The Vlasov-Maxwell system with strong initial magnetic field. Guiding-center approximation

Mihai Bostan *

(22nd June 2007)

Abstract

In this paper we study the asymptotic behavior of the Vlasov-Maxwell equations with strong magnetic field. More precisely we investigate the Cauchy problems associated to strong initial magnetic fields. We justify the convergence towards the so-called "guiding center approximation" when the dynamics is observed on a slower time scale than the plasma frequency. Our proofs rely on the modulated energy method.

Keywords: Vlasov-Maxwell equations, Guiding center approximation, Modulated energy.

AMS classification: 35Q75, 78A35, 82D10.

1 Introduction

The main motivations and applications in plasma physics concern the energy pro-

*Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon Cedex France. E-mail : `mbostan@univ-fcomte.fr`

duction through the thermonuclear fusion process. Two ways are currently explored for this: the inertial confinement fusion (ICF) and the magnetic confinement fusion (MCF). The magnetic confinement is performed in large toroidal devices, called tokamaks, by using strong magnetic fields. Besides studying these phenomena by direct observation and measurements, the numerical simulation of them is of crucial importance.

The dynamics of charged particles is described in terms of a number density by the Vlasov equation, coupled to the Maxwell equations for the electro-magnetic field. Generally the numerical resolution of this model requires important computational efforts, since we are working in a phase space with three spatial dimensions and three momentum dimensions. Moreover new difficulties appear when studying strong magnetic field regimes: large magnetic fields introduce a new time scale, related to the period of rotation of the particles around the magnetic field lines. Since the cyclotron period is proportional to the inverse of the magnitude of the magnetic field, the above time scale is very restrictive from the numerical point of view. Hence it is worth looking for simpler approximate models, like the gyro-kinetic model or the guiding center model [15], [19]. The limits of the Vlasov or Vlasov-Poisson equations with strong external magnetic fields have been investigated recently [7], [11], [8], [3]. For related works we refer to [16], [17].

For understanding the effects of strong magnetic fields let us start by analyzing the motion of individual charged particles under the action of constant electro-magnetic field (E, B). The motion equations of a particle of mass m and charge q are given by

$$\frac{dX}{ds} = V(s), \quad \frac{dV}{ds} = \frac{q}{m}(E + V(s) \wedge B),$$

where $(X(s), V(s))$ represent the position and velocity at time s. Projecting on the direction of B it is easily seen that

$$\frac{d}{ds}\left(V(s) \cdot \frac{B}{|B|}\right) = \frac{q}{m}\frac{E \cdot B}{|B|},$$

saying that the particle is advected with the acceleration $\frac{q}{m}\frac{E \cdot B}{|B|}$ in the direction of

B. Note that this acceleration do not depend on the magnitude of B. For analyzing the motion in the plane orthogonal to B it is convenient to represent the velocity as $V(s) = \frac{E\wedge B}{|B|^2} + U(s)$ where U satisfies

$$\frac{dU}{ds} = \frac{q}{m}\left((E\cdot B)\frac{B}{|B|^2} + U(s)\wedge B\right).$$

We denote by $U_\perp$ the projection of U on the plane orthogonal to B

$$U_\perp(s) = \left(\frac{B}{|B|}\wedge U(s)\right)\wedge\frac{B}{|B|}.$$

A straightforward computation shows that $U_\perp$ verifies

$$\frac{d^2}{ds^2}U_\perp + \frac{q^2}{m^2}|B|^2 U_\perp(s) = 0,$$

implying that

$$U_\perp(s) = \mathcal{R}(-\omega_c s)U_\perp(0) = \mathcal{R}(-\omega_c s)\left(V_\perp(0) - \frac{E\wedge B}{|B|^2}\right),$$

where $\omega_c = \frac{|q|}{m}|B|$ is the cyclotron frequency and for any $\theta\in\mathbb{R}$ we denote by $\mathcal{R}(\theta)$ the rotation of angle θ in the plane orthogonal to B, oriented by qB. We deduce that

$$X_\perp(t) \;=\; X_\perp(0) - \frac{1}{\omega_c}\mathcal{R}\left(\frac{\pi}{2}\right)U_\perp(0) + t\frac{E\wedge B}{|B|^2} + \frac{1}{\omega_c}\mathcal{R}\left(-\omega_c t + \frac{\pi}{2}\right)U_\perp(0).$$

The particles move on a helix with axis parallel to B and radius (called the Larmor radius) proportional to $\frac{1}{\omega_c} = \frac{m}{|q||B|}$. Therefore, when the magnetic field is large, the Larmor radius goes to zero and the particle motion can be approximated by the motion of the axis, whose velocity in the plane orthogonal to B, given by $\frac{E\wedge B}{|B|^2}$, is called the drift velocity. Notice that the drift velocity associated to strong magnetic fields $B = \mathcal{O}(1/\varepsilon)$ is small $\frac{E\wedge B}{|B|^2} = \mathcal{O}(\varepsilon)$. Hence the motion of the axis becomes significant only for large observation time $\mathcal{O}(1/\varepsilon)$.

We consider a population of relativistic electrons whose density in the phase space is denoted by f. We neglect the collisions between particles assuming that they interact only by electro-magnetic fields created collectively. The particle density depends on time $t \in \mathbb{R}_+$, position $x \in \mathbb{R}^3$, momentum $p \in \mathbb{R}^3$ and satisfies the Vlasov equation

$$\partial_t f + v(p) \cdot \nabla_x f - e(E(t,x) + v(p) \wedge B(t,x)) \cdot \nabla_p f = 0, \tag{1}$$

where $-e < 0$ is the electron charge, $v(p)$ is the relativistic velocity associated to the momentum p

$$v(p) = \frac{p}{m_e} \left(1 + \frac{|p|^2}{m_e^2 c_0^2}\right)^{-\frac{1}{2}},$$

m_e is the electron mass and c_0 is the vacuum light speed. The electro-magnetic field is defined in a self-consistent way by the Maxwell equations

$$\partial_t E - c_0^2 \operatorname{curl}_x B = \frac{e}{\varepsilon_0} \int_{\mathbb{R}^3} v(p) f(t,x,p)\, dp, \tag{2}$$

$$\partial_t B + \operatorname{curl}_x E = 0, \tag{3}$$

$$\operatorname{div}_x E = \frac{e}{\varepsilon_0} \left(n - \int_{\mathbb{R}^3} f(t,x,p)\, dp\right), \quad \operatorname{div}_x B = 0, \tag{4}$$

where ε_0 is the vacuum permittivity and n is the concentration of a background ion distribution (*i.e.*, the number of ions per volume unit). Let us write the equations in dimensionless form. We define the thermal potential by $U_{\text{th}} = \frac{K_B T_{\text{th}}}{e}$ where K_B is the Boltzmann constant and T_{th} is the temperature. The thermal momentum p_{th} is given by

$$m_e c_0^2 \left(\left(1 + \frac{p_{\text{th}}^2}{m_e^2 c_0^2}\right)^{\frac{1}{2}} - 1 \right) = K_B T_{\text{th}},$$

which leads to

$$p_{\text{th}} = \left((K_B T_{\text{th}})^2 / c_0^2 + 2 K_B T_{\text{th}} m_e \right)^{\frac{1}{2}}.$$

We introduce a length unit L and a time unit T. As momentum unit we set $P = p_{\text{th}}$. We define dimensionless variables and unknowns by the relations

$$t = Tt', \quad x = Lx', \quad p = p_{\text{th}} p',$$

$$f(t,x,p) = \frac{n_e}{p_{\text{th}}^3} f'(\frac{t}{T}, \frac{x}{L}, \frac{p}{p_{\text{th}}}), \quad E(t,x) = \frac{U_{\text{th}}}{L} E'(\frac{t}{T}, \frac{x}{L}), \quad B(t,x) = \frac{1}{\varepsilon} \frac{m_e}{e T_p} B'(\frac{t}{T}, \frac{x}{L}),$$

where $T_p = \left(\frac{m_e \varepsilon_0}{e^2 n_e}\right)^{1/2}$ is the inverse of the plasma frequency, n_e is the average of the electron concentration and $\varepsilon > 0$ is a small parameter. We assume that the plasma is globally neutral and therefore we have $n_e = n$. We set

$$v'(p') = \frac{p_{\mathrm{th}}^2}{m_e K_B T_{\mathrm{th}}} p' \left(1 + \frac{p_{\mathrm{th}}^2}{m_e^2 c_0^2} |p'|^2\right)^{-\frac{1}{2}}.$$

As a matter of fact note that $v(p) = \frac{K_B T_{\mathrm{th}}}{p_{\mathrm{th}}} v'(p/p_{\mathrm{th}})$. We also introduce the Debye length

$$\lambda_D = \left(\frac{\varepsilon_0 K_B T_{\mathrm{th}}}{e^2 n}\right)^{\frac{1}{2}}.$$

Notice that we have $K_B T_{\mathrm{th}}/m_e = (\lambda_D/T_p)^2$. Then the equations become having dropped the primes

$$\partial_t f + \frac{K_B T_{\mathrm{th}}}{p_{\mathrm{th}}} \frac{T}{L} v(p) \cdot \nabla_x f - \frac{K_B T_{\mathrm{th}}}{p_{\mathrm{th}}} \frac{T}{L} \left(E(t,x) + \frac{L m_e}{T_p p_{\mathrm{th}}} v(p) \wedge \frac{B(t,x)}{\varepsilon}\right) \cdot \nabla_p f = 0,$$

$$\partial_t E - \frac{T}{T_p} \frac{m_e c_0^2}{K_B T_{\mathrm{th}}} \mathrm{curl}_x \left(\frac{B}{\varepsilon}\right) = \left(\frac{L}{\lambda_D}\right)^2 \frac{K_B T_{\mathrm{th}}}{p_{\mathrm{th}}} \frac{T}{L} j(t,x),$$

$$\partial_t \left(\frac{B}{\varepsilon}\right) + \frac{T}{T_p} \left(\frac{\lambda_D}{L}\right)^2 \mathrm{curl}_x E = 0,$$

$$\mathrm{div}_x E = \left(\frac{L}{\lambda_D}\right)^2 (1 - \rho(t,x)), \quad \mathrm{div}_x B = 0,$$

where $\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\; dp$, $j(t,x) = \int_{\mathbb{R}^3} v(p) f(t,x,p)\; dp$. We take as length unit $L = \lambda_D$ and as time unit $T = \frac{T_p}{\varepsilon}$. Observe that

$$\frac{K_B T_{\mathrm{th}}}{p_{\mathrm{th}}} \frac{T_p}{\lambda_D} = \frac{\lambda_D m_e}{T_p p_{\mathrm{th}}} = \left(\frac{K_B T_{\mathrm{th}}}{m_e c_0^2} + 2\right)^{-\frac{1}{2}} =: \alpha.$$

Finally we obtain the equations

$$\partial_t f + \frac{\alpha}{\varepsilon} v(p) \cdot \nabla_x f - \frac{\alpha}{\varepsilon} \left(E(t,x) + \alpha\, v(p) \wedge \frac{B(t,x)}{\varepsilon}\right) \cdot \nabla_p f = 0, \tag{5}$$

$$\partial_t E - \frac{1}{\varepsilon \beta} \mathrm{curl}_x \left(\frac{B}{\varepsilon}\right) = \frac{\alpha}{\varepsilon} j(t,x), \tag{6}$$

$$\partial_t \left(\frac{B}{\varepsilon}\right) + \frac{1}{\varepsilon} \mathrm{curl}_x E = 0, \tag{7}$$

$$\mathrm{div}_x E = 1 - \rho(t,x), \quad \mathrm{div}_x B = 0, \tag{8}$$

with $\beta = \frac{K_B T_{\mathrm{th}}}{m_e c_0^2}$ and $v(p) = \frac{p}{\alpha^2}\left(1 + \frac{\beta}{\alpha^2}|p|^2\right)^{-1/2}$. We are concerned with the asymptotic behavior of (5), (6), (7), (8) when $\varepsilon \searrow 0$, $\beta = \mathcal{O}(1)$ and therefore $\alpha = \mathcal{O}(1)$. In order to simplify our computations we will study the systems

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v(p) \cdot \nabla_x f^\varepsilon - \frac{1}{\varepsilon}\left(E^\varepsilon(t,x) + v(p) \wedge \frac{B^\varepsilon(t,x)}{\varepsilon}\right) \cdot \nabla_p f^\varepsilon = 0, \tag{9}$$

$$\partial_t E^\varepsilon - \frac{1}{\varepsilon}\mathrm{curl}_x\left(\frac{B^\varepsilon}{\varepsilon}\right) = \frac{1}{\varepsilon} j^\varepsilon(t,x), \tag{10}$$

$$\partial_t\left(\frac{B^\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon}\mathrm{curl}_x E^\varepsilon = 0, \tag{11}$$

$$\mathrm{div}_x E^\varepsilon = 1 - \rho^\varepsilon(t,x), \quad \mathrm{div}_x B^\varepsilon = 0, \tag{12}$$

$$\rho^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon \, dp, \quad j^\varepsilon = \int_{\mathbb{R}^3} v(p) f^\varepsilon \, dp, \quad v(p) = \frac{p}{(1+|p|^2)^{\frac{1}{2}}}, \tag{13}$$

which has the same structure as (5), (6), (7), (8). We prescribe also the initial conditions

$$f^\varepsilon(0,x,p) = f_0^\varepsilon(x,p), \quad (E^\varepsilon, B^\varepsilon)(0,x) = (E_0^\varepsilon, B_0^\varepsilon)(x). \tag{14}$$

We assume also periodicity in the space variable $x \in \mathbb{T}^d$ where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, equipped with the restriction of the Lebesgue measure of $\mathbb{R}^d$ on $[0,1[^d$, $d \in \{1,2,3\}$. The subject matter of this paper concerns the stability of the solutions $(f^\varepsilon, E^\varepsilon, B^\varepsilon)_{\varepsilon>0}$ for well prepared initial conditions $(f_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)_{\varepsilon>0}$, where $\varepsilon > 0$ is a small parameter. In particular we are looking for problems with initial magnetic fields B_0^ε close to some constant magnetic field B_0. In this case it is easily seen that at any time $t \in \mathbb{R}_+$ the magnetic field B^ε remains close to B_0. Indeed, consider a constant magnetic field $B_0 \in \mathbb{R}^3$ and observe that (10), (11) can be written in the form

$$\partial_t E^\varepsilon - \frac{1}{\varepsilon}\mathrm{curl}_x\left(\frac{B^\varepsilon - B_0}{\varepsilon}\right) = \frac{j^\varepsilon(t,x)}{\varepsilon}, \tag{15}$$

$$\partial_t\left(\frac{B^\varepsilon - B_0}{\varepsilon}\right) + \frac{1}{\varepsilon}\mathrm{curl}_x E^\varepsilon = 0. \tag{16}$$

Multiplying (9) by $(1+|p|^2)^{\frac{1}{2}}-1$, (15) by E^ε and (16) by $\left(\frac{B^\varepsilon - B_0}{\varepsilon}\right)$ one gets as usual the conservation

$$\frac{d}{dt}\left\{\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}((1+|p|^2)^{\frac{1}{2}}-1)f^\varepsilon \; dp\; dx + \frac{1}{2}\int_{\mathbb{T}^3}\left(|E^\varepsilon|^2 + \left|\frac{B^\varepsilon - B_0}{\varepsilon}\right|^2\right)\; dx\right\} = 0, \quad (17)$$

implying that

$$\begin{aligned}\int_{\mathbb{T}^3}\left|\frac{B^\varepsilon(t,x) - B_0}{\varepsilon}\right|^2 \; dx \;\; &\leq \;\; 2\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}((1+|p|^2)^{\frac{1}{2}}-1)f_0^\varepsilon(x,p)\; dp\; dx \\ &+ \;\; \int_{\mathbb{T}^3}|E_0^\varepsilon(x)|^2\; dx + \int_{\mathbb{T}^3}\left|\frac{B_0^\varepsilon(x) - B_0}{\varepsilon}\right|^2 \; dx.\end{aligned}$$

In particular we deduce that $\sup_{\varepsilon>0, t\in\mathbb{R}_+}\int_{\mathbb{T}^3}\varepsilon^{-2}|B^\varepsilon(t,x)-B_0|^2\; dx < +\infty$ for initial conditions satisfying

$$\sup_{\varepsilon>0}\left\{\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}((1+|p|^2)^{\frac{1}{2}}-1)f_0^\varepsilon \; dp\; dx + \frac{1}{2}\int_{\mathbb{T}^3}|E_0^\varepsilon|^2\; dx + \frac{1}{2}\int_{\mathbb{T}^3}\left|\frac{B_0^\varepsilon - B_0}{\varepsilon}\right|^2 \; dx\right\} < +\infty.$$

Recalling that the unit for the magnetic field was chosen proportional to $1/\varepsilon$, the above arguments say that if initially the (unscaled) magnetic field is close to $\frac{B_0}{\varepsilon}$, then at any time $t>0$ the (unscaled) magnetic field remains close to $\frac{B_0}{\varepsilon}$; we are dealing with a strong magnetic field regime. As a matter of fact this regime is consistent with the Vlasov-Poisson equations with strong external magnetic field. This asymptotic regime has been investigated in [11] by appealing to compactness methods. In the two dimensional case the authors justified the convergence towards the vorticity formulation of the incompressible Euler equations with a right-hand side involving a defect measure. Another approach uses modulated energy (or relative entropy) methods, as introduced in [24]. By this technique one gets strong convergences, provided that the solution of the limit system is smooth. Results for the Vlasov-Poisson equations with strong magnetic field have been obtained recently in [3], [12]. More generally the relative entropy method allows the treatement of various asymptotic questions in plasma physics [4], [14], [2], gas dynamics [22], [1], fluid-particles interaction [13].

We intend to address the Vlasov-Maxwell system with strong initial magnetic field by the method of relative entropy. We follow the ideas in [3] by adapting the arguments to the relativistic case with self-consistent magnetic field. This generalization is important from the physical point of view since we are dealing with a more realistic model. Besides, this work shows how robust the relative entropy method is, which is interesting from the mathematical point of view. A complete convergence result is obtained in the two dimensional case, see Theorem 2.1. As in [11] we obtain as limit model the vorticity formulation of the incompressible Euler equations, this time without any defect measure since the modulated energy method provides strong convergences. Both the relativistic and non relativistic cases are treated by this method. The computations are basically the same, the only main difference concerning the definition of the modulated energy. We highlight that it is also possible to handle measure solutions of the Vlasov equation. For example we obtain a convergence result for particle densities depending on macroscopic charge densities and mean velocities. We remind that the method in [11] do not allow the treatment of such situations since it relies on the uniform boundedness in L^∞ of the family of particle densities. Another original point of this paper is that more accurate limit models can be derived by using relative entropy techniques. We identify higher limit models by standard Hilbert expansions. Surely the difficult task is to check the accuracy of these models and this can be done by defining suitable modulated energy versions, see Theorem 4.1.

The paper is organized as follows. The relativistic Vlasov-Maxwell system in two dimensions is treated in Section 2. After a formal derivation of the limit system we introduce the modulated energy. We study the time evolution of it and we deduce strong convergence for the electro-magnetic field. We obtain also convergence in the distribution sense for the macroscopic quantities like the charge and current densities. Section 3 is devoted to other systems, as the non relativistic case or cases with particle densities depending on macroscopic charge densities and bulk velocities. In the last section we justify the second order approximation. We construct a more detailed version for the modulated energy by taking into account the first order correction terms.

2 The two dimensional case

We consider the Vlasov-Maxwell system (9), (10), (11), (12) in two dimensions. For any $\varepsilon > 0$ we are looking for a solution with particle density $f^\varepsilon = f^\varepsilon(t,x,p)$, $(t,x,p) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2$ and electro-magnetic field of the form $((E_1^\varepsilon, E_2^\varepsilon, 0), (0,0,B_3^\varepsilon))$. It is convenient to introduce the new momentum variable $u = \frac{p}{\varepsilon}$ and the new density function

$$F^\varepsilon(t,x,u) = \varepsilon^2 f^\varepsilon(t,x,\varepsilon u), \;\; (t,x,u) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2.$$

Observe that these distributions have the same charge densities

$$\rho^\varepsilon(t,x) = \int_{\mathbb{R}^2} f^\varepsilon(t,x,p)\,dp = \int_{\mathbb{R}^2} F^\varepsilon(t,x,u)\,du,$$

and that the current densities are related by

$$j^\varepsilon(t,x) = \int_{\mathbb{R}^2} v(p) f^\varepsilon(t,x,p)\,dp = \varepsilon \int_{\mathbb{R}^2} v^\varepsilon(u) F^\varepsilon(t,x,u)\,du = \varepsilon J^\varepsilon(t,x),$$

where the velocity v^ε is given by $v^\varepsilon(u) = u/(1+\varepsilon^2|u|^2)^{\frac{1}{2}}$. We use the notation ${}^\perp v = (v_2, -v_1)$, $\forall\; v = (v_1, v_2) \in \mathbb{R}^2$. With these notations the two dimensional Vlasov-Maxwell system becomes

$$\partial_t F^\varepsilon + v^\varepsilon(u)\cdot\nabla_x F^\varepsilon - \frac{1}{\varepsilon^2}(E^\varepsilon(t,x) + B_3^\varepsilon(t,x)\, {}^\perp v^\varepsilon(u))\cdot\nabla_u F^\varepsilon = 0, \;\; (t,x,u) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2, \tag{18}$$

$$\partial_t E_1^\varepsilon - \frac{1}{\varepsilon}\partial_{x_2}\left(\frac{B_3^\varepsilon}{\varepsilon}\right) = J_1^\varepsilon(t,x), \;\; (t,x) \in \mathbb{R}_+ \times \mathbb{T}^2, \tag{19}$$

$$\partial_t E_2^\varepsilon + \frac{1}{\varepsilon}\partial_{x_1}\left(\frac{B_3^\varepsilon}{\varepsilon}\right) = J_2^\varepsilon(t,x), \;\; (t,x) \in \mathbb{R}_+ \times \mathbb{T}^2, \tag{20}$$

$$\partial_t \left(\frac{B_3^\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon}(\partial_{x_1} E_2^\varepsilon - \partial_{x_2} E_1^\varepsilon) = 0, \;\; (t,x) \in \mathbb{R}_+ \times \mathbb{T}^2, \tag{21}$$

$$\partial_{x_1} E_1^\varepsilon + \partial_{x_2} E_2^\varepsilon = 1 - \rho^\varepsilon(t,x), \;\; (t,x) \in \mathbb{R}_+ \times \mathbb{T}^2, \tag{22}$$

with the initial conditions

$$F^\varepsilon(0,x,u) = \varepsilon^2 f_0^\varepsilon(x,\varepsilon u) =: F_0^\varepsilon(x,u), \;\; (t,x,u) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2, \tag{23}$$

$$(E_1^\varepsilon, E_2^\varepsilon, B_3^\varepsilon)(0,x) = (E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon)(x), \;\; x \in \mathbb{T}^2. \tag{24}$$

We make the following hypotheses on the initial conditions $(f_0^\varepsilon, E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon)$

H1) $f_0^\varepsilon \geq 0, \;\; \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f_0^\varepsilon(x,p)\;dp\;dx = 1$;

H2) $\lim_{\varepsilon \searrow 0} \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} ((1+|p|^2)^{\frac{1}{2}} - 1) f_0^\varepsilon(x,p)\;dp\;dx = 0$;

H3) $(E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon) \in L^2(\mathbb{T}^2)^3$ and $\mathrm{div}_x E_0^\varepsilon = 1 - \rho_0^\varepsilon$ where $\rho_0^\varepsilon = \int_{\mathbb{R}^2} f_0^\varepsilon \;dp$;

H4) there are $E_0 = (E_{0,1}, E_{0,2}) \in L^2(\mathbb{T}^2)^2$ verifying $\partial_{x_1} E_{0,2} - \partial_{x_2} E_{0,1} = 0$ and a constant magnetic field $(0,0,B_{0,3})$ with $B_{0,3} \neq 0$ such that

$$\lim_{\varepsilon \searrow 0} \left\{ \frac{1}{2}\int_{\mathbb{T}^2} |E_0^\varepsilon(x) - E_0(x)|^2\;dx + \frac{1}{2}\int_{\mathbb{T}^2} \left(\frac{B_{0,3}^\varepsilon(x) - B_{0,3}}{\varepsilon}\right)^2\;dx \right\} = 0.$$

Since $\mathrm{div}_x E_0^\varepsilon = 1 - \rho_0^\varepsilon$ and $\lim_{\varepsilon \searrow 0} E_0^\varepsilon = E_0$ in $L^2(\mathbb{T}^2)^2$ we deduce that $\lim_{\varepsilon\searrow 0} \rho_0^\varepsilon = 1 - \mathrm{div}_x E_0$ in $\mathcal{D}'(\mathbb{T}^2)$ and therefore the electric field $E_0 \in L^2(\mathbb{T}^2)^2$ in H4 solves the problem

$$\mathrm{div}_x^\perp E_0 = 0, \;\; \mathrm{div}_x E_0 = 1 - \lim_{\varepsilon \searrow 0} \rho_0^\varepsilon \;\text{ in }\; \mathcal{D}'(\mathbb{T}^2).$$

Assume that the initial charge densities $(\rho_0^\varepsilon)_{\varepsilon > 0}$ are bounded in $L^r(\mathbb{T}^2)$ for some finite $r > 1$ and consider a sequence $(\varepsilon_k)_k$ converging towards zero such that $\lim_{k \to +\infty} \rho_0^{\varepsilon_k} = \rho_0$ weakly in $L^r(\mathbb{T}^2)$. In this case the electric field appearing in H4 is unique up to two constants $e_0 = (e_{0,1}, e_{0,2}) \in \mathbb{R}^2$, $E_0 = \nabla_x \phi_0 + e_0$ where $\phi_0 \in W^{2,r}(\mathbb{T}^2)$ is the unique solution of

$$-\Delta_x \phi_0 = \rho_0(x) - 1, \;\; x \in \mathbb{T}^2, \;\; \int_{\mathbb{T}^2} \phi_0(x)\;dx = 0.$$

Notice that H1, H2 are equivalent to

$$F_0^\varepsilon \geq 0, \;\; \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} F_0^\varepsilon(x,u)\;du\;dx = 1, \;\; \lim_{\varepsilon \searrow 0} \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} ((1+\varepsilon^2|u|^2)^{\frac{1}{2}} - 1) F_0^\varepsilon(x,u)\;du\;dx = 0.$$

The theory for the existence and uniqueness of global classical solution for the relativistic Vlasov-Maxwell system is now well developed in two dimensions cf. [10].

2.1 Analysis of the limit system

Let $(F^\varepsilon, E_1^\varepsilon, E_2^\varepsilon, B_3^\varepsilon)_{\varepsilon>0}$ be smooth solutions for (18), (19), (20), (21), (22) with smooth initial conditions (23), (24). The conservation of the total energy implies

$$\varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F^\varepsilon(t,x,u)\;du\;dx + \frac{1}{2}\int_{\mathbb{T}^2}\left\{ |E^\varepsilon(t,x)|^2 + \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon}\right)^2\right\}\;dx$$
$$= \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F_0^\varepsilon(x,u)\;du\;dx + \frac{1}{2}\int_{\mathbb{T}^2}\left\{ |E_0^\varepsilon(x)|^2 + \left(\frac{B_{0,3}^\varepsilon(x) - B_{0,3}}{\varepsilon}\right)^2\right\}\;dx,$$

where $\mathcal{E}^\varepsilon(u) = \varepsilon^{-2}((1+\varepsilon^2|u|^2)^{\frac{1}{2}} - 1)$ is the energy associated to the velocity $v^\varepsilon(u)$ (*i.e.*, $\nabla_u \mathcal{E}^\varepsilon = v^\varepsilon(u)$) and $B_{0,3}$ is the constant appearing in H4. We deduce that

$$\sup_{\varepsilon>0, t\in\mathbb{R}_+} \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F^\varepsilon\;du\;dx + \frac{1}{2}\int_{\mathbb{T}^2}\left\{ |E^\varepsilon(t,x)|^2 + \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon}\right)^2\right\}\;dx < +\infty. \tag{25}$$

In particular there is a sequence $(\varepsilon_k)_k$ converging towards zero such that

$$\lim_{k\to+\infty}\left(E_1^{\varepsilon_k}, E_2^{\varepsilon_k}, \frac{B_3^{\varepsilon_k} - B_{0,3}}{\varepsilon_k}\right) = (E_1, E_2, b_3),$$

weakly in $L^2(]0,T[\times\mathbb{T}^2)^3$, $\forall\; T>0$. We use also the conservations of the mass and momentum

$$\partial_t \rho^\varepsilon + \mathrm{div}_x J^\varepsilon = 0, \tag{26}$$

$$\varepsilon^2\partial_t \int_{\mathbb{R}^2} uF^\varepsilon\;du + \varepsilon^2 \mathrm{div}_x \int_{\mathbb{R}^2} (u\otimes v^\varepsilon(u))F^\varepsilon\;du + \rho^\varepsilon(t,x)E^\varepsilon(t,x) + B_3^\varepsilon(t,x)^\perp J^\varepsilon(t,x) = 0. \tag{27}$$

By equation (25) we deduce that $\lim_{\varepsilon\searrow 0} B_3^\varepsilon(t,\cdot) = B_{0,3}$ in $L^2(\mathbb{T}^2)$ uniformly with respect to $t\in\mathbb{R}_+$ and thus from (27) we expect that at the limit for $\varepsilon\searrow 0$ one gets

$$\rho(t,x)E(t,x) + B_{0,3}{}^\perp J(t,x) = 0.$$

Moreover from (21) and (25) we deduce that

$$\partial_{x_1} E_2 - \partial_{x_2} E_1 = 0. \tag{28}$$

Combining with the continuity equation (26) and (22) we obtain the limit system

$$J = \rho\frac{{}^\perp E}{B_{0,3}},\;\; \mathrm{div}_x^\perp E = 0, \tag{29}$$

$$\partial_t \rho + \mathrm{div}_x \left(\rho \frac{{}^\perp E}{B_{0,3}} \right) = 0, \tag{30}$$

$$\mathrm{div}_x E = 1 - \rho(t,x). \tag{31}$$

The above equations can be written

$$\partial_t(\rho - 1) + \frac{{}^\perp E}{B_{0,3}} \cdot \nabla_x(\rho - 1) = 0,$$

$$\rho(t,x) - 1 = -\mathrm{div}_x E = \partial_{x_1}({}^\perp E)_2 - \partial_{x_2}({}^\perp E)_1, \;\; \mathrm{div}_x {}^\perp E = 0.$$

We recognize here the Euler equations written in the so-called vorticity formulation with $\rho - 1$ standing for the vorticity and the velocity ${}^\perp E$. For the existence theory of classical solutions to the equations of ideal fluid flow we refer to [18], [23], [5], [20], [21].

The previous equations are supplemented with the initial condition E_0 given in H4, the initial condition for ρ being $\rho_0 = 1 - \mathrm{div}_x E_0$. It is easily seen by standard computation that $\frac{1}{2}\int_{\mathbb{T}^2} |E(t,x)|^2 \, dx$ is preserved in time. Notice also that when ε goes to zero we expect that the total kinetic and electric energy $\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} ((1+|p|^2)^{\frac{1}{2}} - 1) f^\varepsilon \, dp \, dx + \frac{1}{2}\int_{\mathbb{T}^2} |E^\varepsilon|^2 \, dx$ is conserved since $\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \approx 0$ (actually this happens for the Vlasov-Poisson equations with strong external magnetic field $\frac{B_{0,3}}{\varepsilon}$). Therefore we can interpret the hypotheses H2, H4 as follows: as ε goes to zero the total kinetic and electric energy of the conditions $(f_0^\varepsilon, E_0^\varepsilon)$ converges towards the electric energy of E_0 and the magnetic energy of $\frac{B_{0,3}^\varepsilon - B_{0,3}}{\varepsilon}$ goes to zero. We will see that under these hypotheses we can prove strong convergences in L^2 for the fields and also convergences in distributions sense for the charge and current densities. The same limit system has been obtained in [11], [3].

Theorem 2.1 *Assume that the initial conditions* $(f_0^\varepsilon, E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon)_{\varepsilon>0}$ *are smooth and satisfy the hypotheses H1-H4. We denote by* $(f^\varepsilon, E_1^\varepsilon, E_2^\varepsilon, B_3^\varepsilon)_{\varepsilon>0}$ *the solutions of the two dimensional problems* (9), (10), (11), (12), (13), (14) *and we suppose that the limit system* (29), (30), (31) *corresponding to the initial electric field* E_0

appearing in H4 has a smooth solution (ρ, J, E). Then for any $T \in \mathbb{R}_+$ we have

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} ((1+|p|^2)^{\frac{1}{2}} - 1) f^\varepsilon(t,x,p)\, dp\, dx + \frac{1}{2} \int_{\mathbb{T}^2} |E^\varepsilon(t,x) - E(t,x)|^2\, dx$$

$$+ \quad \frac{1}{2} \int_{\mathbb{T}^2} \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon} \right)^2 dx = 0, \quad \textit{uniformly for } t \in [0,T],$$

$$\lim_{\varepsilon \searrow 0} \rho^\varepsilon = \rho, \quad \lim_{\varepsilon \searrow 0} \frac{j^\varepsilon}{\varepsilon} = J \quad \textit{in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2),$$

$$\lim_{\varepsilon \searrow 0} f^\varepsilon(t,x,p) = \rho(t,x)\delta(p), \quad \textit{vaguely in } \mathcal{M}_+^1(\mathbb{T}^2 \times \mathbb{R}^2), \quad \textit{uniformly for } t \in [0,T],$$

where we denote by $\mathcal{M}_+^1(X)$ the set of bounded Radon non negative measures on X and by δ the Dirac mass located at the origin of $\mathbb{R}^2$.

One of the hypotheses of the above statement concerns the existence of smooth solutions for the limit system (29), (30), (31). A very simple situation is that when the initial conditions depend only on x_1. In this case the limit system can be solved explicitly and it is easily seen that the smoothness of the initial conditions propagates globally in time. This situation arises when studying the Vlasov-Maxwell equations in the one and one-half dimensional setting [9] that is, the particle density depends on t, x_1, p_1, p_2 and the fields depend on t, x_1. We are looking now for solutions of the limit system depending only on t and x_1. For any 1-periodic function $u = u(x_1)$ we denote by $\langle u \rangle$ its average over one period $\langle u \rangle := \int_{\mathbb{T}^1} u(x_1)\, dx_1$. From (28) we deduce that $\partial_{x_1} E_2 = 0$. Integrating now (20) with respect to $x_1 \in \mathbb{T}^1$ yields

$$\frac{d}{dt} \int_{\mathbb{T}^1} E_2^\varepsilon(t,x_1)\, dx_1 = \int_{\mathbb{T}^1} J_2^\varepsilon(t,x_1)\, dx_1 = \langle J_2^\varepsilon(t) \rangle,$$

and after passing to the limit we expect that

$$\frac{d}{dt} E_2(t) = \int_{\mathbb{T}^1} J_2(t,x_1)\, dx_1 = -\frac{\langle \rho(t) E_1(t) \rangle}{B_{0,3}}. \tag{32}$$

Combining (19), (22) we find

$$\partial_t E_1 = \lim_{\varepsilon \searrow 0} J_1^\varepsilon(t,x_1) = \frac{\rho(t,x_1) E_2(t)}{B_{0,3}} = (1 - \partial_{x_1} E_1) \frac{E_2(t)}{B_{0,3}},$$

implying that

$$\partial_t E_1 + \frac{E_2(t)}{B_{0,3}} \partial_{x_1} E_1 = \frac{E_2(t)}{B_{0,3}}. \tag{33}$$

In this case the continuity equation becomes

$$\partial_t \rho + \frac{E_2(t)}{B_{0,3}} \partial_{x_1} \rho = 0. \tag{34}$$

Multiplying (33) by ρ and (34) by E_1 one gets

$$\partial_t(\rho E_1) + \frac{E_2(t)}{B_{0,3}} \partial_{x_1}(\rho E_1) = \frac{E_2(t)}{B_{0,3}} \rho(t, x_1),$$

and therefore by taking the average we obtain

$$\frac{d}{dt}\langle \rho(t) E_1(t)\rangle = \frac{E_2(t)}{B_{0,3}} \langle \rho(t)\rangle = \frac{E_2(t)}{B_{0,3}}. \tag{35}$$

Therefore (32), (35) can be solved with respect to E_2 and $\langle \rho E_1\rangle$ and thus

$$E_2(t) = E_{0,2} \cos\left(\frac{t}{B_{0,3}}\right) - \langle \rho_0 E_{0,1}\rangle \sin\left(\frac{t}{B_{0,3}}\right).$$

By H3, H4 E_0 satisfies $\mathrm{div}_x^{\perp} E_0 = 0$, saying that indeed $E_{0,2}$ do not depend on x_1, and $\mathrm{div}_x E_0 = 1 - \rho_0$ which becomes

$$\partial_{x_1} E_{0,1} = 1 - \rho_0(x_1), \quad x_1 \in \mathbb{T}^1.$$

Multiplying by $E_{0,1}$ and integrating over $\mathbb{T}^1$ yields $\langle \rho_0 E_{0,1}\rangle = \langle E_{0,1}\rangle$ and we can eliminate the function ρ_0 in the expression of E_2

$$E_2(t) = E_{0,2} \cos\left(\frac{t}{B_{0,3}}\right) - \langle E_{0,1}\rangle \sin\left(\frac{t}{B_{0,3}}\right).$$

The other unknowns can be easily expressed in terms of the characteristics $X(s; t, x_1)$ associated to $\frac{E_2}{B_{0,3}}$

$$\frac{d}{ds} X(s; t, x_1) = \frac{E_2(s)}{B_{0,3}}, \quad X(t; t, x_1) = x_1,$$

given by

$$X(s; t, x_1) = x_1 + E_{0,2}\left\{\sin\left(\frac{s}{B_{0,3}}\right) - \sin\left(\frac{t}{B_{0,3}}\right)\right\} + \langle E_{0,1}\rangle \left\{\cos\left(\frac{s}{B_{0,3}}\right) - \cos\left(\frac{t}{B_{0,3}}\right)\right\}.$$

Finally we obtain from (34), (33) $\rho(t, x_1) = \rho_0(X(0; t, x_1))$ and

$$E_1(t, x_1) = E_{0,1}(X(0; t, x_1)) + E_{0,2} \sin\left(\frac{t}{B_{0,3}}\right) + \langle E_{0,1}\rangle \left(\cos\left(\frac{t}{B_{0,3}}\right) - 1\right).$$

2.2 Evolution of the modulated energy

In this paragraph we consider smooth solutions $(f^\varepsilon, E_1^\varepsilon, E_2^\varepsilon, B_3^\varepsilon)_{\varepsilon>0}$ for the two dimensional relativistic Vlasov-Maxwell system associated to smooth initial conditions $(F_0^\varepsilon, E_{0,1}^\varepsilon, E_{0,2}^\varepsilon, B_{0,3}^\varepsilon)_{\varepsilon>0}$ satisfying the hypotheses H1-H4. We assume also that the limit system (29), (30), (31) has a smooth solution (ρ, J, E). Notice that the solution of the limit system satisfies $\mathrm{div}_x(\partial_t E - J) = 0$ and therefore there is a periodic function $A_3 = A_3(t,x)$ such that

$$\partial_t E_1 - \partial_{x_2} A_3 = J_1, \quad \partial_t E_2 + \partial_{x_1} A_3 = J_2. \tag{36}$$

Actually A_3 solves the elliptic space periodic problem

$$-\Delta_x A_3 = \partial_{x_2} J_1 - \partial_{x_1} J_2,$$

which has a unique periodic solution, up to an additive constant. In order to fix the constant we choose the solution with zero space average

$$\int_{\mathbb{T}^2} A_3(t,x)\, dx = 0, \quad t \in \mathbb{R}_+.$$

We assume that A_3 is smooth. The proof of Theorem 2.1 relies essentially on the following proposition.

Proposition 2.1 *There is a constant C depending on $\|E\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)}$, $\|A_3\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)}$ such that for any $0 < \varepsilon < \varepsilon(C)$, $t \in [0,T]$ we have*

$$\varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F^\varepsilon(t,x,u)\, du\, dx + \frac{1}{2}\int_{\mathbb{T}^2} \left\{ |E^\varepsilon(t,x) - E(t,x)|^2 + \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon}\right)^2 \right\} dx$$
$$\leq C_1(t) \left\{ \varepsilon + \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F_0^\varepsilon\, du\, dx + \frac{1}{2}\int_{\mathbb{T}^2} \left\{ |E_0^\varepsilon - E_0|^2 + \left(\frac{B_{0,3}^\varepsilon - B_{0,3}}{\varepsilon}\right)^2 \right\} dx \right\},$$

where $C_1(t) = (3 + 2C(4+t))e^{2Ct}$, $t \in [0,T]$.

The above proposition is the consequence of several lemmas which are postponed to the end of this section.

Proof. (of Theorem 2.1) The first statement comes by Proposition 2.1. The convergence of the charge densities $(\rho^\varepsilon)_{\varepsilon>0}$ follows easily by (22), (31) since for any $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{T}^2)$ we have

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \int_{\mathbb{T}^2} (\rho^\varepsilon - \rho)(t,x)\varphi(t,x)\ dxdt = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \int_{\mathbb{T}^2} (E^\varepsilon - E)(t,x) \cdot \nabla_x \varphi\ dxdt = 0.$$

For the convergence of the current densities $(J^\varepsilon)_{\varepsilon>0} = \left(\frac{j^\varepsilon}{\varepsilon}\right)_{\varepsilon>0}$ we use the momentum conservation (27). It is easily seen by Proposition 2.1 and the inequalities

$$\varepsilon|u| \le \varepsilon^2 \mathcal{E}^\varepsilon(u) + 1, \ \ \forall\, \varepsilon > 0, \tag{37}$$

$$\mathcal{E}^\varepsilon(u) \ge \frac{|u|^2}{2(1+\varepsilon^2|u|^2)^{\frac{1}{2}}} = \frac{|u||v^\varepsilon(u)|}{2}, \ \ \forall\, \varepsilon > 0, \tag{38}$$

that

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \partial_t \int_{\mathbb{R}^2} u F^\varepsilon\ du = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2)^2,$$

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \mathrm{div}_x \int_{\mathbb{R}^2} (u \otimes v^\varepsilon(u)) F^\varepsilon\ du = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2)^2.$$

We introduce the quadratic form $\mathcal{F}(w) = \mathrm{div}_x w\ w - \mathrm{div}_x^\perp w\ {}^\perp w$. By using (19), (20), (21), (22) one gets

$$\mathrm{div}_x E^\varepsilon E^\varepsilon - (B_3^\varepsilon - B_{0,3})^\perp J^\varepsilon = \mathcal{F}(E^\varepsilon) - \frac{1}{2}\nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)^2 - \varepsilon \partial_t \left\{ \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) {}^\perp E^\varepsilon \right\}.$$

Notice that $\mathcal{F}(E^\varepsilon) = \mathrm{div}_x(E^\varepsilon \otimes E^\varepsilon) - \frac{1}{2}\nabla_x |E^\varepsilon|^2$ and since we know that $(E^\varepsilon)_{\varepsilon>0}$ converges strongly towards E in $L^2(]0,T[\times \mathbb{T}^2)^2$ we deduce that

$$\lim_{\varepsilon \searrow 0} \mathcal{F}(E^\varepsilon) = \mathcal{F}(E) = \mathrm{div}_x E\ E \ \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2)^2.$$

Similarly, as $\lim_{\varepsilon \searrow 0} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) = 0$ in $L^2(]0,T[\times \mathbb{T}^2)$ we have

$$\lim_{\varepsilon \searrow 0} \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)^2 = \lim_{\varepsilon \searrow 0} \varepsilon \partial_t \left\{ \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) {}^\perp E^\varepsilon \right\} = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2),$$

and therefore

$$\lim_{\varepsilon \searrow 0} \{\mathrm{div}_x E^\varepsilon\ E^\varepsilon - (B_3^\varepsilon - B_{0,3})\, {}^\perp J^\varepsilon\} = \mathrm{div}_x E\ E \ \ \text{ in } \ \mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2)^2.$$

Passing now to the limit in (27) one gets in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2)^2$

$$\begin{aligned}
\lim_{\varepsilon \searrow 0} B_{0,3} {}^\perp J^\varepsilon &= -\lim_{\varepsilon \searrow 0}\{\rho^\varepsilon E^\varepsilon + (B_3^\varepsilon - B_{0,3}) {}^\perp J^\varepsilon\} \\
&= -\lim_{\varepsilon \searrow 0} E^\varepsilon + \lim_{\varepsilon \searrow 0}\{\mathrm{div}_x E^\varepsilon \ E^\varepsilon - (B_3^\varepsilon - B_{0,3}) {}^\perp J^\varepsilon\} \\
&= -E + \mathrm{div}_x E \ E = -\rho E,
\end{aligned}$$

saying that $\lim_{\varepsilon \searrow 0} J^\varepsilon = \rho \frac{{}^\perp E}{B_{0,3}} = J$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2)^2$. Take $\psi \in C_c^0(\mathbb{T}^2 \times \mathbb{R}^2)$, $\eta > 0$ and consider $r = r(\eta) > 0$ such that

$$|\psi(x,p) - \psi(x,0)| \le \eta, \ \forall \ (x,p) \in \mathbb{T}^2 \times \mathbb{R}^2, \ \ |p| \le r.$$

Obviously for any $|p| > r$ we have the inequality

$$|\psi(x,p) - \psi(x,0)| \le C(\eta,\psi)\left((1+|p|^2)^{\frac{1}{2}} - 1\right),$$

with $C(\eta,\psi) = 2\|\psi\|_{C^0}\left((1+r^2)^{\frac{1}{2}} - 1\right)^{-1}$ and therefore we can write

$$\begin{aligned}
\left|\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f^\varepsilon \psi \ dp \ dx - \int_{\mathbb{T}^2} \rho\psi(x,0) \ dx\right| \ &\le \ \left|\int_{\mathbb{T}^2} (\rho^\varepsilon(t,x) - \rho(t,x))\psi(x,0) \ dx\right| \\
&+ \ \eta \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f^\varepsilon \mathbf{1}_{\{|p| \le r\}} \ dp \ dx \\
&+ C(\eta,\psi)\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\left((1+|p|^2)^{\frac{1}{2}} - 1\right) f^\varepsilon \mathbf{1}_{\{|p|>r\}} dp \ dx.
\end{aligned}$$

Combining the previous assertions of Theorem 2.1 yields the convergence

$$\lim_{\varepsilon \searrow 0} f^\varepsilon(t,x,p) = \rho(t,x)\delta(p), \ \text{ vaguely in } \mathcal{M}_+^1(\mathbb{T}^2 \times \mathbb{R}^2), \text{ uniformly for } t \in [0,T].$$

$\square$

We detail now some lemmas necessary in the proof of Proposition 2.1. We introduce the modulated energy

$$\begin{aligned}
\mathcal{H}^\varepsilon(t) \ &= \ \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\left(\mathcal{E}^\varepsilon(u) - D \cdot u + \frac{|D|^2}{2}\right) F^\varepsilon(t,x,u) \ du \ dx \\
&+ \ \frac{1}{2}\int_{\mathbb{T}^2}\left\{|E^\varepsilon(t,x) - E(t,x)|^2 + \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon}\right)^2\right\} \ dx,
\end{aligned}$$

where $D(t,x) = \frac{J(t,x)}{\rho(t,x)} = \frac{{}^\perp E(t,x)}{B_{0,3}}$. We give some explanations concerning the construction of this modulated energy. Clearly the second integral measures the distance between the electro-magnetic fields $(E^\varepsilon, \varepsilon^{-1}B_3^\varepsilon)$ and $(E, \varepsilon^{-1}B_{0,3})$. The first integral represents the kinetic energy of the particles, with velocities computed with respect to the mean velocity. Actually this is the standard choice for the modulated kinetic energy cf. [3]. Indeed, at least in the non relativistic case (*i.e.*, $h^\varepsilon(t,x,u) = \frac{1}{2}|u - D(t,x)|^2$), changing back u with respect to p yields

$$\varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} h^\varepsilon(t,x,u)F^\varepsilon(t,x,u)\,du\,dx = \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{1}{2}|p - \varepsilon D(t,x)|^2 f^\varepsilon(t,x,p)\,dp\,dx,$$

and our claim follows by observing that for ε small one has

$$\frac{\int_{\mathbb{R}^2} p f^\varepsilon(t,x,p)\,dp}{\int_{\mathbb{R}^2} f^\varepsilon(t,x,p)\,dp} = \frac{j^\varepsilon(t,x)}{\rho^\varepsilon(t,x)} = \varepsilon\frac{J^\varepsilon(t,x)}{\rho^\varepsilon(t,x)} \approx \varepsilon\frac{J(t,x)}{\rho(t,x)} = \varepsilon D(t,x).$$

Generally, further computations (see Lemma 2.4) will show that the modulated energy $\mathcal{H}^\varepsilon(t)$ has a good behavior if the function h^ε satisfies $\nabla_u h^\varepsilon = v^\varepsilon(u) - D(t,x)$. Therefore we take $h^\varepsilon = \frac{1}{2}|u|^2 - D(t,x)\cdot u + \frac{1}{2}|D(t,x)|^2$ in the non relativistic case and $h^\varepsilon = \mathcal{E}^\varepsilon(u) - D(t,x)\cdot u + \frac{1}{2}|D(t,x)|^2$ in the relativistic case.

We intend to study the time evolution of $\mathcal{H}^\varepsilon$. For this we multiply the Vlasov equation (18) by the smooth function $h^\varepsilon(t,x,u) = \mathcal{E}^\varepsilon(u) - D(t,x)\cdot u + \frac{|D(t,x)|^2}{2}$. We perform our computations in several steps by observing that the Vlasov equation can be written

$$\begin{aligned}\varepsilon^2(\partial_t F^\varepsilon + \operatorname{div}_x(v^\varepsilon(u)F^\varepsilon)) - \operatorname{div}_u\left((E(t,x) + B_{0,3}\,{}^\perp v^\varepsilon(u))F^\varepsilon\right) \\ - \operatorname{div}_u\left((E^\varepsilon(t,x) - E(t,x) + (B_3^\varepsilon(t,x) - B_{0,3})\,{}^\perp v^\varepsilon(u))F^\varepsilon\right) = 0.\end{aligned} \tag{39}$$

Lemma 2.1 *For any $0 < \varepsilon < 1$, $T \in \mathbb{R}_+$, $t \in [0,T]$ we have the inequality*

$$\begin{aligned}(1-\varepsilon C)\varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u)F^\varepsilon\,du\,dx - \varepsilon C &\le \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} h^\varepsilon F^\varepsilon\,du\,dx \\ &\le (1+\varepsilon C)\varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u)F^\varepsilon\,du\,dx + \varepsilon C,\end{aligned} \tag{40}$$

where C is a constant depending on $\|D\|_{L^\infty(]0,T[\times\mathbb{T}^2)}$. In particular

$$\varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u)F^\varepsilon\,du\,dx - \varepsilon\tilde{C} \le \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} h^\varepsilon F^\varepsilon\,du\,dx \le \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u)F^\varepsilon\,du\,dx + \varepsilon\tilde{C}.$$

Proof. We use the inequality (37) and therefore we can write

$$\begin{aligned}
\varepsilon^2 \left| \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} D \cdot u F^\varepsilon \, du \, dx \right| &\leq \varepsilon \|D\|_{L^\infty} \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (\varepsilon^2 \mathcal{E}^\varepsilon(u) + 1) F^\varepsilon \, du \, dx \\
&= \varepsilon \|D\|_{L^\infty} \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F^\varepsilon \, du \, dx + \varepsilon \|D\|_{L^\infty},
\end{aligned}$$

implying that

$$\begin{aligned}
\left| \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} h^\varepsilon F^\varepsilon \, du \, dx - \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F^\varepsilon \, du \, dx \right| &\leq \varepsilon^2 \left| \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} D \cdot u F^\varepsilon \, du \, dx \right| \\
&+ \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{|D|^2}{2} F^\varepsilon \, du \, dx \\
&\leq \varepsilon \|D\|_{L^\infty} \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F^\varepsilon \, du \, dx \\
&+ \varepsilon \|D\|_{L^\infty} + \frac{\varepsilon^2}{2} \|D\|^2_{L^\infty},
\end{aligned}$$

and the first statement follows. The second one comes easily by using also the total energy conservation (25). □

Lemma 2.2 *For any* $0 < \varepsilon < 1$, $T \in \mathbb{R}_+$, $t \in [0, T]$ *we have*

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \varepsilon^2 (\partial_t F^\varepsilon + \mathrm{div}_x (v^\varepsilon(u) F^\varepsilon)) h^\varepsilon \, du \, dx = \varepsilon^2 \frac{d}{dt} \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} h^\varepsilon F^\varepsilon \, du \, dx - Q_1(t), \quad (41)$$

where

$$|Q_1(t)| \leq C \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F^\varepsilon(t, x, u) \, du \, dx + C\varepsilon,$$

for some constant depending on $\|D\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)}$.

Proof. Integrating by parts with respect to x we deduce that the term $Q_1(t)$ in (41) has the form

$$Q_1(t) = \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} F^\varepsilon (\partial_t h^\varepsilon + v^\varepsilon(u) \cdot \nabla_x h^\varepsilon) \, du \, dx.$$

Using the inequality (38) we deduce that

$$
\begin{aligned}
|Q_1(t)| &\le \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (|\partial_t D| + |(\nabla_x D) v^\varepsilon(u)|)(|u| + |D|) F^\varepsilon \, du \, dx \\
&\le \varepsilon^2 \|\partial_t D\|_{L^\infty} \|D\|_{L^\infty} + \varepsilon \|\partial_t D\|_{L^\infty} \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (\varepsilon^2 \mathcal{E}^\varepsilon(u) + 1) F^\varepsilon \, du \, dx \\
&+ \varepsilon \|\nabla_x D\|_{L^\infty} \|D\|_{L^\infty} + 2\varepsilon^2 \|\nabla_x D\|_{L^\infty} \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F^\varepsilon \, du \, dx \\
&\le C\varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u) F^\varepsilon \, du \, dx + C\varepsilon.
\end{aligned}
$$

□

Lemma 2.3 *For any $\varepsilon > 0$, $t \in \mathbb{R}_+$ we have*

$$
-\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathrm{div}_u \left((E + B_{0,3} \, {}^\perp v^\varepsilon(u)) F^\varepsilon \right) h^\varepsilon(t, x, u) \, du \, dx = 0. \tag{42}
$$

Proof. We have

$$
\nabla_u h^\varepsilon = v^\varepsilon(u) - D = {}^\perp ({}^\perp D - {}^\perp v^\varepsilon(u)) = {}^\perp \left(-\frac{E}{B_{0,3}} - {}^\perp v^\varepsilon(u) \right) = -\frac{1}{B_{0,3}} {}^\perp (E + B_{0,3} \, {}^\perp v^\varepsilon(u)),
$$

and our conclusion follows easily by integration by parts. □

Lemma 2.4 *For any $0 < \varepsilon < 1$, $T \in \mathbb{R}_+$, $t \in [0, T]$ we have*

$$
\begin{aligned}
& -\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathrm{div}_u \left[(E^\varepsilon - E + (B_3^\varepsilon - B_{0,3}) \, {}^\perp v^\varepsilon(u)) F^\varepsilon \right] h^\varepsilon \, du \, dx \\
&= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \left\{ |E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right)^2 \right\} dx \\
&- \varepsilon \frac{d}{dt} \int_{\mathbb{T}^2} [A_3 + D \cdot {}^\perp (E^\varepsilon - E)] \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right) dx - Q_2(t),
\end{aligned} \tag{43}
$$

where Q_2 satisfies

$$
|Q_2(t)| \le C\varepsilon^2 + C \int_{\mathbb{T}^2} \frac{1}{2} \left\{ |E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right)^2 \right\} dx,
$$

for some constant C depending on $\|A_3\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)}$, $\|D\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)}$.

Proof. After integration by parts with respect to u and by taking into account that $\nabla_u h^\varepsilon = v^\varepsilon(u) - D(t,x)$ one gets

$$
\begin{aligned}
&-\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \operatorname{div}_u \left[(E^\varepsilon - E + (B_3^\varepsilon - B_{0,3})\, {}^\perp v^\varepsilon(u))F^\varepsilon\right] h^\varepsilon \, du\, dx \qquad (44)\\
&= \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} F^\varepsilon (E^\varepsilon - E + (B_3^\varepsilon - B_{0,3})\, {}^\perp v^\varepsilon(u))(v^\varepsilon(u) - D)\, du\, dx\\
&= \int_{\mathbb{T}^2} (E^\varepsilon - E)(J^\varepsilon - \rho^\varepsilon D)\, dx - \int_{\mathbb{T}^2} {}^\perp J^\varepsilon \cdot D\, (B_3^\varepsilon - B_{0,3})\, dx\\
&= \int_{\mathbb{T}^2} (E^\varepsilon - E)\cdot(J^\varepsilon - J)\, dx - \int_{\mathbb{T}^2} D\cdot[(E^\varepsilon - E)(\rho^\varepsilon - \rho) +^\perp (J^\varepsilon - J)(B_3^\varepsilon - B_{0,3})]\, dx,
\end{aligned}
$$

since $D\cdot {}^\perp J = 0$. Combining (19), (20), (21), (36) yields

$$
\partial_t(E_1^\varepsilon - E_1) - \frac{1}{\varepsilon}\partial_{x_2}\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) + \partial_{x_2} A_3 = J_1^\varepsilon - J_1, \qquad (45)
$$

$$
\partial_t(E_2^\varepsilon - E_2) + \frac{1}{\varepsilon}\partial_{x_1}\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) - \partial_{x_1} A_3 = J_2^\varepsilon - J_2, \qquad (46)
$$

$$
\partial_t\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) + \frac{1}{\varepsilon}\partial_{x_1}(E_2^\varepsilon - E_2) - \frac{1}{\varepsilon}\partial_{x_2}(E_1^\varepsilon - E_1) = 0. \qquad (47)
$$

Notice that in the last equation we have used $\partial_{x_1} E_2 - \partial_{x_2} E_1 = 0$. Multiplying (45) by $E_1^\varepsilon - E_1$, (46) by $E_2^\varepsilon - E_2$ and (47) by $\varepsilon^{-1}(B_3^\varepsilon - B_{0,3})$ implies

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\left\{|E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)^2\right\} dx + \int_{\mathbb{T}^2}\{(E_1^\varepsilon - E_1)\partial_{x_2} A_3 - (E_2^\varepsilon - E_2)\partial_{x_1} A_3\}\, dx&\\
= \int_{\mathbb{T}^2} (J^\varepsilon - J)\cdot(E^\varepsilon - E)\, dx.&
\end{aligned}
$$

Using one more time (21) we can write

$$
\begin{aligned}
\int_{\mathbb{T}^2} (E_1^\varepsilon \partial_{x_2} A_3 - E_2^\varepsilon \partial_{x_1} A_3)\, dx &= \int_{\mathbb{T}^2} A_3(\partial_{x_1} E_2^\varepsilon - \partial_{x_2} E_1^\varepsilon)\, dx\\
&= -\int_{\mathbb{T}^2} A_3 \partial_t (B_3^\varepsilon - B_{0,3})\, dx\\
&= -\varepsilon\frac{d}{dt}\int_{\mathbb{T}^2} A_3\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) dx + \varepsilon\int_{\mathbb{T}^2}\partial_t A_3\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) dx.
\end{aligned}
$$

Since $\int_{\mathbb{T}^2}(E_1\partial_{x_2} A_3 - E_2\partial_{x_1} A_3)\, dx = 0$ finally one gets

$$
\begin{aligned}
\int_{\mathbb{T}^2} (E^\varepsilon - E)\cdot(J^\varepsilon - J)\, dx &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\left\{|E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)^2\right\} dx \qquad (48)\\
&\quad - \varepsilon\frac{d}{dt}\int_{\mathbb{T}^2} A_3\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) dx + \varepsilon\int_{\mathbb{T}^2}\partial_t A_3\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) dx.
\end{aligned}
$$

We transform now the last term in (44) using (45), (46), (47) and (22), (31). We have

$$
\begin{aligned}
&-(\rho^\varepsilon-\rho)(E^\varepsilon-E)-(B_3^\varepsilon-B_{0,3})^\perp(J^\varepsilon-J)=\mathrm{div}_x(E^\varepsilon-E)\,(E^\varepsilon-E)\\
-&\quad(B_3^\varepsilon-B_{0,3})\left\{\partial_t^\perp(E^\varepsilon-E)+\frac{1}{\varepsilon}\nabla_x\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)-\nabla_x A_3\right\}\\
=&\quad\mathrm{div}_x(E^\varepsilon-E)\,(E^\varepsilon-E)+\partial_t(B_3^\varepsilon-B_{0,3})\,{}^\perp(E^\varepsilon-E)-\partial_t((B_3^\varepsilon-B_{0,3})^\perp(E^\varepsilon-E))\\
-&\quad\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)\nabla_x\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)+(B_3^\varepsilon-B_{0,3})\nabla_x A_3\\
=&\quad\mathrm{div}_x(E^\varepsilon-E)\,(E^\varepsilon-E)-\mathrm{div}_x^\perp(E^\varepsilon-E)\,{}^\perp(E^\varepsilon-E)-\frac{1}{2}\nabla_x\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)^2\\
-&\quad\varepsilon\partial_t\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\,{}^\perp(E^\varepsilon-E)\right)+\varepsilon\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)\nabla_x A_3.
\end{aligned}\tag{49}
$$

We have the identity

$$
\mathrm{div}_x w\;w-\mathrm{div}_x^\perp w\;{}^\perp w=\mathrm{div}_x(w\otimes w)-\frac{1}{2}\nabla_x|w|^2,
$$

for any $w=(w_1,w_2)\in C^1(\mathbb{R}^2)^2$. Multiplying (49) by D and integrating by parts with respect to x yields

$$
\begin{aligned}
&-\int_{\mathbb{T}^2}D\cdot[(\rho^\varepsilon-\rho)(E^\varepsilon-E)+(B_3^\varepsilon-B_{0,3})\,{}^\perp(J^\varepsilon-J)]\,dx\\
&=-\int_{\mathbb{T}^2}(\nabla_x D(E^\varepsilon-E))(E^\varepsilon-E)\,dx+\frac{1}{2}\int_{\mathbb{T}^2}(\mathrm{div}_x D)\left\{|E^\varepsilon-E|^2+\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)^2\right\}\,dx\\
-&\quad\varepsilon\frac{d}{dt}\int_{\mathbb{T}^2}D\cdot\,{}^\perp(E^\varepsilon-E)\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)\,dx+\varepsilon\int_{\mathbb{T}^2}\partial_t D\cdot^\perp(E^\varepsilon-E)\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)\,dx\\
+&\quad\varepsilon\int_{\mathbb{T}^2}(D\cdot\nabla_x A_3)\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)\,dx.
\end{aligned}\tag{50}
$$

Combining (44), (48), (50) and observing that $\mathrm{div}_x D=0$ we deduce that the term $Q_2(t)$ in (43) has the form

$$
\begin{aligned}
-Q_2(t)=&\;\varepsilon\int_{\mathbb{T}^2}(\partial_t A_3+D\cdot\nabla_x A_3)\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)dx+\varepsilon\int_{\mathbb{T}^2}\partial_t D\cdot^\perp(E^\varepsilon-E)\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)dx\\
&-\int_{\mathbb{T}^2}(\nabla_x D(E^\varepsilon-E))\cdot(E^\varepsilon-E)dx.
\end{aligned}
$$

By using the inequality

$$
\varepsilon\left|(\partial_t A_3+D\cdot\nabla_x A_3)\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)\right|\le\frac{\varepsilon^2}{2}|\partial_t A_3+D\cdot\nabla_x A_3|^2+\frac{1}{2}\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}\right)^2,
$$

we obtain

$$|Q_2(t)| \leq C\varepsilon^2 + C\int_{\mathbb{T}^2} \frac{1}{2}\left\{|E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)^2\right\}\, dx,$$

for some constant depending on $\|A_3\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)}$, $\|D\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)}$. $\square$

Proof. (of Proposition 2.1) Using the Lemmas 2.2, 2.3, 2.4 in (39) yields

$$\frac{d}{dt}\mathcal{H}^\varepsilon(t) - \varepsilon\frac{d}{dt}\mathcal{R}^\varepsilon(t) = Q_1(t) + Q_2(t) \leq C\varepsilon + C\mathcal{W}^\varepsilon(t), \tag{51}$$

where

$$\mathcal{R}^\varepsilon(t) = \int_{\mathbb{T}^2} (A_3 + D \cdot^\perp (E^\varepsilon - E))\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)\, dx,$$

and

$$\mathcal{W}^\varepsilon(t) = \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathcal{E}^\varepsilon(u)F^\varepsilon\, du\, dx + \frac{1}{2}\int_{\mathbb{T}^2}\left\{|E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)^2\right\}\, dx.$$

By Lemma 2.1 we deduce that $|\mathcal{H}^\varepsilon(t) - \mathcal{W}^\varepsilon(t)| \leq C\varepsilon$. Observe also that we have $|\mathcal{R}^\varepsilon(t)| \leq C(1 + \mathcal{W}^\varepsilon(t))$. Integrating (51) over $[0,t]$ one gets

$$\mathcal{H}^\varepsilon(t) - \varepsilon\mathcal{R}^\varepsilon(t) \leq \mathcal{H}^\varepsilon(0) - \varepsilon\mathcal{R}^\varepsilon(0) + C\varepsilon t + C\int_0^t \mathcal{W}^\varepsilon(s), \quad t \in [0,T]. \tag{52}$$

Notice that for any $\varepsilon < 1/(2C)$ we have

$$\mathcal{H}^\varepsilon(t) - \varepsilon\mathcal{R}^\varepsilon(t) \geq \mathcal{W}^\varepsilon(t) - 2C\varepsilon - \varepsilon C\mathcal{W}^\varepsilon(t) \geq \frac{1}{2}\mathcal{W}^\varepsilon(t) - 2C\varepsilon,$$

and

$$\mathcal{H}^\varepsilon(0) - \varepsilon\mathcal{R}^\varepsilon(0) \leq \mathcal{W}^\varepsilon(0) + 2C\varepsilon + \varepsilon C\mathcal{W}^\varepsilon(0) \leq \frac{3}{2}\mathcal{W}^\varepsilon(0) + 2C\varepsilon.$$

Combining the above inequalities with (52) implies

$$\frac{1}{2}\mathcal{W}^\varepsilon(t) \leq \frac{3}{2}\mathcal{W}^\varepsilon(0) + C\varepsilon(4+t) + C\int_0^t \mathcal{W}^\varepsilon(s)\, ds,$$

and the conclusion follows easily by Gronwall lemma. $\square$

We end this section with a convergence result in distribution sense for $\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)_{\varepsilon>0}$.

Corollary 2.1 *Besides the hypotheses of Theorem* 2.1 *assume that the following condition holds*

H5) $\lim_{\varepsilon\searrow 0} \int_{\mathbb{T}^2} \frac{B^\varepsilon_{0,3}(x)-B_{0,3}}{\varepsilon^2}\ dx = 0$.

Then we have the convergence $\lim_{\varepsilon\searrow 0} \frac{B^\varepsilon_3 - B_{0,3}}{\varepsilon^2} = A_3$ *in* $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2)$.

Proof. Combining (45), (46) we have

$$\partial_t(E^\varepsilon - E) -^\perp \nabla_x(A^\varepsilon_3 - A_3) = J^\varepsilon - J,$$

where $A^\varepsilon_3 := \frac{B^\varepsilon_3 - B_{0,3}}{\varepsilon^2}$. We deduce easily that

$$\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}_+}\int_{\mathbb{T}^2} (A^\varepsilon_3 - A_3)\nabla_x\varphi\ dxdt = 0,\ \forall\ \varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{T}^2).$$

In particular we have $\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}_+}\int_{\mathbb{T}^2} (A^\varepsilon_3 - A_3)\mathrm{div}_x\varphi\ dxdt = 0$ for any $\varphi \in C^1_c(\mathbb{R}_+ \times \mathbb{T}^2)^2$. Take now $\psi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{T}^2)$ satisfying $\int_{\mathbb{T}^2}\psi(t,x)\ dx = 0,\ t \in \mathbb{R}_+$ and denote by u the solution of $-\Delta_x u(t) = \psi(t,x),\ x \in \mathbb{T}^2, t \in \mathbb{R}_+$, verifying $\int_{\mathbb{T}^2} u(t,x)\ dx = 0, t \in \mathbb{R}_+$. We have

$$\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}_+}\int_{\mathbb{T}^2} (A^\varepsilon_3 - A_3)\psi(t,x)\ dxdt = -\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}_+}\int_{\mathbb{T}^2} (A^\varepsilon_3 - A_3)\mathrm{div}_x(\nabla_x u)\ dxdt = 0.$$

Take now $\psi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{T}^2)$ and observing that $\psi - \int_{\mathbb{T}^2}\psi\ dx$ has zero space average we obtain

$$\begin{aligned}\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}_+}\int_{\mathbb{T}^2} (A^\varepsilon_3 - A_3)\psi(t,x)\ dxdt &= \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}_+}\int_{\mathbb{T}^2} (A^\varepsilon_3 - A_3)(\psi - \langle\psi\rangle + \langle\psi\rangle)\ dxdt \\ &= \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}_+} \langle\psi(t)\rangle \int_{\mathbb{T}^2} (A^\varepsilon_3 - A_3)\ dxdt.\end{aligned}$$

Recall that by definition $\int_{\mathbb{T}^2} A_3(t,x)\ dx = 0,\ t \in \mathbb{R}_+$ and by integrating (47) we deduce that $\frac{d}{dt}\int_{\mathbb{T}^2} A^\varepsilon_3(t,x)\ dx = 0$. Therefore the hypothesis H5 yields

$$\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}_+}\int_{\mathbb{T}^2} (A^\varepsilon_3 - A_3)\psi(t,x)\ dxdt = \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}_+} \langle\psi(t)\rangle \int_{\mathbb{T}^2} A^\varepsilon_3(0,x)\ dxdt = 0.$$

$\square$

3 Other systems

The same method applies for studying other models. We consider here the Vlasov-Maxwell system in the non relativistic setting and mono-kinetic models.

3.1 Non relativistic model

In the two dimensional case the non relativistic Vlasov equation (9) becomes

$$\partial_t f^\varepsilon + \frac{p}{\varepsilon} \cdot \nabla_x f^\varepsilon - \frac{1}{\varepsilon}\left(E^\varepsilon(t,x) + \frac{B_3^\varepsilon(t,x)}{\varepsilon}\, {}^\perp p\right) \cdot \nabla_p f^\varepsilon = 0, \;\; (t,x,p) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2, \tag{53}$$

where $(E_1^\varepsilon, E_2^\varepsilon, B_3^\varepsilon)_{\varepsilon>0}$ solve the two dimensional Maxwell equations with the charge density $\int_{\mathbb{R}^2} f^\varepsilon \, dp$ and the current density $\int_{\mathbb{R}^2} p f^\varepsilon \, dp$. Rescaling the momentum by $p = \varepsilon u$ and the particle density by $F^\varepsilon(t,x,u) = \varepsilon^2 f^\varepsilon(t,x,\varepsilon u)$ leads to the same equations as those in (18), (19), (20), (21), (22) with $v^\varepsilon(u)$ replaced by u and $J^\varepsilon(t,x)$ replaced by $\int_{\mathbb{R}^2} u F^\varepsilon(t,x,u)\, du$. We assume that the hypotheses H1, H3, H4 hold and we replace H2 by

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{|p|^2}{2} f_0^\varepsilon(x,p)\, dp\, dx = 0,$$

or equivalently by

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{|u|^2}{2} F_0^\varepsilon(x,u)\, du\, dx = 0.$$

Following the previous method we show the convergence towards a solution (ρ, J, E) of (29), (30), (31). The modulated energy is given by

$$\begin{aligned}\mathcal{H}_2^\varepsilon(t) &= \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{1}{2}|u - D(t,x)|^2 F^\varepsilon(t,x,u)\, du\, dx \\ &+ \frac{1}{2}\int_{\mathbb{T}^2}\left\{ |E^\varepsilon(t,x) - E(t,x)|^2 + \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon}\right)^2 \right\} dx,\end{aligned}$$

where $D(t,x) = \frac{J(t,x)}{\rho(t,x)} = \frac{{}^\perp E(t,x)}{B_{0,3}}$.

Proposition 3.1 *There is a constant C such that for any $\varepsilon > 0$ small enough and $t \in [0,T]$ we have*

$$\mathcal{H}_2^\varepsilon(t) \le C(\varepsilon^2 + \mathcal{H}_2^\varepsilon(0)). \tag{54}$$

Moreover if

$$\sup_{\varepsilon>0} \varepsilon^{-2} \left\{ \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{|p|^2}{2} f_0^\varepsilon(x,p)\, dp\, dx + \frac{1}{2}\int_{\mathbb{T}^2} |E_0^\varepsilon - E_0|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)^2 dx \right\} < +\infty,$$

then

$$\sup_{\varepsilon>0, t\in[0,T]} \varepsilon^{-2}\mathcal{H}_2^\varepsilon(t) < +\infty,\ \forall\, T \in \mathbb{R}_+. \tag{55}$$

In particular

$$\sup_{\varepsilon>0} \varepsilon^{-1}\|E^\varepsilon - E\|_{L^\infty(]0,T[;L^2(\mathbb{T}^2)^2)} + \sup_{\varepsilon>0}\varepsilon^{-1}\left\|\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right\|_{L^\infty(]0,T[;L^2(\mathbb{T}^2))} < +\infty,$$

and

$$\sup_{\varepsilon>0}\varepsilon^{-1}\|\rho^\varepsilon - \rho\|_{L^\infty(]0,T[;H^{-1}(\mathbb{T}^2))} + \sup_{\varepsilon>0}\varepsilon^{-1}\|J^\varepsilon - J\|_{W^{-1,1}(]0,T[\times\mathbb{T}^2)^2} < +\infty.$$

Proof. Let us give some details. As in Lemma 2.2, by using the inequality $|u-D| \leq 1/2 + |u-D|^2/2$ we have

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{\varepsilon^2}{2}(\partial_t F^\varepsilon + \mathrm{div}_x(uF^\varepsilon))|u-D|^2\, du\, dx = \frac{\varepsilon^2}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} F^\varepsilon |u-D|^2\, du\, dx - \tilde{Q}_1(t),$$

where

$$|\tilde{Q}_1(t)| \leq C\varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\frac{1}{2}|u-D|^2 F^\varepsilon\, du\, dx + C\varepsilon^2,$$

for some constant depending on $\|D\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)^2}$. Exactly as in the relativistic case (see Lemmas $2.3, 2.4$) we have

$$-\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathrm{div}_u\left((E + B_{0,3}{}^\perp u)F^\varepsilon\right)\frac{1}{2}|u-D|^2\, du\, dx = 0,$$

and

$$\begin{aligned}
&-\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\mathrm{div}_u\left((E^\varepsilon - E + (B_3^\varepsilon - B_{0,3})^\perp u)F^\varepsilon\right)\frac{1}{2}|u-D|^2\, du\, dx \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\left(|E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)^2\right) dx \\
&- \varepsilon\frac{d}{dt}\int_{\mathbb{T}^2}(A_3 + D\cdot^\perp(E^\varepsilon - E))\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) dx - \tilde{Q}_2(t),
\end{aligned}$$

where

$$|\tilde{Q}_2(t)| \le C \int_{\mathbb{T}^2} \frac{1}{2} \left(|E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right)^2 \right) \, dx + C\varepsilon^2,$$

for some constant C depending on $\|A_3\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)}$, $\|D\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)^2}$. Combining the above computations yields

$$\frac{d}{dt}\mathcal{H}_2^\varepsilon(t) - \varepsilon \frac{d}{dt}\mathcal{R}_2^\varepsilon \le C\varepsilon^2 + C\mathcal{H}_2^\varepsilon(t),$$

where $\mathcal{R}_2^\varepsilon(t) = \int_{\mathbb{T}^2} (A_3 + D \cdot^\perp (E^\varepsilon - E)) \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right) \, dx$. The inequality (59) follows immediately by Gronwall lemma, using that

$$\begin{aligned} \varepsilon |\mathcal{R}_2^\varepsilon(t)| &\le \varepsilon^2 \int_{\mathbb{T}^2} |A_3(t,x)|^2 \, dx + \frac{1}{4} \int_{\mathbb{T}^2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right)^2 \, dx + C\varepsilon \mathcal{H}_2^\varepsilon(t) \\ &\le \left(\frac{1}{2} + C\varepsilon \right) \mathcal{H}_2^\varepsilon(t) + C\varepsilon^2 < \frac{3}{4}\mathcal{H}_2^\varepsilon(t) + C\varepsilon^2, \end{aligned}$$

for ε small enough. The bound of $\left(\frac{\rho^\varepsilon - \rho}{\varepsilon}\right)_{\varepsilon>0}$ in $L^\infty(]0,T[; H^{-1}(\mathbb{T}^2))$ is obvious. The estimate for $\left(\frac{J^\varepsilon - J}{\varepsilon}\right)_{\varepsilon>0}$ follows by combining the arguments in Theorem 2.1 and (55). Indeed, by the non relativistic version of (27) we have

$$\begin{aligned} {B_{0,3}}^\perp (J^\varepsilon - J) &= -\varepsilon^2 \left(\partial_t \int_{\mathbb{R}^2} uF^\varepsilon \, du + \operatorname{div}_x \int_{\mathbb{R}^2} (u \otimes u) F^\varepsilon \, du \right) \\ &- (E^\varepsilon - E) + \operatorname{div}_x E^\varepsilon E^\varepsilon - (B_3^\varepsilon - B_{0,3})^\perp J^\varepsilon - \operatorname{div}_x EE. \end{aligned}$$

For any $\varphi \in W^{1,\infty}(]0,T[\times\mathbb{T}^2)^2$ we have by (55)

$$\sup_{\varepsilon>0} \varepsilon^2 \left| \langle \partial_t \int_{\mathbb{R}^2} uF^\varepsilon \, du + \operatorname{div}_x \int_{\mathbb{R}^2} (u \otimes u) F^\varepsilon \, du, \varphi \rangle \right| \le C\varepsilon^2 \|\varphi\|_{W^{1,\infty}(\mathbb{R}_+\times\mathbb{T}^2)^2},$$

and $|\langle E^\varepsilon - E, \varphi \rangle| \le C\varepsilon \|\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{T}^2)^2}$. As in the proof of Theorem 2.1 we can write

$$\begin{aligned} S^\varepsilon &:= \operatorname{div}_x E^\varepsilon E^\varepsilon - (B_3^\varepsilon - B_{0,3})^\perp J^\varepsilon - \operatorname{div}_x EE \\ &= \mathcal{F}(E^\varepsilon) - \mathcal{F}(E) - \frac{1}{2}\nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right)^2 - \varepsilon \partial_t \left\{ \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right) {}^\perp E^\varepsilon \right\}. \end{aligned}$$

It is easily seen that $|\langle S^\varepsilon, \varphi \rangle| \le C\varepsilon \|\varphi\|_{W^{1,\infty}(\mathbb{R}_+\times\mathbb{T}^2)^2}$. Finally one gets

$$|B_{0,3} \langle {}^\perp (J^\varepsilon - J), \varphi \rangle| \le C\varepsilon \|\varphi\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)^2},$$

saying that $\sup_{\varepsilon>0} \varepsilon^{-1} \|J^\varepsilon - J\|_{W^{-1,1}(]0,T[\times\mathbb{T}^2)^2} < +\infty$. $\square$

Remark 3.1 *The previous result says that the solution of the limit system is a first order approximation for the non relativistic system* (53), (19), (20), (21), (22), *i.e.,*

$$E^\varepsilon = E + \varepsilon\mathcal{O}(\varepsilon), \quad \frac{B_3^\varepsilon}{\varepsilon} = \frac{B_{0,3}}{\varepsilon} + \varepsilon\mathcal{O}(\varepsilon), \quad \rho^\varepsilon = \rho + \varepsilon\mathcal{O}(\varepsilon), \quad J^\varepsilon = J + \varepsilon\mathcal{O}(\varepsilon),$$

in the corresponding spaces.

3.2 Mono-kinetic model

As in [16] we analyze also the case of distribution functions of the form

$$F^\varepsilon(t,x,u) = \rho^\varepsilon(t,x)\delta(u - u^\varepsilon(t,x)), \quad (t,x,u) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2,$$

with a macroscopic density $\rho^\varepsilon(t,x)$ and a bulk velocity $u^\varepsilon(t,x)$, or equivalently

$$f^\varepsilon(t,x,p) = \rho^\varepsilon(t,x)\delta(p - \varepsilon u^\varepsilon(t,x)), \quad (t,x,p) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2.$$

Following [6] the mass and momentum conservations lead to the equations

$$\partial_t \rho^\varepsilon + \mathrm{div}_x(\rho^\varepsilon u^\varepsilon) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}^2, \tag{56}$$

$$\partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}_x(\rho^\varepsilon(u^\varepsilon \otimes u^\varepsilon)) + \frac{1}{\varepsilon^2}\rho^\varepsilon(E^\varepsilon(t,x) + B_3^\varepsilon(t,x)\ {}^\perp u^\varepsilon(t,x)) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}^2, \tag{57}$$

coupled to the Maxwell equations (19), (20), (21), (22) with $J^\varepsilon(t,x) = \rho^\varepsilon(t,x)u^\varepsilon(t,x)$. By standard computations we obtain the conservation of the total energy

$$\frac{d}{dt}\left\{\int_{\mathbb{T}^2} \frac{\varepsilon^2}{2}|u^\varepsilon(t,x)|^2\rho^\varepsilon(t,x)\,dx + \frac{1}{2}\int_{\mathbb{T}^2}\left(|E^\varepsilon(t,x)|^2 + \left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon}\right)^2\right)\,dx\right\} = 0. \tag{58}$$

We obtain the same limit system

$$u(t,x) = \frac{{}^\perp E(t,x)}{B_{0,3}}, \quad \mathrm{div}_x^\perp E = 0, \quad \partial_t\rho + \frac{{}^\perp E}{B_{0,3}}\cdot\nabla_x\rho = 0, \quad \mathrm{div}_x E = 1 - \rho, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}^2.$$

We work with smooth solutions $(\rho^\varepsilon, u^\varepsilon, E_1^\varepsilon, E_2^\varepsilon, B_3^\varepsilon)_{\varepsilon>0}$, (ρ, u, E) and we define the modulated energy

$$\begin{aligned}\mathcal{H}_3^\varepsilon(t) &= \int_{\mathbb{T}^2}\frac{\varepsilon^2}{2}|u^\varepsilon(t,x) - u(t,x)|^2\rho^\varepsilon(t,x)\,dx + \frac{1}{2}\int_{\mathbb{T}^2}|E^\varepsilon(t,x) - E(t,x)|^2\,dx \\ &+ \frac{1}{2}\int_{\mathbb{T}^2}\left(\frac{B_3^\varepsilon(t,x) - B_{0,3}}{\varepsilon}\right)^2\,dx.\end{aligned}$$

We assume that the initial conditions satisfy the hypotheses

$$\rho_0^\varepsilon \geq 0, \quad \int_{\mathbb{T}^2} \rho_0^\varepsilon(x)\, dx = 1, \quad \lim_{\varepsilon \searrow 0} \varepsilon^2 \int_{\mathbb{T}^2} \frac{|u_0^\varepsilon(x)|^2}{2} \rho_0^\varepsilon(x)\, dx = 0,$$

and H3, H4.

Proposition 3.2 *There is a constant C such that for any $\varepsilon > 0$ small enough and $t \in [0, T]$ we have*

$$\mathcal{H}_3^\varepsilon(t) \leq C(\varepsilon^2 + \mathcal{H}_3^\varepsilon(0)). \tag{59}$$

In particular

$$\lim_{\varepsilon \searrow 0} \|E^\varepsilon - E\|_{L^\infty(]0,T[;L^2(\mathbb{T}^2)^2)} = \lim_{\varepsilon \searrow 0} \left\| \frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right\|_{L^\infty(]0,T[;L^2(\mathbb{T}^2))} < +\infty = 0,$$

and

$$\lim_{\varepsilon \searrow 0} \rho^\varepsilon = \rho, \quad \lim_{\varepsilon \searrow 0} (\rho^\varepsilon u^\varepsilon) = \rho u = \rho \frac{{}^\perp E}{B_{0,3}} \quad in \ \mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2).$$

Proof. We study the time evolution of $\mathcal{H}_3^\varepsilon$ by using the equations for $(\rho^\varepsilon, u^\varepsilon, E_1^\varepsilon, E_2^\varepsilon, B_3^\varepsilon)$ and (ρ, u, E). By using (56) notice that (57) can be written

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla_x) u^\varepsilon + \frac{1}{\varepsilon^2}(E^\varepsilon(t,x) + B_3^\varepsilon(t,x)^\perp u^\varepsilon(t,x)) = 0. \tag{60}$$

We deduce that

$$\partial_t(u^\varepsilon - u) + (u^\varepsilon \cdot \nabla_x)(u^\varepsilon - u) + \frac{1}{\varepsilon^2}(E^\varepsilon(t,x) + B_3^\varepsilon(t,x)^\perp u^\varepsilon(t,x)) = -\partial_t u - (u^\varepsilon \cdot \nabla_x) u.$$

Multiplying by $\rho^\varepsilon(u^\varepsilon - u)$ yields

$$\begin{aligned} \frac{\rho^\varepsilon}{2} \partial_t |u^\varepsilon - u|^2 \ &+\ \frac{\rho^\varepsilon}{2}(u^\varepsilon \cdot \nabla_x)|u^\varepsilon - u|^2 + \frac{\rho^\varepsilon}{\varepsilon^2}(E^\varepsilon + B_3^{\varepsilon\perp} u^\varepsilon) \cdot (u^\varepsilon - u) \\ &=\ -\rho^\varepsilon(\partial_t u + (u^\varepsilon \cdot \nabla_x) u) \cdot (u^\varepsilon - u). \end{aligned} \tag{61}$$

Adding to the above equation the equation (56) multiplied by $|u^\varepsilon - u|^2/2$ we deduce that

$$\begin{aligned} \frac{1}{2}\partial_t(\rho^\varepsilon |u^\varepsilon - u|^2) \ &+\ \frac{1}{2}\mathrm{div}_x(\rho^\varepsilon |u^\varepsilon - u|^2 u^\varepsilon) + \frac{\rho^\varepsilon}{\varepsilon^2}(E^\varepsilon + B_3^{\varepsilon\perp} u^\varepsilon) \cdot (u^\varepsilon - u) \\ &=\ -\rho^\varepsilon(\partial_t u + (u^\varepsilon \cdot \nabla_x) u) \cdot (u^\varepsilon - u). \end{aligned} \tag{62}$$

Notice that ${}^{\perp}(u^\varepsilon - u) = {}^{\perp} u^\varepsilon + \frac{E}{B_{0,3}}$ and thus $(E + B_{0,3}\,{}^{\perp}u^\varepsilon) \cdot (u^\varepsilon - u) = 0$, implying that

$$\begin{aligned}\rho^\varepsilon(E^\varepsilon + B_3^\varepsilon\,{}^{\perp}u^\varepsilon) \cdot (u^\varepsilon - u) \ &= \ (E^\varepsilon - E) \cdot (\rho^\varepsilon u^\varepsilon - \rho u) + \mathrm{div}_x(E^\varepsilon - E)\ (E^\varepsilon - E) \cdot u \\ &- \ (B_3^\varepsilon - B_{0,3})\,{}^{\perp}(\rho^\varepsilon u^\varepsilon - \rho u) \cdot u. \end{aligned} \tag{63}$$

Using now the equations

$$\partial_t(E_1^\varepsilon - E_1) - \frac{1}{\varepsilon}\partial_{x_2}\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) + \partial_{x_2}A_3 = \rho^\varepsilon u_1^\varepsilon - \rho u_1,$$

$$\partial_t(E_2^\varepsilon - E_2) + \frac{1}{\varepsilon}\partial_{x_1}\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) - \partial_{x_1}A_3 = \rho^\varepsilon u_2^\varepsilon - \rho u_2,$$

$$\partial_t\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right) + \frac{1}{\varepsilon}\partial_{x_1}(E_2^\varepsilon - E_2) - \frac{1}{\varepsilon}\partial_{x_2}(E_1^\varepsilon - E_1) = 0,$$

one gets as before

$$\begin{aligned}\int_{\mathbb{T}^2}(E^\varepsilon - E) \cdot (\rho^\varepsilon u^\varepsilon - \rho u)\,dx &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\left(|E^\varepsilon - E|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)^2\right)\,dx \\ - \ \ \varepsilon\frac{d}{dt}\int_{\mathbb{T}^2}A_3\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)\,dx &+ \varepsilon\int_{\mathbb{T}^2}\partial_t A_3\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)\,dx, \end{aligned} \tag{64}$$

and

$$\begin{aligned}&\int_{\mathbb{T}^2}\{\mathrm{div}_x(E^\varepsilon - E)\ (E^\varepsilon - E) - (B_3^\varepsilon - B_{0,3})^{\perp}(\rho^\varepsilon u^\varepsilon - \rho u)\} \cdot u\,dx \\ = \ &- \int_{\mathbb{T}^2}((\nabla_x u)(E^\varepsilon - E)) \cdot (E^\varepsilon - E)\,dx - \varepsilon\frac{d}{dt}\int_{\mathbb{T}^2}\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right){}^{\perp}(E^\varepsilon - E) \cdot u\,dx \\ + \ \ &\varepsilon\int_{\mathbb{T}^2}\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right){}^{\perp}(E^\varepsilon - E) \cdot \partial_t u\,dx + \varepsilon\int_{\mathbb{T}^2}\left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon}\right)\nabla_x A_3 \cdot u\,dx. \end{aligned} \tag{65}$$

Combining (62), (63), (64), (65) and the energy conservation (58) we obtain

$$\mathcal{H}_3^\varepsilon(t) \leq C(\varepsilon^2 + \mathcal{H}_3^\varepsilon(0)) + C\int_0^t \mathcal{H}_3^\varepsilon(s)\,ds,$$

implying by Gronwall lemma that $\lim_{\varepsilon \searrow 0}\mathcal{H}_3^\varepsilon(t) = 0$ uniformly on compact subsets of $\mathbb{R}_+$. Therefore we deduce the convergences

$$\lim_{\varepsilon \searrow 0}\left(E_1^\varepsilon(t) - E_1(t), E_2^\varepsilon(t) - E_2(t), \frac{B_3^\varepsilon(t) - B_{0,3}}{\varepsilon}\right) = (0,0,0) \ \text{ strongly in } L^2(\mathbb{T}^2)^3,$$

uniformly for t in compact subsets of $\mathbb{R}_+$. We can show as before the convergence of the charge and current densities in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{T}^2)$

$$\lim_{\varepsilon \searrow 0} \rho^\varepsilon = \rho, \quad \lim_{\varepsilon \searrow 0} (\rho^\varepsilon u^\varepsilon) = \rho u = \rho \frac{{}^\perp E}{B_{0,3}}.$$

$\square$

4 Second order approximation

It was shown that the guiding-center approximation applies for very large initial magnetic fields, *i.e.,* for very small values of the parameter ε. But situations with ε not so small could occur and in these cases the above approximations are not good enough; a higher order analysis is required. In this section we discuss the second order approximation. In order to simplify the computations we consider the non relativistic case

$$\partial_t F^\varepsilon + u \cdot \nabla_x F^\varepsilon - \frac{1}{\varepsilon^2}(E^\varepsilon(t,x) + B_3^\varepsilon(t,x)\,{}^\perp u) \cdot \nabla_u F^\varepsilon = 0, \tag{66}$$

$$\partial_t E^\varepsilon - \frac{1}{\varepsilon}{}^\perp \nabla_x \left(\frac{B_3^\varepsilon}{\varepsilon} \right) = J^\varepsilon(t,x), \tag{67}$$

$$\partial_t \left(\frac{B_3^\varepsilon}{\varepsilon} \right) + \frac{1}{\varepsilon} \mathrm{div}_x {}^\perp E^\varepsilon = 0, \tag{68}$$

$$\mathrm{div}_x E^\varepsilon = 1 - \rho^\varepsilon(t,x), \tag{69}$$

where $F^\varepsilon(t,x,u) = \varepsilon^2 f^\varepsilon(t,x,p)$, $p = \varepsilon u$. As usual we start with a formal analysis. Let us search for

$$F^\varepsilon = F + \varepsilon F^{(1)} + \varepsilon^2 F^{(2)} + ...,$$

$$E^\varepsilon = E + \varepsilon E^{(1)} + \varepsilon^2 E^{(2)} + ...,$$

$$B_3^\varepsilon = B_{0,3} + \varepsilon^2 A_3 + \varepsilon^3 A_3^{(1)} + ...$$

Notice that since $\lim_{\varepsilon \searrow 0} \frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} = 0$ in $L^2(]0,T[; L^2(\mathbb{T}^2))$ for any $T \in \mathbb{R}_+$ there is no first order term in the expansion of B_3^ε. We denote by (ρ, J), $(\rho^{(k)}, J^{(k)})_{k \geq 1}$

the charge and current densities of F, $(F^{(k)})_{k\geq 1}$. Since $\partial_t \rho^\varepsilon + \mathrm{div}_x J^\varepsilon = 0$ we have $\partial_t \rho + \mathrm{div}_x J = 0$ and $\partial_t \rho^{(k)} + \mathrm{div}_x J^{(k)} = 0$ for any $k \geq 1$. Plugging these ansatz into the Vlasov equation (66) yields

$$-(E + B_{0,3}{}^\perp u) \cdot \nabla_u F = 0, \tag{70}$$

$$-(E + B_{0,3}{}^\perp u) \cdot \nabla_u F^{(1)} - E^{(1)} \cdot \nabla_u F = 0, \tag{71}$$

$$\partial_t F + u \cdot \nabla_x F - (E + B_{0,3}{}^\perp u) \cdot \nabla_u F^{(2)} - E^{(1)} \cdot \nabla_u F^{(1)} - (E^{(2)} + A_3{}^\perp u) \cdot \nabla_u F = 0. \tag{72}$$

Multiplying (70), (71) by u and integrating with respect to u yields

$$\rho E + B_{0,3}{}^\perp J = 0, \;\; \rho^{(1)} E + B_{0,3}{}^\perp J^{(1)} + \rho E^{(1)} = 0,$$

which is equivalent to

$$J = \rho \frac{{}^\perp E}{B_{0,3}}, \;\; J^{(1)} = \frac{\rho^{(1)\perp} E + \rho\, {}^\perp E^{(1)}}{B_{0,3}}. \tag{73}$$

Multiplying now (72) by u we obtain after integration with respect to u

$$\partial_t J + \mathrm{div}_x \int_{\mathbb{R}^2} (u \otimes u) F \, du + \rho^{(2)} E + B_{0,3}{}^\perp J^{(2)} + \rho^{(1)} E^{(1)} + \rho E^{(2)} + A_3{}^\perp J = 0,$$

which is equivalent to

$$\partial_t {}^\perp J + \mathrm{div}_x \int_{\mathbb{R}^2} ({}^\perp u \otimes u) F \, du + \rho^{(2)\perp} E - B_{0,3} J^{(2)} + \rho^{(1)\perp} E^{(1)} + \rho^\perp E^{(2)} - A_3 J = 0. \tag{74}$$

Multiplying now (70) by u_1^2 and $u_1 u_2$ implies

$$\int_{\mathbb{R}^2} u_1 u_2 F \, du = -\frac{E_1 J_1}{B_{0,3}} = -\rho \frac{E_1 E_2}{(B_{0,3})^2}, \tag{75}$$

and

$$\int_{\mathbb{R}^2} (u_2^2 - u_1^2) F \, du = -\frac{E_1 J_2 + E_2 J_1}{B_{0,3}} = -\rho \frac{(E_2)^2 - (E_1)^2}{(B_{0,3})^2}. \tag{76}$$

We deduce that

$$(\nabla_x \otimes \nabla_x) : \int_{\mathbb{R}^2} ({}^\perp u \otimes u) F \, du = -\frac{1}{(B_{0,3})^2} (\nabla_x \otimes \nabla_x) : (\rho^\perp E \otimes E), \tag{77}$$

and finally by taking the divergence in (74) we find

$$
\begin{aligned}
\partial_t \mathrm{div}_x^\perp J - \frac{1}{(B_{0,3})^2}(\nabla_x \otimes \nabla_x) : (\rho^\perp E \otimes E) + \mathrm{div}_x(\rho^{(2)\perp} E) - B_{0,3} \mathrm{div}_x J^{(2)} \\
+ \ \mathrm{div}_x(\rho^{(1)\perp} E^{(1)}) + \mathrm{div}_x(\rho^\perp E^{(2)}) - \mathrm{div}_x(A_3 J) = 0. \qquad (78)
\end{aligned}
$$

Plugging now the asymptotic expansions into the Maxwell equations (67), (68), (69) and combining with the previous equations yields the systems

$$
J = \rho \frac{{}^\perp E}{B_{0,3}}, \ \ \partial_t E -^\perp \nabla_x A_3 = J, \ \ \mathrm{div}_x^\perp E = 0, \ \ \mathrm{div}_x E = 1 - \rho, \qquad (79)
$$

$$
J^{(1)} = \frac{\rho^{(1)\perp} E + \rho^\perp E^{(1)}}{B_{0,3}}, \ \partial_t E^{(1)} -^\perp \nabla_x A_3^{(1)} = J^{(1)}, \ \mathrm{div}_x^\perp E^{(1)} = 0, \ \mathrm{div}_x E^{(1)} = -\rho^{(1)}, \qquad (80)
$$

$$
\begin{cases}
\partial_t A_3 + \mathrm{div}_x^\perp E^{(2)} = 0, \ \ \mathrm{div}_x E^{(2)} = -\rho^{(2)}, \\
B_{0,3}\partial_t \rho^{(2)} +^\perp E \cdot \nabla_x \rho^{(2)} +^\perp E^{(2)} \cdot \nabla_x \rho \ = -\partial_t \mathrm{div}_x{}^\perp J + \dfrac{1}{(B_{0,3})^2}(\nabla_x \otimes \nabla_x) : (\rho^\perp E \otimes E) \\
\qquad -^\perp E^{(1)} \cdot \nabla_x \rho^{(1)} + \mathrm{div}_x(A_3 J) + \rho \partial_t A_3.
\end{cases} \qquad (81)
$$

Obviously the system (79) is exactly the limit system (29), (30), (31). In order to solve the second system it is convenient to eliminate $J^{(1)}$ by taking the divergence of the time evolution equation for $E^{(1)}$ (or by using the continuity equation $\partial_t \rho^{(1)} + \mathrm{div}_x J^{(1)} = 0$). We obtain

$$
B_{0,3}\partial_t \rho^{(1)} +^\perp E \cdot \nabla_x \rho^{(1)} +^\perp E^{(1)} \cdot \nabla_x \rho = 0, \ \ \mathrm{div}_x{}^\perp E^{(1)} = 0, \ \ \mathrm{div}_x E^{(1)} = -\rho^{(1)}.
$$

The last equation of the third system was obtained by eliminating $J^{(2)}$ in (78) using the continuity equation $\partial_t \rho^{(2)} + \mathrm{div}_x J^{(2)} = 0$. The equations $\mathrm{div}_x{}^\perp E = \mathrm{div}_x{}^\perp E^{(1)} = 0$ and $\mathrm{div}_x{}^\perp E^{(2)} = -\partial_t A_3$ have been used as well. We assume that all these systems have smooth solutions (essentially we need that these solutions belong to $W^{2,\infty}(]0,T[\times \mathbb{T}^2)$ for any $T \in \mathbb{R}_+$). We define the modulated energy

$$
\begin{aligned}
\mathcal{H}_4^\varepsilon(t) \ &= \ \varepsilon^2 \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{1}{2}\left| u - \frac{{}^\perp(E + \varepsilon E^{(1)})}{B_{0,3}} \right|^2 F^\varepsilon(t,x,u)\, du\, dx \\
&+ \ \frac{1}{2}\int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2(t,x)\, dx + \frac{1}{2}\int_{\mathbb{T}^2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 (t,x)\, dx.
\end{aligned}
$$

We assume that $\sup_{\varepsilon>0} \varepsilon^{-4}\mathcal{H}_4^\varepsilon(0) < +\infty$, which is equivalent to

$$H6)\quad \sup_{\varepsilon>0} \varepsilon^{-4}\{\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{\varepsilon^2}{2}\left|u - \frac{{}^\perp E_0}{B_{0,3}}\right|^2 F_0^\varepsilon \, du \, dx + \frac{1}{2}\int_{\mathbb{T}^2} |E_0^\varepsilon - E_0 - \varepsilon E_0^{(1)}|^2 \, dx + \frac{1}{2}\int_{\mathbb{T}^2}\left(\frac{B_{0,3}^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_{0,3}\right)^2 \, dx\} < +\infty.$$

The following results establishes that $\left(E + \varepsilon E^{(1)}, \frac{B_{0,3}}{\varepsilon} + \varepsilon A_3\right)$, $(\rho + \varepsilon\rho^{(1)}, J + \varepsilon J^{(1)})$ are second order approximations for $\left(E^\varepsilon, \frac{B_3^\varepsilon}{\varepsilon}\right)$, $(\rho^\varepsilon, J^\varepsilon)$ where (ρ, J, E), $(\rho^{(1)}, J^{(1)}, E^{(1)})$ solve

$$\begin{cases} \partial_t\rho + \frac{{}^\perp E}{B_{0,3}}\cdot\nabla_x\rho = 0, & \mathrm{div}_x{}^\perp E = 0, & \mathrm{div}_x E = 1-\rho, \\ \partial\rho^{(1)} + \frac{{}^\perp E}{B_{0,3}}\cdot\nabla_x\rho^{(1)} + \frac{{}^\perp E^{(1)}}{B_{0,3}}\cdot\nabla_x\rho = 0, & \mathrm{div}_x{}^\perp E^{(1)} = 0, & \mathrm{div}_x E^{(1)} = -\rho^{(1)}, \\ J = \rho\frac{{}^\perp E}{B_{0,3}}, & J^{(1)} = \frac{\rho^{(1)\perp}E + \rho^\perp E^{(1)}}{B_{0,3}}, & \end{cases} \tag{82}$$

with the initial conditions $E(0,\cdot) = E_0$, $\;E^{(1)}(0,\cdot) = E_0^{(1)}$.

Theorem 4.1 *Assume that the initial conditions are smooth and satisfy the hypotheses H1, H3. We suppose that the limit systems* (79), (80), (81) *have smooth solutions. Then for any $T \in \mathbb{R}_+$ there is a constant C such that for $\varepsilon > 0$ small enough, $t \in [0,T]$ we have $\mathcal{H}_4^\varepsilon(t) \le C(\varepsilon^4 + \mathcal{H}_4^\varepsilon(0))$. Moreover if the hypotheses H6 holds then $\sup_{\varepsilon>0, t\in[0,T]} \varepsilon^{-4}\mathcal{H}_4^\varepsilon(t) < +\infty$ for any $T \in \mathbb{R}_+$. In particular*

$$\sup_{\varepsilon>0}\varepsilon^{-2}\|E^\varepsilon - E - \varepsilon E^{(1)}\|_{L^\infty(]0,T[;L^2(\mathbb{T}^2)^2)} + \sup_{\varepsilon>0}\varepsilon^{-2}\left\|\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3\right\|_{L^\infty(]0,T[;L^2(\mathbb{T}^2))} < +\infty,$$

and

$$\sup_{\varepsilon>0}\varepsilon^{-2}\|\rho^\varepsilon - \rho - \varepsilon\rho^{(1)}\|_{L^\infty(]0,T[;H^{-1}(\mathbb{T}^2))} + \sup_{\varepsilon>0}\varepsilon^{-2}\|J^\varepsilon - J - \varepsilon J^{(1)}\|_{W^{-1,1}(]0,T[\times\mathbb{T}^2)^2} < +\infty.$$

Before starting the proof of Theorem 4.1 we give some preliminary results. We write the Vlasov equation (66) in the form

$$\varepsilon^2(\partial_t F^\varepsilon + \mathrm{div}_x(F^\varepsilon u)) - \mathrm{div}_u\left(F^\varepsilon(E + \varepsilon E^{(1)} + (B_{0,3} + \varepsilon^2 A_3)^\perp u)\right) - \mathrm{div}_u\left(F^\varepsilon(E^\varepsilon - E - \varepsilon E^{(1)} + (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp u)\right) = 0.$$

We multiply the above equation by $h_4^\varepsilon(t,x,u) = \frac{1}{2}\left|u - (D+\varepsilon D^{(1)})(t,x)\right|^2$, where $D = \frac{{}^\perp E}{B_{0,3}}$, $D^{(1)} = \frac{{}^\perp E^{(1)}}{B_{0,3}}$, and we perform integration by parts. The computations are standard but long. Therefore we split them into three lemmas. The notation C stands for generic constants depending only on the $W^{2,\infty}(]0,T[\times\mathbb{T}^2)$ norms of the solutions to the limit systems and T. These constants are allowed to change from line to line.

Lemma 4.1 *For any $0<\varepsilon<1$, $T\in\mathbb{R}_+$, $t\in[0,T]$ we have*

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\varepsilon^2(\partial_t F^\varepsilon + \mathrm{div}_x(F^\varepsilon u))h_4^\varepsilon\;du\;dx = -Q_3(t) + \varepsilon^2\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} h_4^\varepsilon F^\varepsilon\;du\;dx$$
$$-\frac{\varepsilon^2}{B_{0,3}}\frac{d}{dt}\int_{\mathbb{T}^2}\{\varepsilon^2\int_{\mathbb{R}^2} uF^\varepsilon du + (B_3^\varepsilon - B_{0,3})^\perp(E^\varepsilon - E - \varepsilon E^{(1)})\}\cdot^\perp(\partial_t D^\varepsilon + (\nabla_x D^\varepsilon)D^\varepsilon)\;dx,$$

where $D^\varepsilon = D + \varepsilon D^{(1)}$ and $|Q_3(t)| \le C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t)$.

Proof. We have

$$\begin{aligned}
Q_3 + \frac{\varepsilon^2}{B_{0,3}}\frac{d}{dt}&\int_{\mathbb{T}^2}\{\varepsilon^2\int_{\mathbb{R}^2} uF^\varepsilon du + (B_3^\varepsilon - B_{0,3})^\perp(E^\varepsilon - E - \varepsilon E^{(1)})\}\cdot^\perp(\partial_t D^\varepsilon + (\nabla_x D^\varepsilon)D^\varepsilon)\;dx \\
&= \varepsilon^2\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} F^\varepsilon(\partial_t h_4^\varepsilon + u\cdot\nabla_x h_4^\varepsilon)\;du\;dx \\
&= -\varepsilon^2\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} F^\varepsilon(u - D^\varepsilon)\cdot(\partial_t D^\varepsilon + (\nabla_x D^\varepsilon)D^\varepsilon)\;du\;dx \\
&- \varepsilon^2\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} F^\varepsilon(u-D^\varepsilon)\cdot(\nabla_x D^\varepsilon)(u-D^\varepsilon)\;du\;dx \\
&=: Q_4(t) + Q_5(t).
\end{aligned} \tag{83}$$

It is easily seen that

$$|Q_5(t)| \le C\varepsilon^2\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} h_4^\varepsilon F^\varepsilon\;du\;dx. \tag{84}$$

Observe that we have

$$Q_4(t) = -\varepsilon^2\int_{\mathbb{T}^2}(J^\varepsilon - \rho^\varepsilon D^\varepsilon)\cdot(\partial_t D^\varepsilon + (\nabla_x D^\varepsilon)D^\varepsilon)\;dx.$$

We check easily that

$$J^\varepsilon - \rho^\varepsilon D^\varepsilon = J^\varepsilon - J - \varepsilon J^{(1)} - (\rho^\varepsilon - \rho - \varepsilon\rho^{(1)})D^\varepsilon - \varepsilon^2\rho^{(1)}D^{(1)}. \tag{85}$$

Using now the equation $\mathrm{div}_x(E^\varepsilon - E - \varepsilon E^{(1)}) = -(\rho^\varepsilon - \rho - \varepsilon\rho^{(1)})$ we deduce that

$$\varepsilon^2 \left| \int_{\mathbb{T}^2} (\rho^\varepsilon - \rho - \varepsilon\rho^{(1)}) D^\varepsilon \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon)\, dx \right| \leq C\varepsilon^2 \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|\, dx \qquad (86)$$
$$\leq C\varepsilon^4 + C\int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 dx.$$

Obviously we have

$$\varepsilon^2 \left| \int_{\mathbb{T}^2} \varepsilon^2 \rho^{(1)} D^{(1)} \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon)\, dx \right| \leq C\varepsilon^4. \qquad (87)$$

It remains to analyze the term $\varepsilon^2 \int_{\mathbb{T}^2} (J^\varepsilon - J - \varepsilon J^{(1)}) \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon)\, dx$. Using the momentum conservation

$$\varepsilon^2 \left(\partial_t \int_{\mathbb{R}^2} F^\varepsilon u\, du + \mathrm{div}_x \int_{\mathbb{R}^2} F^\varepsilon (u \otimes u)\, du \right) + \rho^\varepsilon E^\varepsilon +^\perp J^\varepsilon B_3^\varepsilon = 0,$$

we obtain

$$B_{0,3}{}^\perp (J^\varepsilon - J - \varepsilon J^{(1)}) = -T_6 - T_7 + T_8, \qquad (88)$$

where

$$T_6 = \varepsilon^2 \left(\partial_t \int_{\mathbb{R}^2} F^\varepsilon u\, du + \mathrm{div}_x \int_{\mathbb{R}^2} F^\varepsilon (u \otimes u)\, du \right), \quad T_7 = E^\varepsilon - E - \varepsilon E^{(1)},$$
$$T_8 = \mathrm{div}_x E^\varepsilon E^\varepsilon - \mathrm{div}_x E E - \varepsilon(\mathrm{div}_x E E^{(1)} + \mathrm{div}_x E^{(1)} E) - (B_3^\varepsilon - B_{0,3})^\perp J^\varepsilon.$$

We can write by (88)

$$\varepsilon^2 B_{0,3} \int_{\mathbb{T}^2} (J^\varepsilon - J - \varepsilon J^{(1)}) \cdot (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon)\, dx = -Q_6(t) - Q_7(t) + Q_8(t),$$

where $Q_j = \varepsilon^2 \int_{\mathbb{T}^2} T_j \cdot {}^\perp(\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon)\, dx$, $j \in \{6, 7, 8\}$. It is easily seen that

$$\left| Q_6(t) - \varepsilon^4 \frac{d}{dt} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} F^\varepsilon u \cdot {}^\perp(\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon)\, du\, dx \right| \leq C\varepsilon^4 + C\varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} h_4^\varepsilon F^\varepsilon\, du\, dx, \qquad (89)$$

and

$$|Q_7(t)| \leq C\varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2\, dx. \qquad (90)$$

For any $w \in C^1(\mathbb{T}^2)^2$ we use the notation $\mathcal{F}(w) = \mathrm{div}_x w\, w - \mathrm{div}_x{}^\perp w\, {}^\perp w$. We have $T_8 = T_9 + T_{10}$ with

$$\begin{aligned} T_9 &= \mathcal{F}(E^\varepsilon - E - \varepsilon E^{(1)}) + \mathrm{div}_x(E^\varepsilon - E - \varepsilon E^{(1)})(E + \varepsilon E^{(1)}) \\ &+ \mathrm{div}_x(E + \varepsilon E^{(1)})(E^\varepsilon - E - \varepsilon E^{(1)}) + \varepsilon^2 \mathrm{div}_x E^{(1)} E^{(1)}, \end{aligned}$$

and

$$T_{10} = \mathrm{div}_x {}^\perp E^\varepsilon\, {}^\perp(E^\varepsilon - E - \varepsilon E^{(1)}) - (B_3^\varepsilon - B_{0,3})^\perp J^\varepsilon.$$

Since for any $w \in C^1(\mathbb{T}^2)^2$ we have $\mathcal{F}(w) = \mathrm{div}_x(w \otimes w) - \frac{1}{2}\nabla_x |w|^2$ we check easily that

$$\begin{aligned} |Q_9(t)| &= \varepsilon^2 \left| \int_{\mathbb{T}^2} T_9 \cdot {}^\perp(\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon)\, dx \right| \\ &\leq C\varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2\, dx. \end{aligned} \tag{91}$$

For estimating the term T_{10} we use the equations

$$\partial_t (E^\varepsilon - E - \varepsilon E^{(1)}) - \frac{1}{\varepsilon} {}^\perp\nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right) = J^\varepsilon - J - \varepsilon J^{(1)} - \varepsilon {}^\perp\nabla_x A_3^{(1)},$$

and

$$\partial_t \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} \right) + \frac{1}{\varepsilon} \mathrm{div}_x^\perp E^\varepsilon = 0.$$

We obtain

$$\begin{aligned} T_{10} + (B_3^\varepsilon - B_{0,3})^\perp (J + \varepsilon J^{(1)} + \varepsilon {}^\perp\nabla_x A_3^{(1)}) &= -\partial_t \{ (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \} \\ &- \frac{1}{2} \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 \\ &- \varepsilon A_3 \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right). \end{aligned}$$

Therefore we deduce that the term $Q_{10} := \varepsilon^2 \int_{\mathbb{T}^2} T_{10} \cdot {}^\perp(\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon)\, dx$ satisfies

$$\begin{aligned} |Q_{10}(t) &+ \varepsilon^2 \frac{d}{dt} \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot^\perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon)\, dx| \leq C\varepsilon^4 \\ &+ C \int_{\mathbb{T}^2} \frac{1}{2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 dx. \end{aligned} \tag{92}$$

Combining the partial computations $(83 - 92)$ we deduce that $|Q_3(t)| \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t)$. $\square$

Remark 4.1 *Using the computations of the previous proof yields also the inequalities*

$$\|\rho^\varepsilon - \rho - \varepsilon \rho^{(1)}\|_{L^\infty(]0,T[;H^{-1}(\mathbb{T}^2))} \leq C \|E^\varepsilon - E - \varepsilon E^{(1)}\|_{L^\infty(]0,T[;L^2(\mathbb{T}^2)^2)},$$

and

$$\|J^\varepsilon - J - \varepsilon J^{(1)}\|_{W^{-1,1}(]0,T[\times\mathbb{T}^2)^2} \leq C\varepsilon^2 + C\|E^\varepsilon - E - \varepsilon E^{(1)}\|_{L^\infty(]0,T[;L^2(\mathbb{T}^2)^2)} + C\|\mathcal{H}_4^\varepsilon\|_{L^\infty(]0,T[)}.$$

For further computations we retain also the following estimate. Consider a smooth function $\varphi \in W^{1,\infty}(]0,T[\times\mathbb{T}^2)^2$. Then there is a constant C, depending also on $\|\varphi\|_{W^{1,\infty}(]0,T[\times\mathbb{T}^2)^2}$ such that for any $t \in [0,T]$ we have

$$\begin{aligned}
&\left|\varepsilon^2 \int_{\mathbb{T}^2} {}^\perp(J^\varepsilon - J - \varepsilon J^{(1)}) \cdot \varphi \, dx + \frac{\varepsilon^2}{B_{0,3}} \frac{d}{dt} \int_{\mathbb{T}^2} (\varepsilon^2 \int_{\mathbb{R}^2} u F^\varepsilon du + (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)})) \cdot \varphi \, dx\right| \\
&\leq \; C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t). \qquad (93)
\end{aligned}$$

The above inequality follows by similar computations as those in the proof of Lemma 4.1: start with the equality (88) and then transform the terms T_6, T_7, T_9, T_{10} performing eventually integration by parts.

Lemma 4.2 *For any $\varepsilon > 0, T \in \mathbb{R}_+, t \in [0,T]$ we have*

$$\begin{aligned}
&- \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathrm{div}_u \left(F^\varepsilon (E + \varepsilon E^{(1)} + (B_{0,3} + \varepsilon^2 A_3)^\perp u)\right) h_4^\varepsilon \, du \, dx + Q_{13}(t) \\
&= \; \varepsilon^2 \frac{d}{dt} \int_{\mathbb{T}^2} \frac{A_3 D^\varepsilon}{B_{0,3}} \cdot \left(\varepsilon^2 \int_{\mathbb{R}^2} F^\varepsilon u \, du + (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)})\right) \, dx,
\end{aligned}$$

where $|Q_{13}(t)| \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t)$.

Proof. Observe that for any $u \in \mathbb{R}^2$ we have

$$(E + \varepsilon E^{(1)} + B_{0,3}{}^\perp u) \cdot (u - D^\varepsilon) = 0.$$

Thus integrating by parts and using (85) we obtain

$$\begin{aligned}
&- \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \mathrm{div}_u (F^\varepsilon (E + \varepsilon E^{(1)} + (B_{0,3} + \varepsilon^2 A_3)^\perp u)) h_4^\varepsilon \, du \, dx \\
&\qquad = \; -\varepsilon^2 \int_{\mathbb{T}^2} A_3 D^\varepsilon \cdot {}^\perp (J^\varepsilon - \rho^\varepsilon D^\varepsilon) \, dx \\
&\qquad = \; -Q_{11}(t) + Q_{12}(t),
\end{aligned}$$

where

$$Q_{11}(t) = \varepsilon^2 \int_{\mathbb{T}^2} A_3 D^\varepsilon \cdot^\perp (J^\varepsilon - J - \varepsilon J^{(1)}) \, dx, \;\; Q_{12}(t) = \varepsilon^4 \int_{\mathbb{T}^2} A_3 D^\varepsilon \cdot {}^\perp (\rho^{(1)} D^{(1)}) \, dx.$$

Obviously we have $|Q_{12}(t)| \le C\varepsilon^4$ and by (93) we deduce that

$$\left| Q_{11} + \varepsilon^2 \frac{d}{dt}\int_{\mathbb{T}^2} \frac{A_3 D^\varepsilon}{B_{0,3}} \cdot \left(\varepsilon^2 \int_{\mathbb{R}^2} F^\varepsilon u \, du + (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right) dx \right| \le C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t),$$

which implies our conclusion. □

Lemma 4.3 *For any* $0 < \varepsilon < 1$, $T \in \mathbb{R}_+$, $t \in [0, T]$ *we have*

$$\begin{aligned}
&- \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \operatorname{div}_u \left((E^\varepsilon - E - \varepsilon E^{(1)} + (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp u) F^\varepsilon \right) h_4^\varepsilon \, du \, dx \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} \left(|E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)} \right)^2 \right) dx \\
&- \frac{d}{dt}\int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot \left(D^\varepsilon + \varepsilon^2 \frac{{}^\perp E^{(2)}}{B_{0,3}} \right) dx \\
&- \frac{\varepsilon^4}{B_{0,3}} \frac{d}{dt}\int_{\mathbb{T}^2} {}^\perp E^{(2)} \cdot \left(\int_{\mathbb{R}^2} F^\varepsilon u \, du + A_3{}^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right) dx - Q_{22}(t),
\end{aligned}$$

where $|Q_{22}(t)| \le C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t)$.

Proof. Integrating by parts with respect to u and using (85) yields

$$\begin{aligned}
&- \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \operatorname{div}_u \left((E^\varepsilon - E - \varepsilon E^{(1)} + (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp u) F^\varepsilon \right) h_4^\varepsilon \, du \, dx \\
&= \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \left(E^\varepsilon - E - \varepsilon E^{(1)} + (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp u \right) \cdot (u - D^\varepsilon) F^\varepsilon \, du \, dx \\
&= \int_{\mathbb{T}^2} (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot (J^\varepsilon - \rho^\varepsilon D^\varepsilon) \, dx - \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3) D^\varepsilon \cdot {}^\perp J^\varepsilon \, dx \\
&= Q_{14}(t) - Q_{15}(t) + Q_{16}(t), \qquad (94)
\end{aligned}$$

where

$$Q_{14} = \int_{\mathbb{T}^2} (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot (J^\varepsilon - J - \varepsilon J^{(1)}) \, dx, \quad Q_{15}(t) = \varepsilon^2 \int_{\mathbb{T}^2} \rho^{(1)} (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot D^{(1)} \, dx,$$

$$Q_{16} = -\int_{\mathbb{T}^2} D^\varepsilon \cdot \left((\rho^\varepsilon - \rho - \varepsilon\rho^{(1)})(E^\varepsilon - E - \varepsilon E^{(1)}) + (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp (J^\varepsilon - \rho^\varepsilon D^\varepsilon) \right) dx.$$

It is easily seen that

$$|Q_{15}(t)| \le C\varepsilon^4 + C\int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2 \, dx. \qquad (95)$$

We use now the equations

$$\partial_t(E^\varepsilon-E-\varepsilon E^{(1)}-\varepsilon^2E^{(2)})-\frac{{}^\perp\nabla_x}{\varepsilon}\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}-\varepsilon A_3-\varepsilon^2A_3^{(1)}\right)=J^\varepsilon-J-\varepsilon J^{(1)}-\varepsilon^2\partial_tE^{(2)}, \tag{96}$$

$$\partial_t\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}-\varepsilon A_3-\varepsilon^2A_3^{(1)}\right)+\frac{1}{\varepsilon}\mathrm{div}_x^\perp(E^\varepsilon-E-\varepsilon E^{(1)}-\varepsilon^2E^{(2)})=-\varepsilon^2\partial_tA_3^{(1)}, \tag{97}$$

and we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}\left(|E^\varepsilon-E-\varepsilon E^{(1)}-\varepsilon^2E^{(2)}|^2+\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}-\varepsilon A_3-\varepsilon^2A_3^{(1)}\right)^2\right)\,dx\\
=\;&\int_{\mathbb{T}^2}(J^\varepsilon-J-\varepsilon J^{(1)}-\varepsilon^2\partial_tE^{(2)})\cdot(E^\varepsilon-E-\varepsilon E^{(1)}-\varepsilon^2E^{(2)})\,dx\\
-\;&\varepsilon^2\int_{\mathbb{T}^2}\partial_tA_3^{(1)}\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}-\varepsilon A_3-\varepsilon^2A_3^{(1)}\right)\,dx=:Q_{17}(t).
\end{aligned}\tag{98}$$

By using (93) we obtain

$$\begin{aligned}
Q_{17}(t)\;&=\;Q_{14}(t)+\frac{\varepsilon^4}{B_{0,3}}\frac{d}{dt}\int_{\mathbb{T}^2}{}^\perp E^{(2)}\int_{\mathbb{R}^2}F^\varepsilon u\,du\,dx\\
&+\;\frac{\varepsilon^2}{B_{0,3}}\frac{d}{dt}\int_{\mathbb{T}^2}(B_3^\varepsilon-B_{0,3})^\perp(E^\varepsilon-E-\varepsilon E^{(1)})\cdot{}^\perp E^{(2)}\,dx+Q_{18}(t),
\end{aligned}\tag{99}$$

where $|Q_{18}(t)|\le C\varepsilon^4+C\mathcal{H}_4^\varepsilon(t)$. Using now (85) yields

$$\begin{aligned}
Q_{16}(t)\;&=\;\int_{\mathbb{T}^2}D^\varepsilon(\mathrm{div}_x(E^\varepsilon-E-\varepsilon E^{(1)})(E^\varepsilon-E-\varepsilon E^{(1)})\\
&-\;(B_3^\varepsilon-B_{0,3}-\varepsilon^2A_3)^\perp(J^\varepsilon-J-\varepsilon J^{(1)}))\,dx\\
&+\;\varepsilon^2\int_{\mathbb{T}^2}(B_3^\varepsilon-B_{0,3}-\varepsilon^2A_3)D^\varepsilon\cdot{}^\perp(\rho^{(1)}D^{(1)})\,dx\\
&=\;Q_{19}(t)+Q_{20}(t).
\end{aligned}\tag{100}$$

Taking into account that

$${}^\perp(J^\varepsilon-J-\varepsilon J^{(1)})=\partial_t{}^\perp(E^\varepsilon-E-\varepsilon E^{(1)})+\frac{\nabla_x}{\varepsilon}\left(\frac{B_3^\varepsilon-B_{0,3}}{\varepsilon}-\varepsilon A_3\right)-\varepsilon\nabla_xA_3^{(1)},$$

we deduce that

$$
\begin{aligned}
& \operatorname{div}_x(E^\varepsilon - E - \varepsilon E^{(1)})(E^\varepsilon - E - \varepsilon E^{(1)}) - (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp (J^\varepsilon - J - \varepsilon J^{(1)}) \\
= \; & \mathcal{F}(E^\varepsilon - E - \varepsilon E^{(1)}) + \operatorname{div}_x{}^\perp (E^\varepsilon - E - \varepsilon E^{(1)})\, {}^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \\
- \; & (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3) \left(\partial_t{}^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) + \frac{1}{\varepsilon} \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right) - \varepsilon \nabla_x A_3^{(1)} \right) \\
= \; & \mathcal{F}(E^\varepsilon - E - \varepsilon E^{(1)}) - \frac{1}{2} \nabla_x \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 + \varepsilon^2 \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right) \nabla_x A_3^{(1)} \\
- \; & \partial_t \left\{ (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right\} - \varepsilon^2 \partial_t A_3{}^\perp (E^\varepsilon - E - \varepsilon E^{(1)}),
\end{aligned}
$$

and therefore we obtain

$$
Q_{19}(t) = -\frac{d}{dt} \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot D^\varepsilon \, dx + Q_{21}(t), \quad (101)
$$

where

$$
|Q_{21}(t)| \leq C\varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} \left\{ |E^\varepsilon - E - \varepsilon E^{(1)}|^2 + \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 \right\} dx.
$$

Observe also that we have

$$
|Q_{20}(t)| \leq C\varepsilon^4 + C \int_{\mathbb{T}^2} \frac{1}{2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 \right)^2 dx,
$$

and finally combining (94), (95), (99 − 101) yields our conclusion. $\square$

The previous lemmas allow us to justify the Theorem 4.1

Proof. (of Theorem 4.1) By Lemmas 4.1, 4.2, 4.3 we deduce that

$$
\begin{aligned}
\frac{d}{dt} \varepsilon^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} h_4^\varepsilon F^\varepsilon \, du \, dx \; & + \; \frac{d}{dt} \int_{\mathbb{T}^2} \frac{1}{2} |E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}|^2 \, dx \\
& + \; \int_{\mathbb{T}^2} \frac{1}{2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)} \right)^2 dx - \frac{d}{dt} \mathcal{R}_4^\varepsilon \\
& \leq \; C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(t),
\end{aligned}
$$

where

$$
\begin{aligned}
B_{0,3} \mathcal{R}_4^\varepsilon(t) \; = \; & \int_{\mathbb{T}^2} \left(\varepsilon^4 \int_{\mathbb{R}^2} F^\varepsilon u \, du + \varepsilon^2 (B_3^\varepsilon - B_{0,3})^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right) \\
\cdot \; & {}^\perp (\partial_t D^\varepsilon + (\nabla_x D^\varepsilon) D^\varepsilon + A_3{}^\perp D^\varepsilon) \, dx \\
+ \; & \int_{\mathbb{T}^2} (B_3^\varepsilon - B_{0,3} - \varepsilon^2 A_3)^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \cdot (B_{0,3} D^\varepsilon + \varepsilon^{2\perp} E^{(2)}) \, dx \\
+ \; & \varepsilon^4 \int_{\mathbb{T}^2} {}^\perp E^{(2)} \left(\int_{\mathbb{R}^2} F^\varepsilon u \, du + A_3{}^\perp (E^\varepsilon - E - \varepsilon E^{(1)}) \right) dx.
\end{aligned}
$$

We check easily that $|\mathcal{R}_4^\varepsilon(t)| \leq C\varepsilon^4 + C\varepsilon \mathcal{H}_4^\varepsilon(t)$. Observe also that

$$
\begin{aligned}
|\int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}|^2\,dx - \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2\,dx| \leq 3\varepsilon^4 \int_{\mathbb{T}^2} |E^{(2)}|^2\,dx \\
+ \quad \frac{1}{2}\int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2\,dx,
\end{aligned}
$$

implying that

$$
\begin{aligned}
-C\varepsilon^4 + \frac{1}{2}\int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2\,dx \quad &\leq \quad \int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)} - \varepsilon^2 E^{(2)}|^2\,dx \\
&\leq \quad C\varepsilon^4 + \frac{3}{2}\int_{\mathbb{T}^2} |E^\varepsilon - E - \varepsilon E^{(1)}|^2\,dx.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
-C\varepsilon^4 + \frac{1}{2}\int_{\mathbb{T}^2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3\right)^2\,dx \quad &\leq \quad \int_{\mathbb{T}^2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3 - \varepsilon^2 A_3^{(1)}\right)^2\,dx \\
&\leq \quad C\varepsilon^4 + \frac{3}{2}\int_{\mathbb{T}^2} \left(\frac{B_3^\varepsilon - B_{0,3}}{\varepsilon} - \varepsilon A_3\right)^2\,dx.
\end{aligned}
$$

Finally we deduce that

$$
\mathcal{H}_4^\varepsilon(t) \leq C\varepsilon^4 + C\mathcal{H}_4^\varepsilon(0) + \int_0^t \mathcal{H}_4^\varepsilon(s)\,ds, \quad t \in [0,T].
$$

Our conclusion follows immediately by Gronwall lemma and Remark 4.1. $\square$

Acknowledgement: The author is thankful to Prof. E. Sonnendrücker for helpful remarks and advices.

References

[1] F. Berthelin, A. Vasseur, From kinetic equations to multidimensional isentropic gas dynamics before shocks, SIAM J. Math. Anal. 36(2005) 1807-1835.

[2] M. Bostan, T. Goudon, High-electric-field limit for the Vlasov-Maxwell-Fokker-Planck system, preprint 2005.

[3] Y. Brenier, Convergence of the Vlasov-Poisson system to the incompressible Euler equations, Comm. Partial Differential Equations 25(2000) 737-754.

[4] Y. Brenier, N. Mauser, M. Puel, Incompressible Euler and e-MHD as scaling limits of the Vlasov-Maxwell system, Commun. Math. Sci. 1(2003) 437-447.

[5] J.-Y. Chemin, Fluides parfaits incompressibles, Astérisque 230(1995).

[6] P. Degond, M. Pulvirenti, Personal communication, 1994.

[7] E. Frénod, E. Sonnendrücker, Homogenization of the Vlasov equation and of the Vlasov-Poisson system with strong external magnetic field, Asymptotic Anal. 18(1998) 193-213.

[8] E. Frénod, E. Sonnendrücker, Long time behavior of the Vlasov equation with a strong external magnetic field, Math. Models Methods Appl. Sci. 10(2000) 539-553.

[9] R. Glassey, J. Schaeffer, On the 'one and one-half dimensional' relativistic Vlasov-Maxwell system, Math. Methods Appl. Sci. 13(1990) 169-179.

[10] R. Glassey, J. Schaeffer, The relativistic Vlasov-Maxwell system in two space dimensions, Part I and II, Arch. Ration. Mech. Anal. 141(1998) 331-354 and 355-374.

[11] F. Golse, L. Saint-Raymond, The Vlasov-Poisson system with strong magnetic field, J. Math. Pures Appl. 78(1999) 791-817.

[12] F. Golse, L. Saint-Raymond, The Vlasov-Poisson system with strong magnetic field in quasineutral regime, Math. Models Methods Appl. Sci. 13(2003) 661-714.

[13] T. Goudon, P.-E. Jabin, A. Vasseur, Hydrodynamic limits for the Vlasov-Navier-Stokes equations. Part II: Fine particles regime, Indiana Univ. Math. J. 53(2004) 1517-1536.

[14] T. Goudon, J. Nieto, F. Poupaud, J. Soler, Multidimensional high-field limit of the electrostatic Vlasov-Poisson-Fokker-Planck system, J. Differential Equations 213(2005) 418–442.

[15] H. Grad, The guiding center plasma, Proc. Symp. Appl. Math. XVIII, Amer. Math. Soc., Providence R.I.(1967) 162-248.

[16] E. Grenier, Pseudo-differential energy estimates of singular perturbations, Comm. Pure Appl. Math. 50(1997) 821-865.

[17] P.-E. Jabin, Large time concentrations for solutions to kinetic equations with energy dissipation, Comm. Partial Differential Equations 25(2000) 541-557.

[18] T. Kato, Nonstationary flows of viscous and ideal fluid in $\mathbb{R}^3$, J. Funct. Anal. 9(1972) 296-305.

[19] E.M. Lifshitz, L.P. Pitaevskii, Course of theoretical physics, Vol. 10, translated from Russian by J.B. Sykes and R.N. Franklin, Pergamon Int. Library of Sci., Technology, Engineering and Social Studies, Pergamon Press, Oxford-Elmsford, N.Y. 1981.

[20] P.-L. Lions, Mathematical topics in fluid mechanics, Vol.1 Incompressible models, Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, New-York, 1996.

[21] C. Marchioro, M. Pulvirenti, Mathematical theory of incompressible nonviscous fluids, Springer, New-York, 1994.

[22] L. Saint-Raymond, Convergence of solutions to the Boltzmann equation in the incompressible Euler limit, Arch. Ration. Mech. Anal. 166(2003) 47-80.

[23] R. Temam, Local existence of C^∞ solutions of the Euler equations of incompressible perfect fluids, in Turbulence and the Navier-Stokes equations, Springer-Verlag, New-York, 1976 184-194.

[24] H.T. Yau, Relative entropy and hydrodynamics of Ginzburg-Landau models, Lett. Math. Phys. 22(1991) 63-80.