# Stationary solutions of the 1 D Vlasov-Maxwell equations for laser-plasma interaction

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**Abstract**

We study a stationary 1 D Vlasov-Maxwell system which describes the laser-plasma interaction. Three cases are analyzed : the classical case, the quasi-relativistic case and the relativistic case. We prove the existence of stationary solution and we establish estimates for the charge and current densities.

**Keywords:** Kinetic equations, Vlasov-Maxwell system, weak/mild solutions, characteristics.

**AMS classification:** 35A05, 82D10.

## 1 Introduction

We consider a population of relativistic electrons with mass $m > 0$ and charge $-e < 0$. We denote by $\mathcal{E}(p)$, $v(p) = \nabla_p \mathcal{E}(p)$ the kinetic energy and the velocity

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associated to a given momentum $p \in \mathbb{R}^3$

$$\mathcal{E}(p) = mc^2 \left( \sqrt{1 + \frac{|p|^2}{m^2c^2}} - 1 \right), \quad v(p) = \frac{p}{m\sqrt{1 + \frac{|p|^2}{m^2c^2}}},$$

where $c$ is the speed of light. The electrons move under the action of an electromagnetic field verifying the Maxwell equations and their distribution function $F = F(t, x, p)$ satisfies the Vlasov equation. Recently a reduced 1D Vlasov-Maxwell system was introduced for studying laser-plasma interactions. This model was studied by Carrillo and Labrunie [5]. We distinguish three cases : the nonrelativistic model NR, the quasi-relativistic model QR and the original fully relativistic model FR. After introducing dimensionless unknowns and variables we obtain (see [5] for more details)

$$\partial_t f + \frac{p}{\gamma_1}\partial_x f - \left( E(t,x) + \frac{A(t,x)}{\gamma_2}\partial_x A \right) \partial_p f = 0, \tag{1}$$

$$\partial_t^2 A - \partial_x^2 A = -\rho_{\gamma_2}(t,x)A(t,x), \tag{2}$$

$$\partial_t E = j(t,x), \tag{3}$$

$$\partial_x E = \rho_{ext}(x) - \rho(t,x), \tag{4}$$

where $\{\rho, \rho_{\gamma_2}, j\}(t,x) = \int_{\mathbb{R}} \{1, \frac{1}{\gamma_2}, \frac{p}{\gamma_1}\} f(t,x,p)\ dp$, $\gamma_1 = \gamma_2 = 1$ in the NR case, $\gamma_1 = (1 + |p|^2)^{1/2}$, $\gamma_2 = 1$ in the QR case and $\gamma_1 = \gamma_2 = (1 + |p|^2 + |A(t,x)|^2)^{1/2}$ in the FR case. Here $\rho_{ext}$ is the density of a background population of ions which are supposed at rest. We supplement these equations with initial conditions

$$f(0,x,p) = f_0(x,p),\ (x,p) \in \mathbb{R}^2,\ \ (E, A, \partial_t A)(0,x) = (E_0, A_0, A_1)(x),\ x \in \mathbb{R}. \tag{5}$$

In [5] the authors investigated the existence of space periodic solutions and free-space solutions for the system (1), (2), (3), (4), (5). They studied the existence and uniqueness of weak solution in the NR and QR cases. The FR case is more delicate. In this article we concentrate our attention on the stationary solutions of the system (1), (2), (4) on the slab $]0,1[$ with boundary conditions. The same method applies for the NR, QR and FR cases. We use the Schauder fixed point theorem. One of the key points is to obtain estimates for the charge and current densities. Our main result is

**Theorem 1.1** *Assume that* $g_0 \in L^1(]0,+\infty[) \cap L^\infty(]0,+\infty[)$, $g_1 \in L^1(]-\infty,0[) \cap L^\infty(]-\infty,0[)$, $g_0, g_1 \geq 0$, $\rho_{ext} \in L^\infty(]0,1[)$, $\rho_{ext} \geq 0$, $\varphi_0, \varphi_1, A_0, A_1 \in \mathbb{R}$. *Then, there is a stationary solution* $(f \geq 0, E, A) \in L^1(]0,1[\times\mathbb{R}) \cap L^\infty(]0,1[\times\mathbb{R}) \times W^{1,\infty}(]0,1[) \times W^{2,\infty}(]0,1[)$ *of*

$$\frac{p}{\gamma_1}\partial_x f - (E(x) + \frac{A(x)}{\gamma_2}A'(x))\partial_p f = 0, \quad (x,p) \in ]0,1[\times\mathbb{R}, \tag{6}$$

$$f(x=0, p>0) = g_0(p), \quad f(x=1, p<0) = g_1(p), \tag{7}$$

$$E'(x) = \rho_{ext}(x) - \rho(x), \quad x \in ]0,1[, \quad \int_0^1 E(x)\,dx = \varphi_1 - \varphi_0, \tag{8}$$

$$-A''(x) + \rho_{\gamma_2}(x)A(x) = 0, \quad x \in ]0,1[, \quad A(0) = A_0, A(1) = A_1, \tag{9}$$

*where* $(\rho, \rho_{\gamma_2}) = \int_{\mathbb{R}} \{1, \frac{1}{\gamma_2}\} f(\cdot,p)\,dp$. *Moreover* $\rho$ *belongs to* $L^\infty(]0,1[)$ *and we have* $\lim_{R\to+\infty} \left\| \int_{|p|>R} f(\cdot,p)\,dp \right\|_{L^\infty(]0,1[)} = 0$.

The considered system provides particular measure solutions for the Vlasov-Maxwell system in a very specific geometry, but at the price of nonlinearities that look stronger. The analysis of the Cauchy problem for the Vlasov-Maxwell system (weak solutions or classical solutions) can be dealt with by using different methods as done by DiPerna and Lions [6], Glassey and Schaeffer [8], Glassey and Strauss [9], Klainerman and Staffilani [11], Bouchut, Golse and Pallard [4]. For applications (tube discharges, cold plasma, solar wind, satellite ionization, thruster, ...) boundary conditions have to be taken into account. The Vlasov-Maxwell initial-boundary value problem was studied by Guo [10]. The three dimensional stationary Vlasov-Maxwell system was analyzed by Poupaud [12]. Results for the time periodic case can be found in [3].

The paper is organized as follows. In Section 2 we recall the notion of weak and mild solutions for the Vlasov problem and several properties of such solutions. Some technical proofs are postponed to the Appendix. In Section 3 we construct a fixed point application for the reduced 1 D Vlasov-Maxwell system and the existence of weak solution follows by Schauder fixed point theorem. We investigate also the propagation of the impulsion moments.

## 2 The Vlasov problem

In this section we assume that the fields $E = E(x), A = A(x)$ are given and we introduce the notions of mild solution (or solution by characteristics) and weak solution for the stationary Vlasov equation

$$\frac{p}{\gamma_1}\partial_x f - \left(E(x) + \frac{A(x)}{\gamma_2}A'(x)\right)\partial_p f = 0, \quad (x,p) \in ]0,1[\times\mathbb{R}, \tag{10}$$

with the boundary conditions

$$f(x=0,p>0) = g_0(p), \quad f(x=1,p<0) = g_1(p). \tag{11}$$

The study of the linear Vlasov problem is motivated by the construction of a fixed point application at the level of the fields $(E, A)$, see Section 3. We need to estimate the moments of the particle distribution. The main ingredients for such computations are the conservation of the particle energy along characteristics and a technical result on bounds for the impulsion variation along characteristics.

Assume that $E \in W^{1,\infty}(]0,1[)$, $A \in W^{2,\infty}(]0,1[)$ and consider the system of characteristics associated to (10)

$$\frac{dX}{ds} = \frac{P(s)}{\gamma_1(s)}, \quad \frac{dP}{ds} = -E(X(s)) - \frac{A(X(s))}{\gamma_2(s)}A'(X(s)), \tag{12}$$

with the conditions

$$X(0) = x, \quad P(0) = p. \tag{13}$$

Recall that $\gamma_1(s) = \gamma_2(s) = 1$ in the NR case, $\gamma_1(s) = (1 + |P(s)|^2)^{\frac{1}{2}}, \gamma_2(s) = 1$ in the QR case and $\gamma_1(s) = \gamma_2(s) = (1 + |P(s)|^2 + |A(s, X(s))|^2)^{\frac{1}{2}}$ in the FR case. Observe that in all cases, under the above regularity hypotheses for $E, A$, for all

$(x,p) \in (]0,1[\times\mathbb{R}) \cup (\{0\} \times [0,+\infty[) \cup (\{1\}\times]-\infty,0])$ there is a unique solution for (12), (13) denoted $(X(s),P(s)) = (X(s;x,p),P(s;x,p))$. We introduce also the entry/exit times $s_{in}(x,p) = \inf\{\tau \le 0 \;:\; X(s;x,p) \in ]0,1[,\ \forall s \in ]\tau,0[\}$, respectively $s_{out}(x,p) = \sup\{\tau \ge 0 \;:\; X(s;x,p) \in ]0,1[,\ \forall s \in ]0,\tau[\}$. The Vlasov equation says that $f$ is constant along characteristics, $\frac{d}{ds}f(X(s),P(s)) = 0$ and therefore we construct as usual the solution by characteristics.

**Definition 2.1** *Assume that $E \in W^{1,\infty}(]0,1[)$, $A \in W^{2,\infty}(]0,1[)$. The mild solution of the stationary Vlasov problem* (10), (11) *is given by*

$$f(x,p) = g_k(P(s_{in}(x,p);x,p)) \text{ if } s_{in} > -\infty,\ X(s_{in}(x,p);x,p) = k,\ k \in \{0,1\},$$

$$f(x,p) = 0 \text{ if } s_{in}(x,p) = -\infty.$$

When $E, A$ are less regular we can introduce the notion of weak solution. For this observe that in all three cases we have $\operatorname{div}_{(x,p)}\left(\frac{p}{\gamma_1}, -\left(E(x) + \frac{A(x)}{\gamma_2}A'(x)\right)\right) = 0$, and therefore the Vlasov equation can be written

$$\partial_x\left(\frac{p}{\gamma_1}f\right) - \partial_p\left(\left(E(x) + \frac{A(x)}{\gamma_2}A'(x)\right)f\right) = 0, \quad (x,p) \in ]0,1[\times\mathbb{R}.$$

**Definition 2.2** *Assume that $E \in L^\infty(]0,1[)$, $A \in W^{1,\infty}(]0,1[)$, $g_0 \in L^1_{\text{loc}}([0,+\infty[)$, $g_1 \in L^1_{\text{loc}}(]-\infty,0])$. We say that $f \in L^1_{\text{loc}}([0,1]\times\mathbb{R})$ is a weak solution for the stationary Vlasov problem* (10), (11) *iff*

$$\begin{aligned} -\int_0^1\!\!\int_{\mathbb{R}} f(x,p)\left(\frac{p}{\gamma_1}\partial_x\varphi - \left(E(x) + \frac{A(x)}{\gamma_2}A'(x)\right)\partial_p\varphi\right)\,dp\,dx \\ = \int_{p>0}\frac{p}{\gamma_1}\,g_0(p)\,\varphi(0,p)\,dp - \int_{p<0}\frac{p}{\gamma_1}\,g_1(p)\,\varphi(1,p)\,dp, \end{aligned} \tag{14}$$

*for any test function $\varphi \in C^1_c([0,1]\times\mathbb{R})$ satisfying $\varphi(0,p<0) = \varphi(1,p>0) = 0$.*

Unfortunately in general there is no uniqueness for the weak solution because $f$ can take arbitrary values on the characteristics such that $s_{in} = -\infty$. Nevertheless as in [12] we can define the minimal weak solution which coincides with the mild solution. The key point here is the following classical comparison result (see [1], [7]).

**Proposition 2.1** *Assume that $E, A$ are smooth (for example $E \in W^{1,\infty}(]0,1[), A \in W^{2,\infty}(]0,1[)$), $\alpha > 0$, $S_1, S_2 \in L^\infty(]0,1[\times\mathbb{R})$. Consider $(f_k)_{k\in\{1,2\}}$ two bounded weak solutions for*

$$\alpha\, f_k + \frac{p}{\gamma_1}\,\partial_x f_k - \left(E(x) + \frac{A(x)}{\gamma_2}\,A'(x)\right)\,\partial_p f_k = S_k(x,p),\ \ (x,p) \in ]0,1[\times\mathbb{R},$$

*satisfying $f_1(0,p) = f_2(0,p)$ for any $p > 0$ and $f_1(1,p) = f_2(1,p)$ for any $p < 0$. If $S_1 \le S_2$ then we have $f_1 \le f_2$.*

**Remark 2.1** *The previous comparison result guarantees the uniqueness of the bounded weak solution for the problem*

$$\alpha\, f + \frac{p}{\gamma_1}\,\partial_x f - \left(E(x) + \frac{A(x)}{\gamma_2}\,A'(x)\right)\,\partial_p f = 0,\ \ (x,p) \in ]0,1[\times\mathbb{R}, \tag{15}$$

$$f(x=0,p>0) = g_0(p),\ \ f(x=1,p<0) = g_1(p),$$

*with $\alpha > 0$. Actually this solution coincides with the solution by characteristics*

$$f(x,p) = \mathrm{e}^{\alpha\, s_{in}(x,p)}\, g_k(P(s_{in}(x,p);x,p)) \text{ if } s_{in} > -\infty,\ X(s_{in}(x,p);x,p) = k,\ k \in \{0,1\}, \tag{16}$$

$$f(x,p) = 0 \text{ if } s_{in}(x,p) = -\infty.$$

Suppose that $E, A$ are smooth (for example $E \in W^{1,\infty}(]0,1[), A \in W^{2,\infty}(]0,1[)$) and $g_0 \in L^\infty(]0,+\infty[)$, $g_1 \in L^\infty(]-\infty,0[)$, $g_0, g_1 \ge 0$. We construct now the minimal weak solution for (10), (11). For any $\alpha > 0$ we denote by $f_\alpha$ the unique bounded weak solution of (15), (11). In view of Remark 2.1 we have $0 \le f_\alpha \le \max\{\|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}\}$. A straightforward application of Proposition 2.1 yields $f_\alpha \ge f_\beta$ for any $0 < \alpha \le \beta$. We denote $f(x,p) = \lim_{\alpha\searrow 0} f_\alpha(x,p) = \sup_{\alpha>0} f_\alpha(x,p)$, $\forall\ (x,p) \in [0,1]\times\mathbb{R}$. Obviously we have $0 \le f \le \max\{\|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}\}$ and we check easily that $f$ is a weak solution for (10), (11). Proposition 2.1 implies easily

**Proposition 2.2** *The function $f$ satisfies the following minimality property : if $h$ is a bounded nonnegative weak solution of* (10), (11) *such that $h \le f$, then $h = f$.*

We call the solution $f = \sup_{\alpha>0} f_\alpha$ the minimal solution of (10), (11). By passing to the limit for $\alpha \searrow 0$ in (16) we deduce that the minimal solution coincides with the mild solution.

## 2.1 Properties of the characteristics

In this section we assume that $E, A$ are smooth such that the solutions $(X(s), P(s))$ of (12), (13) are well defined. An important property is that the total energy is conserved along characteristics. As a direct consequence we obtain useful informations about the geometry of characteristics. At the end of this paragraph we state our technical lemma concerning the bounds of the impulsion variation along characteristics.

**Proposition 2.3** *Assume that $E \in W^{1,\infty}(]0,1[)$, $A \in W^{2,\infty}(]0,1[)$. Consider $\Phi$ a primitive of $E$, i.e., $\Phi' = E$ and denote by $W$ the total energy*

$$W(x,p) = \frac{|p|^2}{2} + \frac{|A(x)|^2}{2} + \Phi(x), \ \textit{in the NR case},$$

$$W(x,p) = \left(1+|p|^2\right)^{\frac{1}{2}} + \frac{|A(x)|^2}{2} + \Phi(x), \ \textit{in the QR case},$$

$$W(x,p) = \left(1+|p|^2+|A(x)|^2\right)^{\frac{1}{2}} + \Phi(x), \ \textit{in the FR case}.$$

*Then for any solution $(X(s), P(s))$ of (12) we have*

$$\frac{d}{ds}\{W(X(s),P(s))\} = 0, \ \ \forall\, s_{in} < s < s_{out}.$$

**Proof.** Compute the derivative of $W$ with respect to $s$ and use (12). The conclusion follows immediately in all three cases. □

We summarize below several properties of the function $W$.

**Proposition 2.4** *With the notations of Proposition 2.3 we have*

*1) $W(x,p_1) < W(x,p_2)$ iff $|p_1| < |p_2|$ ;*

*2) For any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $W(x, p_2) - W(x, p_1) \geq \varepsilon$ for some $(x, p_1, p_2) \in [0, 1] \times \mathbb{R}^2$ then $|p_2|^k - |p_1|^k \geq \delta$ with $k = 2$ in the NR case and $k = 1$ in the QR and FR cases.*

**Proof.** The first statement is obvious. We prove the second one. In the NR case we have $|p_2|^2 - |p_1|^2 \geq 2\,\varepsilon =: \delta(\varepsilon)$. In the QR and FR cases we obtain

$$|p_2|^2 \geq |p_1|^2 + \varepsilon^2 + 2\,\varepsilon\ \left(1 + |p_1|^2 + \alpha^2\right)^{\frac{1}{2}},$$

where $\alpha = 0$ in the QR case and $\alpha = A(x)$ in the FR case. In both cases we deduce

$$\begin{aligned}
|p_2| - |p_1| \quad &\geq \quad \left(|p_1|^2 + \varepsilon^2 + 2\,\varepsilon\,(1 + |p_1|^2)^{\frac{1}{2}}\right)^{\frac{1}{2}} - |p_1| \\
&= \quad \frac{\varepsilon^2 + 2\,\varepsilon\,(1 + |p_1|^2)^{\frac{1}{2}}}{\left\{\left((1 + |p_1|^2)^{\frac{1}{2}} + \varepsilon\right)^2 - 1\right\}^{\frac{1}{2}} + |p_1|} \\
&\geq \quad \frac{\varepsilon^2 + 2\,\varepsilon\,(1 + |p_1|^2)^{\frac{1}{2}}}{(1 + |p_1|^2)^{\frac{1}{2}} + \varepsilon + |p_1|} \\
&\geq \quad \frac{2\varepsilon\,(1 + |p_1|^2)^{\frac{1}{2}}}{\varepsilon + 2(1 + |p_1|^2)^{\frac{1}{2}}} \\
&\geq \quad \frac{\varepsilon(1 + |p_1|^2)^{\frac{1}{2}}}{\varepsilon + (1 + |p_1|^2)^{\frac{1}{2}}} \\
&\geq \quad \frac{1}{2}\min\{1, \varepsilon\},
\end{aligned}$$

and therefore we can take $\delta(\varepsilon) := \frac{1}{2}\min\{1, \varepsilon\}$. $\square$

In the following we perform a phase plane analysis. We have similar behaviors in all three cases, due to the conservation of the total energy along characteristics. We introduce the notations

$$p_0 := \inf\{p \geq 0\ :\ W(0, p) \geq W(x, 0),\ \forall\, x \in [0, 1]\}, \tag{17}$$

$$p_1 := \sup\{p \leq 0\ :\ W(1, p) \geq W(x, 0),\ \forall\, x \in [0, 1]\}. \tag{18}$$

By using the continuity of $W$ with respect to $p$ we deduce easily that

$$W(0, p_0) \geq W(x, 0),\ \ W(1, p_1) \geq W(x, 0),\ \forall\, x \in [0, 1].$$

The above definitions are motivated by the following result

**Proposition 2.5** *Assume that* $E \in W^{1,\infty}(]0,1[)$, $A \in W^{2,\infty}(]0,1[)$.

*1) For any* $0 < p < p_0$ *there is* $x_0 \in ]0,1[$, $0 < s_0 \le s_{out}(0,p) \le +\infty$ *such that*

$$X(s;0,p) < x_0, \;\; P(s;0,p) > 0, \; \forall\, 0 < s < s_0, \lim_{s \nearrow s_0} X(s;0,p) = x_0, \;\; \lim_{s \nearrow s_0} P(s;0,p) = 0.$$

*Moreover, if* $s_{out}(0,p) < +\infty$ *then*

$$s_0 = \frac{s_{out}(0,p)}{2}, \;\; X(s;0,p) = X(2s_0 - s;0,p), \; P(s;0,p) = -P(2s_0 - s;0,p), \forall s \in [0,2s_0].$$

*In particular* $X(s_{out}(0,p);0,p) = 0$.

*2) For any* $p > p_0$ *we have* $s_{out}(0,p) < +\infty$, $P(s;0,p) > 0$, $\forall\, 0 \le s \le s_{out}(0,p)$ *and* $X(s_{out}(0,p);0,p) = 1$.

*3) For any* $p_1 < p < 0$ *there is* $x_1 \in ]0,1[$, $0 < s_1 \le s_{out}(1,p) \le +\infty$ *such that*

$$X(s;1,p) > x_1, \;\; P(s;1,p) < 0, \; \forall\, 0 < s < s_1, \lim_{s \nearrow s_1} X(s;1,p) = x_1, \;\; \lim_{s \nearrow s_1} P(s;1,p) = 0.$$

*Moreover, if* $s_{out}(1,p) < +\infty$ *then*

$$s_1 = \frac{s_{out}(1,p)}{2}, \;\; X(s;1,p) = X(2s_1 - s;1,p), \; P(s;1,p) = -P(2s_1 - s;1,p), \forall s \in [0,2s_1].$$

*In particular* $X(s_{out}(1,p);1,p) = 1$.

*4) For any* $p < p_1$ *we have* $s_{out}(1,p) < +\infty$, $P(s;1,p) < 0$, $\forall\, 0 \le s \le s_{out}(1,p)$ *and* $X(s_{out}(1,p);1,p) = 0$.

**Proof.** We justify only the first two statements. The other ones follow in similar manner.

1) Assume that $p_0 > 0$ and consider $p \in ]0, p_0[$. By the definition of $p_0$ we deduce that there is $\tilde{x} \in [0,1]$ such that $W(0,p) < W(\tilde{x},0)$. Actually, by the continuity of $W$ we can suppose that $\tilde{x} \in ]0,1[$. We claim that $X(s;0,p) \in [0,\tilde{x}[$ for any $s \in [0, s_{out}(0,p)[$. Indeed, if there is $\tilde{\tilde{s}}$ such that $X(\tilde{\tilde{s}};0,p) \ge \tilde{x} > 0 = X(0;0,p)$ then there is $\tilde{s} \in ]0,\tilde{\tilde{s}}]$ such that $X(\tilde{s};0,p) = \tilde{x}$ and by the conservation of the total energy one gets a contradiction

$$W(\tilde{x}, P(\tilde{s})) = W(X(\tilde{s}), P(\tilde{s})) = W(0,p) < W(\tilde{x},0) \le W(\tilde{x}, P(\tilde{s})).$$

We introduce now the notations $s_0 = \sup\{0 < \tau \leq s_{out}(0,p) \ : \ P(s;0,p) > 0, \ \forall \ s \in [0,\tau[\}$, and $x_0 = \lim_{s \nearrow s_0} X(s;0,p)$. Since for any $s \in ]0,s_0[$ we have $0 < X(s;0,p) < \tilde{x}$ and $X(\cdot;0,p)$ is strictly increasing on $]0,s_0[$ we deduce that $0 < x_0 \leq \tilde{x} < 1$ and $X(s;0,p) < x_0 \ \forall \ s \in ]0,s_0[$. By construction we have $P(s;0,p) > 0 \ \forall \ s \in ]0,s_0[$. We have $W(X(s),P(s)) = W(0,p)$ for any $s \in ]0,s_0[$ and therefore $P(s;0,p)$ has a finite limit as $s$ tends to $s_0$ : $\lim_{s \nearrow s_0} P(s;0,p) = \eta \geq 0$. We claim that $\eta = 0$. Indeed, in the case $s_0 = +\infty$, if $\eta > 0$ then for $s$ large enough we have $P(s;0,p) > \frac{\eta}{2}$, $\forall \ s > s_1$. Taking into account that $\frac{P(s;0,p)}{\gamma_1(s)} \geq C(\eta)$, $\forall s > s_1$ where $C(\eta) = \frac{\eta}{2}$ in the NR case, $C(\eta) = \frac{\eta}{2}(1 + (\frac{\eta}{2})^2)^{-\frac{1}{2}}$ in the QR case and $C(\eta) = \frac{\eta}{2}(1 + (\frac{\eta}{2})^2 + \|A\|^2_{L^\infty})^{-\frac{1}{2}}$ in the FR case, we obtain immediately a contradiction since $1 > X(s) - X(s_1) > C(\eta) \ (s - s_1)$, $\forall \ s > s_1$. In the case $s_0 < +\infty$ if $\eta > 0$ then $P(s_0) = \lim_{s \nearrow s_0} P(s) = \eta > 0$ and there is $\tilde{s}_0 > s_0$ such that $P(\tau) > 0$ for any $\tau \in [s_0, \tilde{s}_0]$ which is in contradiction with the definition of $s_0$. Therefore in both cases $\lim_{s \nearrow s_0} P(s) = 0$. Notice also that in the case $s_0 < +\infty$, $(X^\pm(s), P^\pm(s)) := (X(s_0 \pm s), \pm P(s_0 \pm s))$ verify (12) and the condition $(X^\pm(0), P^\pm(0)) = (x_0, 0)$. By the uniqueness of the characteristics we deduce that $(X^+, P^+) = (X^-, P^-)$ saying that $s_{out} = 2s_0$ and $X(s;0,p) = X(2s_0 - s;0,p)$, $P(s;0,p) = -P(2s_0 - s;0,p)$ for any $s \in [0, 2s_0]$.
2) Take now $p > p_0$ and let $\varepsilon = W(0,p) - W(0,p_0) > 0$. By Proposition 2.4 there is $\delta = \delta(\varepsilon) > 0$ such that $|p_2|^k - |p_1|^k \geq \delta$ for any $(x, p_1, p_2) \in [0,1] \times \mathbb{R}^2$ satisfying $W(x,p_2) - W(x,p_1) \geq \varepsilon$, where $k = 2$ in the NR case and $k = 1$ in the QR and FR cases. Observe that

$$W(X(s),P(s)) = W(0,p) = \varepsilon + W(0,p_0) \geq \varepsilon + W(X(s),0), \ \forall \ s \in [0, s_{out}(0,p)[,$$

and therefore $|P(s)|^k \geq \delta$ for any $s \in [0, s_{out}(0,p)[$. In particular we deduce that $s_{out}(0,p) < +\infty$, $P(s) > 0$ for any $s \in [0, s_{out}(0,p)]$ and $X(s_{out}(0,p);0,p) = 1$. □

We call $p_0, p_1$ introduced in (17), (18) the critical impulsion of the left, respectively right boundary.

Another important property is that the variation of the impulsion $p$ along any characteristic is bounded by some constant depending on the fields $E, A$. It holds also true for nonstationary fields with basically the same proof, and thus, for the sake of generality, we state this result for time dependent fields $E, A$. This can be useful when studying the time periodic case (see for example [2]). We introduce the notations

$$D_{NR} := (2\ \|E\|_{L^\infty} + 2\ \|A\|_{L^\infty}\ \|\partial_x A\|_{L^\infty})^{\frac{1}{2}}\,, \tag{19}$$

$$D_{QR} := (\beta_{QR}(1+\beta_{QR}))^{\frac{1}{2}}\,,\ \ \beta_{QR} = 4\ (\|E\|_{L^\infty} + \|A\|_{L^\infty}\ \|\partial_x A\|_{L^\infty}), \tag{20}$$

$$D_{FR} := \max\left\{\|A\|_{L^\infty}, \left(\beta_{FR}\ (\beta_{FR} + (1+\|A\|^2_{L^\infty})^{\frac{1}{2}})\right)^{\frac{1}{2}}\right\}, \tag{21}$$

where $\beta_{FR} = 8\sqrt{2}\ (\|E\|_{L^\infty} + \|\partial_x A\|_{L^\infty} + \|\partial_t A\|_{L^\infty})$.

We have the following results in the NR, QR and FR cases (the details of the proof can be found in the Appendix).

**Lemma 2.1** *Assume that $E \in L^\infty(\mathbb{R}; W^{1,\infty}(]0,1[))$, $A \in L^\infty(\mathbb{R}; W^{2,\infty}(]0,1[))$, $D = D_{NR}$ in the NR case, $D = D_{QR}$ in the QR case, $D = D_{FR}$ in the FR case. In the FR case we suppose moreover that $\partial_t A$ belongs to $L^\infty(\mathbb{R}\times]0,1[)$. Consider $(X(s), P(s)), s_{in} \le s \le s_{out}$ an arbitrary solution of (12). Then*
*1) if there is $t \in [s_{in}, s_{out}]$ such that $|P(t)| > D$, therefore we have*

$$s_{out} - s_{in} \le 4\,\frac{\gamma_1(t)}{|P(t)|},\ \ |P(s)-P(t)| \le D,\ \ \left|\frac{P(s)}{\gamma_1(s)} - \frac{P(t)}{\gamma_1(t)}\right| \le \frac{D}{\gamma_1(t)},\ \forall\ s_{in} \le s \le s_{out}\ ;$$

*2) for any $s_{in} \le s_1 \le s_2 \le s_{out}$ we have $|P(s_1) - P(s_2)| \le 2\ D$.*

**Remark 2.2** *When $(E, A) \in W^{1,\infty}(]0,1[) \times W^{2,\infty}(]0,1[)$ are stationary fields the previous lemma holds with*

$$D_{NR} := (2\ \|E\|_{L^\infty} + 2\ \|A\|_{L^\infty}\ \|A'\|_{L^\infty})^{\frac{1}{2}}\,,$$

$$D_{QR} := (\beta_{QR}(1+\beta_{QR}))^{\frac{1}{2}}\,,\ \ \beta_{QR} = 4\ (\|E\|_{L^\infty} + \|A\|_{L^\infty}\ \|A'\|_{L^\infty}),$$

$$D_{FR} := \max\left\{\|A\|_{L^\infty}, \left(\beta_{FR}\ (\beta_{FR} + (1+\|A\|^2_{L^\infty})^{\frac{1}{2}})\right)^{\frac{1}{2}}\right\},\ \ \beta_{FR} = 8\sqrt{2}\ (\|E\|_{L^\infty} + \|A'\|_{L^\infty}).$$

## 2.2 Properties of the mild solution

In this paragraph we give some properties of the mild solution of the linear problem (10), (11). We recall the formulation by characteristics and we estimate the moments of $f$ by using the geometry of characteristics. One of the key points for establishing $L^\infty$ bounds is to use duality computations involving $L^1$ test functions.

**Proposition 2.6** *Assume that* $E \in W^{1,\infty}(]0,1[)$, $A \in W^{2,\infty}(]0,1[)$, $g_0 \in L^\infty_{\rm loc}([0,+\infty[)$, $g_1 \in L^\infty_{\rm loc}(]-\infty,0])$*. Denote by* $f$ *the mild solution of* (10), (11)*. Then*
*1) if* $g_0, g_1$ *are nonnegative,* $f$ *is nonnegative ;*
*2)* $f$ *belongs to* $L^\infty_{\rm loc}([0,1]\times\mathbb{R})$ *; moreover if* $g_0 \in L^\infty(]0,+\infty[), g_1 \in L^\infty(]-\infty,0[)$ *then* $f \in L^\infty(]0,1[\times\mathbb{R})$ *and* $\|f\|_{L^\infty} \le \max\{\|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}\}$ *;*
*3) for any test function* $\psi \in C^0_c([0,1]\times\mathbb{R})$ *we have*

$$\int_0^1\int_{\mathbb{R}} f(x,p)\psi(x,p)\;dp\;dx \;\;=\;\; \sum_{k=0}^{1}\int_{(-1)^k p>0}\frac{|p|}{\gamma_1}g_k(p)\int_0^{s_{out}(k,p)}\psi(X(s;k,p),P(s;k,p))\;ds\;dp\;;$$

*4) if* $g_0 \in L^1(]0,+\infty[), g_1 \in L^1(]-\infty,0[)$ *then* $f \in L^1(]0,1[\times\mathbb{R})$ *and* $\int_{\mathbb{R}}|f(\cdot,p)|\;dp$ *belongs to* $L^\infty(]0,1[)$ *;*
*5) if* $g_0 \in L^1(]0,+\infty[), g_1 \in L^1(]-\infty,0[)$ *then* $\lim_{R\to+\infty}\left\|\int_{|p|>R}|f(\cdot,p)|\;dp\right\|_{L^\infty(]0,1[)} = 0.$

**Proof of 1) and 2)** The first statement and the last part of the second one follow immediately by the definition of the mild solution. Take $R>0$ and consider $C = C(R,D) > 0$ such that $|g_0(p)| \le C$, $|g_1(-p)| \le C$ for any $0 < p < R+2D$ where $D = D_{NR}$ in the NR case, $D = D_{QR}$ in the QR case and $D = D_{FR}$ in the FR case. For any $(x,p) \in [0,1]\times[-R,R]$ such that $s_{in}(x,p) > -\infty$ we have (cf. Lemma 2.1) $|P(s_{in}(x,p);x,p) - p| \le 2D$, and therefore we obtain $|P(s_{in}(x,p);x,p)| \le R+2D$. We deduce that $\|f\|_{L^\infty(]0,1[\times]-R,R[)} \le C$ saying that $f$ is locally bounded.
**Proof of 3)** In order to prove the third statement we assume for the moment that $g_0, g_1 \ge 0$. We obtain the equality (22) for any $\psi \in C^0_c([0,1]\times\mathbb{R})$, $\psi \ge 0$ by performing the change of variables $(s,p) \to (X(s;0,p), P(s;0,p))$ for $p > 0, s \in$

$]0, s_{out}(0,p)[$, respectively $(s,p) \to (X(s;1,p), P(s;1,p))$ for $p < 0, s \in ]0, s_{out}(1,p)[$ and by taking into account that $f$ is constant along characteristics. Notice that we have the equalities

$$\left|\det\left(\frac{\partial(X(s;k,p),P(s;k,p))}{\partial(s,p)}\right)\right| = \frac{|p|}{\gamma_1},\ (-1)^k p > 0,\ s \in ]0, s_{out}(k,p)[,\ k \in \{0,1\}.$$

Since $f$ is nonnegative and locally bounded we deduce that for any $\psi \in C_c^0([0,1] \times \mathbb{R})$, $\psi \geq 0$, the functions $p \to \frac{p}{\gamma_1} g_0(p) \int_0^{s_{out}(0,p)} \psi(X(s;0,p), P(s;0,p))\ ds$ and $p \to -\frac{p}{\gamma_1} g_1(p) \int_0^{s_{out}(1,p)} \psi(X(s;1,p), P(s;1,p))\ ds$ are integrable on $]0, +\infty[$, respectively $]-\infty, 0[$. Actually by using the decomposition into positive and negative parts $\psi = \psi^+ - \psi^-$ we obtain that formula (22) holds for any $\psi \in C_c^0([0,1] \times \mathbb{R})$ (in fact for any $\psi$ compactly supported in $[0,1] \times \mathbb{R}$ and integrable). The general case follows by decomposing $g_{0,1} = g_{0,1}^+ - g_{0,1}^-$ and by observing that $f^\pm$ are the mild solutions corresponding to the boundary conditions $g_0^\pm, g_1^\pm$.

**Proof of 4)** Suppose now that $g_0, g_1$ are integrable on $]0, +\infty[$, respectively $]-\infty, 0[$. We have

$$\int_0^1\!\!\int_{\mathbb{R}} f^\pm\ dp\ dx \ \ = \ \ \int_0^1\!\!\int_{\mathbb{R}} f^\pm \mathbf{1}_{\{|p| \leq 3D\}}\ dp\ dx + \int_0^1\!\!\int_{\mathbb{R}} f^\pm \mathbf{1}_{\{|p| > 3D\}}\ dp\ dx =: I_1^\pm + I_2^\pm.$$

We know that $f$ is locally bounded and we have

$$\|f\|_{L^\infty(]0,1[\times]-3D,3D[)} \leq \max\left(\|g_0\|_{L^\infty(]0,5D[)}, \|g_1\|_{L^\infty(]-5D,0[)}\right).$$

Therefore $I_1^+ + I_1^- \leq 6D\ \|f\|_{L^\infty(]0,1[\times]-3D,3D[)}$. In order to estimate $I_2^\pm$ we use the formula (22) with the function $\psi(x,p) = \mathbf{1}_{\{3D<|p|<R_1\}}$, for $R_1 > 3D$. We obtain

$$\int_0^1\!\!\int_{\mathbb{R}} f^\pm \mathbf{1}_{\{3D<|p|<R_1\}}\ dp\ dx = \sum_{k=0}^{1} \int_{(-1)^k p > 0} \frac{|p|}{\gamma_1} g_k^\pm(p) \int_0^{s_{out}(k,p)} \mathbf{1}_{\{3D<|P(s;k,p)|<R_1\}}\ ds\ dp. \quad (22)$$

By applying Lemma 2.1 we deduce that $|P(s;0,p) - p| \leq 2D$ for any $p > 0$ and $|P(s;1,p) - p| \leq 2D$ for any $p < 0$. In particular if $0 < p \leq D$ we have $|P(s;0,p)| \leq 3D$ and therefore

$$\int_0^{s_{out}(0,p)} \mathbf{1}_{\{3D<|P(s;0,p)|<R_1\}}\ ds = 0, \quad 0 < p \leq D. \quad (23)$$

Similarly we obtain

$$\int_0^{s_{out}(1,p)} \mathbf{1}_{\{3D<|P(s;1,p)|<R_1\}}\, ds = 0, \;\; -D \leq p < 0. \tag{24}$$

Using one more time Lemma 2.1 we deduce that

$$s_{out}(0,p) = s_{out}(0,p) - s_{in}(0,p) \leq 4\frac{\gamma_1}{p}, \;\; p > D, \tag{25}$$

and

$$s_{out}(1,p) = s_{out}(1,p) - s_{in}(1,p) \leq -4\frac{\gamma_1}{p}, \;\; p < -D. \tag{26}$$

Combining (22), (23), (24), (25), (26) yields

$$\begin{aligned}
\int_0^1\!\!\int_{\mathbb{R}} f^{\pm}\mathbf{1}_{\{3D<|p|<R_1\}}\, dp\, dx &= \sum_{k=0}^{1}\int_{(-1)^k p>D} \frac{|p|}{\gamma_1} g_k^{\pm}(p) \int_0^{s_{out}(k,p)} \mathbf{1}_{\{3D<|P(s;k,p)|<R_1\}}\, ds\, dp \\
&\leq \int_{p>D} \frac{p}{\gamma_1} g_0^{\pm}(p) s_{out}(0,p)\, dp - \int_{p<-D} \frac{p}{\gamma_1} g_1^{\pm}(p) s_{out}(1,p)\, dp \\
&\leq 4\int_{p>0} g_0^{\pm}(p)\, dp + 4\int_{p<0} g_1^{\pm}(p)\, dp, \; \forall\, R_1 > 3D. \qquad (27)
\end{aligned}$$

After letting $R_1 \to +\infty$ one gets $I_2^{\pm} \leq 4\int_{p>0} g_0^{\pm}(p)\, dp + 4\int_{p<0} g_1^{\pm}(p)\, dp$, and therefore

$$\int_0^1\!\!\int_{\mathbb{R}} f^{\pm}(x,p)\, dp\, dx \leq \int_0^1\!\!\int_{\mathbb{R}} f^{\pm}(x,p)\mathbf{1}_{\{|p|\leq 3D\}}\, dp\, dx + 4\int_{p>0} g_0^{\pm}(p)\, dp + 4\int_{p<0} g_1^{\pm}(p)\, dp.$$

Finally one gets

$$\begin{aligned}
\int_0^1\!\!\int_{\mathbb{R}} |f(x,p)|\, dp\, dx &\leq I_1^+ + I_1^- + 4\int_{p>0} |g_0(p)|\, dp + 4\int_{p<0} |g_1(p)|\, dp \\
&\leq 4\int_{p>0} |g_0(p)|\, dp + 4\int_{p<0} |g_1(p)|\, dp \\
&+ 6\, D \max\left(\|g_0\|_{L^\infty(]0,5D[)}, \|g_1\|_{L^\infty(]-5D,0[)}\right). \qquad (28)
\end{aligned}$$

In order to estimate the $L^\infty$ norm of $\int_{\mathbb{R}} |f(\cdot,p)|\, dp$ we write for any nonnegative function $\varphi \in L^1(]0,1[)$

$$\int_0^1\!\!\int_{\mathbb{R}} f^{\pm}\, \varphi\, dp\, dx = \int_0^1\!\!\int_{\mathbb{R}} f^{\pm}\mathbf{1}_{\{|p|\leq 4D\}}\varphi\, dp\, dx + \int_0^1\!\!\int_{\mathbb{R}} f^{\pm}\mathbf{1}_{\{|p|>4D\}}\varphi\, dp\, dx =: I_3^{\pm} + I_4^{\pm}.$$

We know that $f$ is locally bounded and

$$\|f\|_{L^\infty(]0,1[\times]-4D,4D[)} \leq \max\{\|g_0\|_{L^\infty(]0,6D[)}, \|g_1\|_{L^\infty(]-6D,0[)}\}.$$

Therefore we have

$$\begin{aligned} I_3^+ + I_3^- &= \int_0^1\!\!\int_{\mathbb{R}} |f(x,p)| \mathbf{1}_{\{|p|\le 4D\}} \varphi(x)\; dp\; dx \\ &\le 8D \max\{\|g_0\|_{L^\infty(]0,6D[)}, \|g_1\|_{L^\infty(]-6D,0[)}\}\; \|\varphi\|_{L^1}. \end{aligned} \tag{29}$$

In order to estimate $I_4^\pm$ we use formula (22) with the function $\psi(x,p) = \mathbf{1}_{\{|p|>4D\}}$ (actually we have to consider first $\psi_{R_1}(x,p) = \mathbf{1}_{\{4D<|p|<R_1\}}$ for any $R_1 > 4D$ and then let $R_1 \to +\infty$, but we skip these arguments). We obtain

$$\int_0^1\!\!\int_{\mathbb{R}} f^\pm \mathbf{1}_{\{|p|>4D\}} \varphi\; dp\; dx = \sum_{k=0}^{1} \int_{(-1)^k p>0} \frac{|p|}{\gamma_1} g_k^\pm(p) \int_0^{s_{out}(k,p)} \mathbf{1}_{\{|P(s;k,p)|>4D\}} \varphi(X(s;k,p))\; ds\; dp. \tag{30}$$

By Lemma 2.1 we deduce that $|P(s;0,p)-p| \le 2D$ for any $p > 0$ and $|P(s;1,p)-p| \le 2D$ for any $p < 0$. In particular if $0 < p \le 2D$ we have $|P(s;0,p)| \le 4D$ for any $s \in ]0, s_{out}(0,p)[$ and therefore

$$\int_0^{s_{out}(0,p)} \mathbf{1}_{\{|P(s;0,p)|>4D\}} \varphi(X(s;0,p))\; ds = 0,\ \forall\, 0 < p \le 2D. \tag{31}$$

Similarly we obtain

$$\int_0^{s_{out}(1,p)} \mathbf{1}_{\{|P(s;1,p)|>4D\}} \varphi(X(s;1,p))\; ds = 0,\ \forall\, -2D \le p < 0. \tag{32}$$

Notice also that by Lemma 2.1 we have

$$\left| \frac{P(s;0,p)}{\gamma_1(s)} - \frac{p}{\gamma_1} \right| \le \frac{D}{\gamma_1},\ \ P(s;0,p) \ge p - D > 0,\ \forall\, p > 2D,\ 0 < s < s_{out}(0,p),$$

respectively

$$\left| \frac{P(s;1,p)}{\gamma_1(s)} - \frac{p}{\gamma_1} \right| \le \frac{D}{\gamma_1},\ \ P(s;1,p) \le p + D < 0,\ \forall\, p < -2D,\ 0 < s < s_{out}(1,p).$$

In particular $s_{out}(0,p) < +\infty$, $X(s_{out}(0,p);0,p) = 1$, $s_{out}(1,p) < +\infty$, $X(s_{out}(1,p);1,p) = 0$. We deduce that for any $p > 2D$, $s \in [0, s_{out}(0,p)]$ we have

$$0 < \frac{\frac{p}{\gamma_1}}{\frac{P(s;0,p)}{\gamma_1(s)}} \le \frac{\frac{p}{\gamma_1}}{\frac{p}{\gamma_1} - \frac{D}{\gamma_1}} = \frac{p}{p-D} \le 2,$$

and for any $p < -2D$, $s \in [0, s_{out}(1,p)]$ we have

$$0 < \frac{-\frac{p}{\gamma_1}}{-\frac{P(s;1,p)}{\gamma_1(s)}} \le \frac{-\frac{p}{\gamma_1}}{-\frac{p}{\gamma_1} - \frac{D}{\gamma_1}} = \frac{-p}{-p-D} \le 2.$$

One gets for any $p > 2D$

$$\begin{aligned}
\frac{p}{\gamma_1}\int_0^{s_{out}(0,p)} \mathbf{1}_{\{|P(s;0,p)|>4D\}}\varphi(X(s;0,p))\,ds &\le \frac{p}{\gamma_1}\int_0^{s_{out}(0,p)} \frac{1}{\frac{P(s;0,p)}{\gamma_1(s)}}\varphi(X(s;0,p))\frac{dX}{ds}\,ds \\
&\le 2\int_0^{s_{out}(0,p)} \varphi(X(s;0,p))\frac{dX}{ds}\,ds \\
&= 2\|\varphi\|_{L^1(]0,1[)}. \qquad (33)
\end{aligned}$$

Similarly we obtain for any $p < -2D$

$$-\frac{p}{\gamma_1}\int_0^{s_{out}(1,p)} \mathbf{1}_{\{|P(s;1,p)|>4D\}}\varphi(X(s;1,p))\,ds \le 2\|\varphi\|_{L^1(]0,1[)}. \qquad (34)$$

Combining (30), (31), (32), (33), (34) yields

$$I_4^+ + I_4^- \le 2\left(\int_{p>0}|g_0(p)|\,dp + \int_{p<0}|g_1(p)|\,dp\right)\|\varphi\|_{L^1(]0,1[)}, \qquad (35)$$

and thus (29), (35) imply

$$\begin{aligned}
\int_0^1\int_{\mathbb{R}}|f(x,p)|\varphi(x)\,dp\,dx &\le 8D\max\{\|g_0\|_{L^\infty(]0,6D[)}, \|g_1\|_{L^\infty(]-6D,0[)}\}\|\varphi\|_{L^1(]0,1[)} \\
&+ 2\left(\int_{p>0}|g_0(p)|\,dp + \int_{p<0}|g_1(p)|\,dp\right)\|\varphi\|_{L^1(]0,1[)},
\end{aligned}$$

for any nonnegative function $\varphi \in L^1(]0,1[)$. We deduce that

$$\begin{aligned}
\left\|\int_{\mathbb{R}}|f(\cdot,p)|\,dp\right\|_{L^\infty(]0,1[)} &\le 8D\max\{\|g_0\|_{L^\infty(]0,6D[)}, \|g_1\|_{L^\infty(]-6D,0[)}\} \\
&+ 2\left(\int_{p>0}|g_0(p)|\,dp + \int_{p<0}|g_1(p)|\,dp\right). \qquad (36)
\end{aligned}$$

**Proof of 5)** Take now $R > 4D$ and denote $\rho_R^\pm(x) = \int_{|p|>R} f^\pm(x,p)\,dp,\ x \in [0,1]$. We have for any nonnegative function $\varphi \in L^1(]0,1[)$

$$\int_0^1 \rho_R^\pm(x)\varphi(x)\,dx = \sum_{k=0}^{1}\int_{(-1)^k p>0}\frac{|p|}{\gamma_1}g_k^\pm(p)\int_0^{s_{out}(k,p)}\mathbf{1}_{\{|P(s;k,p)|>R\}}\varphi(X(s;k,p))\,ds\,dp.$$

As before observe that for any $0 < p \le R - 2D$ we have $|P(s;0,p)| \le p + 2D \le R$ and thus $\int_0^{s_{out}(0,p)} \mathbf{1}_{\{|P(s;0,p)|>R\}}\varphi(X(s;0,p))\ ds = 0$. Similarly, for any $-R+2D \le p < 0$ we have $\int_0^{s_{out}(1,p)} \mathbf{1}_{\{|P(s;1,p)|>R\}}\varphi(X(s;1,p))\ ds = 0$. Notice also that for any $p > R - 2D > 2D$ we have,

$$\left|\frac{P(s;0,p)}{\gamma_1(s)} - \frac{p}{\gamma_1}\right| \le \frac{D}{\gamma_1},\quad P(s;0,p) \ge p - D > 0, s_{out} < +\infty, X(s_{out};0,p) = 1$$

and therefore

$$\begin{aligned}
\frac{p}{\gamma_1}\int_0^{s_{out}(0,p)} \mathbf{1}_{\{|P(s;0,p)|>R\}}\varphi(X(s;0,p))\ ds &\le \frac{p}{\gamma_1}\int_0^{s_{out}(0,p)} \varphi(X(s;0,p))\frac{\frac{dX}{ds}}{\frac{P(s;0,p)}{\gamma_1}}\ ds \\
&\le \int_0^{s_{out}(0,p)} \frac{\frac{p}{\gamma_1}}{\frac{p}{\gamma_1} - \frac{D}{\gamma_1}}\varphi(X(s;0,p))\frac{dX}{ds}\ ds \\
&\le 2\|\varphi\|_{L^1(]0,1[)}.
\end{aligned}$$

Similarly one gets for any $p < -R + 2D$ that

$$-\frac{p}{\gamma_1}\int_0^{s_{out}(1,p)} \mathbf{1}_{\{|P(s;1,p)|>R\}}\varphi(X(s;1,p))\ ds \le 2\|\varphi\|_{L^1(]0,1[)}.$$

Finally we obtain

$$\int_0^1 \rho_R^{\pm}(x)\varphi(x)\ dx \le 2\left(\int_{p>R-2D} g_0^{\pm}(p)\ dp + \int_{p<-R+2D} g_1^{\pm}(p)\ dp\right)\ \|\varphi\|_{L^1(]0,1[)},$$

which implies that

$$\int_{|p|>R} f^{\pm}(x,p)\ dp \le 2\left(\int_{p>R-2D} g_0^{\pm}(p)\ dp + \int_{p<-R+2D} g_1^{\pm}(p)\ dp\right),\quad \text{a.e. } x \in ]0,1[.$$

Therefore we deduce that

$$\left\|\int_{|p|>R} |f(\cdot,p)|\ dp\right\|_{L^\infty(]0,1[)} \le 2\left(\int_{p>R-2D} |g_0(p)|\ dp + \int_{p<-R+2D} |g_1(p)|\ dp\right). \quad (37)$$

**Remark 2.3** *Under the hypotheses of Proposition* 2.6 *denote by $f_\alpha$ the unique solution of* (15), (11) *and let $f$ be the mild solution of* (10), (11). *Since we have $|f_\alpha| \le |f|$ for any $\alpha > 0$, we deduce that the statements* 1), 2), 4), 5) *of Proposition* 2.6 *hold also for $f_\alpha$. Moreover, by change of variables, we obtain the analogous formula (see* (22)*)*

$$\int_0^1\int_{\mathbb{R}} f_\alpha\psi\ dp\ dx = \sum_{k=0}^{1}\int_{(-1)^k p>0}\frac{|p|}{\gamma_1}g_k(p)\int_0^{s_{out}(k,p)} \mathrm{e}^{-\alpha s}\psi(X(s;k,p),P(s;k,p))\ ds\ dp,$$

*for any function $\psi \in C_c^0([0,1]\times\mathbb{R})$.*

We estimate now the current densities $j^\pm(x) := \int_{\mathbb{R}} \frac{p^\pm}{\gamma_1} f(x,p)\, dp$ where $p^\pm = \max(0, \pm p)$. The interesting point is that these estimates do not depend on the fields $E, A$.

**Proposition 2.7** *Assume that $E \in W^{1,\infty}(]0,1[)$, $A \in W^{2,\infty}(]0,1[)$, $g_0, g_1 \geq 0$ and*

$$G_1 := \int_{p>0} \frac{p}{\gamma_1} g_0(p)\, dp - \int_{p<0} \frac{p}{\gamma_1} g_1(p)\, dp < +\infty.$$

*Denote by $f$ the mild solution of* (10), (11). *Then for a.a. $x \in ]0,1[$ we have*

$$\int_{\mathbb{R}} \frac{|p|}{\gamma_1} f(x,p)\, dp = j^+(x) + j^-(x) \leq 2G_1.$$

**Proof.** We use the formula (22) with $\psi(x,p) = \frac{p^+}{\gamma_1}\varphi(x)$, where $\varphi \in L^1(]0,1[)$, $\varphi \geq 0$. We obtain

$$\begin{aligned}
\int_0^1 j^+(x)\varphi(x)\, dx &= \int_0^1 \int_{\mathbb{R}} f(x,p)\varphi(x) \frac{p^+}{\gamma_1}\, dp\, dx \\
&= \sum_{k=0}^{1} \int_{(-1)^k p>0} \frac{|p|}{\gamma_1} g_k(p) \int_0^{s_{out}(k,p)} \left( \frac{P(s;k,p)}{\gamma_1(s)} \right)^+ \varphi(X(s;k,p))\, ds\, dp \\
&= I_0 + I_1. \qquad (38)
\end{aligned}$$

We introduce now $p_0$ the critical impulsion corresponding to the left boundary $x = 0$. By Proposition 2.5 we know that for any $p > p_0$ we have $s_{out}(0,p) < +\infty$, $P(s;0,p) > 0$ for any $s \in [0, s_{out}(0,p)]$, $X(s_{out}(0,p);0,p) = 1$. In this case we obtain

$$\begin{aligned}
\int_0^{s_{out}(0,p)} \left( \frac{P(s;0,p)}{\gamma_1(s)} \right)^+ \varphi(X(s;0,p))\, ds &= \int_0^{s_{out}(0,p)} \frac{dX}{ds} \varphi(X(s;0,p))\, ds \\
&= \int_0^1 \varphi(u)\, du. \qquad (39)
\end{aligned}$$

Consider now $0 < p < p_0$. If $s_{out}(0,p) = +\infty$ we know by Proposition 2.5 that there is $x_0 \in ]0,1[$ such that $\lim_{s \to +\infty} X(s;0,p) = x_0$, $P(s;0,p) > 0$, $\forall\ s > 0$ and therefore we find as above that

$$\begin{aligned}
\int_0^{s_{out}(0,p)} \left( \frac{P(s;0,p)}{\gamma_1(s)} \right)^+ \varphi(X(s;0,p))\, ds &= \int_0^{+\infty} \frac{dX}{ds} \varphi(X(s;0,p))\, ds \\
&= \int_0^{x_0} \varphi(u)\, du. \qquad (40)
\end{aligned}$$

It remains to analyze the case $0 < p < p_0$, $s_{out}(0,p) < +\infty$. We know that there is $x_0 \in ]0,1[$ such that $X(s_{out}(0,p)/2; 0,p) = x_0$, $P(s;0,p) > 0$, $\forall\ s \in [0, s_{out}(0,p)/2[$, $P(s_{out}(0,p)/2;0,p) = 0$, $P(s;0,p) < 0$, $\forall s \in ]s_{out}(0,p)/2, s_{out}(0,p)]$. We find easily that

$$\begin{aligned}\int_0^{s_{out}(0,p)} \left(\frac{P(s;0,p)}{\gamma_1(s)}\right)^+ \varphi(X(s;0,p))\ ds &= \int_0^{s_{out}(0,p)/2} \frac{dX}{ds}\varphi(X(s;0,p))\ ds \\ &= \int_0^{x_0} \varphi(u)\ du. \end{aligned} \tag{41}$$

Combining (39), (40), (41) yields $I_0 \leq \int_{p>0} \frac{p}{\gamma_1} g_0(p)\ dp\ \|\varphi\|_{L^1(]0,1[)}$. Similarly, by introducing the critical impulsion $p_1$ corresponding to the right boundary $x = 1$ and by using Proposition 2.5 we deduce that $I_1 \leq -\int_{p<0} \frac{p}{\gamma_1} g_1(p)\ dp\ \|\varphi\|_{L^1(]0,1[)}$, and finally (38) implies $\int_0^1 j^+(x)\varphi(x)\ dx \leq G_1 \|\varphi\|_{L^1(]0,1[)}$, for any nonnegative function $\varphi \in L^1(]0,1[)$. Therefore $\|j^+\|_{L^\infty(]0,1[)} \leq G_1$. By similar computations we obtain $\|j^-\|_{L^\infty(]0,1[)} \leq G_1$ and thus $\int_{\mathbb{R}} \frac{|p|}{\gamma_1} f(x,p)\ dp \leq 2G_1$, for a.a. $x \in ]0,1[$. $\square$

**Remark 2.4** *Under the hypotheses of Proposition* 2.7 *we have the estimate*

$$\int_{\mathbb{R}} \frac{|p|}{\gamma_1} f_\alpha(x,p)\ dp \leq 2G_1,\ \text{a.e. } x \in ]0,1[,\ \forall\ \alpha > 0,$$

*where $f_\alpha$ is the solution of* (15), (11).

# 3 Fixed point application for the Vlasov-Maxwell equations

We intend to apply the Schauder fixed point theorem. We will construct a fixed point application $(E,A) \to (\tilde{E}, \tilde{A}) =: \mathcal{F}(E,A)$ for $(E,A)$ in some compact subset of $C^0([0,1]) \times C^1([0,1])$. The estimates obtained in the previous section allows us to construct such a subset which is left invariant by $\mathcal{F}$. We need to study the continuity of $\mathcal{F}$ with respect to the topology of $C^0([0,1]) \times C^1([0,1])$. Let us start by analyzing the equation satisfied by $A$ for a given density $n$.

**Proposition 3.1** *Assume that $n \in L^\infty(]0,1[)$, $n \geq 0$, $A_0, A_1 \in \mathbb{R}$. Then there is a unique solution $A \in W^{2,\infty}(]0,1[)$ for the problem*

$$-A''(x) + n(x)\ A(x) = 0, \quad x \in ]0,1[, \tag{42}$$

$$A(0) = A_0, \quad A(1) = A_1, \tag{43}$$

*satisfying the estimates*

$$\|A\|_{L^\infty(]0,1[)} \leq \max\{|A_0|, |A_1|\}, \ \|A'\|_{L^\infty(]0,1[)} \leq |A_1 - A_0| + \|n\|_{L^\infty(]0,1[)} \max\{|A_0|, |A_1|\},$$

$$\|A''\|_{L^\infty(]0,1[)} \leq \|n\|_{L^\infty(]0,1[)}\ \max\{|A_0|, |A_1|\}.$$

**Proof.** By performing the change of unknown $A(x) = \tilde{A}(x) + (1-x)A_0 + xA_1$, $x \in [0,1]$ the problem (42), (43) becomes

$$-\tilde{A}''(x) + n(x)\ \tilde{A}(x) = -n(x)\ [(1-x)A_0 + xA_1] =: F(x), \quad x \in ]0,1[, \tag{44}$$

$$\tilde{A}(0) = 0, \quad \tilde{A}(1) = 0. \tag{45}$$

Since $n$ is nonnegative and bounded there is a unique solution $\tilde{A} \in H_0^1(]0,1[)$ for (44), (45). Observe also that the function $x \to \frac{1}{2}|A(x)|^2$ is convex since $\frac{d^2}{dx^2}\frac{1}{2}|A|^2 \geq 0$. Therefore one gets $\|A\|_{L^\infty(]0,1[)} \leq \max\{|A_0|, |A_1|\}$ and we deduce also that

$$\|A''\|_{L^\infty(]0,1[)} \leq \|n\|_{L^\infty(]0,1[)}\ \max\{|A_0|, |A_1|\}.$$

Taking into account that $\int_0^1 A'(x)\ dx = A_1 - A_0$ we deduce that there is $x_0 \in [0,1]$ such that $A'(x_0) = A_1 - A_0$ and we obtain

$$|A'(x)| \leq |A'(x_0)| + \left| \int_{x_0}^x A''(y)\ dy \right| \leq |A_1 - A_0| + \|n\|_{L^\infty(]0,1[)}\ \max\{|A_0|, |A_1|\}.$$

$\square$

We define now the fixed point application. Since there is no uniqueness result for the weak solution of (10), (11) it is convenient to use the problem (15), (11). Therefore, for any $\alpha > 0$ consider $\tilde{\mathcal{F}}_\alpha(E,A) = (\tilde{E},\tilde{A})$ where $E \in W^{1,\infty}(]0,1[)$, $A \in W^{2,\infty}(]0,1[)$, $f_\alpha$ is the unique solution of (15), (11) and

$$\rho(\cdot) = \int_{\mathbb{R}} f_\alpha(\cdot,p)\, dp, \quad \rho_{\gamma_2}(\cdot) = \int_{\mathbb{R}} \gamma_2^{-1} f(\cdot,p)\, dp,$$

$$\tilde{E} = \Phi', \ \Phi'' = \rho_{ext} - \rho, \ \Phi(0) = \varphi_0, \ \Phi(1) = \varphi_1,$$

$$-\tilde{A}'' + \rho_{\gamma_2}\tilde{A} = 0, \ \tilde{A}(0) = A_0, \ \tilde{A}(1) = A_1.$$

**Proposition 3.2** *Assume that $g_0 \in L^1(]0,+\infty[) \cap L^\infty(]0,+\infty[)$, $g_1 \in L^1(]-\infty,0[) \cap L^\infty(]-\infty,0[)$, $g_0, g_1 \geq 0$, $\rho_{ext} \in L^\infty(]0,1[)$, $\rho_{ext} \geq 0$, $\varphi_0,\varphi_1,A_0,A_1 \in \mathbb{R}$. For any $R > 0$ we consider the set*

$$\tilde{\mathcal{D}}_R := \{\ (E,A) \in W^{1,\infty}(]0,1[) \times W^{2,\infty}(]0,1[)\ :\ \|E\|_{L^\infty(]0,1[)} + \|A'\|_{L^\infty(]0,1[)} \leq R,$$
$$\|A\|_{L^\infty(]0,1[)} \leq \max\{|A_0|,|A_1|\}\}. \tag{46}$$

*Then there are $R_0 > 0$ and $C_0 > 0$ such that $\tilde{\mathcal{F}}_\alpha(\tilde{\mathcal{D}}_{R_0}) \subset \tilde{\mathcal{D}}_{R_0}$ and*

$$\sup_{(E,A)\in\tilde{\mathcal{D}}_{R_0}} \max\{\|\tilde{E}\,'\|_{L^\infty(]0,1[)}, \|\tilde{A}''\|_{L^\infty(]0,1[)}\} \leq C_0.$$

**Proof.** Since $\int_0^1 \tilde{E}(x)\, dx = \varphi_1 - \varphi_0$ we deduce that there is $x_0 \in [0,1]$ such that $\tilde{E}(x_0) = \varphi_1 - \varphi_0$. Therefore one gets

$$\|\tilde{E}\|_{L^\infty(]0,1[)} \leq |\tilde{E}(x_0)| + \|\tilde{E}\,'\|_{L^\infty(]0,1[)} \leq c_1 + \|\rho\|_{L^\infty(]0,1[)}, \tag{47}$$

where $c_1 := |\varphi_1 - \varphi_0| + \|\rho_{ext}\|_{L^\infty(]0,1[)}$. By Proposition 3.1 we have

$$\|\tilde{A}\|_{L^\infty(]0,1[)} \leq c_2, \ \|\tilde{A}'\|_{L^\infty(]0,1[)} \leq c_3 + c_2\, \|\rho_{\gamma_2}\|_{L^\infty(]0,1[)} \leq c_3 + c_2\, \|\rho\|_{L^\infty(]0,1[)}, \tag{48}$$

where $c_2 := \max\{|A_0|,|A_1|\}$, $c_3 := |A_1 - A_0|$. By Proposition 2.6 (see formula (36)) and Remark 2.3 we know that in the NR case we have

$$\|\rho\|_{L^\infty(]0,1[)} \leq 2G_0 + 8G_\infty D_{NR}, \tag{49}$$

where $G_0 := \|g_0\|_{L^1(]0,+\infty[)} + \|g_1\|_{L^1(]-\infty,0[)}$, $G_\infty := \max\{\|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}\}$ and

$$D_{NR} = (2(\|E\|_{L^\infty} + \|A\|_{L^\infty}\|A'\|_{L^\infty}))^{\frac{1}{2}} \leq \sqrt{2}\,(\|E\|_{L^\infty} + c_2\|A'\|_{L^\infty})^{\frac{1}{2}}\,,$$

for any $A$ satisfying $\|A\|_{L^\infty} \le c_2$. Combining (47), (48), (49) implies

$$\begin{aligned}
\|\tilde{E}\|_{L^\infty} + \|\tilde{A}'\|_{L^\infty} &\le c_1 + c_3 + (1+c_2)\|\rho\|_{L^\infty} \\
&\le c_1 + c_3 + (1+c_2)\left\{2G_0 + 8G_\infty\sqrt{2(1+c_2)}\,(\|E\|_{L^\infty} + \|A'\|_{L^\infty})^{\frac{1}{2}}\right\} \\
&= c_4(\|E\|_{L^\infty} + \|A'\|_{L^\infty})^{\frac{1}{2}} + c_5, \qquad (50)
\end{aligned}$$

where $c_4 = 8G_\infty(1+c_2)_\infty\sqrt{2(1+c_2)}$ and $c_5 = c_1+c_3+2G_0(1+c_2)$. We deduce easily that if $\|E\|_\infty + \|A'\|_{L^\infty} \le (c_4+\sqrt{c_5})^2$ then we have $\|\tilde{E}\|_\infty + \|\tilde{A}'\|_{L^\infty} \le (c_4+\sqrt{c_5})^2$ and therefore in the NR case we can take $R_0 = (c_4+\sqrt{c_5})^2$. Now if $(E,A) \in \tilde{\mathcal{D}}_{R_0}$ we have

$$\begin{aligned}
\|\tilde{E}\,'\|_{L^\infty} &\le \|\rho_{ext}\|_{L^\infty} + \|\rho\|_{L^\infty} \le \|\rho_{ext}\|_{L^\infty} + 2G_0 + 8G_\infty\sqrt{2(1+c_2)}\,(\|E\|_{L^\infty} + \|A'\|_{L^\infty})^{\frac{1}{2}} \\
&\le \|\rho_{ext}\|_{L^\infty} + 2G_0 + 8G_\infty\sqrt{2(1+c_2)}\,(c_4+\sqrt{c_5}).
\end{aligned}$$

By Proposition 3.1 we have also for any $(E,A) \in \tilde{\mathcal{D}}_{R_0}$

$$\|\tilde{A}''\|_{L^\infty} \le c_2\|\rho\|_{L^\infty} \le c_2\{\,2G_0 + 8G_\infty\sqrt{2(1+c_2)}\,(c_4+\sqrt{c_5})\}, \qquad (51)$$

and finally we proved that $\sup_{(E,A)\in\tilde{\mathcal{D}}_{R_0}} \max\{\|\tilde{E}\,'\|_{L^\infty(]0,1[)}, \|\tilde{A}''\|_{L^\infty(]0,1[)}\} < +\infty$. Let us analyze now the QR and FR cases. We will use Proposition 2.7. Observe that in both cases for any $|p| > \left(\frac{1+|c_2|^2}{3}\right)^{\frac{1}{2}}$ and $A$ verifying $\|A\|_{L^\infty} \le c_2$ we have the inequality $\frac{|p|}{\gamma_1} \ge \frac{1}{2}$. Therefore by Proposition 2.7 we deduce

$$\begin{aligned}
\rho(x) &= \int_{\mathbb{R}} f_\alpha(x,p)\,dp \le 2\left(\frac{1+|c_2|^2}{3}\right)^{\frac{1}{2}} G_\infty + \int_{\mathbb{R}} 2\frac{|p|}{\gamma_1} f_\alpha(x,p)\,\mathbf{1}_{\{|p|>(\frac{1+|c_2|^2}{3})^{1/2}\}}\,dp \\
&\le 2\left(\frac{1+|c_2|^2}{3}\right)^{\frac{1}{2}} G_\infty + 4\left(\int_{p>0}\frac{p}{\gamma_1}g_0(p)\,dp - \int_{p<0}\frac{p}{\gamma_1}g_1(p)\,dp\right) \\
&\le 2\left(\frac{1+|c_2|^2}{3}\right)^{\frac{1}{2}} G_\infty + 4G_0 =: c_6.
\end{aligned}$$

We obtain from (47), (48) that $\|\tilde{E}\|_{L^\infty} \le c_1 + c_6$, $\|\tilde{A}'\|_{L^\infty} \le c_3 + c_2c_6$, and in these cases we can take $R_0 = c_1 + c_3 + c_6(1+c_2)$. We check easily that we have also $\sup_{(E,A)\in\tilde{\mathcal{D}}_{R_0}} \max\{\|\tilde{E}\,'|_{L^\infty(]0,1[)}, \|\tilde{A}''\|_{L^\infty(]0,1[)}\} < +\infty$. $\square$

**Remark 3.1** *We introduce now the set*

$$\begin{aligned}\mathcal{D} \;=\; & \{(E, A) \in C^0([0,1]) \times C^1([0,1]) \;:\; \|E\|_{L^\infty} + \|A'\|_{L^\infty} \leq R_0, \|A\|_{L^\infty} \leq c_2, \\ & |E(x) - E(y)| \leq C_0\, |x-y|, \;\; |A'(x) - A'(y)| \leq C_0\, |x-y|, \forall\, x, y \in [0,1]\}.\end{aligned}$$

*The previous proposition implies that* $\tilde{\mathcal{F}}_\alpha(\mathcal{D}) \subset \mathcal{D}$ *and we check easily by using Arzela-Ascoli theorem that* $\mathcal{D}$ *is a compact set in* $C^0([0,1]) \times C^1([0,1])$.

**Proposition 3.3** *Under the hypotheses of Proposition* 3.2 *and with the notations of Remark* 3.1 *for any* $\alpha > 0$ *consider* $\mathcal{F}_\alpha = \tilde{\mathcal{F}}_\alpha|_{\mathcal{D}}$. *Then* $\mathcal{F}_\alpha$ *is a continuous map with respect to the topology of* $C^0([0,1]) \times C^1([0,1])$.

**Proof.** The arguments are standard. The uniqueness of the weak solution for (15), (11) is crucial here. Take $(E_n, A_n)_n \subset \mathcal{D}$ converging towards $(E, A)$ in $C^0([0,1]) \times C^1([0,1])$. Denote by $(f_{\alpha,n})_n$ the sequence of mild solutions of (15), (11) associated to $(E_n, A_n)$ and by $f_\alpha$ the mild solution of (15), (11) associated to $(E, A)$. Since $0 \leq f_{\alpha,n} \leq \max\{\|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}\}$ we can extract a subsequence $(f_{\alpha,n_k})_k$ converging weakly $\star$ in $L^\infty$ towards some function $f$ satisfying $0 \leq f \leq \max\{\|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}\}$. We check immediately that $f$ is weak solution for (15), (11) and by the uniqueness of the weak solution we deduce that $f = f_\alpha$. Actually all the sequence $(f_{\alpha,n})_n$ converges towards $f_\alpha$ weakly $\star$ in $L^\infty(]0,1[\times\mathbb{R})$. For any $n \geq 1$ consider $D_n$ given by (19), (20), (21) in the NR, QR, respectively FR case, corresponding to the stationary fields $(E_n, A_n)$. Obviously the sequence $(D_n)_n$ is bounded. By Proposition 2.6 (see formula (36)) and Remark 2.3 we have

$$\|\rho_n\|_{L^\infty} = \left\| \int_{\mathbb{R}} f_{\alpha,n}(\cdot, p)\; dp \right\|_{L^\infty} \leq 8 D_n G_\infty + 2 G_0, \; \forall\, n,$$

and (see formula (37))

$$\left\| \int_{|p|>R} f_{\alpha,n}(\cdot, p)\; dp \right\|_{L^\infty} \leq 2 \int_{p > R - 2D_n} g_0(p)\; dp + 2 \int_{p < -R + 2D_n} g_1(p)\; dp, \; \forall\, n.$$

Therefore $(\|\rho_n\|_{L^\infty})_n$ is bounded and $\lim_{R \to +\infty} \left\| \int_{|p|>R} f_{\alpha,n}(\cdot, p)\; dp \right\|_{L^\infty} = 0$, uniformly with respect to $n$. We deduce easily that $(\rho_n)_n$ converges towards $\rho := \int_{\mathbb{R}} f_\alpha(\cdot, p)\; dp$

weakly $\star$ in $L^\infty(]0,1[)$. Since $\lim_{n\to+\infty} A_n = A$ in $C^0([0,1])$ we obtain also that in the FR case we have

$$\rho_{\gamma_2,n} := \int_{\mathbb{R}} \frac{f_{\alpha,n}(\cdot,p)}{\sqrt{1+|p|^2+|A_n(x)|^2}}\, dp \rightharpoonup \int_{\mathbb{R}} \frac{f_\alpha(\cdot,p)}{\sqrt{1+|p|^2+|A(x)|^2}}\, dp =: \rho_{\gamma_2},$$

weakly $\star$ in $L^\infty(]0,1[)$. Then in all three cases we have $\lim_{n\to+\infty} \rho_{\gamma_2,n} = \rho_{\gamma_2}$ weakly $\star$ in $L^\infty(]0,1[)$. Since $(\tilde{E}_n, \tilde{A}_n) = \mathcal{F}_\alpha(E_n, A_n)$ remains in a compact set of $C^0([0,1]) \times C^1([0,1])$ we can extract a subsequence $(E_{n_k}, A_{n_k})_k$ such that $\lim_{k\to+\infty} (\tilde{E}_{n_k}, \tilde{A}_{n_k}) = (e,a)$ in $C^0([0,1]) \times C^1([0,1])$. In order to identify the limit we can pass to the limit in distribution sense in the equations

$$\tilde{E}'_{n_k}(x) = \rho_{ext}(x) - \rho_{n_k}(x), \quad -\tilde{A}''_{n_k}(x) + \rho_{\gamma_2,n_k}(x)\tilde{A}_{n_k}(x) = 0, \; x \in ]0,1[.$$

We obtain

$$e'(x) = \rho_{ext}(x) - \rho(x), \quad -a''(x) + \rho_{\gamma_2}(x) a(x) = 0, \; x \in ]0,1[,$$

and since $\int_0^1 e(x)\, dx = \lim_{k\to+\infty} \int_0^1 \tilde{E}_{n_k}(x)\, dx = \varphi_1 - \varphi_0$, $a(0) = \lim_{k\to+\infty} \tilde{A}_{n_k}(0) = A_0$, $a(1) = \lim_{k\to+\infty} \tilde{A}_{n_k}(1) = A_1$, we deduce that $(e,a) = \mathcal{F}_\alpha(E,A)$. Actually all the sequence $(\tilde{E}_n, \tilde{A}_n)$ converges towards $\mathcal{F}_\alpha(E,A)$ in $C^0([0,1]) \times C^1([0,1])$, saying that $\mathcal{F}_\alpha$ is continuous. $\square$

We obtain now the existence result for the stationary solution of the reduced 1 D Vlasov-Maxwell system (NR, QR and FR cases) as stated in Theorem 1.1.

**Proof.** (of Theorem 1.1) Consider $(\alpha_n)_n$ a real sequence of positive numbers decreasing to 0. Observe that for any $n \geq 1$ the map $\mathcal{F}_{\alpha_n} : \mathcal{D} \to \mathcal{D}$ satisfies the hypotheses of Schauder fixed point theorem and therefore there are $(E_n, A_n) \in \mathcal{D}$ such that $\mathcal{F}_{\alpha_n}(E_n, A_n) = (E_n, A_n)$. Denote also by $f_n$ the mild solution of (15), (11) associated to $(E_n, A_n)$. As in the proof of Proposition 3.3 we can extract subsequences $(f_{n_k})_k$, $(E_{n_k})_k$, $(A_{n_k})_k$ such that $f_{n_k} \rightharpoonup f$ weakly $\star$ in $L^\infty(]0,1[\times\mathbb{R})$, $(E_{n_k}, A_{n_k}) \to (E,A) \in \mathcal{D}$ in $C^0([0,1]) \times C^1([0,1])$ and we deduce that $(f,E,A)$ solves (6), (7), (8), (9). Note that $f$ is only weak solution. The other estimates for $f$ follow by passing to the limit for $k \to +\infty$ in the analogous estimates for $f_{n_k}$. $\square$

We can prove that the solution constructed above propagates the impulsion moments. The computations are very similar to those in the proof of 4) and 5), Proposition 2.6 and they are left to the reader.

**Proposition 3.4** *Assume that the hypotheses of Theorem* 1.1 *hold. Moreover suppose that for some* $m \geq 2$ *we have*

$$\int_{p>0} \frac{|p|^m}{\gamma_1} g_0(p)\, dp + \int_{p<0} \frac{|p|^m}{\gamma_1} g_1(p)\, dp < +\infty,$$

*and denote by* $(f, E, A)$ *the solution of* (6), (7), (8), (9) *constructed in Theorem* 1.1. *Then* $\int_{\mathbb{R}} \frac{|p|^m}{\gamma_1} f(\cdot, p)\, dp$ *belongs to* $L^\infty(]0,1[)$ *and*

$$\lim_{R \to +\infty} \left\| \int_{|p|>R} \frac{|p|^m}{\gamma_1} f(\cdot, p)\, dp \right\|_{L^\infty(]0,1[)} = 0.$$

# 4 Appendix

We give here the proof of the impulsion variation lemma for the QR and FR cases. The NR case was analyzed in [2]. We have the following easy results, cf. [2]

**Lemma 4.1** *Consider the quadratic function* $F : \mathbb{R} \to \mathbb{R}$ *given by* $F(s) = \frac{1}{2}a(s - s_1)^2 - b(s - s_1) + c$, *with* $a, b, c > 0, \Delta = b^2 - 2ac > 0$ *and* $s_1 \leq s_2$ *such that* $F(s) \geq 0\ \forall s_1 \leq s \leq s_2$. *Then we have* $s_2 - s_1 \leq (b - \sqrt{\Delta})/a \leq 2c/b$.

**Remark 4.1** *If* $a = 0$ *we still have the inequalities* $s_2 - s_1 \leq c/b < 2c/b$.

**Corollary 4.1** *Consider the function* $F_1 : \mathbb{R} \to \mathbb{R}$ *given by* $F_1(s) = \frac{1}{2}a(s-t)^2 - b|s - t| + c$ *with* $a \geq 0, b, c > 0, \Delta = b^2 - 2ac > 0$ *and* $s_1 \leq t \leq s_2$ *such that* $F_1(s) \geq 0\ \forall s_1 \leq s \leq s_2$. *Then we have* $\max\{t - s_1, s_2 - t\} \leq 2c/b$ *and* $s_2 - s_1 \leq 4c/b$.

For checking the impulsion variation lemma in the QR and FR cases we will use the following immediate results

**Lemma 4.2** *1) Denote by $v : \mathbb{R} \to \mathbb{R}$ the function $v(p) = \frac{p}{\sqrt{1+|p|^2}}$. Then we have*

$$|v(p_1) - v(p_2)| \leq 2\frac{|p_1 - p_2|}{\sqrt{1+|p_1|^2}}, \ \forall\, p_1, p_2 \in \mathbb{R},\ |p_1 - p_2| \leq \frac{|p_1|}{2}.$$

*2) Denote by $w : \mathbb{R}^2 \to \mathbb{R}$ the function $w(p,a) = \frac{p}{\sqrt{1+|p|^2+|a|^2}}$. Then we have*

$$|w(p_1,a_1) - w(p_2,a_2)| \leq \frac{2\sqrt{2}}{\sqrt{1+|p_1|^2+|a_1|^2}}(|p_1-p_2|+|a_1-a_2|),\ \forall\, (p_1,a_1),(p_2,a_2) \in \mathbb{R}^2,$$

*satisfying $|p_1 - p_2| \leq \frac{|p_1|}{2}$, $|a_1| \leq |p_1|$.*

**Proof.** 1) We write

$$|v(p_1) - v(p_2)| = \left| \int_0^1 \frac{(p_1-p_2)d\tau}{(1+|\tau p_1 + (1-\tau)p_2|^2)^{\frac{3}{2}}} \right| \leq |p_1 - p_2| \int_0^1 \frac{d\tau}{(1+|\tau p_1 + (1-\tau)p_2|^2)^{\frac{1}{2}}}.$$

Notice that for any $\tau \in [0,1]$ we have

$$|\tau p_1 + (1-\tau)p_2| \geq |p_1| - (1-\tau)|p_1 - p_2| \geq |p_1| - |p_1 - p_2| \geq \frac{|p_1|}{2},$$

and therefore

$$|v(p_1) - v(p_2)| \leq \frac{|p_1 - p_2|}{\sqrt{1+\frac{|p_1|^2}{4}}} \leq \frac{2|p_1 - p_2|}{\sqrt{1+|p_1|^2}}.$$

2) With the notation $(p_\tau, a_\tau) = \tau(p_1,a_1) + (1-\tau)(p_2,a_2)$ for any $\tau \in [0,1]$, we have

$$|w(p_1,a_1) - w(p_2,a_2)| = \left| \int_0^1 \left\{ (p_1-p_2)\frac{\partial w}{\partial p}(p_\tau,a_\tau) + (a_1-a_2)\frac{\partial w}{\partial a}(p_\tau,a_\tau) \right\} \,d\tau \right| \tag{52}$$

As before we have $|p_\tau| \geq \frac{|p_1|}{2}$ for any $\tau \in [0,1]$ and by taking into account that $|p_1| \geq |a_1|$ we can write

$$\left| \frac{\partial w}{\partial p}(p_\tau,a_\tau) \right| \leq \frac{1}{(1+|p_\tau|^2)^{\frac{1}{2}}} \leq \frac{1}{(1+\frac{|p_1|^2}{4})^{\frac{1}{2}}} \leq \frac{2\sqrt{2}}{(1+|p_1|^2+|a_1|^2)^{\frac{1}{2}}}, \tag{53}$$

and

$$\left| \frac{\partial w}{\partial a}(p_\tau,a_\tau) \right| \leq \frac{1}{(1+|p_\tau|^2)^{\frac{1}{2}}} \leq \frac{2\sqrt{2}}{(1+|p_1|^2+|a_1|^2)^{\frac{1}{2}}}. \tag{54}$$

The conclusion follows by combining (52), (53), (54). $\square$

**Proof.** (Lemma 2.1, QR case)

1) By using the equation $\frac{dP}{ds} = -(E(s,X(s)) + A(s,X(s))\partial_x A(s,X(s)))$ we obtain $|P(s) - P(t)| \le |s-t| \; \|F\|_{L^\infty} \le \frac{|P(t)|}{2}$, for any $s \in [s_{in}, s_{out}] \cap [t - \frac{|P(t)|}{2\|F\|_{L^\infty}}, t + \frac{|P(t)|}{2\|F\|_{L^\infty}}]$ if $\|F\|_{L^\infty} > 0$ and $s \in [s_{in}, s_{out}]$ if $\|F\|_{L^\infty} = 0$, where $F(t,x) = -E(t,x) - A(t,x)\partial_x A(t,x)$, $\forall\ (t,x) \in \mathbb{R}\times]0,1[$. By Lemma 4.2 we have

$$|v(P(s)) - v(P(t))| \le 2\frac{|P(s)-P(t)|}{(1+|P(t)|^2)^{\frac{1}{2}}} \le 2\frac{|s-t| \; \|F\|_{L^\infty}}{(1+|P(t)|^2)^{\frac{1}{2}}},\ \forall\ s \in [r_1, r_2], \tag{55}$$

where $r_1 = \max\{s_{in}, t - \frac{|P(t)|}{2\|F\|_{L^\infty}}\}$, $r_2 = \min\{s_{out}, t + \frac{|P(t)|}{2\|F\|_{L^\infty}}\}$ if $\|F\|_{L^\infty} > 0$ and $r_1 = s_{in}$, $r_2 = s_{out}$ if $\|F\|_{L^\infty} = 0$. By using the equation $\frac{dX}{ds} = \frac{P(s)}{\gamma_1(s)} = v(P(s))$ and (55) we find for any $r_1 \le s \le r_2$

$$\begin{aligned} 1 &\ge |X(s) - X(t)| \ge |s-t| \; |v(P(t))| - \left| \int_t^s \{v(P(\tau)) - v(P(t))\} d\tau \right| \\ &\ge |s-t| \; |v(P(t))| - |s-t|^2 \frac{\|F\|_{L^\infty}}{(1+|P(t)|^2)^{\frac{1}{2}}}. \end{aligned}$$

We consider the function

$$F_1(s) = \frac{1}{2}|s-t|^2 \frac{2(\|E\|_{L^\infty} + \|A\|_{L^\infty}\|\partial_x A\|_{L^\infty})}{(1+|P(t)|^2)^{\frac{1}{2}}} - |s-t|\frac{|P(t)|}{(1+|P(t)|^2)^{\frac{1}{2}}} + 1.$$

By the above computations we have $F_1(s) \ge 0$ for any $s \in [r_1, r_2]$. Moreover, the condition $\Delta > 0$ is equivalent to $|P(t)|^2 > 4(\|E\|_{L^\infty} + \|A\|_{L^\infty}\|\partial_x A\|_{L^\infty})(1 + |P(t)|^2)^{\frac{1}{2}}$, which can be written $(|P(t)|^2 - \beta^2/2)^2 > \beta^2 + \beta^4/4$, where $\beta = 4(\|E\|_{L^\infty} + \|A\|_{L^\infty}\|\partial_x A\|_{L^\infty})$. By the hypothesis we have

$$|P(t)|^2 - \frac{\beta^2}{2} > \beta(1+\beta) - \frac{\beta^2}{2} = \beta + \frac{\beta^2}{2},$$

and thus

$$\left(|P(t)|^2 - \frac{\beta^2}{2}\right)^2 > \beta^2\left(1 + \frac{\beta}{2}\right)^2 \ge \beta^2 + \frac{\beta^4}{4}.$$

Therefore the condition $\Delta > 0$ is satisfied. By Corollary 4.1 we deduce that

$$\max\{t - r_1, r_2 - t\} \le \frac{2}{|P(t)|}(1+|P(t)|^2)^{\frac{1}{2}}. \tag{56}$$

Suppose that $\|F\|_{L^\infty} > 0$ and $t + \frac{|P(t)|}{2\|F\|_{L^\infty}} < s_{out}$, or $r_2 = t + \frac{|P(t)|}{2\|F\|_{L^\infty}}$. By using (56) we have

$$\frac{|P(t)|}{2\|F\|_{L^\infty}} \leq \frac{2}{|P(t)|}\left(1+|P(t)|^2\right)^{\frac{1}{2}},$$

which is equivalent to $|P(t)|^2 \leq 4\|F\|_{L^\infty}\left(1+|P(t)|^2\right)^{\frac{1}{2}}$, and thus $|P(t)|^2 \leq 4(\|E\|_{L^\infty} + \|A\|_{L^\infty}\|\partial_x A\|_{L^\infty})\left(1+|P(t)|^2\right)^{\frac{1}{2}}$, saying that $\Delta \leq 0$. But we have already proved that $\Delta > 0$ and finally we deduce that $s_{out} \leq t + \frac{|P(t)|}{2\|F\|_{L^\infty}}$ and similarly we have $s_{in} \geq t - \frac{|P(t)|}{2\|F\|_{L^\infty}}$ if $\|F\|_{L^\infty} > 0$. It follows that $r_1 = s_{in}$, $r_2 = s_{out}$ and

$$\max\{t - s_{in}, s_{out} - t\} \leq \frac{2}{|P(t)|}\left(1+|P(t)|^2\right)^{\frac{1}{2}}, \quad s_{out} - s_{in} \leq \frac{4}{|P(t)|}\left(1+|P(t)|^2\right)^{\frac{1}{2}}.$$

We check easily that if $|P(t)| > D_{QR} = \sqrt{\beta(1+\beta)}$ then

$$|P(t)|\ \left(1+|P(t)|^2\right)^{-\frac{1}{2}} > \frac{\sqrt{\beta(1+\beta)}}{\sqrt{1+\beta(1+\beta)}}.$$

We obtain that

$$\max\{t - s_{in}, s_{out} - t\} \leq 2\frac{\sqrt{1+\beta(1+\beta)}}{\sqrt{\beta(1+\beta)}}.$$

Finally we find for any $s \in [s_{in}, s_{out}]$ that

$$|P(s) - P(t)| \leq \frac{\beta}{2}\frac{\sqrt{1+\beta(1+\beta)}}{\sqrt{\beta(1+\beta)}} \leq \frac{1}{2}\sqrt{\beta(1+\beta)} = \frac{1}{2}D_{QR} \leq D_{QR}.$$

By using (55) we deduce also that for any $s \in [s_{in}, s_{out}]$ we have

$$\left|\frac{P(s)}{\gamma_1(s)} - \frac{P(t)}{\gamma_1(t)}\right| \leq \frac{2}{\gamma_1(t)}\max\{t - s_{in}, s_{out} - t\}\frac{\beta}{4} \leq \frac{\beta}{\gamma_1(t)}\frac{\sqrt{1+\beta(1+\beta)}}{\sqrt{\beta(1+\beta)}} \leq \frac{D_{QR}}{\gamma_1(t)}.$$

2) If $|P(s_1)| \leq D_{QR}$ and $|P(s_2)| \leq D_{QR}$ we have $|P(s_1) - P(s_2)| \leq 2D_{QR}$. If $|P(s_1)| > D_{QR}$, by applying the previous point with $t = s_1$ one gets $|P(s_2) - P(s_1)| \leq D_{QR} \leq 2D_{QR}$. If $|P(s_2)| > D_{QR}$ we apply the previous point with $t = s_2$. $\square$

The proof in the FR case is very similar to those of the QR case. For the sake of completeness we give some details.

**Proof.** (Lemma 2.1, FR case)

1) Consider $\beta = 8\sqrt{2}\,(\|E\|_{L^\infty} + \|\partial_t A\|_{L^\infty} + \|\partial_x A\|_{L^\infty})$. By using the equation $\frac{dP}{ds} = -\left(E(s,X(s)) + \frac{A(s,X(s))}{\gamma_2(s)}\partial_x A(s,X(s))\right)$ we obtain

$$|P(s)-P(t)| \le |s-t|\,(\|E\|_{L^\infty} + \|\partial_x A\|_{L^\infty}) \le |s-t|\frac{\beta}{8\sqrt{2}} \le \frac{|P(t)|}{2}, \tag{57}$$

for any $s \in [s_{in}, s_{out}] \cap [t - \frac{4\sqrt{2}|P(t)|}{\beta}, t + \frac{4\sqrt{2}|P(t)|}{\beta}]$ if $\beta > 0$ and $s \in [s_{in}, s_{out}]$ if $\beta = 0$. Since $|P(t)| > D_{FR} \ge \|A\|_{L^\infty}$ we have $|P(t)| > |A(t,X(t))|$ and therefore Lemma 4.2 implies

$$\begin{aligned}
\left|\frac{P(s)}{\gamma_1(s)} - \frac{P(t)}{\gamma_1(t)}\right| &\le \frac{2\sqrt{2}}{\gamma_1(t)}(|P(s)-P(t)| + |A(s,X(s)) - A(t,X(t))|) \\
&\le \frac{2\sqrt{2}}{\gamma_1(t)}\,(|s-t|\,(\|E\|_{L^\infty} + \|\partial_x A\|_{L^\infty}) + |s-t|(\|\partial_t A\|_{L^\infty} + \|\partial_x A\|_{L^\infty})) \\
&\le \frac{\beta}{2\gamma_1(t)}|s-t|,\ \forall\, s \in [r_1, r_2],
\end{aligned} \tag{58}$$

where $r_1 = \max\{s_{in}, t - \frac{4\sqrt{2}|P(t)|}{\beta}\}$, $r_2 = \min\{s_{out}, t + \frac{4\sqrt{2}|P(t)|}{\beta}\}$ if $\beta > 0$ and $r_1 = s_{in}$, $r_2 = s_{out}$ if $\beta = 0$. Notice that in the second line of the above formula we used the inequality $|X(s) - X(t)| \le |s-t|,\ \ \forall\, s \in [s_{in}, s_{out}]$. By using now the equation $\frac{dX}{ds} = \frac{P(s)}{\gamma_1(s)}$ we find for any $s \in [r_1, r_2]$

$$1 \ \ge \ \left|\int_t^s \frac{P(t)}{\gamma_1(t)}\,d\tau\right| - \left|\int_t^s \left\{\frac{P(\tau)}{\gamma_1(\tau)} - \frac{P(t)}{\gamma_1(t)}\right\} d\tau\right| \ge |s-t|\,\frac{|P(t)|}{\gamma_1(t)} - \frac{1}{2}|s-t|^2\frac{\beta}{2\gamma_1(t)}.$$

We consider the function $F_1(s) = \frac{1}{2}|s-t|^2\frac{\beta}{2\gamma_1(t)} - |s-t|\frac{|P(t)|}{\gamma_1(t)} + 1$. By the above computations we have $F_1(s) \ge 0$ for any $s \in [r_1, r_2]$. Moreover, the condition $\Delta > 0$ is equivalent to $|P(t)|^2 > \beta(1 + |P(t)|^2 + |A(t,X(t))|^2)^{\frac{1}{2}}$ which can be written $\left(|P(t)|^2 - \frac{\beta^2}{2}\right)^2 > \beta^2(1 + |A(t,X(t))|^2) + \frac{\beta^4}{4}$. By the hypothesis we have $|P(t)|^2 > \beta(\beta + \sqrt{1 + |A(t,X(t))|^2})$ and therefore

$$\left(|P(t)|^2 - \frac{\beta^2}{2}\right)^2 > \left(\frac{\beta^2}{2} + \beta\sqrt{1 + |A(t,X(t))|^2}\right)^2 \ge \beta^2(1 + |A(t,X(t))|^2) + \frac{\beta^4}{4},$$

which says that the condition $\Delta > 0$ is satisfied. By Corollary 4.1 we deduce that

$$\max\{t - r_1, r_2 - t\} \le \frac{2\gamma_1(t)}{|P(t)|} = \frac{2}{|P(t)|}(1 + |P(t)|^2 + |A(t,X(t))|^2)^{\frac{1}{2}}. \tag{59}$$

Suppose that $\beta > 0$ and $t + \frac{4\sqrt{2}}{\beta}|P(t)| < s_{out}$, or $r_2 = t + \frac{4\sqrt{2}}{\beta}|P(t)|$. The inequality (59) implies $\frac{4\sqrt{2}}{\beta}|P(t)| \le \frac{2\gamma_1(t)}{|P(t)|}$, and we deduce that

$$|P(t)|^2 \le \frac{\beta}{2\sqrt{2}}\gamma_1(t) \le \beta\left(1 + |P(t)|^2 + |A(t,X(t))|^2\right)^{\frac{1}{2}}, \tag{60}$$

saying that $\Delta \le 0$. But we have already proved that $\Delta > 0$ and finally we deduce that $s_{out} \le t + \frac{4\sqrt{2}}{\beta}|P(t)|$ and similarly we have $s_{in} \ge t - \frac{4\sqrt{2}}{\beta}|P(t)|$ if $\beta > 0$. It follows that $r_1 = s_{in}$, $r_2 = s_{out}$ and

$$\max\{t - s_{in}, s_{out} - t\} \le \frac{2\gamma_1(t)}{|P(t)|}, \quad s_{out} - s_{in} \le \frac{4\gamma_1(t)}{|P(t)|}. \tag{61}$$

Since the function $p \to \frac{p}{\sqrt{1+|p|^2+|a|^2}}$ is nondecreasing with respect to $p$ for any $a \in \mathbb{R}$ and $|P(t)|^2 > \beta(\beta + \sqrt{1+|A(t,X(t))|^2})$ we have

$$\begin{aligned}
\beta\frac{\gamma_1(t)}{|P(t)|} &\le \beta\frac{\sqrt{1+|A(t,X(t))|^2+\beta(\beta+\sqrt{1+|A(t,X(t))|^2})}}{\sqrt{\beta(\beta+\sqrt{1+|A(t,X(t))|^2})}} \\
&\le \sqrt{\beta(\beta+\sqrt{1+|A(t,X(t))|^2})} \\
&\le \sqrt{\beta(\beta+\sqrt{1+\|A\|_{L^\infty}^2})} \le D_{FR}.
\end{aligned} \tag{62}$$

Combining (57), (61), (62) yields for any $s \in [s_{in}, s_{out}]$

$$|P(s) - P(t)| \le \frac{\beta}{4\sqrt{2}}\frac{\gamma_1(t)}{|P(t)|} \le \frac{D_{FR}}{4\sqrt{2}} \le D_{FR}.$$

We deduce from (58), (61), (62)

$$\left|\frac{P(s)}{\gamma_1(s)} - \frac{P(t)}{\gamma_1(t)}\right| \le \frac{\beta}{\gamma_1(t)}\frac{\gamma_1(t)}{|P(t)|} \le \frac{D_{FR}}{\gamma_1(t)}, \ \forall\, s \in [s_{in}, s_{out}].$$

2) The second statement follows as in the QR case. □

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