WEAK SOLUTIONS FOR THE VLASOV-POISSON INITIAL-BOUNDARY VALUE PROBLEM WITH BOUNDED ELECTRIC FIELD

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Abstract. *The aim of this work is to construct weak solutions for the three dimensional Vlasov-Poisson initial-boundary value problem with bounded electric field. The main ingredient consists of estimating the change in momentum along characteristics of regular electric fields inside bounded spatial domains. As direct consequences we obtain the propagation of the momentum moments and the existence of weak solution satisfying the balance of total energy.*

Key words. Vlasov-Poisson equations, Vlasov-Maxwell equations.

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1. Introduction.

The Vlasov equation gives a kinetic description of the motion of charged particles under the action of the electro-magnetic field in the collisionless case. This equation is coupled to the Maxwell equations for the electro-magnetic field ; we obtain the Vlasov-Maxwell system. When the magnetic field is neglected, the system obtained is called the Vlasov-Poisson system.

Consider Ω an open bounded subset of $\mathbb{R}^3_x$ with boundary $\partial\Omega$ regular. We introduce the notations $\Sigma = \partial\Omega \times \mathbb{R}^3_p$, $\Sigma_R = \partial\Omega \times B_R$ where $B_R = \{p \in \mathbb{R}^3_p \mid |p| \le R\}$ and :

$$\Sigma^{\pm} = \{(x,p) \in \partial\Omega \times \mathbb{R}^3_p \mid \pm (v(p)\cdot n(x)) > 0\}, \;\; \Sigma^{\pm}_R = \Sigma^{\pm} \cap \Sigma_R, \tag{1.1}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at x and $v(p)$ is the velocity associated with some energy function $\mathcal{E}(p)$ by $v(p) = \nabla_p \mathcal{E}(p)$, $\forall p \in \mathbb{R}^3_p$. The functions to be considered are :

$$\mathcal{E}(p) = \frac{|p|^2}{2m}, \;\; v(p) = \frac{p}{m}, \tag{1.2}$$

for the classical case and :

$$\mathcal{E}(p) = mc_0^2 \left(\left(1 + \frac{|p|^2}{m^2 c_0^2}\right)^{1/2} - 1 \right), \;\; v(p) = \frac{p}{m}\left(1 + \frac{|p|^2}{m^2 c_0^2}\right)^{-1/2}, \tag{1.3}$$

for the relativistic case, where m is the mass of particles, c_0 is the light speed in the vacuum. We denote by $f(t,x,p)$ the particles distribution depending on the time $t \in]0,T[$, position $x \in \Omega$ and momentum $p \in \mathbb{R}^3_p$ and by $F(t,x,p)$ the electro-magnetic force :

$$F(t,x,p) = q(E(t,x) + v(p) \wedge B(t,x)), \;\; (t,x,p) \in]0,T[\times\Omega\times\mathbb{R}^3_p, \tag{1.4}$$

where (E,B) is the electro-magnetic field and q is the charge of particles. The Vlasov-Maxwell system is given by :

$$\partial_t f + v(p)\cdot\nabla_x f + F(t,x,p)\cdot\nabla_p f = 0, \;\; (t,x,p) \in]0,T[\times\Omega\times\mathbb{R}^3_p, \tag{1.5}$$

$$\partial_t E - c_0^2 \cdot \text{rot } B = -\frac{j}{\varepsilon_0}, \;\; \partial_t B + \text{rot } E = 0, \;\; \text{div } E = \frac{\rho}{\varepsilon_0}, \;\; \text{div } B = 0, \;\; (t,x) \in]0,T[\times\Omega, \tag{1.6}$$

where $\rho(t,x) = q\int_{\mathbb{R}^3_p} f(t,x,p)\,dp$, $j(t,x) = q\int_{\mathbb{R}^3_p} v(p) f(t,x,p)\,dp$ are the charge and current densities respectively, ε_0 is the permittivity of the vacuum, μ_0 is the permeability of the vacuum ($\varepsilon_0 \cdot \mu_0 \cdot c_0^2 = 1$). The above equations are completed with the initial conditions :

$$f(0,x,p) = f_0(x,p), \;\; (x,p) \in \Omega\times\mathbb{R}^3_p, \;\; E(0,x) = E_0(x), \;\; B(0,x) = B_0(x), \;\; x \in \Omega, \tag{1.7}$$

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and the boundary conditions :

$$f(t,x,p) = g(t,x,p), \;\; (t,x) \in]0,T[\times \Sigma^-, \tag{1.8}$$

$$n \wedge E(t,x) + c_0 \cdot n \wedge (n \wedge B(t,x)) = h(t,x), \;\; (t,x) \in]0,T[\times \partial\Omega. \tag{1.9}$$

Some other boundary conditions can be analyzed. When neglecting the magnetic field, $B = 0$, the electric field derives from a potential $E = -\nabla_x \Phi$, the electric force is given by $F(t,x) = -q\nabla_x \Phi$ and we obtain the Vlasov-Poisson system :

$$\partial_t f + v(p) \cdot \nabla_x f + F(t,x) \cdot \nabla_p f = 0, \;\; (t,x,p) \in]0,T[\times \Omega \times \mathbb{R}^3_p, \tag{1.10}$$

$$-\Delta_x \Phi = \frac{\rho}{\varepsilon_0}, \;\; (t,x) \in]0,T[\times \Omega, \tag{1.11}$$

$$f(0,x,p) = f_0(x,p), \;\; (x,p) \in \Omega \times \mathbb{R}^3_p, \;\; f(t,x,p) = g(t,x,p), \;\; (t,x) \in]0,T[\times \Sigma^-, \tag{1.12}$$

$$\Phi(t,x) = \varphi_0(t,x), \;\; (t,x) \in]0,T[\times \partial\Omega. \tag{1.13}$$

This model can be derived from the relativistic Vlasov-Maxwell system by letting $c_0 \to +\infty$, see [11], [9].

Various results were obtained for the free space Vlasov-Poisson system. Weak solutions were constructed by Arseneev [1], Horst and Hunze [22]. The existence of classical solutions has been studied by Ukai and Okabe [29], Horst [21], Batt [3], Pfaffelmoser [25]. The existence of global classical solutions for the Vlasov-Poisson equations was proved by Bardos and Degond [5], Schaeffer [27], [28]. The propagation of the moments for the three dimensional Vlasov-Poisson system was studied by Lions and Perthame in [24]. The existence of global weak solution for the Vlasov-Maxwell system in three dimensions was obtained by DiPerna and Lions [13]. Results for the relativistic case were proved by Glassey and Schaeffer [14], [15], Glassey and Strauss [16], [17], Klainerman and Staffilani [23], Bouchut, Golse and Pallard [10].

Results for the initial-boundary value problem were obtained by Ben Abdallah [6] for the Vlasov-Poisson system in three dimensions and Guo [19] for the Vlasov-Maxwell system. The stationary problem for the Vlasov-Poisson equations was studied by Greengard and Raviart [18] in one dimension and by Poupaud [26] in three dimensions for the Vlasov-Maxwell system. An asymptotic analysis of the Vlasov-Poisson system was done by Degond and Raviart [12] in the case of the plane diode. The regularity of the solutions for the Vlasov-Maxwell system has been studied by Guo [20]. Results for the time periodic case can be found in [7], [8].

The aim of this paper is to construct weak solutions for the three dimensional Vlasov-Poisson initial-boundary value problem with bounded electric field. As usual we start by analyzing a regularized system for which the existence of solution follows by a fixed point method. Next we find uniform a priori bounds for these solutions by using the physical conservation laws, under the natural hypotheses

$$\int_\Omega \int_{\mathbb{R}^3_p} (1+\mathcal{E}(p)) f_0(x,p)\, dxdp + \int_\Omega |\nabla_x \Phi(0,x)|^2\, dx + \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p)) g\, dtd\sigma dp < +\infty,$$

and φ_0 smooth. Finally we construct a weak solution by taking a weak limit of the sequence of smooth solutions (see Theorem 5.1 for exact statements). Of coarse, such a construction is standard (see [6]). The new results of this work consists of establishing L^∞ bounds for the electric field (see Section 4.2) and the derivation of some important consequences. One of the crucial points is to observe that the change in momentum along characteristics inside a bounded spatial domain can be estimated in term of the L^∞ norm of the electric field. This idea has been already used in [7]. For example, in the classical case we prove that for all characteristic

$$\frac{dX}{ds} = \frac{P(s)}{m}, \quad \frac{dP}{ds} = qE(s, X(s)),$$

we have

$$|P(s_1) - P(s_2)| \leq 2 \cdot (2 \cdot |q| \cdot \|E\|_{L^\infty} \cdot m \cdot diam(\Omega))^{\frac{1}{2}},$$

for all $s_{in} \leq s_1 \leq s_2 \leq s_{out}$ (here s_{in}, s_{out} denote the incoming and outgoing times, respectively). Combining the above result with Sobolev inequalities and standard bounds for the total mass and energy yields a L^∞ estimate for the electric field. As direct consequences of the L^∞ bound for the electric field we mention the propagation of the momentum moments and also the existence of weak solutions (f, E) for the Vlasov-Poisson system with particle distribution f compactly supported in momentum when the initial-boundary conditions have compact support in momentum. Another consequence is that the weak solution obtained as limit of smooth solutions exactly verifies the energy conservation law (generally by weak limit only inequalities are preserved). For example, if the potential vanishes on the boundary we construct a weak solution (f, E) satisfying

$$\frac{d}{dt}\left\{\int_\Omega \int_{\mathbb{R}^N_p} \mathcal{E}(p) f \, dxdp + \frac{\varepsilon_0}{2}\int_\Omega |E|^2 \, dx\right\} + \int_\Sigma (v(p)\cdot n(x))\mathcal{E}(p)\gamma f \, d\sigma dp = 0, \quad \text{a.e. } t \in]0, T[,$$

where γf is the trace of f on Σ.

The content of this paper is organized as follows. We recall some standard definitions and results about the Vlasov problem. We remind the notion of weak/mild solution for this problem with initial-boundary conditions or only boundary conditions (the time periodic case). We state the momentum change lemma for the classical and relativistic cases (the details of proofs can be found in the Appendix) and we apply the above lemma in order to construct weak solutions uniformly compactly supported in momentum for the Vlasov problem with initial-boundary conditions or time periodic boundary conditions. In section 3 we prove the existence of weak solution for a regularized Vlasov-Poisson system by using a fixed point method. In the next section we establish a priori estimates for the total energy and the L^∞ norm of the electric field, uniformly with respect to the regularization parameters. In the last section we construct solutions for the Vlasov-Poisson system by weak stability arguments. We end this paper with some properties of the solutions constructed above. We present also the time periodic case.

2. The Vlasov equation.

In this section we recall the basic definitions and results on the Vlasov equation. For the completeness of the presentation we consider the case of electro-magnetic forces. Later on the magnetic field will be neglected in order to study the Vlasov-Poisson system. We assume that the electro-magnetic field is given and bounded. We introduce the notion of weak solution for the initial-boundary value problem :

$$\partial_t f + v(p)\cdot\nabla_x f + F(t,x,p)\cdot\nabla_p f = 0, \quad (t,x,p) \in]0,+\infty[\times\Omega\times\mathbb{R}^3_p, \tag{2.1}$$

$$f(0,x,p) = f_0(x,p), \quad (x,p) \in \Omega\times\mathbb{R}^3_p, \tag{2.2}$$

$$f(t,x,p) = g(t,x,p), \quad (t,x,p) \in]0,+\infty[\times\Sigma^-. \tag{2.3}$$

Remark 2.1. *Note that in both classical and relativistic case we have* $\nabla_x \cdot v(p) = 0$, $\nabla_p \cdot F = 0$ *and thus* (2.1) *can be written also :*

$$\partial_t f + \nabla_x\cdot(v(p)f) + \nabla_p\cdot(F(t,x,p)f) = 0, \quad (t,x,p) \in]0,+\infty[\times\Omega\times\mathbb{R}^3_p.$$

DEFINITION 2.2. *Assume that* $E, B \in L^\infty(]0,T_1[\times\Omega)^3$, $f_0 \in L^1(\Omega\times B_R)$, $(v(p)\cdot n(x))g \in L^1(]0,T_1[\times\Sigma_R^-), \forall T_1>0, \forall R>0$. *We say that* $f \in L^1(]0,T_1[\times\Omega\times B_R)$, $\forall T_1>0, \forall R>0$ *is a weak solution for the problem* (2.1), (2.2), (2.3) *iff :*

$$-\int_0^\infty\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(\partial_t\varphi + v(p)\cdot\nabla_x\varphi + F(t,x,p)\cdot\nabla_p\varphi)dtdxdp = \int_\Omega\int_{\mathbb{R}^3_p} f_0(x,p)\varphi(0,x,p)dxdp$$
$$-\int_0^{+\infty}\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi(t,x,p)dtd\sigma dp, \quad (2.4)$$

for all test function which belongs to $\mathcal{T}_w = \{\varphi \in C^1_c([0,+\infty[\times\overline{\Omega}\times\mathbb{R}^3_p) \mid \varphi|_{[0,+\infty[\times\Sigma^+} = 0\}$.

Suppose now that $E, B \in L^\infty_{loc}(]0,+\infty[;W^{1,\infty}(\Omega))^3$ and introduce the characteristic equations :

$$\frac{dX}{ds} = v(P(s;t,x,p)), \quad \frac{dP}{ds} = F(s, X(s;t,x,p), P(s;t,x,p)), \quad s_{in}(t,x,p) \le s \le s_{out}(t,x,p),$$

with the conditions $X(s=t;t,x,p) = x, \quad P(s=t;t,x,p) = p$. Here $s_{in}(t,x,p), s_{out}(t,x,p)$ denote the incoming, respectively outgoing time, given by :

$$s_{in}(t,x,p) = \max\{0, \sup\{s\le t \mid X(s;t,x,p)\in\partial\Omega\}\}, s_{out}(t,x,p) = \inf\{s \ge t \mid X(s;t,x,p)\in\partial\Omega\}.$$

The mild formulation follows now formally by solving :

$$-\partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x,p)\cdot\nabla_p\varphi = \psi, \quad (t,x,p)\in]0,+\infty[\times\Omega\times\mathbb{R}^3_p,$$

with the boundary condition $\varphi|_{[0,+\infty[\times\Sigma^+} = 0$, which gives after integration along the characteristic curves :

$$\varphi_\psi(t,x,p) = \int_t^{s_{out}(t,x,p)} \psi(s, X(s;t,x,p), P(s;t,x,p))ds.$$

DEFINITION 2.3. *Assume that* $E, B \in L^\infty_{loc}(]0,+\infty[;W^{1,\infty}(\Omega))^3, f_0 \in L^1(\Omega\times B_R), (v(p)\cdot n(x))g \in L^1(]0,T_1[\times\Sigma_R^-)$ $\forall T_1>0, \forall R>0$. *We say that* $f\in L^1(]0,T_1[\times\Omega\times B_R)$, $\forall T_1>0,\forall R>0$ *is a mild solution for* (2.1), (2.2), (2.3) *iff :*

$$\int_0^{+\infty}\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)\psi(t,x,p)dtdxdp = \int_\Omega\int_{\mathbb{R}^3_p} f_0(x,p)\varphi_\psi(0,x,p)dxdp$$
$$-\int_0^{+\infty}\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi_\psi(t,x,p)dtd\sigma dp, \quad (2.5)$$

for all test function which belongs to $\mathcal{T}_m = \{\psi \in C^0_c([0,+\infty[\times\overline{\Omega}\times\mathbb{R}^3_p)\}$.

Note that for all $\psi\in\mathcal{T}_m$ the function φ_ψ has compact support in $[0,+\infty[\times\overline{\Omega}\times\mathbb{R}^3_p$ and is bounded. Thus the above definition makes sense. Indeed suppose that $\psi = \psi\cdot\mathbf{1}_{\{0\le t\le T_1\}}\cdot\mathbf{1}_{\{|p|\le R\}}$. Therefore when $t > T_1$ we have $\varphi_\psi = 0$ and for $t\le T_1$:

$$\varphi_\psi(t,x,p) = \int_t^{\min\{T_1, s_{out}(t,x,p)\}} \psi(s, X(s;t,x,p), P(s;t,x,p))ds.$$

By writing for $t\le s\le \min\{T_1, s_{out}(t,x,p)\}$:

$$\frac{1}{2}|P(s;t,x,p)|^2 = \frac{1}{2}|p|^2 + \int_t^s qE(\tau, X(\tau))\cdot P(\tau)d\tau \ge \frac{1}{2}|p|^2 - \int_t^s |q|\cdot\|E\|_{L^\infty}\cdot|P(\tau)|d\tau,$$

we deduce by using Bellman's lemma that $|P(s;t,x,p)| \ge |p| - (s-t)\cdot|q|\cdot\|E\|_{L^\infty} \ge |p| - T_1\cdot|q|\cdot\|E\|_{L^\infty}$ and thus we have $\varphi_\psi(t,x,p) = 0$ if $|p| > R + T_1\cdot|q|\cdot\|E\|_{L^\infty}$. Moreover we have also that

$\|\varphi_\psi\|_{L^\infty} \leq T_1 \cdot \|\psi\|_{L^\infty}$.

REMARK 2.4. *It is well known that the mild solution is unique and is given by :*

$$f(t,x,p) = f_0(X(0;t,x,p), P(0;t,x,p)) \text{ if } s_{in}(t,x,p) = 0,$$

and :

$$f(t,x,p) = g(s_{in}, X(s_{in};t,x,p), P(s_{in};t,x,p)) \text{ if } s_{in}(t,x,p) > 0.$$

REMARK 2.5. *We check easily that the mild solution is also weak solution. Moreover the mild solution verifies the following Green formula :*

$$\begin{aligned}
-\int_0^{T_1}\!\!\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)(\partial_t\varphi + v(p)\cdot\nabla_x\varphi + F(t,x,p)\cdot\nabla_p\varphi)dtdxdp + \int_\Omega\int_{\mathbb{R}^3_p}\gamma f(T_1,x,p)\varphi(T_1,x,p)dxdp \\
+\int_0^{T_1}\!\!\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(t,x,p)\varphi(t,x,p)dtd\sigma dp \\
=\int_\Omega\int_{\mathbb{R}^3_p} f_0(x,p)\varphi(0,x,p)dxdp - \int_0^{T_1}\!\!\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi(t,x,p)dtd\sigma dp, \qquad (2.6)
\end{aligned}$$

$\forall \varphi \in C^1_c([0,+\infty[\times\overline{\Omega}\times\mathbb{R}^3_p), \forall T_1 > 0$, *where the traces* $\gamma f(T_1,\cdot,\cdot), \gamma^+ f$ *are defined as in the Remark* 2.4 *and belong to* $L^1(\Omega \times B_R)$, *respectively* $L^1(]0,T_1[\times\Sigma^+_R)$ $\forall R > 0, \forall T_1 > 0$.

REMARK 2.6. *By using the Remark* 2.4 *we check easily that the mild solution* f *verifies :*

$$\min\{\inf_{\Omega\times\mathbb{R}^3_p} f_0, \inf_{]0,+\infty[\times\Sigma^-} g\} \leq f \leq \max\{\sup_{\Omega\times\mathbb{R}^3_p} f_0, \sup_{]0,+\infty[\times\Sigma^-} g\},$$

with the same inequalities for the traces $\gamma f(T_1,\cdot,\cdot), \gamma^+ f$. *In particular if* $f_0 \geq 0, g \geq 0$ *then* $f \geq 0, \gamma^+ f \geq 0, \gamma f(T_1,\cdot,\cdot) \geq 0$, $\forall T_1 > 0$.

2.1. The momentum change in the classical case.

In this section we set $\mathcal{E}(p) = |p|^2/(2m)$, $v(p) = p/m$, $\forall p \in \mathbb{R}^3_p$. In this case the characteristic system is given by :

$$\frac{dX}{ds} = \frac{P(s)}{m}, \ \frac{dP}{ds} = q(E(s,X(s)) + \frac{P(s)}{m}\wedge B(s,X(s))), \ s_{in} \leq s \leq s_{out}, \qquad (2.7)$$

where the electro-magnetic field is regular $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$. We state the momentum change lemma for the classical case. The proof details can be found in the Appendix.

LEMMA 2.7. *Assume that* $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$ *and consider* $(X(s), P(s))$, $s_{in} \leq s \leq s_{out}$ *an arbitrary solution for* (2.7). *Denote by* D_{cla} *the quantity :*

$$D_{cla} = (2|q|\cdot\|E\|_\infty\cdot m\cdot diam(\Omega))^{1/2} + 2\cdot|q|\cdot\|B\|_\infty\cdot diam(\Omega).$$

Then :
(1) *if there is* $t \in [s_{in}, s_{out}]$ *such that* $|P(t)| > D_{cla}$, *therefore we have :*

$s_{out} - s_{in} \leq 4\cdot diam(\Omega)/|v(P(t))| \leq 4m\cdot diam(\Omega)/D_{cla}$, *and* $|P(s) - P(t)| \leq D_{cla}$, $\forall s_{in} \leq s \leq s_{out}$;

(2) *for all* $s_{in} \leq s_1 \leq s_2 \leq s_{out}$ *we have* $|P(s_1) - P(s_2)| \leq 2D_{cla}$.

Note that the previous estimate for the momentum change is optimal. Indeed, let us analyze a particular case. We consider that the electro-magnetic field is constant $E = (0, 0, E_3)$, $B = (0, 0, B_3)$. The characteristic equations in the classical case are given by :

$$\frac{dX_1}{ds} = \frac{P_1(s)}{m}, \quad \frac{dX_2}{ds} = \frac{P_2(s)}{m}, \quad \frac{dX_3}{ds} = \frac{P_3(s)}{m},$$

$$\frac{dP_1}{ds} = \frac{q}{m} \cdot B_3 \cdot P_2(s), \quad \frac{dP_2}{ds} = -\frac{q}{m} \cdot B_3 \cdot P_1(s), \quad \frac{dP_3}{ds} = q \cdot E_3.$$

By writing $q \cdot B_3 \frac{dX_1}{ds} = -\frac{dP_2}{ds}$, $q \cdot B_3 \frac{dX_2}{ds} = \frac{dP_1}{ds}$, we obtain after integration in respect to $s \in]s_1, s_2[$ that :

$$\begin{aligned} |q| \cdot |B_3| \cdot diam(\Omega) \geq & |q| \cdot |B_3| \cdot [(X_1(s_2) - X_1(s_1))^2 + (X_2(s_2) - X_2(s_1))^2]^{1/2} \\ \geq & [(P_1(s_2) - P_1(s_1))^2 + (P_2(s_2) - P_2(s_1))^2]^{1/2}. \end{aligned}$$

We want to estimate also the change of P_3 on the interval $]s_1, s_2[$. In order to simplify the computations we suppose that $P_3(s_1) = 0$ and thus we obtain that $P_3(s) = (s - s_1)qE_3$ and hence $X_3(s_2) - X_3(s_1) = \frac{q \cdot E_3}{2 \cdot m}(s_2 - s_1)^2$. We deduce that $s_2 - s_1 \leq (2 \cdot m \cdot diam(\Omega)/(|q| \cdot |E_3|))^{1/2}$ and that $|P_3(s_2) - P_3(s_1)| \leq (2 \cdot |q| \cdot |E_3| \cdot m \cdot diam(\Omega))^{1/2}$. Finally the change in momentum along the characteristic is bounded by :

$$|P(s_2) - P(s_1)| \leq (2 \cdot |q| \cdot |E_3| \cdot m \cdot diam(\Omega))^{1/2} + |q| \cdot |B_3| \cdot diam(\Omega). \tag{2.8}$$

The Lemma 2.7 holds true in two dimensional spatial domain $\Omega \subset \mathbb{R}^2_x$ for orthogonal electric and magnetic fields $E = (E_1, E_2, 0), B = (0, 0, B_3)$. In this case the system of characteristics is given by :

$$\frac{dX_1}{ds} = \frac{P_1(s)}{m}, \quad \frac{dP_1}{ds} = q\left(E_1(s, X_1(s), X_2(s)) + \frac{P_2(s)}{m} \cdot B_3(s, X_1(s), X_2(s))\right),$$

$$\frac{dX_2}{ds} = \frac{P_2(s)}{m} \quad \frac{dP_2}{ds} = q\left(E_2(s, X_1(s), X_2(s)) - \frac{P_1(s)}{m} \cdot B_3(s, X_1(s), X_2(s))\right).$$

Remark also that in the purely electric case ($B = 0$) the Lemma 2.7 is valid in any dimension.

2.2. The momentum change lemma in the relativistic case.

We analyze also the relativistic case. In this section we set $\mathcal{E}(p) = mc_0^2\left(\left(1 + \frac{|p|^2}{(mc_0)^2}\right)^{\frac{1}{2}} - 1\right)$ with the corresponding velocity $v(p) = (p/m) \cdot (1 + |p|^2/(mc_0)^2)^{-1/2}$. We start with the purely electric system of characteristics which is given by :

$$\frac{dX}{ds} = \frac{P(s)}{m}\left(1 + \frac{|P(s)|^2}{m^2c_0^2}\right)^{-1/2}, \quad \frac{dP}{ds} = qE(s, X(s)), \quad s_{in} \leq s \leq s_{out}. \tag{2.9}$$

We will analyze (2.9) for any dimension $N \geq 1$. We state the momentum change lemma for the relativistic case (see the Appendix for proof details).

LEMMA 2.8. *Assume that $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^N$ and consider $(X(s), P(s))$, $s_{in} \leq s \leq s_{out}$ an arbitrary solution for* (2.9). *Denote by D^{ele}_{rel} the quantity :*

$$D^{ele}_{rel} = mc_0\sqrt{\beta(1 + \beta)}, \quad \textit{with } \beta = \frac{4\sqrt{N} \cdot diam(\Omega) \cdot |q| \cdot \|E\|_\infty}{mc_0^2}.$$

Then :
(1) *if there is* $t \in [s_{in}, s_{out}]$ *such that* $|P(t)| > D^{ele}_{rel}$ *therefore :*

$$s_{out} - s_{in} \leq 4 \cdot diam(\Omega)/|v(P(t))| \text{ and } |P(s) - P(t)| \leq D^{ele}_{rel}, \ \forall s_{in} \leq s \leq s_{out} \ ;$$

(2) *for all* $s_{in} \leq s_1 \leq s_2 \leq s_{out}$ *we have* $|P(s_1) - P(s_2)| \leq 2D^{ele}_{rel}$.

Consider now the relativistic characteristic system with $N = 3$:

$$\frac{dX}{ds} = v(P(s)), \ \ \frac{dP}{ds} = q(E(s, X(s)) + v(P(s)) \wedge B(s, X(s))), \ \ s_{in} \leq s \leq s_{out}. \tag{2.10}$$

By observing that $|q(E + v(p) \wedge B)| \leq |q| \cdot (\|E\|_\infty + c_0 \cdot \|B\|_\infty)$ we deduce also that :

Lemma 2.9. *Assume that* $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$ *and consider* $(X(s), P(s))$, $s_{in} \leq s \leq s_{out}$ *an arbitrary solution for* (2.10). *Then the conclusions of Lemma* 2.8 *hold true with :*

$$D_{rel} = mc_0\sqrt{\beta_1(1 + \beta_1)}, \quad \text{with } \beta_1 = \frac{4\sqrt{3} \cdot |q| \cdot diam(\Omega) \cdot (\|E\|_\infty + c_0\|B\|_\infty)}{mc_0^2}.$$

Note also that the above estimate for the momentum change is optimal. For this, consider first the relativistic case with $E = (0, 0, E_3)$ and $B = (0, 0, 0)$. In order to simplify we take $P(s_1) = 0$. By using the relativistic characteristic system we obtain :

$$P_1(s) = 0, \ \ P_2(s) = 0, \ \ \frac{dX_3}{ds} = \frac{P_3(s)}{m}\left(1 + \left(\frac{P_3(s)}{mc_0}\right)^2\right)^{-1/2}, \ \ \frac{dP_3}{ds} = q \cdot E_3.$$

We deduce that :

$$\frac{q \cdot E_3}{mc_0^2} \cdot \frac{dX_3}{ds} = \frac{d}{ds}\left(1 + \left(\frac{P_3(s)}{mc_0}\right)^2\right)^{1/2},$$

and after integration we obtain also a bound for the momentum change along this characteristic :

$$|P(s_2) - P(s_1)| = |P_3(s_2)| \leq mc_0\sqrt{\beta(\beta + 2)}, \text{ with } \beta = \frac{|q| \cdot |E_3| \cdot diam(\Omega)}{mc_0^2}. \tag{2.11}$$

Secondly consider the relativistic case with $E = (0, 0, 0)$ and $B = (0, 0, B_3)$, $P_3(s_1) = 0$. The characteristic system is given by :

$$\frac{dX_i}{ds} = \frac{P_i(s)}{m} \cdot \left(1 + \frac{|P(s)|^2}{m^2c_0^2}\right)^{-1/2} = v_i(P(s)), \ \ 1 \leq i \leq 3,$$

and :

$$\frac{dP_1}{ds} = q \cdot B_3 \cdot v_2(P(s)), \ \ \frac{dP_2}{ds} = -q \cdot B_3 \cdot v_1(P(s)), \ \ \frac{dP_3}{ds} = 0.$$

By writing $\frac{dP_1}{ds} = q \cdot B_3 \cdot \frac{dX_2}{ds}$, $\frac{dP_2}{ds} = -q \cdot B_3 \cdot \frac{dX_1}{ds}$ we find after integration that :

$$|P(s_2) - P(s_1)| \leq |q| \cdot |B_3| \cdot diam(\Omega). \tag{2.12}$$

2.3. Estimate of the momentum support for the initial-boundary value problem .

Generally we will assume that the electro-magnetic field is bounded $(E, B) \in L^\infty(]0, +\infty[\times\Omega)^6$ and that the initial-boundary conditions are compactly supported in momentum, uniformly in

t,x : $\exists R>0$ such that $f_0(x,p)=0\ \forall (x,p)\in\Omega\times\mathbb{R}^3_p, |p|>R$ and $g(t,x,p)=0\ \forall (t,x,p)\in]0,+\infty[\times\Sigma^-, |p|>R$. In this case, at least for regular electro-magnetic field it is easy to see that f has compact support in momentum, uniformly with respect to $(t,x)\in]0,T_1[\times\Omega, \forall T_1>0$. Indeed, by using the characteristic equations

$$\frac{dX}{ds}=v(P(s)),\quad \frac{dP}{ds}=F(s,X(s),P(s)),$$

we deduce that

$$\frac{1}{2}\frac{d}{ds}|P(s)|^2=q\cdot E(s,X(s))\cdot P(s),$$

and by Bellman's lemma we obtain that the change of the momentum norm along any characteristic included in $]0,T_1[\times\Omega\times\mathbb{R}^3_p$ is bounded by $T_1\cdot|q|\cdot\|E\|_{L^\infty}$ and thus we have :

$$f=f\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad (t,x,p)\in]0,T_1[\times\Omega\times\mathbb{R}^3_p, \forall T_1>0, \tag{2.13}$$

where $R_1=R+T_1\cdot|q|\cdot\|E\|_{L^\infty}$. The situation is very different when considering boundary value problems (for example stationary or time periodic problems). In this case we don't know if the solution of the Vlasov equation remains compactly supported in momentum (think that the life time of the characteristics inside the bounded domain Ω can be arbitrarily large). The natural question arising from the above observations is : can we deduce that $f=f\cdot\mathbf{1}_{\{|p|\le R_1\}}$ with R_1 not depending on $(t,x)\in]0,+\infty[\times\Omega$ respectively $(t,x)\in\mathbb{R}_t\times\Omega$? The motivation for finding globally in time estimate for the momentum support comes for numerical considerations. Clearly, if a bound R_1 of the momentum support is available, the computation domain can be restricted to $\Omega\times B_{R_1}$.

THEOREM 2.10. *Assume that* $E,B\in L^\infty(]0,+\infty[;W^{1,\infty}(\Omega))^3$, $f_0\in L^1(\Omega\times\mathbb{R}^3_p)$, $(v(p)\cdot n(x))g\in L^1(]0,T_1[\times\Sigma^-)$, $\forall T_1>0$ *with* $f_0=f_0\cdot\mathbf{1}_{\{|p|\le R\}}$, $g=g\cdot\mathbf{1}_{\{|p|\le R\}}$, *for some* $R>0$. *Then the mild solution for* (2.1), (2.2), (2.3) *is compactly supported in momentum uniformly in* $(t,x)\in]0,+\infty[\times\Omega$ *and we have :*

$$f=f\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad \gamma^+f=\gamma^+f\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad \gamma f(T_1,\cdot,\cdot)=\gamma f(T_1,\cdot,\cdot)\cdot\mathbf{1}_{\{|p|\le R_1\}}, \forall T_1>0,$$

where $R_1=R+2D_{cla/rel}$.

Proof. Take $p\in\mathbb{R}^3_p$ with $|p|>R_1$. By the Lemmas 2.7, 2.9 we deduce that $|P(s;t,x,p)-p|\le 2D_{cla/rel}$, $\forall s_{in}\le s\le t$ and therefore $|P(s;t,x,p)|\ge|p|-|P(s;t,x,p)-p|>R_1-2D_{cla/rel}=R$, $\forall s_{in}\le s\le t$. By the Remark 2.4 we deduce that $f(t,x,p)=0$. The same arguments apply for the traces $\gamma^+f,\gamma f(T_1,\cdot,\cdot)$, $\forall T_1>0$. □

We can construct also weak solutions for (2.1), (2.2), (2.3) with compact support in momentum :

THEOREM 2.11. *Assume that* $E,B\in L^\infty(]0,T_1[;\Omega)^3$, $|f_0|^r\in L^1(\Omega\times\mathbb{R}^3_p)$, $(v(p)\cdot n(x))|g|^r\in L^1(]0,T_1[\times\Sigma^-)$, *for some* $T_1>0$, $1<r<+\infty$ *with* $f_0=f_0\cdot\mathbf{1}_{\{|p|\le R\}}$, $g=g\cdot\mathbf{1}_{\{|p|\le R\}}$. *Then there is a weak solution for* (2.1), (2.2), (2.3) *on* $]0,T_1[\times\Omega\times\mathbb{R}^3_p$ *such that :*

$$f=f\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad \gamma^+f=\gamma^+f\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad \gamma f(T_1,\cdot,\cdot)=\gamma f(T_1,\cdot,\cdot)\cdot\mathbf{1}_{\{|p|\le R_1\}},$$

where $R_1=R+2D_{cla/rel}$.

Proof. Regularize the electro-magnetic field by convolution in respect to x (extend E,B by 0 outside Ω). Denote by f_ε the mild solution for (2.1), (2.2), (2.3) corresponding to the regularized field $E_\varepsilon,B_\varepsilon$. As in [4] we obtain :

$$\partial_t|f_\varepsilon|^r+v(p)\cdot\nabla_x|f_\varepsilon|^r+F_\varepsilon\cdot\nabla_p|f_\varepsilon|^r=0,$$

where $F_\varepsilon = q(E_\varepsilon(t,x) + v(p) \wedge B_\varepsilon(t,x))$. After integration on $]0,T_1[\times\Omega\times\mathbb{R}^3_p$ we find :

$$\int_\Omega\int_{\mathbb{R}^3_p}|\gamma f_\varepsilon|^r(T_1,x,p)dxdp+\int_0^{T_1}\int_{\Sigma^+}(v(p)\cdot n(x))|\gamma^+ f_\varepsilon|^r(t,x,p)dtd\sigma dp$$
$$=\int_\Omega\int_{\mathbb{R}^3_p}|f_0|^r(x,p)dxdp-\int_0^{T_1}\int_{\Sigma^-}(v(p)\cdot n(x))|g|^r(t,x,p)dtd\sigma dp,$$

which gives uniform estimates in L^r for $\varepsilon>0$:

$$\sup_{0\le t\le T_1}\int_\Omega\int_{\mathbb{R}^3_p}|\gamma f_\varepsilon|^r(t,x,p)dxdp+\int_0^{T_1}\int_{\Sigma^+}(v(p)\cdot n(x))|\gamma^+ f_\varepsilon|^r(t,x,p)dtd\sigma dp$$
$$\le 2\left(\int_\Omega\int_{\mathbb{R}^3_p}|f_0|^r(x,p)dxdp-\int_0^{T_1}\int_{\Sigma^-}(v(p)\cdot n(x))|g|^r(t,x,p)dtd\sigma dp\right).$$

We can extract subsequences $f_{\varepsilon_k}\rightharpoonup f$ weakly in $L^r(]0,T_1[\times\Omega\times\mathbb{R}^3_p)$, $\gamma f_{\varepsilon_k}(T_1,\cdot,\cdot)\rightharpoonup\gamma f(T_1,\cdot,\cdot)$ weakly in $L^r(\Omega\times\mathbb{R}^3_p)$, $\gamma^+ f_{\varepsilon_k}\rightharpoonup\gamma^+ f$ weakly in $L^r(]0,T_1[\times\Sigma^+,(v(p)\cdot n(x))dtd\sigma dp)$. By standard arguments we deduce that f is a weak solution for $(2.1),(2.2),(2.3)$ associated to the electro-magnetic field (E,B) with traces $\gamma^+ f,\ \gamma f(T_1,\cdot,\cdot)$. On the other hand, for $|p|>R_1=R+2D_{cla/rel}\ge R+2D^{\varepsilon_k}_{cla/rel}=R_1^{\varepsilon_k}$ we have $f_{\varepsilon_k}=0,\gamma f_{\varepsilon_k}(T_1)=0,\gamma^+ f_{\varepsilon_k}=0$ and by weak limit we deduce that $\int_0^{T_1}\int_\Omega\int_{\mathbb{R}^3_p}f\psi dtdxdp=\lim_{k\to+\infty}\int_0^{T_1}\int_\Omega\int_{\mathbb{R}^3_p}f_{\varepsilon_k}\psi dtdxdp=0,\ \ \forall\psi\in C^0_c([0,T_1]\times\overline{\Omega}\times(\mathbb{R}^3_p-B_{R_1}))$ which implies that $f=0$ a.e. in $]0,T_1[\times\Omega\times(\mathbb{R}^3_p-B_{R_1})$ or supp $f\subset]0,T_1[\times\Omega\times B_{R_1}$. Similarly we deduce that supp $\gamma^+ f\subset]0,T_1[\times\Sigma^+_{R_1}$ and supp $\gamma f(T_1,\cdot,\cdot)\subset\Omega\times B_{R_1}$. Note that if $E,B\in L^\infty(]0,+\infty[\times\Omega)^3$, then $R_1=R+2D_{cla/rel}$ doesn't depend on T_1 and therefore the solution is compactly supported in momentum uniformly with respect to $T_1>0$. □

REMARK 2.12. *By using the Remark* 2.6 *we can prove that the conclusion of the above theorem holds also in the case* $r=+\infty$.

2.4. Estimate of the momentum support for the time periodic problem.

An application of the momentum change lemma could be the estimate of the momentum support for time periodic solutions of the Vlasov problem. First we introduce the perturbed time periodic Vlasov problem :

$$\alpha f+\partial_t f+v(p)\cdot\nabla_x f+F(t,x,p)\cdot\nabla_p f=0,\ \ (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p, \tag{2.14}$$

with the boundary condition :

$$g(t,x,p)=f(t,x,p),\ \ (t,x,p)\in\mathbb{R}_t\times\Sigma^-, \tag{2.15}$$

where this time g,E,B are supposed T periodic in time, $T>0,\alpha>0$ fixed. The definition of T periodic weak solution is given by :

DEFINITION 2.13. *Assume that* $E,B\in L^\infty(\mathbb{R}_t\times\Omega)^3$ *and* g *are* T *periodic with* $(v(p)\cdot n(x))g\in L^1(]0,T[\times\Sigma^-_R),\ \ \forall R>0$. *We say that* $f\in L^1(]0,T[\times\Omega\times B_R)\ \forall R>0$ *is a* T *periodic weak solution for the problem* $(2.14),(2.15)$ *iff :*

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}f(t,x,p)(\alpha\varphi-\partial_t\varphi-v(p)\cdot\nabla_x\varphi-F(t,x,p)\cdot\nabla_p\varphi)dtdxdp=-\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g\varphi dtd\sigma dp,$$

for all test function which belongs to $\mathcal{T}^{per}_w=\{\varphi\in C^1(\mathbb{R}_t\times\overline{\Omega}\times\mathbb{R}^3_p)\mid\exists R>0\ :\ \varphi=\varphi\cdot\mathbf{1}_{\{|p|\le R\}},\ \varphi|_{\mathbb{R}_t\times\Sigma^+}=0,\ \varphi(\cdot+T)=\varphi\}$.

Note also that in the periodic case the definition for s_{in} is :

$$s_{in}(t,x,p) = \sup\{s \le t \mid X(s;t,x,p) \in \partial\Omega\}.$$

It may happen that $s_{in} = -\infty$. Let us give now the definition for time periodic mild solution.

DEFINITION 2.14. *Assume that $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$ and g are T periodic with $(v(p)\cdot n(x))g \in L^1(]0,T[\times\Sigma_R^-)$, $\forall R > 0$. We say that $f \in L^1(]0,T[\times\Omega\times B_R)$, $\forall R>0$ is a T periodic mild solution for* (2.14), (2.15) *iff :*

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)\psi(t,x,p)dtdxdp = -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi^\alpha_\psi(t,x,p)dtd\sigma dp,$$

for all test function which belongs to :

$$\mathcal{T}_m^{per} = \{\psi \in C^0(\mathbb{R}_t\times\overline{\Omega}\times\mathbb{R}^3_p) \mid \exists R>0\ :\ \psi = \psi\cdot\mathbf{1}_{\{|p|\le R\}},\ \psi(\cdot+T)=\psi\},$$

where :

$$\varphi^\alpha_\psi(t,x,p) = \int_t^{s_{out}(t,x,p)} e^{-\alpha(s-t)}\psi(s, X(s;t,x,p), P(s;t,x,p))ds.$$

REMARK 2.15. *Observe that by the Lemmas* 2.7, 2.9 *the function φ^α_ψ has also compact support in momentum (if $\psi = \psi\cdot\mathbf{1}_{\{|p|\le R\}}$ then $\varphi^\alpha_\psi = \varphi^\alpha_\psi\cdot\mathbf{1}_{\{|p|\le R+2D_{cla/rel}\}}$) and that for $\alpha>0$ the function φ^α_ψ is bounded : $\|\varphi^\alpha_\psi\|_\infty \le \|\psi\|_\infty/\alpha$. Therefore the above definition makes sense.*

REMARK 2.16. *In this case the mild solution is given by $f(t,x,p) = 0$ if $s_{in} = -\infty$ and $f(t,x,p) = e^{-\alpha(t-s_{in})}g(s_{in}, X(s_{in};t,x,p), P(s_{in};t,x,p))$ if $s_{in} > -\infty$.*

REMARK 2.17. *The mild T periodic solution is also a T periodic weak solution and verifies the following Green formula :*

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x,p)\cdot\nabla_p\varphi)dtdxdp$$
$$= -\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\varphi dtd\sigma dp - \int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g\varphi dtd\sigma dp,$$

for all $\varphi \in C^1(\mathbb{R}_t\times\overline{\Omega}\times\mathbb{R}^3_p)$, compactly supported in momentum and T periodic, where the trace function $\gamma^+ f$ is defined as in the Remark 2.16.

REMARK 2.18. *Suppose that g is bounded. Then the T periodic mild solution of problem* (2.14), (2.15) *verifies :*

$$\max\{\|f\|_\infty, \|\gamma^+ f\|_\infty\} \le \|g\|_\infty.$$

In particular, if $g \ge 0$ then $f, \gamma^+ f \ge 0$.

THEOREM 2.19. *Assume that $\alpha > 0, E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3, g \in L^\infty(\mathbb{R}_t\times\Sigma^-)$ are T periodic with $g = g\cdot\mathbf{1}_{\{|p|\le R\}}$ for some $R>0$. Then the T periodic mild solution f for* (2.14), (2.15) *verifies :*

$$\max\{\|f\|_\infty, \|\gamma^+ f\|_\infty\} \le \|g\|_\infty,\ \ f = f\cdot\mathbf{1}_{\{|p|\le R_1\}},\ \ \gamma^+ f = \gamma^+ f\cdot\mathbf{1}_{\{|p|\le R_1\}},$$

with $R_1 = R + 2D_{cla/rel}$.

Proof. Take $\psi \in C^0(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p)$, T periodic, with compact support in momentum in $\mathbb{R}^3_p - B_{R_1}$. By the mild formulation we have :

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)\psi(t,x,p)dtdxdp = -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi^\alpha_\psi(t,x,p)dtd\sigma dp.$$

If $|p| > R$, then $g = 0$ and $g \cdot \varphi^\alpha_\psi = 0$. If $|p| \leq R$, then by the Lemmas 2.7, 2.9 we deduce that $|P(s)| \leq |p| + 2D_{cla/rel} \leq R_1$ and thus $\varphi^\alpha_\psi = 0$ or $g \cdot \varphi^\alpha_\psi = 0$. We deduce that $\int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f\psi dtdxdp = 0$, or supp $f \subset \mathbb{R}_t \times \Omega \times B_{R_1}$. Now, by using the Green formula we have :

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x,p)\cdot\nabla_p\varphi)dtdxdp$$
$$= -\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\varphi dtd\sigma dp - \int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g\varphi dtd\sigma dp,$$

for any function $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p)$, T periodic, with compact support in momentum in $\mathbb{R}^3_p - B_{R_1}$. Therefore we have $\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\varphi dtd\sigma dp = 0$ which implies that supp $\gamma^+ f \subset \mathbb{R}_t \times \Sigma^+_{R_1}$. □

By regularization we can prove the existence of T periodic weak solution with compact support in momentum.

THEOREM 2.20. *Assume that* $\alpha = 0, E, B \in L^\infty(\mathbb{R}_t \times \Omega)^3, g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$ *are* T *periodic with* $g = g \cdot \mathbf{1}_{\{|p|\leq R\}}$ *for some* $R > 0$*. Then there is a* T *periodic weak solution* f *for* (2.14), (2.15) *which verifies :*

$$\max\{\|f\|_\infty, \|\gamma^+ f\|_\infty\} \leq \|g\|_\infty, \;\; f = f\cdot\mathbf{1}_{\{|p|\leq R_1\}}, \;\; \gamma^+ f = \gamma^+ f \cdot \mathbf{1}_{\{|p|\leq R_1\}},$$

with $R_1 = R + 2D_{cla/rel}$.

Proof. Regularize the electro-magnetic field and take f_ε the T periodic mild solutions constructed in the previous theorem with $\alpha = \varepsilon$ and the electro-magnetic field $(E_\varepsilon, B_\varepsilon)$. We have $\max\{\|f_\varepsilon\|_\infty, \|\gamma^+ f_\varepsilon\|_\infty\} \leq \|g\|_\infty$, $f_\varepsilon = f_\varepsilon \cdot \mathbf{1}_{\{|p|\leq R_1\}}$, $\gamma^+ f_\varepsilon = \gamma^+ f_\varepsilon \cdot \mathbf{1}_{\{|p|\leq R_1\}}$ since $R_1^\varepsilon = R + 2D^\varepsilon_{cla/rel} \leq R + 2D_{cla/rel} = R_1$. We can extract sequences such that $f_{\varepsilon_k} \rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$, $\gamma^+ f_{\varepsilon_k} \rightharpoonup \gamma^+ f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Sigma^+)$. By passing to the limit for $k \to \infty$ in the weak formulation, we deduce that f is periodic weak solution corresponding to the electro-magnetic field (E, B) and $\varepsilon = 0$. Also by passing to the limit in the Green formula for $k \to +\infty$ we deduce that $\gamma^+ f$ is the trace of f. By weak $\star$ limit we have $\max\{\|f\|_\infty, \|\gamma^+ f\|_\infty\} \leq \liminf_{k\to+\infty} \max\{\|f_{\varepsilon_k}\|_\infty, \|\gamma^+ f_{\varepsilon_k}\|_\infty\} \leq \|g\|_\infty$ and also $f = f\cdot\mathbf{1}_{\{|p|\leq R_1\}}$ and $\gamma^+ f = \gamma^+ f\cdot\mathbf{1}_{\{|p|\leq R_1\}}$. □

3. The regularized Vlasov-Poisson system.

We consider $\Omega \subset \mathbb{R}^N_x$ an open, regular bounded set. We denote by $E_0 = -\nabla_x\Phi_0$ the exterior electric field :

$$-\Delta_x\Phi_0(t,x) = 0, \;\; (t,x) \in]0,T[\times\Omega, \;\; \Phi_0(t,x) = \varphi_0(t,x), \;\; (t,x) \in]0,T[\times\partial\Omega.$$

In this section we construct solutions for the following regularized Vlasov-Poisson system (classical or relativistic case) :

$$
\begin{cases}
\partial_t f + v(p)\cdot\nabla_x f + qE_\varepsilon\cdot\nabla_p f = 0, \ \ (t,x,p)\in]0,T[\times\Omega\times\mathbb{R}^N_p,\\
f(0,x,p)=f_0(x,p), \ \ (x,p)\in\Omega\times\mathbb{R}^N_p, \ \ f(t,x,p)=g(t,x,p), \ \ (t,x,p)\in]0,T[\times\Sigma^-,\\
-(1-\alpha\Delta_x)^{2m}\Delta_x\Phi = \dfrac{\rho_\varepsilon}{\varepsilon_0}, \ \ (t,x)\in]0,T[\times\Omega,\\
\Phi=\Delta_x\Phi=...=\Delta_x^{2m}\Phi=0, \ \ (t,x)\in]0,T[\times\partial\Omega,
\end{cases}
\tag{3.1}
$$

where $E_\varepsilon = \overline{E}\star\zeta_\varepsilon$, $\overline{E}$ is the extension by 0 outside $]0,T[\times\Omega$ of $E=-\nabla_x\Phi-\nabla_x\Phi_0$ and $\zeta_\varepsilon(t,x)=\frac{1}{\varepsilon^{N+1}}\zeta(\frac{t}{\varepsilon},\frac{x}{\varepsilon})$ is a mollifier *i.e.,* $\zeta\in C^\infty_c(\mathbb{R}\times\mathbb{R}^N_x)$, $\zeta\geq 0$, $\int_{\mathbb{R}^{N+1}}\zeta(s,y)\ dsdy=1$ and $\alpha,\varepsilon>0$ are small parameters. Regularized systems of this type have been used in previous works (see [6]). We recall here the following result :

LEMMA 3.1. *Let $\rho\in L^p(\Omega)$ for some $1<p<+\infty$ and suppose that $\partial\Omega$ is smooth. Then the solution Φ of the regularized Poisson problem :*

$$
-(1-\alpha\Delta_x)^{2m}\Delta_x\Phi = \frac{\rho}{\varepsilon_0}, \ \ x\in\Omega,
$$
$$
\Phi=\Delta_x\Phi=...=\Delta_x^{2m}\Phi=0, \ \ x\in\partial\Omega,
$$

verifies :

$$
\|\Phi\|_{W^{4m+2,p}(\Omega)}\leq C(p,\alpha,\Omega)\cdot\|\rho\|_{L^p(\Omega)}, \ \ \|\Phi\|_{W^{2,p}(\Omega)}\leq C(p,\Omega)\cdot\|\rho\|_{L^p(\Omega)}.
$$

By using the fixed point method we prove the existence of solution for the regularized Vlasov-Poisson system. For the sake of the presentation we give a sketch of the proof. For more details the reader can refer to [6]. We consider the set $\chi=L^2(]0,T[;H^1(\Omega))$ and define the application $\mathcal{F}:\chi\to\chi$ by :

$$
\Phi\to E=-\nabla_x\Phi-\nabla_x\Phi_0\to E_\varepsilon=\overline{E}\star\zeta_\varepsilon\to f\to\rho=q\int_{\mathbb{R}^N_p} f\ dp\to\rho_\varepsilon\to\Phi_1=\mathcal{F}\Phi,
$$

where :
- f is the mild solution of the Vlasov problem associated with the regularized field $E_\varepsilon(t,x) = -\int_0^T\int_\Omega(\nabla_x\Phi(s,y)+\nabla_x\Phi_0(s,y))\zeta_\varepsilon(t-s,x-y)\ dsdy$;
- ρ_ε is the regularized charge density $\rho_\varepsilon=\int_0^T\int_\Omega\rho(s,y)\zeta_\varepsilon(t-s,x-y)\ dsdy$;
- Φ_1 is the solution of the regularized Poisson problem associated with the charge density ρ_ε.

PROPOSITION 3.2. *Under the hypotheses $M_0+M^- := \int_\Omega\int_{\mathbb{R}^N_p} f_0(x,p)\ dxdp + \int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|g(t,x,p)\ dtd\sigma dp<+\infty$, $\varphi_0\in L^2(]0,T[;H^{\frac{1}{2}}(\partial\Omega))$ we have :*

$$
\mathcal{F}(\chi)\subset\{\Phi\in L^2(]0,T[;H^1(\Omega))\ |\ \|\Phi\|_{L^2(]0,T[;H^1(\Omega))}\leq M_\varepsilon\},
$$

where $M_\varepsilon = C(\Omega)\cdot\frac{T}{\varepsilon_0}\cdot(M_0+M^-)\cdot\|\zeta\|_{L^2(\mathbb{R}^{N+1})}\cdot\varepsilon^{-\frac{N+1}{2}}$.

Proof. As usual we have :

$$
\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)\ dxdp + \int_0^t\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(s,x,p)\ dsd\sigma dp = \int_\Omega\int_{\mathbb{R}^N_p} f_0(x,p)\ dxdp + \int_0^t\int_{\Sigma^-}|(v(p)\cdot n(x))|g\ dsd\sigma dp,
$$

and therefore $\|f\|_{L^1(]0,T[\times\Omega\times\mathbb{R}^N_p)}\leq T\cdot(M_0+M^-)$. We have the inequalities :

$$
\|\Phi_1\|_{L^2(]0,T[;H^1(\Omega))}\leq C(\Omega)\left\|\frac{\rho_\varepsilon}{\varepsilon_0}\right\|_{L^2(]0,T[\times\Omega)}\leq\frac{C(\Omega)}{\varepsilon_0}\cdot\|\rho\|_{L^1(]0,T[\times\Omega)}\cdot\|\zeta_\varepsilon\|_{L^2}
$$
$$
\leq C(\Omega)\cdot\frac{T}{\varepsilon_0}\cdot(M_0+M^-)\cdot\|\zeta\|_{L^2(\mathbb{R}^{N+1})}\cdot\varepsilon^{-\frac{N+1}{2}}.
$$

□

In the following proposition we prove the continuity of the application $\mathcal{F}$ with respect to the weak topology of $L^2(]0,T[;H^1(\Omega))$.

PROPOSITION 3.3. *Assume that* $0 \le f_0 \in L^\infty(\Omega \times \mathbb{R}^N_p)$, $0 \le g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$, $\int_\Omega \int_{\mathbb{R}^N_p} (1 + \mathcal{E}(p)) f_0(x,p)\, dxdp + \int_0^T\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p)) g(t,x,p)\, dtd\sigma dp < \infty$, $\varphi_0 \in L^2(]0,T[;H^{\frac{1}{2}}(\partial\Omega))$. *Then the application* $\mathcal{F}$ *is continuous with respect to the weak topology of* $L^2(]0,T[;H^1(\Omega))$.

Proof. Consider $(\Phi_k)_k$ such that $\Phi_k \rightharpoonup \Phi$ weakly in $L^2(]0,T[;H^1(\Omega))$, which implies that $\nabla_x\Phi_k \rightharpoonup \nabla_x\Phi$ weakly in $L^2(]0,T[;L^2(\Omega)^N)$ and $E_k := -\nabla_x\Phi_k - \nabla_x\Phi_0 \rightharpoonup -\nabla_x\Phi - \nabla_x\Phi_0 := E$ weakly in $L^2(]0,T[;L^2(\Omega)^N)$. By regularization we deduce that $E_{k,\varepsilon} = \overline{E}_k \star \zeta_\varepsilon \to \overline{E} \star \zeta_\varepsilon = E_\varepsilon$ strongly in $L^2(]0,T[;L^2(\Omega)^N)$. Denote by f_k, f the mild solutions of the Vlasov problem associated with the fields $E_{k,\varepsilon}$ and E_ε respectively. By standard arguments we prove that $f_k \rightharpoonup f$ weakly $\star$ in $L^\infty(]0,T[\times\Omega\times\mathbb{R}^N_p)$. In order to pass to the limit in the regularized Poisson equation we need to prove that $(\int_0^T\!\int_\Omega\int_{\mathbb{R}^N_p} f_k(t,x,p)|p|\, dtdxdp)_k$ is bounded. Indeed, by using the weak formulation of the Vlasov problem with the test function $|p|$ we have :

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^N_p} |p| f_k(t,x,p)\, dxdp + \int_0^t\!\!\int_{\Sigma^+} (v(p)\cdot n(x)) \cdot |p| \gamma^+ f_k(s,x,p)\, dsd\sigma dp = \int_\Omega \int_{\mathbb{R}^N_p} |p| f_0(x,p)\, dxdp \\
&\quad + \int_0^t\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))| \cdot |p| g(s,x,p)\, dsd\sigma dp + \int_0^t\!\!\int_\Omega \int_{\mathbb{R}^N_p} q f_k(s,x,p) E_{k,\varepsilon}(s,x)\cdot \frac{p}{|p|}\, dsdxdp \\
&\quad \le C\cdot \int_\Omega \int_{\mathbb{R}^N_p} (1+\mathcal{E}(p)) f_0(x,p)\, dxdp + C \cdot \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p)) g(t,x,p)\, dtd\sigma dp \\
&\quad + C\cdot \|E_{k,\varepsilon}\|_{L^\infty(]0,T[\times\Omega)} \cdot \|f_k\|_{L^1(]0,T[\times\Omega\times\mathbb{R}^N_p)}. \qquad (3.2)
\end{aligned}
$$

By taking into account that $(E_{k,\varepsilon})$ is bounded in $L^\infty(]0,T[\times\Omega)$:

$$
\begin{aligned}
\|E_{k,\varepsilon}\|_{L^\infty} &\le \varepsilon^{-\frac{N+1}{2}} \cdot \|\zeta\|_{L^2(\mathbb{R}^{N+1})} \cdot \left\{ \|\nabla_x\Phi_k\|_{L^2(]0,T[\times\Omega)} + \|\nabla_x\Phi_0\|_{L^2(]0,T[\times\Omega)} \right\} \\
&\le \varepsilon^{-\frac{N+1}{2}} \cdot \|\zeta\|_{L^2(\mathbb{R}^{N+1})} \cdot C \cdot (1 + \|\varphi_0\|_{L^2(]0,T[;H^{\frac{1}{2}}(\partial\Omega))}),
\end{aligned}
$$

and that $\|f_k\|_{L^1(]0,T[\times\Omega\times\mathbb{R}^N_p)} \le T\cdot(M_0 + M^-)$, we deduce that $(\int_\Omega \int_{\mathbb{R}^N_p} |p|\cdot f_k(t,x,p)\, dxdp)_k$ is uniformly bounded with respect to $t \in]0,T[$, which implies that $\rho_k \rightharpoonup \rho := q\int_{\mathbb{R}^N_p} f(t,x,p)\, dp$ weakly in $L^1(]0,T[\times\Omega)$ and $\rho_{k,\varepsilon} = \overline{\rho}_k \star \zeta_\varepsilon \to \overline{\rho}\star\zeta_\varepsilon := \rho_\varepsilon$ strongly in $L^2(]0,T[\times\Omega)$. Finally $\mathcal{F}(\Phi_k) = \Phi_{k,1} \to \Phi_1 = \mathcal{F}(\Phi)$ strongly in $L^2(]0,T[;H^1(\Omega))$ and our conclusion follows.

□

By applying the Schauder fixed point theorem we deduce that :

PROPOSITION 3.4. *Under the hypotheses of Propositions* 3.2, 3.3 *there is at least one weak solution for the regularized Vlasov-Poisson system.*

We denote by (f, Φ_s) the solution constructed above :

$$
\begin{cases}
\partial_t f + v(p)\cdot\nabla_x f + q(\overline{E}\star\zeta_\varepsilon)\cdot\nabla_p f = 0, \ \ (t,x,p) \in]0,T[\times\Omega\times\mathbb{R}^N_p, \\
f(0,x,p) = f_0(x,p), \ \ (x,p)\in\Omega\times\mathbb{R}^N_p, \ \ f(t,x,p) = g(t,x,p), \ \ (t,x,p)\in]0,T[\times\Sigma^-, \\
-(1-\alpha\Delta_x)^{2m}\Delta_x\Phi_s = \dfrac{\overline{\rho}\star\zeta_\varepsilon}{\varepsilon_0}, \ \ (t,x)\in]0,T[\times\Omega, \ \ E = -\nabla_x\Phi_s - \nabla_x\Phi_0, \ \ (t,x)\in]0,T[\times\Omega, \\
\Phi_s = \Delta_x\Phi_s = ... = \Delta^{2m}\Phi_s = 0, \ \ (t,x)\in]0,T[\times\partial\Omega.
\end{cases}
$$

Following the idea of [6] we can pass to the limit for $\varepsilon \searrow 0$ when $\alpha > 0$ is fixed. We obtain the result :

PROPOSITION 3.5. *Assume that $\Omega \subset \mathbb{R}^N_x$ is open and bounded, with $\partial\Omega$ smooth. Consider $p_0 = \frac{2N}{2N-1}$, $p_0' = 2N$, $(\frac{1}{p_0} + \frac{1}{p_0'} = 1)$ and m such that $W^{4m,p_0}(\Omega) \to L^\infty(\Omega)$ is continuous $(\frac{1}{p_0} - \frac{4m}{N} < 0)$. We suppose also that the initial-boundary conditions verify $0 \le f_0 \in L^\infty(\Omega \times \mathbb{R}^N_p)$, $0 \le g \in L^\infty(]0,T[\times\Sigma^-)$, $\exists R > 0$ such that $f_0 = f_0 \cdot \mathbf{1}_{\{|p|\le R\}}$, $g = g \cdot \mathbf{1}_{\{|p|\le R\}}$, $\varphi_0 \in L^\infty(]0,T[; W^{4m+2-\frac{1}{p_0},p_0}(\partial\Omega))$, $\partial_t\varphi_0 \in L^\infty(]0,T[; W^{4m+1-\frac{1}{p_0},p_0}(\partial\Omega))$. Then there is at least one solution for the Vlasov problem (classical or relativistic case) coupled to the regularized Poisson problem :*

$$\left\{\begin{array}{r} \partial_t f + v(p)\cdot\nabla_x f + q(-\nabla_x\Phi_s - \nabla_x\Phi_0)\cdot\nabla_p f = 0, \;\; (t,x,p) \in]0,T[\times\Omega\times\mathbb{R}^N_p, \\ f(0,x,p) = f_0(x,p), \;\; (x,p)\in\Omega\times\mathbb{R}^N_p, \;\; f(t,x,p) = g(t,x,p), \;\; (t,x,p)\in]0,T[\times\Sigma^-, \\ -(1-\alpha\Delta_x)^{2m}\Delta_x\Phi_s = \dfrac{\rho}{\varepsilon_0}, \;\; (t,x)\in]0,T[\times\Omega, \\ \Phi_s = \Delta_x\Phi_s = ... = \Delta^{2m}\Phi_s = 0, \;\; (t,x)\in]0,T[\times\partial\Omega. \end{array}\right. \tag{3.3}$$

The particle densities f, $\gamma^+ f$ have compact support in momentum and the self consistent potential Φ_s verifies $\partial_t\Phi_s \in L^\infty(]0,T[; W^{1,\infty}(\Omega))$, $\nabla_x\Phi_s \in L^\infty(]0,T[; W^{1,\infty}(\Omega))^N$. In particular the electric field $E = -\nabla_x\Phi_s - \nabla_x\Phi_0$ belongs to $W^{1,\infty}(]0,T[\times\Omega)^N$.

Proof. The proof follows by standard arguments (see [6]). The main idea is to estimate the L^∞ norm of the electric field uniformly with respect to $\varepsilon > 0$, when $\alpha > 0$ is fixed. Denote by $(f_\varepsilon, \Phi_{s,\varepsilon})$ the solutions of (3.1) constructed above. First, since the initial-boundary conditions have momentum support contained in $B(0,R)$, we deduce that f_ε has momentum support contained in $B(0,R_1)$, with $R_1 = R + |q|\cdot T\cdot(\|\nabla_x\Phi_0\|_{L^\infty} + \|\nabla_x\Phi_{s,\varepsilon}\|_{L^\infty})$. We deduce that $\|\rho_\varepsilon\|_{L^\infty} \le C\cdot(1 + \|\nabla_x\Phi_{s,\varepsilon}\|^N_{L^\infty})$. By elliptic regularity result (see Lemma 3.1) we can write :

$$\begin{aligned} \|\nabla_x\Phi_{s,\varepsilon}\|_{L^\infty} &\le C\cdot\|\Phi_{s,\varepsilon}\|_{L^\infty(]0,T[;W^{4m+2,p_0}(\Omega))} \le C\cdot\|\rho_\varepsilon\|_{L^\infty(]0,T[;L^{p_0}(\Omega))} \\ &\le C\cdot\|\rho_\varepsilon\|^{\frac{1}{p_0}}_{L^\infty(]0,T[;L^1(\Omega))}\cdot\|\rho_\varepsilon\|^{\frac{1}{p_0'}}_{L^\infty(]0,T[;L^\infty(\Omega))} \\ &\le C\cdot(1 + \|\nabla_x\Phi_{s,\varepsilon}\|^N_{L^\infty})^{\frac{1}{p_0'}}, \end{aligned} \tag{3.4}$$

which gives the desired estimate for the L^∞ norm of the electric field $E_\varepsilon = -\nabla_x\Phi_{s,\varepsilon} - \nabla_x\Phi_0$. The existence of solution follows by passing to the limit for $\varepsilon \searrow 0$ in (3.1). For the other statements use the inclusion $W^{4m,p_0}(\Omega) \to L^\infty(\Omega)$, the elliptic regularity result and the continuity equation $\partial_t\rho + \mathrm{div}\,_x j = 0$.
□

4. A priori estimates.

In this section we establish uniform estimates with respect to $\alpha > 0$ for the solutions of (3.3). First we recall the classical estimates for the total mass and energy. Secondly we deduce also an estimate for the L^∞ norm of the electric field. We assume that the hypotheses of Proposition 3.5 are verified and we denote by (f, Φ_s) the solution of (3.3). We recall that $\partial_t\Phi_0$, $\partial_t\Phi_s \in L^\infty(]0,T[; W^{1,\infty}(\Omega))$, $\nabla_x\Phi_0$, $\nabla_x\Phi_s \in L^\infty(]0,T[; W^{1,\infty}(\Omega))^N$ and $f, \gamma^+ f$ have compact support in momentum.

4.1. The mass and energy estimates.

We introduce the notations :

$$M_0 := \int_\Omega\int_{\mathbb{R}^N_p} f_0(x,p)\,dxdp, \;\; M(t) := \int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)\,dxdp,$$

$$M^{\pm}(t) := \int_{\Sigma^{\pm}} |(v(p)\cdot n(x))|\gamma^{\pm} f(t,x,p)\; d\sigma dp,\;\; M^{\pm} := \int_0^T M^{\pm}(t)\; dt,$$

$$K_0 := \int_{\Omega}\int_{\mathbb{R}^N_p} \mathcal{E}(p) f_0(x,p)\; dxdp,\;\; K(t) := \int_{\Omega}\int_{\mathbb{R}^N_p} \mathcal{E}(p) f(t,x,p)\; dxdp,$$

$$K^{\pm}(t) := \int_{\Sigma^{\pm}} |(v(p)\cdot n(x))|\mathcal{E}(p)\gamma^{\pm} f(t,x,p)\; d\sigma dp,\;\; K^{\pm} := \int_0^T K^{\pm}(t)\; dt,$$

$$V_s(t) := \frac{1}{2}\int_{\Omega} \rho(t,x)\Phi_s(t,x)\; dx,\;\; V_0(t) := \frac{1}{2}\int_{\Omega} \rho(t,x)\Phi_0(t,x)\; dx.$$

The estimate for the total mass follows by using the weak formulation of the Vlasov problem with the test function $\theta = 1$:

$$\frac{d}{dt}M(t) + M^+(t) = M^-(t),\;\; t\in]0,T[. \tag{4.1}$$

We deduce that :

$$M(t) + \int_0^t M^+(s)\; ds = M_0 + \int_0^t M^-(s)\; ds,\;\; t\in]0,T[, \tag{4.2}$$

which implies :

$$\sup_{0\le t\le T}\{M(t)\} + M^+ \le 2(M_0 + M^-). \tag{4.3}$$

The estimate for the total energy follows by using the test functions $\mathcal{E}(p)$ and $q\Phi_s$. We have :

$$\frac{d}{dt}K(t) + K^+(t) = K^-(t) + \int_{\Omega} E(t,x)\cdot j(t,x)\; dx,\;\; t\in]0,T[. \tag{4.4}$$

We deduce that :

$$K(t) + \int_0^t K^+(s)\; ds = K_0 + \int_0^t K^-(s)\; ds + \int_0^t\int_{\Omega} E(s,x)\cdot j(s,x)\; dsdx,\;\; t\in]0,T[. \tag{4.5}$$

By using as test function the potential Φ_s one gets :

$$\frac{d}{dt}\int_{\Omega} \rho(t,x)\Phi_s(t,x)\; dx = \int_{\Omega} \{\rho(t,x)\partial_t\Phi_s + j(t,x)\cdot\nabla_x\Phi_s\}\; dx,\;\; t\in]0,T[. \tag{4.6}$$

By using the regularized Poisson equation, after multiplication by Φ_s and integration by parts we obtain :

$$V_s(t) = \frac{1}{2}\int_{\Omega} \rho(t,x)\Phi_s(t,x)\; dx = \frac{\varepsilon_0}{2}\int_{\Omega} |(1-\alpha\Delta_x)^m\nabla_x\Phi_s|^2\; dx, \tag{4.7}$$

and we deduce that :

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}\frac{1}{2}\rho(t,x)\Phi_s(t,x)\; dx &= \varepsilon_0\int_{\Omega}(1-\alpha\Delta_x)^m\nabla_x\Phi_s\cdot(1-\alpha\Delta_x)^m\nabla_x\partial_t\Phi_s\; dx\\ &= \int_{\Omega}\rho(t,x)\partial_t\Phi_s\; dx.\end{aligned} \tag{4.8}$$

Now, by combining (4.6) and (4.8) we have

$$\frac{d}{dt}V_s(t) = \int_\Omega j(t,x)\cdot\nabla_x\Phi_s \, dx, \quad t\in[0,T]. \tag{4.9}$$

Finally, by using (4.4), (4.9) one gets :

$$\frac{d}{dt}\{K(t)+V_s(t)\} + K^+(t) = K^-(t) - \int_\Omega \nabla_x\Phi_0\cdot j(t,x)\, dx, \quad t\in]0,T[, \tag{4.10}$$

which implies :

$$K(t)+V_s(t)+\int_0^t K^+(s)\, ds = K_0+V_s(0)+\int_0^t K^-(s)\, ds-\int_0^t\int_\Omega \nabla_x\Phi_0(s,x)\cdot j(s,x)\, dsdx, \quad t\in]0,T[. \tag{4.11}$$

By interpolation inequalities we have :

$$\left|\int_\Omega \nabla_x\Phi_0\cdot j(s,x)\, dx\right| \le \|\nabla_x\Phi_0(s)\|_{L^\infty}\cdot\|j(s)\|_{L^1(\Omega)} \le C\cdot\|\nabla_x\Phi_0(s)\|_{L^\infty}\cdot\|j(s)\|_{L^\beta(\Omega)}$$
$$\le C\cdot\|\nabla_x\Phi_0(s)\|_{L^\infty}\cdot(M(s)+K(s))^{\frac{1}{\beta}},$$

where $\beta=\frac{N+2}{N+1}$ in the classical case and $\beta=\frac{N+1}{N}$ in the relativistic case. From (4.2), (4.11) we obtain that :

$$M(t)+K(t)+V_s(t)+\int_0^t\{M^+(s)+K^+(s)\}\, ds \le M_0+K_0+V_s(0)+\int_0^t\{M^-(s)+K^-(s)\}\, ds$$
$$+C\cdot\|\nabla_x\Phi_0\|_{L^\infty}\cdot\int_0^t (M(s)+K(s))^{\frac{1}{\beta}}\, ds, \tag{4.12}$$

which implies easily that there is a constant depending on the initial-boundary conditions and T but not on the size of the momentum support R and α such that :

$$\sup_{0\le t\le T}\{M(t)+K(t)+V_s(t)\}+M^++K^+ \le C(M_0,K_0,V_s(0),M^-,K^-,\|\nabla_x\Phi_0\|_{L^\infty},T). \tag{4.13}$$

4.2. The L^∞ estimate for the electric field.

We want to estimate uniformly with respect to $\alpha>0$ the L^∞ norm of the electric field $E=-\nabla_x\Phi_s-\nabla_x\Phi_0$, where (f,Φ_s) is solution of (3.3). In the one dimensional case such a bound follows immediately from the estimate (4.13). Consider now the cases $N\ge 2$. We assume that there are $F_0, G:[0,+\infty[\to\mathbb{R}^+$ non increasing functions such that :

$$f_0(x,p)\le F_0(|p|), \ \ \forall(x,p)\in\Omega\times\mathbb{R}^N_p, \ \ g(t,x,p)\le G(|p|), \ \ \forall(t,x,p)\in]0,T[\times\Sigma^-, \tag{4.14}$$

and :

$$\tilde{M}_0 := \int_{\mathbb{R}^N_p} F_0(|p|)\, dp + \int_{\mathbb{R}^N_p} G(|p|)\, dp < +\infty. \tag{4.15}$$

Roughly speaking, the above hypotheses say that the initial-boundary conditions have charge densities in L^∞ :

$$\rho_0(x) = \int_{\mathbb{R}^N_p} f_0(x,p)\, dp \le \int_{\mathbb{R}^N_p} F_0(|p|)\, dp, \ \ x\in\Omega,$$

$$\rho^-(t,x) = \int_{(v(p)\cdot n(x))<0} g(t,x,p)\,dp \le \int_{\mathbb{R}^N_p} G(|p|)\,dp,\quad (t,x)\in]0,T[\times\Omega.$$

Note that E is smooth and therefore f can be calculated by using characteristics. The idea is to separate the charge density into two parts corresponding to small and large momentum and to use the momentum change lemma which says that $|P(s_1)-P(s_2)| \le 2D_{cal/rel}$, $\forall s_{in}\le s_1\le s_2\le s_{out}$ where $D_{cla}\sim \|E\|^{\frac{1}{2}}_{L^\infty}$ and $D_{rel}\sim\|E\|_{L^\infty}$. Let us decompose :

$$\rho(t,x) = \rho_1+\rho_2 = q\int_{\mathbb{R}^N_p} f(t,x,p)\mathbf{1}_{\{|p|\le 4D\}}\,dp + q\int_{\mathbb{R}^N_p} f(t,x,p)\mathbf{1}_{\{|p|>4D\}}\,dp,$$

with $D=D_{cla/rel}$ and estimate separately ρ_1,ρ_2. For $\eta>0$ we can write :

$$\begin{aligned} q^{-1}\rho_1(t,x) =& \int_{|p|\le 4D} f^{\frac{1}{N+\eta}}\cdot|p|^{\frac{r}{N+\eta}}\cdot f^{\frac{1}{(N+\eta)'}}\cdot|p|^{-\frac{r}{N+\eta}}\,dp\\ &\le \left(\int_{|p|\le 4D} f(t,x,p)\cdot|p|^r\,dp\right)^{\frac{1}{N+\eta}}\cdot\left(\int_{|p|\le 4D} f(t,x,p)\cdot|p|^{-\frac{r\cdot(N+\eta)'}{N+\eta}}\,dp\right)^{\frac{1}{(N+\eta)'}}, \end{aligned}$$

where $\frac{1}{N+\eta}+\frac{1}{(N+\eta)'}=1$, $r=2$ in the classical case and $r=1$ in the relativistic case. We deduce that :

$$\int_\Omega (|q|^{-1}\rho_1(t,x))^{N+\eta}\,dx \le C\cdot\|f\|^{\frac{N+\eta}{(N+\eta)'}}_{L^\infty}\cdot D^{[N-\frac{r\cdot(N+\eta)'}{N+\eta}]\cdot\frac{N+\eta}{(N+\eta)'}}\cdot\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}(p))f(t,x,p)\,dxdp,$$

which implies by using the estimate (4.13) :

$$\|\rho_1(t)\|_{L^{N+\eta}} \le C\cdot\|f\|^{\frac{1}{(N+\eta)'}}_{L^\infty}\cdot D^{[N-\frac{r\cdot(N+\eta)'}{N+\eta}]\cdot\frac{1}{(N+\eta)'}}\cdot(M(t)+K(t))^{\frac{1}{N+\eta}} \le C\cdot D^{[\frac{N}{(N+\eta)'}-\frac{r}{N+\eta}]}.$$

Notice that the above estimate is valid for $\eta>0$ such that $\frac{N}{(N+\eta)'}-\frac{r}{N+\eta}>0$. For ρ_2 it is possible to find a L^∞ bound. We have :

$$\begin{aligned} q^{-1}\rho_2(t,x) =& \int_{|p|>4D} f(t,x,p)\,dp = \int_{|p|>4D} f_0(X(0;t,x,p),P(0;t,x,p))\cdot\mathbf{1}_{\{s_{in}(t,x,p)=0\}}\,dp\\ &+\int_{|p|>4D} g(s_{in}(t,x,p),X(s_{in};t,x,p),P(s_{in};t,x,p))\cdot\mathbf{1}_{\{s_{in}(t,x,p)>0\}}\,dp. \end{aligned}$$

By using the momentum change lemma we have $|P(s;t,x,p)|\ge|p|-2D$, $\forall s_{in}(t,x,p)\le s\le t$ and therefore we have the inequalities :

$$\begin{aligned} q^{-1}\rho_2 \le& \int_{|p|>4D} F_0(|p|-2D)\,dp + \int_{|p|>4D} G(|p|-2D)\,dp\\ &\le C\cdot\int_{4D}^{+\infty}\{F_0(u-2D)\cdot u^{N-1}+G(u-2D)\cdot u^{N-1}\}\,du\\ &= C\cdot\int_{2D}^{+\infty}\{F_0(w)+G(w)\}\cdot(2D+w)^{N-1}\,dw\\ &\le C\cdot\int_{2D}^{+\infty}\{F_0(w)+G(w)\}\cdot(2\cdot w)^{N-1}\,dw \le C\cdot\int_{\mathbb{R}^N_p}\{F_0(|p|)+G(|p|)\}\,dp = C\cdot\tilde{M}_0<+\infty. \end{aligned}$$

The L^∞ bound for E follows by Sobolev inequalities and Lemma 3.1 :

$$\begin{aligned} \|\nabla_x\Phi_s(t)\|_{L^\infty(\Omega)} &\le \|\nabla_x\Phi_s(t)\|_{W^{1,N+\eta}(\Omega)} \le \|\Phi_s(t)\|_{W^{2,N+\eta}(\Omega)} \le C\cdot\|\rho(t)\|_{L^{N+\eta}(\Omega)}\\ &\le C\cdot\|\rho_1(t)\|_{L^{N+\eta}(\Omega)}+C\cdot\|\rho_2(t)\|_{L^{N+\eta}(\Omega)}\\ &\le C\cdot D^{[\frac{N}{(N+\eta)'}-\frac{r}{N+\eta}]}+C. \end{aligned}$$

In the classical case we have $D \sim \|E\|_{L^\infty}^{\frac{1}{2}}$, $r = 2$ and thus we deduce that :

$$\|E\|_{L^\infty(]0,T[\times\Omega)} \leq \|\nabla_x \Phi_0\|_{L^\infty(]0,T[\times\Omega)} + C(T) \cdot \left(1 + \|E\|_{L^\infty(]0,T[\times\Omega)}^{\frac{1}{2}[\frac{N}{(N+\eta)'} - \frac{2}{N+\eta}]}\right),$$

which gives a L^∞ bound for E as soon as there is $\eta > 0$ such that $0 < \frac{1}{2}[\frac{N}{(N+\eta)'} - \frac{2}{N+\eta}] < 1$, or $N(N+\eta) > N+2$ and $N-2 < \frac{N+2}{N+\eta}$. This is possible for $N \in \{2,3\}$. In the relativistic case we have $D \sim \|E\|_{L^\infty}$, $r = 1$ and :

$$\|E\|_{L^\infty(]0,T[\times\Omega)} \leq \|\nabla_x \Phi_0\|_{L^\infty(]0,T[\times\Omega)} + C(T) \cdot \left(1 + \|E\|_{L^\infty(]0,T[\times\Omega)}^{[\frac{N}{(N+\eta)'} - \frac{1}{N+\eta}]}\right),$$

which gives a L^∞ bound for E if there is $\eta > 0$ such that $0 < \frac{N}{(N+\eta)'} - \frac{1}{N+\eta} < 1$, or $N(N+\eta) > N+1$ and $N-1 < \frac{N+1}{N+\eta}$. This is possible for $N = 2$. Note that once we have a bound for the L^∞ norm of E we can estimate the L^∞ norm of the charge density $\|\rho\|_{L^\infty} \leq \|\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty}$. It is sufficient to estimate ρ_1. We have :

$$|\rho_1(t,x)| = |q| \cdot \int_{|p|\leq 4D} f(t,x,p)\ dp \leq C \cdot D^N \cdot \|f\|_{L^\infty} \leq C,$$

since $D \sim \|E\|_{L^\infty}^{\frac{1}{2}}$ in the classical case, $D \sim \|E\|_{L^\infty}$ in the relativistic case and E is bounded. Similar computations show that $\partial_t \Phi_s$ belongs to $L^\infty(]0,T[\times\Omega)$. For this we need to assume that the current densities of the initial-boundary conditions belong to L^∞ :

$$\tilde{M}_1 := \int_{\mathbb{R}^N_p} F_0(|p|)|v(p)|\ dp + \int_{\mathbb{R}^N_p} G(|p|)|v(p)|\ dp < +\infty. \tag{4.16}$$

Note also that in the relativistic case (4.15) implies (4.16). Indeed, by using elliptic regularity results and the continuity equation $\partial_t \rho + \mathrm{div}\ _x j = 0$ we have :

$$\begin{aligned}\|\partial_t \Phi_s(t)\|_{L^\infty(\Omega)} &\leq C \cdot \|\partial_t \Phi_s(t)\|_{W^{1,N+\eta}(\Omega)} \leq C \cdot \|\partial_t \rho(t)\|_{W^{-1,N+\eta}(\Omega)} \\ &= C \cdot \|\mathrm{div}\ _x j(t)\|_{W^{-1,N+\eta}(\Omega)} \leq C \cdot \|j(t)\|_{L^{N+\eta}(\Omega)}.\end{aligned} \tag{4.17}$$

As before we decompose :

$$j(t,x) = j_1 + j_2 = q \int_{\mathbb{R}^N_p} v(p) f(t,x,p) \mathbf{1}_{\{|p|\leq 4D\}}\ dp + q \int_{\mathbb{R}^N_p} v(p) f(t,x,p) \mathbf{1}_{\{|p|> 4D\}}\ dp. \tag{4.18}$$

For the first current density we can write :

$$|j_1(t,x)| \leq |q| \cdot \|f\|_{L^\infty} \cdot \int_{\mathbb{R}^N_p} |v(p)| \mathbf{1}_{\{|p|\leq 4D\}}\ dp \leq C. \tag{4.19}$$

For the second current density we have :

$$\begin{aligned}q^{-1} j_2(t,x) &= \int_{|p|>4D} v(p) f(t,x,p)\ dp = \int_{|p|>4D} v(p) f_0(X(0;t,x,p), P(0;t,x,p)) \cdot \mathbf{1}_{\{s_{in}(t,x,p)=0\}}\ dp \\ &+ \int_{|p|>4D} v(p) g(s_{in}(t,x,p), X(s_{in};t,x,p), P(s_{in};t,x,p)) \cdot \mathbf{1}_{\{s_{in}(t,x,p)>0\}}\ dp.\end{aligned} \tag{4.20}$$

We deduce that :

$$|q^{-1} \cdot j_2(t,x)| \leq \int_{|p|>4D} |v(p)| \cdot F_0(|p| - 2D)\ dp + \int_{|p|>4D} |v(p)| \cdot G(|p| - 2D)\ dp. \tag{4.21}$$

In the classical case $v(p) = \frac{p}{m}$ and therefore we have :

$$|q^{-1} \cdot j_2(t,x)| \le C \int_{4D}^{+\infty} \{F_0(u-2D) + G(u-2D)\} \cdot u^N \, du = C \int_{2D}^{+\infty} \{F_0(u) + G(u)\} \cdot (u+2D)^N \, du$$
$$\le C \cdot \int_{\mathbb{R}_p^N} \{F_0(|p|) + G(|p|)\} \cdot |p| \, dp = C \cdot \tilde{M}_1. \tag{4.22}$$

In the relativistic case we write :

$$|q^{-1} \cdot j_2(t,x)| \le c_0 \cdot \int_{|p|>4D} \{F_0(|p|-2D) + G(|p|-2D)\} \cdot \, dp \le C \cdot \tilde{M}_0. \tag{4.23}$$

We deduce from (4.18), (4.19), (4.22), (4.23) that $j \in L^\infty(]0,T[\times\Omega)$. By using now (4.17) we obtain that $\partial_t \Phi_s \in L^\infty(]0,T[\times\Omega)$.

5. The Vlasov-Poisson system.

We can prove now the existence of weak solution for the Vlasov-Poisson system.

THEOREM 5.1. *Assume that $\Omega \subset \mathbb{R}_x^N$ is open and bounded, with $\partial\Omega$ smooth. We suppose that the initial-boundary conditions verify :*

(i) $0 \le f_0 \in L^\infty(\Omega \times \mathbb{R}_p^N)$, $0 \le g \in L^\infty(]0,T[\times\Sigma^-)$;

(ii) $M_0 + K_0 + M^- + K^- + V_{s,0} = \int_\Omega \int_{\mathbb{R}_p^N} (1+\mathcal{E}(p)) f_0 \, dxdp + \int_0^T \int_{\Sigma^-} |(v(p)\cdot n(x))| \cdot (1+\mathcal{E}(p)) g \, dtd\sigma dp + \frac{q}{2} \int_\Omega \int_{\mathbb{R}_p^N} f_0 \Phi_{s,0}(x) \, dxdp < +\infty$ *(here $\Phi_{s,0}(\cdot)$ is the solution for* $-\Delta_x \Phi_{s,0} = \frac{\rho_0(x)}{\varepsilon_0}$, $x \in \Omega$, $\Phi_{s,0}(x) = 0$, $x \in \partial\Omega$*)* ;

(iii) $\nabla_x \Phi_0$ *belongs to* $L^\infty(]0,T[\times\Omega)^N$ *(here Φ_0 is the solution of* $-\Delta_x \Phi_0(t,x) = 0$, $(t,x) \in]0,T[\times\Omega$, $\Phi_0(t,x) = \varphi_0(t,x)$, $(t,x) \in]0,T[\times\partial\Omega$*)* .

Then there is at least one weak solution $(f, \Phi = \Phi_s + \Phi_0)$ for the Vlasov-Poisson system verifying :

$$0 \le f \le \max\{\|f_0\|_{L^\infty}, \|g\|_{L^\infty}\}, \quad 0 \le \gamma^+ f \le \max\{\|f_0\|_{L^\infty}, \|g\|_{L^\infty}\}, \tag{5.1}$$

$$\operatorname{ess\,sup}_{0<t<T} \left\{ \int_\Omega \int_{\mathbb{R}_p^N} (1+\mathcal{E}(p)) f(t,x,p) \, dxdp + \frac{\varepsilon_0}{2} \int_\Omega |\nabla_x \Phi_s(t,x)|^2 \, dx \right\} \tag{5.2}$$
$$+ \int_0^T \int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p)) \gamma^+ f \, dtd\sigma dp \le C(M_0, K_0, V_s(0), M^-, K^-, \|\nabla_x \Phi_0\|_{L^\infty}, T).$$

Moreover, in the classical case with $N \in \{2,3\}$ or in the relativistic case with $N = 2$ if there are $F_0, G : [0,+\infty[\to [0,+\infty[$ non increasing functions such that

(iv) $f_0(x,p) \le F_0(|p|)$, $\forall (x,p) \in \Omega \times \mathbb{R}_p^N$, $g(t,x,p) \le G(|p|)$, $\forall (t,x,p) \in]0,T[\times\Sigma^-$,

(v) $\tilde{M}_0 = \int_{\mathbb{R}_p^N} F_0(|p|) \, dp + \int_{\mathbb{R}_p^N} G(|p|) \, dp < +\infty$,

then $E \in L^\infty(]0,T[\times\Omega)^N$, $\rho \in L^\infty(]0,T[\times\Omega)$. *If*

(vi) $\partial_t \Phi_0 \in L^\infty(]0,T[\times\Omega)$,

(vii) $\tilde{M}_1 = \int_{\mathbb{R}_p^N} |v(p)| \cdot F_0(|p|) \, dp + \int_{\mathbb{R}_p^N} |v(p)| \cdot G(|p|) \, dp < +\infty$,

then $\partial_t \Phi \in L^\infty(]0,T[\times\Omega)$, $j \in L^\infty(]0,T[\times\Omega)^N$.

Proof. We truncate the initial-boundary conditions by taking $f_{0,R} = f_0 \cdot \mathbf{1}_{\{|p|\le R\}}$, $g_R = g \cdot \mathbf{1}_{\{|p|\le R\}}$ and regularize the potential on the boundary such that $\varphi_{0,\alpha}\in L^\infty(]0,T[;W^{4m+2-\frac{1}{p_0},p_0}(\partial\Omega))$ $\partial_t\varphi_{0,\alpha}\in L^\infty(]0,T[;W^{4m+1-\frac{1}{p_0},p_0}(\partial\Omega))$ (here $p_0 = \frac{2N}{2N-1}$, $p_0' = 2N$, $\frac{1}{p_0} - \frac{4m}{N} < 0$), $\|\nabla_x\Phi_{0,\alpha}\|_{L^\infty} \le \|\nabla_x\Phi_0\|_{L^\infty}$, $\nabla_x\Phi_{0,\alpha} \rightharpoonup \nabla_x\Phi_0$ weakly $\star$ in $L^\infty(]0,T[\times\Omega)^N$ as $\alpha \searrow 0$, $\nabla_x\Phi_{0,\alpha} \to \nabla_x\Phi_0$ strongly in $L^p(]0,T[\times\Omega)^N$, $1 \le p < +\infty$ as $\alpha \searrow 0$. We denote by $(f_\alpha, \Phi_\alpha = \Phi_{s,\alpha} + \Phi_{0,\alpha})$ the solution of (3.3) constructed at the Proposition 3.5. We have for all $\alpha > 0$:

$$M_\alpha(t) + K_\alpha(t) + V_{s,\alpha}(t) + \int_0^t \{M_\alpha^+(s) + K_\alpha^+(s)\}ds \le M_{0,\alpha} + K_{0,\alpha} + V_{s,\alpha}(0) + \int_0^t \{M_\alpha^-(s) + K_\alpha^-(s)\}ds$$
$$+ C \cdot \|\nabla_x\Phi_0\|_{L^\infty} \cdot \int_0^t (M_\alpha(s) + K_\alpha(s))^{\frac{1}{\beta}}\, ds,\ \ 0 \le t \le T, \tag{5.3}$$

with $\beta = \frac{N+2}{N+1}$ in the classical case and $\beta = \frac{N+1}{N}$ in the relativistic case. Consider $(\alpha_k)_k$ a sequence such that $\lim_{k\to+\infty} \alpha_k = 0$ and keep $R > 0$ fixed. Obviously we have $M_{0,\alpha_k} \le M_0$, $K_{0,\alpha_k} \le K_0$, $M_{\alpha_k}^-(s) \le M^-(s)$, $K_{\alpha_k}^-(s) \le K^-(s)$, $\forall 0 \le s \le T$. Observe that

$$V_{s,\alpha_k}(0) = \frac{1}{2}\int_\Omega \rho_{0,R}(x)\Phi_{s,0}^R(x)dx =: V_{s,0}^R,$$

where $-\Delta_x\Phi_{s,0}^R = \frac{\rho_{0,R}(x)}{\varepsilon_0}$, $x \in \Omega$, $\Phi_{s,0}^R(x) = 0$, $x \in \partial\Omega$. Note also that $0 \le q^{-1}\rho_{0,R} \le q^{-1}\rho_0$ and by the maximum principle we have $0 \le q^{-1}\Phi_{s,0}^R \le q^{-1}\Phi_{s,0}$, $x \in \Omega$ where $-\Delta_x\Phi_{s,0} = \frac{\rho_0}{\varepsilon_0}$, $x \in \Omega$, $\Phi_{s,0}(x) = 0$, $x \in \partial\Omega$. Finally one gets :

$$V_{s,\alpha_k}(0) = \frac{1}{2}\int_\Omega \rho_{0,R}(x)\Phi_{s,0}^R(x)dx \le \frac{1}{2}\int_\Omega \rho_0(x)\Phi_{s,0}(x)dx$$
$$= \frac{1}{2}\int_\Omega\int_{\mathbb{R}_p^N} f_0(x,p)\Phi_{s,0}(x)\, dxdp = V_{s,0} < +\infty,\ \ \forall R > 0. \tag{5.4}$$

From the inequality (5.3) written for $\alpha = \alpha_k$ we deduce that :

$$\limsup_{k\to+\infty} \left\{ \sup_{0\le t\le T} \{M_{\alpha_k}(t) + K_{\alpha_k}(t) + V_{s,\alpha_k}(t)\} + M_{\alpha_k}^+ + K_{\alpha_k}^+ \right\}$$
$$\le C(M_0, K_0, V_{s,0}, M^-, K^-, \|\nabla_x\Phi_0\|_{L^\infty}, T). \tag{5.5}$$

Observe also that we have the following estimates : $(\rho_{\alpha_k})_k$ is bounded in $L^\infty(]0,T[;L^\gamma(\Omega))$, $(j_{\alpha_k})_k$ is bounded in $L^\infty(]0,T[;L^\beta(\Omega))$, $(\Phi_{s,\alpha_k})_k$ is bounded in $L^\infty(]0,T[;W^{2,\gamma}(\Omega))$, $(\partial_t\Phi_{s,\alpha_k})_k$ is bounded in $L^\infty(]0,T[;W^{1,\beta}(\Omega))$, with $\gamma = \frac{N+2}{N} > \frac{N+2}{N+1} = \beta$ in the classical case and $\gamma = \frac{N+1}{N} = \beta$ in the relativistic case. After extraction of subsequences if necessary we deduce that :

$$f_{\alpha_k} \rightharpoonup f,\ \ \text{weakly}\ \ \star\ \ \text{in}\ \ L^\infty(]0,T[\times\Omega\times\mathbb{R}_p^N),$$

$$\gamma^+ f_{\alpha_k} \rightharpoonup \gamma^+ f,\ \ \text{weakly}\ \ \star\ \ \text{in}\ \ L^\infty(]0,T[\times\Sigma^+).$$

By using also a result due to Aubin [2] we can assume that :

$$\nabla_x\Phi_{s,\alpha_k} \to \nabla_x\Phi_s,\ \ \text{strongly in}\ \ L^2(]0,T[;L^\gamma(\Omega)). \tag{5.6}$$

By using the above convergence we can pass easily to the limit for $k \to +\infty$ in the Vlasov equation and we deduce that f is weak solution for :

$$\partial_t f + v(p)\cdot\nabla_x f + q(-\nabla_x\Phi_s - \nabla_x\Phi_0)\cdot\nabla_p f = 0,\ \ (t,x,p) \in]0,T[\times\Omega\times\mathbb{R}_p^N,$$

$$f(0,x,p) = f_{0,R}(x,p),\ \ (x,p) \in \Omega \times \mathbb{R}^N_p,\ \ f(t,x,p) = g_R(t,x,p),\ \ (t,x,p) \in]0,T[\times \Sigma^-.$$

Moreover, the trace of f on $]0,T[\times\Sigma^+$ is $\gamma^+ f$. The passing to the limit for $k \to +\infty$ in the regularized Poisson equation follows immediately by observing that $\rho_{\alpha_k} \rightharpoonup \rho = q\int_{\mathbb{R}^N_p} f(t,x,p)\ dp$ weakly in $L^1(]0,T[\times\Omega)$. Indeed, for $R_1 > 0$, $k \geq 1$ we have :

$$\int_0^T\int_\Omega\int_{|p|>R_1} f_{\alpha_k}\ dtdxdp \leq \frac{1}{R_1}\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} |p|\cdot f_{\alpha_k}\ dtdxdp \leq \frac{C}{R_1}\int_0^T\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}(p))f_{\alpha_k}\ dtdxdp \leq \frac{C}{R_1},$$

and the weak L^1 convergence of $(\rho_{\alpha_k})_k$ follow from the weak $\star$ L^∞ convergence of $(f_k)_k$. The estimates (5.1), (5.2) follows by standard arguments. Note that these estimates are uniform with respect to $R > 0$ and thus it is possible to pass to the limit for $R \to +\infty$ in order to solve the Vlasov-Poisson equations with the initial-boundary conditions f_0 and g. The L^∞ bounds for $\nabla_x\Phi, \partial_t\Phi, \rho$ and j follow by using the L^∞ estimates proved in the paragraph 4.2 for smooth solutions (f_α, Φ_α) and by passing to the limit for $\alpha \searrow 0$, $R \to +\infty$ weakly $\star$ in L^∞.
□

In the following let us give some immediate properties of the solution constructed above.

PROPOSITION 5.2. *Under the hypotheses* $(i), (ii), (iii), (iv), (v)$ *of Theorem* 5.1 *the weak solution constructed before satisfies*
(1) *the application* $t \to \int_\Omega\int_{\mathbb{R}^N_p} f\ dxdp$ *is absolutely continuous for* $t \in [0,T]$ *and :*

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^N_p} f\ dxdp + \int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\ d\sigma dp = \int_{\Sigma^-}|(v(p)\cdot n(x))|g\ d\sigma dp,\quad a.e.\ t \in]0,T[\ ; \quad (5.7)$$

(2) *the application* $t \to \int_\Omega\int_{\mathbb{R}^N_p}\mathcal{E}(p)f\ dxdp + \frac{\varepsilon_0}{2}\int_\Omega|\nabla_x\Phi_s|^2\ dx$ *is absolutely continuous for* $t \in [0,T]$ *and :*

$$\frac{d}{dt}\left\{\int_\Omega\int_{\mathbb{R}^N_p}\mathcal{E}(p)f\ dxdp + \frac{\varepsilon_0}{2}\int_\Omega|\nabla_x\Phi_s|^2\ dx\right\} + \int_{\Sigma^+}(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f\ d\sigma dp = -\int_\Omega\nabla_x\Phi_0\cdot j\ dx$$
$$+\int_{\Sigma^-}|(v(p)\cdot n(x))|\mathcal{E}(p)g\ d\sigma dp,\quad a.e.\ t \in]0,T[. \quad (5.8)$$

Proof. Indeed, recall that the weak solution (f,E) was obtained as $(f,E) = \lim_{R\to+\infty}(f_R, E_R)$ with $(f_R, E_R) = (f_R, -\nabla_x\Phi_s^R - \nabla_x\Phi_0) = \lim_{\alpha\searrow 0}(f_{\alpha,R}, E_{\alpha,R})$, where $(f_{\alpha,R}, E_{\alpha,R})$ is solution of (3.3) with the initial-boundary conditions $f_{0,R}, g_R, \varphi_{0,\alpha}$ (observe that (f_R, E_R) is solution of the Vlasov-Poisson system with the initial-boundary conditions $f_{0,R}, g_R, \varphi_0$). For the moment we keep $R > 0$ fixed and write the analogous of (5.7), (5.8) for the smooth solutions $(f_{\alpha,R}, E_{\alpha,R}) = (f_\alpha, E_\alpha)$ which are uniformly compactly supported in momentum with respect to $\alpha > 0$:

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^N_p} f_\alpha\ dxdp + \int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f_\alpha\ d\sigma dp = \int_{\Sigma^-}|(v(p)\cdot n(x))|g_R\ d\sigma dp,\ \ \text{a.e. } t \in]0,T[. \quad (5.9)$$

Similarly the application $t \to \int_\Omega\int_{\mathbb{R}^N_p}\mathcal{E}(p)f_\alpha\ dxdp + \frac{q}{2}\int_\Omega\int_{\mathbb{R}^N_p} f_\alpha(t,x,p)\Phi_{s,\alpha}(t,x)\ dxdp$ is absolutely continuous for $t \in [0,T]$ and :

$$\frac{d}{dt}\left\{\int_\Omega\int_{\mathbb{R}^N_p}(\mathcal{E}(p) + \frac{q}{2}\Phi_{s,\alpha}(t,x))f_\alpha\ dxdp\right\} + \int_{\Sigma^+}(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f_\alpha\ d\sigma dp = -\int_\Omega\nabla_x\Phi_{0,\alpha}\cdot j_\alpha\ dx$$
$$+\int_{\Sigma^-}|(v(p)\cdot n(x))|\mathcal{E}(p)g_R\ d\sigma dp,\ \ \text{a.e. } t \in]0,T[. \quad (5.10)$$

By passing to the limit for $\alpha \searrow 0$ in (5.9) we deduce that :

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^N_p} f_R\ dxdp + \int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f_R\ d\sigma dp = \int_{\Sigma^-}|(v(p)\cdot n(x))|g_R\ d\sigma dp,\ \ \text{a.e. } t \in]0,T[. \quad (5.11)$$

The passing to the limit for $\alpha \searrow 0$ in (5.10) is a little more complicated. For $\theta \in \mathcal{D}([0,T[)$ we have :

$$-\theta(0) \quad \int_\Omega \int_{\mathbb{R}^N_p} (\mathcal{E}(p) + \frac{q}{2}\Phi^R_{s,0}(x)) f_{0,R}\, dxdp - \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} \theta'(t)(\mathcal{E}(p) + \frac{q}{2}\Phi_{s,\alpha}(t,x)) f_\alpha\, dtdxdp$$
$$+ \int_0^T\!\!\int_{\Sigma^+} \theta(t)(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f_\alpha\, dtd\sigma dp$$
$$= \int_0^T\!\!\int_{\Sigma^-} \theta(t)|(v(p)\cdot n(x))|\mathcal{E}(p) g_R\, dtd\sigma dp - \int_0^T \int_\Omega \theta(t)\nabla_x \Phi_{0,\alpha}\cdot j_\alpha(t,x)\, dtdx. \qquad (5.12)$$

Since $(f_\alpha)_{\alpha>0}$ are uniformly compactly supported in momentum we deduce also that :

$$\lim_{\alpha\searrow 0} \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} \theta'(t)\mathcal{E}(p) f_\alpha\, dtdxdp = \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} \theta'(t)\mathcal{E}(p) f_R\, dtdxdp,$$

$$\lim_{\alpha\searrow 0} \int_0^T\!\!\int_{\Sigma^+} \theta(t)(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f_\alpha\, dtd\sigma dp = \int_0^T\!\!\int_{\Sigma^+} \theta(t)(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f_R\, dtd\sigma dp.$$

In order to pass to the limit in the term $\int_0^T\!\int_\Omega \int_{\mathbb{R}^N_p} \theta(t)\nabla_x\Phi_{0,\alpha}\cdot j_\alpha\, dtdxdp$ we can combine the weak convergence $j_\alpha \rightharpoonup j_R$ weakly in $L^1(]0,T[\times\Omega)^N$, the uniform bound of j_α in $L^\infty(]0,T[;L^\beta(\Omega))^N$ and the strong convergence $\nabla_x\Phi_{0,\alpha} \to \nabla_x\Phi_0$ strongly in $L^r(]0,T[\times\Omega)^N$, $\forall 1 < r < +\infty$ (for example $r = \beta'$). In order to pass to the limit in the term $\int_0^T\!\int_\Omega \int_{\mathbb{R}^N_p} \theta'(t)\Phi_{s,\alpha} q f_\alpha\, dtdxdp = \int_0^T \int_\Omega \theta'(t)\Phi_{s,\alpha}\rho_\alpha\, dtdx$ combine the weak convergence $\rho_\alpha \rightharpoonup \rho_R$ in $L^1(]0,T[\times\Omega)$, the uniform bounds of ρ_α, Φ_α in $L^\infty(]0,T[\times\Omega)$ and the strong convergence $\Phi_{s,\alpha} \to \Phi^R_s$ in $L^2(]0,T[;W^{1,\gamma}(\Omega))$. After passing to the limit in (5.12) we deduce that :

$$-\theta(0) \quad \int_\Omega \int_{\mathbb{R}^N_p} (\mathcal{E}(p) + \frac{q}{2}\Phi^R_{s,0}(x)) f_{0,R}\, dxdp - \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} \theta'(t)(\mathcal{E}(p) + \frac{q}{2}\Phi^R_s(t,x)) f_R\, dtdxdp$$
$$+ \int_0^T\!\!\int_{\Sigma^+} \theta(t)(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f_R\, dtd\sigma dp$$
$$= \int_0^T\!\!\int_{\Sigma^-} \theta(t)|(v(p)\cdot n(x))|\mathcal{E}(p) g_R\, dtd\sigma dp - \int_0^T \int_\Omega \theta(t)\nabla_x \Phi_0\cdot j_R(t,x)\, dtdx. \qquad (5.13)$$

In order to prove (5.7), (5.8) we need to pass to the limit for $R \to +\infty$ in (5.11), (5.13). The proof is similar and is left to the reader. Note that $(f_R)_{R>0}$ are not anymore uniformly compactly supported in momentum but we can prove that :

$$\sup_{0\le t\le T} \int_\Omega \int_{\mathbb{R}^N_p} (1+\mathcal{E}(p)) f_R \cdot \mathbf{1}_{\{|p|>R_1\}}\, dxdp \to 0, \;\; \text{as } R_1 \to +\infty, \qquad (5.14)$$

$$\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f_R \cdot \mathbf{1}_{\{|p|>R_1\}}\, dtd\sigma dp \to 0, \;\; \text{as } R_1 \to +\infty, \qquad (5.15)$$

uniformly with respect to the solution f_R. For this take $\chi \in C^\infty_c([0,+\infty[)$, $0 \le \chi \le 1$, $\chi(u) = 1, 0 \le u \le \frac{1}{2}$, $\chi(u) = 0, u \ge 1$ and multiply the Vlasov equation by $(1-\chi_{R_1}(|p|))\cdot(1+\mathcal{E}(p))$, where $\chi_{R_1}(\cdot) = \chi(\cdot/R_1)$. After easy computations (involving the L^∞ bound for the electric field E_R) we find (5.14), (5.15) which implies that

$$\lim_{R\to+\infty} (1+\mathcal{E}(p)) f_R = (1+\mathcal{E}(p)) f, \;\; \text{weakly in } \; L^1(]0,T[\times\Omega\times\mathbb{R}^N_p),$$

and

$$\lim_{R\to+\infty} (v(p)\cdot n(x))(1+\mathcal{E}(p))f_R = (v(p)\cdot n(x))(1+\mathcal{E}(p))f, \text{ weakly in } L^1(]0,T[\times\Sigma^+).$$

The passing to the limit for $R\to+\infty$ in (5.11), (5.13) follows now easily by using the above weak convergence. Observe also that by passing to the L^∞ weak $\star$ limit $f=\lim_{R\to+\infty} f_R$ we have :

$$\lim_{R_1\to+\infty} \operatorname{ess\,sup}_{0<t<T} \int_\Omega\int_{\mathbb{R}^N_p} (1+\mathcal{E}(p))f\cdot \mathbf{1}_{\{|p|>R_1\}}\, dxdp = 0. \tag{5.16}$$

□

Another direct consequence of Theorem 5.1 is the propagation of the moments.

PROPOSITION 5.3. *Under the hypotheses* $(i),(ii),(iii),(iv),(v)$ *of Theorem* 5.1 *with* $1\le N\le 3$ *in the classical case and* $1\le N\le 2$ *in the relativistic case denote by* $(f, E=-\nabla_x\Phi_s-\nabla_x\Phi_0)$ *the solution constructed previously. Suppose also that for some* m *such that* $m>2$ *in the classical case and* $m>1$ *in the relativistic case the initial-boundary conditions verify :*

$$\int_\Omega\int_{\mathbb{R}^N_p} |p|^m\cdot f_0(x,p)\, dxdp + \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|\cdot|p|^m\cdot g(t,x,p)\, dtd\sigma dp < +\infty. \tag{5.17}$$

Then we have :

$$\left\|\int_\Omega\int_{\mathbb{R}^N_p} |p|^m\cdot f(\cdot,x,p)\, dxdp\right\|_{L^\infty(]0,T[)} + \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\cdot|p|^m\cdot\gamma^+ f(t,x,p)\, dtd\sigma dp < +\infty. \tag{5.18}$$

Proof. It is sufficient to prove (5.18) for smooth solutions. The conclusion follows easily by observing that for $r=m, m-1, \ldots$ we have :

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^N_p} |p|^r\cdot f_\alpha(t,x,p)\, dxdp + \int_\Sigma (v(p)\cdot n(x))\cdot|p|^r\cdot\gamma f_\alpha\, d\sigma dp\\
&= \int_\Omega\int_{\mathbb{R}^N_p} q\cdot f_\alpha(t,x,p)\cdot r\cdot|p|^{r-2}(E_\alpha(t,x)\cdot p)\, dxdp\\
&\le |q|\cdot r\cdot\|E_\alpha\|_{L^\infty}\int_\Omega\int_{\mathbb{R}^N_p} |p|^{r-1}\cdot f_\alpha(t,x,p)\, dxdp.
\end{aligned}$$

□

PROPOSITION 5.4. *Under the hypotheses* $(i),(ii),(iii),(iv),(v)$ *of Theorem* 5.1 *with* $1\le N\le 3$ *in the classical case and* $1\le N\le 2$ *in the relativistic case we suppose also that for some* $m>0$ *we have:*

$$\tilde{M}_m := \int_{\mathbb{R}^N_p} |p|^m\cdot F_0(|p|)\, dp + \int_{\mathbb{R}^N_p} |p|^m\cdot G(|p|)\, dp < +\infty. \tag{5.19}$$

Then we have :

$$\left\|\int_{\mathbb{R}^N_p} |p|^m\cdot f(\cdot,\cdot,p)\, dp\right\|_{L^\infty(]0,T[\times\Omega)} + \left\|\int_{\mathbb{R}^N_p} |p|^m\cdot\gamma f(\cdot,\cdot,p)\, dp\right\|_{L^\infty(]0,T[\times\partial\Omega)} < +\infty. \tag{5.20}$$

Proof. Write $\int_{\mathbb{R}^N_p} |p|^m\cdot f(t,x,p)\, dp = \int_{|p|\le 4D}\{\ldots\, dp\} + \int_{|p|>4D}\{\ldots\, dp\}$ and continue as it was done for the cases $m=0$, $m=1$.

□

5.1. The time periodic case.

We end this paper by considering permanent regimes. We assume that the boundary data g, φ_0 are T periodic and under natural hypotheses we construct weak solutions for the Vlasov-Poisson system with bounded electric field. We consider the classical case. First of all let us deduce bounds for the total mass and energy by performing formal computations (for more details see [8]). We assume that the boundary conditions verify :

$$0 \leq g \in L^\infty(\mathbb{R}_t \times \Sigma^-), \quad \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\ dtd\sigma dp < +\infty,$$

$$\varphi_0 \in L^2(]0,T[;H^1(\partial\Omega)), \quad \nabla_x \Phi_0 \in L^\infty(\mathbb{R}_t \times \Omega),$$

where Φ_0 is the exterior potential ($-\Delta_x \Phi_0 = 0, \quad (t,x) \in \mathbb{R}_t \times \Omega,\ \Phi_0 = \varphi_0, \quad (t,x) \in \mathbb{R}_t \times \partial\Omega$). Consider $(f, \Phi = \Phi_s + \Phi_0)$ a T periodic smooth solution with compact support in momentum. The conservations of the mass and kinetic energy give :

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)\ dxdp + \int_\Sigma (v(p)\cdot n(x))\gamma f(t,x,p)\ d\sigma dp = 0, \quad t \in \mathbb{R}_t, \tag{5.21}$$

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^N_p} \mathcal{E}(p) f(t,x,p)\ dxdp + \int_\Sigma (v(p)\cdot n(x))\mathcal{E}(p)\gamma f(t,x,p)\ d\sigma dp = \int_\Omega\int_{\mathbb{R}^N_p} q(E(t,x)\cdot v(p))f\ dxdp$$
$$= -\int_\Omega j(t,x)\cdot(\nabla_x\Phi_s + \nabla_x\Phi_0)\ dx, \quad t\in\mathbb{R}_t. \tag{5.22}$$

After multiplying the Vlasov equation by $q\Phi_s$ and by using the Poisson equation we find as before

$$\frac{d}{dt}\left\{\int_\Omega\int_{\mathbb{R}^N_p} \mathcal{E}(p)f(t,x,p)\ dxdp + \frac{1}{2}\int_\Omega \rho(t,x)\Phi_s(t,x)\ dx\right\} + \int_\Sigma (v(p)\cdot n(x))\mathcal{E}(p)\gamma f(t,x,p)\ d\sigma dp$$
$$= -\int_\Omega j(t,x)\cdot\nabla_x\Phi_0\ dx. \tag{5.23}$$

After integration on $]0,T[$ we deduce that :

$$\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f(t,x,p)\ dtd\sigma dp = \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))| g(t,x,p)\ dtd\sigma dp, \tag{5.24}$$

and :

$$\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f\ dtd\sigma dp = \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|\mathcal{E}(p) g\ dtd\sigma dp - \int_0^T\!\!\int_\Omega j(t,x)\cdot\nabla_x\Phi_0\ dtdx. \tag{5.25}$$

We multiply the Vlasov equation by $(p\cdot x)$ and we suppose that $\partial\Omega$ is strictly star-shaped with respect to $0\in\Omega$ *i.e.,* $\exists r>0$ such that $r \leq (n(x)\cdot x)\ \forall x\in\partial\Omega$. We obtain :

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^N_p}(p\cdot x) f\ dxdp + \int_\Sigma (v(p)\cdot n(x))(p\cdot x)\gamma f\ d\sigma dp = \int_\Omega\int_{\mathbb{R}^N_p}(v(p)\cdot p) f\ dxdp + \int_\Omega\int_{\mathbb{R}^N_p} q(E\cdot x) f\ dxdp$$
$$= \int_\Omega\int_{\mathbb{R}^N_p}(v(p)\cdot p)f\ dxdp + \int_\Omega \rho(E\cdot x)\ dx. \tag{5.26}$$

We use the identity :

$$E_i \mathrm{div}\ E = \sum_{j=1}^N \frac{\partial}{\partial x_j}(E_iE_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|E|^2, \quad \forall 1\leq i\leq N, \tag{5.27}$$

if $\frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$, $\forall 1 \leq i,j \leq N$. After integration by parts and by using the decomposition $E = (E\cdot n)n + E_\tau$, $(t,x) \in \mathbb{R}_t \times \partial\Omega$ we find :

$$\begin{aligned}\int_\Omega (E\cdot x)\text{div } E \, dx &= \int_\Omega \sum_{i=1}^N x_i \left\{ \sum_{j=1}^N \frac{\partial}{\partial x_j}(E_i E_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|E|^2 \right\} dx \\ &= \left(\frac{N}{2} - 1\right)\int_\Omega |E(t,x)|^2 \, dx + \frac{1}{2}\int_{\partial\Omega}(n(x)\cdot x)(E\cdot n)^2 \, d\sigma \\ &+ \int_{\partial\Omega}(E_\tau \cdot x)\cdot (E\cdot n(x)) \, d\sigma - \frac{1}{2}\int_{\partial\Omega}(n(x)\cdot x)\cdot |E_\tau|^2 \, d\sigma. \qquad (5.28)\end{aligned}$$

By using (5.26), (5.28) we deduce that :

$$\begin{aligned}&\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} \mathcal{E}(p)f \, dtdxdp + \varepsilon_0\left(\frac{N}{2}-1\right)\int_0^T\int_\Omega |E|^2 \, dtdx + \frac{\varepsilon_0 r}{2}\int_0^T\int_{\partial\Omega}(E\cdot n(x))^2 \, dtd\sigma \\ &\leq \int_0^T\int_\Sigma (v(p)\cdot n(x))(p\cdot x)\gamma f \, dtd\sigma dp + \frac{\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}(n(x)\cdot x)\cdot |E_\tau|^2 \, dtd\sigma \\ &- \varepsilon_0 \int_0^T\int_{\partial\Omega}(E_\tau\cdot x)\cdot(E\cdot n(x)) \, dtd\sigma. \qquad (5.29)\end{aligned}$$

Observe that $\|E_\tau\|_{L^2(]0,T[\times\partial\Omega)} \leq C\cdot\|\varphi_0\|_{L^2(]0,T[;H^1(\partial\Omega))}$ and from (5.24), (5.25) note that :

$$\begin{aligned}\left|\int_0^T\int_\Sigma (v(p)\cdot n(x))(p\cdot x)\gamma f \, dtd\sigma dp\right| &\leq C\cdot \int_0^T\int_\Sigma |(v(p)\cdot n(x))|(1+\mathcal{E}(p))\gamma f \, dtd\sigma dp \\ &\leq C\cdot\int_0^T\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p))g \, dtd\sigma dp \\ &+ C\cdot\|\nabla_x\Phi_0\|_{L^\infty}\cdot\int_0^T\int_\Omega |j(t,x)| \, dtdx. \qquad (5.30)\end{aligned}$$

By using interpolation inequalities and (5.29), (5.30) we obtain bounds for :

$$\begin{aligned}\int_0^T\int_\Omega\int_{\mathbb{R}_p^N}\mathcal{E}(p)f \, dtdxdp + \frac{\varepsilon_0}{2}\int_0^T\int_\Omega |E|^2 \, dtdx + \int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f \, dtd\sigma dp \\ + \frac{\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}(E\cdot n)^2 \, dtd\sigma \leq C,\end{aligned}$$

for the case $N > 2$. In the case $N = 2$ we obtain bounds only for

$$W = \int_0^T\int_\Omega\int_{\mathbb{R}_p^N}\mathcal{E}(p)f \, dtdxdp + \frac{\varepsilon_0}{2}\int_0^T\int_{\partial\Omega}(E\cdot n)^2 \, dtd\sigma + \int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f \, dtd\sigma dp.$$

By interpolation inequalities we have $\|\rho\|_{L^2(]0,T[\times\Omega)} \leq C$ and therefore

$$\int_0^T\int_\Omega |\nabla_x\Phi_s|^2 \, dtdx \leq C\cdot\int_0^T\int_\Omega \rho^2 \, dtdx \leq C.$$

In fact the total energy is uniformly bounded in time. Indeed, since $\int_0^T\{K(t) + V_s(t)\} \, dt \leq C$, there is t_0 such that $K(t_0) + V_s(t_0) \leq \frac{C}{T}$ and we can propagate the total energy for $t \in [t_0, t_0 + T]$. Suppose also that there is $G : [0,+\infty[\to [0,+\infty[$ non increasing such that

$$g(t,x,p) \leq G(|p|), \quad \forall (t,x,p) \in \mathbb{R}_t \times \Sigma^-,$$

and

$$\tilde{M}^- := \int_{\mathbb{R}^N_p} G(|p|)\,dp < +\infty.$$

By using the method presented at the paragraph 4.2 we deduce a bound for the L^∞ norm of the electric field and the charge density in the cases $N \in \{2,3\}$. The one dimensional case was studied in [7]. In this case we write :

$$\|E_s(t)\|_{L^\infty} \le C \cdot \|\rho(t)\|_{L^1} \le C \cdot \|\rho_1(t)\|_{L^1} + C \cdot \|\rho_2(t)\|_{L^1}, \tag{5.31}$$

where $\rho_1(t,x) = q \cdot \int_{|p|\le 4D} f(t,x,p)\,dp$ and $\rho_2(t,x) = q \cdot \int_{|p|>4D} f(t,x,p)\,dp$. For the first charge density we have :

$$\|\rho_1(t)\|_{L^1} \le C \cdot \|\rho_1(t)\|_{L^\infty} \le C \cdot \|f\|_{L^\infty} \cdot D \le C \cdot \|E\|^{\frac{1}{2}}_{L^\infty}, \tag{5.32}$$

and for the second charge density we have as usual :

$$\|\rho_2(t)\|_{L^1} \le C \cdot \|\rho_2(t)\|_{L^\infty} \le C \cdot \int_{\mathbb{R}_p} G(|p|)\,dp. \tag{5.33}$$

From (5.31), (5.32), (5.33) we obtain a bound for the L^∞ norm of E and ρ.
A direct consequence of the L^∞ bound for the electric field is the existence of weak solution for the time periodic Vlasov-Poisson system with particle distribution compactly supported in momentum, when the boundary condition has compact support in momentum *i.e.,* $\exists R > 0$ such that $g = g \cdot \mathbf{1}_{\{|p|\le R\}}$ (cf. Theorem 2.20).

6. Appendix.

We give here the proof of momentum change lemmas for the classical and relativistic cases.

6.1. The classical case.

We will need the following easy lemma :

LEMMA 6.1. *Consider the quadratic function $F : \mathbb{R} \to \mathbb{R}$ given by $F(s) = \frac{1}{2}a(s-s_1)^2 - b(s-s_1) + c$, with $a, b, c > 0, \Delta = b^2 - 2ac > 0$ and $s_1 \le s_2$ such that $F(s) \ge 0\ \forall s_1 \le s \le s_2$. Then we have $s_2 - s_1 \le (b - \sqrt{\Delta})/a \le 2c/b$.*

Proof. Without loss of generality we can suppose that $s_1 = 0$. The equation $F(s) = 0$ has two positive real roots $r_{1,2} = (b \mp \sqrt{\Delta})/a,\ 0 < r_1 < r_2$. Since $a > 0$ we have $F(s) < 0\ \forall r_1 < s < r_2$. Suppose that $s_2 > r_1$ and consider $s_0 \in [0, s_2] \cap]r_1, r_2[\ne \emptyset$. Thus, since $0 \le s_0 \le s_2$ by the hypothesis we have $F(s_0) \ge 0$. On the other hand, since $r_1 < s_0 < r_2$ we have $F(s_0) < 0$. Therefore $s_2 > r_1$ is not possible and we get that $s_2 \le r_1 = (b - \sqrt{\Delta})/a \le 2c/b$. □

REMARK 6.2. *If $a = 0$ we still have the inequalities $s_2 - s_1 \le c/b < 2c/b$.*

COROLLARY 6.3. *Consider the function $F_1 : \mathbb{R} \to \mathbb{R}$ given by $F_1(s) = \frac{1}{2}a(s-t)^2 - b|s-t| + c$ with $a \ge 0, b, c > 0, \Delta = b^2 - 2ac > 0$ and $s_1 \le t \le s_2$ such that $F_1(s) \ge 0\ \forall s_1 \le s \le s_2$. Then we have* $\max\{t - s_1, s_2 - t\} \le 2c/b$ *and* $s_2 - s_1 \le 4c/b$.

Proof. Consider $F(r) = \frac{1}{2}ar^2 - br + c$. Observe that $F(r) \ge 0\ \forall\ 0 \le r \le \max\{t - s_1, s_2 - t\}$. The conclusion follows by applying the Lemma 6.1. □

Proof. (Lemma 2.7)
(1) Let us consider for $s_{in} \le s \le s_{out}$:

$$M(s) = \frac{q}{m}\begin{pmatrix} 0 & B_3(s,X(s)) & -B_2(s,X(s)) \\ -B_3(s,X(s)) & 0 & B_1(s,X(s)) \\ B_2(s,X(s)) & -B_1(s,X(s)), & 0 \end{pmatrix}$$

We have :

$$\|M(s)\| = \sup_{p\in\mathbb{R}^3_p} \frac{|M(s)\cdot p|}{|p|} = \frac{|q|}{m}\sup_{p\in\mathbb{R}^3_p}\frac{|p\wedge B(s)|}{|p|} \le \frac{|q|}{m}\|B\|_\infty, \ \ \forall s_{in} \le s \le s_{out}.$$

Denote by $R(s;t)$ the resolvent for $\frac{\partial R}{\partial s}(s;t) = M(s)R(s;t)$ with $R(s=t;t) = I$. Since $M(s)$ is antisymmetric we have $\|R(s;t)\| = 1$, $\forall s_{in} \le s \le s_{out}$ (in fact $R(s;t)$ is orthogonal) and therefore we have :

$$\|R(s;t) - I\| \le |s-t|\cdot\|M(\cdot)\|_\infty \le |s-t|\cdot\frac{|q|}{m}\cdot\|B\|_\infty.$$

By (2.7) we have $P(s) = R(s;t)P(t) + q\int_t^s R(s;\tau)E(\tau,X(\tau))d\tau$, $\forall s_{in} \le s \le s_{out}$, and therefore we obtain that :

$$|P(s) - P(t)| \le |s-t|\cdot\frac{|q|}{m}\cdot\|B\|_\infty\cdot|P(t)| + |q|\cdot|s-t|\cdot\|E\|_\infty. \tag{6.1}$$

We use now the equation $\frac{dX}{ds} = \frac{P(s)}{m}$ and (6.1) to obtain :

$$\begin{aligned} diam(\Omega) \ge & \left|\left(X(s)-X(t), \frac{P(t)}{|P(t)|}\right)\right| = \left|\int_t^s \left(\frac{P(\tau)}{m}, \frac{P(t)}{|P(t)|}\right) d\tau\right| \\ \ge & \left|\int_t^s \frac{|P(t)|}{m} d\tau\right| - \left|\int_t^s \left(\frac{P(\tau)-P(t)}{m}, \frac{P(t)}{|P(t)|}\right) d\tau\right| \\ \ge & \frac{1}{m}|s-t|\cdot|P(t)| - \frac{1}{2m}\cdot|s-t|^2\left(\frac{|q|}{m}\cdot\|B\|_\infty\cdot|P(t)| + |q|\cdot\|E\|_\infty\right). \end{aligned} \tag{6.2}$$

Denote by $F_1 : \mathbb{R} \to \mathbb{R}$ the function given by :

$$F_1(s) = \frac{1}{2}|s-t|^2\left(\frac{|q|}{m}\cdot\|B\|_\infty\cdot|P(t)| + |q|\cdot\|E\|_\infty\right) - |s-t|\cdot|P(t)| + m\cdot diam(\Omega).$$

The discriminant is :

$$\begin{aligned} \Delta = & |P(t)|^2 - 2\cdot\left(\frac{|q|}{m}\cdot\|B\|_\infty\cdot|P(t)| + |q|\cdot\|E\|_\infty\right)\cdot m\cdot diam(\Omega) \\ = & (|P(t)| - |q|\cdot\|B\|_\infty\cdot diam(\Omega))^2 - \left(|q|^2\cdot\|B\|^2_\infty\cdot diam(\Omega)^2 + 2|q|\cdot\|E\|_\infty\cdot m\cdot diam(\Omega)\right) > 0, \end{aligned}$$

since $|P(t)| > D_{cla}$. By (6.2) we have that $F_1(s) \ge 0$, $\forall s_{in} \le s \le s_{out}$ and thus by applying the *Corollary* 6.3 we deduce that $\max\{t - s_{in}, s_{out} - t\} \le 2\cdot m\cdot diam(\Omega)/|P(t)|$ and $s_{out} - s_{in} \le 4\cdot m\cdot diam(\Omega)/|P(t)| \le 4\cdot m\cdot diam(\Omega)/D_{cla}$. Using one more time (6.1) we deduce that for all $s_{in} \le s \le s_{out}$ we have :

$$\begin{aligned} |P(s) - P(t)| \le & |s-t|\left(\frac{|q|}{m}\cdot\|B\|_\infty\cdot|P(t)| + |q|\cdot\|E\|_\infty\right) \\ \le & \frac{2\cdot m\cdot diam(\Omega)}{|P(t)|}\left(\frac{|q|}{m}\cdot\|B\|_\infty\cdot|P(t)| + |q|\cdot\|E\|_\infty\right) \\ \le & 2|q|\cdot\|B\|_\infty\cdot diam(\Omega) + \frac{2}{D_{cla}}\cdot|q|\cdot\|E\|_\infty\cdot m\cdot diam(\Omega) \\ \le & D_{cla}. \end{aligned}$$

We deduce that $|P(s_1) - P(s_2)| \le 2D_{cla}, \ \forall s_{in} \le s_1 \le s_2 \le s_{out}$.

(2) If $|P(s_1)| \le D_{cla}$ and $|P(s_2)| \le D_{cla}$ we have $|P(s_1) - P(s_2)| \le 2D_{cla}$. If $|P(s_1)| > D_{cla}$, by applying the previous point for $t = s_1$ we deduce that $|P(s_2) - P(s_1)| \le D_{cla} \le 2D_{cla}, \ \forall s_2$. If $|P(s_2)| > D_{cla}$ we apply the previous point with $t = s_2$. □

6.2. The relativistic case.

Let us establish some preliminary properties concerning the function $v(p), \ p \in \mathbb{R}^N_p$.

LEMMA 6.4. *Consider* $v : \mathbb{R}^N_p \to \mathbb{R}^N$ *given by* $v(p) = (p/m) \cdot (1 + |p|^2/(mc_0)^2)^{-1/2}$. *Then we have :*
(1) $|v(p)| \le c_0, \ \forall p \in \mathbb{R}^N_p$;
(2) $(v(p_1) - v(p_2), p_1 - p_2) > 0, \ \forall p_1 \ne p_2$;
(3) $|v(p_1) - v(p_2)|^2 \le \frac{N}{m^2}|p_1 - p_2|^2 \int_0^1 (1 + |tp_1 + (1-t)p_2|^2/(mc_0)^2)^{-1} dt, \ \forall p_1, p_2 \in \mathbb{R}^N_p$;
(4) $|v(p_1) - v(p_2)| \le \frac{2\sqrt{N}}{m} \cdot |p_1 - p_2| \cdot (1 + |p_1|^2/(mc_0)^2)^{-1/2}$, *if* $|p_1 - p_2| \le |p_1|/2, \ \forall p_1, p_2 \in \mathbb{R}^N_p$.

Proof. (1) is obvious. For the point (2) consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(u) = mc_0^2((1 + u^2/(m^2c_0^2))^{1/2} - 1)$ and check that φ is strictly convex on $\mathbb{R}$ and strictly increasing on $[0, +\infty[$. We deduce that $\mathcal{E}(p)$ is strictly convex on $\mathbb{R}^N_p$. Indeed, for $\lambda \in]0, 1[$ we have :

$$
\begin{aligned}
\mathcal{E}(\lambda p_1 + (1-\lambda)p_2) &= \varphi(|\lambda p_1 + (1-\lambda)p_2|) \\
&\le \varphi(\lambda|p_1| + (1-\lambda)|p_2|) \le \lambda\varphi(|p_1|) + (1-\lambda)\varphi(|p_2|) \\
&\le \lambda\mathcal{E}(p_1) + (1-\lambda)\mathcal{E}(p_2),
\end{aligned}
$$

with equality iff $|\lambda p_1 + (1-\lambda)p_2| = \lambda|p_1| + (1-\lambda)|p_2|$ and $|p_1| = |p_2|$, which means iff $p_1 = p_2$. Therefore we have for $p_1 \ne p_2$ that $(\nabla_p\mathcal{E}(p_1) - \nabla_p\mathcal{E}(p_2), p_1 - p_2) > 0$ or $(v(p_1) - v(p_2), p_1 - p_2) > 0$. The point (3) follows by direct computation by writing $v(p_1) - v(p_2) = \int_0^1 \nabla_p v(tp_1 + (1-t)p_2) \cdot (p_1 - p_2) dt$. For (4) we write :

$$
|tp_1 + (1-t)p_2| \ge |p_1| - (1-t)|p_1 - p_2| \ge |p_1| - |p_1 - p_2| \ge \frac{|p_1|}{2}, \ \forall t \in [0, 1],
$$

and the conclusion follows by (3). □

Proof. (Lemma 2.8)
We have $P(s) = P(t) + q\int_t^s E(\tau, X(\tau)) d\tau$ and we deduce that :

$$
|P(s) - P(t)| \le |q| \cdot |s - t| \cdot \|E\|_\infty \le \frac{|P(t)|}{2}, \ \ s_{in} \le s \le s_{out}, \ |s - t| \le \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty}.
$$

Note that if $\|E\|_\infty = 0$ the above inequality holds $\forall s \in [s_{in}, s_{out}]$. By Lemma 6.4 we have :

$$
\begin{aligned}
|v(P(s)) - v(P(t))| &\le \frac{2\sqrt{N}}{m} \cdot |P(s) - P(t)| \cdot \left(1 + \frac{|P(t)|^2}{m^2c_0^2}\right)^{-1/2}, \\
&\le \frac{2\sqrt{N}}{m} \cdot |q| \cdot \|E\|_\infty \cdot |s - t| \cdot \left(1 + \frac{|P(t)|^2}{m^2c_0^2}\right)^{-1/2}, \ \ \forall r_1 \le s \le r_2, \qquad (6.3)
\end{aligned}
$$

where $r_1 = \max\{s_{in}, t - \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty}\}$, $r_2 = \min\{s_{out}, t + \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty}\}$ if $\|E\|_\infty > 0$ and $r_1 = s_{in}, \ r_2 =$

s_{out} if $\|E\|_\infty = 0$. By using the equation $\frac{dX}{ds} = v(P(s))$ and (6.3) we find for $r_1 \leq s \leq r_2$ that :

$$\begin{aligned} diam(\Omega) \geq& \left| \left(X(s) - X(t), \frac{v(P(t))}{|v(P(t))|} \right) \right| = \left| \int_t^s \left(v(P(\tau)), \frac{v(P(t))}{|v(P(t))|} \right) d\tau \right| \\ \geq& \left| \int_t^s \left(v(P(t)), \frac{v(P(t))}{|v(P(t))|} \right) d\tau \right| - \left| \int_t^s \left(v(P(\tau)) - v(P(t)), \frac{v(P(t))}{|v(P(t))|} \right) d\tau \right| \\ \geq& |s-t| \cdot |v(P(t))| - \left| \int_t^s |v(P(\tau)) - v(P(t))| d\tau \right| \\ \geq& |s-t| \cdot |v(P(t))| - \frac{\sqrt{N} \cdot |q| \cdot \|E\|_\infty}{m} |s-t|^2 \left(1 + \frac{|P(t)|^2}{m^2 c_0^2} \right)^{-1/2}. \end{aligned}$$

We consider also the function $F_1(s) = \frac{1}{2}|s-t|^2 \cdot 2\sqrt{N} \cdot |q| \cdot \|E\|_\infty \left(1 + \frac{|P(t)|^2}{m^2 c_0^2}\right)^{-1/2} - |s-t| \cdot |P(t)| \cdot \left(1 + \frac{|P(t)|^2}{m^2 c_0^2}\right)^{-1/2} + m \cdot diam(\Omega)$. By the above computations we have $F_1(s) \geq 0$, $\forall r_1 \leq s \leq r_2$. Moreover, the condition $\Delta > 0$ is equivalent to $\alpha^2 > \beta\sqrt{1+\alpha^2}$ where $\alpha = |P(t)|/(mc_0)$. The previous inequality can be written also $(\alpha^2 - \beta^2/2)^2 > \beta^2 + \beta^4/4$ and thus $\Delta > 0$ if $\alpha^2 > \beta + \beta^2 > \beta^2/2 + \sqrt{\beta^2 + \beta^4/4}$. But $\alpha = |P(t)|/(mc_0) > (\beta + \beta^2)^{1/2}$ is satisfied by hypothesis. By the *Corollary* 6.3 we deduce that :

$$\max\{t - r_1, r_2 - t\} \leq \frac{2m \cdot diam(\Omega)}{|P(t)|} \left(1 + \frac{|P(t)|^2}{m^2 c_0^2} \right)^{1/2}. \tag{6.4}$$

Suppose that $t + \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty} < s_{out}$, or $r_2 = t + \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty}$. We have by (6.4) that :

$$\frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty} \leq \frac{2m \cdot diam(\Omega)}{|P(t)|} \left(1 + \frac{|P(t)|^2}{m^2 c_0^2} \right)^{1/2},$$

which is equivalent to $\alpha^2/\sqrt{1+\alpha^2} \leq \beta/\sqrt{N}$ with the previous notations. Since $N \geq 1$ we would deduce that $\alpha^2/\sqrt{1+\alpha^2} \leq \beta$ or $\Delta \leq 0$ but we have proved that $\Delta > 0$. Finally we deduce that $s_{out} \leq t + \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty}$ and similarly we have $t - \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty} \leq s_{in}$. It follows that $r_1 = s_{in}$, $r_2 = s_{out}$, $\max\{t - s_{in}, s_{out} - t\} \leq \frac{2diam(\Omega)}{|v(P(t))|}$ and $s_{out} - s_{in} \leq \frac{4diam(\Omega)}{|v(P(t))|}$. We check easily that if $|P(t)| > D_{rel}^{ele}$, then $|v(P(t))| = c_0 \frac{|P(t)|}{mc_0} \left(1 + \frac{|P(t)|^2}{m^2 c_0^2}\right)^{-1/2} > c_0\sqrt{\beta(1+\beta)}/\sqrt{1+\beta(1+\beta)}$ and thus we obtain that $\max\{t - s_{in}, s_{out} - t\} < \frac{2diam(\Omega)}{c_0} \cdot \sqrt{1+\beta(1+\beta)}/\sqrt{\beta(1+\beta)}$. Finally we find for $s_{in} \leq s \leq s_{out}$:

$$\begin{aligned} |P(s) - P(t)| \leq& |q| \cdot \|E\|_\infty \cdot |s-t| < \frac{2|q| \cdot \|E\|_\infty \cdot diam(\Omega)}{c_0} \cdot \frac{\sqrt{1+\beta(1+\beta)}}{\sqrt{\beta(1+\beta)}} \\ =& \frac{\beta m c_0}{2\sqrt{N}} \cdot \frac{\sqrt{1+\beta(1+\beta)}}{\sqrt{\beta(1+\beta)}} < mc_0\sqrt{\beta(1+\beta)} = D_{rel}^{ele}. \end{aligned} \tag{6.5}$$

(2) If $\max\{|P(s_1)|, |P(s_2)|\} \leq D_{rel}^{ele}$, then we have $|P(s_1) - P(s_2)| \leq 2D_{rel}^{ele}$. If $|P(s_1)| > D_{rel}^{ele}$, by the point (1) with $t = s_1$ we deduce that $|P(s_2) - P(s_1)| \leq D_{rel}^{ele} \leq 2D_{rel}^{ele}$ and the same if $|P(s_2)| > D_{rel}^{ele}$ by taking $t = s_2$. □

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