{7}$$

Theorem 5.2 *Assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing and $f : \mathbb{R} \to \mathbb{R}$ is almost periodic such that $\langle f \rangle \in g(\mathbb{R})$ and $\sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right) < +\infty$. Then there is at least one almost periodic solution for (7) and the solution is unique iff*

$$diam(g^{-1}\langle f \rangle) \le \sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right).$$

The content of this paper is organized as follows. We start our analysis by studying the existence and uniqueness of bounded solution for first order differential equations. We recall the notions of sub/supersolutions and we introduce the concept of minimal/maximal solutions. In Section 3 we prove our first existence result of almost periodic solution (see Theorem 3.3). Actually we prove that the minimal solution is almost periodic. Moreover we deduce that all bounded solution is almost periodic. In the next section we prove our second existence result of almost periodic solution (see Theorem 4.1). In Section 5 we study the asymptotic behavior of almost periodic solutions for large frequencies. We end with some uniqueness and stability results for almost periodic solutions.

2 Bounded solutions for first order differential equations

As we will see later, under appropriate hypotheses, the classes of bounded solutions and almost periodic solutions of first order differential equations coincide. In this section we analyze the existence and uniqueness of bounded solution for the equation

$$x'(t) + g(t, x(t)) = 0, \quad t \in \mathbb{R}, \tag{8}$$

where $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, nondecreasing wrt x

$$g(t, x) \le g(t, y), \quad \forall t \in \mathbb{R}, \quad \forall\, x \le y. \tag{9}$$

Note that it is also possible to study the equation (8) when g is nonincreasing wrt x. For this observe that $x(\cdot)$ is solution of (8) iff $y(t) = x(-t)$ is solution of

$$y'(t) - g(-t, y(t)) = 0, \quad t \in \mathbb{R}.$$

In the following we always suppose that g is nondecreasing wrt x.

2.1 Sub/supersolutions for first order differential equations

In this paragraph we study the properties of sub/supersolutions of (8).

Definition 2.1 *Let $x : \mathbb{R} \to \mathbb{R}$ be a C^1 function.*
1) We say that $x(\cdot)$ is subsolution of (8) *iff*

$$x'(t) + g(t, x(t)) \le 0, \quad \forall t \in \mathbb{R}.$$

2) We say that $y(\cdot)$ is supersolution of (8) *iff*

$$y'(t) + g(t, y(t)) \ge 0, \quad \forall t \in \mathbb{R}.$$

The main tool is the following classical comparison result for bounded sub/supersolutions. We assume that g satisfies the hypothesis

$$\exists \gamma > 0 \; : \; g(t,x) - \gamma x \le g(t,y) - \gamma y, \quad \forall t \in \mathbb{R}, \quad \forall x \le y. \tag{10}$$

Proposition 2.1 *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies* (10)*. Let $x(\cdot)$ be a bounded (from above) subsolution of* (8) *and $y(\cdot)$ be a bounded (from below) supersolution of* (8)*. Then we have the inequality*

$$x(t) \le y(t), \quad \forall t \in \mathbb{R}.$$

Proof. The arguments are standard. Consider $z_\delta(t) = x(t) - y(t) - \delta(t^2+1)^{\frac{1}{2}}$, $t \in \mathbb{R}, \delta > 0$. Since $x(\cdot)$ is bounded from above and $y(\cdot)$ is bounded from below we deduce that $\lim_{|t| \to +\infty} z_\delta(t) = -\infty$ and therefore there is $t_\delta \in \mathbb{R}$ such that

$$x(t) - y(t) - \delta(t^2+1)^{\frac{1}{2}} = z_\delta(t) \le z_\delta(t_\delta) = x(t_\delta) - y(t_\delta) - \delta(t_\delta^2+1)^{\frac{1}{2}}, \quad \forall t \in \mathbb{R}. \tag{11}$$

In particular we obtain

$$x'(t_\delta) - y'(t_\delta) - \delta \frac{t_\delta}{(t_\delta^2+1)^{\frac{1}{2}}} = 0. \tag{12}$$

Since $x(\cdot)$ is subsolution and $y(\cdot)$ is supersolution we have

$$x'(t_\delta) - y'(t_\delta) + g(t_\delta, x(t_\delta)) - g(t_\delta, y(t_\delta)) \le 0. \tag{13}$$

The hypothesis (10) implies that

$$g(t_\delta, x(t_\delta)) - g(t_\delta, y(t_\delta)) = (\gamma + r_\delta) \cdot (x(t_\delta) - y(t_\delta)), \tag{14}$$

where $r_\delta \ge 0$. Combining (12), (13), (14) yields

$$\delta \frac{t_\delta}{(t_\delta^2+1)^{\frac{1}{2}}} + (\gamma + r_\delta) \cdot (x(t_\delta) - y(t_\delta)) \le 0, \tag{15}$$

which implies

$$x(t_\delta) - y(t_\delta) \le \frac{\delta}{\gamma + r_\delta} \le \frac{\delta}{\gamma}. \tag{16}$$

By using (11), (16) we deduce that

$$x(t) - y(t) \le \delta(t^2+1)^{\frac{1}{2}} + \frac{\delta}{\gamma}, \quad \forall t \in \mathbb{R}, \forall \delta > 0.$$

The conclusion follows by keeping t fixed and by passing $\delta \searrow 0$. $\square$

As a direct consequence of the above comparison result we obtain the uniqueness of bounded solution for (8).

Corollary 2.1 *Assume that g is continuous and satisfies* (10)*. Then the equation* (8) *has at most one bounded solution.*

When g satisfies only (9) it is possible to show that the difference between two solutions of (8) keeps constant sign on $\mathbb{R}$.

Proposition 2.2 *Assume that g is continuous and satisfies* (9)*. Consider x, y two solutions of* (8)*. Then $x(t) \leq y(t), \forall t \in \mathbb{R}$ or $x(t) \geq y(t), \forall t \in \mathbb{R}$.*

Proof. Suppose that there is t_1, t_2 such that $(x(t_1) - y(t_1)) \cdot (x(t_2) - y(t_2)) < 0$. For example consider the case $x(t_1) < y(t_1)$ and $x(t_2) > y(t_2)$. We have

$$\frac{1}{2}\frac{d}{dt}|x - y|^2 + (g(t, x(t)) - g(t, y(t))) \cdot (x(t) - y(t)) = 0, \;\; t \in \mathbb{R},$$

and thus $t \rightarrow |x(t) - y(t)|$ is nonincreasing. There is $t_3 = \theta t_1 + (1-\theta)t_2$, $\theta \in]0,1[$ such that $x(t_3) = y(t_3)$. We deduce that $x(t) = y(t)$, $\forall t \geq t_3$ which is impossible since we have $x(t_4) \neq y(t_4)$ with $t_4 = \max(t_1, t_2) > t_3$. Hence the difference $x(t) - y(t)$ keeps constant sign for $t \in \mathbb{R}$. $\square$

2.2 Existence of bounded solution

We prove now the existence of bounded solution for (8) when g satisfies hypothesis (10). In this case the proof is standard.

Proposition 2.3 *Assume that g is continuous satisfying* (10) *and* $\sup_{t \in \mathbb{R}} |g(t, 0)| = C < +\infty$*. Then there is a unique bounded solution for* (8)*.*

Proof. The uniqueness of bounded solution was already proved. Let us construct a bounded solution. For all $n \geq 0$ we consider x_n the unique classical solution for

$$x_n'(t) + g(t, x_n(t)) = 0, \;\; t > -n, \;\; x_n(-n) = 0. \tag{17}$$

Note that the existence and uniqueness for the solution of (17) hold for every continuous, nondecreasing wrt x function g. Indeed, the uniqueness follows easily by using the monotony of g and the initial condition. In order to prove the existence, construct $\tilde{x}_n : [-n, \tau_n[$ the maximal solution of (17). We show that $\tau_n = +\infty$. Suppose that $\tau_n < +\infty$ and observe that

$$\frac{1}{2}\frac{d}{dt}|\tilde{x}_n(t)|^2 + (g(t, \tilde{x}_n(t)) - g(t, 0)) \cdot \tilde{x}_n(t) = -g(t, 0) \cdot \tilde{x}_n(t), \;\; -n \leq t < \tau_n,$$

which implies

$$\frac{1}{2}|\tilde{x}_n(t)|^2 \leq \int_{-n}^{t} |g(s,0)| \cdot |\tilde{x}_n(s)| \, ds, \ \ -n \leq t < \tau_n.$$

By using Bellman's lemma we obtain

$$|\tilde{x}_n(t)| \leq \int_{-n}^{t} |g(s,0)| \, ds \leq C(\tau_n + n), \ \ \forall -n \leq t < \tau_n,$$

and therefore the maximal solution remains bounded on $[-n, \tau_n[$, which is not possible. Thus $\tau_n = +\infty$. We prove now that $(\|x_n\|_{L^\infty(]-n,+\infty[)})_n$ is bounded. We can write

$$\frac{1}{2}\frac{d}{dt}|x_n(t)|^2 + (g(t, x_n(t)) - g(t,0)) \cdot x_n(t) = -g(t,0) \cdot x_n(t), \ \ t \geq -n. \tag{18}$$

By hypothesis (10) we deduce that

$$(g(t, x_n(t)) - g(t,0)) \cdot x_n(t) \geq \gamma |x_n(t)|^2,$$

and therefore we obtain

$$\frac{1}{2}\frac{d}{dt}|x_n(t)|^2 + \gamma |x_n(t)|^2 \leq |g(t,0)| \cdot |x_n(t)|, \ \ t \geq -n.$$

By using Bellman's lemma we have

$$|x_n(t)| e^{\gamma t} \leq \int_{-n}^{t} |g(s,0)| e^{\gamma s} \, ds \leq \frac{C}{\gamma}(e^{\gamma t} - e^{-\gamma n}),$$

and finally one gets

$$|x_n(t)| \leq \frac{C}{\gamma}, \ \ \forall t \geq -n, \forall n \geq 0.$$

We can prove also that $(x_n)_n$ converges uniformly on every interval $[\tau, +\infty[$, $\tau \in \mathbb{R}$. Indeed, as before we obtain

$$\frac{1}{2}\frac{d}{dt}|x_n(t) - x_m(t)|^2 + \gamma |x_n(t) - x_m(t)|^2 \leq 0, \ \ t \geq \max(-n, -m),$$

which implies

$$|x_n(t) - x_m(t)| \leq e^{-\gamma(t - t_0)} |x_n(t_0) - x_m(t_0)| \leq \frac{2C}{\gamma} e^{-\gamma(t-t_0)}, \ \ \forall t \geq t_0 \geq \max(-n, -m).$$

Assume that $m \geq n$ and take $t_0 = -n$. We have for all $t \geq \tau \geq -n$

$$|x_n(t) - x_m(t)| \leq \frac{2C}{\gamma} e^{-\gamma(t+n)} \leq \frac{2C}{\gamma} e^{-\gamma(\tau+n)}.$$

We deduce that $(x_n)_n$ is a Cauchy sequence in $C^0([\tau,+\infty[)$ and therefore converges uniformly on $[\tau,+\infty[$. We denote by x the limit function. Since $|x_n(t)| \le \frac{C}{\gamma}$, $\forall t \ge -n, \forall n \ge 0$ we deduce that $|x(t)| \le \frac{C}{\gamma}$, $\forall t \in \mathbb{R}$. Let us check that x is solution for (8). For n large enough we have

$$x_n(t) - x_n(\tau) + \int_\tau^t g(s, x_n(s))\, ds = 0, \ \ \forall t \ge \tau \ge -n.$$

By passing to the limit for $n \to +\infty$ we obtain

$$x(t) - x(\tau) + \int_\tau^t g(s, x(s))\, ds = 0, \ \ \forall t \ge \tau,$$

and therefore $x \in C^1(\mathbb{R})$ and $x'(t) + g(t, x(t)) = 0, \forall t \in \mathbb{R}$. $\square$

2.3 Minimal/maximal solutions

In this section we suppose that the function g is only nondecreasing wrt x. Generally, in this case, we can not prove the uniqueness of solution for (8). We need to distinguish some particular solutions. We suppose that g satisfies the hypothesis

$$\forall R > 0, \exists C_R > 0 \ : \ |g(t,x)| \le C_R, \ \forall (t,x) \in \mathbb{R} \times [-R, R]. \tag{19}$$

Proposition 2.4 *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (9), (19). *Consider x_0 a bounded subsolution of* (8) *and y_0 a bounded supersolution of* (8) *such that*

$$x_0(t) \le y_0(t), \ \ \forall t \in \mathbb{R}.$$

For $\alpha > 0$ we denote by x_α the unique bounded solution of the equation

$$\alpha(x_\alpha(t) - x_0(t)) + x_\alpha'(t) + g(t, x_\alpha(t)) = 0, \ \ t \in \mathbb{R}. \tag{20}$$

Then the family $(x_\alpha)_\alpha$ converges uniformly on compact sets towards a solution x of (8) *verifying*

$$x_0(t) \le x(t) \le y_0(t), \ \ \forall t \in \mathbb{R}.$$

In particular there is at least one bounded solution for (8).

Proof. Note that the hypotheses of Proposition 2.3 hold for the function

$$g_\alpha(t,x) = \alpha x + g(t,x) - \alpha x_0(t), \ \ (t,x) \in \mathbb{R} \times \mathbb{R},$$

and therefore x_α is well defined for all $\alpha > 0$. Moreover x_0 is bounded subsolution of (20) and by Proposition 2.1 we obtain $x_0 \le x_\alpha$ (*i.e.*, $x_0(t) \le x_\alpha(t), \ \ \forall t \in \mathbb{R}$). Since $x_0 \le y_0$ we check easily that y_0 is bounded supersolution for (20) and thus by

Proposition 2.1 we have $x_\alpha \le y_0$. In fact it is possible to prove that if $0 < \alpha \le \beta$ then

$$x_0 \le x_\beta \le x_\alpha \le y_0.$$

Indeed, it is sufficient to prove that x_β is subsolution for (20). We can write

$$\begin{aligned}\alpha(x_\beta(t) - x_0(t)) + x_\beta'(t) + g(t, x_\beta(t)) &= (\alpha - \beta)\cdot(x_\beta(t) - x_0(t)) + \beta(x_\beta(t) - x_0(t)) \\ &+ x_\beta'(t) + g(t, x_\beta(t)) \\ &\le \beta(x_\beta(t) - x_0(t)) + x_\beta'(t) + g(t, x_\beta(t)) \\ &= 0.\end{aligned}$$

We denote $x(t) = \sup_{\alpha>0} x_\alpha(t) = \lim_{\alpha \searrow 0} x_\alpha(t)$. Obviously we have

$$x_0 \le x_\alpha \le x \le y_0.$$

Take $R_0 = \max(\|x_0\|_{L^\infty(\mathbb{R})}, \|y_0\|_{L^\infty(\mathbb{R})})$ and note that we have the inequalities

$$|g(t, x_\alpha(t))| \le \max(|g(t, x_0(t))|, |g(t, y_0(t))|) \le C_{R_0}, \quad \forall t \in \mathbb{R}, \alpha > 0.$$

We deduce that the solutions $(x_\alpha)_\alpha$ are uniformly lipschitz

$$|x_\alpha'(t)| \le y_0(t) - x_0(t) + C_{R_0} \le 2R_0 + C_{R_0}, \quad \forall t \in \mathbb{R}, 0 < \alpha \le 1,$$

and therefore $(x_\alpha)_\alpha$ converges to x uniformly on compact sets. It follows easily that x is a classical solution of (8) and $|x'(t)| \le C_{R_0}$, $\forall t \in \mathbb{R}$. □

Definition 2.2 *Under the hypotheses of Proposition* 2.4 *we say that* $x = \sup_{\alpha>0} x_\alpha$ *is the minimal solution of* (8).

Proposition 2.5 *Under the hypotheses of Proposition* 2.4*, the minimal solution verifies the following minimal property : if* z *is a supersolution of* (8) *such that* $x_0 \le z \le x$ *then* $z = x$.

Proof. We need to prove the inequality $z \ge x$ which is equivalent to $z \ge x_\alpha$, $\forall \alpha > 0$. For this it is sufficient to show that z is a supersolution for (20). We have

$$\alpha(z(t) - x_0(t)) + z'(t) + g(t, z(t)) \ge z'(t) + g(t, z(t)) \ge 0, \quad \forall t \in \mathbb{R},$$

and therefore, by Proposition 2.1 we obtain $z \ge x_\alpha$, $\forall \alpha > 0$. □

Similarly we define the notion of maximal solution.

Proposition 2.6 *Let* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* (9), (19). *Consider* x_0 *a bounded subsolution of* (8) *and* y_0 *a bounded supersolution of* (8) *such that*

$$x_0(t) \le y_0(t), \quad \forall t \in \mathbb{R}.$$

For $\alpha > 0$ we denote by y_α the unique bounded solution of the equation

$$\alpha(y_\alpha(t) - y_0(t)) + y'_\alpha(t) + g(t, y_\alpha(t)) = 0, \quad t \in \mathbb{R}. \tag{21}$$

Then the family $(y_\alpha)_\alpha$ converges uniformly on compact sets towards a solution y of (8) *verifying*

$$x_0(t) \le y(t) \le y_0(t), \quad \forall t \in \mathbb{R}.$$

Proof. Similar to those of Proposition 2.4. In this case we prove that if $0 < \alpha \le \beta$ then

$$x_0 \le y_\alpha \le y_\beta \le y_0,$$

and therefore $y = \lim_{\alpha \searrow 0} y_\alpha = \inf_{\alpha > 0} y_\alpha$. □

Definition 2.3 *Under the hypotheses of Proposition* 2.6 *we say that $y = \inf_{\alpha>0} y_\alpha$ is the maximal solution of* (8)*.*

Proposition 2.7 *Under the hypotheses of Proposition* 2.6*, the maximal solution verifies the following maximal property : if z is a subsolution of* (8) *such that $y \le z \le y_0$ then $z = y$.*

For all x bounded subsolution of (8) and y bounded supersolution of (8) such that $x \le y$ we denote by $x_{min}(x, y)$, $y_{max}(x, y)$ the minimal, resp. maximal solutions constructed before. Let us give some easy properties of the minimal/maximal solutions. The proof are immediate and are left to the reader.

Proposition 2.8 *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (9)*,* (19)*.*
1) If x is bounded subsolution, y is bounded supersolution such that $x \le y$, then $x_{min}(x, y) \le y_{max}(x, y)$;
2) If x is bounded subsolution and y_1, y_2 are bounded supersolutions such that $x \le \min(y_1, y_2)$ then $x_{min}(x, y_1) = x_{min}(x, y_2)$;
3) If x_1, x_2 are bounded subsolutions and y is bounded supersolution such that $x_1 \le x_2 \le y$ then $x_{min}(x_1, y) \le x_{min}(x_2, y)$;
4) If x_1, x_2 are bounded subsolutions and y is bounded supersolution such that $y \ge \max(x_1, x_2)$ then $y_{max}(x_1, y) = y_{max}(x_2, y)$;
5) If x is bounded subsolution and y_1, y_2 are bounded supersolutions such that $x \le y_1 \le y_2$ then $y_{max}(x, y_1) \le y_{max}(x, y_2)$.

3 Almost periodic solutions for first order differential equations

In the previous section we studied the existence and uniqueness of bounded solution for (8) when g is continuous function satisfying (10) or only (9). Now we are interested on almost periodic solutions for (8). We recall here the notions of almost periodic function and normal function.

Definition 3.1 *Consider $f : \mathbb{R} \to \mathbb{R}$ a continuous function. We say that f is almost periodic iff $\forall \varepsilon > 0$, $\exists\, l(\varepsilon) > 0$ such that all interval of length $l(\varepsilon)$ contains a number τ satisfying*

$$|f(t+\tau) - f(t)| \le \varepsilon, \ \ \forall t \in \mathbb{R}.$$

The number τ is called ε-almost period of the function f. We deduce immediately the following properties of almost periodic functions.

Proposition 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be almost periodic function. Then f is bounded and uniformly continuous on $\mathbb{R}$.*

Another important property of almost periodic functions is the existence of the average (see [4]).

Proposition 3.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be almost periodic function. Then $\frac{1}{T}\int_a^{a+T} f(t)\, dt$ converges as $T \to +\infty$ uniformly wrt $a \in \mathbb{R}$. Moreover the limit does not depend on a and is called the average of f*

$$\exists\, \langle f \rangle := \lim_{T\to+\infty} \frac{1}{T}\int_a^{a+T} f(t)\, dt, \textit{ uniformly wrt } a \in \mathbb{R}.$$

Definition 3.2 *Consider $f : \mathbb{R} \to \mathbb{R}$ a continuous function. We say that f is a normal function iff for all real sequence $(h_n)_n$ there is a subsequence $(h_{n_k})_k$ such that $(f(\cdot + h_{n_k}))_k$ converges uniformly on $\mathbb{R}$.*

We have the following result (see [4]).

Theorem 3.1 *(Bochner) Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Then f is almost periodic function iff f is a normal function.*

We introduce also the notion of almost periodic function of t, uniformly wrt x on compact sets.

Definition 3.3 *Consider $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function. We say that g is almost periodic of t, uniformly wrt x on compact sets iff $\forall R > 0$, $\forall \varepsilon > 0$, $\exists\, l(\varepsilon, R) > 0$ such that all interval of length $l(\varepsilon, R)$ contains a number τ satisfying*

$$|g(t+\tau, x) - g(t,x)| \le \varepsilon, \ \ \forall (t,x) \in \mathbb{R} \times [-R, R]. \tag{22}$$

We have the analogous result.

Proposition 3.3 *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be almost periodic of t uniformly wrt x on compact sets. Then g is bounded and uniformly continuous on $\mathbb{R} \times [-R, R]$, $\forall R > 0$.*

Definition 3.4 *Consider $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function. We say that g is a normal function of t uniformly wrt x on compact sets iff for all real sequence $(h_n)_n$ there is a subsequence $(h_{n_k})_k$ such that $(g(\cdot + h_{n_k}, \cdot))_k$ converges uniformly on $\mathbb{R} \times [-R, R]$, $\forall R > 0$.*

By adapting the proof of Bochner's theorem we obtain.

Theorem 3.2 *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a real function. Then g is almost periodic of t uniformly wrt x on compact sets iff g is normal of t uniformly wrt x on compact sets.*

3.1 Existence and uniqueness of almost periodic solution when $\gamma > 0$

In this paragraph we establish the existence and uniqueness of the almost periodic solution when g satisfies (10).

Proposition 3.4 *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (10)*,* (22)*. Then there is a unique almost periodic solution for* (8)*.*

Proof. The uniqueness follows from Corollary 2.1 since all almost periodic function is bounded function. For the existence use Proposition 2.3. Indeed, by Proposition 3.3 we deduce that $\sup_{t\in\mathbb{R}} |g(t, 0)| < +\infty$ and therefore there is a unique bounded solution x for (8). It remains to prove that x is almost periodic. Take $R_0 = \|x\|_{L^\infty(\mathbb{R})}$, $\varepsilon > 0$ and $l = l(\varepsilon\gamma, R_0) > 0$. All interval of length l contains a number τ such that

$$|g(t + \tau, x) - g(t, x)| \le \varepsilon\gamma, \ \ \forall (t, x) \in \mathbb{R} \times [-R_0, R_0].$$

We have for all $t \in \mathbb{R}$

$$x'(t + \tau) - x'(t) + g(t + \tau, x(t + \tau)) - g(t + \tau, x(t)) = -(g(t + \tau, x(t)) - g(t, x(t))),$$

and after multiplication by $x(t + \tau) - x(t)$ one gets

$$\frac{1}{2}\frac{d}{dt}|x(t+\tau)-x(t)|^2+\gamma|x(t+\tau)-x(t)|^2 \le |g(t+\tau, x(t))-g(t, x(t))|\cdot|x(t+\tau)-x(t)|, t \in \mathbb{R}.$$

We obtain by using Bellman's lemma

$$\begin{aligned} e^{\gamma t}|x(t + \tau) - x(t)| \ &\le \ e^{\gamma t_0}|x(t_0 + \tau) - x(t_0)| \\ &+ \ \left(\int_{t_0}^{t} e^{\gamma s} ds\right) \sup_{(r,y)\in\mathbb{R}\times[-R_0,R_0]} |g(r + \tau, y) - g(r, y)|, \ \ \forall t \ge t_0. \end{aligned}$$

Finally we deduce

$$|x(t + \tau) - x(t)| \le e^{-\gamma(t-t_0)}|x(t_0 + \tau) - x(t_0)| + \varepsilon, \ \ \forall t \ge t_0.$$

By keeping t fixed and passing $t_0 \to -\infty$ we obtain

$$|x(t + \tau) - x(t)| \le \varepsilon, \ \ \forall t \in \mathbb{R},$$

and thus x is almost periodic function. $\square$

We mention here the following result which will be used in the next paragraph.

Proposition 3.5 *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (10), $\sup_{t\in\mathbb{R}} |g(t, 0)| = C < +\infty$ *and consider* $(h_n)_n$ *a real sequence such that* $(g(t+h_n, x))_n$ *converges uniformly on* $\mathbb{R} \times [-R, R]$, $\forall R > 0$

$$\lim_{n\to+\infty} g(t + h_n, x) =: \tilde{g}(t, x), \quad \textit{uniformly on } \mathbb{R} \times [-R, R],\ \forall R > 0.$$

Denote by x *the unique bounded solution of* (8)*. Then* $(x(\cdot + h_n))_n$ *converges uniformly on* $\mathbb{R}$ *towards the unique bounded solution of* (8) *associated to the function* $\tilde{g}$*.*

Proof. Note that the function $\tilde{g}$ satisfies (10) and $\sup_{t\in\mathbb{R}} |\tilde{g}(t, 0)| = C < +\infty$ and therefore, by Proposition 2.3, we deduce that there is a unique bounded solution $\tilde{x}$ for

$$\tilde{x}' + \tilde{g}(t, \tilde{x}(t)) = 0, \quad \forall t \in \mathbb{R}.$$

We introduce the notations $x_n(t) = x(t + h_n), g_n(t, x) = g(t + h_n, x)$. By the computations in the proof of Proposition 2.3 we know that

$$\max(\|x\|_{L^\infty(\mathbb{R})}, \|x_n\|_{L^\infty(\mathbb{R})}, \|\tilde{x}\|_{L^\infty(\mathbb{R})}) \le \frac{C}{\gamma} =: R_0.$$

We obtain as before that

$$\frac{1}{2}\frac{d}{dt}|x_n - \tilde{x}|^2 + \gamma|x_n(t) - \tilde{x}(t)|^2 \le \|g_n - \tilde{g}\|_{L^\infty(\mathbb{R}\times[-R_0,R_0])} \cdot |x_n(t) - \tilde{x}(t)|, \forall t \in \mathbb{R}, \forall n,$$

which implies that

$$\|x_n - \tilde{x}\|_{L^\infty(\mathbb{R})} \le \frac{1}{\gamma}\|g_n - \tilde{g}\|_{L^\infty(\mathbb{R}\times[-R_0,R_0])}, \quad \forall n.$$

$\square$

3.2 Existence of almost periodic solution when $\gamma = 0$

We want to establish now the existence of almost periodic solution for (8) when g satisfies (9), (22). We assume also that there are constant sub/supersolutions for (8)

$$\exists M > 0 \ :\ g(t, -M) \le 0 \le g(t, M), \quad \forall t \in \mathbb{R}. \tag{23}$$

We need several easy lemmas concerning almost periodic functions.

Lemma 3.1 *Let* $f : \mathbb{R} \to \mathbb{R}^+$ *be a nonnegative almost periodic function such that* $\langle f\rangle = 0$*. Then* $f = 0$*.*

Proof. Suppose that there is $t_0 \in \mathbb{R}$ such that $f(t_0) > 0$. We deduce that there is $\delta_0 > 0$ such that $f(t) \geq 2\varepsilon_0$, $\forall t \in [t_0 - \delta_0, t_0 + \delta_0]$. Consider $l_0 = l(\varepsilon) > 0$ such that all interval of length l_0 contains an ε_0-almost period of f. Without loss of generality we can assume that $\delta_0 < l_0$. For all $k \geq 0$ the interval $[2kl_0 + \delta_0, 2(k+1)l_0[$ contains an ε_0-almost period of f denoted τ_k. Observe that we have the inclusion

$$[t_0 - \delta_0 + \tau_k, t_0 + \tau_k[\subset [t_0 + 2kl_0, t_0 + (2k+2)l_0[,$$

and the inequality

$$f(t + \tau_k) = f(t) + f(t + \tau_k) - f(t) \geq 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0, \quad \forall t \in [t_0 - \delta_0, t_0 + \delta_0].$$

We have

$$\begin{aligned}
\frac{1}{2Nl_0} \int_{t_0}^{t_0 + 2Nl_0} f(t)\, dt &= \frac{1}{2Nl_0} \sum_{k=0}^{N-1} \int_{t_0 + 2kl_0}^{t_0 + (2k+2)l_0} f(t)\, dt \\
&\geq \frac{1}{2Nl_0} \sum_{k=0}^{N-1} \int_{t_0 - \delta_0 + \tau_k}^{t_0 + \tau_k} f(t)\, dt \\
&= \frac{1}{2Nl_0} \sum_{k=0}^{N-1} \int_{t_0 - \delta_0}^{t_0} f(t + \tau_k)\, dt \\
&\geq \frac{1}{2Nl_0} \sum_{k=0}^{N-1} \delta_0 \varepsilon_0 \\
&= \frac{\delta_0 \varepsilon_0}{2l_0}, \quad \forall N \geq 1.
\end{aligned}$$

By passing to the limit for $N \to +\infty$ we deduce that $\langle f \rangle \geq \frac{\delta_0 \varepsilon_0}{2l_0}$ which contradicts our hypothesis. Therefore $f = 0$. $\Box$

Generally when g satisfies (9) we have no uniqueness for the almost periodic solution of (8). Nevertheless it is possible to show that the difference between two almost periodic solutions is a constant.

Proposition 3.6 *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies (9), (22) and consider x, y two almost periodic solutions of (8). Then there is a constant $C \in \mathbb{R}$ such that $x(t) - y(t) = C, \;\; \forall t \in \mathbb{R}$.*

Proof. We have

$$\frac{1}{2} \frac{d}{dt} |x(t) - y(t)|^2 + (g(t, x(t)) - g(t, y(t))) \cdot (x(t) - y(t)) = 0, \quad \forall t \in \mathbb{R}. \tag{24}$$

Since g satisfies (22) and x, y are almost periodic, we check easily that the functions $t \to g(t, x(t))$, $t \to g(t, y(t))$ are almost periodic and thus $t \to H(t) := (g(t, x(t)) -$

$g(t,y(t)))\cdot(x(t)-y(t))$ is almost periodic and nonnegative. After integration of (24) we deduce

$$\frac{1}{2T}|x(T)-y(T)|^2 - \frac{1}{2T}|x(0)-y(0)|^2 + \frac{1}{T}\int_0^T H(t)\,dt = 0.$$

Since x, y are bounded we deduce that $\langle H\rangle = 0$ and by Lemma 3.1 one gets $H = 0$ which implies that

$$g(t,x(t)) = g(t,y(t)), \quad \forall t \in \mathbb{R}.$$

Therefore we obtain

$$x'(t) = -g(t,x(t)) = -g(t,y(t)) = y'(t), \quad \forall t \in \mathbb{R},$$

and thus there is a constant $C \in \mathbb{R}$ such that $x(t) - y(t) = C$, $\forall t \in \mathbb{R}$. □

In the following propositions we establish two important properties of the minimal/maximal solutions.

Proposition 3.7 *Assume that $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies* (9), (22), (23)*. We denote by $x = x_{min}(-M,M)$, $y = y_{max}(-M,M)$ the minimal and maximal solutions of* (8) *respectively. Then there is a constant $C \in \mathbb{R}$ such that*

$$y(t) - x(t) = C, \quad \forall t \in \mathbb{R}.$$

Proof. We have $(x(t),y(t)) = (\sup_{\alpha>0} x_\alpha(t), \inf_{\alpha>0} y_\alpha(t))\ \forall\ t \in \mathbb{R}$, where x_α, y_α solve

$$\alpha(x_\alpha(t)+M) + x_\alpha'(t) + g(t,x_\alpha(t)) = 0, \quad t \in \mathbb{R},$$
$$\alpha(y_\alpha(t)-M) + y_\alpha'(t) + g(t,y_\alpha(t)) = 0, \quad t \in \mathbb{R}.$$

By Proposition 3.4 we know that x_α, y_α are almost periodic functions. We introduce the notations $z_\alpha(t) = y_\alpha(t) - x_\alpha(t)$, $z(t) = y(t) - x(t)$, $t \in \mathbb{R}$. Obviously we have $\lim_{\alpha\searrow 0} z_\alpha = z$. We have the inequalities

$$-M \le x_\alpha \le x \le y \le y_\alpha \le M, \quad \forall \alpha > 0,$$

which implies $z_\alpha \ge z, \forall \alpha > 0$. By the hypothesis (9) we deduce easily that z is nonincreasing. Assume that there is $t_1 < t_2$ such that $z(t_1) > z(t_2)$ and denote $\eta = z(t_1) - z(t_2) > 0$. For α small enough we have

$$z_\alpha(t_2) \le z(t_2) + \frac{\eta}{2}.$$

Observe that for $t \le t_1$ we have

$$z_\alpha(t) \ge z(t) \ge z(t_1) = \eta + z(t_2) \ge z_\alpha(t_2) + \frac{\eta}{2}. \tag{25}$$

Take now τ a $\frac{\eta}{4}$-almost period of z_α such that $\tau \ge t_2 - t_1$. We have

$$z_\alpha(t_2 - \tau) \le z_\alpha(t_2) + \frac{\eta}{4}. \tag{26}$$

Combining (25) and (26) one gets

$$z_\alpha(t_2) + \frac{\eta}{2} \le z_\alpha(t_2 - \tau) \le z_\alpha(t_2) + \frac{\eta}{4},$$

which gives a contradiction. Therefore z is a constant function. □

Proposition 3.8 *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies* (9), (22), (23) *and consider $(h_n)_n$ a real sequence such that $(g(\cdot + h_n, \cdot))_n$ converges uniformly on $\mathbb{R} \times [-R, R]$, $\forall R > 0$. We denote by $x = x_{min}(-M, M)$, $y = y_{max}(-M, M)$ the minimal and maximal solutions of* (8) *respectively. Then we have the convergence*

$$\lim_{n \to +\infty} (x(t + h_n), y(t + h_n)) = (\tilde{x}(t), \tilde{y}(t)), \quad \textit{uniformly on } \mathbb{R},$$

where $\tilde{x} = \tilde{x}_{min}(-M, M)$ and $\tilde{y} = \tilde{y}_{max}(-M, M)$ are respectively the minimal and maximal solutions of (8) *associated to the function $\tilde{g}(t, x) := \lim_{n \to +\infty} g(t + h_n, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}$.*

Proof. Notice that the function $\tilde{g}$ satisfies the same hypotheses as g. In particular the solutions $\tilde{x}, \tilde{y}$ are well defined (cf. Propositions 2.4, 2.6). Recall that

$$x = \sup_{\alpha > 0} x_\alpha, \quad y = \inf_{\alpha > 0} y_\alpha, \quad \tilde{x} = \sup_{\alpha > 0} \tilde{x}_\alpha, \quad \tilde{y} = \inf_{\alpha > 0} \tilde{y}_\alpha,$$

where

$$\alpha(x_\alpha(t) + M) + x'_\alpha(t) + g(t, x_\alpha(t)) = 0, \quad \alpha(\tilde{x}_\alpha(t) + M) + \tilde{x}'_\alpha(t) + \tilde{g}(t, \tilde{x}_\alpha(t)) = 0, \quad t \in \mathbb{R},$$

$$\alpha(y_\alpha(t) - M) + y'_\alpha(t) + g(t, y_\alpha(t)) = 0, \quad \alpha(\tilde{y}_\alpha(t) - M) + \tilde{y}'_\alpha(t) + \tilde{g}(t, \tilde{y}_\alpha(t)) = 0, \quad t \in \mathbb{R}.$$

By Proposition 3.5 we have the convergence

$$\lim_{n \to +\infty} (x_\alpha(t + h_n), y_\alpha(t + h_n)) = (\tilde{x}_\alpha(t), \tilde{y}_\alpha(t)), \quad \text{uniformly on } \mathbb{R}.$$

By Proposition 3.7 there are the constants $C, \tilde{C} \in \mathbb{R}$ such that

$$y(t) - x(t) = C, \quad \tilde{y}(t) - \tilde{x}(t) = \tilde{C}, \quad \forall t \in \mathbb{R}.$$

We prove that $C = \tilde{C}$. Indeed, we can write

$$C \quad = \quad y(t + h_n) - x(t + h_n) \leq y_\alpha(t + h_n) - x_\alpha(t + h_n),$$

and by passing to the limit for $n \to +\infty$ one gets

$$C \leq \tilde{y}_\alpha(t) - \tilde{x}_\alpha(t), \forall t \in \mathbb{R}, \alpha > 0.$$

We deduce that

$$C \leq \lim_{\alpha \searrow 0} (\tilde{y}_\alpha(t) - \tilde{x}_\alpha(t)) = \tilde{y}(t) - \tilde{x}(t) = \tilde{C}.$$

For the reverse inequality observe that since $(g(\cdot + h_n, \cdot))_n$ converges towards $\tilde{g}$ uniformly on $\mathbb{R} \times [-R, R]$, $\forall R > 0$, then $(\tilde{g}(\cdot - h_n, \cdot))_n$ converges towards g uniformly on $\mathbb{R} \times [-R, R]$, $\forall R > 0$ and we deduce as before that $\tilde{C} \leq C$. We prove now that

$(x(t+h_n), y(t+h_n))$ converges towards $(\tilde{x}(t), \tilde{y}(t))$, $\forall t \in \mathbb{R}$. Indeed, we have for all $\alpha > 0$

$$\begin{aligned}\limsup_{n\to+\infty} y(t+h_n) - \liminf_{n\to+\infty} x(t+h_n) &\leq \lim_{n\to+\infty} y_\alpha(t+h_n) - \lim_{n\to+\infty} x_\alpha(t+h_n) \\ &= \tilde{y}_\alpha(t) - \tilde{x}_\alpha(t).\end{aligned}$$

By taking the limit as $\alpha \searrow 0$ we obtain the inequality

$$\limsup_{n\to+\infty} y(t+h_n) - \liminf_{n\to+\infty} x(t+h_n) \leq C. \tag{27}$$

By using the equality $y(t+h_n) - x(t+h_n) = C, \forall t \in \mathbb{R}, \forall n$ we obtain also

$$\limsup_{n\to+\infty} y(t+h_n) = \limsup_{n\to+\infty} x(t+h_n) + C. \tag{28}$$

Combining (27), (28) yields

$$\limsup_{n\to+\infty} x(t+h_n) \leq \liminf_{n\to+\infty} x(t+h_n), \quad \forall t \in \mathbb{R}, \tag{29}$$

and therefore $(x(t+h_n))_n, (y(t+h_n))_n$ converge for all $t \in \mathbb{R}$. Now we can write

$$\lim_{n\to+\infty} y(t+h_n) \leq \inf_{\alpha>0} \lim_{n\to+\infty} y_\alpha(t+h_n) = \inf_{\alpha>0} \tilde{y}_\alpha(t) = \tilde{y}(t), \tag{30}$$

and

$$\lim_{n\to+\infty} x(t+h_n) \geq \sup_{\alpha>0} \lim_{n\to+\infty} x_\alpha(t+h_n) = \sup_{\alpha>0} \tilde{x}_\alpha(t) = \tilde{x}(t). \tag{31}$$

Note also that

$$\lim_{n\to+\infty} y(t+h_n) - \lim_{n\to+\infty} x(t+h_n) = \lim_{n\to+\infty} (y(t+h_n) - x(t+h_n)) = C = \tilde{y}(t) - \tilde{x}(t). \tag{32}$$

We deduce from (30), (31), (32) that $\lim_{n\to+\infty}(x(t+h_n), y(t+h_n)) = (\tilde{x}(t), \tilde{y}(t))$. It remains to prove that the above convergence is uniform on $\mathbb{R}$. We use the same method as in [8]. Since $y - x = C$ it is sufficient to treat only the convergence $\lim_{n\to+\infty} x(t+h_n) = \tilde{x}(t)$. Assume that the convergence is not uniform

$$\exists \varepsilon_0 > 0 \ : \ |x(t_k + h_{n_k}) - \tilde{x}(t_k)| \geq \varepsilon_0, \quad \forall k, \tag{33}$$

for a sequence $(t_k)_k$ and a subsequence $(h_{n_k})_k$. The Theorem 3.2 implies (after extraction if necessary) that

$$\lim_{k\to+\infty} g(t + t_k + h_{n_k}, x) = \tilde{\tilde{g}}(t,x), \text{uniformly for } (t,x) \in \mathbb{R} \times [-R,R], \forall R > 0.$$

Therefore for all $R > 0, \varepsilon > 0, \exists k_1 = k_1(R, \varepsilon)$ such that

$$|g(t + t_k + h_{n_k}, x) - \tilde{\tilde{g}}(t,x)| < \varepsilon, \forall (t,x) \in \mathbb{R} \times [-R,R], \forall k \geq k_1. \tag{34}$$

Recall that for all $R > 0, \varepsilon > 0, \exists k_2 = k_2(R, \varepsilon)$ such that

$$|g(t + h_{n_k}, x) - \tilde{g}(t, x)| < \varepsilon, \forall (t, x) \in \mathbb{R} \times [-R, R], \forall k \geq k_2.$$

In particular we obtain

$$|g(t + t_k + h_{n_k}, x) - \tilde{g}(t + t_k, x)| < \varepsilon, \forall (t, x) \in \mathbb{R} \times [-R, R], \forall k \geq k_2. \tag{35}$$

From (34), (35) we obtain the convergence

$$\lim_{k \to +\infty} \tilde{g}(t + t_k, x) = \tilde{\tilde{g}}(t, x), \text{uniformly on } \mathbb{R} \times [-R, R], \forall R > 0.$$

We denote by $\tilde{x}, \tilde{\tilde{x}}$ the minimal solutions associated to the functions $\tilde{g}, \tilde{\tilde{g}}$. The convergence $\lim_{k \to +\infty} g(t + t_k + h_{n_k}, x) = \tilde{\tilde{g}}(t, x)$ uniformly on $\mathbb{R} \times [-R, R]$, $\forall R > 0$ implies the convergence $\lim_{k \to +\infty} x(t + t_k + h_{n_k}) = \tilde{\tilde{x}}(t)$, $\forall t \in \mathbb{R}$. In particular we have

$$\lim_{k \to +\infty} x(t_k + h_{n_k}) = \tilde{\tilde{x}}(0). \tag{36}$$

The convergence $\lim_{k \to +\infty} \tilde{g}(t + t_k, x) = \tilde{\tilde{g}}(t, x)$ uniformly on $\mathbb{R} \times [-R, R]$, $\forall R > 0$ implies the convergence $\lim_{k \to +\infty} \tilde{x}(t + t_k) = \tilde{\tilde{x}}(t)$, $\forall t \in \mathbb{R}$. In particular we have

$$\lim_{k \to +\infty} \tilde{x}(t_k) = \tilde{\tilde{x}}(0). \tag{37}$$

From (36), (37) we deduce that

$$\lim_{k \to +\infty} (x(t_k + h_{n_k}) - \tilde{x}(t_k)) = 0,$$

which contradicts the assumption (33). $\square$

Now we can prove the following existence result for almost periodic solutions.

Theorem 3.3 *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (9), (22), (23). *Then there is at least one almost periodic solution* x *for* (8) *verifying*

$$-M \leq x(t) \leq M, \quad \forall t \in \mathbb{R}.$$

Proof. Consider $x = x_{min}(-M, M)$ the minimal solution of (8), cf. Proposition 2.4. By construction we have $-M \leq x(t) \leq M$, $\forall t \in \mathbb{R}$. We need to check that x is almost periodic. We use Bochner's theorem. Consider $(h_n)_n$ a real sequence. After extraction of a subsequence we have

$$\lim_{k \to +\infty} g(t + h_{n_k}, x) = \tilde{g}(t, x), \quad \text{uniformly on } \mathbb{R} \times [-R, R], \forall R > 0.$$

By Proposition 3.8 we have

$$\lim_{k \to +\infty} x(t + h_{n_k}) = \tilde{x}(t), \quad \text{uniformly on } \mathbb{R},$$

where $\tilde{x} = \tilde{x}_{min}(-M, M)$ is the minimal solution associated to $\tilde{g}$. Therefore $(x(\cdot + h_{n_k}))_k$ converges uniformly and hence the minimal solution is almost periodic. $\square$

Remark 3.1 *In the above theorem we can replace* (9) *by* $g|_{\mathbb{R}\times[-M,M]}$ *nondecreasing wrt* x *(apply the previous theorem with the function* $g_1(t,x) = \mathbf{1}_{]-\infty,-M[}(x)g(t,-M) + \mathbf{1}_{[-M,M]}(x)g(t,x) + \mathbf{1}_{]M,+\infty[}(x)g(t,M)$, $\forall (t,x) \in \mathbb{R}\times\mathbb{R}$*).*

In fact it is possible to show that all bounded solution of (8) is almost periodic. For the completeness of the presentation we recall here the following result (cf. [8]).

Proposition 3.9 *Assume that* $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *satisfies* (9), (22) *and consider* x *an almost periodic solution and* y *a bounded solution for* (8)*. Then there is a constant* $C \in \mathbb{R}$ *such that* $x(t) - y(t) = C$, $\forall t \in \mathbb{R}$*. In particular* y *is almost periodic solution.*

Proof. From Proposition 2.2 we deduce that $x(t)-y(t) \geq 0, \forall t \in \mathbb{R}$ or $x(t)-y(t) \leq 0, \forall t \in \mathbb{R}$. We analyze the case $x(t) - y(t) \leq 0, \forall t \in \mathbb{R}$, the other case follows in similar way. As usual we obtain

$$\frac{1}{2}\frac{d}{dt}|x(t) - y(t)|^2 + (g(t, x(t)) - g(t, y(t))) \cdot (x(t) - y(t)) = 0, \quad \forall t \in \mathbb{R},$$

and therefore $y(t) - x(t)$ is nonincreasing on $\mathbb{R}$ which implies that $y'(t) - x'(t) \leq 0, \forall t \in \mathbb{R}$. Assume that there is $t_0 \in \mathbb{R}$ such that $y'(t_0) - x'(t_0) < 0$. We deduce that there is $k > 0$ and $[a, b] \subset \mathbb{R}$ such that $x'(t) - y'(t) = g(t, y(t)) - g(t, x(t)) \geq 2k, \forall t \in [a, b]$. For $t \leq a$ we have

$$y(t) - x(t) \geq y(a) - x(a) = \eta \geq 0, \tag{38}$$

and thus

$$x'(t) - y'(t) = g(t, y(t)) - g(t, x(t)) \geq g(t, x(t) + \eta) - g(t, x(t)), \quad \forall t \leq a. \tag{39}$$

It is easy to check that $h(t) = g(t, x(t) + \eta) - g(t, x(t)) \geq 0$ is almost periodic function. Moreover, for all $t \in [a, b]$ we have

$$y(t) - x(t) \leq y(a) - x(a) = \eta,$$

and therefore

$$h(t) \geq g(t, y(t)) - g(t, x(t)) = x'(t) - y'(t) \geq 2k. \tag{40}$$

After integration of (39) one gets

$$0 \geq x(a) - y(a) \geq x(t) - y(t) + \int_t^a h(s)\, ds, \quad \forall t \leq a,$$

which implies

$$\int_t^a h(s)\, ds \leq \|x\|_{L^\infty(\mathbb{R})} + \|y\|_{L^\infty(\mathbb{R})}, \quad \forall t \leq a. \tag{41}$$

Combining (40), (41) and the almost periodicity of $h(t)$ provides a contradiction, since $\lim_{t\to-\infty} \int_t^a h(s)\, ds = +\infty$. Therefore $x'(t) - y'(t) = 0, \forall t \in \mathbb{R}$ and there is a constant $C \in \mathbb{R}$ such that $x(t) - y(t) = C, \forall t \in \mathbb{R}$. $\square$

4 Necessary and sufficient conditions for the existence of almost periodic solution

In the previous section we indicated sufficient conditions for the existence of almost periodic solution for (8) when g is nondecreasing wrt x (see Theorem 3.3). In fact the hypothesis (23) is not necessary for the existence of almost periodic solution. For this we can take the easy example

$$x'(t) + g(t, x(t)) = 0, \quad \forall t \in \mathbb{R}, \tag{42}$$

where $g(t,x) = \sin t, \forall t \in \mathbb{R}$. Note that $x(t) = \cos t, \forall t \in \mathbb{R}$ is periodic solution for (42) but that the hypothesis (23) is not satisfied. In this section we analyze other sufficient conditions and we give also a necessary condition for the existence of almost periodic solution for (8). We start with the following necessary condition.

Proposition 4.1 *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (9), (22). *If there is a bounded solution for* (8), *then there is* $X \in \mathbb{R}$ *such that*

$$\langle g(\cdot, X)\rangle := \lim_{T\to+\infty} \frac{1}{T}\int_0^T g(t,X)\,dt = 0. \tag{43}$$

Proof. Let us denote by $G : \mathbb{R} \to \mathbb{R}$ the average function

$$G(x) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T g(t,x)\,dt, \quad \forall x \in \mathbb{R}.$$

We check easily that G is continuous nondecreasing function. Consider x a bounded solution for (8) and $m, M \in \mathbb{R}$ such that

$$m \le x(t) \le M, \quad \forall t \in \mathbb{R}.$$

We have

$$g(t,m) \le g(t,x(t)) \le g(t,M), \quad \forall t \in \mathbb{R},$$

and thus

$$\frac{1}{T}\int_0^T g(t,m)\,dt \le \frac{1}{T}(x(0) - x(T)) \le \frac{1}{T}\int_0^T g(t,M)\,dt, \quad \forall T > 0.$$

After passing to the limit for $T \to +\infty$ we find

$$G(m) \le 0 \le G(M),$$

and hence $\exists X \in [m, M]$ such that $G(X) = 0$. $\square$

We denote by C the convex closed set $C = \{X \in \mathbb{R} \mid G(X) = 0\}$. We intend to construct an almost periodic solution for (8) by taking the limit $x = \lim_{\alpha \searrow 0} x_\alpha^X$ where $x_\alpha^X = x_\alpha$ is the unique almost periodic solution for

$$\alpha(x_\alpha(t) - X) + x_\alpha'(t) + g(t, x_\alpha(t)) = 0, \quad t \in \mathbb{R}, \tag{44}$$

for all $\alpha > 0$, $\forall X \in C$ (cf. Proposition 3.4). We need to find uniform estimates wrt $\alpha > 0$. We use the following hypothesis

$$\exists X \in \mathbb{R} \ : \ \sup_{s,t \in \mathbb{R}} \left(- \int_s^t g(\sigma, X)\, d\sigma \right) < +\infty. \tag{45}$$

Note that if $X \in \mathbb{R}$ satisfies (45), then $X \in C$. Moreover, if $Y \in C$ then

$$\langle g(\cdot, X) \rangle - \langle g(\cdot, Y) \rangle = 0.$$

Assume that $X \geq Y$ which implies $g(t, X) \geq g(t, Y), \forall t \in \mathbb{R}$. By Lemma 3.1 we obtain that $g(t, X) = g(t, Y), \forall t \in \mathbb{R}$ and therefore the hypothesis (45) holds for all $Y \in C$. We establish now a priori estimates for $(x_\alpha)_\alpha$.

Proposition 4.2 *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (9), (22) *and* (45). *Then*

$$\| x_\alpha^X - X \|_{L^\infty(\mathbb{R})} \leq \sup_{s,t \in \mathbb{R}} \left(- \int_s^t g(\sigma, X)\, d\sigma \right), \quad \forall \alpha > 0, \forall X \in C.$$

Proof. We write

$$\alpha(x_\alpha(t) - X) + \frac{d}{dt}(x_\alpha(t) - X) + g(t, x_\alpha(t)) - g(t, X) = -g(t, X), \quad \forall t \in \mathbb{R}, \forall \alpha > 0.$$

Since $g(t, \cdot)$ is nondecreasing we have for some $r_\alpha(t) \geq 0$

$$g(t, x_\alpha(t)) - g(t, X) = r_\alpha(t)(x_\alpha(t) - X), \quad \forall t \in \mathbb{R}, \alpha > 0,$$

and thus we obtain

$$(\alpha + r_\alpha(t))(x_\alpha(t) - X) + \frac{d}{dt}(x_\alpha(t) - X) = -g(t, X).$$

After integration we find for all $t_0 \leq t$

$$x_\alpha(t) - X = e^{-(A_\alpha(t) - A_\alpha(t_0))}(x_\alpha(t_0) - X) - e^{-A_\alpha(t)} \int_{t_0}^t g(s, X) e^{A_\alpha(s)}\, ds = I_1 + I_2, \tag{46}$$

where $A_\alpha(t) = \int_0^t \{\alpha + r_\alpha(s)\}\, ds$, $\forall t \in \mathbb{R}$. By Proposition 2.3 we have

$$\| x_\alpha \|_{L^\infty(\mathbb{R})} \leq \frac{\sup_{t \in \mathbb{R}} |g(t, 0)|}{\alpha} + |X|,$$

and thus we obtain

$$|I_1| \le e^{-\alpha(t-t_0)}(\alpha^{-1} \sup_{t\in\mathbb{R}} |g(t,0)| + 2|X|). \tag{47}$$

In order to estimate the second term I_2 we introduce the function

$$F(s) = -\int_s^t g(\sigma, X)\ d\sigma, \ \ \forall s \in \mathbb{R}.$$

We have

$$\begin{aligned} I_2 &= -e^{-A_\alpha(t)} \int_{t_0}^t F'(s) e^{A_\alpha(s)}\ ds \\ &= e^{-(A_\alpha(t)-A_\alpha(t_0))} F(t_0) + e^{-A_\alpha(t)} \int_{t_0}^t F(s) e^{A_\alpha(s)} A'_\alpha(s)\ ds \\ &\le e^{-(A_\alpha(t)-A_\alpha(t_0))} F(t_0) + (1 - e^{-(A_\alpha(t)-A_\alpha(t_0))}) \sup_{t_0\le s\le t} F(s) \\ &\le \sup_{t_0\le s\le t} F(s) \\ &\le \sup_{s\le t} F(s). \end{aligned} \tag{48}$$

Similarly we can prove that

$$I_2 \ge \inf_{t_0\le s\le t} F(s) \ge \inf_{s\le t} F(s). \tag{49}$$

Combining (46), (47), (48), (49) we deduce that for all $t_0 \le t$ we have

$$-e^{-\alpha(t-t_0)}(\|x_\alpha\|_{L^\infty}+|X|)+\inf_{s\le t} F(s) \le x_\alpha(t)-X \le e^{-\alpha(t-t_0)}(\|x_\alpha\|_{L^\infty}+|X|)+\sup_{s\le t} F(s).$$

After passing to the limit for $t_0 \to -\infty$ we obtain for all $t \in \mathbb{R}$

$$-\sup_{t_1,s\in\mathbb{R}} \left(-\int_{t_1}^s g(\sigma,X) d\sigma\right) \le \inf_{s\le t} F(s) \le x_\alpha(t)-X \le \sup_{s\le t} F(s) \le \sup_{s,t_1\in\mathbb{R}} \left(-\int_s^{t_1} g(\sigma,X) d\sigma\right),$$

and therefore the sequence $(x_\alpha)_\alpha$ is bounded

$$\|x_\alpha - X\|_{L^\infty(\mathbb{R})} \le \sup_{s,t\in\mathbb{R}} \left(-\int_s^t g(\sigma,X)\ d\sigma\right), \ \ \forall \alpha > 0.$$

$\square$

Remark 4.1 *In the above proof we have used the following important inequalities*

$$\inf_{t_0\le s\le t} \int_s^t h(\sigma)\ d\sigma \le e^{-A_\alpha(t)} \int_{t_0}^t e^{A_\alpha(s)} h(s)\ ds \le \sup_{t_0\le s\le t} \int_s^t h(\sigma)\ d\sigma.$$

We state now the following existence result for almost periodic solution.

Theorem 4.1 *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies* (9), (22), (45). *Then there is at least one almost periodic solution for* (8).

Proof. Take $X \in C$ and consider the unique almost periodic solution x_α for (44), $\forall \alpha > 0$. By Proposition 4.2 we have the inequalities

$$-D \leq x_\alpha(t) - X \leq D, \forall \alpha > 0, \forall t \in \mathbb{R},$$

where $D = \sup_{s,t \in \mathbb{R}} \left(-\int_s^t g(\sigma, X)\, d\sigma \right)$. We check easily that $(x'_\alpha)_{0<\alpha\leq 1}$ is bounded since

$$\begin{aligned} |x'_\alpha(t)| &= |\alpha(x_\alpha(t) - X) + g(t, x_\alpha(t))| \\ &\leq D + \sup_{(t,x) \in \mathbb{R} \times [X-D, X+D]} |g(t,x)| < +\infty. \end{aligned}$$

By using the theorem of Arzela-Ascoli we can extract a sequence $(x_{\alpha_n})_n$ which converges uniformly on compact sets to a bounded solution x of (8), satisfying

$$-D \leq x(t) - X \leq D, \quad \forall t \in \mathbb{R}.$$

It remains to prove that x is almost periodic. We consider also the function

$$g_1(t,x) = g(t,x) + (x + D - X)\mathbf{1}_{\{x \leq -D+X\}} + (x - D - X)\mathbf{1}_{\{x \geq D+X\}}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Note that the function g_1 satisfies the hypotheses (9), (22), (23) with $M = |X| + D + \sup_{t \in \mathbb{R}} |g(t,0)|$. By Theorem 3.3 we know that there is at least one almost periodic solution x_1 for the equation

$$x'_1(t) + g_1(t, x_1(t)) = 0, \quad t \in \mathbb{R}. \tag{50}$$

Note that since $-D \leq x(t) - X \leq D, \forall t \in \mathbb{R}$ we have

$$g(t, x(t)) = g_1(t, x(t)), \quad t \in \mathbb{R},$$

and therefore x is also bounded solution for (50). By applying Proposition 3.9 we deduce that there is a constant $K \in \mathbb{R}$ such that $x(t) = x_1(t) + K, \forall t \in \mathbb{R}$. Hence x is almost periodic function. $\square$

Remark 4.2 *In the previous theorem we can replace hypothesis* (9) *by $g|_{\mathbb{R} \times [X-D, X+D]}$ nondecreasing.*

Remark 4.3 *The above theorem contains the well-known result concerning the almost periodicity of the primitive of almost periodic functions. Indeed, take $g(t,x) = -f(t)$ where f is almost periodic such that $F(t) = \int_0^t f(s)\, ds$ is bounded. Therefore we have for all $X \in \mathbb{R}$ $\left| \int_s^t g(\sigma, X)\, d\sigma \right| = \left| \int_s^t f(\sigma)\, d\sigma \right| \leq 2\|F\|_{L^\infty(\mathbb{R})}$ and thus there is at least one almost periodic solution x for $x'(t) = f(t), t \in \mathbb{R}$. Since $F(t)$ satisfies also $F'(t) = f(t), \forall t \in \mathbb{R}$ we deduce that $F(t) = x(t) + K, \forall t \in \mathbb{R}$ for some constant $K \in \mathbb{R}$. Thus F is almost periodic.*

5 Other properties of almost periodic solutions

In this section we analyze the asymptotic behavior of almost periodic solutions for large frequencies. We present also uniqueness and stability results.

5.1 Homogenization

Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies (9), (22), (45). By Theorem 4.1 we know that there is at least one almost periodic solution, for example $x = \lim_{\alpha \searrow 0} x_\alpha^X$ where $x_\alpha^X = x_\alpha$ are the unique almost periodic solution of (44). Consider the function $g^\varepsilon(t,x) = g\left(\frac{t}{\varepsilon}, x\right)$, $(t,x) \in \mathbb{R} \times \mathbb{R}, \varepsilon > 0$. Note that the functions g^ε satisfy the same hypotheses as g and $C^\varepsilon = \{X \in \mathbb{R} \mid \langle g^\varepsilon(\cdot, X)\rangle = 0\} = \{X \in \mathbb{R} \mid \langle g(\cdot, X)\rangle = 0\} = C$. Therefore the equation

$$x'(t) + g^\varepsilon(t, x(t)) = 0, \quad t \in \mathbb{R}, \tag{51}$$

has at least one almost periodic solution, for example $x^\varepsilon = \lim_{\alpha \searrow 0} x_\alpha^{\varepsilon, X}$, where $x_\alpha^{\varepsilon,X} = x_\alpha^\varepsilon$ is the unique almost periodic solution of

$$\alpha(x(t) - X) + x'(t) + g^\varepsilon(t, x(t)) = 0, \quad t \in \mathbb{R}. \tag{52}$$

We want to establish the convergence of $(x^\varepsilon)_\varepsilon$ as $\varepsilon \searrow 0$.

Theorem 5.1 *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies (9), (22), (45). With the previous notations we have*

$$\lim_{\varepsilon \searrow 0} x^\varepsilon(t) = X, \quad \textit{uniformly for } t \in \mathbb{R},$$

and

$$\|x^\varepsilon - X\|_{L^\infty(\mathbb{R})} \le \varepsilon \sup_{s,t \in \mathbb{R}} \left(- \int_s^t g(\sigma, X)\, d\sigma \right).$$

Proof. Consider the function $y_\alpha^\varepsilon(t) = x_\alpha^\varepsilon(\varepsilon t), \forall t \in \mathbb{R}$. We deduce that y_α^ε is solution for

$$\alpha\varepsilon(y_\alpha^\varepsilon(t) - X) + \frac{d}{dt} y_\alpha^\varepsilon(t) + \varepsilon g(t, y_\alpha^\varepsilon(t)) = 0, \quad t \in \mathbb{R}.$$

From Proposition 4.2 we deduce that

$$\|y_\alpha^\varepsilon - X\|_{L^\infty(\mathbb{R})} \le \varepsilon \sup_{s,t \in \mathbb{R}} \left(- \int_s^t g(\sigma, X)\, d\sigma \right),$$

which implies that

$$|x_\alpha^\varepsilon(\tau) - X| \le \varepsilon \sup_{s,t \in \mathbb{R}} \left(- \int_s^t g(\sigma, X)\, d\sigma \right), \quad \forall \tau \in \mathbb{R}, \alpha > 0, \varepsilon > 0.$$

After passing to the limit for $\alpha \searrow 0$ we deduce

$$|x^\varepsilon(\tau) - X| \le \varepsilon \sup_{s,t \in \mathbb{R}} \left(- \int_s^t g(\sigma, X)\, d\sigma \right), \quad \forall \tau \in \mathbb{R}, \varepsilon > 0.$$

$\square$

5.2 Uniqueness

In this paragraph we consider the following equation

$$x'(t) + g(x(t)) = f(t), \quad t \in \mathbb{R}, \tag{53}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing and $f : \mathbb{R} \to \mathbb{R}$ is almost periodic. Note that the function $(t, x) \to g(x) - f(t), \forall (t, x) \in \mathbb{R} \times \mathbb{R}$ satisfies the hypotheses (9), (22). In this case the set C is given by

$$C = \{X \in \mathbb{R} \mid g(X) = \langle f \rangle\} = g^{-1}\langle f \rangle.$$

We suppose that the function f satisfies the hypothesis

$$\sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\}\, d\sigma \right) < +\infty. \tag{54}$$

From Theorem 4.1 we deduce that under the hypotheses (54) there is at least one almost periodic solution for (53). In this case it is possible to give also necessary and sufficient conditions for the uniqueness of the almost periodic solution. We use the following easy lemma.

Lemma 5.1 *Assume that $h : \mathbb{R} \to \mathbb{R}$ is almost periodic. Then we have the equalities*

$$\sup_{s\le t}\{h(t) - h(s)\} = \sup_{s,t\in\mathbb{R}}\{h(t) - h(s)\} = \sup_{s\ge t}\{h(t) - h(s)\} = D,$$

$$\inf_{s\le t}\{h(t) - h(s)\} = \inf_{s,t\in\mathbb{R}}\{h(t) - h(s)\} = \inf_{s\ge t}\{h(t) - h(s)\} = d,$$

$$d + D = 0.$$

Proof. We show only the first and last equality. Obviously we have

$$\sup_{s\le t}\{h(t) - h(s)\} \le \sup_{s,t\in\mathbb{R}}\{h(t) - h(s)\}.$$

For all $\varepsilon > 0$ take $s_\varepsilon, t_\varepsilon \in \mathbb{R}$ such that

$$\sup_{s,t\in\mathbb{R}}\{h(t) - h(s)\} \le \frac{\varepsilon}{2} + h(t_\varepsilon) - h(s_\varepsilon).$$

Take τ large enough (such that $t_\varepsilon + \tau \ge s_\varepsilon$) a $\frac{\varepsilon}{2}$-almost period for h. We have

$$\sup_{s,t\in\mathbb{R}}\{h(t) - h(s)\} \le \frac{\varepsilon}{2} + h(t_\varepsilon + \tau) + \frac{\varepsilon}{2} - h(s_\varepsilon) \le \varepsilon + \sup_{s\le t}\{h(t) - h(s)\}.$$

Finally one gets that $\sup_{s,t\in\mathbb{R}}\{h(t) - h(s)\} \le \sup_{s\le t}\{h(t) - h(s)\}$ and the first equality follows. For the last equality we write

$$\begin{aligned} D &= \sup_{s,t\in\mathbb{R}}\{h(t) - h(s)\} = \sup_{s\le t}\{h(t) - h(s)\} \\ &= -\inf_{s\le t}\{h(s) - h(t)\} = -\inf_{s,t\in\mathbb{R}}\{h(t) - h(s)\} \\ &= -d. \end{aligned}$$

$\square$

Theorem 5.2 *Assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous nondecreasing and $f : \mathbb{R} \to \mathbb{R}$ is almost periodic satisfying* (54) *and such that $\langle f \rangle \in g(\mathbb{R})$. Then, for all $X \in g^{-1}\langle f \rangle$ there is at least one almost periodic solution x for* (53) *verifying*

$$\|x - X\|_{L^\infty(\mathbb{R})} \le \sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right). \tag{55}$$

Moreover the almost periodic solution is unique iff

$$diam(g^{-1}\langle f \rangle) \le \sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right).$$

Proof. The existence of almost periodic solution follows from Theorem 4.1. Suppose now that $diam(C) > \sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right)$ and let us construct two different almost periodic solutions. Denote $F(t) = \int_0^t \{f(\sigma) - \langle f \rangle\} \, d\sigma, \forall t \in \mathbb{R}$, which is also almost periodic function. We take ε small enough such that $diam(C) > \sup_{s,t\in\mathbb{R}}\{F(t) - F(s)\} + \varepsilon = \sup F - \inf F + \varepsilon$. Consider $t_\varepsilon \in \mathbb{R}$ such that

$$F(t_\varepsilon) \le \inf F + \frac{\varepsilon}{2},$$

and $X, Y \in C$ such that $Y - X > diam(C) - \frac{\varepsilon}{4}$. Let

$$x_1(t) = X + \int_{t_\varepsilon}^t \{f(\sigma) - \langle f \rangle\} \, d\sigma + \frac{\varepsilon}{2} = X + F(t) - F(t_\varepsilon) + \frac{\varepsilon}{2}, \;\; \forall t \in \mathbb{R}.$$

Observe that

$$x_1(t) \ge X + \inf F - F(t_\varepsilon) + \frac{\varepsilon}{2} \ge X, \forall t \in \mathbb{R},$$

and that

$$x_1(t) \le X + \sup F - \inf F + \frac{\varepsilon}{2} < X - \frac{\varepsilon}{2} + diam(C) < Y - \frac{\varepsilon}{4},$$

which implies that $x_1(t) \in C, \forall t \in C$. Therefore we have $g(x_1(t)) = \langle f \rangle, \forall t \in \mathbb{R}$ and thus x_1 is almost periodic solution of (53)

$$x_1'(t) + g(x_1(t)) = f(t) - \langle f \rangle + \langle f \rangle = f(t), \;\; \forall t \in \mathbb{R}.$$

Consider now the function $x_2(t) = x_1(t) + \frac{\varepsilon}{4}$. As before we have

$$X + \frac{\varepsilon}{4} \le x_2(t) < Y, \;\; \forall t \in \mathbb{R},$$

and therefore x_2 is another almost periodic solution for (53). Suppose now that

$$diam(C) \le \sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right), \tag{56}$$

and let us prove that we have uniqueness of the almost periodic solution. Take x, y two almost periodic solutions. There is a constant $K_1 \in \mathbb{R}$ such that $x(t) - y(t) = K_1, \forall t \in \mathbb{R}$ and $g(x(t)) = g(y(t)), \forall t \in \mathbb{R}$. If $K_1 \neq 0$ we deduce easily by using the monotony of g that $g(x(t)) = g(y(t)) = K_2, \forall t \in \mathbb{R}$, for some constant $K_2 \in \mathbb{R}$. In fact K_2 must be the average of f and thus $x(t), y(t) \in C, \forall t \in \mathbb{R}$ and

$$x(t) - x(s) = \int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma = y(t) - y(s), \ \ \forall s, t \in \mathbb{R}.$$

We deduce that

$$\sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right) \leq diam(C),$$

and by (56) we obtain

$$\sup_{s,t\in\mathbb{R}} \{x(t) - x(s)\} \ \ = \ \ \sup_{s,t\in\mathbb{R}} \{y(t) - y(s)\} = \sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right) = diam(C).$$

Take $0 < \varepsilon < \frac{|K_1|}{2}$ and $s_\varepsilon, t_\varepsilon \in \mathbb{R}$ such that

$$\inf(C) + \varepsilon > x(s_\varepsilon), \ \ \sup(C) - \varepsilon < x(t_\varepsilon).$$

If $K_1 > 0$ we obtain that $y(s_\varepsilon) = x(s_\varepsilon) - K_1 < \inf(C) - \frac{K_1}{2} < \inf(C)$ and if $K_1 < 0$ we obtain $y(t_\varepsilon) = x(t_\varepsilon) - K_1 > \sup(C) - \frac{K_1}{2} > \sup(C)$. In both cases we obtained a contradiction since we have already proved that $y(t) \in C, \forall t \in \mathbb{R}$. Therefore $K_1 = 0$ and the almost periodic solution is unique. $\square$

5.3 Stability

We consider the equations

$$x'(t) + g(x(t)) = f_1(t), \ \ t \in \mathbb{R}, \tag{57}$$

and

$$x'(t) + g(x(t)) = f_2(t), \ \ t \in \mathbb{R}, \tag{58}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous nondecreasing and $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are almost periodic satisfying $\langle f_1 \rangle = \langle f_2 \rangle \in g(\mathbb{R})$ and

$$\sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f_1(\sigma) - f_2(\sigma)\} \, d\sigma \right) < +\infty. \tag{59}$$

By Theorem 4.1 we know that for all $X \in g^{-1}\langle f_1 \rangle = g^{-1}\langle f_2 \rangle$ we can construct the solutions $x_1 = \lim_{\alpha \searrow 0} x_{1\alpha}^X$ of (57) and $x_2 = \lim_{\alpha \searrow 0} x_{2\alpha}^X$ of (58) where $x_{k\alpha} = x_{k\alpha}^X, k \in \{1, 2\}$ are the unique almost periodic solutions of

$$\alpha(x_{k\alpha}(t) - X) + x_{k\alpha}'(t) + g(x_{k\alpha}(t)) = f_k(t), \ \ t \in \mathbb{R}, \ k \in \{1, 2\}.$$

We can prove the following stability result.

Proposition 5.1 *Assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous nondecreasing and $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are almost periodic satisfying $\langle f_1 \rangle = \langle f_2 \rangle \in g(\mathbb{R})$ and (59). Then, with the previous notations, we have the inequality*

$$\|x_1 - x_2\|_{L^\infty(\mathbb{R})} \le \sup_{s,t\in\mathbb{R}} \left(\int_s^t \{f_1(\sigma) - f_2(\sigma)\}\, d\sigma \right). \tag{60}$$

Proof. We have the equality

$$\alpha(x_{1\alpha}(t) - x_{2\alpha}(t)) + x'_{1\alpha}(t) - x'_{2\alpha}(t) + g(x_{1\alpha}(t)) - g(x_{2\alpha}(t)) = f_1(t) - f_2(t), \;\; t \in \mathbb{R}.$$

Since g is nondecreasing we can write

$$g(x_{1\alpha}(t)) - g(x_{2\alpha}(t)) = r_\alpha(t)(x_{1\alpha}(t) - x_{2\alpha}(t)), \;\; t \in \mathbb{R},$$

where $r_\alpha(t) \ge 0, \forall t \in \mathbb{R}$. With the notations $z_\alpha = x_{1\alpha} - x_{2\alpha}$, $h = f_1 - f_2$ one gets

$$(\alpha + r_\alpha(t))z_\alpha(t) + z'_\alpha(t) = h(t), \;\; t \in \mathbb{R}.$$

As in the proof of Proposition 4.2 we obtain

$$\|z_\alpha\|_{L^\infty(\mathbb{R})} \le \sup_{s,t\in\mathbb{R}} \left(\int_s^t h(\sigma)\, d\sigma \right).$$

The conclusion follows easily by passing to the limit for $\alpha \searrow 0$ in the previous inequality. $\square$

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