Mild solutions for the relativistic Vlasov-Maxwell system for laser-plasma interaction

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Abstract

We study a reduced 1 D Vlasov-Maxwell system which describes the laser-plasma interaction. The unknowns of this system are the distribution function of charged particles, satisfying a Vlasov equation, the electrostatic field, verifying a Poisson equation and a vector potential term solving a nonlinear wave equation. The nonlinearity in the wave equation is due to the coupling with the Vlasov equation through the charge density. We prove here the existence and uniqueness of the mild solution (*i.e.*, solution by characteristics) in the relativistic case by using the iteration method.

Keywords: Kinetic equations, Vlasov-Maxwell system, weak/mild solution, characteristics.

AMS classification: 35A05, 35B35, 82D10.

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1 Introduction

We consider a population of relativistic electrons with mass $m > 0$ and charge $-e < 0$. We denote by $v(p) = \frac{p}{m}\left(1 + \frac{|p|^2}{m^2c^2}\right)^{-1/2}$ the velocity associated to a given momentum $p \in \mathbb{R}^3$, where c is the speed of light. The electrons move under the action of an electric field E and a magnetic field B. Their distribution function $F = F(t, x, p)$ satisfies the Vlasov equation

$$\partial_t F + v(p)\cdot\nabla_x F - e(E(t,x) + v(p)\wedge B(t,x))\cdot\nabla_p F = 0, \;\; (t,x,p) \in]0,T[\times\mathbb{R}^3\times\mathbb{R}^3. \quad (1)$$

The electro-magnetic field verifies the Maxwell equations

$$\partial_t E - c^2 \mathrm{curl} B = \frac{e}{\varepsilon_0} j, \; \partial_t B + \mathrm{curl} E = 0, \; \mathrm{div} E = \frac{e}{\varepsilon_0}(\rho_{ext} - \rho), \; \mathrm{div} B = 0, \qquad (2)$$

where ε_0 is the dielectric permittivity of vacuum, ρ_{ext} is the density of a background population of ions which are supposed at rest and the electron density ρ and current j are given by

$$\rho(t,x) = \int_{\mathbb{R}^3} F(t,x,p)\, dp, \;\; j(t,x) = \int_{\mathbb{R}^3} v(p) F(t,x,p)\, dp, \;\; \forall (t,x) \in [0,T] \times \mathbb{R}^3.$$

We intend to analyze here a reduced 1D Vlasov-Maxwell system introduced recently in the physical literature for studying laser-plasma interactions. The assumptions of this model are the following : all unknowns depend on only one space variable, for example x_1 and the electrons are monokinetic in the directions transversal to x_1. The distribution function becomes

$$F(t,x,p) = f(t,x_1,p_1)\delta(p_2 - p_2(t,x_1))\delta(p_3 - p_3(t,x_1)). \qquad (3)$$

We consider the initial condition

$$F(0,x,p) = F_0(x,p) := f_0(x_1,p_1)\delta(p_2 - p_2^0(x_1))\;\delta(p_3 - p_3^0(x_1)).$$

Let us introduce also the vector and scalar potentials A, Φ such that

$$B = \mathrm{curl} A, \;\; E = -\partial_t A - \nabla_x \Phi.$$

Under the hypotheses of our model $\partial_{x_2} = \partial_{x_3} = 0$ and thus the previous equalities become

$$B_1 = 0, \ \ B_2 = -\partial_{x_1} A_3, \ \ B_3 = \partial_{x_1} A_2, \tag{4}$$

$$E_1 = -\partial_t A_1 - \partial_{x_1}\Phi, \ \ E_2 = -\partial_t A_2, \ \ E_3 = -\partial_t A_3. \tag{5}$$

The distribution function f satisfies also a Vlasov equation in the phase space (x_1, p_1). For checking this consider the characteristic system associated to (1)

$$\frac{dX}{ds} = v(P(s)), \ \ \frac{dP}{ds} = -e(E(s, X(s)) + v(P(s)) \wedge B(s, X(s))), \tag{6}$$

$$X(s = t) = x, \ \ P(s = t) = p. \tag{7}$$

We denote by $(X(s; t, x, p), P(s; t, x, p))$ the solution of (6), (7) (we suppose that the electro-magnetic field is smooth). By using (4), (5) the last three equations in (6) become

$$\frac{dP_1}{ds} = -e(-\partial_t A_1 - \partial_{x_1}\Phi + v_2(P(s))\, \partial_{x_1} A_2 + v_3(P(s))\partial_{x_1} A_3), \tag{8}$$

$$\frac{dP_2}{ds} = -e(-\partial_t A_2 - v_1(P(s))\, \partial_{x_1} A_2) = e\frac{d}{ds} A_2(s, X_1(s)), \tag{9}$$

$$\frac{dP_3}{ds} = -e(-\partial_t A_3 - v_1(P(s))\, \partial_{x_1} A_3) = e\frac{d}{ds} A_3(s, X_1(s)). \tag{10}$$

Therefore $P_2 - eA_2$ and $P_3 - eA_3$ are constant along the characteristics. By a suitable change of referential we can suppose that $e\ A_2(0, x_1) = p_2^0(x_1)$, $e\ A_3(0, x_1) = p_3^0(x_1)$. By imposing the gauge $\mathrm{div} A = 0$, we can assume also that $A_1 = 0$. After some computations (see [8]) we deduce that f satisfies the kinetic equation

$$\partial_t f + \frac{p_1}{m\gamma}\partial_{x_1} f - e(E_1(t, x_1) + \frac{e}{m\gamma} A_2(t, x_1)\partial_{x_1} A_2 + \frac{e}{m\gamma} A_3(t, x_1)\partial_{x_1} A_3)\partial_{p_1} f = 0, \tag{11}$$

with the initial condition

$$f(0, x_1, p_1) = f_0(x_1, p_1), \ \ (x_1, p_1) \in \mathbb{R}^2,$$

and we have the equality

$$F(t,x,p) = f(t,x_1,p_1)\delta(p_2 - e\ A_2(t,x_1))\delta(p_3 - e\ A_3(t,x_1)). \tag{12}$$

The function γ is given by

$$\gamma(t,x_1,p_1) = \left(1 + \frac{|p_1|^2}{m^2c^2} + \frac{e^2}{m^2c^2}(|A_2(t,x_1)|^2 + |A_3(t,x_1)|^2)\right)^{\frac{1}{2}}.$$

Notice that the field $(\frac{p_1}{m\gamma}, -e(E_1(t,x_1) + \frac{e}{m\gamma}A_2(t,x_1)\partial_{x_1}A_2 + \frac{e}{m\gamma}A_3(t,x_1)\partial_{x_1}A_3))$ is divergence free with respect to (x_1,p_1). For the sake of simplicity we assume that $A_3 = 0$. Under these circumstances, by adding the first and second Maxwell equations one gets the system

$$\partial_t f + \frac{p_1}{m\gamma}\partial_{x_1} f - e(E_1(t,x_1) + \frac{e}{m\gamma}A_2(t,x_1)\partial_{x_1}A_2)\partial_{p_1} f = 0, \tag{13}$$

$$\partial_t^2 A_2 - c^2\partial_{x_1}^2 A_2 = -\frac{e^2}{m\varepsilon_0}\rho_\gamma(t,x_1)A_2(t,x_1), \tag{14}$$

$$\partial_t E_1 = \frac{e}{\varepsilon_0} j_1(t,x_1), \tag{15}$$

$$\partial_{x_1} E_1 = \frac{e}{\varepsilon_0}(\rho_{ext}(x_1) - \rho(t,x_1)), \tag{16}$$

where

$$\{\rho, \rho_\gamma, j_1\}(t,x_1) = \int_{\mathbb{R}} \left\{1, \frac{1}{\gamma}, \frac{p_1}{\gamma}\right\} f(t,x_1,p_1)\ dp_1,$$

and $\gamma = \left(1 + \frac{|p_1|^2}{m^2c^2} + \frac{e^2|A_2(t,x_1)|^2}{m^2c^2}\right)^{\frac{1}{2}}$. Observe also that the total energy at the moment t is

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} F(t,x,p)mc^2\left(1 + \frac{|p|^2}{m^2c^2}\right)^{\frac{1}{2}}\ dp\ dx + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^3}\{|E(t,x)|^2 + c^2|B(t,x)|^2\}\ dx \\
= \quad & \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x_1,p_1)mc^2\left(1 + \frac{|p_1|^2}{m^2c^2} + \frac{e^2|A_2(t,x_1)|^2}{m^2c^2}\right)^{\frac{1}{2}}\ dp_1 dx_1 \\
+ \quad & \frac{\varepsilon_0}{2}\int_{\mathbb{R}}\{|E_1(t,x_1)|^2 + |\partial_t A_2|^2 + c^2|\partial_{x_1}A_2|^2\}\ dx_1.
\end{aligned} \tag{17}$$

The above model describes the interaction of the electro-magnetic field created by a laser wave (called pump wave) with a population of charged particles. It was studied recently by Carrillo and Labrunie in [8]. The strong nonlinear coupling through the Lorentz factor γ makes this system difficult to study theoretically but also numerically. Other reduced models have been considered by physicists.

1) The nonrelativistic model NR is obtained by setting $\gamma = 1$ everywhere. It is physically justified when the temperature is low enough, so that the proportion of relativistic electrons is negligible and the intensity of the pump is small.

2) The quasi-relativistic model (also called semi-relativistic by some authors) denoted QR consists in approximating γ by $\left(1 + \frac{|p_1|^2}{m^2c^2}\right)^{\frac{1}{2}}$ in the second term of (13) and in the definition of j_1, and setting $\gamma = 1$ in the third term of (13) and in the definition of ρ_γ (which means $\rho_\gamma = \rho$). It is acceptable when the proportion of ultra-relativistic electrons ($v \approx c$) is negligible and the pump intensity is moderate.

3) The original model with $\gamma = \left(1 + \frac{|p_1|^2}{m^2c^2} + \frac{e^2|A_2(t,x_1)|^2}{m^2c^2}\right)^{\frac{1}{2}}$ will be referred to as fully relativistic FR.

Notice that $(F = f(t, x_1, p_1)\delta(p_2 - e\ A_2(t, x_1))\delta(p_3), E, B)$, where (f, E_1, A_2) solves the NR model (13), (14), (15), (16), is a class of exact solutions for the nonrelativistic Vlasov-Maxwell system, *i.e.,* (1), (2) with $v(p) = \frac{p}{m}$. Similarly, when (f, E_1, A_2) solves the FR model, then $(F = f(t, x_1, p_1)\delta(p_2 - e\ A_2(t, x_1))\delta(p_3), E, B)$ is a class of exact solutions for the relativistic Vlasov-Maxwell system (1), (2). Nevertheless, the QR model is only an approximation of the FR model.

The equations (13), (14), (15), (16) can be simplified by introducing dimensionless unknowns and variables. If we omit the subscripts of x_1, p_1, E_1, A_2, j_1 and keep the same notations for the rescaled unknowns and variables we obtain (think that $m = 1, c = 1, e = 1, \varepsilon_0 = 1$)

$$\partial_t f + \frac{p}{\gamma_1}\partial_x f - \left(E(t,x) + \frac{A(t,x)}{\gamma_2}\partial_x A\right)\partial_p f = 0, \tag{18}$$

$$\partial_t^2 A - \partial_x^2 A = -\rho_{\gamma_2}(t,x)A(t,x), \tag{19}$$

$$\partial_t E = j(t,x), \tag{20}$$

$$\partial_x E = \rho_{ext}(x) - \rho(t,x), \tag{21}$$

where $\{\rho, \rho_{\gamma_2}, j\}(t,x) = \int_{\mathbb{R}} \{1, \frac{1}{\gamma_2}, \frac{p}{\gamma_1}\} f(t,x,p)\ dp$, $\gamma_1 = \gamma_2 = 1$ in the NR case, $\gamma_1 = (1+|p|^2)^{1/2}$, $\gamma_2 = 1$ in the QR case and $\gamma_1 = \gamma_2 = (1+|p|^2+|A(t,x)|^2)^{1/2}$ in the FR case. We supplement these equations with initial conditions

$$f(0,x,p) = f_0(x,p),\ (x,p) \in \mathbb{R}^2,\ \ (E, A, \partial_t A)(0,x) = (E_0, A_0, A_1)(x),\ x \in \mathbb{R}. \quad (22)$$

In [8] the authors investigated the existence of space periodic solutions and free-space solutions of the system $(13), (14), (15), (16)$. They proved the existence of weak and characteristic solutions in the NR and QR cases. In this article we concentrate our attention on the FR case. Actually the same method applies to the QR case and some arguments can be also used for analyzing the NR case. We are able to construct globally in time solutions by characteristics in the QR and FR cases, whereas only locally in time solutions by characteristics are available in the NR case (see also [8]). The arguments relie on iterative procedure (cf. [9]). The main idea consists in using the formulation by characteristics to obtain L^∞ estimates for the electro-magnetic field and the spacial derivatives by duality computations involving L^1 test functions. This method has been already used in [4] to prove the existence and uniqueness of the solution by characteristics for the 1D Vlasov-Poisson initial-boundary value problem.

To our knowledge this is the first theoretical work on the FR reduced Vlasov-Maxwell model $(13), (14), (15), (16)$. It has common features with the Nordström-Vlasov system, studied recently by Calogero and Rein [6], [7].

The Cauchy problem for the Vlasov-Maxwell system was analyzed by using different methods by DiPerna and Lions [10], Glassey and Schaeffer [11], [12], Glassey and Strauss [14], [15], Klainerman and Staffilani [17], Bouchut, Golse and Pallard [5]. For applications (tube discharges, cold plasma, solar wind, satellite ionization, thruster, ...) boundary conditions have to be taken into account. The Vlasov-Maxwell initial-boundary value problem was studied by Guo [16]. The three dimensional stationary Vlasov-Maxwell system was analyzed by Poupaud [18]. Results for the time periodic case can be found in [1], [2], [3].

The paper is organized as follows. In Section 2 we recall the notion of weak and mild solutions for the Vlasov problem and several main properties of such solutions. We establish the continuous dependence of the characteristics upon the electro-magnetic field and the initial conditions. In Section 3 we define the fixed point application $(E, A) \to \mathcal{F}(E, A)$ for regular fields (E, A) and we construct a domain D which is left invariant by this application. The main ingredient for using the iteration method is to estimate $\mathcal{F}(E_1, A_1) - \mathcal{F}(E_2, A_2)$ in terms of $(E_1, A_1) - (E_2, A_2)$ for pairs $(E_1, A_1), (E_2, A_2) \in D$. In the next section we prove the existence of a unique fixed point for $\mathcal{F}$ which guarantees the existence and uniqueness of solution for the reduced Vlasov-Maxwell system. We show also that the solution constructed preserves the total energy.

2 The Vlasov problem

In this paragraph we assume that the fields E, A are given and we introduce the notions of weak and mild (by characteristics) solution. We check easily that in all three cases we have

$$\operatorname{div}_{(x,p)} \left(\frac{p}{\gamma_1}, -E(t,x) - \frac{A(t,x)}{\gamma_2} \partial_x A \right) = 0,$$

and therefore the Vlasov equation (18) can be written also

$$\partial_t f + \partial_x \left(\frac{p}{\gamma_1} f \right) - \partial_p \left(\left(E(t,x) + \frac{A(t,x)}{\gamma_2} \partial_x A \right) f \right) = 0, \quad (t,x,p) \in]0,T[\times \mathbb{R}^2. \quad (23)$$

Consider also the initial condition

$$f(0,x,p) = f_0(x,p), \quad (x,p) \in \mathbb{R}^2. \quad (24)$$

Definition 2.1 *Assume that $E \in L^\infty(]0,T[\times\mathbb{R})$, $A \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}))$, $f_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. We say that $f \in L^1_{\mathrm{loc}}([0,T[\times\mathbb{R}^2)$ is a weak solution for the Vlasov problem*

(23), (24) *iff*

$$-\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x,p)\left(\partial_t\varphi+\frac{p}{\gamma_1}\partial_x\varphi-\left(E(t,x)+\frac{A(t,x)}{\gamma_2}\partial_x A\right)\partial_p\varphi\right)\,dp\,dx\,dt$$
$$=\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\varphi(0,x,p)\,dp\,dx, \quad (25)$$

for all test function $\varphi \in C_c^1([0,T[\times\mathbb{R}^2)$.

We need to consider also some special solutions of (23), (24) which are called mild solutions or solutions by characteristics. These solutions require more regularity for E, A. Assume that $E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}))$, $A \in L^\infty(]0,T[;W^{2,\infty}(\mathbb{R}))$ and let us introduce the system of characteristics associated to (18)

$$\frac{dX}{ds}=\frac{P(s)}{\gamma_1},\ \ \frac{dP}{ds}=-E(x,X(s))-\frac{A(s,X(s))}{\gamma_2}\partial_x A(s,X(s)), \quad (26)$$

with the initial conditions

$$X(s=t)=x,\ \ P(s=t)=p. \quad (27)$$

Observe that in all cases, under the above regularity hypotheses for E, A, for all $(t,x,p) \in [0,T[\times\mathbb{R}^2$ there is a unique solution for (26), (27) denoted

$$(X(s),P(s))=(X(s;t,x,p),P(s;t,x,p)).$$

The definition of the solution by characteristics can be obtained by replacing the transport term of equation (25) by a test function ψ

$$\partial_t\varphi+\frac{p}{\gamma_1}\partial_x\varphi-\left(E(t,x)+\frac{A(t,x)}{\gamma_2}\partial_x A\right)\partial_p\varphi=-\psi.$$

After integration along the characteristics and by imposing $\varphi(T,\cdot,\cdot)=0$ we find formally that

$$\varphi(t,x,p)=\int_t^T \psi(s,X(s;t,x,p),P(s;t,x,p))\,ds.$$

Definition 2.2 *Assume that* $E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}))$, $A \in L^\infty(]0,T[;W^{2,\infty}(\mathbb{R}))$, $f_0 \in L^1_{\text{loc}}(\mathbb{R}^2)$. *We say that* $f \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^2)$ *is a mild solution for the Vlasov problem* (23), (24) *iff*

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} f\psi \, dp \, dx \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x,p) \int_0^T \psi(s, X(s;0,x,p), P(s;0,x,p)) \, ds \, dp \, dx, \tag{28}$$

for all test function $\psi \in C^0_c([0,T] \times \mathbb{R}^2)$.

It is well known that the mild solution is unique and is given by

$$f(t,x,p) = f_0(X(0;t,x,p), P(0;t,x,p)), \ \ \forall\, (t,x,p) \in [0,T] \times \mathbb{R}^2. \tag{29}$$

We check easily that any mild solution is also weak solution. By performing the change of variables $(x,p) \to (X(t;0,x,p), P(t;0,x,p))$ we verify that if $f_0 \in L^1(\mathbb{R}^2)$, then the mild solution belongs to $L^\infty(]0,T[;L^1(\mathbb{R}^2))$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(t,x,p)| \, dp \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} |f_0(x,p)| \, dp \, dx, \ \ \forall\, t \in]0,T[.$$

Obviously, if f_0 is nonnegative, f remains nonnegative. Note also that if f_0 belongs to $L^1(\mathbb{R}^2)$ then the mild formulation holds true for any continuous bounded test function $\psi \in C^0([0,T] \times \mathbb{R}^2) \cap L^\infty(]0,T[\times\mathbb{R}^2)$.
Generally, the existence of weak solution for the Vlasov problem with initial condition $f_0 \in L^r(\mathbb{R}^2)$, $1 < r \le +\infty$ follows by regularization of the fields E, A and by passing to the limit for $\varepsilon \searrow 0$ in the weak formulation of f^ε, the mild solution associated with the regular fields $E^\varepsilon, A^\varepsilon$.

2.1 Continuous dependence of characteristics

We estimate here the difference between two solutions of the characteristic system (26), (27). We start with a general result. The proof is immediate and is left to the reader.

Proposition 2.1 *Assume that* $F, \tilde{F} \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}^N))^N$ *and that* $X, \tilde{X}$ *are solutions for*

$$\frac{dX}{ds} = F(s, X(s)), \ \ 0 < s < T, \ \ X(s = t) = x,$$

and

$$\frac{d\tilde{X}}{ds} = \tilde{F}(s, \tilde{X}(s)), \ \ 0 < s < T, \ \ \tilde{X}(s = t) = \tilde{x},$$

for some $(t, x, \tilde{x}) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$. *Then we have for all* $s \in [0,T]$

$$\begin{aligned} |X(s) - \tilde{X}(s)| \ \ &\leq \ \ \left(|x - \tilde{x}| + \left| \int_t^s \|F(\tau, \cdot) - \tilde{F}(\tau, \cdot)\|_{L^\infty(\mathbb{R}^N)} \, d\tau \right| \right) \\ &\times \ \ \exp \left(\left| \int_t^s \left(\sum_{i=1}^N \|\nabla F_i(\tau, \cdot)\|^2_{L^\infty(\mathbb{R}^N)} \right)^{\frac{1}{2}} \, d\tau \right| \right). \end{aligned}$$

By applying the general result to our three cases we obtain

Proposition 2.2 *Assume that* $E, \tilde{E} \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}))$, $A, \tilde{A} \in L^\infty(]0,T[; W^{2,\infty}(\mathbb{R}))$ *and consider* $(t, x, p), (t, \tilde{x}, \tilde{p}) \in [0,T] \times \mathbb{R}^2$. *We denote by by* $(X, P)(s; t, x, p)$, *resp.* $(\tilde{X}, \tilde{P})(s; t, \tilde{x}, \tilde{p})$ *the solution of* (26), (27) *corresponding to* (E, A), $(\tilde{E}, \tilde{A})$.

1) In the NR and QR cases we have for all $s \in [0,T]$

$$\begin{aligned} &\left(|X(s) - \tilde{X}(s)|^2 + |P(s) - \tilde{P}(s)|^2 \right)^{\frac{1}{2}} \leq ((|x - \tilde{x}|^2 + |p - \tilde{p}|^2)^{\frac{1}{2}} \\ + \ \ &\left| \int_t^s \{ \|(E - \tilde{E})(\tau)\|_{L^\infty(\mathbb{R})} + \|(A - \tilde{A})\partial_x A(\tau)\|_{L^\infty(\mathbb{R})} + \|\tilde{A}(\partial_x A - \partial_x \tilde{A})(\tau)\|_{L^\infty(\mathbb{R})} \} d\tau \right|) \\ \times \ \ &\exp \left(\left| \int_t^s \{ 1 + \|\partial_x E(\tau)\|_{L^\infty(\mathbb{R})} + \|\partial_x A(\tau)\|^2_{L^\infty(\mathbb{R})} + \|A \partial_x^2 A(\tau)\|_{L^\infty(\mathbb{R})} \} \, d\tau \right| \right). \end{aligned}$$

2) In the FR case we have for all $s \in [0,T]$

$$\begin{aligned} &\left(|X(s) - \tilde{X}(s)|^2 + |P(s) - \tilde{P}(s)|^2 \right)^{\frac{1}{2}} \leq ((|x - \tilde{x}|^2 + |p - \tilde{p}|^2)^{\frac{1}{2}} \\ + \ \ &C \left| \int_t^s \{ \|(E - \tilde{E})(\tau)\|_\infty + (1 + \|\partial_x A(\tau)\|_\infty) \|(A - \tilde{A})(\tau)\|_\infty + \|(\partial_x A - \partial_x \tilde{A})(\tau)\|_\infty \} d\tau \right|) \\ \times \ \ &\exp \left(C \left| \int_t^s \{ 1 + \|\partial_x E(\tau)\|_{L^\infty(\mathbb{R})} + \|\partial_x A(\tau)\|^2_{L^\infty(\mathbb{R})} + \|\partial_x^2 A(\tau)\|_{L^\infty(\mathbb{R})} \} \, d\tau \right| \right). \end{aligned}$$

3 Existence and uniqueness for the reduced Vlasov-Maxwell system

We intend to prove the existence and uniqueness of the mild solution for the system (18), (19), (20), (21), (22) in the FR case by using the iterated approximation method. We assume that $\gamma_1 = \gamma_2 = (1 + |p|^2 + |A(t,x)|^2)^{1/2}$ everywhere from now on if nothing else specified. Nevertheless we prefer to distinguish the Lorentz factors γ_1, γ_2 ; the reader can try to adapt the proofs in order to treat the NR and QR cases. We consider the application $\mathcal{F}$ defined for regular fields $E \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}))$ and $A \in L^\infty(]0,T[; W^{2,\infty}(\mathbb{R}))$ as follows

$$(E, A) \to f_{E,A} \to (\tilde{E}, \tilde{A}) =: \mathcal{F}(E, A), \tag{30}$$

where $f_{E,A}$ is the mild solution of the Vlasov problem (23), (24) associated with the fields E, A and $\tilde{E}, \tilde{A}$ are given by

$$\int_{\mathbb{R}} \tilde{E}(t,x)\varphi(x)\,dx = \int_{\mathbb{R}} E_0(x)\varphi(x)\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_x^{X(t;0,x,p)} \varphi(u)\,du\,dp\,dx \tag{31}$$

for any function $\varphi \in L^1(\mathbb{R})$, where (X, P) are the characteristics associated with (E, A), respectively

$$\begin{aligned} \tilde{A}(t,x) &= \frac{1}{2}(A_0(x+t) + A_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} A_1(y)\,dy \\ &- \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} (\rho_{\gamma_2} A)(s,y)\,dy\,ds, \quad (t,x) \in [0,T] \times \mathbb{R}, \end{aligned} \tag{32}$$

where $\rho_{\gamma_2}(t,x) = \int_{\mathbb{R}} \frac{f_{E,A}(t,x,p)}{\gamma_2}\,dp$.

Obviously the expression of $\tilde{A}$ comes from the Duhamel representation formula of the solution for the wave equation in one dimension with the source term $-\rho_{\gamma_2} A$ and the initial conditions A_0, A_1. Note that if $A_1 \in L^\infty(\mathbb{R})$, $f_0 \in L^1(\mathbb{R}^2)$, then $\rho_{\gamma_2} \in L^\infty(]0,T[; L^1(\mathbb{R}))$ which implies that $\rho_{\gamma_2} A \in L^\infty(]0,T[; L^1(\mathbb{R}))$ and thus $\tilde{A}$ is well defined. Let us explain now our choice for the definition of $\tilde{E}$. The Maxwell equations involving the electrostatic field $\tilde{E}$ are

$$\partial_t \tilde{E} = j_{E,A} = \int_{\mathbb{R}} \frac{p}{\gamma_1} f_{E,A}\,dp, \quad \partial_x \tilde{E} = \rho_{ext} - \rho_{E,A} = \rho_{ext} - \int_{\mathbb{R}} f_{E,A}\,dp, \quad (t,x) \in]0,T[\times\mathbb{R},$$

with the initial condition $\tilde{E}(0,x) = E_0(x), \quad x \in \mathbb{R}$. By using the continuity equation $\partial_t \rho_{E,A} + \partial_x j_{E,A} = 0$ it is sufficient to impose $E_0' = \rho_{ext} - \rho_0$ where $\rho_0 = \int_{\mathbb{R}} f_0\,dp$ and to solve $\partial_t \tilde{E} = j_{E,A}$, which gives $\tilde{E}(t,x) = E_0 + \int_0^t j_{E,A}(s,x)\,ds$. After multiplication by a test function $\varphi \in L^1(\mathbb{R})$ one gets by formal computations using the mild formulation (28)

$$\begin{aligned}
\int_{\mathbb{R}} \tilde{E}(t,x)\varphi(x)\,dx &= \int_{\mathbb{R}} E_0(x)\varphi(x)\,dx + \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} f_{E,A}(s,x,p)\frac{p}{\gamma_1}\varphi(x)\,dp\,dx\,ds \\
&= \int_{\mathbb{R}} E_0(x)\varphi(x)\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^t \frac{dX}{ds}\varphi(X(s))\,ds\,dp\,dx \\
&= \int_{\mathbb{R}} E_0(x)\varphi(x)\,dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_x^{X(t;0,x,p)} \varphi(u)\,du\,dp\,dx.
\end{aligned}$$

Note that if $E_0 \in L^\infty(\mathbb{R})$ and $f_0 \in L^1(\mathbb{R}^2)$ the formula (31) defines a unique $\tilde{E} \in L^\infty(]0,T[\times\mathbb{R})$ and

$$\|\tilde{E}(t)\|_{L^\infty(\mathbb{R})} \le \|E_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)}, \quad t \in]0,T[.$$

As usual, the idea is to study the existence and uniqueness of fixed point for $\mathcal{F}$. We introduce some notations. If $u : \mathbb{R} \to [0,+\infty[$ is a bounded function nondecreasing on $\mathbb{R}^-$ and nonincreasing on $\mathbb{R}^+$ and $R > 0$, we denote by $u^R : \mathbb{R} \to [0,+\infty[$ the function given by $u^R(t) = u(0)$ if $|t| \le R$, $u^R(t) = u(t-R)$ if $t > R$ and $u^R(t) = u(t+R)$ if $t < -R$. If u belongs to $L^1(\mathbb{R})$ then $u^R \in L^1(\mathbb{R})$ and

$$\|u^R\|_{L^1(\mathbb{R})} = 2R\|u\|_{L^\infty(\mathbb{R})} + \|u\|_{L^1(\mathbb{R})}.$$

Following the ideas in [8] we obtain L^∞ bounds for $\tilde{E}, \tilde{A}$ and their first derivatives.

3.1 Estimates for $\tilde{E}$

We assume that f_0 verifies the following hypotheses. There is $n_0 : \mathbb{R} \to [0,+\infty[$

nondecreasing on $\mathbb{R}^-$ and nonincreasing on $\mathbb{R}^+$ such that

$$(H) \quad f_0(x,p) \le n_0(p), \quad \forall\, (x,p) \in \mathbb{R}^2, \tag{33}$$

$$(H_0) \quad M_0 := \int_{\mathbb{R}} n_0(p)\, dp < +\infty, \tag{34}$$

$$(H_\infty) \quad M_\infty := \|n_0\|_{L^\infty(\mathbb{R})} < +\infty. \tag{35}$$

We can prove the following L^∞ bounds for $\tilde{E}$, $\partial_x \tilde{E}$.

Proposition 3.1 *Assume that f_0 is nonnegative, belongs to $L^1(\mathbb{R}^2)$ and satisfies (H), (H_0), (H_∞). We suppose also that ρ_{ext} is a given nonnegative function in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and that E_0 is a primitive of $\rho_{ext} - \rho_0$, where $\rho_0 = \int_{\mathbb{R}} f_0\, dp$*

$$E_0{}' = \rho_{ext} - \rho_0.$$

Then for all regular fields $E \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}))$, $A \in L^\infty(]0,T[; W^{2,\infty}(\mathbb{R}))$ we have $\tilde{E} \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}))$ and the following estimates hold for all $t \in [0,T]$

$$\|\tilde{E}\|_{L^\infty(\mathbb{R})} \le \|E_0\|_{L^\infty(\mathbb{R})} + \|f_0\|_{L^1(\mathbb{R}^2)}, \tag{36}$$

$$\|\partial_x \tilde{E}\|_{L^\infty(\mathbb{R})} \le \|\rho_{ext}\|_{L^\infty(\mathbb{R})} + M_0 + 2M_\infty \int_0^t \{\|E(s)\|_{L^\infty(\mathbb{R})} + \|A(s)\partial_x A(s)\|_{L^\infty(\mathbb{R})}\}\, ds. \tag{37}$$

Proof. The estimate (36) follows immediately from our definition for $\tilde{E}$. We introduce the charge density $\rho_{E,A} = \int_{\mathbb{R}} f_{E,A}\, dp$. By using (29) we can write

$$\rho_{E,A}(t,x) = \int_{\mathbb{R}} f_0(X(0;t,x,p), P(0;t,x,p))\, dp \le \int_{\mathbb{R}} n_0(P(0;t,x,p))\, dp.$$

Using now the second equation in (26) yields

$$|P(0;t,x,p) - p| \le \int_0^t \{\|E(s)\|_{L^\infty(\mathbb{R})} + \|A(s)\partial_x A(s)\|_{L^\infty(\mathbb{R})}\}\, ds =: R(t).$$

Notice that if $p > R(t)$ then $P(0;t,x,p) \ge p - R(t) > 0$ and thus $n_0(P(0;t,x,p)) \le n_0(p - R(t)) = n_0^{R(t)}(p)$. Similarly, if $p < -R(t)$ then $P(0;t,x,p) \le p + R(t) < 0$ and thus $n_0(P(0;t,x,p)) \le n_0(p+R(t)) = n_0^{R(t)}(p)$. When $|p| \le R(t)$ we have obviously $n_0(P(0;t,x,p)) \le n_0(0) = n_0^{R(t)}(p)$. Therefore we obtain

$$\rho_{E,A}(t,x) \le \int_{\mathbb{R}} n_0^{R(t)}(p)\ dp = 2R(t)M_\infty + M_0, \ \ (t,x) \in [0,T] \times \mathbb{R}. \tag{38}$$

In order to estimate $\partial_x \tilde{E}$ we prove that $\tilde{E}$ satisfies $\partial_x \tilde{E} = \rho_{ext} - \rho_{E,A}$. Indeed, take $\varphi \in C_c^1(\mathbb{R})$ and let us calculate

$$\begin{aligned}
\int_{\mathbb{R}} \tilde{E}(t,x)\frac{d\varphi}{dx}\ dx &= \int_{\mathbb{R}} E_0(x)\frac{d\varphi}{dx}\ dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\int_x^{X(t;0,x,p)} \frac{d\varphi}{du}\ du\ dp\, dx \\
&= -\int_{\mathbb{R}} E_0'\varphi(x)\ dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(\varphi(X(t;0,x,p)) - \varphi(x))\ dp\, dx \\
&= -\int_{\mathbb{R}} \rho_{ext}(x)\varphi(x)\ dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_{E,A}(t,x,p)\varphi(x)\ dp\, dx \\
&= -\int_{\mathbb{R}} (\rho_{ext}(x) - \rho_{E,A}(t,x))\varphi(x)\ dx.
\end{aligned} \tag{39}$$

Therefore we have $\partial_x \tilde{E}(t) = \rho_{ext} - \rho_{E,A}(t)$ and (37) follows by (38). $\square$

3.2 Estimates for $\tilde{A}$

We establish now L^∞ bounds for $\tilde{A}, \partial_x \tilde{A}$ and $\partial_t \tilde{A}$.

Proposition 3.2 *Assume that f_0 is nonnegative, belongs to $L^1(\mathbb{R}^2)$ and satisfies (H),(H_0), (H_∞). We suppose also that $A_0 \in W^{1,\infty}(\mathbb{R}), A_1 \in L^\infty(\mathbb{R})$. Then for all regular fields $E \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R})), A \in L^\infty(]0,T[;W^{2,\infty}(\mathbb{R}))$ we have $\tilde{A} \in W^{1,\infty}(]0,T[\times\mathbb{R})$ and*

$$\|\tilde{A}(t)\|_{L^\infty(\mathbb{R})} \le \|A_0\|_{L^\infty(\mathbb{R})} + t\ \|A_1\|_{L^\infty(\mathbb{R})} + \frac{\|f_0\|_{L^1(\mathbb{R}^2)}}{2}\int_0^t \|A(s)\|_{L^\infty(\mathbb{R})}\ ds, \ \ t \in [0,T], \tag{40}$$

$$\max\{\|\partial_x \tilde{A}(t)\|_{L^\infty}, \|\partial_t \tilde{A}\|_{L^\infty}\} \le \|A_0'\|_{L^\infty} + \|A_1\|_{L^\infty} + \int_0^t \|A(s)\|_{L^\infty}(M_0 + 2R(s)M_\infty)\ ds, \tag{41}$$

where $R(t) = \int_0^t \{\|E(s)\|_{L^\infty(\mathbb{R})} + \|A(s)\partial_x A(s)\|_{L^\infty(\mathbb{R})}\}\ ds, \ \ t \in [0,T]$.

Proof. From (32) we deduce easily that

$$\begin{aligned}\|\tilde{A}(t)\|_{L^\infty} &\le \|A_0\|_{L^\infty} + t\,\|A_1\|_{L^\infty} + \frac{1}{2}\int_0^t \|A(s)\|_{L^\infty}\|\rho_{E,A}(s)\|_{L^1}\,ds \\ &\le \|A_0\|_{L^\infty} + t\,\|A_1\|_{L^\infty} + \frac{\|f_0\|_{L^1(\mathbb{R}^2)}}{2}\int_0^t \|A(s)\|_{L^\infty(\mathbb{R})}\,ds. \qquad (42)\end{aligned}$$

We have the following representation formula for the space derivative of $\tilde{A}$

$$\begin{aligned}\partial_x \tilde{A}(t,x) &= \frac{1}{2}\{A_0'(x+t)+A_0'(x-t)\} + \frac{1}{2}\{A_1(x+t)-A_1(x-t)\} \\ &- \frac{1}{2}\int_0^t \{(\rho_{\gamma_2}A)(s,x+t-s) - (\rho_{\gamma_2}A)(s,x-t+s)\}\,ds, \qquad (43)\end{aligned}$$

and therefore, by using (38) we obtain the estimate

$$\begin{aligned}\|\partial_x \tilde{A}(t)\|_{L^\infty} &\le \|A_0'\|_{L^\infty} + \|A_1\|_{L^\infty} + \int_0^t \|A(s)\rho_{E,A}(s)\|_{L^\infty}\,ds \\ &\le \|A_0'\|_{L^\infty} + \|A_1\|_{L^\infty} + \int_0^t \|A(s)\|_{L^\infty}\{M_0 + 2R(s)M_\infty\}\,ds. \qquad (44)\end{aligned}$$

The time derivative of $\tilde{A}$ is given by

$$\begin{aligned}\partial_t \tilde{A}(t,x) &= \frac{1}{2}\{A_0'(x+t)-A_0'(x-t)\} + \frac{1}{2}\{A_1(x+t)+A_1(x-t)\} \\ &- \frac{1}{2}\int_0^t \{(\rho_{\gamma_2}A)(s,x+t-s) + (\rho_{\gamma_2}A)(s,x-t+s)\}\,ds, \qquad (45)\end{aligned}$$

and we obtain the same estimate for $\partial_t \tilde{A}$ as for $\partial_x \tilde{A}$. □

We construct now a domain D_T for the application $\mathcal{F}$ such that

$$\|\tilde{E}\|_{L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}))} + \|\tilde{A}\|_{W^{1,\infty}(]0,T[\times\mathbb{R})} \le C, \ \ \forall (E,A)\in D_T,$$

for some constant depending only on the initial conditions and T.

Proposition 3.3 *Assume that the hypotheses of Propositions* 3.1, 3.2 *hold. We consider the set*

$$D_T = \{(E,A)\in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R}))\times L^\infty(]0,T[;W^{2,\infty}(\mathbb{R}))\ |\ \|E\|_{L^\infty}\le e,$$
$$\|\partial_x E\|_{L^\infty}\le e_1,\ \|A(t)\|_{L^\infty}\le a(t),\ \|\partial_x A(t)\|_{L^\infty}\le a_1(t),\ t\in[0,T]\},$$

where $e = \|E_0\|_{L^\infty} + \|f_0\|_{L^1}$,

$$a(t) = (\|A_0\|_{L^\infty} + t\ \|A_1\|_{L^\infty})\ \exp\left(\frac{t\ \|f_0\|_{L^1}}{2}\right),\ t \in [0,T],$$

$$a_1(t) = \{\|A_0'\|_{L^\infty} + \|A_1\|_{L^\infty} + M_0\ t\ a(t) + M_\infty e\ t^2 a(t)\}\exp(2M_\infty t^2 a(t)^2),\ t \in [0,T],$$

and

$$e_1 = \|\rho_{ext}\|_{L^\infty} + M_0 + 2\ M_\infty\ e\ T + 2\ M_\infty \int_0^T a(t)a_1(t)\ dt.$$

Then for all $(E,A) \in D_T$ *and any* $t \in [0,T]$ *we have the inequalities*

$$\|\tilde{E}\|_{L^\infty} \le e,\ \ \|\partial_x \tilde{E}\|_{L^\infty} \le e_1,$$

$$\|\tilde{A}(t)\|_{L^\infty} \le a(t),\ \ \max\{\|\partial_x \tilde{A}(t)\|_{L^\infty}, \|\partial_t \tilde{A}(t)\|_{L^\infty}\} \le a_1(t),\ \ t \in [0,T].$$

Proof. From (36) we have $\|\tilde{E}\|_{L^\infty} \le e$. From (40) we obtain

$$\begin{aligned}\|\tilde{A}(t)\|_{L^\infty} &\le (\|A_0\|_{L^\infty} + t\ \|A_1\|_{L^\infty})\left(1 + \frac{\|f_0\|_{L^1}}{2}\int_0^t \exp\left(\frac{s\ \|f_0\|_{L^1}}{2}\right)\ ds\right) \\ &= a(t).\end{aligned}$$

We introduce the notation $c(t) = \|A_0'\|_{L^\infty} + \|A_1\|_{L^\infty} + M_0\ ta(t) + M_\infty\ e\ t^2 a(t)$. The formula (41) yields

$$\begin{aligned}\|\partial_x \tilde{A}(t)\|_{L^\infty} &\le c(t) + 2M_\infty\ ta(t)^2 \int_0^t \|\partial_x A(\tau)\|_{L^\infty}\ d\tau \\ &\le c(t)\left(1 + 2M_\infty\ ta(t)^2 \int_0^t \exp(2M_\infty\ \tau^2 a(\tau)^2)\ d\tau\right) \\ &\le c(t)\left(1 + 2M_\infty\ ta(t)^2 \int_0^t \exp(2M_\infty\ ta(t)^2 \tau)\ d\tau\right) \\ &= a_1(t). \end{aligned} \tag{46}$$

Similarly we have $\|\partial_t \tilde{A}(t)\|_{L^\infty} \le a_1(t)$, $t \in [0,T]$ and from (37) one gets also

$$\|\partial_x \tilde{E}\|_{L^\infty(]0,T[\times\mathbb{R})} \le \|\rho_{ext}\|_{L^\infty} + M_0 + 2M_\infty\ e\ T + 2M_\infty \int_0^T a(s)a_1(s)\ ds = e_1.$$

$\square$

In view of use of iterative procedure it is convenient to restrict the domain D_T to

$$D_T = \{(E, A) \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R})) \times L^\infty(]0,T[; W^{2,\infty}(\mathbb{R})) \mid \|E\|_{L^\infty} \le e, \|\partial_x E\|_{L^\infty} \le e_1,$$
$$\|A(t)\|_{L^\infty} \le a(t),\ \max\{\|\partial_x A(t)\|_{L^\infty}, \|\partial_t A(t)\|_{L^\infty}\} \le a_1(t),\ t \in [0,T]\},$$

For further computations we need to estimate also the L^∞ norm of the second space derivative $\partial_x^2 \tilde{A}$. This type of estimate has been obtained in [8] locally in time for the NR case and globally in time for the QR case. We will show that this is possible globally in time in the FR case. We need the following easy lemmas.

Lemma 3.1 *Assume that f_0 is nonnegative satisfying*

$$(\tilde{H}_k) \qquad \int_{\mathbb{R}}\int_{\mathbb{R}} (1 + |p|^k) f_0(x,p)\, dp\, dx < +\infty.$$

Then there is a constant C such that for all regular fields $(E, A) \in D_T$ we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |p|^k f_{E,A}(t,x,p)\, dp\, dx \le C \int_{\mathbb{R}}\int_{\mathbb{R}} (1 + |p|^k) f_0(x,p)\, dp\, dx, \quad t \in [0,T].$$

Proof. For any $t \in [0,T]$ we can write

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |p|^k f_{E,A}(t,x,p)\, dp\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} |p|^k f_0(X(0;t,x,p), P(0;t,x,p))\, dp\, dx,$$

where (X, P) are the characteristics associated to (E, A). By taking into account that

$$\begin{aligned} |p - P(0;t,x,p)| &\le R(t) = \int_0^t \{\|E(s)\|_{L^\infty} + \|A(s)\partial_x A(s)\|_{L^\infty}\}\, ds \\ &\le T\ (\ e + a(T)a_1(T)\ ) =: R, \quad t \in [0,T], \end{aligned}$$

we deduce that $|p|^k \le C\ (1 + |P(0;t,x,p)|^k)$ and the conclusion follows easily since we have for any $t \in [0,T]$

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} |p|^k f_0(X(0), P(0))\, dp\, dx &\le C \int_{\mathbb{R}}\int_{\mathbb{R}} (1 + |P(0)|^k) f_0(X(0), P(0))\, dp\, dx \\ &= C \int_{\mathbb{R}}\int_{\mathbb{R}} (1 + |p|^k) f_0(x,p)\, dp\, dx. \end{aligned}$$

$\square$

Lemma 3.2 *Assume that the hypotheses of Proposition* 3.1 *hold and suppose that*

$$(H_k) \qquad M_k := \int_{\mathbb{R}} |p|^k n_0(p)\, dp < +\infty.$$

Then for any $(E, A) \in D_T$ *we have*

$$\left\| \int_{\mathbb{R}} |p|^k f_{E,A}(t,\cdot,p)\, dp \right\|_{L^\infty} \leq C\,(M_0 + M_k + M_\infty),$$

for a constant C *depending on* T *and the initial conditions.*

Proof. We have for $(t, x) \in [0, T] \times \mathbb{R}$

$$\begin{aligned} \int_{\mathbb{R}} |p|^k f_{E,A}(t,x,p)\, dp &= \int_{\mathbb{R}} |p|^k f_0(X(0;t,x,p), P(0;t,x,p))\, dp \\ &\leq \int_{\mathbb{R}} |p|^k n_0(P(0;t,x,p))\, dp \\ &\leq \int_{\mathbb{R}} |p|^k n_0^{R(t)}(p)\, dp, \end{aligned}$$

and the conclusion follows easily since $R(t) \leq t(\ e + a(t)a_1(t)\ ), \forall\, t \in [0, T]$. □

Proposition 3.4 *Assume that the hypotheses of Proposition* 3.1 *hold. Then for any regular fields* $(E, A) \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R})) \times L^\infty(]0,T[; W^{2,\infty}(\mathbb{R}))$ *we have the estimate* $\|\partial_t \tilde{E}(t)\|_{L^\infty} \leq M_0 + 2\ M_\infty R(t)$ *where* $R(t) = \int_0^t \{\|E(s)\|_{L^\infty} + \|A(s)\partial_x A(s)\|_{L^\infty}\}\, ds$, $t \in [0, T]$.

Proof. Observe that under the above hypotheses $j_{E,A} = \int_{\mathbb{R}} \frac{p}{\gamma_1} f_{E,A}\, dp$ is well defined. We prove that $\partial_t \tilde{E} = j_{E,A}$. For this pick a function $\varphi \in C_c^1(]0, T[\times\mathbb{R})$ and calculate

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \tilde{E}(t,x)\partial_t\varphi\, dx\, dt &= \int_0^T \int_{\mathbb{R}} E_0(x)\partial_t\varphi\, dx\, dt + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} f_0 \int_x^{X(t;0,x,p)} \partial_t\varphi\, du\, dp\, dx\, dt \\ &= \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} f_0 \left\{ \frac{d}{dt} \int_x^{X(t;0,x,p)} \varphi(t,u)\, du - \varphi(t, X(t)) \frac{P(t)}{\gamma_1(t)} \right\} dp\, dx\, dt \\ &= -\int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x,p) \int_0^T \varphi(t, X(t;0,x,p)) \frac{P(t;0,x,p)}{\gamma_1(t)}\, dt\, dp\, dx \\ &= -\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} f_{E,A}(t,x,p)\varphi(t,x) \frac{p}{\gamma_1}\, dp\, dx\, dt \\ &= -\int_0^T \int_{\mathbb{R}} j_{E,A}(t,x)\varphi(t,x)\, dx\, dt, \end{aligned} \tag{47}$$

which implies that $\partial_t \tilde{E} = j_{E,A}$. Therefore we obtain

$$\begin{aligned}
\|\partial_t \tilde{E}(t)\|_{L^\infty} &= \|j_{E,A}\|_{L^\infty} \le \|\rho_{E,A}\|_{L^\infty} \\
&\le M_0 + 2\, M_\infty \int_0^t \{\|E(s)\|_{L^\infty} + \|A(s)\partial_x A(s)\|_{L^\infty}\}\, ds.
\end{aligned}$$

$\square$

Proposition 3.5 *Assume that the hypotheses of Propositions* 3.1, 3.2 *hold,* $A_0 \in W^{2,\infty}(\mathbb{R}), A_1 \in W^{1,\infty}(\mathbb{R})$. *Moreover we suppose that* $M_1 = \int_{\mathbb{R}} |p|\; n_0(p)\; dp < +\infty$. *Then for any* $T > 0$ *there is a constant* C *depending on* T *and the initial conditions such that*

$$\max\{\|\partial^2_{xt}\tilde{A}\|_{L^\infty(]0,T[\times\mathbb{R})}, \|\partial^2_x \tilde{A}\|_{L^\infty(]0,T[\times\mathbb{R})}, \|\partial^2_t \tilde{A}\|_{L^\infty(]0,T[\times\mathbb{R})}\} \le C, \;\; \forall\, (E,A) \in D_T.$$

Proof. Recall that the first space derivative of $\tilde{A}$ is given by

$$\partial_x \tilde{A}(t,x) = D^0(t,x) + \frac{1}{2}D^-(t,x) - \frac{1}{2}D^+(t,x),$$

where

$$D^0(t,x) = \frac{1}{2}\{A_0'(x+t) + A_0'(x-t)\} + \frac{1}{2}\{A_1(x+t) - A_1(x-t)\},$$

$$D^\pm(t,x) = \int_0^t (\rho_{\gamma_2}\; A)(s, x \pm (t-s))\; ds.$$

Obviously we have

$$\|\partial_x D^0\|_{L^\infty(]0,T[\times\mathbb{R})} \le \|A_0''\|_{L^\infty} + \|A_1'\|_{L^\infty},$$

and it remains to estimate the space derivatives of $D^\pm(t,\cdot)$ for $t \in [0,T]$. Pick a test function $\varphi \in C^1_c(\mathbb{R})$ and by using (28) one gets

$$\begin{aligned}
\int_{\mathbb{R}} D^\pm(t,x)\varphi'(x)\; dx &= \int_0^t \int_{\mathbb{R}} (\rho_{\gamma_2}\; A)(s, x \pm (t-s))\varphi'(x)\; dx\; ds \\
&= \int_0^t \int_{\mathbb{R}} (\rho_{\gamma_2}\; A)(s,x)\varphi'(x \mp (t-s))\; dx\; ds \\
&= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(s,x,p)}{\gamma_2} A(s,x)\varphi'(x \mp (t-s))\; dp\, dx\, ds \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x,p) I^\pm(t,x,p)\; dp\, dx, \qquad (48)
\end{aligned}$$

where $I^{\pm}(t,x,p) = \int_0^t \frac{A(s,X(s;0,x,p))}{\gamma_2(s)} \varphi'(X(s;0,x,p) \mp (t-s))\ ds, \forall\ (t,x,p) \in [0,T] \times \mathbb{R}^2$, $\gamma_2(s) = (1+|P(s;0,x,p)|^2 + |A(s,X(s;0,x,p))|^2)^{1/2}$. We need to evaluate $I^{\pm}(t,x,p)$. The crucial point here is that the velocities of any characteristic (26) remain below the characteristic's speed of (19)

$$\left|\frac{dX}{ds}\right| = \left|\frac{P(s;0,x,p)}{\gamma_1(s)}\right| < 1, \ \ \forall\ (s,x,p) \in [0,T] \times \mathbb{R}^2,$$

where $\gamma_1(s) = (1+|P(s;0,x,p)|^2 + |A(s,X(s;0,x,p))|^2)^{1/2}$. This fact was also one of the key points of the proofs in [8], [12], [13]. Therefore the following computations are valid

$$\begin{aligned}
I^{\pm}(t,x,p) &= \int_0^t \frac{A(s,X(s))}{\gamma_2(s)(X'(s) \pm 1)} \frac{d}{ds}\varphi(X(s) \mp (t-s))\ ds \\
&= \frac{A(s,X(s))}{\gamma_2(s)(X'(s) \pm 1)}\varphi(X(s) \mp (t-s))|_{s=0}^{s=t} \\
&- \int_0^t \frac{d}{ds}\left\{\frac{A(s,X(s))}{\gamma_2(s)(X'(s) \pm 1)}\right\}\varphi(X(s) \mp (t-s))\ ds \\
&= I_1\ \varphi(X(t)) - I_2\ \varphi(x \mp t) - \int_0^t I_3(s)\ \varphi(X(s) \mp (t-s))\ ds.
\end{aligned}$$

Note that we have

$$\begin{aligned}
\frac{1}{|\gamma_2(s)(X'(s) \pm 1)|} &= \frac{\sqrt{1+|P(s)|^2 + |A(s,X(s))|^2} \mp P(s)}{1+|A(s,X(s))|^2} \\
&\leq 2\frac{\sqrt{1+|P(s)|^2 + |A(s,X(s))|^2}}{1+|A(s,X(s))|^2} \\
&\leq 2(1+|P(s)|).
\end{aligned}$$

With the previous notations we obtain

$$\begin{aligned}
\int_{\mathbb{R}} D^{\pm}(t,x)\varphi'(x)\ dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) I_1\ \varphi(X(t))\ dp\, dx - \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) I_2\ \varphi(x \mp t)\ dp\, dx \\
&- \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^t I_3(s)\ \varphi(X(s) \mp (t-s))\ ds\ dp\, dx \\
&= Q_1 + Q_2 + Q_3. \qquad (49)
\end{aligned}$$

Estimate of Q_1

$$
\begin{aligned}
|Q_1| &\le \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\frac{|A(t,X(t))|}{|\gamma_2(t)(X'(t)\pm 1)|}|\varphi(X(t))|\,dp\,dx \\
&\le 2a(t)\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(1+|P(t)|)\;|\varphi(X(t))|\,dp\,dx \\
&= 2a(t)\int_{\mathbb{R}}\int_{\mathbb{R}} f_{E,A}(t,X(t),P(t))(1+|P(t)|)\;|\varphi(X(t))|\,dp\,dx \\
&= 2a(t)\int_{\mathbb{R}}\int_{\mathbb{R}} f_{E,A}(t,x,p)(1+|p|)\;|\varphi(x)|\,dp\,dx \\
&\le 2a(t)\left(\left\|\int_{\mathbb{R}} f_{E,A}(t,\cdot,p)\;dp\right\|_{L^\infty}+\left\|\int_{\mathbb{R}} f_{E,A}(t,\cdot,p)|p|\;dp\right\|_{L^\infty}\right)\|\varphi\|_{L^1} \\
&\le C_1\|\varphi\|_{L^1}, \qquad (50)
\end{aligned}
$$

where $C_1 = 2a(T)(M_0 + 2M_\infty\;T\;(e + a(T)a_1(T)\;) + C\;(M_0 + M_1 + M_\infty)\;)$. $\square$

Estimate of Q_2

$$
\begin{aligned}
|Q_2| &\le \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\frac{|A(0,x)|}{|\gamma_2(0)(X'(0)\pm 1)|}|\varphi(x\mp t)|\,dp\,dx \\
&\le 2\|A_0\|_{L^\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(1+|p|)\;|\varphi(x\mp t)|\,dp\,dx \\
&\le 2\|A_0\|_{L^\infty}\left(\left\|\int_{\mathbb{R}} f_0(\cdot,p)\;dp\right\|_{L^\infty}+\left\|\int_{\mathbb{R}} f_0(\cdot,p)|p|\;dp\right\|_{L^\infty}\right)\|\varphi\|_{L^1} \\
&\le C_2\;\|\varphi\|_{L^1}, \qquad (51)
\end{aligned}
$$

with $C_2 = 2\|A_0\|_{L^\infty}(M_0 + M_1)$.

Estimate of Q_3

We obtain

$$
\begin{aligned}
|I_3(s)| &= \left|\frac{d}{ds}\left\{\frac{A(s,X(s))}{P(s)\pm\sqrt{1+|P(s)|^2+|A(s,X(s))|^2}}\right\}\right| \\
&= \left|\frac{d}{ds}\left\{\frac{A(s,X(s))}{1+|A(s,X(s))|^2}(\sqrt{1+|P(s)|^2+|A(s,X(s))|^2}\mp P(s))\right\}\right| \\
&\le C(1+|P(s)|).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
|Q_3| &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^t C(1+|P(s)|)|\varphi(X(s) \mp (t-s))| \, ds \, dp \, dx \\
&= C \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} f_{E,A}(s,x,p)(1+|p|)|\varphi(x \mp (t-s))| \, dp \, dx \, ds \\
&\leq C \int_0^t \int_{\mathbb{R}} |\varphi(x \mp (t-s))| \left(\left\| \int_{\mathbb{R}} f_{E,A}(s,\cdot,p) \, dp \right\|_{L^\infty} + \left\| \int_{\mathbb{R}} f_{E,A}(s,\cdot,p)|p| \, dp \right\|_{L^\infty} \right) dx \\
&\leq C_3 \, \|\varphi\|_{L^1}. \qquad (52)
\end{aligned}
$$

The equality (49) and the inequalities (50), (51), (52) imply

$$
\left| \int_{\mathbb{R}} D^{\pm}(t,x)\varphi'(x) \, dx \right| \leq (C_1 + C_2 + C_3) \, \|\varphi\|_{L^1},
$$

and therefore $\partial_x D^{\pm} \in L^\infty(]0,T[\times\mathbb{R})$, $\|\partial_x D^{\pm}\|_{L^\infty} \leq C_1 + C_2 + C_3 := C_4$. We obtain finally that

$$
\|\partial_x^2 \tilde{A}\|_{L^\infty} \leq \|A_0''\|_{L^\infty} + \|A_1'\|_{L^\infty} + C_4 := C_5, \;\; \forall \, (E,A) \in D_T.
$$

The second derivative $\partial_{xt}^2 \tilde{A}$ satisfies the same estimate since we have

$$
\partial_t \tilde{A}(t,x) = \frac{1}{2}\{A_0'(x+t) - A_0'(x-t)\} + \frac{1}{2}\{A_1(x+t) + A_1(x-t)\} - \frac{1}{2}D^+(t,x) - \frac{1}{2}D^-(t,x),
$$

and therefore

$$
\begin{aligned}
\|\partial_{xt}^2 \tilde{A}\|_{L^\infty} &\leq \|A_0''\|_{L^\infty} + \|A_1'\|_{L^\infty} + \frac{1}{2}(\|\partial_x D^+\|_{L^\infty} + \|\partial_x D^-\|_{L^\infty}) \\
&\leq \|A_0''\|_{L^\infty} + \|A_1'\|_{L^\infty} + C_4 \\
&= C_5.
\end{aligned}
$$

By using the wave equation (19), we obtain also an estimate for $\partial_t^2 \tilde{A}$

$$
\begin{aligned}
\|\partial_t^2 \tilde{A}(t)\|_{L^\infty} &\leq \|\partial_x^2 \tilde{A}(t)\|_{L^\infty} + \|\rho_{\gamma_2}(t)\|_{L^\infty} \, \|A(t)\|_{L^\infty} \\
&\leq C_5 + \{M_0 + 2 \, M_\infty \, T \, (e + a(T) \, a_1(T))\} \, a(T) =: C_6.
\end{aligned}
$$

We restrict one more time the domain D_T to the set

$$
\begin{aligned}
D_T \;=\; & \{(E,A) \in W^{1,\infty}(]0,T[\times\mathbb{R}) \times W^{2,\infty}(]0,T[\times\mathbb{R}) \mid \|E\|_{L^\infty} \leq e, \|\partial_x E\|_{L^\infty} \leq e_1, \\
& \|\partial_t E\|_{L^\infty} \leq e_2, \; \max\{\|\partial_x^2 A\|_{L^\infty}, \|\partial_{xt}^2 A\|_{L^\infty}\} \leq C_5, \; \|\partial_t^2 A\|_{L^\infty} \leq C_6, \\
& \|A(t)\|_{L^\infty} \leq a(t), \; \max\{\|\partial_x A(t)\|_{L^\infty}, \|\partial_t A(t)\|_{L^\infty}\} \leq a_1(t), \; t \in [0,T]\},
\end{aligned}
$$

where $e_2 = M_0 + 2\ M_\infty\ T\ (\ e + a(T)\ a_1(T)\ )$. Observe that this set is left invariant by $\mathcal{F}$.

3.3 Estimates for $\mathcal{F}(E_1, A_1) - \mathcal{F}(E_2, A_2)$

In the previous paragraph we constructed the domain D_T such that $\mathcal{F}(D_T) \subset D_T$. Observe that D_T is a closed bounded set of $X_T = W^{1,\infty}(]0,T[\times\mathbb{R}) \times W^{2,\infty}(]0,T[\times\mathbb{R})$. Our goal now is to evaluate the difference $\mathcal{F}(E_1, A_1) - \mathcal{F}(E_2, A_2)$ in terms of $(E_1 - E_2, A_1 - A_2)$ when $(E_k, A_k)_{1\le k\le 2}$ belong to D_T. We suppose that f_0 is nonnegative, belongs to $L^1(\mathbb{R}^2)$ and satisfies (H), (H_0), $(\tilde{H}_1)$, (H_∞), ρ_{ext} is nonnegative, $\rho_{ext} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $E_0' = \rho_{ext} - \int_\mathbb{R} f_0(\cdot, p)\ dp$, $A_0 \in W^{2,\infty}(\mathbb{R})$, $A_1 \in W^{1,\infty}(\mathbb{R})$. We use the notation

$$|||(E,A)(t)||| = \|E(t)\|_{L^\infty} + \|A(t)\|_{L^\infty} + \|\partial_x A(t)\|_{L^\infty} + \|\partial_t A(t)\|_{L^\infty}.$$

Estimate for $\tilde{E}_1 - \tilde{E}_2$

Consider a test function $\varphi \in L^1(\mathbb{R})$. From the definitions of $\tilde{E}_1, \tilde{E}_2$ one gets

$$\begin{aligned} |Q_4| &:= \left| \int_\mathbb{R} (\tilde{E}_1(t,x) - \tilde{E}_2(t,x))\varphi(x)\ dx \right| \le \int_\mathbb{R}\int_\mathbb{R} f_0(x,p) \left| \int_{X_2(t;0,x,p)}^{X_1(t;0,x,p)} |\varphi(u)|\ du \right|\ dp\, dx \\ &\le \int_\mathbb{R}\int_\mathbb{R} f_0(x,p)|\varphi(u)|\ \mathbf{1}_{\{|u-X_1(t)|\le |X_2(t)-X_1(t)|\}}\ du\ dp\, dx, \end{aligned} \tag{53}$$

where $(X_k, P_k)_{1\le k\le 2}$ are the characteristics associated with the fields $(E_k, A_k)_{1\le k\le 2}$. By Proposition 2.2 we deduce that

$$\begin{aligned} |X_1(t;0,x,p) - X_2(t;0,x,p)| &+ |P_1(t;0,x,p) - P_2(t;0,x,p)| \\ &\le C \int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\ ds =: \delta(t), \end{aligned}$$

and therefore we can write

$$
\begin{aligned}
|Q_4| &\le \int_{\mathbb{R}} |\varphi(u)| \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\mathbf{1}_{\{|u-X_1(t)|\le\delta(t)\}}\, dp\, dx\; du \\
&= \int_{\mathbb{R}} |\varphi(u)| \int_{\mathbb{R}}\int_{\mathbb{R}} f_{E_1,A_1}(t,x,p)\mathbf{1}_{\{|u-x|\le\delta(t)\}}\, dp\, dx\; du \\
&= \int_{\mathbb{R}} |\varphi(u)| \int_{\mathbb{R}} \rho_{E_1,A_1}(t,x)\mathbf{1}_{\{|u-x|\le\delta(t)\}}\, dx\; du \\
&\le 2\delta(t)\; \|\rho_{E_1,A_1}(t)\|_{L^\infty}\; \|\varphi\|_{L^1}, \quad \forall\, \varphi \in L^1(\mathbb{R}). \qquad (54)
\end{aligned}
$$

We deduce that for any $T > 0$ there is a constant depending on T and the initial conditions such that for any $(E_k, A_k) \in D_T, k \in \{1,2\}$ we have

$$
\|(\tilde{E}_1 - \tilde{E}_2)(t)\|_{L^\infty} \le C \int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\; ds, \;\; \forall\, t \in [0,T]. \qquad (55)
$$

Remark 3.1 *We retain also the inequality*

$$
\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \left| \int_{X_2(t;0,x,p)}^{X_1(t;0,x,p)} |\varphi(u)|\; du \right|\; dp\, dx \le C\; \|\varphi\|_{L^1} \int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\; ds,
$$

for all $t \in [0,T]$.

Estimate for $\tilde{A}_1 - \tilde{A}_2$

For any test function $\varphi \in L^1(\mathbb{R})$ we have

$$
\begin{aligned}
|Q_5| &:= \left| \int_{\mathbb{R}} (\tilde{A}_1(t,x) - \tilde{A}_2(t,x))\varphi(x)\; dx \right| \\
&= \frac{1}{2} \left| \int_{\mathbb{R}}\int_0^t \int_{x-(t-s)}^{x+(t-s)} (\rho_{1,\gamma_2}A_1 - \rho_{2,\gamma_2}A_2)(s,y)\; dy\; ds\; \varphi(x)\; dx \right| \\
&\le \frac{1}{2} \left| \int_{\mathbb{R}}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \{\rho_{1,\gamma_2}(A_1 - A_2)\}(s,y)\; dy\; ds\; \varphi(x)\; dx \right| \\
&+ \frac{1}{2} \left| \int_{\mathbb{R}}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \{(\rho_{1,\gamma_2} - \rho_{2,\gamma_2})A_2)\}(s,y)\; dy\; ds\; \varphi(x)\; dx \right| \\
&= \frac{1}{2}(|I_1| + |I_2|). \qquad (56)
\end{aligned}
$$

We check easily that

$$
|I_1| \le \int_0^t \|A_1(s) - A_2(s)\|_{L^\infty(\mathbb{R})}\; ds\; \|f_0\|_{L^1(\mathbb{R}^2)}\; \|\varphi\|_{L^1(\mathbb{R})}. \qquad (57)
$$

For analyzing the second term I_2 we introduce the notation $\psi(t,s,y) = \int_{y-(t-s)}^{y+(t-s)} \varphi(x)\, dx$. We have

$$
\begin{aligned}
|I_2| &= \left| \int_0^t \int_{\mathbb{R}} \{(\rho_{1,\gamma_2} - \rho_{2,\gamma_2})A_2\}(s,y)\psi(t,s,y)\, dy\, ds \right| \\
&= \left| \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{f_{E_1,A_1}(s,y,p)}{\sqrt{1+|p|^2+|A_1(s,y)|^2}} - \frac{f_{E_2,A_2}(s,y,p)}{\sqrt{1+|p|^2+|A_2(s,y)|^2}} \right) A_2\psi\, dp\, dy\, ds \right| \\
&= \left| \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^t \sum_{k=1}^{2} (-1)^k \frac{A_2(s,X_k(s))\ \psi(t,s,X_k(s))}{\sqrt{1+|P_k(s)|^2+|A_k(s,X_k(s))|^2}}\, ds\, dp\, dx \right| \\
&\leq \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)|I_3(s,x,p)|\, dp\, dx\, ds, \qquad (58)
\end{aligned}
$$

where

$$
I_3(s,x,p) = \sum_{k=1}^{2} (-1)^k \frac{A_2(s,X_k(s))\ \psi(t,s,X_k(s))}{\sqrt{1+|P_k(s)|^2+|A_k(s,X_k(s))|^2}}.
$$

Observe that

$$
\begin{aligned}
|I_3(s,x,p)| &\leq |A_2(s,X_2(s)) - A_2(s,X_1(s))|\ |\psi(t,s,X_2(s))| \qquad (59) \\
&+ |A_2(s,X_1(s))|\ |\psi(t,s,X_2(s)) - \psi(t,s,X_1(s))| \\
&+ \frac{|\psi(t,s,X_2(s))|\ \{|P_1(s) - P_2(s)| + |A_1(s,X_1(s)) - A_2(s,X_2(s))|\}}{\sqrt{1+|P_1(s)|^2+|A_1(s,X_1(s))|^2}} \\
&\leq 2|A_2(s,X_2(s)) - A_2(s,X_1(s))|\ |\psi(t,s,X_2(s))| \\
&+ |A_2(s,X_1(s))|\ |\psi(t,s,X_2(s)) - \psi(t,s,X_1(s))| \\
&+ \|A_1(s) - A_2(s)\|_{L^\infty} |\psi(t,s,X_2(s))| + |P_1(s) - P_2(s)|\ |\psi(t,s,X_2(s))| \\
&\leq 2\|\partial_x A_2(s)\|_{L^\infty} |X_1(s) - X_2(s)|\ \|\varphi\|_{L^1} \\
&+ |P_1(s) - P_2(s)|\ \|\varphi\|_{L^1} + \|A_1(s) - A_2(s)\|_{L^\infty} \|\varphi\|_{L^1} \\
&+ \|A_2(s)\|_{L^\infty} \left\{ \left| \int_{X_2(s)+(t-s)}^{X_1(s)+(t-s)} \varphi(u)\, du \right| + \left| \int_{X_2(s)-(t-s)}^{X_1(s)-(t-s)} \varphi(u)\, du \right| \right\}.
\end{aligned}
$$

Combining (58), (59) and using Remark 3.1 yields

$$
|I_2| \leq C \int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\, ds\ \|\varphi\|_{L^1}. \qquad (60)
$$

Finally (56), (57) and (60) imply that for all $t \in [0,T], (E_1,A_1), (E_2,A_2) \in D_T$ we have

$$
\|(\tilde{A}_1 - \tilde{A}_2)(t)\|_{L^\infty(\mathbb{R})} \leq C \int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\, ds, \qquad (61)
$$

for some constant depending on T and the initial conditions.

Estimate for $\partial_x \tilde{A}_1 - \partial_x \tilde{A}_2$

For any test function $\varphi \in C_c^0(\mathbb{R})$ we need to estimate

$$\begin{aligned} Q_6 &:= \int_{\mathbb{R}} (\partial_x \tilde{A}_1 - \partial_x \tilde{A}_2)(t,x)\ \varphi(x)\ dx \\ &= -\frac{1}{2}\int_0^t \int_{\mathbb{R}} (\rho_{1,\gamma_2} A_1 - \rho_{2,\gamma_2} A_2)(s,x)\varphi(x-t+s)\ dx\ ds \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{R}} (\rho_{1,\gamma_2} A_1 - \rho_{2,\gamma_2} A_2)(s,x)\varphi(x+t-s)\ dx\ ds \\ &= -\frac{1}{2} Q_6^+ + \frac{1}{2} Q_6^- . \end{aligned} \tag{62}$$

We have

$$\begin{aligned} Q_6^{\pm} &= \sum_{k=1}^{2} (-1)^k \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f_{E_k,A_k}(s,x,p)}{\gamma_2} A_k(s,x)\varphi(x \mp (t-s))\ dp\,dx\ ds \qquad (63) \\ &= \sum_{k=1}^{2} (-1)^k \int_{\mathbb{R}}\int_{\mathbb{R}} f_0 \int_0^t \frac{A_k(s,X_k(s))}{P_k(s) \pm \gamma_1(P_k(s),A_k(s,X_k(s)))} \frac{d}{ds}\int_0^{X_k(s)\mp(t-s)} \varphi(u)\ du\ ds\ dp\,dx \\ &= \sum_{k=1}^{2} (-1)^k \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^t \{G^{\pm}(P_k(s),A_k(s,X_k(s)))\frac{d}{ds}\int_0^{X_k(s)\mp(t-s)} \varphi(u)\ du\ \} ds\ dp\,dx, \end{aligned}$$

where $G^{\pm}(P,A) = \frac{A}{P \pm \gamma_1(P,A)}$. Observe that the functions $G^{\pm}$ are well defined since $P + \gamma_1(P,A) > 0$ and $P - \gamma_1(P,A) < 0$, $\forall (P,A) \in \mathbb{R}^2$. In order to simplify our further computations we introduce some notations. Consider $G = G(P,A)$ a smooth function (C^2). For any pair of regular fields $(E,A) \in X_T$ we construct the derivative of G along the characteristics curves corresponding to (E,A), *i.e.,* for all $(t,x,p) \in [0,T]\times\mathbb{R}^2$ we compute

$$\begin{aligned} &\lim_{s\to t} \frac{G(P(s;t,x,p),A(s,X(s;t,x,p))) - G(p,A(t,x))}{s-t} \\ &= -\partial_P G(p,A(t,x))\left(E(t,x) + \frac{A(t,x)\partial_x A}{\gamma_2(p,A(t,x))}\right) + \partial_A G(p,A(t,x))\left(\partial_t A + \frac{p\ \partial_x A}{\gamma_1(p,A(t,x))}\right) \\ &= H(p,A(t,x),\partial_x A(t,x),\partial_t A(t,x),E(t,x)), \end{aligned} \tag{64}$$

with the notation

$$H(P,A,B,C,E) = -\partial_P G(P,A)\left(E + \frac{A\ B}{\gamma_2(P,A)}\right) + \partial_A G(P,A)\left(C + \frac{P\ B}{\gamma_1(P,A)}\right),$$

for all $(P, A, B, C, E) \in \mathbb{R}^5$. For $l \in \{1, 2\}$ we introduce the class $\mathcal{C}_l$ of smooth functions $G \in C^2$ such that

$$\max\{|G|, |\partial_P G|, |\partial_A G|\}(P, A) \le g(A)(1 + |P|^l), \ \ \forall\ (P, A) \in \mathbb{R}^2,$$

for some continuous function g and

$$\max\{|H|, |\partial_P H|, |\partial_A H|, |\partial_B H|, |\partial_C H|, |\partial_E H|\}(P, A, B, C, E) \le h(A, B, C, E)(1+|P|^l),$$

for all $(P, A, B, C, E) \in \mathbb{R}^5$ and some continuous function h.

Lemma 3.3 *Assume that $(E_k, A_k) \in D_T, k \in \{1, 2\}$, $\varphi \in C_c^0(\mathbb{R})$ and $G \in \mathcal{C}_1$. We denote by $(X_k, P_k)_{k\in\{1,2\}}$ the characteristics corresponding to $(E_k, A_k)_{k\in\{1,2\}}$. Then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=1}^{2}(-1)^k \int_0^t G(P_k(s; 0, x, p), A_k(s, X_k(s; 0, x, p)))\frac{d}{ds}\left(\int_0^{X_k(s)\mp(t-s)}\varphi(u)du\right) ds\right| \\
\le\ & C(1 + |P_1(t)| + |P_2(t)|)\,(\|A_1(t) - A_2(t)\|_{L^\infty} + |X_1(t) - X_2(t)| + |P_1(t) - P_2(t)|)\,\|\varphi\|_{L^1} \\
+\ & C(1 + |P_2(t)|)\left|\int_{X_2(t)}^{X_1(t)} \varphi(u)\ du\right| + \int_0^t \{C(1 + |P_1(s)| + |P_2(s)|) \\
\times\ & (|||(E_1 - E_2, A_1 - A_2)(s)||| + |X_1(s) - X_2(s)| + |P_1(s) - P_2(s)|)\|\varphi\|_{L^1} \\
+\ & C(1 + |P_2(s)|)\left|\int_{X_2(s)\mp(t-s)}^{X_1(s)\mp(t-s)} \varphi(u)\ du\right|\}\ ds, \qquad (65)
\end{aligned}$$

for some constant depending on T and the initial conditions.

Proof. After integration by parts we have for $k \in \{1, 2\}$

$$\int_0^t G(P_k(s), A_k(s, X_k(s)))\frac{d}{ds}\left(\int_0^{X_k(s)\mp(t-s)}\varphi(u)\ du\right) ds = I_k^t - I_k^0 - \int_0^t I_k(s)\ ds, \quad (66)$$

where $I_k^t = G(P_k(t), A_k(t, X_k(t))) \int_0^{X_k(t)}\varphi(u)\ du$, $I_k^0 = G(p, A_k(0, x)) \int_0^{x\mp t}\varphi(u)\ du$ and

$$\begin{aligned}
I_k(s) \ =\ & H(P_k(s), A_k(s, X_k(s)), \partial_x A_k(s, X_k(s)), \partial_t A_k(s, X_k(s)), E_k(s, X_k(s))) \\
\times\ & \int_0^{X_k(s)\mp(t-s)} \varphi(u)\ du.
\end{aligned}$$

Estimate of $I_1^t - I_2^t$

$$
\begin{aligned}
|I_1^t - I_2^t| &\le \left| \sum_{k=1}^{2} (-1)^k G(P_k(t), A_k(t, X_k(t))) \right| \left| \int_0^{X_1(t)} \varphi(u)\, du \right| \\
&+ |G(P_2(t), A_2(t, X_2(t)))| \left| \int_{X_1(t)}^{X_2(t)} \varphi(u)\, du \right|. \qquad (67)
\end{aligned}
$$

Since $(E_k, A_k) \in D_T$ we have $\|A_k(t)\|_{L^\infty} \le a(T), k \in \{1,2\}, t \in [0,T]$ and by using the fact that $G \in \mathcal{C}_1$ we have

$$
\begin{aligned}
&\left| \sum_{k=1}^{2} (-1)^k G(P_k(t), A_k(t, X_k(t))) \right| = \left| \int_0^1 \frac{d}{d\tau} G(P_2 + \tau(P_1 - P_2), A_2 + \tau(A_1 - A_2))\, d\tau \right| \\
\le\ & \int_0^1 g(\tau A_1 + (1-\tau) A_2)(1 + |\tau P_1 + (1-\tau) P_2|\ ) d\tau \\
\times\ & (\ |P_1(t) - P_2(t)| + |A_1(t, X_1(t)) - A_2(t, X_2(t))|\ ) \qquad (68) \\
\le\ & \sup_{|A| \le a(T)} g(A)\ (1 + |P_1(t)| + |P_2(t)|\ )(|P_1(t) - P_2(t)| + |A_1(t, X_1(t)) - A_2(t, X_2(t))|).
\end{aligned}
$$

Since $\|\partial_x A_k\|_{L^\infty} \le a_1(T)$, $k \in \{1,2\}$ we have

$$
\begin{aligned}
|A_1(t, X_1(t)) - A_2(t, X_2(t))| &\le |A_1(t, X_1(t)) - A_1(t, X_2(t))| + |A_1(t, X_2(t)) - A_2(t, X_2(t))| \\
&\le a_1(T)|X_1(t) - X_2(t)| + \|A_1(t) - A_2(t)\|_{L^\infty}. \qquad (69)
\end{aligned}
$$

By collecting the inequalities (67), (68), (69) we obtain

$$
\begin{aligned}
|I_1^t - I_2^t| &\le C(1 + |P_1(t)| + |P_2(t)|\ )(|X_1(t) - X_2(t)| + |P_1(t) - P_2(t)| \\
&+ \|A_1(t) - A_2(t)\|_{L^\infty})\|\varphi\|_{L^1} + C(1 + |P_2(t)|\ ) \left| \int_{X_2(t)}^{X_1(t)} \varphi(u)\, du \right|. \quad (70)
\end{aligned}
$$

Estimate of $I_1^0 - I_2^0$

The term $I_1^0 - I_2^0$ vanishes since $A_1(0,x) = A_2(0,x) = A_0(x),\ \ x \in \mathbb{R}$

$$
I_1^0 - I_2^0 = 0. \qquad (71)
$$

Estimate of $I_1(s) - I_2(s)$

We use the notation

$$
Z_k(s) = (P_k(s), A_k(s, X_k(s)), \partial_x A_k(s, X_k(s)), \partial_t A_k(s, X_k(s)), E_k(s, X_k(s))) \in \mathbb{R}^5,
$$

for $k \in \{1,2\}$ and $s \in [0,T]$. As before we can write

$$\begin{aligned} |I_1(s) - I_2(s)| &\leq \left| (H(Z_1(s)) - H(Z_2(s))) \int_0^{X_1(s) \mp (t-s)} \varphi(u)\, du \right| \\ &+ |H(Z_2(s))| \left| \int_{X_2(s)\mp(t-s)}^{X_1(s)\mp(t-s)} \varphi(u)\, du \right|. \end{aligned} \tag{72}$$

Using the hypothesis $G \in \mathcal{C}_1$ yields

$$\begin{aligned} |H(Z_1(s)) - H(Z_2(s))| &\leq \int_0^1 |\nabla H(\tau Z_1(s) + (1-\tau) Z_2(s))|\, d\tau\, |Z_1(s) - Z_2(s)| \\ &\leq C(1 + |P_1(s)| + |P_2(s)|\,)|Z_1(s) - Z_2(s)|. \end{aligned} \tag{73}$$

It remains to estimate the difference $|Z_1(s) - Z_2(s)|$. We have

$$\begin{aligned} |Z_1(s) - Z_2(s)| &\leq |P_1(s) - P_2(s)| + \|\partial_x A_1(s)\|_{L^\infty} |X_1(s) - X_2(s)| + \|(A_1 - A_2)(s)\|_{L^\infty} \\ &+ \|\partial_x^2 A_1(s)\|_{L^\infty} |X_1(s) - X_2(s)| + \|\partial_x (A_1 - A_2)(s)\|_{L^\infty} \\ &+ \|\partial_{xt}^2 A_1(s)\|_{L^\infty} |X_1(s) - X_2(s)| + \|\partial_t (A_1 - A_2)(s)\|_{L^\infty} \\ &+ \|\partial_x E_1(s)\|_{L^\infty} |X_1(s) - X_2(s)| + \|(E_1 - E_2)(s)\|_{L^\infty} \qquad (74) \\ &\leq C(|||(E_1 - E_2, A_1 - A_2)(s)||| + |X_1(s) - X_2(s)| + |P_1(s) - P_2(s)|). \end{aligned}$$

Finally one gets from (72), (73), (74)

$$\begin{aligned} |I_1(s) - I_2(s)| &\leq C(1 + |P_1(s)| + |P_2(s)|\,)(|||(E_1 - E_2, A_1 - A_2)(s)||| \\ &+ |X_1(s) - X_2(s)| + |P_1(s) - P_2(s)|)\|\varphi\|_{L^1} \\ &+ C(1 + |P_2(s)|\,) \left| \int_{X_2(s)\mp(t-s)}^{X_1(s)\mp(t-s)} \varphi(u)\, du \right|. \end{aligned} \tag{75}$$

Our conclusion follows from (66), (70), (71), (75). $\square$

Lemma 3.4 *Assume that the hypotheses of Lemma* 3.3 *hold and* $\int_{\mathbb{R}}\int_{\mathbb{R}} |p| f_0(x,p)\, dp\, dx < +\infty$. *Then there is a constant* C *depending on* T *and the initial conditions such that for any function* $\varphi \in C_c^0(\mathbb{R})$ *and* $(E_k, A_k) \in D_T, k \in \{1,2\}$ *we have*

$$\begin{aligned} &\left| \sum_{k=1}^{2} (-1)^k \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p) \int_0^t G(P_k(s), A_k(s, X_k(s))) \frac{d}{ds} \left(\int_0^{X_k(s)\mp(t-s)} \varphi(u)\, du \right) ds\, dp\, dx \right| \\ &\leq C \left(\|(A_1 - A_2)(t)\|_{L^\infty} + \int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\, ds \right) \|\varphi\|_{L^1}. \end{aligned} \tag{76}$$

Proof. By Lemma 3.1 we deduce that

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(1+|P_1(t)|+|P_2(t)|\ )\ dp\,dx &= \|f_0\|_{L^1} + \sum_{k=1}^{2}\int_{\mathbb{R}}\int_{\mathbb{R}} f_{E_k,A_k}(t,x,p)|p|\ dp\,dx \\
&\leq C\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|p|)f_0(x,p)\ dp\,dx. \qquad (77)
\end{aligned}$$

By performing similar computations as before (see also Remark 3.1) we obtain the estimate

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(1+|P_2(t)|\ )\left|\int_{X_2(t)}^{X_1(t)}\varphi(u)\ du\right|\ dp\,dx &\leq C\int_0^t |||(E_1-E_2,A_1-A_2)(s)|||\ ds \\
&\times \|\varphi\|_{L^1}, \qquad (78)
\end{aligned}$$

and also

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(1+|P_2(s)|\ )&\left|\int_{X_2(s)\mp(t-s)}^{X_1(s)\mp(t-s)}\varphi(u)\ du\right|\ dp\,dx \leq C\int_0^s |||(E_1-E_2,A_1-A_2)(\tau)|||\ d\tau\|\varphi\|_{L^1} \\
&\leq C\int_0^t |||(E_1-E_2,A_1-A_2)(\tau)|||\ d\tau\ \|\varphi\|_{L^1},\ \forall\ s\in[0,t]. \qquad (79)
\end{aligned}$$

The conclusion follows by combining (65), (77), (78), (79) and using Proposition 2.2.

□

We intend to apply Lemma 3.4 for estimating $Q_6 = \int_{\mathbb{R}}(\partial_x\tilde{A}_1 - \partial_x\tilde{A}_2)\varphi(x)\ dx = -\frac{1}{2}(Q_6^+ - Q_6^-)$ (see (62)). All we need to do is to check that the functions $G^\pm$ belong to the class $\mathcal{C}_1$. We have

$$G^\pm(P,A) = \frac{A}{P\pm\sqrt{1+|P|^2+|A|^2}} = \frac{A}{1+|A|^2}(\pm\sqrt{1+|P|^2+|A|^2}-P),$$

and we check easily that

$$\max\{|G^\pm|,|\partial_P G^\pm|,|\partial_A G^\pm|\}(P,A)\leq g(A)(1+|P|\ ),\ \forall\ (P,A)\in\mathbb{R}^2,$$

for some continuous function g. We have

$$\begin{aligned}
H^\pm(P,A,B,C,E) &= -\partial_P G^\pm(P,A)\left(E+\frac{AB}{\sqrt{1+|P|^2+|A|^2}}\right) \\
&+ \partial_A G^\pm(P,A)\left(C+\frac{PB}{\sqrt{1+|P|^2+|A|^2}}\right), \qquad (80)
\end{aligned}$$

and we check by direct computation that

$$\max\{|H^{\pm}|, |\partial_P H^{\pm}|, |\partial_A H^{\pm}|, |\partial_B H^{\pm}|, |\partial_C H^{\pm}|, |\partial_E H^{\pm}|\} \leq h(A, B, C, E)(1 + |P|\ ),$$

for some continuous function h. Therefore Lemma 3.4 applies and by combining (62), (63) one gets for any function $\varphi \in C_c^0(\mathbb{R})$

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_x(\tilde{A}_1 - \tilde{A}_2)\varphi(x)\ dx\right| &\leq C\|A_1(t) - A_2(t)\|_{L^\infty}\|\varphi\|_{L^1} \\ &+ C\int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\ ds\ \|\varphi\|_{L^1}. \quad (81)\end{aligned}$$

Since we already know that $\partial_x \tilde{A}_1,\ \partial_x \tilde{A}_2 \in L^\infty(\mathbb{R})$ we deduce by density that the previous inequality holds true for any function $\varphi \in L^1(\mathbb{R})$ and therefore we obtain

$$\begin{aligned}\|\partial_x \tilde{A}_1(t) - \partial_x \tilde{A}_2(t)\|_{L^\infty} &\leq C\ \|A_1(t) - A_2(t)\|_{L^\infty} \\ &+ C\int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\ ds,\ t \in [0, T]. \quad (82)\end{aligned}$$

Estimate for $\partial_t \tilde{A}_1 - \partial_t \tilde{A}_2$

With the notations introduced in (62) we have for any function $\varphi \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}} (\partial_t \tilde{A}_1 - \partial_t \tilde{A}_2)\varphi(x)\ dx = -\frac{1}{2}Q_6^+ - \frac{1}{2}Q_6^-, \quad (83)$$

and therefore we obtain exactly as before that

$$\begin{aligned}\|\partial_t \tilde{A}_1(t) - \partial_t \tilde{A}_2(t)\|_{L^\infty} &\leq C\ \|A_1(t) - A_2(t)\|_{L^\infty} \\ &+ C\int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\ ds,\ \forall\ t \in [0, T]. \quad (84)\end{aligned}$$

Collecting now all the partial estimates of (55), (61), (82), (84) we deduce that there is a constant C depending on T and the initial conditions such that for any $(E_k, A_k) \in D_T, k \in \{1, 2\}$ we have

$$\begin{aligned}|||\mathcal{F}(E_1, A_1)(t) - \mathcal{F}(E_2, A_2)(t)||| &\leq C\ \|A_1(t) - A_2(t)\|_{L^\infty} \quad (85) \\ &+ C\int_0^t |||(E_1 - E_2, A_1 - A_2)(s)|||\ ds,\ \forall\ t \in [0, T].\end{aligned}$$

Remark 3.2 *A similar inequality holds in the QR case. We need to assume that* $M_2 = \int_{\mathbb{R}} |p|^2 n_0(p)\ dp < +\infty$, $\int_{\mathbb{R}}\int_{\mathbb{R}} |p|^2 f_0(x, p)\ dp\, dx < +\infty$ *and to work with the class* $\mathcal{C}_2$.

4 Existence and uniqueness of fixed point for $\mathcal{F}$

We start with a very easy lemma.

Lemma 4.1 *Assume that $(z_n)_n \subset L^\infty(]0,T[)$ is a sequence of nonnegative functions satisfying*

$$z_{n+2}(t) \le \alpha \int_0^t z_n(s)\,ds + \beta \int_0^t z_{n+1}(s)\,ds,\ t \in [0,T],\ n \ge 0. \tag{86}$$

Then $\sum_{n\ge 0} z_n(t)$ converges uniformly on $[0,T]$ and we have

$$\sum_{n\ge 0} z_n(t) \le (\|z_0\|_{L^\infty} + \|z_1\|_{L^\infty})\,\mathrm{e}^{(\alpha+\beta)\,t},\ t \in [0,T]. \tag{87}$$

Proof. Denote $S_m(t) = \sum_{n=0}^m z_n(t)$. By adding the inequalities (86) written for $n \in \{0,1,2,...,m\}$ we obtain

$$\begin{aligned} S_{m+2}(t) - z_0(t) - z_1(t) &\le \alpha \int_0^t S_m(s)\,ds + \beta \int_0^t (S_{m+1}(s) - z_0(s))\,ds \\ &\le (\alpha+\beta) \int_0^t S_{m+1}(s)\,ds, \end{aligned} \tag{88}$$

and therefore

$$S_{m+1}(t) \le \|z_0\|_{L^\infty} + \|z_1\|_{L^\infty} + (\alpha+\beta)\int_0^t S_{m+1}(s)\,ds,\ \ t\in[0,T],\ m \ge 0.$$

By Gronwall lemma we deduce that

$$S_{m+1}(t) \le (\|z_0\|_{L^\infty} + \|z_1\|_{L^\infty})\,\mathrm{e}^{(\alpha+\beta)\,t},\ t \in [0,T],\ m \ge 0,$$

and the conclusion follows by letting $m \to +\infty$. $\square$

By using successive approximations we prove the existence of a fixed point for $\mathcal{F}$ and we obtain the existence of solution for the system (18), (19), (20), (21), (22).

Theorem 4.1 *Assume that f_0 in nonnegative, $(1+|p|)f_0$ belongs to $L^1(\mathbb{R}^2)$ and satisfies (H), (H_0), (H_1), (H_∞), ρ_{ext} is nonnegative, belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$,*

$E_0' = \rho_{ext} - \int_{\mathbb{R}} f_0 \, dp$, $A_0 \in W^{2,\infty}(\mathbb{R})$, $A_1 \in W^{1,\infty}(\mathbb{R})$. *Then for any* $T > 0$ *there is at least one solution* (f, E, A) *for the system* (18), (19), (20), (21), (22) *in the FR case, verifying* $f \geq 0$, $(1 + |p|)f \in L^\infty(]0, T[; L^1(\mathbb{R}^2))$, $\int_{\mathbb{R}} f(\cdot, \cdot, p)(1 + |p|) \, dp \in L^\infty(]0, T[\times\mathbb{R})$, $(E, A) \in D_T$.

Proof. We consider $(E_0, A_0) = (0, 0) \in D_T$ and define $(E_n, A_n) = \mathcal{F}(E_{n-1}, A_{n-1})$, for any $n \geq 1$ and $z_n(t) = |||(E_{n+1} - E_n, A_{n+1} - A_n)(t)|||$, $t \in [0, T]$, $n \geq 0$. Observe that all functions z_n are bounded on $[0, T]$

$$\|z_n\|_{L^\infty} \leq 2\,(e + a(T) + 2\,a_1(T)), \;\; \forall\, n \geq 0.$$

The inequality (85) implies

$$\begin{aligned} z_{n+2}(t) &= |||(E_{n+3}, A_{n+3})(t) - (E_{n+2}, A_{n+2})(t)||| \\ &= |||\mathcal{F}(E_{n+2}, A_{n+2})(t) - \mathcal{F}(E_{n+1}, A_{n+1})(t)||| \\ &\leq C\left(\|A_{n+2}(t) - A_{n+1}(t)\|_{L^\infty} + \int_0^t |||(E_{n+2}, A_{n+2})(s) - (E_{n+1}, A_{n+1})(s)|||\, ds\right) \\ &= C\left(\|A_{n+2}(t) - A_{n+1}(t)\|_{L^\infty} + \int_0^t z_{n+1}(s)\, ds\right). \end{aligned}$$

But (61) yields

$$\begin{aligned} \|A_{n+2}(t) - A_{n+1}(t)\|_{L^\infty} &= \|\tilde{A}_{n+1}(t) - \tilde{A}_n(t)\|_{L^\infty} \\ &\leq C\int_0^t |||(E_{n+1}, A_{n+1})(s) - (E_n, A_n)(s)|||\, ds \\ &= C\int_0^t z_n(s)\, ds. \end{aligned}$$

Finally one gets that there is a constant C depending on T and the initial conditions such that

$$z_{n+2}(t) \leq C\int_0^t (z_n(s) + z_{n+1}(s))\, ds, \;\; \forall\, t \in [0, T], \; n \geq 0.$$

Lemma 4.1 implies that $(E_n, A_n)_n$ is a Cauchy sequence in $Y_T = L^\infty(]0, T[\times\mathbb{R}) \times W^{1,\infty}(]0, T[\times\mathbb{R})$ and therefore converges to some fields (E, A) in Y_T. Actually since $(E_n, A_n)_n \subset D_T$ we obtain immediately that $(E, A) \in D_T$. We check easily that

(E,A) is a fixed point of $\mathcal{F}$. We take $f = f_{E,A}$ and thus (f,E,A) solve (18), (19). As in the proofs of Propositions 3.1, 3.4 we check that (21), (20) hold. The estimates for $\|(1+|p|)f\|_{L^\infty(]0,T[;L^1(\mathbb{R}^2))}$ and $\|\int_{\mathbb{R}} f(\cdot,\cdot,p)(1+|p|)\,dp\|_{L^\infty(]0,T[\times\mathbb{R})}$ follow from Lemmas 3.1, 3.2.

$\square$

Theorem 4.2 *Assume that the hypotheses of Theorem* 4.1 *are satisfied. Then there is at most one mild solution (f,E,A) (i.e., $(E,A) \in L^\infty(]0,T[;W^{1,\infty}(\mathbb{R})) \times L^\infty(]0,T[;W^{2,\infty}(\mathbb{R}))$ and f solution by characteristics) for the system* (18)*,* (19)*,* (20)*,* (21)*,* (22) *in the FR case.*

Proof. Suppose that $(f_k,E_k,A_k)_{k\in\{1,2\}}$ are two mild solutions, which means that $\mathcal{F}(E_k,A_k) = (E_k,A_k)$, $k \in \{1,2\}$. By computations similar to those in the proofs of Propositions 3.1, 3.2 (see also Propositions 3.4, 3.5) we show that $(E_k,A_k) \in D_T$, $k \in \{1,2\}$. Using (85) yields

$$\begin{aligned} |||(E_1,A_1)(t) - (E_2,A_2)(t)||| \;&\le\; C\int_0^t |||(E_1,A_1)(s) - (E_2,A_2)(s)|||\,ds \\ &+\; C\,\|A_1(t) - A_2(t)\|_{L^\infty}, \end{aligned} \tag{89}$$

for some constant C depending on T and the initial conditions. But from (61) we have also

$$\|A_1(t) - A_2(t)\|_{L^\infty} = \|\tilde{A}_1(t) - \tilde{A}_2(t)\|_{L^\infty} \le C\int_0^t |||(E_1,A_1)(s) - (E_2,A_2)(s)|||\,ds,$$

and thus we obtain

$$|||(E_1,A_1)(t) - (E_2,A_2)(t)||| \le C\int_0^t |||(E_1,A_1)(s) - (E_2,A_2)(s)|||\,ds.$$

The conclusion follows immediately by Gronwall lemma.

$\square$

Notice that the hypotheses under which we have proved the existence and uniqueness of the solution for (18), (19), (20), (21), (22) are closely related to the boundedness of the initial kinetic energy

$$K_0 := \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\sqrt{1+|p|^2+|A_0(x)|^2}\,dp\,dx.$$

Actually the solution constructed above preserves the total energy.

Proposition 4.1 *Assume that the hypotheses of Theorem* 4.1 *are satisfied and denote by* (f, E, A) *the unique solution of the system* (18), (19), (20), (21), (22).
If the initial energy is finite i.e.,

$$\begin{aligned} W_0 &:= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\sqrt{1+|p|^2+|A_0(x)|^2}\,dp\,dx \\ &+ \frac{1}{2}\int_{\mathbb{R}} \{|E_0(x)|^2 + |A_0'(x)|^2 + |A_1(x)|^2\}\,dx < +\infty, \end{aligned}$$

then we have

$$\begin{aligned} W(t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x,p)\sqrt{1+|p|^2+|A(t,x)|^2}\,dp\,dx \\ &+ \frac{1}{2}\int_{\mathbb{R}} \{|E(t,x)|^2 + |\partial_x A(t,x)|^2 + |\partial_t A(t,x)|^2\}\,dx \\ &= W_0, \quad t \in [0,T]. \end{aligned} \tag{90}$$

Proof. The proof is standard. Since $E, A, \partial_x A$ are bounded, we have $|P(t;0,x,p) - p| \le T(\|E\|_{L^\infty} + \|A\,\partial_x A\|_{L^\infty}) = C$, for any $(t,x,p) \in [0,T] \times \mathbb{R}^2$ and therefore

$$\begin{aligned} \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x,p)|p|\,dp\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(X(0;t,x,p), P(0;t,x,p))|p|\,dp\,dx \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(X(0;t,x,p), P(0;t,x,p))(|P(0;t,x,p)| + C)\,dp\,dx \\ &= C\|f_0\|_{L^1} + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)|p|\,dp\,dx < +\infty, \end{aligned}$$

which implies that the kinetic energy $t \to \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x,p)\sqrt{1+|p|^2+|A(t,x)|^2}\,dp\,dx$ is bounded on $[0,T]$. Since f is solution by characteristics, it is also weak solution for the Vlasov problem. By using the weak formulation with the test function $\eta(t)\chi_R(x)\chi_R(p)\sqrt{1+|p|^2+|A(t,x)|^2}$ where $\chi \in C^1(\mathbb{R})$, $\chi(u) = 1$ if $|u| \le 1$, $\chi(u) = 0$ if $|u| \ge 2$, $0 \le \chi \le 1$, $\chi_R(\cdot) = \chi(\frac{\cdot}{R})$, $\eta \in C_c^1(]0,T[)$ one gets after letting $R \to +\infty$

$$\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t,x,p)\sqrt{1+|p|^2+|A(t,x)|^2}\,dp\,dx = \int_{\mathbb{R}} \rho_{\gamma_2} A\partial_t A\,dx - \int_{\mathbb{R}} j(t,x)E(t,x)\,dx. \tag{91}$$

From equation (20) we deduce that

$$\frac{1}{2}|E(t,x)|^2 = \frac{1}{2}|E_0(x)|^2 + \int_0^t j(s,x)E(s,x)\,ds.$$

It follows that $E(t) \in L^2(\mathbb{R})$ and

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |E(t,x)|^2 \, dx = \int_{\mathbb{R}} j(t,x)E(t,x) \, dx. \tag{92}$$

We multiply now (19) by $\chi_R(x)\partial_t A(t,x)$ and after integration with respect to x we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |\partial_t A|^2 \chi_R(x) \, dx \quad &+ \quad \int_{\mathbb{R}} \frac{1}{R}\chi'\left(\frac{x}{R}\right) \partial_x A \, \partial_t A \, dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \chi_R(x)|\partial_x A|^2 \, dx \\ &= \quad -\int_{\mathbb{R}} \chi_R(x)\rho_{\gamma_2}(t,x)A(t,x)\partial_t A \, dx. \end{aligned} \tag{93}$$

Taking into account that $\partial_x A, \partial_t A \in L^\infty(]0,T[\times\mathbb{R})$ and that $\chi'(x/R) = 0$ for $|x| \leq R$ and $|x| \geq 2R$ we deduce that

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}} \chi_R(x)\{|\partial_t A|^2 + |\partial_x A|^2\} \, dx \quad &\leq \quad \frac{1}{2}\int_{\mathbb{R}} \chi_R(x)\{|A_1(x)|^2 + |A_0'(x)|^2\} \, dx \\ &+ \quad 2\, t \, \|\chi'\|_{L^\infty} \|\partial_x A\|_{L^\infty} \|\partial_t A\|_{L^\infty} \\ &+ \quad t \, \|f_0\|_{L^1} \|A\|_{L^\infty} \|\partial_t A\|_{L^\infty}. \end{aligned} \tag{94}$$

By letting $R \to +\infty$ we obtain that $t \to \int_{\mathbb{R}} \{|\partial_t A|^2 + |\partial_x A|^2\} \, dx$ is bounded on $[0,T]$. The equality (93) implies easily that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \{|\partial_x A|^2 + |\partial_t A|^2\} \, dx = -\int_{\mathbb{R}} \rho_{\gamma_2}(t,x)A(t,x)\partial_t A \, dx, \quad t \in [0,T]. \tag{95}$$

Combining (91), (92), (95) yields the conservation of the total energy (90). $\square$

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