BOUNDARY VALUE PROBLEM FOR THE N DIMENSIONAL TIME PERIODIC VLASOV-POISSON SYSTEM

M. BOSTAN †

Abstract. *In this work we study the existence of time periodic weak solution for the N dimensional Vlasov-Poisson system with boundary conditions. We start by constructing time periodic solutions with compact support in momentum and bounded electric field for a regularized system. Then, the a priori estimates follow by computations involving the conservation laws of mass, momentum and energy. One of the key point is to impose a geometric hypothesis on the domain : we suppose that its boundary is strictly star-shaped with respect to some point of the domain. These results apply for both classical or relativistic case and for systems with several species of particles.*

Key words. Vlasov-Poisson equations, weak/mild formulation, regularization.

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1. Introduction.

The motion of charged particles can be described, in the collisionless case by the Vlasov-Poisson system. More generally, a lot of phenomena in the physics of charged particles are modeled by kinetic equations (Vlasov, Boltzmann) coupled to the electro-magnetism equations (Maxwell).

Consider Ω an open bounded subset of $\mathbb{R}^N_x$, $N \geq 2$ with regular boundary $\partial\Omega$. We introduce the notations $\Sigma = \partial\Omega \times \mathbb{R}^N_p$ and :

$$\Sigma^{\pm} = \{(x,p) \in \partial\Omega \times \mathbb{R}^N_p \mid \pm (v(p) \cdot n(x)) > 0\}, \tag{1.1}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at x and $v(p)$ is the velocity associated to some energy function $\mathcal{E}(p)$ by $v(p) = \nabla_p \mathcal{E}(p),\ p \in \mathbb{R}^N_p$. The functions to be considered are :

$$\mathcal{E}(p) = \frac{|p|^2}{2m},\ \ v(p) = \frac{p}{m}, \tag{1.2}$$

in the classical case and :

$$\mathcal{E}(p) = mc_0^2 \left(\left(1 + \frac{|p|^2}{m^2 c_0^2}\right)^{1/2} - 1 \right),\ \ v(p) = \frac{p}{m}\left(1 + \frac{|p|^2}{m^2 c_0^2}\right)^{-1/2}, \tag{1.3}$$

in the relativistic case, where m is the mass of particles, c_0 is the light speed in the vacuum. We denote by $f(t,x,p)$ the particles distribution depending on the time t, the position $x \in \Omega$ and momentum $p \in \mathbb{R}^N_p$ and by $E(t,x) = -\nabla_x \Phi$ the electric field which derives from a potential $\Phi(t,x)$ depending on t and x. If we denote by $F(t,x) = -q \cdot \nabla_x \Phi$ the electric force, the Vlasov problem

†Université de Franche-Comté, 16 route de Gray F-25030 Besançon Cedex, tel : (33).(0)3.81.66.63.38, fax : (33).(0)3.81.66.66.23, mbostan@math.univ-fcomte.fr

is given by :

$$\partial_t f + v(p) \cdot \nabla_x f + F(t,x) \cdot \nabla_p f = 0, \quad (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}_p^N, \tag{1.4}$$

$$f(t,x,p) = g(t,x,p), \quad (t,x,p) \in \mathbb{R}_t \times \Sigma^-, \tag{1.5}$$

where q is the charge of particles and g represents the distribution of the incoming particles. Some other boundary conditions can be considered as we will see later on. The problem (1.4), (1.5) is coupled to the Poisson problem for the potential :

$$-\Delta_x \Phi = \frac{1}{\varepsilon_0}\rho(t,x), \quad (t,x) \in \mathbb{R}_t \times \Omega, \tag{1.6}$$

$$\Phi(t,x) = \varphi_0(t,x), \quad (t,x) \in \mathbb{R}_t \times \partial\Omega, \tag{1.7}$$

where ε_0 is the permittivity of the vacuum, $\rho(t,x) = q\int_{\mathbb{R}_p^N} f(t,x,p)dp$ is the charge density and φ_0 is a given potential on the boundary $\mathbb{R}_t \times \partial\Omega$. We denote also by $j(t,x) = q\int_{\mathbb{R}_p^N} v(p)f(t,x,p)dp$ the current density.

The aim of this paper is to prove the existence of time T periodic solution for the $N \geq 2$ dimensional Vlasov-Poisson system (1.4), (1.5), (1.6), (1.7) when the boundary conditions are supposed T periodic, with $T > 0$ fixed. The hypotheses on the boundary conditions will be precised later on, generally we suppose that the incoming energy is bounded $\int_0^T\int_{\Sigma^-} |(v(p)\cdot n(x))| g\mathcal{E}(p)dtd\sigma dp < +\infty$ and we impose some hypotheses on the tangential derivatives of φ_0 with respect to $(t,x) \in \mathbb{R}_t \times \partial\Omega$.

Various results were obtained for the free space system of Vlasov-Poisson. Weak solutions were constructed by Arseneev [2], Horst and Hunze [24]. The existence of classical solutions has been studied by Ukai and Okabe [30], Horst [23], Batt [4]. The existence of global classical solutions for the Vlasov-Poisson equations with small initial data is a result of Bardos and Degond [6], see also Schaeffer [28], [29]. The propagation of the moments for the three dimensional Vlasov-Poisson system was studied by Lions and Perthame in [26]. The existence of global weak solution for the Vlasov-Maxwell system in three dimensions was obtained by DiPerna and Lions [14]. The relativistic case was studied by Glassey and Schaeffer [16], [17], Glassey and Strauss [18], [19], Klainerman and Staffilani [25], Bouchut, Golse and Pallard [11].

Results for the initial-boundary value problem were obtained by Abdallah [1] for the Vlasov-Poisson system in three dimensions and Guo [21] for the Vlasov-Maxwell system. The stationary problem for the Vlasov-Poisson equations was studied by Greengard and Raviart [20] in one dimension and by Poupaud [27] in three dimensions for the Vlasov-Maxwell system. An asymptotic analysis of the Vlasov-Poisson system was done by Degond and Raviart [13] in the case of the plane diode. The regularity of the solutions for the Vlasov-Maxwell system in a half line has been studied by Guo [22]. Results for the time periodic case can be found in [9] for the one dimensional

Vlasov-Poisson system and in [8], [10] for the one and three dimensional Vlasov-Maxwell system respectively.

As usual we start by analyzing a regularized system for which the existence of solution follows by fixed point method (the Schauder theorem). Secondly we need to establish a priori estimates for the regularized solutions, namely to find uniform bounds for the total (kinetic and electric) energy. If for the initial-boundary value problem of the Vlasov-Poisson system this is an immediate consequence of the mass and energy conservation laws, the situation is quite different in the case of T periodic regimes. In fact in this case the mass conservation

$$\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)dxdp+\int_0^t\int_{\Sigma^+}(v(p)\cdot n(x))f(s,x,p)dsd\sigma dp$$
$$=\int_\Omega\int_{\mathbb{R}^N_p} f(0,x,p)dxdp-\int_0^t\int_{\Sigma^-}(v(p)\cdot n(x))g(s,x,p)dsd\sigma dp,\quad t>0,$$

doesn't provide any estimate of $f(t)$ in $L^1(\Omega\times\mathbb{R}^N_p)$ since initial data is not available. Nevertheless, by using the T periodicity we can estimate the time average of the outgoing mass. For example we can write :

$$\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))f(t,x,p)dtd\sigma dp=-\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)dtd\sigma dp.$$

The second estimate comes by using the energy conservation law: by multiplying the Vlasov equation by $\mathcal{E}(p)+q\Phi(t,x)$ we obtain a bound for the outgoing kinetic energy averaged over a period $\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\mathcal{E}(p)f(t,x,p)dtd\sigma dp$, but this is not enough to conclude about an estimate for the total energy. If in addition we suppose that the boundary $\partial\Omega$ is strictly star-shaped (for example with respect to $0\in\Omega$) then the momentum conservation law provides the desired estimate for the total energy averaged over a period, as well as a $L^2(]0,T[\times\partial\Omega)$ estimate for the normal trace of the electric field

$$\int_0^T\int_\Omega\int_{\mathbb{R}^N_p}\mathcal{E}(p)f(t,x,p)dtdxdp+\frac{\varepsilon_0}{2}\int_0^T\int_\Omega|\nabla_x\Phi|^2(t,x)dtdx+\int_0^T\int_{\partial\Omega}|\partial_n\Phi|^2(t,x)dtd\sigma\le C.$$

Once we have established a priori estimates in time average, we deduce easily that this estimates hold uniformly in time (for this it is sufficient to consider $t_0\in]0,T[$ such that the corresponding total energy is below to the average of the energy over a period and to use this level as an initial condition). The passing to the limit for the sequence of regularized solutions follows easily by elliptic regularity.

There is also another important point to be discussed. In order to perform all these computations it is very convenient to work with regularized solutions with particle distribution compactly supported in momentum ($\exists R>0$ such that $f(t,x,p)=0,\ \forall(t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^N_p,|p|>R$) and with bounded electric field. For example, in the case of initial-boundary value problems, if we suppose that the initial-boundary conditions are compactly supported in momentum $f_0=f_0\cdot\mathbf{1}_{\{|p|\le R\}}$,

$g = g \cdot \mathbf{1}_{\{|p| \leq R\}}$ and E is bounded, we check easily that the solution on $]0, t[\times \Omega \times \mathbb{R}_p^N$ is also compactly supported in momentum and $f = f \cdot \mathbf{1}_{\{|p| \leq R + t|q| \cdot \|E\|_\infty\}}$. For permanent regimes, if $g = g \cdot \mathbf{1}_{\{|p| \leq R\}}$, it is not of all evident that the solution remains compactly supported in momentum since the life time of the characteristics into the domain Ω can be arbitrarly large. Actually this result still holds true. In fact in [9] it was proved that in the classical case the change in momentum along any characteristic is bounded uniformly

$$|P(s_1) - P(s_2)| \leq C(\Omega) \cdot \|E\|_\infty^{1/2},$$

where $C(\Omega) \sim diam(\Omega)^{1/2}$, and we can deduce that $f = f \cdot \mathbf{1}_{\{|p| \leq R + C(\Omega) \cdot \|E\|_\infty^{1/2}\}}$. Moreover, if $N = 1$, $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$ thus $\|f\|_\infty \leq \|g\|_\infty$ and by using the Poisson equation we deduce that the regularized solution verifies $\|E\|_\infty \leq C(1 + \|\rho\|_\infty) \leq C_1(1 + \|E\|_\infty^{1/2})$ which allows us to conclude about the boundedness of E (the bound will depend on Ω and the support of g). In fact we can prove similar results for any dimension if the velocity function appearing in the characteristic system :

$$\frac{dX}{ds} = v(P(s)), \;\; \frac{dP}{ds} = qE(s, X(s)),$$

satisfies $v(p) = \frac{p}{|p|} \cdot w(|p|)$, with $w : [0, +\infty[\to \mathbb{R}$ nondecreasing such that $\inf_{t>0} \frac{w(t)}{t^\lambda} > 0$ for some $\lambda \geq 0$. In this case it can be shown that the change in momentum along any characteristic remains bounded :

$$|P(s_1) - P(s_2)| \leq C \cdot (diam(\Omega) \cdot \|E\|_\infty)^{\frac{1}{1+\lambda}}, \;\; \forall s_1, s_2.$$

Therefore we have $\|\rho\|_\infty \leq C \cdot (1 + \|E\|_\infty^{\frac{N}{1+\lambda}})$ and thus we deduce that $\|E\|_\infty \leq C \cdot (1 + \|E\|_\infty^{\frac{N}{1+\lambda}})$ which provides a L^∞ bound for the electric field if $\lambda + 1 > N$. This is why we consider a perturbed energy function for the regularized system, namely $\mathcal{E}_\delta(p) = \mathcal{E}(p) \cdot (1 + \delta |p|^\gamma)$ and $v_\delta(p) = \nabla_p \mathcal{E}_\delta$ with $\delta > 0$ small and γ large enough but fixed. Our main result is

THEOREM 1.1. *Consider an open bounded set Ω of $\mathbb{R}_x^N, N \geq 2$ with the boundary strictly star-shaped with respect to some point of Ω. Assume that $(g \geq 0, \varphi_0) \in L^\infty(\mathbb{R}_t \times \Sigma^-) \times L^\infty(\mathbb{R}_t \times \partial\Omega)$ are time T periodic , $T > 0$ given, such that :*

$$W_0 = \int_0^T \!\! \int_{\Sigma^-} |(v(p) \cdot n(x))|(1 + \mathcal{E}(p)) g \; dt d\sigma dp + \|\varphi_0\|^2_{L^2(]0,T[; H^1(\partial\Omega))} + \|\partial_t \varphi_0\|^2_{L^2(]0,T[\times\partial\Omega)} < +\infty.$$

Then there is a T periodic weak solution $(f, \Phi) \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}_p^N) \times L^2(]0, T[; H^1(\Omega))$ for the Vlasov-Poisson system (1.4), (1.6), (1.5), (1.7) *(classical or relativistic), with traces $(\gamma^+ f, \partial_n \Phi) \in L^\infty(\mathbb{R}_t \times \Sigma^+) \times L^2(]0, T[\times \partial\Omega)$ such that :*

$$\int_0^T \!\! \int_\Omega \int_{\mathbb{R}_p^N} f(t,x,p)(1 + \mathcal{E}(p)) \; dt dx dp + \int_0^T \!\! \int_\Omega |\nabla_x \Phi|^2(t,x) \; dt dx + \int_0^T \!\! \int_{\partial\Omega} |\partial_n \Phi|^2(t,x) \; dt d\sigma$$
$$+ \int_0^T \!\! \int_{\Sigma^+} (v(p) \cdot n(x)) \gamma^+ f (1 + \mathcal{E}(p)) \; dt d\sigma dp \leq C(\Omega, \|g\|_\infty, \|\varphi_0\|_\infty) \cdot F(W_0), \qquad (1.8)$$

where $F(W) = 1 + W$ *in the classical case with* $N \geq 2$ *and relativistic case with* $N > 2$ *and* $F(W) = 1 + W + (1+W)^{4/3}$ *in the relativistic case with* $N = 2$*. Moreover, the solution verifies :*

$$\int_0^T\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f(t,x,p)\; dtd\sigma dp = -\int_0^T\int_{\Sigma^-} (v(p)\cdot n(x))g(t,x,p)\; dtd\sigma dp, \tag{1.9}$$

and :

$$\int_0^T\int_{\Sigma^+} (v(p)\cdot n(x))(\mathcal{E}(p) + q\cdot\varphi_0(t,x))\gamma^+ f\; dtd\sigma dp \leq \int_0^T\int_{\Sigma^-} |(v(p)\cdot n(x))|(\mathcal{E}(p) + q\cdot\varphi_0(t,x))g\; dtd\sigma dp$$
$$-\varepsilon_0\int_0^T\int_{\partial\Omega} \partial_n\Phi\cdot\partial_t\varphi_0\; dtd\sigma, \tag{1.10}$$

and the charge and current densities satisfy $\rho \in L^{\frac{N+2}{N}}(]0,T[\times\Omega)), j \in L^{\frac{N+2}{N+1}}(]0,T[\times\Omega))$ *in the classical case and* $\rho, j \in L^{\frac{N+1}{N}}(]0,T[\times\Omega))$ *in the relativistic case.*

The content of this paper is organized as follows. In Section 2 we recall some basic definitions and results concerning the Vlasov problem. We state also the technical lemma about the change in momentum along characteristics. The details of proof can be found in Appendix. In Section 3 we prove the existence for the regularized system by using a fixed point technique. In the next section we obtain a priori estimates by using the conservation laws of the mass, momentum and energy. In Section 5 we perform the passing to the limit for the sequence of regularized solutions. The time periodic Vlasov-Poisson system with specular boundary condition is treated in Section 6. In the last section we indicate formally how to obtain a priori estimates for the time periodic Vlasov-Poisson-Fokker-Planck system.

2. The Vlasov equation.

In this section we suppose that the electric field is a given T periodic function E. The time periodic Vlasov problem is given by :

$$\partial_t f + v(p)\cdot\nabla_x f + F(t,x)\cdot\nabla_p f = 0,\;\; (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}_p^N, \tag{2.1}$$

$$f(t,x,p) = g(t,x,p),\;\; (t,x,p)\in\mathbb{R}_t\times\Sigma^-, \tag{2.2}$$

where $F(t,x) = q\cdot E(t,x)$ is the electric force. By taking into account that $\nabla_{(x,p)}\cdot(v(p),F(t,x)) = 0$ the equation (2.1) can be written also :

$$\partial_t f + \nabla_x\cdot(v(p)f) + \nabla_p\cdot(F(t,x)f) = 0,\;\; (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}_p^N.$$

Since there is no uniqueness for the Vlasov problem (2.1), (2.2), it is convenient to consider also the perturbed problem :

$$\alpha f + \partial_t f + v(p)\cdot\nabla_x f + F(t,x)\cdot\nabla_p f = 0,\;\; (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}_p^N, \tag{2.3}$$

with the boundary condition (2.2), where $\alpha > 0$ is fixed. We introduce the definitions of weak/mild solution for the perturbed Vlasov problem :

DEFINITION 2.1. *Assume that E, g are T periodic such that $E \in L^\infty(\mathbb{R}_t \times \Omega)^N$ and $(v(p) \cdot n(x))g \in L^1_{loc}(\mathbb{R}_t \times \Sigma^-)$. We say that $f \in L^1_{loc}(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$ is a T periodic weak solution for the perturbed Vlasov problem* (2.3), (2.2) *iff :*

$$\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)(\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x)\cdot\nabla_p\varphi)dtdxdp$$
$$= -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi(t,x,p)dtd\sigma dp, \tag{2.4}$$

for all test function which belongs to :

$$\mathcal{T}_w = \{\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^N_p) \mid \exists R > 0 : \varphi = \varphi \cdot \mathbf{1}_{\{|p|\le R\}}, \varphi|_{\mathbb{R}_t\times\Sigma^+} = 0, \varphi(\cdot + T) = \varphi\}.$$

REMARK 2.2. *In the above definition we can assume that E is only in $L^q(]0,T[\times\Omega)^N$ by requiring more regularity on f, namely $f \in L^p_{loc}(\mathbb{R}_t\times\Omega\times\mathbb{R}^N_p)$, where q is the conjugate exponent of p.*

Suppose now that E is T periodic and belongs to $L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^N$. In this case we can define the notion of solution by characteristics or mild solution. First of all let us introduce the characteristics : for $(t,x,p) \in \mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^N_p$ we denote by $(X(s), P(s)) = (X(s;t,x,p), P(s;t,x,p))$ the unique solution of the system :

$$\frac{dX}{ds} = v(P(s;t,x,p)),\;\; \frac{dP}{ds} = F(s, X(s;t,x,p)),\;\; s_{in}(t,x,p) \le s \le s_{out}(t,x,p), \tag{2.5}$$

with the conditions $X(s=t;t,x,p) = x, P(s=t;t,x,p) = p$. Here s_{in}, s_{out} represent the incoming, respectively outgoing time given by :

$$s_{in}(t,x,p) = \sup\{s \le t \mid X(s;t,x,p) \in \partial\Omega\},$$

$$s_{out}(t,x,p) = \inf\{s \ge t \mid X(s;t,x,p) \in \partial\Omega\}.$$

The mild formulation follows formally by solving :

$$\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x)\cdot\nabla_p\varphi = \psi(t,x,p),\; (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^N_p,$$

with the boundary condition $\varphi|_{\mathbb{R}_t\times\Sigma^+} = 0$. By integration along the characteristic curves we obtain :

$$\varphi^\alpha_\psi(t,x,p) = \int_t^{s_{out}(t,x,p)} e^{-\alpha(s-t)}\psi(s, X(s;t,x,p), P(s;t,x,p))ds,$$

and we define the mild solution by :

DEFINITION 2.3. *Assume that E, g are T periodic such that $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^N$ and $(v(p)\cdot n(x))g \in L^1_{loc}(\mathbb{R}_t \times \Sigma^-)$. For any $\alpha > 0$ we say that $f \in L^1_{loc}(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$ is a T periodic mild solution for the perturbed Vlasov problem* (2.3), (2.2) *iff :*

$$\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)\psi(t,x,p)dtdxdp = -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi^\alpha_\psi(t,x,p)dtd\sigma dp, \quad (2.6)$$

for all test function which belongs to :

$$\mathcal{T}_m = \{\psi \in C^0(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^N_p) \mid \exists R > 0 : \psi = \psi \cdot \mathbf{1}_{\{|p|\le R\}}, \ \ \psi(\cdot + T) = \psi\}.$$

For $\alpha = 0$ one gets the definitions of the weak/mild solution for the Vlasov problem (2.1), (2.2). The existence of the T periodic mild solution is a standard result and follows by change of variables along characteristics (see also Remark 2.6).

PROPOSITION 2.4. *Assume that $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^N$ and $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$ are T periodic, $\alpha > 0$. Then the perturbed Vlasov problem* (2.3), (2.2) *has a unique T periodic mild solution $f \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$, verifying $\|f\|_\infty \le \|g\|_\infty$. Moreover, if $g \ge 0$ then $f \ge 0$.*

REMARK 2.5. *It is easy to check that all T periodic mild solution is also T periodic weak solution.*

REMARK 2.6. *It is well known that the T periodic mild solution is given by $f(t,x,p) = e^{-\alpha(t-s_{in}(t,x,p))}g(s_{in}, X(s_{in};t,x,p), P(s_{in};t,x,p))$ if $s_{in}(t,x,p) > -\infty$ and $f(t,x,p) = 0$ otherwise.*

REMARK 2.7. *Under the hypotheses of Proposition* 2.4*, the T periodic mild solution f has a trace $\gamma^+ f \in L^\infty(\mathbb{R}_t \times \Sigma^+)$ verifying the following Green formula :*

$$\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)(\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x)\cdot\nabla_p\varphi)dtdxdp \quad (2.7)$$

$$= -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi(t,x,p)dtd\sigma dp - \int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(t,x,p)\varphi(t,x,p)dtd\sigma dp,$$

for all $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^N_p)$ with compact support in momentum and T periodic in time. The trace $\gamma^+ f$ is given by the same formula as those of Remark 2.6 *and we have $\|\gamma^+ f\|_\infty \le \|g\|_\infty$. Moreover, if $g \ge 0$ then $\gamma^+ f \ge 0$.*

PROPOSITION 2.8. *Under the hypotheses of Proposition* 2.4*, there is a unique T periodic bounded weak solution for the perturbed Vlasov problem* (2.3), (2.2) *and therefore coincides with the T periodic mild solution.*

Proof. Assume that $f \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$ is a T periodic weak solution with boundary data $g = 0$. We have $\partial_t f + v(p) \cdot \nabla_x f + F(t,x) \cdot \nabla_p f = -\alpha f \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$ and therefore (cf. [5], [15]) we obtain :

$$\frac{1}{2} \cdot (\partial_t f^2 + v(p) \cdot \nabla_x f^2 + F \cdot \nabla_p f^2) = -\alpha f^2.$$

After integration on $]0,T[\times \Omega \times \mathbb{R}^N_p$ we deduce that :

$$\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} f^2(t,x,p)\ dtdxdp + \frac{1}{2}\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))(\gamma^+ f)^2(t,x,p)\ dtd\sigma dp = 0,$$

or $f = 0$, $\gamma^+ f = 0$. □

We state now the momentum change lemma. The proof is postponed to the Appendix.

LEMMA 2.9. *Assume that Ω is bounded, $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^N$, $\|E\|_\infty > 0$ and consider a velocity function $v : \mathbb{R}^N_p \to \mathbb{R}^N$ given by $v(p) = \frac{p}{|p|} \cdot w(|p|)$, $\forall p \in \mathbb{R}^N_p - \{0\}$, $v(0) = 0$, where $w : [0, +\infty[\to \mathbb{R}$ is a nondecreasing function such that $w(t) \geq C \cdot t^\lambda, \forall t > 0$ for some constants $C > 0, \lambda \geq 0$. Consider $(X(s), P(s)), s_{in} \leq s \leq s_{out}$ an arbitrary solution for :*

$$\frac{dX}{ds} = v(P(s)),\ \ \frac{dP}{ds} = q \cdot E(s, X(s)),\ \ s_{in} \leq s \leq s_{out}.$$

Denote by D the quantity :

$$D = \left(\frac{3(\lambda+1)\cdot diam(\Omega)\cdot |q| \cdot \|E\|_\infty}{C \cdot (1 - 2^{-(\lambda+1)})}\right)^{\frac{1}{\lambda+1}}.$$

Then :

(1) *if there is $t \in [s_{in}, s_{out}]$ such that $|P(t)| > D$, we have :*

$$s_{out} - s_{in} \leq \frac{6\cdot(\lambda+1)diam(\Omega)}{C \cdot |P(t)|^\lambda} \quad and \quad |P(s) - P(t)| \leq D,\ \forall s_{in} \leq s \leq s_{out}\ ;$$

(2) *for all $s_{out} \leq s_1 \leq s_2 \leq s_{in}$ we have $|P(s_1) - P(s_2)| \leq 2D$.*

REMARK 2.10. *We can check easily that the conclusions of the previous lemma still hold if $\|E\|_\infty = 0$.*

Now we can prove the following proposition :

PROPOSITION 2.11. *Under the hypotheses of Lemma 2.9 we suppose that $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$, $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^N$ are T periodic such that $g = g \cdot \mathbf{1}_{\{|p| \leq R\}}$ for some $R > 0$. Then for every $\alpha > 0$, the T periodic mild solution for :*

$$\alpha f + \partial_t f + w(|p|)\frac{p}{|p|} \cdot \nabla_x f + F(t,x) \cdot \nabla_p f = 0,\ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^N_p, \tag{2.8}$$

with the boundary condition :

$$f(t,x,p) = g(t,x,p), \ \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-, \tag{2.9}$$

verifies $f = f \cdot \mathbf{1}_{\{|p| \le R_1\}}, \ \gamma^+ f = \gamma^+ f \cdot \mathbf{1}_{\{|p|\le R_1\}}$ *where* $R_1 = R + 2D$.

Proof. It is a direct consequence of Remarks 2.6, 2.7 and Lemma 2.9. □

REMARK 2.12. *By an easy argument of regularization we can construct a T periodic weak solution for the problem* (2.8), (2.9) *verifying* $f = f \cdot \mathbf{1}_{\{|p|\le R_1\}}, \ \gamma^+ f = \gamma^+ f \cdot \mathbf{1}_{\{|p|\le R_1\}}$ *with* $R_1 = R + 2D$, *when* $g \in L^\infty(\mathbb{R}_t \times \Sigma^-), \ g = g \cdot \mathbf{1}_{\{|p|\le R\}}, \ E \in L^\infty(\mathbb{R}_t \times \Omega)^N$.

We introduce now the energy $\mathcal{E}_\delta(p) = \mathcal{E}(p)(1 + \delta \cdot |p|^\gamma), \ \delta > 0, \gamma \ge 1$ and the corresponding velocity function :

$$v_\delta(p) = \nabla_p \mathcal{E}(p) = v(p)(1 + \delta \cdot |p|^\gamma) + \mathcal{E}(p) \cdot \delta \cdot \gamma \cdot |p|^{\gamma - 2} \cdot p = \frac{p}{|p|} \cdot w_\delta(|p|),$$

where the function w_δ is given in the classical case by :

$$w_\delta(t) = \frac{t}{m} + \frac{\delta \cdot (\gamma + 2)}{2m} \cdot t^{\gamma+1} \ge \frac{\delta \cdot (\gamma+2)}{2m} \cdot t^{\gamma+1}, \forall t \ge 0,$$

and in the relativistic case by :

$$w_\delta(t) = \frac{t}{m}\left(1 + \frac{t^2}{m^2 c_0^2}\right)^{-1/2} + \delta \cdot \left[\left(1 + \frac{t^2}{m^2 c_0^2}\right)^{-1/2} \cdot \frac{t^{\gamma+1}}{m} + \gamma m c_0^2 \cdot \left(\left(1 + \frac{t^2}{m^2 c_0^2}\right)^{1/2} - 1\right) t^{\gamma - 1}\right].$$

Note that in the relativistic case we have for $t \ge m c_0$ that $w_\delta(t) \ge \delta \cdot \gamma \cdot m c_0^2 \left(\left(1 + \frac{t^2}{m^2 c_0^2}\right)^{1/2} - 1\right) \cdot t^{\gamma-1} \ge \delta \cdot \gamma \cdot c_0(\sqrt{2} - 1) \cdot t^\gamma$ and for $t < m c_0$ we can write :

$$w_\delta(t) \ge \frac{t}{m} \cdot \left(1 + \frac{t^2}{m^2 c_0^2}\right)^{-1/2} \ge \frac{c_0}{\sqrt{2}} \cdot \frac{t}{m c_0} \ge \frac{c_0}{\sqrt{2}} \cdot \left(\frac{t}{m c_0}\right)^\gamma.$$

One gets that $w_\delta(t) \ge \min\{\delta \cdot \gamma \cdot c_0(\sqrt{2} - 1), \frac{c_0}{\sqrt{2} \cdot (m c_0)^\gamma}\} \cdot t^\gamma, \forall t \ge 0, \gamma \ge 1$. We have proved that $w_\delta(t) \ge C_\delta \cdot t^\lambda, \ \forall t > 0$ for some constant $C_\delta > 0$ where $\lambda = \gamma + 1$ in the classical case and $\lambda = \gamma$ in the relativistic case. Actually C_δ depends also on λ but this parameter is fixed.

3. The modified Vlasov-Poisson system.

We intend now to analyze the Vlasov-Poisson problem (2.1), (2.2), (1.6), (1.7). We suppose that Ω is an open bounded set of $\mathbb{R}^N_x$, with regular boundary $\partial\Omega \in C^2$. As usual we start by solving a regularized system. We consider the following perturbation of the Vlasov equation :

$$\alpha f + \partial_t f + v_\delta(p) \cdot \nabla_x f - q \cdot (\nabla_x \Phi \star \zeta_\varepsilon) \cdot \nabla_p f = 0, \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^N_p, \tag{3.1}$$

where $\alpha, \delta, \varepsilon > 0$ are small parameters (destinated to be passed to the limit to 0) and γ is a fixed parameter. We denote by C all various constants appearing in our computations, depending on Ω, m, c_0, γ, etc. but not on $\alpha, \delta, \varepsilon$. The term αf ensures the uniqueness of the T periodic weak solution for the problem $(3.1), (2.2)$. In order to construct solutions by characteristics we need also to regularize the electric field by convolution with some mollifier :

$$\zeta_\varepsilon(t,x) = \zeta_{1,\varepsilon}^{per}(t) \cdot \zeta_{2,\varepsilon}(x) = \left[\frac{1}{\varepsilon} \sum_{k \in \mathbb{Z}} \zeta_1 \left(\frac{t - kT}{\varepsilon} \right) \right] \cdot \frac{1}{\varepsilon^N} \zeta_2 \left(\frac{x}{\varepsilon} \right),$$

where $\zeta_1 \in C_c^\infty(\mathbb{R}), \zeta_2 \in C_c^\infty(\mathbb{R}^N), \zeta_1, \zeta_2 \geq 0, \operatorname{supp} \zeta_1 \subset [-1,1], \operatorname{supp} \zeta_2 \subset B(0,1), \int_{\mathbb{R}} \zeta_1(u) du = 1, \int_{\mathbb{R}^N} \zeta_2(v) dv = 1$. It is convenient to work with solutions f compactly supported in momentum p. This is why we replace the energy $\mathcal{E}(p)$ by $\mathcal{E}_\delta(p) = \mathcal{E}(p)(1 + \delta \cdot |p|^\gamma)$ and we consider the corresponding velocity $v_\delta(p) = \nabla_p \mathcal{E}_\delta(p)$ for γ large enough. This point will be clarified below and we will establish an estimate for the momentum support.

PROPOSITION 3.1. *Assume that* $g \in L^\infty(\mathbb{R}_t \times \Sigma^-), \varphi_0 \in L^2(]0,T[; H^{1/2}(\partial\Omega))$ *are* T *periodic and* $g = g \cdot \mathbf{1}_{\{|p| \leq R\}}$ *for some* $R > 0$*. Then, if* $\lambda + 1 > N$ *(which means that* $\gamma + 2 > N$ *in the classical case and* $\gamma + 1 > N$ *in the relativistic case), there is a* T *periodic solution for the modified Vlasov-Poisson problem* $(3.1), (2.2), (1.6), (1.7)$ *with compact support in momentum, uniformly in* $(t,x) \in \mathbb{R}_t \times \Omega$*.*

Proof. We consider the set $\chi = \{\Phi \in L^2(]0,T[; H^1(\Omega)) \mid \Phi(\cdot + T) = \Phi\}$ and we define the application $\mathcal{F} : \chi \to \chi$ by :

$$\Phi \to E = -\nabla_x \Phi \to E_\varepsilon = \overline{E} \star \zeta_\varepsilon \to f \to \rho \to \Phi_1 = \mathcal{F}(\Phi),$$

where :

- E_ε is the convolution in time (T periodic) and space by ζ_ε of $\overline{E}$, the extension of E by 0 outside Ω :

$$E_\varepsilon(t,x) = (\overline{E} \star \zeta_\varepsilon)(t,x) = \int_0^T \int_\Omega E(s,y) \zeta_\varepsilon(t-s, x-y) ds dy, \quad (t,x) \in \mathbb{R}_t \times \Omega \ ;$$

- f is the T periodic mild solution of the perturbed Vlasov problem $(3.1), (2.2)$ associated to the electric field E_ε ;
- $\rho(t,x) = q \int_{\mathbb{R}_p^N} f(t,x,p) dp$ is the charge density of f ;
- Φ_1 is the solution of the Poisson problem associated to the second member $\frac{1}{\varepsilon_0} \rho$.

By regularization we have :

$$\begin{aligned}|E_\varepsilon(t,x)| &= \left| \int_0^T\int_\Omega E(s,y)\zeta_{1,\varepsilon}^{per}(t-s)\cdot\zeta_{2,\varepsilon}(x-y)dsdy \right| \\ &\leq \left(\int_0^T\int_\Omega |E(s,y)|^2 dsdy\right)^{1/2}\cdot\left(\int_0^T\int_\Omega |\zeta_{1,\varepsilon}^{per}(t-s)\cdot\zeta_{2,\varepsilon}(x-y)|^2 dsdy\right)^{1/2} \\ &\leq \|\Phi\|_{L^2(]0,T[;H^1(\Omega))}\frac{\|\zeta_1\|_\infty^{1/2}\cdot\|\zeta_2\|_\infty^{1/2}}{\varepsilon^{(N+1)/2}},\end{aligned}$$

and thus $\|E_\varepsilon\|_\infty \leq \|\Phi\|_{L^2(]0,T[;H^1(\Omega))}\frac{\|\zeta_1\|_\infty^{1/2}\cdot\|\zeta_2\|_\infty^{1/2}}{\varepsilon^{(N+1)/2}}$. By Proposition 2.11 we deduce that $f = f\cdot\mathbf{1}_{\{|p|\leq R+2D_\delta^\varepsilon\}}$ with :

$$D_\delta^\varepsilon = \left(\frac{3(\lambda+1)\cdot diam(\Omega)\cdot|q|\cdot\|E_\varepsilon\|_\infty}{C_\delta\cdot(1-2^{-(\lambda+1)})}\right)^{\frac{1}{\lambda+1}}.$$

We deduce that $\|\rho(t)\|_{L^2(\Omega)} \leq |q|\cdot(meas(\Omega)^{1/2}\cdot\|g\|_\infty\cdot meas(B_{\mathbb{R}_p^N}(0,1))\cdot(R+2D_\delta^\varepsilon)^N$ which implies that :

$$\begin{aligned}\|\Phi_1(t)\|_{H^1(\Omega)} &\leq C(\Omega)\left(\frac{1}{\varepsilon_0}\|\rho(t)\|_{L^2(\Omega)} + \|\varphi_0(t)\|_{H^{1/2}(\partial\Omega)}\right) \\ &\leq C\cdot\left((R+2D_\delta^\varepsilon)^N + \|\varphi_0(t)\|_{H^{1/2}(\partial\Omega)}\right).\end{aligned}$$

Finally one gets :

$$\begin{aligned}\|\mathcal{F}(\Phi)\|_{L^2(]0,T[;H^1(\Omega))} &= \|\Phi_1\|_{L^2(]0,T[;H^1(\Omega))} \leq C(\Omega,T,R,\delta,\gamma,\varepsilon)\cdot(1+\|\Phi\|_{L^2(]0,T[;H^1(\Omega))}^{\frac{N}{\lambda+1}} \\ &\quad + \|\varphi_0\|_{L^2(]0,T[;H^{1/2}(\partial\Omega))}).\end{aligned}$$

Since $\lambda+1 > N$ we can take M large enough such that :

$$C(\Omega,T,R,\delta,\gamma,\varepsilon)\cdot(1+M^{N/(\lambda+1)} + \|\varphi_0\|_{L^2(]0,T[;H^{1/2}(\partial\Omega))}) \leq M,$$

and thus the convex subset $\mathcal{C} = \{\Phi\in\chi \mid \|\Phi\|_{L^2(]0,T[;H^1(\Omega))}\leq M\}$ is left invariant by $\mathcal{F}$, $\mathcal{F}(\mathcal{C})\subset\mathcal{C}$. In order to apply the Schauder fixed point theorem it remains to prove that $\mathcal{F}$ is continuous with respect to the weak topology of $L^2(]0,T[;H^1(\Omega))$ (since $\mathcal{C}$ is weakly compact in $L^2(]0,T[;H^1(\Omega))$). Consider $\Phi_k \rightharpoonup \Phi$ weakly in $L^2(]0,T[;H^1(\Omega))$ and thus $E_k = -\nabla_x\Phi_k \rightharpoonup -\nabla_x\Phi = E$ weakly in $L^2(]0,T[;L^2(\Omega)^N)$. We deduce the pointwise convergence :

$$(\overline{E_k}\star\zeta_\varepsilon)(t,x)\to(\overline{E}\star\zeta_\varepsilon)(t,x),\ \ \forall(t,x)\in\mathbb{R}_t\times\Omega,$$

and by dominated convergence theorem we have :

$$\overline{E_k}\star\zeta_\varepsilon\to\overline{E}\star\zeta_\varepsilon,\ \ \text{in } L^2(]0,T[;L^2(\Omega)^N).$$

Denote by f_k, f the T periodic mild solutions for :

$$\alpha f_k + \partial_t f_k + v_\delta(p)\cdot\nabla_x f_k + q\cdot(\overline{E_k}\star\zeta_\varepsilon)\cdot\nabla_p f_k = 0,\ \ (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}_p^N,$$

and

$$\alpha f + \partial_t f + v_\delta(p) \cdot \nabla_x f + q \cdot (\overline{E} \star \zeta_\varepsilon) \cdot \nabla_p f = 0, \;\; (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^N_p,$$

with the boundary conditions $f_k = g, \; (t,x,p) \in \mathbb{R}_t \times \Sigma^-$ respectively $f = g, \; (t,x,p) \in \mathbb{R}_t \times \Sigma^-$. Since $(f_k)_k$ is bounded in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$, $\|f_k\|_\infty \leq \|g\|_\infty \; \forall k$ we can suppose after extraction that $f_k \rightharpoonup \tilde{f}$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$. Take $\varphi \in \mathcal{T}_w$ and observe that :

$$\int_{\mathbb{R}^N_p} f_k(t,x,p) \nabla_p \varphi \; dp \rightharpoonup \int_{\mathbb{R}^N_p} \tilde{f}(t,x,p) \nabla_p \varphi \; dp, \;\; \text{weakly in } L^2(]0,T[; L^2(\Omega)^N).$$

By combining with the strong convergence of $(\overline{E_k} \star \zeta_\varepsilon)_k$ we deduce that $\tilde{f}$ is the unique T periodic weak solution (and therefore the T periodic mild solution) of the perturbed Vlasov problem associated to the field $\overline{E} \star \zeta_\varepsilon$ and thus $\tilde{f} = f$. Moreover, all the sequence $(f_k)_k$ converges weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$ to f. In order to pass to the limit in the Poisson equation, observe that since $(\overline{E_k} \star \zeta_\varepsilon)_k$ is uniformly bounded in $L^\infty(\mathbb{R}_t \times \Omega)^N$:

$$\|\overline{E_k} \star \zeta_\varepsilon\|_\infty \leq \frac{M \cdot \|\zeta_1\|_\infty^{1/2} \cdot \|\zeta_2\|_\infty^{1/2}}{\varepsilon^{(N+1)/2}},$$

then $(f_k)_k$ is uniformly compactly supported in momentum and therefore $\rho_k \rightharpoonup \rho$ weakly in $L^r(]0,T[\times\Omega)$, $\forall \; 1 < r < +\infty$. In particular, by taking $r = 2$ we deduce that $\mathcal{F}(\Phi_k) \rightharpoonup \mathcal{F}(\Phi)$ weakly in $L^2(]0,T[; H^1(\Omega))$. By applying the Schauder fixed point theorem, we deduce that there is (f, Φ) a T periodic solution for the perturbed Vlasov-Poisson system. Since $\overline{E} \star \zeta_\varepsilon$ is bounded, $\|\overline{E} \star \zeta_\varepsilon\|_\infty \leq M \cdot \|\zeta_1\|_\infty^{1/2} \cdot \|\zeta_2\|_\infty^{1/2} \cdot \varepsilon^{-(N+1)/2}$, we deduce that f is compactly supported in momentum. □

In order to pass to the limit for $\varepsilon \to 0$ we prove that $E_\varepsilon = -\nabla_x \Phi_\varepsilon$ are uniformly bounded in $L^\infty(\mathbb{R}_t \times \Omega)^N$ and f_ε are compactly supported in momentum, uniformly in $(t,x) \in \mathbb{R}_t \times \Omega$ and $\varepsilon > 0$. We use the decomposition $\Phi = \Phi_s + \Phi_0$ where Φ_s is the self-consistent potential solving $-\Delta \Phi_s = \frac{1}{\varepsilon_0} \rho$, $(t,x) \in \mathbb{R}_t \times \Omega$ and $\Phi_s|_{\mathbb{R}_t \times \partial\Omega} = 0$ and Φ_0 is the potential induced by the boundary condition : $-\Delta \Phi_0 = 0$, $(t,x) \in \mathbb{R}_t \times \Omega$ and $\Phi_0|_{\mathbb{R}_t \times \partial\Omega} = \varphi_0$. Sometimes we write $E = E_s + E_0$ with $E_s = -\nabla_x \Phi_s$, $E_0 = -\nabla_x \Phi_0$.

PROPOSITION 3.2. *Assume that $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$, $\varphi_0 \in L^\infty(\mathbb{R}_t; W^{2-1/p,p}(\partial\Omega))$ for some $p > N$ are T periodic and that $g = g \cdot \mathbf{1}_{\{|p| \leq R\}}$ for some $R > 0$. Then, if $\lambda + 1 > N$, the T periodic solutions $(f_\varepsilon, E_\varepsilon = -\nabla_x \Phi_\varepsilon)$ constructed in Proposition* 3.1 *verify :*

$$\sup_{\varepsilon > 0} \|E_\varepsilon\|_{L^\infty(\mathbb{R}_t \times \Omega)} < +\infty, \;\; f_\varepsilon = f_\varepsilon \cdot \mathbf{1}_{\{|p| \leq R + 2D_\delta\}}, \;\; \gamma^+ f_\varepsilon = \gamma^+ f_\varepsilon \cdot \mathbf{1}_{\{|p| \leq R + 2D_\delta\}},$$

where $D_\delta = \sup_{\varepsilon > 0} D_\delta^\varepsilon < +\infty$ and :

$$\alpha \rho_\varepsilon + \partial_t \rho_\varepsilon + \nabla_x \cdot j_\varepsilon = 0, \quad \textit{in } \mathcal{D}'(]0,T[\times\Omega).$$

Proof. By elliptic regularity we have :

$$\|\nabla_x \Phi_0\|_{L^\infty(\mathbb{R}_t;W^{1,p}(\Omega))} \leq C(p,\Omega)\cdot \|\varphi_0\|_{L^\infty(\mathbb{R}_t;W^{2-1/p,p}(\partial\Omega))},$$

and by Sobolev imbeddings, since $p > N$ we deduce that $\|E_0\|_\infty \leq C \cdot \|\varphi_0\|_{L^\infty(\mathbb{R}_t;W^{2-1/p,p}(\partial\Omega))}$. For the self-consistent field we have also :

$$\begin{aligned}\|E_{\varepsilon,s}\|_{L^\infty(\mathbb{R}_t;W^{1,p}(\Omega))} \leq & C(p,\Omega)\cdot \varepsilon_0^{-1}\cdot \|\rho_\varepsilon\|_{L^\infty(\mathbb{R}_t;L^p(\Omega))}\\ \leq & \frac{C(p,\Omega)\cdot |q|\cdot meas(\Omega)^{1/p}}{\varepsilon_0}\cdot \|g\|_\infty \cdot meas(B_{\mathbb{R}_p^N}(0,1))\cdot (R+2D_\delta^\varepsilon)^N,\end{aligned}$$

and by using one more time Sobolev imbeddings we find that :

$$\begin{aligned}\|E_{\varepsilon,s}\|_\infty \leq & C(p,\Omega,\varepsilon_0,g,\delta,\gamma,R)\cdot \left(1+\|\overline{E_\varepsilon}\star\zeta_\varepsilon\|_\infty^{\frac{N}{\lambda+1}}\right)\\ \leq & C\cdot\left(1+\|E_\varepsilon\|_\infty^{\frac{N}{\lambda+1}}\right)\\ \leq & C\cdot\left(1+\|E_0\|_\infty^{\frac{N}{\lambda+1}}+\|E_{\varepsilon,s}\|_\infty^{\frac{N}{\lambda+1}}\right).\end{aligned}$$

Since $\lambda+1 > N$ we deduce that $(E_{\varepsilon,s})_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\mathbb{R}_t\times\Omega)^N$ as well as $E_\varepsilon = E_{\varepsilon,s}+E_0$. Moreover, if $|p| > R+2D_\delta \geq R+2D_\delta^\varepsilon$ we have $f_\varepsilon(t,x,p)=0,\ \forall (t,x)\in\mathbb{R}_t\times\Omega,\ \forall\varepsilon>0$. Similarly we have $\gamma^+ f_\varepsilon = \gamma^+ f_\varepsilon\cdot \mathbf{1}_{\{|p|\leq R+2D_\delta\}}$. In order to prove the continuity equation take $\varphi\in C_c^1(]0,T[\times\Omega)$, and by applying the weak formulation with the test function $\varphi(t,x)\cdot\chi(|p|/R_2)$ where $R_2 > R+2D_\delta,\ \chi\in C_c^\infty(\mathbb{R}),\ \chi(u)=1, |u|\leq 1, \chi(u)=0,\ |u|\geq 2,\ 0\leq\chi(u)\leq 1,\ \forall u\in\mathbb{R}$ we find :

$$\alpha\int_0^T\!\!\int_\Omega \rho_\varepsilon\varphi\ dtdx = \int_0^T\!\!\int_\Omega \rho_\varepsilon\partial_t\varphi\ dtdx + \int_0^T\!\!\int_\Omega j_\varepsilon\cdot\nabla_x\varphi\ dtdx, \tag{3.2}$$

which means that $\alpha\rho_\varepsilon+\partial_t\rho_\varepsilon+\nabla_x\cdot j_\varepsilon = 0$ in $\mathcal{D}'(]0,T[\times\Omega)$ (here $j_\varepsilon = q\int_{\mathbb{R}_p^N} f_\varepsilon v_\delta(p)\ dp$). Actually the equality (3.2) holds for all T periodic C^1 test function compactly supported in Ω. □

In the following proposition we justify the limit for $\varepsilon\to 0$.

PROPOSITION 3.3. *Assume that $g\geq 0, \varphi_0$ are T periodic with $g\in L^\infty(\mathbb{R}_t\times\Sigma^-)$, $g = g\cdot\mathbf{1}_{\{|p|\leq R\}}$ for some $R>0$, $\varphi_0\in L^\infty(\mathbb{R}_t;W^{2-1/p,p}(\partial\Omega))$, $\partial_t\varphi_0\in L^\infty(\mathbb{R}_t;W^{1-1/p,p}(\partial\Omega))$ for some $p>N, \lambda+1>N$. Then there is a T periodic solution for the perturbed Vlasov-Poisson problem :*

$$\alpha f+\partial_t f+v_\delta(p)\cdot\nabla_x f - q\nabla_x\Phi\cdot\nabla_p f = 0,\ \ (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}_p^N, \tag{3.3}$$

$$-\Delta_x\Phi = \frac{1}{\varepsilon_0}\rho(t,x),\ \ (t,x)\in\mathbb{R}_t\times\Omega,$$

with the boundary conditions :

$$f(t,x,p) = g(t,x,p),\ (t,x,p) \in \mathbb{R}_t \times \Sigma^-,\ \ \Phi(t,x) = \varphi_0(t,x),\ (t,x) \in \mathbb{R}_t \times \partial\Omega.$$

Moreover the electric field verifies $-\nabla_x \Phi \in L^\infty(\mathbb{R}_t \times \Omega)^N$, $\partial_t \Phi \in L^\infty(\mathbb{R}_t \times \Omega)$ *and* $f = f \cdot \mathbf{1}_{\{|p|\leq R_1\}}$, $\gamma^+ f = \gamma^+ f \cdot \mathbf{1}_{\{|p|\leq R_1\}}$ *for some* $R_1 > 0$.

Proof. Take $(f_\varepsilon, \Phi_\varepsilon)$ the T periodic solutions constructed before, for $\varepsilon > 0$. By Proposition 3.2 we have $\sup_{\varepsilon>0} \|E_\varepsilon\|_\infty = M < +\infty$ and $f_\varepsilon = f_\varepsilon \cdot \mathbf{1}_{\{|p|\leq R_1\}}$ with $R_1 = R + 2\sup_{\varepsilon>0} D^\varepsilon_\delta$. We have also :

$$\sup_{\varepsilon>0} \|\rho_\varepsilon\|_{L^\infty(\mathbb{R}_t;L^p(\Omega))} < +\infty,\ \ \sup_{\varepsilon>0} \|j_\varepsilon\|_{L^\infty(\mathbb{R}_t;L^p(\Omega))} < +\infty,$$

$$\sup_{\varepsilon>0} \|\Phi_\varepsilon\|_{L^\infty(\mathbb{R}_t;W^{2,p}(\Omega))} < +\infty,\ \ \sup_{\varepsilon>0} \|\partial_t\Phi_\varepsilon\|_{L^\infty(\mathbb{R}_t;W^{1,p}(\Omega))} < +\infty.$$

Indeed, for the last estimate we write $\partial_t\Phi_\varepsilon = \partial_t\Phi_{\varepsilon,s} + \partial_t\Phi_0$ with $-\Delta_x\partial_t\Phi_{\varepsilon,s} = \frac{1}{\varepsilon_0}\partial_t\rho_\varepsilon = -\frac{1}{\varepsilon_0}(\alpha\rho_\varepsilon + \nabla_x \cdot j_\varepsilon)$, $(t,x) \in \mathbb{R}_t \times \Omega$, $\partial_t\Phi_{\varepsilon,s} = 0$, $(t,x) \in \mathbb{R}_t \times \partial\Omega$ and thus we deduce that :

$$\begin{aligned}\|\partial_t\Phi_{\varepsilon,s}\|_{L^\infty(\mathbb{R}_t;W^{1,p}(\Omega))} &\leq C \cdot \|\alpha\rho_\varepsilon + \nabla_x \cdot j_\varepsilon\|_{L^\infty(\mathbb{R}_t;W^{-1,p}(\Omega))} \\ &\leq C \cdot \left(\|\rho_\varepsilon\|_{L^\infty(\mathbb{R}_t;L^p(\Omega))} + \|j_\varepsilon\|_{L^\infty(\mathbb{R}_t;L^p(\Omega))}\right). \end{aligned} \tag{3.4}$$

By using $-\Delta_x\partial_t\Phi_0 = 0$, $(t,x) \in \mathbb{R}_t \times \Omega$, $\partial_t\Phi_0 = \partial_t\varphi_0$, $(t,x) \in \mathbb{R}_t \times \partial\Omega$ we have :

$$\|\partial_t\Phi_0\|_{L^\infty(\mathbb{R}_t;W^{1,p}(\Omega))} \leq C \cdot \|\partial_t\varphi_0\|_{L^\infty(\mathbb{R}_t;W^{1-1/p,p}(\partial\Omega))}.$$

We can extract a subsequence $(\varepsilon_k)_k$ such that the following convergences for $(f_k, \Phi_k) := (f_{\varepsilon_k}, \Phi_{\varepsilon_k})$ hold : $f_k \rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$, $\gamma^+ f_k \rightharpoonup \gamma^+ f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Sigma^+)$, $E_k \rightharpoonup E$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega)^N$, $\partial_t\Phi_k \rightharpoonup \partial_t\Phi$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega)$. Since Φ_k and $\partial_t\Phi_k$ are uniformly bounded in $L^2(]0,T[;W^{2,p}(\Omega))$ respectively in $L^2(]0,T[;W^{1,p}(\Omega))$, by using an interpolation lemma due to Aubin [3] we deduce (after extraction) the strong convergence $\Phi_k \to \Phi$ in $L^2(]0,T[;W^{1,p}(\Omega))$ and therefore $E_k \to E$ strongly in $L^2(]0,T[;L^p(\Omega)^N)$ and $L^2(]0,T[;L^2(\Omega)^N)$. We deduce that $\|E\|_\infty \leq \liminf_{k\to+\infty} \|E_k\|_\infty \leq M$ and $\|\partial_t\Phi\|_\infty \leq \liminf_{k\to+\infty} \|\partial_t\Phi_k\|_\infty \leq C \cdot \sup_k \|\partial_t\Phi_k\|_{L^\infty(\mathbb{R}_t;W^{1,p}(\Omega))}$. Since $f_k = f_k \cdot \mathbf{1}_{\{|p|\leq R_1\}}, \gamma^+ f_k = \gamma^+ f_k \cdot \mathbf{1}_{\{|p|\leq R_1\}}$ we deduce also that $f = f \cdot \mathbf{1}_{\{|p|\leq R_1\}}, \gamma^+ f = \gamma^+ f \cdot \mathbf{1}_{\{|p|\leq R_1\}}$. As usual, in order to check that f is T periodic weak solution, take $\varphi \in \mathcal{T}_w$ and remark that :

$$\int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} f_k(\alpha\varphi - \partial_t\varphi - v_\delta(p) \cdot \nabla_x\varphi)dtdxdp \to \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} f(\alpha\varphi - \partial_t\varphi - v_\delta(p) \cdot \nabla_x\varphi)dtdxdp.$$

For the nonlinear term combine the strong convergence $\lim_{k\to\infty}(\overline{E}_k \star \zeta_{\varepsilon_k}) = E$ in $L^2(]0,T[;L^2(\Omega)^N)$ with the weak convergence $\lim_{k\to+\infty} \int_{\mathbb{R}^N_p} f_k \nabla_p\varphi\ dp = \int_{\mathbb{R}^N_p} f\nabla_p\varphi\ dp$ in $L^2(]0,T[;L^2(\Omega)^N)$. Thus

f is T periodic weak solution corresponding to the field $E = -\nabla_x \Phi$. By using the Green formula we deduce that $\gamma^+ f$ is the trace on $\mathbb{R}_t \times \Sigma^+$ of f and by weak $\star$ convergence we have $\|f\|_\infty \leq \liminf_{k\to+\infty} \|f_k\|_\infty \leq \|g\|_\infty$ and $\|\gamma^+ f\|_\infty \leq \liminf_{k\to+\infty} \|\gamma^+ f_k\|_\infty \leq \|g\|_\infty$. We check easily that $\rho_k = q\int_{\mathbb{R}^N_p} f_k \; dp \rightharpoonup q\int_{\mathbb{R}^N_p} f \; dp = \rho$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega)$. In particular $\rho_k \rightharpoonup \rho$ weakly in $L^2(]0,T[\times\Omega)$ and therefore $\Phi_k \rightharpoonup \tilde{\Phi}$ weakly in $L^2(]0,T[;H^1(\Omega))$, where $\tilde{\Phi}$ is the solution of the Poisson problem with second member ρ. Since $\Phi_k \to \Phi$ strongly in $L^2(]0,T[;W^{1,p}(\Omega))$ we deduce finally that $\tilde{\Phi} = \Phi$. $\square$

4. A priori estimates.

In this section we establish a priori estimates for the solutions of $(3.3), (2.2), (1.6), (1.7)$. We suppose that $\alpha, \delta, \gamma > 0$ are fixed and we denote by (f, Φ) the corresponding T periodic solution. The computations follow by multiplying the perturbed Vlasov equation by $1, p$ and $\mathcal{E}_\delta(p)$.

PROPOSITION 4.1. *Under the hypotheses of Proposition* 3.3*, the T periodic solution (f,Φ) of* $(3.3), (2.2), (1.6), (1.7)$ *verifies :*

$$\alpha\rho + \partial_t \rho + \nabla_x \cdot j_\delta = 0, \quad in \;\; \mathcal{D}'(]0,T[\times\Omega),$$

where $\rho = q\int_{\mathbb{R}^N_p} f dp$, $j_\delta = q\int_{\mathbb{R}^N_p} v_\delta(p) f dp$*. Moreover we have :*

$$\alpha \int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\; dx dp + \frac{d}{dt}\int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\; dx dp + \int_\Sigma (v_\delta(p)\cdot n(x))\gamma f \; d\sigma dp = 0, \quad a.e. \; t, \quad (4.1)$$

and :

$$\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\; dt dx dp + \int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))\gamma^+ f \; dt d\sigma dp = -\int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x)) g \; dt d\sigma dp. \quad (4.2)$$

Proof. Take the test function $\varphi \in C^1_c(]0,T[\times\Omega)$ and by applying the Green formula (f has compact support in momentum) we deduce that $\alpha\rho + \partial_t\rho + \nabla_x \cdot j_\delta = 0$ in $\mathcal{D}'$. By considering $\varphi(t,x) = \theta(t)$ with $\theta \in C^1_c(]0,T[)$ we have :

$$\alpha \int_0^T \theta(t) \int_\Omega \rho(t,x) dt dx + \int_0^T \theta(t) \int_\Sigma (v_\delta(p)\cdot n(x))\gamma f dt d\sigma dp = \int_0^T \theta'(t) \int_\Omega \rho(t,x) dt dx,$$

saying that $\alpha\int_\Omega \rho(t)\; dx + \frac{d}{dt}\int_\Omega \rho \; dx + \int_{\Sigma^+} (v_\delta(p)\cdot n(x))\gamma^+ f \; d\sigma dp = -\int_{\Sigma^-} (v_\delta(p)\cdot n(x)) g \; d\sigma dp$ in $\mathcal{D}'(]0,T[)$ and the conclusion follows. $\square$

PROPOSITION 4.2. *Under the hypotheses of Proposition* 3.3 *we have for a.e.* $t \in \mathbb{R}_t$ *:*

$$\alpha \int_\Omega \int_{\mathbb{R}_p^N} f(t,x,p)\mathcal{E}_\delta(p)dxdp + \frac{d}{dt}\int_\Omega \int_{\mathbb{R}_p^N} f(t,x,p)\mathcal{E}_\delta(p)dxdp + \int_{\Sigma^+}(v_\delta(p)\cdot n(x))\gamma^+ f\mathcal{E}_\delta(p)d\sigma dp$$
$$= -\int_{\Sigma^-}(v_\delta(p)\cdot n(x))g\mathcal{E}_\delta(p)d\sigma dp - \int_\Omega \nabla_x\Phi\cdot j_\delta(t,x)dx, \tag{4.3}$$

and :

$$\alpha \int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f(t,x,p)\mathcal{E}_\delta(p)\ dtdxdp + \int_0^T\int_{\Sigma^+}(v_\delta(p)\cdot n(x))\gamma^+ f\mathcal{E}_\delta(p)\ dtd\sigma dp$$
$$= -\int_0^T\int_{\Sigma^-}(v_\delta(p)\cdot n(x))g\mathcal{E}_\delta(p)\ dtd\sigma dp - \int_0^T\int_\Omega \nabla_x\Phi\cdot j_\delta(t,x)\ dtdx. \tag{4.4}$$

Proof. By using the Green formula with the test function $\varphi(t,x) = \theta(t)\mathcal{E}_\delta(p), \theta \in C_c^1(]0,T[)$ we deduce :

$$\alpha \int_0^T \theta(t)\int_\Omega\int_{\mathbb{R}_p^N} f\mathcal{E}_\delta dtdxdp + \int_0^T\theta(t)\int_{\Sigma^+}(v_\delta(p)\cdot n(x))\gamma^+ f\mathcal{E}_\delta dtd\sigma dp = -\int_0^T\theta(t)\int_{\Sigma^-}(v_\delta(p)\cdot n(x))g\mathcal{E}_\delta$$
$$+\int_0^T\theta'(t)\int_\Omega\int_{\mathbb{R}_p^N} f\mathcal{E}_\delta dtdxdp - \int_0^T\theta(t)\int_\Omega \nabla_x\Phi\cdot j_\delta dtdx. \tag{4.5}$$

Therefore we obtain

$$\alpha \int_\Omega\int_{\mathbb{R}_p^N} f\mathcal{E}_\delta dxdp + \frac{d}{dt}\int_\Omega\int_{\mathbb{R}_p^N} f\mathcal{E}_\delta dxdp + \int_{\Sigma^+}(v_\delta(p)\cdot n(x))\gamma^+ f\mathcal{E}_\delta d\sigma dp = -\int_{\Sigma^-}(v_\delta(p)\cdot n(x))g\mathcal{E}_\delta d\sigma dp$$
$$-\int_\Omega \nabla_x\Phi\cdot j_\delta\ dx,$$

in $\mathcal{D}'(]0,T[)$ and the conclusion follows. □

PROPOSITION 4.3. *Under the hypotheses of Proposition* 3.3 *we have :*

$$\alpha\int_\Omega \rho(t,x)\Phi_s(t,x)dx + \frac{d}{dt}\int_\Omega\rho(t,x)\Phi_s(t,x)dx = \int_\Omega \rho(t)\partial_t\Phi_s dx + \int_\Omega j_\delta(t)\cdot\nabla_x\Phi_s dx,\ a.e.\ t\in\mathbb{R}_t, \tag{4.6}$$

$$\alpha\int_0^T\int_\Omega \rho(t,x)\Phi_s(t,x)\ dtdx = \int_0^T\int_\Omega \rho(t,x)\partial_t\Phi_s\ dtdx + \int_0^T\int_\Omega j_\delta(t,x)\cdot\nabla_x\Phi_s\ dtdx, \tag{4.7}$$

and :

$$\alpha\int_\Omega\rho(t,x)\Phi_0(t,x)dx + \frac{d}{dt}\int_\Omega\rho(t,x)\Phi_0(t,x)dx + \int_{\Sigma^+}(v_\delta(p)\cdot n(x))q\gamma^+ f\varphi_0 d\sigma dp$$
$$= -\int_{\Sigma^-}(v_\delta(p)\cdot n(x))qg\varphi_0 d\sigma dp + \int_\Omega\rho(t,x)\partial_t\Phi_0\ dx + \int_\Omega j_\delta(t,x)\cdot\nabla_x\Phi_0\ dx,\quad a.e.\ t\in\mathbb{R}_t, \tag{4.8}$$

$$\alpha\int_0^T\int_\Omega\rho(t,x)\Phi_0(t,x)\ dtdx + \int_0^T\int_{\Sigma^+}(v_\delta(p)\cdot n(x))q\gamma^+ f\varphi_0\ dtd\sigma dp = -\int_0^T\int_{\Sigma^-}(v_\delta(p)\cdot n(x))qg\varphi_0\ dtd\sigma dp$$
$$+\int_0^T\int_\Omega\rho(t,x)\partial_t\Phi_0\ dtdx + \int_0^T\int_\Omega j_\delta(t,x)\cdot\nabla_x\Phi_0\ dtdx. \tag{4.9}$$

Proof. From Proposition 3.3 we know that $\Phi_s, \Phi_0 \in W^{1,\infty}(\mathbb{R}_t \times \Omega)$ and by an easy density argument we can check that the weak formulation and Green formula still hold for T periodic $W^{1,\infty}$ test functions. Take the test function $q\theta(t) \cdot \Phi_s(t,x) \in W^{1,\infty}(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$ where $\theta \in C^1_c(]0,T[)$ and we write :

$$\alpha \int_0^T \theta(t) \int_\Omega \int_{\mathbb{R}^N_p} qf\Phi_s dtdxdp = \int_0^T \theta'(t) \int_\Omega \int_{\mathbb{R}^N_p} qf\Phi_s dtdxdp + \int_0^T \theta(t) \int_\Omega \int_{\mathbb{R}^N_p} qfv_\delta(p) \cdot \nabla_x \Phi_s dtdxdp + \int_0^T \theta(t) \int_\Omega \int_{\mathbb{R}^N_p} qf\partial_t \Phi_s dtdxdp. \quad (4.10)$$

Thus $\alpha \int_\Omega \rho(t,x)\Phi_s(t,x)dx + \frac{d}{dt}\int_\Omega \rho\Phi_s dx = \int_\Omega (\rho(t,x)\partial_t\Phi_s + j_\delta(t,x) \cdot \nabla_x \Phi_s)dx$ in $\mathcal{D}'(]0,T[)$. We deduce also (4.7) by periodicity, after integration of (4.6) on $]0,T[$. The equalities (4.8), (4.9) follow in the same manner.

□

Let us establish now an estimate for the outgoing kinetic energy $\int_0^T \int_\Omega \int_{\mathbb{R}^N_p} (v_\delta(p) \cdot n(x))\gamma^+ f\mathcal{E}_\delta(p)$.

PROPOSITION 4.4. *Under the hypotheses of Proposition* 3.3 *we have :*

$$\int_0^T \int_{\Sigma^+} (v_\delta(p) \cdot n(x))\gamma^+ f\mathcal{E}_\delta(p) dtd\sigma dp \le -\int_0^T \int_{\Sigma^-} (v_\delta(p) \cdot n(x)) g(2|q| \cdot \|\varphi_0\|_\infty + \mathcal{E}_\delta(p))dtd\sigma dp + \varepsilon_0 \cdot \|\partial_n \Phi\|_{L^2(]0,T[\times\partial\Omega)} \cdot (\alpha \|\varphi_0\|_{L^2(]0,T[\times\partial\Omega)} + \|\partial_t\varphi_0\|_{L^2(]0,T[\times\partial\Omega)}). \quad (4.11)$$

Proof. By adding the equalities (4.3), (4.6), (4.8) we deduce that :

$$\alpha \int_\Omega \int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)dxdp + \alpha \int_\Omega \rho(\Phi_s + \Phi_0)dx + \frac{d}{dt}\int_\Omega \int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)dxdp + \frac{d}{dt}\int_\Omega \rho(\Phi_s+\Phi_0)dx + \int_{\Sigma^+} (v_\delta(p)\cdot n(x))\gamma^+ f(\mathcal{E}_\delta(p) + q\varphi_0)d\sigma dp = -\int_{\Sigma^-}(v_\delta(p)\cdot n(x))g(\mathcal{E}_\delta(p)+q\varphi_0)d\sigma dp + \int_\Omega \rho(\partial_t\Phi_s + \partial_t\Phi_0)dx. \quad (4.12)$$

Since $\partial_t\Phi = \partial_t\Phi_0 + \partial_t\Phi_s \in L^\infty(\mathbb{R}_t; W^{1,p}(\Omega))$ and $\Phi \in L^\infty(\mathbb{R}_t; W^{2,p}(\Omega))$ we can write :

$$\int_\Omega \nabla_x\Phi \cdot \nabla_x\partial_t\Phi dx - \int_{\partial\Omega} \partial_n\Phi(t,x)\partial_t\varphi_0 \, d\sigma = \frac{1}{\varepsilon_0}\int_\Omega \rho(t,x)\partial_t\Phi dx.$$

Since $\nabla_x\Phi \in L^2(]0,T[;L^2(\Omega)^N)$, $\partial_t\nabla_x\Phi \in L^2(]0,T[;L^2(\Omega)^N)$ we have for a.e. $t \in \mathbb{R}_t$:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla_x\Phi|^2(t,x)dx = \int_\Omega \nabla_x\Phi(t,x)\cdot\partial_t\nabla_x\Phi(t,x)dx,$$

and thus we deduce that :

$$\int_\Omega \rho(t,x)\partial_t\Phi dx = \varepsilon_0 \int_\Omega \nabla_x\Phi\cdot\nabla_x\partial_t\Phi \, dx - \varepsilon_0\int_{\partial\Omega}\partial_n\Phi(t)\partial_t\varphi_0 \, d\sigma = \frac{\varepsilon_0}{2}\frac{d}{dt}\int_\Omega |\nabla_x\Phi|^2(t,x)\,dx - \varepsilon_0\int_{\partial\Omega}\partial_n\Phi(t)\partial_t\varphi_0 \, d\sigma. \quad (4.13)$$

Note also that :

$$\int_{\Omega} \rho(t,x)\Phi(t,x)\, dx = \varepsilon_0 \int_{\Omega} |\nabla_x \Phi|^2(t,x)\, dx - \varepsilon_0 \int_{\partial\Omega} \partial_n \Phi(t)\varphi_0(t,x)\, d\sigma. \tag{4.14}$$

Finally, by integration of (4.12) on $]0,T[$ and by using (4.13), (4.14) we deduce that :

$$\alpha \int_0^T\!\!\int_{\Omega}\int_{\mathbb{R}_p^N} f\mathcal{E}_\delta(p) dtdxdp + \alpha\varepsilon_0 \int_0^T\!\!\int_{\Omega} |\nabla_x\Phi|^2 dtdx + \int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))\gamma^+ f(\mathcal{E}_\delta(p) + q\cdot\varphi_0) dtd\sigma dp$$
$$= \varepsilon_0 \int_0^T\!\!\int_{\partial\Omega} \partial_n\Phi(\alpha\,\varphi_0 - \partial_t\varphi_0)\, dtd\sigma - \int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x)) g(q\varphi_0 + \mathcal{E}_\delta) dtd\sigma dp, \tag{4.15}$$

and therefore :

$$\int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))(\mathcal{E}_\delta(p) + q\cdot\varphi_0)\gamma^+ f\; dtd\sigma dp \le -\int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x))(\mathcal{E}_\delta(p) + q\cdot\varphi_0) g\; dtd\sigma dp$$
$$-\varepsilon_0 \int_0^T\!\!\int_{\partial\Omega} \partial_n\Phi\cdot\partial_t\varphi_0\; dtd\sigma + \alpha\varepsilon_0 \int_0^T\!\!\int_{\partial\Omega} \partial_n\Phi\cdot\varphi_0(t,x)\; dtd\sigma. \tag{4.16}$$

By using (4.2) we obtain :

$$\int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))\;\; \mathcal{E}_\delta(p)\gamma^+ f\; dtd\sigma dp \le -\int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x)) g(2|q|\cdot\|\varphi_0\|_\infty + \mathcal{E}_\delta) dtd\sigma dp$$
$$+\varepsilon_0 \|\partial_n\Phi\|_{L^2(]0,T[\times\partial\Omega)}(\alpha\;\|\varphi_0\|_{L^2(]0,T[\times\partial\Omega)} + \|\partial_t\varphi_0\|_{L^2(]0,T[\times\partial\Omega)}). \tag{4.17}$$

□

Now we deduce the a priori estimate for the total (kinetic and electric) energy. We use the following lemma, whose proof is immediate and is left to the reader.

Lemma 4.5. *Assume that* $\partial\Omega \in C^1$, $u \in L^2(]0,T[;H^1(\Omega))$ *with* $\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}$, $\forall 1 \le i,j \le N$. *Then :*

$$u_i u_j \in L^1(]0,T[;W^{1,1}(\Omega)),\;\; \frac{\partial}{\partial x_k}(u_i u_j) = \frac{\partial u_i}{\partial x_k} u_j + \frac{\partial u_j}{\partial x_k} u_i,\; \forall 1 \le i,j,k \le N,$$

$$u_i \nabla_x \cdot u = \sum_{j=1}^N \frac{\partial}{\partial x_j}(u_i u_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|u|^2,\;\; \forall 1 \le i \le N,$$

as functions in $L^1(]0,T[\times\Omega)$. *Moreover we have* $\gamma_1(u_i u_j) = \gamma_2(u_i)\gamma_2(u_j)$, $\forall 1 \le i,j \le N$ *where* $\gamma_1 : L^1(]0,T[;W^{1,1}(\Omega)) \to L^1(]0,T[\times\partial\Omega)$ *and* $\gamma_2 : L^2(]0,T[;H^1(\Omega)) \to L^2(]0,T[\times\partial\Omega)$ *are the usual trace operators.*

We need also to estimate the moments $\rho = q\int_{\mathbb{R}_p^N} f dp, j = q\int_{\mathbb{R}_p^N} v(p) f dp, j_\delta = q\int_{\mathbb{R}_p^N} v_\delta(p) f dp$ by interpolation inequalities. We have the following results :

LEMMA 4.6. *The following inequalities hold :*

(1) $\mathcal{E}(p) \le p \cdot v(p) \le 2\mathcal{E}(p), \ \ \forall p \in \mathbb{R}^N_p$;

(2) $\mathcal{E}_\delta(p) \le p \cdot v_\delta(p) \le (2+\gamma)\cdot \mathcal{E}_\delta(p), \ \ \forall p \in \mathbb{R}^N_p$;

(3) $|v_\delta(p) - v(p)| \le C(m, c_0)\cdot \delta \cdot (2+\gamma)\cdot |p|^\lambda, \ \ \forall p \in \mathbb{R}^N_p$, *where* $\lambda = \gamma + 1$ *in the classical case and* $\lambda = \gamma$ *in the relativistic case;*

(4) $|(v_\delta(p) - v(p))\cdot p\,| \le (2+\gamma)\cdot \mathcal{E}_\delta(p), \ \ \forall p \in \mathbb{R}^N_p$;

(5) *in the classical case we have :*

$$\|\rho\|_{L^{\frac{N+2}{N}}} \le C \cdot \|f\|_\infty^{\frac{2}{N+2}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f\mathcal{E}(p)dxdp \right)^{\frac{N}{N+2}} \le C \cdot \|f\|_\infty^{\frac{2}{N+2}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)dxdp \right)^{\frac{N}{N+2}},$$

and in the relativistic case we have :

$$\|\rho\|_{L^{\frac{N+1}{N}}} \le C \cdot \|f\|_\infty^{\frac{1}{N+1}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f(1+\mathcal{E})dxdp \right)^{\frac{N}{N+1}} \le C \cdot \|f\|_\infty^{\frac{1}{N+1}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f(1+\mathcal{E}_\delta)dxdp \right)^{\frac{N}{N+1}};$$

(6) *in the classical case we have :*

$$\|j\|_{L^{\frac{N+2}{N+1}}} \le C \cdot \|f\|_\infty^{\frac{1}{N+2}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f\mathcal{E}(p)dxdp \right)^{\frac{N+1}{N+2}} \le C \cdot \|f\|_\infty^{\frac{1}{N+2}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)dxdp \right)^{\frac{N+1}{N+2}},$$

$$\|j_\delta\|_{L^{\frac{N+\lambda+1}{N+\lambda}}} \le C \cdot \|j\|_{L^{\frac{N+2}{N+1}}} + C\cdot (\delta \cdot \|f\|_\infty)^{\frac{1}{N+\lambda+1}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)dxdp \right)^{\frac{N+\lambda}{N+\lambda+1}},$$

and in the relativistic case we have ;

$$\|j\|_{L^{\frac{N+1}{N}}} \le C \cdot \|f\|_\infty^{\frac{1}{N+1}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f(1+\mathcal{E})dxdp \right)^{\frac{N}{N+1}} \le C \cdot \|f\|_\infty^{\frac{1}{N+1}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f(1+\mathcal{E}_\delta)dxdp \right)^{\frac{N}{N+1}},$$

$$\|j_\delta\|_{L^{\frac{N+\lambda+1}{N+\lambda}}} \le C \cdot \|j\|_{L^{\frac{N+1}{N}}} + C\cdot (\delta \cdot \|f\|_\infty)^{\frac{1}{N+\lambda+1}} \cdot \left(\int_\Omega \int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)dxdp \right)^{\frac{N+\lambda}{N+\lambda+1}}.$$

Proof. (1) In the classical case we have $(p\cdot v(p)) = |p|^2/m = 2\mathcal{E}(p)$ and in the relativistic case we have $(p\cdot v(p)) = \mathcal{E}(p)\cdot(1+(1+|p|^2/(mc_0)^2)^{-1/2}) \in [\mathcal{E}(p), 2\mathcal{E}(p)]$. Similarly the points (2), (3), (4) follow by easy computations. For estimating ρ and j use standard interpolation inequalities. In order to estimate j_δ observe that $|j_\delta(t,x)| \le |j(t,x)| + |\int_{\mathbb{R}^N_p}(v_\delta(p)-v(p))fdp|$ and that :

$$\left|\int_{\mathbb{R}^N_p}(v_\delta(p)-v(p))fdp\right| \le \int_{|p|\le R}|v_\delta(p)-v(p)|fdp + \frac{1}{R}\int_{|p|>R}|(v_\delta(p)-v(p))\cdot p\,|fdp$$
$$\le C\int_{|p|\le R} f\cdot\delta\cdot(2+\gamma)\cdot|p|^\lambda dp + \frac{1}{R}\int_{|p|>R} f(2+\gamma)\mathcal{E}_\delta(p)dp$$
$$\le C\cdot\|f\|_\infty\cdot\delta\cdot(2+\gamma)\cdot R^{N+\lambda} + \frac{2+\gamma}{R}\int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)dp$$
$$\le C\cdot(2+\gamma)\cdot(\delta\cdot\|f\|_\infty)^{\frac{1}{N+\lambda+1}}\cdot\left(\int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)dp\right)^{\frac{N+\lambda}{N+\lambda+1}},$$

and we deduce that :

$$\left\|\int_{\mathbb{R}^N_p}(v_\delta(p)-v(p))fdp\right\|_{L^{\frac{N+\lambda+1}{N+\lambda}}} \le C(\delta\cdot\|f\|_\infty)^{\frac{1}{N+\lambda+1}}\cdot\left(\int_\Omega\int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)dxdp\right)^{\frac{N+\lambda}{N+\lambda+1}}.$$

Since $\frac{N+1}{N} \ge \frac{N+2}{N+1} \ge \frac{N+\lambda+1}{N+\lambda}$ (we take $\lambda+1 > N \ge 2$) the conclusion follows. □

Proposition 4.7. *Under the hypotheses of Proposition* 3.3 *(which imply in particular that $\partial_t\varphi_0 \in L^2(]0,T[\times\partial\Omega)$, $\varphi_0 \in L^\infty(\mathbb{R}_t\times\partial\Omega)\cap L^2(]0,T[;H^1(\partial\Omega))$), and if $\partial\Omega$ is strictly star-shaped with respect to the origin $0\in\Omega$ (i.e., $\exists r>0$ such that $(n(x)\cdot x)\ge r$, $\forall x\in\partial\Omega$) then we have :*

$$\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f(1+\mathcal{E}_\delta(p))dtdxdp + \int_0^T\!\!\int_\Omega|\nabla_x\Phi|^2dtdx + \int_0^T\!\!\int_{\Sigma^+}(v_\delta(p)\cdot n(x))\gamma^+f(1+\mathcal{E}_\delta(p))dtd\sigma dp$$
$$+\int_0^T\!\!\int_{\partial\Omega}|\partial_n\Phi|^2dtd\sigma \le C(\Omega,\|g\|_\infty,\|\varphi_0\|_\infty)F_{\alpha,\delta}(W_0), \tag{4.18}$$

where $W_0 = \int_0^T\!\!\int_{\Sigma^-}|(v_\delta(p)\cdot n(x))|g(1+\mathcal{E}_\delta(p))dtd\sigma dp + \|\varphi_0\|^2_{L^2(]0,T[;H^1(\partial\Omega))} + \|\partial_t\varphi_0\|^2_{L^2(]0,T[\times\partial\Omega)}$ and $F_{\alpha,\delta}(W) = 1+W$ in the classical case with $N\ge 2$ and relativistic case with $N>2$ and $F_{\alpha,\delta}(W) = 1+W+(1+\alpha)^2\cdot(1+W)^{4/3}+\delta^{2/(\lambda+3)}\cdot W^{2(2+\lambda)/(3+\lambda)}$ in the relativistic case with $N=2$.

Proof. The idea is to use also the momentum conservation law. Let us use the Green formula with the test function $\theta(t)\cdot p_i\cdot x_i,\ 1\le i\le N,\ \theta\in C^1_c(]0,T[)$:

$$\alpha\int_0^T\theta(t)\int_\Omega\int_{\mathbb{R}^N_p} fp_i\cdot x_idtdxdp + \int_0^T\theta(t)\int_\Sigma(v_\delta(p)\cdot n(x))\gamma fp_i\cdot x_idtd\sigma dp = \int_0^T\theta'(t)\int_\Omega\int_{\mathbb{R}^N_p} fp_i\cdot x_idtdxdp$$
$$+\int_0^T\theta(t)\int_\Omega\int_{\mathbb{R}^N_p} fv^i_\delta(p)\cdot p_idtdxdp - \int_0^T\theta(t)\int_\Omega\int_{\mathbb{R}^N_p} f\cdot q\cdot\frac{\partial\Phi}{\partial x_i}x_idtdxdp,$$

and thus we deduce that for a.e. $t\in\mathbb{R}_t$ and $\forall 1\le i\le N$:

$$\alpha\int_\Omega\int_{\mathbb{R}^N_p} fp_i\cdot x_idxdp + \frac{d}{dt}\int_\Omega\int_{\mathbb{R}^N_p} fp_i\cdot x_idxdp + \int_\Sigma(v_\delta(p)\cdot n(x))\gamma fp_i\cdot x_id\sigma dp$$
$$= -\int_\Omega\rho\cdot\frac{\partial\Phi}{\partial x_i}x_idx + \int_\Omega\int_{\mathbb{R}^N_p} fv^i_\delta(p)\cdot p_idxdp. \tag{4.19}$$

Since $\varphi_0 \in L^\infty(\mathbb{R}_t; W^{2-1/p,p}(\partial\Omega)),\ p > N \geq 2$ and $\rho \in L^\infty(\mathbb{R}_t \times \Omega)$ (f is compactly supported in momentum) we have $u = \nabla_x \Phi \in L^2(]0,T[; H^1(\Omega))$ and we observe that $\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}$. By Lemma 4.5 we can write :

$$-\frac{\partial \Phi}{\partial x_i} \cdot \frac{\rho}{\varepsilon_0} = u_i \nabla_x \cdot u = \sum_{j=1}^N \frac{\partial}{\partial x_j}(u_i u_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|u|^2, \ \ \forall 1 \leq i \leq N. \tag{4.20}$$

By combining (4.19), (4.20) and taking the sum for all $1 \leq i \leq N$ we obtain after integration on $]0, T[$:

$$\begin{aligned}\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} & f(t,x,p)(p\cdot x) dtdxdp + \int_0^T\!\!\int_\Sigma (v_\delta(p)\cdot n(x))\gamma f(t,x,p)(p\cdot x) dtd\sigma dp \\ &= \varepsilon_0 \int_0^T\!\!\int_\Omega \sum_{i=1}^N \left(\sum_{j=1}^N \frac{\partial}{\partial x_j}(u_i u_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|u|^2 \right) x_i \; dtdx + \int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} f(v_\delta(p)\cdot p) dtdxdp.\end{aligned} \tag{4.21}$$

Let us analyze each term. Observe that $|(p\cdot x)| \leq |p|\cdot|x| \leq C\cdot(1+\mathcal{E}(p)) \leq C\cdot(1+\mathcal{E}_\delta(p))$ and therefore, by using (4.2) we deduce :

$$\alpha \left| \int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} f(p\cdot x)\; dtdxdp \right| \leq C\cdot \left(\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} f\mathcal{E}_\delta(p)\; dtdxdp - \int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x)) g \; dtd\sigma dp \right).$$

In the same manner we deduce that :

$$\begin{aligned}\left| \int_0^T\!\!\int_\Sigma (v_\delta(p)\cdot n(x))\gamma f(p\cdot x)\; dtd\sigma dp \right| \leq & C \int_0^T\!\!\int_{\Sigma^-} |(v_\delta(p)\cdot n(x))| g(1+\mathcal{E}_\delta(p))\; dtd\sigma dp \\ & + C\int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))\gamma^+ f(1+\mathcal{E}_\delta(p))\; dtd\sigma dp.\end{aligned}$$

After integration by parts we have :

$$\begin{aligned}\int_0^T\!\!\int_\Omega \sum_{i=1}^N \Big(\sum_{j=1}^N \frac{\partial}{\partial x_j}(u_i u_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|u|^2\Big) x_i\; dtdx = & \int_0^T\!\!\int_{\partial\Omega} \sum_{i=1}^N \sum_{j=1}^N \gamma_1(u_i u_j) n_j x_i\; dtd\sigma \\ & - \int_0^T\!\!\int_{\partial\Omega} \sum_{i=1}^N \frac{1}{2}\gamma_1(|u|^2)(n\cdot x)\; dtd\sigma + \left(\frac{N}{2}-1\right)\int_0^T\!\!\int_\Omega |u|^2\; dtdx.\end{aligned}$$

By splitting the field $\gamma_2(u)$ into its normal and tangential part $\gamma_2(u) = (\gamma_2(u)\cdot n(x))\cdot n(x) - n(x)\wedge(n(x)\wedge\gamma_2(u)) = (\gamma_2(u)\cdot n(x))\cdot n(x) + u_\tau$, with $|\gamma_2(u)|^2 = |(\gamma_2(u)\cdot n(x))|^2 + |u_\tau|^2$ we can write the boundary terms as :

$$
\begin{aligned}
&\int_0^T\!\!\int_{\partial\Omega} \sum_{i,j=1}^N \gamma_2(u_i)\gamma_2(u_j)n_j(x)x_i \; dtd\sigma - \int_0^T\!\!\int_{\partial\Omega} \frac{1}{2}|\gamma_2(u)|^2(n(x)\cdot x)\; dtd\sigma \\
&\quad = \int_0^T\!\!\int_{\partial\Omega} (\gamma_2(u)\cdot n(x))(\gamma_2(u)\cdot x)\; dtd\sigma - \int_0^T\!\!\int_{\partial\Omega} \frac{1}{2}|\gamma_2(u)|^2(n(x)\cdot x)\; dtd\sigma \\
&\quad = \int_0^T\!\!\int_{\partial\Omega} (\gamma_2(u)\cdot n(x))[(\gamma_2(u)\cdot n(x))(n(x)\cdot x) + (u_\tau\cdot x)]\; dtd\sigma - \int_0^T\!\!\int_{\partial\Omega} \frac{1}{2}|\gamma_2(u)|^2(n(x)\cdot x)\; dtd\sigma \\
&\quad = \frac{1}{2}\int_0^T\!\!\int_{\partial\Omega} |(\gamma_2(u)\cdot n(x))|^2(n(x)\cdot x)\; dtd\sigma + \int_0^T\!\!\int_{\partial\Omega} (\gamma_2(u)\cdot n(x))(u_\tau\cdot x)\; dtd\sigma \\
&\quad - \frac{1}{2}\int_0^T\!\!\int_{\partial\Omega} |u_\tau|^2(n(x)\cdot x)\; dtd\sigma \\
&\quad \geq \frac{r}{2}\|(\gamma_2(u)\cdot n)\|^2_{L^2(]0,T[\times\partial\Omega)} - C\cdot\{\|(\gamma_2(u)\cdot n)\|_{L^2(]0,T[\times\partial\Omega)} + \|u_\tau\|_{L^2(]0,T[\times\partial\Omega)}\}\cdot\|u_\tau\|_{L^2(]0,T[\times\partial\Omega)}.
\end{aligned}
$$

For the last term in (4.21) observe by Lemma 4.6 that $(p\cdot v_\delta(p)) \geq \mathcal{E}_\delta(p)$ and thus we can write $\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)(p\cdot v_\delta(p))\; dtdxdp \geq \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)\mathcal{E}_\delta(p)\; dtdxdp$. Finally, by collecting all these partial computations we deduce that :

$$
\begin{aligned}
&(1-\alpha\cdot C)\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)\; dtdxdp + \varepsilon_0\left(\frac{N}{2}-1\right)\int_0^T\!\!\int_\Omega |\nabla_x\Phi|^2\; dtdx + \frac{r\varepsilon_0}{2}\|\partial_n\Phi\|^2_{L^2(]0,T[\times\partial\Omega)} \\
&\leq C\int_0^T\!\!\int_{\Sigma^-} |(v_\delta(p)\cdot n(x))| g(1+\mathcal{E}_\delta(p))\; dtd\sigma dp + C\int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))\gamma^+ f\mathcal{E}_\delta(p)\; dtd\sigma dp \\
&+C\cdot\{\|\partial_n\Phi\|_{L^2(]0,T[\times\partial\Omega)} + \|\partial_\tau\Phi\|_{L^2(]0,T[\times\partial\Omega)}\}\cdot\|\partial_\tau\Phi\|_{L^2(]0,T[\times\partial\Omega)},
\end{aligned} \tag{4.22}
$$

with $\|\partial_\tau\Phi\|_{L^2(]0,T[\times\partial\Omega)} = \|u_\tau\|_{L^2(]0,T[\times\partial\Omega)} \leq C\|\varphi_0\|_{L^2(]0,T[;H^1(\partial\Omega))}$. By combining (4.11), (4.22) we deduce that :

$$
\begin{aligned}
&(1-\alpha C)\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)\; dtdxdp + \varepsilon_0\left(\frac{N}{2}-1\right)\int_0^T\!\!\int_\Omega |\nabla_x\Phi|^2\; dtdx + \frac{\varepsilon_0 r}{2}\int_0^T\!\!\int_{\partial\Omega} |\partial_n\Phi|^2\; dtd\sigma \\
&+\int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))(1+\mathcal{E}_\delta(p))\gamma^+ f\; dtd\sigma dp \\
&\leq C(\|\varphi_0\|_\infty)\int_0^T\!\!\int_{\Sigma^-} |(v_\delta(p)\cdot n(x))| g(1+\mathcal{E}_\delta(p))\; dtd\sigma dp + C\cdot\|\varphi_0\|^2_{L^2(]0,T[;H^1(\partial\Omega))} \\
&+C\cdot\|\partial_n\Phi\|_{L^2(]0,T[\times\partial\Omega)}\{\|\varphi_0\|_{L^2(]0,T[;H^1(\partial\Omega))} + \|\partial_t\varphi_0\|_{L^2(]0,T[\times\partial\Omega)}\}.
\end{aligned} \tag{4.23}
$$

If $N>2$ it is clear that the previous inequality gives uniform estimates for the total energy, the outgoing kinetic energy and the normal trace of the electric field if $\alpha>0$ is small enough :

$$
\begin{aligned}
\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)\; dtdxdp &+ \int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))(1+\mathcal{E}_\delta(p))\gamma^+ f\; dtd\sigma dp \\
&+\int_0^T\!\!\int_\Omega |\nabla_x\Phi|^2\; dtdx + \int_0^T\!\!\int_{\partial\Omega} |\partial_n\Phi|^2\; dtd\sigma \\
&\leq C(\|\varphi_0\|_\infty)(\int_0^T\!\!\int_{\Sigma^-} |(v_\delta(p)\cdot n(x))| g(1+\mathcal{E}_\delta(p))\; dtd\sigma dp \\
&+\|\varphi_0\|^2_{L^2(]0,T[;H^1(\partial\Omega))} + \|\partial_t\varphi_0\|^2_{L^2(]0,T[\times\partial\Omega)}).
\end{aligned}
$$

The conclusion follows by observing that :

$$\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)\,dtdxdp = \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f\cdot\mathbf{1}_{\{|p|\le 1\}}\,dtdxdp + \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f\cdot\mathbf{1}_{\{|p|>1\}}\,dtdxdp$$
$$\le C(\Omega,\|g\|_\infty)\left(1+\int_0^T\int_\Omega\int_{\mathbb{R}^N_p}\mathcal{E}_\delta(p)f\,dtdxdp\right).$$

Now let us clarify the case $N=2$. As before, by using (4.23) we deduce uniform estimates for the kinetic energy, the outgoing kinetic energy and the normal trace of the electric field (for $\alpha>0$ small enough) :

$$\int_0^T\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}_\delta(p))f\,dtdxdp + \int_0^T\int_{\Sigma^+}(v_\delta(p)\cdot n(x))(1+\mathcal{E}_\delta(p))\gamma^+ f\,dtd\sigma dp + \int_0^T\int_{\partial\Omega}|\partial_n\Phi|^2\,dtd\sigma$$
$$\le C(\Omega,\|g\|_\infty,\|\varphi_0\|_\infty)(1+W_0). \tag{4.24}$$

In the classical case we can write by using interpolation inequalities :

$$\|\rho\|_{L^2(]0,T[\times\Omega)}\le C\|g\|_\infty^{1/2}\left(\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f\mathcal{E}(p)\,dtdxdp\right)^{1/2}\le C\|g\|_\infty^{1/2}\left(\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)\,dtdxdp\right)^{1/2},$$

and thus :

$$\|\nabla_x\Phi\|_{L^2(]0,T[\times\Omega)}\le\|\nabla_x\Phi_0\|_{L^2(]0,T[\times\Omega)}+\|\nabla_x\Phi_s\|_{L^2(]0,T[\times\Omega)}$$
$$\le C\cdot\|\varphi_0\|_{L^2(]0,T[;H^{1/2}(\partial\Omega))}+C\cdot\|\rho\|_{L^2(]0,T[\times\Omega)}$$
$$\le C\cdot\|\varphi_0\|_{L^2(]0,T[;H^1(\partial\Omega))}+C\|g\|_\infty^{1/2}\cdot\left(\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f\mathcal{E}_\delta(p)\,dtdxdp\right)^{1/2}. \tag{4.25}$$

Therefore by using (4.24) we deduce an uniform estimate for the electric energy as well :

$$\int_0^T\int_\Omega|\nabla_x\Phi|^2(t,x)\,dtdx\le C(\Omega,\|g\|_\infty,\|\varphi_0\|_\infty)(1+W_0).$$

For the relativistic case we have :

$$\|\rho\|_{L^{\frac{3}{2}}(]0,T[\times\Omega)}\le C\|g\|_\infty^{1/3}\left(\int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f(1+\mathcal{E}_\delta(p))\,dtdxdp\right)^{2/3}.$$

Note that $\|\nabla_x\Phi_s(t)\|_{W^{1,\frac{3}{2}}(\Omega)}\le C\|\rho(t)\|_{L^{\frac{3}{2}}(\Omega)}$ and thus :

$$\|\nabla_x\Phi_s\|_{L^{\frac{3}{2}}(]0,T[;W^{1,\frac{3}{2}}(\Omega))}\le C\|\rho\|_{L^{\frac{3}{2}}(]0,T[\times\Omega)}.$$

We need also to estimate the time derivative $\partial_t E_s$. We have :

$$\|\partial_t\Phi_s(t)\|_{W^{1,r}(\Omega)}\le C(\Omega,r)\cdot\|\partial_t\rho(t)\|_{W^{-1,r}(\Omega)}=C(\Omega,r)\|-\alpha\rho-\nabla_x\cdot j_\delta\|_{W^{-1,r}(\Omega)}$$
$$\le C(\Omega,r)(\alpha\cdot\|\rho(t)\|_{L^r(\Omega)}+\|j_\delta(t)\|_{L^r(\Omega)}),$$

and by using Lemma 4.6 for $N = 2$, $\lambda > 1$ in the relativistic case with $r = (\lambda+3)/(\lambda+2)$ we deduce that :

$$\begin{aligned}\|\partial_t E_s\|_{L^r(]0,T[\times\Omega)} \leq & C \cdot (\alpha \cdot \|\rho\|_{L^r(]0,T[\times\Omega)} + \|j_\delta\|_{L^r(]0,T[\times\Omega)}) \\ \leq & C \cdot (\alpha+1) \cdot \|g\|_\infty^{\frac{1}{3}} \left(\int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} f(1+\mathcal{E}_\delta(p))\, dtdxdp \right)^{\frac{2}{3}} \\ & + C \cdot (\delta \cdot \|g\|_\infty)^{\frac{1}{3+\lambda}} \left(\int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} f\mathcal{E}_\delta(p)\, dtdxdp \right)^{\frac{2+\lambda}{3+\lambda}}.\end{aligned}$$

We obtain that :

$$\begin{aligned}\|E_s\|_{L^r(]0,T[\times\Omega)} + & \|\nabla_x E_s\|_{L^r(]0,T[\times\Omega)} + \|\partial_t E_s\|_{L^r(]0,T[\times\Omega)} \leq C \cdot (\alpha+1) \cdot \|g\|_\infty^{\frac{1}{3}} \\ & \cdot \left(\int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} f(1+\mathcal{E}_\delta(p))\, dtdxdp \right)^{\frac{2}{3}} + C \cdot (\delta\|g\|_\infty)^{\frac{1}{3+\lambda}} \left(\int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} f\mathcal{E}_\delta(p)\, dtdxdp \right)^{\frac{2+\lambda}{3+\lambda}}.\end{aligned}$$

Finally, by using Sobolev imbeddings one gets that :

$$\begin{aligned}\|E_s\|_{L^s(]0,T[\times\Omega)} \leq & C \cdot \|E_s\|_{W^{1,r}(]0,T[\times\Omega)} \leq C_1(\alpha) \cdot \left(\int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} (1+\mathcal{E}_\delta(p)) f\, dtdxdp \right)^{\frac{2}{3}} \\ & + C_2(\delta) \cdot \left(\int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} \mathcal{E}_\delta(p) f\, dtdxdp \right)^{\frac{2+\lambda}{3+\lambda}},\end{aligned}$$

where $C_1(\alpha) = C \cdot (\alpha+1) \cdot \|g\|_\infty^{1/3}$, $C_2(\delta) = C \cdot (\delta \cdot \|g\|_\infty)^{1/(\lambda+3)}$, for all $s \leq s^\star$ such that $\frac{1}{s^\star} = \frac{1}{r} - \frac{1}{3} = \frac{\lambda+2}{\lambda+3} - \frac{1}{3}$. In particular if we take $\lambda > 1$, close enough to 1 (which is possible, since all previous proofs require only $\lambda + 1 > N = 2$), we have $\frac{1}{s^\star} \searrow \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$ or $s^\star \nearrow \frac{12}{5}$ and thus is possible to take $s = 2$, which implies :

$$\begin{aligned}\|\nabla_x \Phi\|_{L^2(]0,T[\times\Omega)} \leq & \|\nabla_x \Phi_s\|_{L^2(]0,T[\times\Omega)} + \|\nabla_x \Phi_0\|_{L^2(]0,T[\times\Omega)} \leq C_1(\alpha) \left(\int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} (1+\mathcal{E}_\delta(p)) f\, dtdxdp \right)^{\frac{2}{3}} \\ & + C_2(\delta) \left(\int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} \mathcal{E}_\delta(p) f\, dtdxdp \right)^{\frac{2+\lambda}{3+\lambda}} + C \cdot \|\varphi_0\|_{L^2(]0,T[;H^1(\partial\Omega))}.\end{aligned}$$

Now, by using (4.24) we can deduce the estimate for the relativistic case with $N = 2$. □

Actually we can show that the total energy is bounded uniformly with respect to $t \in \mathbb{R}_t$. For the details of proof see the Appendix.

Proposition 4.8. *Assume that the hypotheses of Proposition* 3.3 *hold,* $\partial\Omega$ *is strictly star-shaped and* $\varphi_0 \in L^\infty(\mathbb{R}_t; H^{1/2}(\partial\Omega))$*. Then the solution* (f, Φ) *constructed in Proposition* 3.3 *has total energy uniformly bounded with respect to* $t \in \mathbb{R}_t$*.*

5. Passing to the limit. Existence for the periodic Vlasov-Poisson system.

Now we are ready to prove the existence of T periodic weak solution for the Vlasov-Poisson system.

Proof. (of Theorem 1.1) Take γ such that $\gamma + 2 > N$ in the classical case and $\gamma + 1 > N$ in the relativistic case, with γ close enough to 1 if $N = 2$, in the relativistic case. Consider $(\alpha_k)_k$ a sequence which converges to 0, $\alpha_k > 0$, $\delta_k = \alpha_k$, $\forall k$ and the T periodic boundary conditions g_R, φ_{0k} such that $g_R = g \cdot \mathbf{1}_{\{|p|\leq R\}}$ for some fixed $R > 0$,

$$\varphi_{0k} \in L^\infty(\mathbb{R}_t; W^{2-1/p,p}(\partial\Omega)),\ \ \partial_t\varphi_{0k} \in L^\infty(\mathbb{R}_t; W^{1-1/p,p}(\partial\Omega)) \text{ for some } p > N,$$

$$\|\varphi_{0k}\|_{L^\infty(\mathbb{R}_t\times\partial\Omega)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}_t\times\partial\Omega)},\ \forall\, k,$$

$$\lim_{k\to+\infty} \varphi_{0k} = \varphi_0 \ \text{ in } \ L^2(]0,T[; H^1(\partial\Omega)),$$

$$\lim_{k\to+\infty} \partial_t\varphi_{0k} = \partial_t\varphi_0 \ \text{ in } \ L^2(]0,T[\times\partial\Omega).$$

Denote by (f_k, Φ_k) the corresponding T periodic solutions constructed in Proposition 3.3 and by $(\gamma^+ f_k, \partial_n\Phi_k)$ the associated traces. We suppose that $\partial\Omega$ is strictly star-shaped with respect to $0 \in \Omega$. By Proposition 4.7 we deduce that :

$$\begin{aligned} W_k =& \int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f_k(t,x,p)(1+\mathcal{E}_k(p))\ dtdxdp + \int_0^T\!\!\int_\Omega |\nabla_x\Phi_k|^2\ dtdx \\ &+\int_0^T\!\!\int_{\Sigma^+} (v_k(p)\cdot n(x))\gamma^+ f_k(t,x,p)(1+\mathcal{E}_k(p))\ dtd\sigma dp + \int_0^T\!\!\int_{\partial\Omega} |\partial_n\Phi_k|^2\ dtd\sigma \\ &\leq C(\Omega, \|g_R\|_\infty, \|\varphi_{0k}\|_\infty)F_k(W_{0k}) \leq C(\Omega, \|g\|_\infty, \|\varphi_0\|_\infty)F_k(W_{0k}), \end{aligned}$$

where we have used the notations $\mathcal{E}_k(p) = \mathcal{E}_{\delta_k}(p), v_k(p) = v_{\delta_k}(p),\ F_k = F_{\alpha_k,\delta_k}$ and :

$$W_{0k} = \int_0^T\!\!\int_{\Sigma^-} |(v_k(p)\cdot n(x))|(1+\mathcal{E}_k(p))g_R\ dtd\sigma dp + \|\varphi_{0k}\|^2_{L^2(]0,T[;H^1(\partial\Omega))} + \|\partial_t\varphi_{0k}\|^2_{L^2(]0,T[\times\partial\Omega)}.$$

Since $\mathcal{E}_k(p) \to \mathcal{E}(p), v_k(p) \to v(p)$ uniformly for $|p| \leq R$ we deduce that :

$$\begin{aligned} \lim_{k\to+\infty} \int_0^T\!\!\int_{\Sigma^-} |(v_k(p)\cdot n(x))|(1+\mathcal{E}_k(p))g_R\ dtd\sigma dp =& \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p))g_R\ dtd\sigma dp \\ \leq& \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\ dtd\sigma dp, \end{aligned}$$

and thus we have $\lim_{k\to+\infty} W_{0k} \leq W_0$ which implies that $\sup_k W_k \leq C\cdot\sup_k F_k(W_{0k}) < +\infty$. By using Lemma 4.6 we deduce also that $\sup_k \|\rho_k\|_{L^\mu(]0,T[\times\Omega)} < +\infty$, $\sup_k \|j_k\|_{L^\nu(]0,T[\times\Omega)} < +\infty$ where $\mu = \frac{N+2}{N}$, $\nu = \frac{N+\gamma+2}{N+\gamma+1}$ in the classical case and $\mu = \frac{N+1}{N}, \nu = \frac{N+\gamma+1}{N+\gamma}$ in the relativistic case, $2 \geq \mu > \nu > 1$ (here $\rho_k = q\int_{\mathbb{R}^N_p} f_k\ dp$ and $j_k = q\int_{\mathbb{R}^N_p} v_k(p)f_k\ dp$). As usual we can prove that $(E_{sk})_k$ is uniformly bounded in $W^{1,\nu}(]0,T[\times\Omega)$. Since $\varphi_{0k} \to \varphi_0$ in $L^2(]0,T[;H^1(\partial\Omega))$ we have $\Phi_{0k} \to \Phi_0$ in $L^2(]0,T[;H^1(\Omega))$, implying that $E_{0k} \to E_0$ in $L^2(]0,T[\times\Omega)$ and also $E_{0k} \to E_0$ in $L^\nu(]0,T[\times\Omega)$. We can extract subsequences (still denoted by f_k, Φ_k) such that the following convergences hold as k goes to $+\infty$: $f_k \rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$, $\gamma^+ f_k \rightharpoonup \gamma^+ f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Sigma^+)$, $\Phi_k \to \Phi$ strongly in $L^\nu(]0,T[\times\Omega)$, $E_k = -\nabla_x \Phi_k \to -\nabla_x \Phi = E$ strongly in $L^\nu(]0,T[\times\Omega)$, $\partial_n \Phi_k \rightharpoonup \partial_n \Phi$ weakly in $L^2(]0,T[\times\partial\Omega)$, $E_k \rightharpoonup E$ weakly in $L^2(]0,T[\times\Omega)$. In order to prove that f is T periodic weak solution of the Vlasov problem associated to the electric field E, take $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^N_p)$, T periodic, compactly supported in momentum and write :

$$\alpha_k \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f_k\varphi \; dtdxdp + \int_0^T\int_{\Sigma^-} (v_k(p)\cdot n(x)) g_R \varphi \; dtd\sigma dp + \int_0^T\int_{\Sigma^+} (v_k(p)\cdot n(x))\gamma^+ f_k \varphi \; dtd\sigma dp$$
$$= \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f_k(\partial_t\varphi + v_k(p)\cdot\nabla_x\varphi + qE_k\cdot\nabla_p\varphi) \; dtdxdp. \tag{5.1}$$

We check easily that :

$$\lim_{k\to+\infty} \alpha_k \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f_k\varphi \; dtdxdp = 0,$$

$$\lim_{k\to+\infty} \int_0^T\int_{\Sigma^-} (v_k(p)\cdot n(x)) g_R\varphi \; dtd\sigma dp = \int_0^T\int_{\Sigma^-} (v(p)\cdot n(x)) g_R\varphi \; dtd\sigma dp,$$

$$\lim_{k\to+\infty} \int_0^T\int_{\Sigma^+} (v_k(p)\cdot n(x))\gamma^+ f_k\varphi \; dtd\sigma dp = \int_0^T\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f\varphi \; dtd\sigma dp,$$

$$\lim_{k\to+\infty} \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f_k(\partial_t\varphi + v_k(p)\cdot\nabla_x\varphi) \; dtdxdp = \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f(\partial_t\varphi + v(p)\cdot\nabla_x\varphi) \; dtdxdp.$$

By combining strong and weak convergences we prove also that $\lim_{k\to\infty} \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f_k qE_k\cdot\nabla_p\varphi dtdxdp = \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} fqE\cdot\nabla_p\varphi \; dtdxdp$ and thus by passing to the limit for $k\to+\infty$ in (5.1) we deduce that f is a T periodic weak solution for the Vlasov problem associated to the field E, with trace $\gamma^+ f$. Obviously, by weak $\star$ convergence we have $\|f\|_\infty \le \liminf_{k\to+\infty}\|f_k\|_\infty \le \|g_R\|_\infty \le \|g\|_\infty$ and $\|\gamma^+ f\|_\infty \le \liminf_{k\to+\infty}\|\gamma^+ f_k\|_\infty \le \|g_R\|_\infty \le \|g\|_\infty$. Now, in order to pass to the limit in the Poisson equation, take $\theta \in C^1(\overline{\Omega}), \eta \in C([0,T])$ and write :

$$\int_0^T\int_\Omega \eta(t)\nabla_x\Phi_{sk}\cdot\nabla_x\theta \; dtdx = \frac{1}{\varepsilon_0}\int_0^T\int_\Omega \eta(t)\rho_k(t,x)\theta(x) \; dtdx.$$

Clearly we have $\lim_{k\to+\infty}\int_0^T\int_\Omega \eta(t)\nabla_x\Phi_{sk}\cdot\nabla_x\theta \; dtdx = \int_0^T\int_\Omega \eta(t)\nabla_x\Phi_s\cdot\nabla_x\theta \; dtdx$. For the right hand term observe that :

$$\int_0^T\int_\Omega \rho_k\eta(t)\theta(x) \; dtdx = \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f_k q\eta(t)\theta(x) \; dtdxdp$$
$$= \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f_k q\eta(t)\theta(x)\mathbf{1}_{\{|p|<S\}} \; dtdxdp + \int_0^T\int_\Omega\int_{\mathbb{R}^N_p} f_k q\eta(t)\theta(x)\mathbf{1}_{\{|p|>S\}} \; dtdxdp.$$

But for S large enough we have :

$$\left|\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f_k\eta(t)\theta(x)\mathbf{1}_{\{|p|>S\}}\,dtdxdp\right| \le \|\eta\|_\infty\cdot\|\theta\|_\infty\cdot S^{-1}\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f_k|p|\,dtdxdp$$
$$\le C\|\eta\|_\infty\cdot\|\theta\|_\infty\cdot S^{-1}\cdot\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f_k(1+\mathcal{E}_k(p))\,dtdxdp$$
$$\le C\|\eta\|_\infty\cdot\|\theta\|_\infty\cdot S^{-1}\cdot W_k\to 0,$$

as $S\to+\infty$ and since $f_k\rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t\times\Omega\times\mathbb{R}_p^N)$ as k goes to $+\infty$ we have also for fixed $S>0$:

$$\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f_k\eta(t)\theta(x)\mathbf{1}_{\{|p|<S\}}\,dtdxdp\to\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f\eta(t)\theta(x)\mathbf{1}_{\{|p|<S\}}\,dtdxdp.$$

Finally we have proved that $\lim_{k\to+\infty}\int_0^T\int_\Omega\eta(t)\rho_k(t,x)\theta(x)\,dtdx=\int_0^T\int_\Omega\eta(t)\rho(t,x)\theta(x)\,dtdx$, with $\rho(t,x)=q\int_{\mathbb{R}_p^N} fdp$ (note that $\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f\mathbf{1}_{\{|p|<R_1\}}\,dtdxdp=\lim_{k\to+\infty}\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f_k\mathbf{1}_{\{|p|<R_1\}}\,dtdxdp\le\sup_k W_k$ and thus $f\in L^1(]0,T[\times\Omega\times\mathbb{R}_p^N)$, $\rho\in L^1(]0,T[\times\Omega)$). Therefore we have for all $\eta\in C([0,T])$, $\theta\in C^1(\overline{\Omega})$:

$$\int_0^T\int_\Omega\eta(t)\nabla_x\Phi_s\cdot\nabla_x\theta\,dtdx=\frac{1}{\varepsilon_0}\int_0^T\int_\Omega\eta(t)\rho(t,x)\theta(x)\,dtdx,$$

implying that :

$$\int_\Omega\nabla_x\Phi_s(t)\cdot\nabla_x\theta\,dx=\frac{1}{\varepsilon_0}\int_\Omega\rho(t,x)\theta(x)\,dx,\quad\text{a.e. }t\in\mathbb{R}_t,$$

and thus $\Phi=\Phi_s+\Phi_0$ solves in distributions the Poisson problem $-\Delta_x\Phi=\frac{1}{\varepsilon_0}\rho$ with $\Phi|_{\mathbb{R}_t\times\partial\Omega}=\varphi_0$. Moreover, if $\eta\in C([0,T]),\theta\in C^1(\overline{\Omega})$ we have :

$$\int_0^T\int_\Omega\eta(t)\nabla_x\Phi_k\cdot\nabla_x\theta\,dtdx-\int_0^T\int_{\partial\Omega}\eta(t)\partial_n\Phi_k\theta(x)\,dtd\sigma=\frac{1}{\varepsilon_0}\int_0^T\int_\Omega\eta(t)\rho_k(t,x)\theta(x)\,dtdx,$$

and by passing to the limit for $k\to+\infty$ we find that :

$$\int_0^T\int_\Omega\eta(t)\nabla_x\Phi\cdot\nabla_x\theta\,dtdx-\int_0^T\int_{\partial\Omega}\eta(t)\partial_n\Phi\theta(x)\,dtd\sigma=\frac{1}{\varepsilon_0}\int_0^T\int_\Omega\eta(t)\rho(t,x)\theta(x)\,dtdx,$$

which yields :

$$\int_\Omega\nabla_x\Phi(t)\cdot\nabla_x\theta\,dx-\int_{\partial\Omega}\partial_n\Phi(t)\cdot\theta(x)\,d\sigma=\frac{1}{\varepsilon_0}\int_\Omega\rho(t,x)\theta(x)\,dx,\quad\text{a. e. }\ t\in\mathbb{R}_t,$$

and thus $\partial_n\Phi$ is the normal trace of $\nabla\Phi$. By weak convergence we have :

$$\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f(1+\mathcal{E}(p))\cdot\mathbf{1}_{\{|p|<S\}}\,dtdxdp+\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+f(1+\mathcal{E}(p))\mathbf{1}_{\{|p|<S\}}\,dtd\sigma dp$$
$$+\|\nabla_x\Phi\|^2_{L^2(]0,T[\times\Omega)}+\|\partial_n\Phi\|^2_{L^2(]0,T[\times\partial\Omega)}\le\liminf_{k\to+\infty}W_k$$
$$\le\liminf_{k\to+\infty}C(\Omega,\|g_R\|_\infty,\|\varphi_{0k}\|_\infty)F_k(W_{0k})\le C(\Omega,\|g\|_\infty,\|\varphi_0\|_\infty)F(W_0),$$

with $F(W) = \lim_{k\to+\infty} F_k(W)$ and the inequality (1.8) follows by letting $S \to +\infty$. By using :

$$\alpha_k \int_0^T\!\!\int_\Omega \int_{\mathbb{R}_p^N} f_k \; dtdxdp + \int_0^T\!\!\int_{\Sigma^+} (v_k(p)\cdot n(x))\gamma^+ f_k \; dtd\sigma dp = -\int_0^T\!\!\int_{\Sigma^-} (v_k(p)\cdot n(x)) g_R \; dtd\sigma dp, \quad (5.2)$$

and by taking into account that $\sup_k \int_0^T\!\int_\Omega \int_{\mathbb{R}_p^N} f_k \; dtdxdp < +\infty$ and $\sup_k \int_0^T\!\int_{\Sigma^+} (v_k(p)\cdot n(x))(1+\mathcal{E}_k(p))\gamma^+ f_k \; dtd\sigma dp < +\infty$ we deduce easily, by passing to the limit for $k \to +\infty$ in (5.2) that $\int_0^T\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f \; dtd\sigma dp = -\int_0^T\!\int_{\Sigma^-} (v(p)\cdot n(x)) g_R \; dtd\sigma dp$. By combining the strong convergence of φ_{0k} in $L^2(]0,T[\times\partial\Omega)$ with the weak $\star$ convergence of $\gamma^+ f_k$ in $L^\infty(\mathbb{R}_t \times \Sigma^+)$ and by using (4.16) we have :

$$\begin{aligned}
&\int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\mathcal{E}(p)\mathbf{1}_{\{|p|<S\}}\gamma^+ f \; dtd\sigma dp + \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x)) q\varphi_0 \gamma^+ f \; dtd\sigma dp \\
&= \lim_{k\to+\infty} \int_0^T\!\!\int_{\Sigma^+} (v_k(p)\cdot n(x))\mathcal{E}_k(p)\mathbf{1}_{\{|p|<S\}}\gamma^+ f_k \; dtd\sigma dp + \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x)) q\varphi_0\gamma^+ f \; dtd\sigma dp \\
&\leq \liminf_{k\to+\infty} \int_0^T\!\!\int_{\Sigma^+} (v_k(p)\cdot n(x))\mathcal{E}_k(p)\gamma^+ f_k \; dtd\sigma dp + \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x)) q\varphi_0\gamma^+ f \; dtd\sigma dp \\
&= \liminf_{k\to+\infty} \int_0^T\!\!\int_{\Sigma^+} (v_k(p)\cdot n(x))(\mathcal{E}_k(p) + q\varphi_{0k})\gamma^+ f_k \; dtd\sigma dp \\
&\leq \liminf_{k\to+\infty} \Big(-\int_0^T\!\!\int_{\Sigma^-} (v_k(p)\cdot n(x))(\mathcal{E}_k(p) + q\varphi_{0k}) g_R \; dtd\sigma dp - \varepsilon_0 \int_0^T\!\!\int_{\partial\Omega} \partial_n \Phi_k \cdot \partial_t \varphi_{0k} \; dtd\sigma \\
&+ \alpha_k \varepsilon_0 \int_0^T\!\!\int_{\partial\Omega} \partial_n \Phi_k \cdot \varphi_{0k} \; dtd\sigma\Big) \\
&= -\int_0^T\!\!\int_{\Sigma^-} (v(p)\cdot n(x))(\mathcal{E}(p) + q\varphi_0) g_R \; dtd\sigma dp - \varepsilon_0 \int_0^T\!\!\int_{\partial\Omega} \partial_n \Phi \cdot \partial_t \varphi_0 \; dtd\sigma,
\end{aligned}$$

and the inequality (1.10) follows by letting $S \to +\infty$. At this point we have proved the existence of T periodic weak solution for the Vlasov-Poisson system with the boundary conditions (g_R, φ_0). The moment estimates follow by Lemma 4.6. A second passing to the limit for $R \to +\infty$ allows us to remove the compact support in momentum hypothesis on g, the proof being basically the same. □

If in addition $\varphi_0 \in L^\infty(\mathbb{R}_t; H^{1/2}(\partial\Omega))$, then the total energy is uniformly bounded in time. Indeed, it is sufficient to check this for the boundary conditions (g_R, φ_0), $\forall R > 0$. The general case follows easily by standard arguments. Suppose also that the regularized boundary potentials verify $\limsup_{k\to+\infty} \|\varphi_{0k}\|_{L^\infty(\mathbb{R}_t;H^{1/2}(\partial\Omega))} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}_t;H^{1/2}(\partial\Omega))}$. For $S > 0, \eta \in C([0,T]), \eta \geq 0$

we have by Proposition 4.8 :

$$\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} \eta(t)(1+\mathcal{E}(p))\mathbf{1}_{\{|p|<S\}}f\ dtdxdp + \int_0^T\int_\Omega \eta(t)|\nabla_x\Phi|^2(t,x)\ dtdx$$
$$\leq \lim_{k\to+\infty}\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} \eta(t)(1+\mathcal{E}_k(p))\mathbf{1}_{\{|p|<S\}}f_k\ dtdxdp + \liminf_{k\to+\infty}\int_0^T\int_\Omega \eta(t)|\nabla_x\Phi_k|^2(t,x)\ dtdx$$
$$=\liminf_{k\to+\infty}\left(\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} \eta(t)(1+\mathcal{E}_k(p))\mathbf{1}_{\{|p|<S\}}f_k\ dtdxdp + \int_0^T\int_\Omega \eta(t)|\nabla_x\Phi_k|^2(t,x)\ dtdx\right)$$
$$\leq\liminf_{k\to+\infty}\left(C_1(\Omega,\|g_R\|_\infty,\|\varphi_{0k}\|_\infty)(F_k(W_{0k})+\|\varphi_{0k}\|^2_{L^\infty(\mathbb{R}_t;H^{1/2}(\partial\Omega))})\right)\int_0^T\eta(t)dt$$
$$\leq C_1(\Omega,\|g\|_\infty,\|\varphi_0\|_\infty)(F(W_0)+\|\varphi_0\|^2_{L^\infty(\mathbb{R}_t;H^{1/2}(\partial\Omega))})\int_0^T\eta(t)dt,$$

and thus we deduce that :

$$\int_\Omega\int_{\mathbb{R}_p^N}(1+\mathcal{E}(p))\mathbf{1}_{\{|p|<S\}}f(t,x,p)\ dxdp + \int_\Omega|\nabla_x\Phi|^2(t,x)\ dx$$
$$\leq C_1(\Omega,\|g\|_\infty,\|\varphi_0\|_\infty)(F(W_0)+\|\varphi_0\|^2_{L^\infty(\mathbb{R}_t;H^{1/2}(\partial\Omega))}).$$

Our statement follows by letting $S\to+\infty$.

6. The Vlasov-Poisson system with specular boundary condition.

In this section we analyze the time periodic Vlasov-Poisson system with the boundary condition :

$$f(t,x,p) = g(t,x,p) + a(t,x,p)f(t,x,R(x)p),\ \ (t,x,p)\in\mathbb{R}_t\times\Sigma^-, \tag{6.1}$$

where $R(x):\mathbb{R}_p^N\to\mathbb{R}_p^N,\ R(x)p = p-2(p\cdot n(x))n(x), \forall\ (x,p)\in\Sigma$ and a is a T periodic function satisfying $0\leq a(t,x,p)\leq a_0<1,\ \forall\ (t,x,p)\in\mathbb{R}_t\times\Sigma^-$. Let us give the exact definition for T periodic weak solution for the Vlasov problem in this case. For this we introduce the space of test functions :

$$\mathcal{T}_w^{spec}=\{\varphi\in C^1(\mathbb{R}_t\times\overline{\Omega}\times\mathbb{R}_p^N)|\varphi(\cdot+T)=\varphi(\cdot),\ \varphi(t,x,Rp)=a(t,x,p)\varphi(t,x,p),\forall(t,x,p)\in\mathbb{R}_t\times\Sigma^-,$$
$$\exists R>0:\varphi=\varphi\cdot\mathbf{1}_{\{|p|\leq R\}}\}. \tag{6.2}$$

Definition 6.1. *Assume that E, g are T periodic such that $E\in L^\infty(\mathbb{R}_t\times\Omega)^N$ and $(v(p)\cdot n(x))g\in L^1_{loc}(\mathbb{R}_t\times\Sigma^-)$. We say that $f\in L^1_{loc}(\mathbb{R}_t\times\Omega\times\mathbb{R}_p^N)$ is T periodic weak solution for the Vlasov equation* (2.3) *with the boundary condition* (6.1) *iff for all test function $\varphi\in\mathcal{T}_w^{spec}$ we have :*

$$\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f(t,x,p)(-\alpha\varphi+\partial_t\varphi+v(p)\cdot\nabla_x\varphi+qE(t,x)\cdot\nabla_p\varphi)\ dtdxdp = \int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g\varphi\ dtd\sigma dp.$$

Proposition 6.2. *Assume that* $\alpha > 0, 0 \le a(t,x,p) \le a_0 < 1, \ \forall (t,x,p) \in \mathbb{R}_t \times \Sigma^-$, $E \in L^\infty(\mathbb{R}_t \times \Omega)^N$, $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$ *and* a *are* T *periodic. Then there is a* T *periodic weak solution for the problem* $(2.3), (6.1)$ *which verifies* $\|f\|_\infty \le \|g\|_\infty/(1-a_0)$. *Moreover* f *has traces* $\gamma^\pm f \in L^\infty(\mathbb{R}_t \times \Sigma^\pm)$, $\|\gamma^\pm f\|_\infty \le \|g\|_\infty/(1-a_0)$, $\gamma^- f(t,x,p) = g(t,x,p) + a(t,x,p)\gamma^+ f(t,x,Rp), \ \forall \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-$, *such that the following Green formula holds for all function* $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^N_p)$, T *periodic with compact support in momentum :*

$$-\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f(-\alpha\varphi + \partial_t\varphi + v(p)\cdot\nabla_x\varphi + qE(t,x)\cdot\nabla_p\varphi)\ dtdxdp + \int_0^T\!\!\int_{\Sigma^-}(v(p)\cdot n(x))g\varphi\ dtd\sigma dp$$
$$= -\int_0^T\!\!\int_{\Sigma^-}(v(p)\cdot n(x))\gamma^+ f(t,x,Rp)(a(t,x,p)\cdot\varphi(t,x,p) - \varphi(t,x,Rp))\ dtd\sigma dp.$$

Proof. Consider the sequence $(f_k)_{k\ge 0}$ of T periodic weak solutions for :

$$\alpha f_0 + \partial_t f_0 + v(p)\cdot\nabla_x f_0 + qE\cdot\nabla_p f_0 = 0, \ \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^N_p,$$

$$f_0(t,x,p) = g(t,x,p), \ \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-,$$

and for $k \ge 1$:

$$\alpha f_k + \partial_t f_k + v(p)\cdot\nabla_x f_k + qE\cdot\nabla_p f_k = 0, \ \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^N_p,$$

$$f_k(t,x,p) = g(t,x,p) + a(t,x,p)\cdot\gamma^+ f_{k-1}(t,x,Rp), \ \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-.$$

It is easy to show that $(f_k)_k, (\gamma^\pm f_k)_k$ converge in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$ respectively $L^\infty(\mathbb{R}_t \times \Sigma^\pm)$ to $f, \gamma^\pm f$ which is a T periodic weak solution for the Vlasov problem with the boundary condition (6.1).

□

Remark 6.3. *Assume that* E, g, a *are* T *periodic such that* $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^N$, $0 \le a(t,x,p) \le a_0 < 1, \ \forall (t,x,p) \in \mathbb{R}_t \times \Sigma^-$, $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$. *Then the perturbed Vlasov problem* $(2.3), (6.1)$ *has a unique* T *periodic bounded weak solution.*

Proof. Assume that $f_{1,2} \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$ are T periodic weak solutions for the problem $(2.3), (6.1)$. Denote by f the difference $f = f_1 - f_2$. We have $\partial_t f + v(p)\cdot\nabla_x f + F(t,x)\cdot\nabla_p f = -\alpha f \in L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^N_p)$ and $\gamma^- f(t,x,p) = a(t,x,p)\gamma^+ f(t,x,Rp), \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-$. Therefore (cf. [5], [15]) we obtain :

$$\frac{1}{2}\cdot(\partial_t f^2 + v(p)\cdot\nabla_x f^2 + F\cdot\nabla_p f^2) = -\alpha f^2.$$

After integration on $]0,T[\times\Omega\times\mathbb{R}^N_p$ we deduce that :

$$\alpha\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f^2(t,x,p)\ dtdxdp + \frac{1}{2}\int_0^T\!\!\int_{\Sigma^+}(v(p)\cdot n(x))(1 - a^2(t,x,Rp))(\gamma^+ f)^2(t,x,p)\ dtd\sigma dp = 0.$$

Since $\alpha > 0$ and $1 - a^2(t,x,Rp) \geq 1 - a_0^2 > 0,\ \forall (t,x,p) \in \mathbb{R}_t \times \Sigma^+$ we deduce that $f = 0,\ \gamma^{\pm} f = 0$.

□

The existence of T periodic solution for the Vlasov-Poisson system with boundary condition (6.1) follows by the same method as previous, with minor changes. At the beginning we assume that g and a has compact support in momentum. We start by analyzing the perturbed Vlasov problem :

$$\alpha f + \partial_t f + v_\delta(p) \cdot \nabla_x f + qE \cdot \nabla_p f = 0,\ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^N_p,$$

$$\gamma^- f(t,x,p) = g(t,x,p) + a(t,x,p)\gamma^+ f(t,x,Rp),\ (t,x,p) \in \mathbb{R}_t \times \Sigma^-.$$

Note that if $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^N, g = g \cdot \mathbf{1}_{\{|p|\leq R\}}, a = a \cdot \mathbf{1}_{\{|p|\leq R\}}$, then $\gamma^- f = \gamma^- f \cdot \mathbf{1}_{\{|p|\leq R\}}$, $f = f \cdot \mathbf{1}_{\{|p|\leq R+2D_\delta\}}$, $\gamma^+ f = \gamma^+ f \cdot \mathbf{1}_{\{|p|\leq R+2D_\delta\}}$, with $D_\delta = \left(\frac{3(\lambda+1)\cdot diam(\Omega)\cdot|q|\cdot\|E\|_\infty}{C_\delta\cdot(1-2^{-(\lambda+1)})}\right)^{\frac{1}{\lambda+1}}$. Indeed, it is sufficient to observe that the sequence $(f_k)_{k\geq 0}$ constructed in the proof of Proposition 6.2 verifies $\gamma^- f_k = \gamma^- f_k \cdot \mathbf{1}_{\{|p|\leq R\}}$, $f_k = f_k \cdot \mathbf{1}_{\{|p|\leq R+2D_\delta\}}$, $\gamma^+ f_k = \gamma^+ f_k \cdot \mathbf{1}_{\{|p|\leq R+2D_\delta\}}, \forall k \geq 0$. The existence for the regularized Vlasov-Poisson system (3.1), (6.1), (1.6), (1.7) is obtained by using the Schauder fixed point theorem, as before. The passing to the limit for $\varepsilon \searrow 0$ follows in the same manner and therefore we deduce the existence of T periodic solution for (3.3), (6.1), (1.6), (1.7). The a priori estimates follow by similar computations. For example, by using equality (4.2) we find :

$$\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\ dtdxdp + \int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))\gamma^+ f\ dtd\sigma dp = -\int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x)) g\ dtd\sigma dp$$
$$-\int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x)) a(t,x,p)\gamma^+ f(t,x,Rp)\ dtd\sigma dp,$$

and after using the boundary condition (6.1) we deduce that :

$$\alpha \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^N_p} f\ dtdxdp + \int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))(1 - a(Rp))\gamma^+ f\ dtd\sigma dp = -\int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x)) g\ dtd\sigma dp.$$

Since $a(t,x,p) \leq a_0 < 1,\ \forall (t,x,p) \in \mathbb{R}_t \times \Sigma^-$, one gets :

$$\int_0^T\!\!\int_{\Sigma^\pm} |(v_\delta(p)\cdot n(x))|\gamma^\pm f(t,x,p)\ dtd\sigma dp \leq \frac{1}{1-a_0} \int_0^T\!\!\int_{\Sigma^-} |(v_\delta(p)\cdot n(x))| g(t,x,p)\ dtd\sigma dp.$$

Similarly, by using equality (4.15) we deduce that :

$$\int_0^T\!\!\int_{\Sigma^+} (v_\delta(p)\cdot n(x))(1-a(Rp))(\mathcal{E}_\delta(p) + q\cdot\varphi_0(t,x))\gamma^+ f\ dtd\sigma dp \leq \varepsilon_0 \int_0^T\!\!\int_{\partial\Omega} \partial_n \Phi(\alpha\varphi_0(t,x) - \partial_t\varphi_0)\ dtd\sigma$$
$$-\int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x))(\mathcal{E}_\delta(p) + q\cdot\varphi_0(t,x)) g\ dtd\sigma dp,$$

which implies that :

$$(1-a_0)\int_0^T\!\!\int_{\Sigma^\pm} |(v_\delta(p)\cdot n(x))|\mathcal{E}_\delta(p)\gamma^\pm f\ dtd\sigma dp \leq \varepsilon_0 \|\partial_n\Phi\|_{L^2}(\alpha\ \|\varphi_0\|_{L^2} + \|\partial_t\varphi_0\|_{L^2})$$
$$-\int_0^T\!\!\int_{\Sigma^-} (v_\delta(p)\cdot n(x))\left(\mathcal{E}_\delta(p) + \left(1 + \frac{1}{1-a_0}\right)|q|\ \|\varphi_0\|_\infty\right) g\ dtd\sigma dp.$$

From this point the computations follow exactly as in the case of absorbing boundary conditions. Finally we obtain the existence result :

THEOREM 6.4. *Consider an open bounded set Ω of $\mathbb{R}^N_x$, $N \geq 2$ with the boundary strictly star-shaped with respect to some point of Ω. Assume that $(g \geq 0, \varphi_0) \in L^\infty(\mathbb{R}_t \times \Sigma^-) \times L^\infty(\mathbb{R}_t \times \partial\Omega)$ are T periodic, such that :*

$$W_0 = \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\ dtd\sigma dp + \|\varphi_0\|^2_{L^2(]0,T[;H^1(\partial\Omega))} + \|\partial_t\varphi_0\|^2_{L^2(]0,T[\times\partial\Omega)} < \infty.$$

Then there is a T periodic weak solution $(f,\Phi) \in L^\infty(\mathbb{R}_t\times\Omega\times\mathbb{R}^N_p)\times L^2(]0,T[;H^1(\Omega))$ for the Vlasov-Poisson system (2.1), (1.6), (6.1), (1.7) (classical or relativistic case) with traces $(\gamma^\pm f, \partial_n\Phi) \in L^\infty(\mathbb{R}_t \times \Sigma^\pm) \times L^2(]0,T[\times\partial\Omega)$, $\|f\|_\infty \leq (1-a_0)^{-1}\|g\|_\infty$, $\|\gamma^\pm f\|_\infty \leq (1-a_0)^{-1}\|g\|_\infty$, such that :

$$\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)(1+\mathcal{E}(p))\ dtdxdp + \int_0^T\!\!\int_\Omega |\nabla_x\Phi|^2(t,x)\ dtdx + \int_0^T\!\!\int_{\partial\Omega} |\partial_n\Phi|^2(t,x)\ dtd\sigma$$
$$+\int_0^T\!\!\int_\Sigma |(v(p)\cdot n(x))|\gamma f(t,x,p)(1+\mathcal{E}(p))\ dtd\sigma dp \leq C(\Omega, \|g\|_\infty, \|\varphi_0\|_\infty, a_0)F(W_0).$$

Moreover the solution verifies :

$$\int_0^T\!\!\int_{\Sigma^+} (1-a(t,x,Rp))(v(p)\cdot n(x))\gamma^+ f(t,x,p)\ dtd\sigma dp = -\int_0^T\!\!\int_{\Sigma^-} (v(p)\cdot n(x))g(t,x,p)\ dtd\sigma dp,$$

and :

$$\int_0^T\!\!\int_{\Sigma^+} (1-a(t,x,Rp))(v(p)\cdot n(x))(\mathcal{E}(p)+q\ \varphi_0(t,x))\gamma^+ f\ dtd\sigma dp \leq -\ \varepsilon_0 \int_0^T\!\!\int_{\partial\Omega} \partial_n\Phi\ \partial_t\varphi_0\ dtd\sigma$$
$$-\int_0^T\!\!\int_{\Sigma^-} (v(p)\cdot n(x))(\mathcal{E}(p)+q\ \varphi_0(t,x))g\ dtd\sigma dp.$$

REMARK 6.5. *All these arguments apply for the Vlasov-Poisson system with several species of particles.*

7. A priori estimates for the time periodic Vlasov-Poisson-Fokker-Planck system.

We end this paper by presenting a priori estimates for T periodic solutions for the Vlasov-Poisson-Fokker-Planck system (classical case) obtained by formal computations (details for this system can be found in [7], [12]). The system is given by :

$$\partial_t f + v(p)\cdot\nabla_x f - q\nabla_x\Phi\cdot\nabla_p f = \sigma\Delta_p f + \nabla_p\cdot(\beta v(p) f) = \nabla_p\cdot(\sigma\nabla_p f + \beta v(p) f),\ \ (t,x,p) \in \mathbb{R}_t\times\Omega\times\mathbb{R}^N_p,$$

$$-\Delta_x\Phi = \frac{1}{\varepsilon_0}\rho,\ \ (t,x) \in \mathbb{R}_t \times \Omega,$$

with the boundary conditions $f(t,x,p) = g(t,x,p)$, $(t,x,p) \in \mathbb{R}_t \times \Sigma^-$ and $\varphi(t,x) = \varphi_0(t,x)$, $(t,x) \in \mathbb{R}_t \times \partial\Omega$. The parameters σ, β satisfy $\sigma > 0, \beta \geq 0$. We suppose that $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$, $g \geq 0$ such that $\int_0^T\int_{\Sigma^-} |(v(p)\cdot n(x))| g(t,x,p)\mathcal{E}(p)\ dtd\sigma dp < +\infty$ and in order to simplify we take $\varphi_0 = 0$. Suppose that (f,Φ) is a regular T periodic solution. Multiplying the Vlasov equation by 1 yields after integration :

$$\int_0^T\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f\ dtd\sigma dp = -\int_0^T\int_{\Sigma^-} (v(p)\cdot n(x)) g\ dtd\sigma dp. \tag{7.1}$$

After multiplication by $\mathcal{E}(p) + q\cdot\Phi(t,x)$ we obtain :

$$\begin{aligned}\int_0^T\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f\mathcal{E}(p)\ dtd\sigma dp &= \int_0^T\int_\Omega\int_{\mathbb{R}_p^N} qf(t,x,p)\partial_t\Phi\ dtdxdp \\ &-\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} (\sigma\nabla_p f + \beta v(p)f)\cdot v(p)\ dtdxdp - \int_0^T\int_{\Sigma^-} (v(p)\cdot n(x))\mathcal{E}(p) g\ dtd\sigma dp,\end{aligned}$$

and thus, since $\int_0^T\int_\Omega \rho\partial_t\Phi\ dtdx = -\varepsilon_0\int_0^T\int_\Omega \Delta_x\Phi\partial_t\Phi\ dtdx = \frac{\varepsilon_0}{2}\int_0^T \frac{d}{dt}\left(\int_\Omega |\nabla_x\Phi|^2\ dx\right) dt = 0$ and $\nabla_p\cdot v(p) = \frac{N}{m}$, we deduce that :

$$\begin{aligned}\beta \int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f(t,x,p)|v(p)|^2\ dtdxdp + \int_0^T\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f(t,x,p)\mathcal{E}(p)\ dtd\sigma dp = \\ -\int_0^T\int_{\Sigma^-} (v(p)\cdot n(x)) g(t,x,p)\mathcal{E}(p)\ dtd\sigma dp + \sigma\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f(t,x,p)\nabla_p\cdot v(p)\ dtdxdp \\ = -\int_0^T\int_{\Sigma^-} (v(p)\cdot n(x)) g(t,x,p)\mathcal{E}(p)\ dtd\sigma dp + \frac{N}{m}\sigma\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f(t,x,p)\ dtdxdp.\end{aligned} \tag{7.2}$$

After multiplication by $(p\cdot x)$ one gets :

$$\begin{aligned}\int_0^T\int_\Sigma (v(p)\cdot n(x))\gamma f(t,x,p)(p\cdot x)\ dtd\sigma dp &= \int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f(t,x,p)((v(p)\cdot p) + qE(t,x)\cdot x)\ dtdxdp \\ &-\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} (\sigma\nabla_p f + \beta f(t,x,p)v(p))\cdot x\ dtdxdp,\end{aligned}$$

and thus :

$$\begin{aligned}\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f(t,x,p)(v(p)\cdot p)\ dtdxdp + \varepsilon_0\int_0^T\int_\Omega \sum_{i=1}^N \left(\sum_{j=1}^N \frac{\partial}{\partial x_j}(E_iE_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|E|^2\right) x_i\ dtdx \\ = \int_0^T\int_\Sigma (v(p)\cdot n(x))\gamma f(p\cdot x)\ dtd\sigma dp + \beta\int_0^T\int_\Omega\int_{\mathbb{R}_p^N} f(t,x,p)(v(p)\cdot x)\ dtdxdp.\end{aligned} \tag{7.3}$$

As before if Ω is bounded, $\Omega \subset B(0,R)$, and $\partial\Omega$ is strictly star-shaped with respect to $0 \in \Omega$, since

$(v(p)\cdot p) \geq \mathcal{E}(p)$ we have by (7.3), (7.2), (7.1) :

$$\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)\mathcal{E}(p)\,dtdxdp + \frac{\varepsilon_0\cdot r}{2}\int_0^T\!\!\int_{\partial\Omega}|\partial_n\Phi|^2\,dtd\sigma + \varepsilon_0\left(\frac{N}{2}-1\right)\int_0^T\!\!\int_\Omega|\nabla_x\Phi|^2\,dtdx$$
$$\leq CR\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|g(1+\mathcal{E}(p))\,dtd\sigma dp + CR\int_0^T\!\!\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(1+\mathcal{E}(p))\,dtd\sigma dp$$
$$+\beta R\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)(1+|v(p)|^2)\,dtdxdp,$$
$$\leq(2CR+R)\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g\,dtd\sigma dp + \left(CR\frac{N}{m}\sigma+\beta R+R\frac{N}{m}\sigma\right)\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f.$$

We have obtained :

$$\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}(p))f\,dtdxdp + \varepsilon_0\left(\frac{N}{2}-1\right)\int_0^T\!\!\int_\Omega|\nabla_x\Phi|^2\,dtdx + \frac{\varepsilon_0\cdot r}{2}\int_0^T\!\!\int_{\partial\Omega}|\partial_n\Phi|^2\,dtd\sigma$$
$$+\int_0^T\!\!\int_{\Sigma^+}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\,dtd\sigma dp$$
$$\leq C_1\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g\,dtd\sigma dp + C_2\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f\,dtdxdp, \tag{7.4}$$

where $C_1 = 2CR+R+1, C_2 = \frac{N}{m}\sigma(CR+R+1)+\beta R+1$. If C_3 is a constant such that $1+\mathcal{E}(p)\geq C_3|p|,\ \forall p\in\mathbb{R}^N_p$, we have :

$$\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f\,dtdxdp = \int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f\cdot\mathbf{1}_{\{|p|\leq R_1\}}\,dtdxdp + \int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p} f\cdot\mathbf{1}_{\{|p|>R_1\}}\,dtdxdp$$
$$\leq\|g\|_\infty T\cdot meas(\Omega)\cdot R_1^N\cdot meas(B_{\mathbb{R}^N_p}(0,1)) + \frac{1}{R_1}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p}|p|f\,dtdxdp$$
$$\leq\|g\|_\infty T\cdot meas(\Omega)\cdot R_1^N\cdot meas(B_{\mathbb{R}^N_p}(0,1)) + \frac{1}{C_3R_1}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}(p))f\,dtdxdp.$$

By taking R_1 large enough (such that $\frac{C_2}{C_3\cdot R_1}<1$), from (7.4) we deduce that :

$$\left(1-\frac{C_2}{C_3R_1}\right)\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}(p))f\,dtdxdp + \varepsilon_0\left(\frac{N}{2}-1\right)\int_0^T\!\!\int_\Omega|\nabla_x\Phi|^2\,dtdx + \frac{\varepsilon_0 r}{2}\int_0^T\!\!\int_{\partial\Omega}|\partial_n\Phi|^2\,dtd\sigma$$
$$+\int_0^T\!\!\int_{\Sigma^+}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\,dtd\sigma dp$$
$$\leq C_1\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g\,dtd\sigma dp + C_2T\cdot meas(\Omega)R_1^N meas(B_{\mathbb{R}^N_p}(0,1))\|g\|_\infty,$$

which gives uniform estimates for the kinetic energy, the outgoing kinetic energy and the normal trace of the electric field, when $N\geq 2$ and also for the electric energy if $N>2$. The case $N=2$ can be analyzed as well as it was done in the proof of Proposition 4.7. Notice also that these estimates are uniform with respect to $\beta\in[0,\beta_0]$, $\forall\,\beta_0\geq 0$. In particular we can take $\beta=0$.

8. Appendix.

We detail here the proof of the momentum change lemma

Proof. (of Lemma 2.9) (1) We have $P(s) = P(t) + \int_t^s q \cdot E(\tau, X(\tau))d\tau$, $\forall s_{in} \le s \le s_{out}$, which implies that $|P(s)| \ge |P(t)| - |q| \cdot |s-t| \cdot \|E\|_\infty$, $\forall s_{in} \le s \le s_{out}$. Consider $r_1 = \max\{s_{in}, t - \frac{|P(t)|}{2|q|\cdot\|E\|_\infty}\}$, $r_2 = \min\{s_{out}, t + \frac{|P(t)|}{2|q|\cdot\|E\|_\infty}\}$. By using the monotonicity of w we deduce that :

$$w(|P(s)|) \ge w(|P(t)| - |q| \cdot |s-t| \cdot \|E\|_\infty),\ \forall r_1 \le s \le r_2.$$

Notice that we have :

$$\left(\frac{P(s)}{|P(s)|}, \frac{P(t)}{|P(t)|}\right) = \frac{1}{|P(s)|} \cdot \left(P(s), \frac{P(t)}{|P(t)|}\right) \ge \frac{|P(t)| - |q| \cdot |s-t| \cdot \|E\|_\infty}{|P(t)| + |q| \cdot |s-t| \cdot \|E\|_\infty} \ge \frac{1}{3},\ r_1 \le s \le r_2.$$

By combining with the equation $\frac{dX}{ds} = v(P(s))$ we find that :

$$\begin{aligned}\left(\frac{P(t)}{|P(t)|}, \frac{dX}{ds}\right) =& \left(\frac{P(t)}{|P(t)|}, \frac{P(s)}{|P(s)|}\right) w(|P(s)|) \\ \ge& \frac{1}{3} \cdot w(|P(t)| - |q| \cdot |s-t| \cdot \|E\|_\infty),\ r_1 \le s \le r_2. \end{aligned} \tag{8.1}$$

After integration of (8.1) we deduce that :

$$\begin{aligned} diam(\Omega) \ge& \left|\left(\frac{P(t)}{|P(t)|}, X(s) - X(t)\right)\right| = \left|\int_t^s \left(\frac{P(t)}{|P(t)|}, v(P(\tau))\right) d\tau\right| \\ \ge& \frac{1}{3} \cdot \left|\int_t^s w(|P(t)| - |q| \cdot |\tau - t| \cdot \|E\|_\infty) d\tau\right| \\ =& \frac{1}{3 \cdot |q| \cdot \|E\|_\infty} \int w(u) \cdot \mathbf{1}_{\{|P(t)| - |q|\cdot|s-t|\cdot\|E\|_\infty < u < |P(t)|\}} du,\ r_1 \le s \le r_2. \end{aligned}$$

Suppose now that $s_{out} \ge t + \frac{|P(t)|}{2|q|\cdot\|E\|_\infty}$ or $s_{in} \le t - \frac{|P(t)|}{2|q|\cdot\|E\|_\infty}$. Thus we can take $s = t + \frac{|P(t)|}{2|q|\cdot\|E\|_\infty}$ or $s = t - \frac{|P(t)|}{2|q|\cdot\|E\|_\infty}$ in the previous inequality and we obtain :

$$diam(\Omega) \ge \frac{1}{3|q| \cdot \|E\|_\infty} \int_{|P(t)|/2}^{|P(t)|} C \cdot u^\lambda du = \frac{C \cdot |P(t)|^{\lambda+1} \cdot (1 - 2^{-(\lambda+1)})}{3|q| \cdot \|E\|_\infty \cdot (\lambda + 1)}.$$

We deduce that $|P(t)| \le D$ which contradicts our hypothesis. Therefore we must have $r_1 = s_{in}$, $r_2 = s_{out}$. In particular we have :

$$\begin{aligned} 3 \cdot diam(\Omega) \cdot |q| \cdot \|E\|_\infty \ge& \int C \cdot u^\lambda \cdot \mathbf{1}_{\{|P(t)| - |q|\cdot|s-t|\cdot\|E\|_\infty < u < |P(t)|\}} du \\ =& \frac{C}{\lambda+1} \left(|P(t)|^{\lambda+1} - (|P(t)| - |q| \cdot |s-t| \cdot \|E\|_\infty)^{\lambda+1}\right),\ \forall s_{in} \le s \le s_{out}, \end{aligned}$$

saying that $F(s) \ge 0$, $\forall s_{in} \le s \le s_{out}$ with $F : [t - \frac{|P(t)|}{|q|\cdot\|E\|_\infty}, t + \frac{|P(t)|}{|q|\cdot\|E\|_\infty}] \to \mathbb{R}$ given by :

$$F(s) = (|P(t)| - |q| \cdot |s-t| \cdot \|E\|_\infty)^{\lambda+1} - |P(t)|^{\lambda+1} + \frac{3 \cdot (\lambda+1) \cdot diam(\Omega) \cdot |q| \cdot \|E\|_\infty}{C}.$$

We observe that F is strictly increasing on $[t - \frac{|P(t)|}{|q|\cdot\|E\|_\infty}, t]$ and strictly decreasing on $[t, t + \frac{|P(t)|}{|q|\cdot\|E\|_\infty}]$ and that $F(t) > 0$. Moreover, by the hypothesis we have $F(t \pm \frac{|P(t)|}{2|q|\cdot\|E\|_\infty}) < 0$ and thus there is a unique root u_1 for F on the interval $[t - \frac{|P(t)|}{|q| \cdot \|E\|_\infty}, t]$ and a unique root u_2 for F on the interval $[t, t + \frac{|P(t)|}{|q| \cdot \|E\|_\infty}]$. We have the inequalities :

$$t - \frac{|P(t)|}{2|q| \cdot \|E\|_\infty} < u_1 \le s_{in} \le t \le s_{out} \le u_2 < t + \frac{|P(t)|}{2|q| \cdot \|E\|_\infty}.$$

Indeed, if $s_{in} < u_1$ we can take $s_1 \in]s_{in}, u_1[$ with $F(s_1) < F(u_1) = 0$ (F is strictly increasing on $[t - \frac{|P(t)|}{|q| \cdot \|E\|_\infty}, t]$) which is in contradiction with $F(s) \ge 0,\ \forall s_{in} \le s \le s_{out}$. Similarly we prove that $s_{out} \le u_2$. Now we only need to estimate the roots $u_{1,2}$. We have :

$$|P(t)|^{\lambda+1} - (|P(t)| - |q| \cdot |u_i - t| \cdot \|E\|_\infty)^{\lambda+1} = \frac{3(\lambda+1) \cdot diam(\Omega) \cdot |q| \cdot \|E\|_\infty}{C}.$$

By using the inequality $x^\alpha - y^\alpha \ge x^{\alpha-1}(x-y),\ \forall x \ge y > 0, \forall \alpha \ge 1$ with $x = |P(t)|,\ y = |P(t)| - |q| \cdot |u_i - t| \cdot \|E\|_\infty$, $\alpha = \lambda + 1$, we find that $|u_i - t| \le 3(\lambda+1) \cdot diam(\Omega)/(C \cdot |P(t)|^\lambda)$ and therefore one gets :

$$\max\{t - s_{in}, s_{out} - t\} \le \max\{t - u_1, u_2 - t\} \le \frac{3(\lambda+1) \cdot diam(\Omega)}{C \cdot |P(t)|^\lambda},$$

and also :

$$s_{out} - s_{in} \le \frac{6(\lambda+1) \cdot diam(\Omega)}{C \cdot |P(t)|^\lambda}.$$

By direct computation we find for $s_{in} \le s \le s_{out}$ that :

$$|P(s) - P(t)| \le |q| \cdot \|E\|_\infty \cdot |s - t| \le |q| \cdot \|E\|_\infty \cdot \frac{3(\lambda+1) \cdot diam(\Omega)}{C \cdot |P(t)|^\lambda} \le \frac{D^{\lambda+1}}{|P(t)|^\lambda} \le D.$$

(2) If $|P(s_i)| \le D$ for $i = 1, 2$, then we have $|P(s_1) - P(s_2)| \le 2D$. If $|P(s_1)| > D$ we can apply the previous point with $t = s_1$ and we deduce that $|P(s_1) - P(s_2)| \le D \le 2D,\ \forall s_{in} \le s_1 \le s_2 \le s_{out}$. If $|P(s_2)| > D$ we apply the previous point with $t = s_2$.

□

We justify now the boundedness of the total energy, uniformly with respect to the time.

Proof. (of Proposition 4.8) Indeed, by Proposition 4.7 we deduce that there is $t_0 \in]0, T[$ such that :

$$\int_\Omega \int_{\mathbb{R}^N_p} f(t_0, x, p)(1 + \mathcal{E}_\delta(p))\ dxdp + \int_\Omega |\nabla_x \Phi|^2(t_0, x)\ dx \le \frac{1}{T} \int_0^T \int_\Omega \int_{\mathbb{R}^N_p} (1 + \mathcal{E}_\delta(p)) f\ dtdxdp$$
$$+ \frac{1}{T} \int_0^T \int_\Omega |\nabla_x \Phi|^2\ dtdx \le C \cdot F_{\alpha,\delta}(W_0). \tag{8.2}$$

Now, by using (4.1), after integration on $]t_0, t[$ with $t_0 \le t \le t_0 + T$ we deduce that :

$$\alpha \int_{t_0}^t ds \int_\Omega \int_{\mathbb{R}^N_p} f(s, x, p)\ dxdp + \int_\Omega \int_{\mathbb{R}^N_p} f(t, x, p)\ dxdp + \int_{t_0}^t ds \int_{\Sigma^+} (v_\delta(p) \cdot n(x)) \gamma^+ f\ d\sigma dp$$
$$= \int_\Omega \int_{\mathbb{R}^N_p} f(t_0, x, p)\ dxdp - \int_{t_0}^t ds \int_{\Sigma^-} (v_\delta(p) \cdot n(x)) g(s, x, p)\ d\sigma dp,$$

and thus, by periodicity we obtain for all $t \in \mathbb{R}_t$:

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\ dxdp &\leq \int_\Omega \int_{\mathbb{R}^N_p} f(t_0,x,p)\ dxdp + \int_0^T\!\!\int_{\Sigma^-} |(v_\delta(p)\cdot n(x))| g\ dtd\sigma dp \\ &\leq C\cdot F_{\alpha,\delta}(W_0) + \int_0^T\!\!\int_{\Sigma^-} |(v_\delta(p)\cdot n(x))| g\ dtd\sigma dp.\end{aligned}$$

Observe that for any $t \in \mathbb{R}_t$ we have :

$$\begin{aligned}\left|\int_\Omega \rho(t,x)\Phi_0(t,x)\ dx\right| &= \left|\int_\Omega \int_{\mathbb{R}^N_p} qf(t,x,p)\Phi_0(t,x)\ dxdp\right| \leq \int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\cdot |q|\cdot |\Phi_0(t,x)|\ dxdp \\ &\leq |q|\cdot \|\varphi_0\|_\infty \int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\ dxdp \leq |q|\cdot\|\varphi_0\|_\infty (C\cdot F_{\alpha,\delta}(W_0) \\ &+ \int_0^T\!\!\int_{\Sigma^-} |(v_\delta(p)\cdot n(x))| g\ dtd\sigma dp). \qquad (8.3)\end{aligned}$$

Note also that :

$$\begin{aligned}\int_\Omega \rho(t,x)\Phi(t,x)\ dx &= \int_\Omega \rho(t,x)\Phi_0(t,x)\ dx + \int_\Omega \rho(t,x)\Phi_s(t,x)\ dx \\ &= \int_\Omega \rho(t,x)\Phi_0(t,x)\ dx + \varepsilon_0 \int_\Omega |\nabla_x \Phi_s|^2(t,x)\ dx. \qquad (8.4)\end{aligned}$$

From (4.12) we deduce that :

$$\begin{aligned}\alpha \int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\ \cdot \mathcal{E}_\delta(p)\ dxdp + \alpha \int_\Omega \rho(t,x)\Phi(t,x)\ dx + \frac{d}{dt}\int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\mathcal{E}_\delta(p)\ dxdp \qquad (8.5) \\ + \frac{d}{dt}\int_\Omega \rho(t,x)\Phi(t,x)\ dx + \int_{\Sigma^+} (v_\delta(p)\cdot n(x))\gamma^+ f(\mathcal{E}_\delta(p) + q\cdot\varphi_0(t,x))\ d\sigma dp \\ = -\int_{\Sigma^-} (v_\delta(p)\cdot n(x)) g(t,x,p)(\mathcal{E}_\delta(p) + q\cdot\varphi_0(t,x))\ d\sigma dp + \int_\Omega \rho(t,x)\partial_t \Phi\ dx.\end{aligned}$$

After integration on $]t_0, t[$ with $t \in]t_0, t_0+T[$, by using (4.13), (4.14), (8.4) we obtain :

$$\begin{aligned}&\alpha \int_{t_0}^t ds \int_\Omega \int_{\mathbb{R}^N_p} f(s,x,p)\cdot\mathcal{E}_\delta(p)\ dxdp + \alpha\cdot\varepsilon_0 \int_{t_0}^t ds \int_\Omega |\nabla_x \Phi|^2\ dx - \alpha\cdot\varepsilon_0 \int_{t_0}^t ds \int_{\partial\Omega} \partial_n \Phi\cdot\varphi_0(s,x)\ d\sigma \\ &+ \int_\Omega \int_{\mathbb{R}^N_p} f(s,x,p)\mathcal{E}_\delta(p)\ dxdp\Big|_{s=t_0}^{s=t} + \int_\Omega \rho(s,x)\Phi_0(s,x)\ dx\Big|_{s=t_0}^{s=t} + \varepsilon_0 \int_\Omega |\nabla_x \Phi_s|^2(s,x)\ dx\Big|_{s=t_0}^{s=t} \\ &+ \int_{t_0}^t ds \int_{\Sigma^+} (v_\delta(p)\cdot n(x))(\mathcal{E}_\delta(p) + q\cdot\varphi_0(s,x))\gamma^+ f\ d\sigma dp \\ &= -\int_{t_0}^t ds \int_{\Sigma^-} (v_\delta(p)\cdot n(x))(\mathcal{E}_\delta(p) + q\cdot\varphi_0(s,x)) g\ d\sigma dp + \frac{\varepsilon_0}{2}\int_\Omega |\nabla_x \Phi|^2(s,x)\ dx\Big|_{s=t_0}^{s=t} \\ &- \varepsilon_0 \int_{t_0}^t ds \int_{\partial\Omega} \partial_n \Phi\cdot\partial_t \varphi_0\ d\sigma. \qquad (8.6)\end{aligned}$$

Let us analyze each term in the previous equality. We have $\alpha \int_{t_0}^t ds \int_\Omega \int_{\mathbb{R}^N_p} f(s)\mathcal{E}_\delta(p)\ dxdp \geq 0$ and also $\alpha\cdot\varepsilon_0 \int_{t_0}^t ds \int_\Omega |\nabla_x \Phi|^2(s,x)\ dx \geq 0$. For the third term we write :

$$\left|\alpha\cdot\varepsilon_0 \int_{t_0}^t ds \int_{\partial\Omega} \partial_n \Phi\cdot\varphi_0\ d\sigma\right| \leq \alpha\cdot\varepsilon_0 \|\partial_n \Phi\|_{L^2(]0,T[\times\partial\Omega)}\cdot\|\varphi_0\|_{L^2(]0,T[\times\partial\Omega)}.$$

By using (8.3) we have :

$$\left|\int_\Omega \rho(s,x)\Phi_0(s,x)\,dx|_{s=t_0}^{s=t}\right| \leq 2|q|\cdot\|\varphi_0\|_\infty\left(C\cdot F_{\alpha,\delta}(W_0)+\int_0^T\!\!\int_{\Sigma^-}|(v_\delta(p)\cdot n(x))|g\;dtd\sigma dp\right).$$

We can write :

$$\begin{aligned}\varepsilon_0\int_\Omega|\nabla_x\Phi_s|^2\,dx-\frac{\varepsilon_0}{2}\int_\Omega|\nabla_x\Phi|^2\,dx &\geq \varepsilon_0\left(\|\nabla_x\Phi(t)\|_{L^2(\Omega)}-\|\nabla_x\Phi_0(t)\|_{L^2(\Omega)}\right)^2-\frac{\varepsilon_0}{2}\|\nabla_x\Phi(t)\|^2_{L^2(\Omega)}\\ &=\frac{\varepsilon_0}{2}\left(\|\nabla_x\Phi(t)\|_{L^2(\Omega)}-2\|\nabla_x\Phi_0(t)\|_{L^2(\Omega)}\right)^2-\varepsilon_0\|\nabla_x\Phi_0(t)\|^2_{L^2(\Omega)},\end{aligned}$$

and also :

$$\varepsilon_0\int_\Omega|\nabla_x\Phi_s|^2(t_0,x)\,dx-\frac{\varepsilon_0}{2}\int_\Omega|\nabla_x\Phi|^2(t_0,x)\,dx\leq 2\varepsilon_0\left(\|\nabla_x\Phi(t_0)\|^2_{L^2(\Omega)}+\|\nabla_x\Phi_0(t_0)\|^2_{L^2(\Omega)}\right).$$

For the last term in (8.6) we write :

$$\left|\varepsilon_0\int_{t_0}^t ds\int_{\partial\Omega}\partial_n\Phi\cdot\partial_t\varphi_0\,d\sigma\right|\leq\varepsilon_0\|\partial_n\Phi\|_{L^2(]0,T[\times\partial\Omega)}\cdot\|\partial_t\varphi_0\|_{L^2(]0,T[\times\partial\Omega)}.$$

By using all these computations, the equality (8.6) implies :

$$\begin{aligned}&\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)\mathcal{E}_\delta(p)\,dxdp+\frac{\varepsilon_0}{2}\left(\|\nabla_x\Phi(t)\|_{L^2}-2\cdot\|\nabla_x\Phi_0(t)\|_{L^2}\right)^2\leq\int_\Omega\int_{\mathbb{R}^N_p} f(t_0,x,p)\mathcal{E}_\delta(p)\,dxdp\\ &\quad+2\varepsilon_0\left(\|\nabla_x\Phi(t_0)\|^2_{L^2}+\|\nabla_x\Phi_0(t_0)\|^2_{L^2}\right)+\alpha\cdot\varepsilon_0\|\partial_n\Phi\|_{L^2(]0,T[\times\partial\Omega)}\cdot\|\varphi_0\|_{L^2(]0,T[\times\partial\Omega)}\\ &\quad+2|q|\cdot\|\varphi_0\|_\infty\left(C\cdot F_{\alpha,\delta}(W_0)+\int_0^T\!\!\int_{\Sigma^-}|(v_\delta(p)\cdot n(x))|g\;dtd\sigma dp\right)+\varepsilon_0\|\nabla_x\Phi_0(t)\|^2_{L^2(\Omega)}\\ &\quad+\int_0^T\!\!\int_{\Sigma^-}|(v_\delta(p)\cdot n(x))|(\mathcal{E}_\delta(p)+2\cdot|q|\cdot\|\varphi_0\|_\infty)g\;dtd\sigma dp\\ &\quad+\varepsilon_0\|\partial_n\Phi\|_{L^2(]0,T[\times\partial\Omega)}\cdot\|\partial_t\varphi_0\|_{L^2(]0,T[\times\partial\Omega)}.\end{aligned}$$

Finally, by using that $\|\Phi_0\|_{L^\infty(\mathbb{R}_t;H^1(\Omega))}\leq C(\Omega)\cdot\|\varphi_0\|_{L^\infty(\mathbb{R}_t;H^{1/2}(\partial\Omega))}$, the inequality (8.2) and Lemma 4.7, we deduce that the total energy is uniformly bounded (for $\alpha>0$ small enough) with respect to $t\in\mathbb{R}_t$:

$$\begin{aligned}&\int_\Omega\int_{\mathbb{R}^N_p} f(t,x,p)(1+\mathcal{E}_\delta(p))\,dxdp+\int_\Omega|\nabla_x\Phi|^2(t,x)\,dx\\ &\qquad\leq C_1(\Omega,\|g\|_\infty,\|\varphi_0\|_\infty)\cdot\left(F_{\alpha,\delta}(W_0)+\|\varphi_0\|^2_{L^\infty(\mathbb{R}_t;H^{1/2}(\partial\Omega))}\right).\end{aligned}$$

□

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