Multi-scale analysis for highly anisotropic parabolic problems

Thomas BLANC *, Mihaï BOSTAN †

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Abstract

We focus on the asymptotic behavior of strongly anisotropic parabolic problems. We concentrate on heat equations, whose diffusion matrix fields have disparate eigen-values. We establish strong convergence results toward a profile. Under suitable smoothness hypotheses, by introducing an appropriate corrector term, we estimate the convergence rate. The arguments rely on two-scale analysis, based on average operators with respect to unitary groups.

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1 Introduction

The subject matter of this paper concerns the behavior of the solutions for heat equations whose diffusion becomes very high along some privilegiated directions. This study is motivated by many applications like transport in magnetized plasmas [5], image processing [12, 17], thermal properties of crystals [13]. We consider the parabolic problem

$$\partial_t u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) - \frac{1}{\varepsilon}\mathrm{div}_y(b(y)\otimes b(y)\nabla_y u^\varepsilon) = 0, \ \ (t,y)\in\mathbb{R}_+\times\mathbb{R}^m \tag{1}$$

$$u^\varepsilon(0,y) = u^{\mathrm{in}}(y), \ \ y\in\mathbb{R}^m \tag{2}$$

where $D(y)\in\mathcal{M}_m(\mathbb{R})$ and $b(y)\in\mathbb{R}^m$ are given matrix and vector fields on $\mathbb{R}^m$. For any two vectors $\xi,\eta\in\mathbb{R}^m$, the notations $\xi\otimes\eta$ stands for the matrix whose entry (i,j) is $\xi_i\eta_j$, and for any two matrices $A,B\in\mathcal{M}_m(\mathbb{R})$, the notations $A:B$ stands for trace$({}^tAB) = \sum_{i=1}^m\sum_{j=1}^m A_{ji}B_{ji}$. The matrix field D is assumed symmetric, such that $D+b\otimes b$ is positive definite. We analyse the behavior of the family $(u^\varepsilon)_\varepsilon$ for small ε, let us say $0<\varepsilon\leq 1$, in which case $\left(D+\frac{1}{\varepsilon}b\otimes b\right)_{0<\varepsilon\leq 1}$ remain positive definite. Another motivation for performing this asymptotic analysis comes from the numerical simulation of highly anisotropic parabolic problems. Notice that the explicit methods require very small time steps, through the CFL stability condition $\Delta t\sim\varepsilon|\Delta y|^2$. Therefore implicit methods have been proposed in [2, 15, 16], finite volume methods have been discussed in [9, 1] and asymptotic preserving schemes have been investigated in [8, 10]. For a detailed theoretical study of (1), (2) we refer to [3] where it was shown that, for any initial condition $u^{\mathrm{in}}\in L^2(\mathbb{R}^m)$, the family $(u^\varepsilon)_\varepsilon$ converges weakly $\star$ in $L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))$ toward the solution of another parabolic problem, whose diffusion matrix field appears like an average of the original diffusion matrix field D. The main goal of this work is to go further into this analysis. We intend to give a complete description of the behavior of $(u^\varepsilon)_\varepsilon$, due to the high diffusion anisotropy. We prove a strong convergence result toward a profile, and analyze the well posedness of the corresponding limit model.

*Aix Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, UMR 7373, Château Gombert 39 rue F. Joliot Curie, 13453 Marseille FRANCE. E-mail : thomas.blanc@univ-amu.fr

†Aix Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, UMR 7373, Château Gombert 39 rue F. Joliot Curie, 13453 Marseille FRANCE. E-mail : mihai.bostan@univ-amu.fr

We consider variational solutions for (1), (2). For doing that we introduce a weighted Sobolev space H^1_P see (23) and define the bounded symmetric bilinear form

$$\mathfrak{a}^\varepsilon(u,v) = \int_{\mathbb{R}^m} D(y)\nabla u \cdot \nabla v \;\mathrm{d}y + \frac{1}{\varepsilon}\int_{\mathbb{R}^m} (b\cdot\nabla u)(b\cdot\nabla v)\;\mathrm{d}y, \quad u,v\in H^1_P.$$

The variational formulation for (1), (2) writes

$$u^\varepsilon(0) = u^{\mathrm{in}}, \;\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} u^\varepsilon(t,y)\varphi(y)\;\mathrm{d}y + \mathfrak{a}^\varepsilon(u^\varepsilon(t),\varphi) = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}_+), \;\varphi\in H^1_P.$$

The well posedness of the above problem follows by standard results. Under coercivity assumptions, for any $\varepsilon \in]0,1]$, there is a unique solution $u^\varepsilon \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m))\cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1_P), \varepsilon\in]0,1]$. We consider the second order operator $\mathcal{B} = -\mathcal{T}^2, \mathcal{T} = \mathrm{div}_y(\cdot\, b)$ and the semi-group $(e^{-\tau\mathcal{B}})_{\tau\in\mathbb{R}_+}$. The idea is to search for a solution $v = v(t)$ of another variational problem, such that

$$u^\varepsilon(t) = e^{-\frac{t}{\varepsilon}\mathcal{B}} v(t) + \mathcal{O}(\varepsilon) \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)). \tag{3}$$

The main difficulties are to identify the limit problem satisfied by $v(t)$ and to construct a corrector which will allow us to justify the approximation (3). The limit problem appears as a variational formulation whose bilinear form, denoted by $\mathfrak{m}$, is defined in Proposition 5.5. This bilinear form can be expressed in terms of two C^0-groups of unitary transformations operating on functions and matrix fields. We denote by $Y(s;y)$ the characteristic flow of the vector field b, by $(\zeta(s))_{s\in\mathbb{R}}$ the group of the translations along Y

$$\zeta(s)u = u\circ Y(s;\cdot), \;\; u\in L^2(\mathbb{R}^m), \;\; s\in\mathbb{R}$$

and by $(G(s))_{s\in\mathbb{R}}$ the group acting on the weighted L^2 space of matrix fields H_Q, given by

$$G(s)A = \partial Y^{-1}(s;\cdot)\; A\circ Y(s;\cdot)\; {}^t\partial Y^{-1}(s;\cdot), \;\; A\in H_Q, \;\; s\in\mathbb{R}$$

see Proposition 3.3. With these notations, the bilinear form $\mathfrak{m}$ writes cf. Proposition 5.5

$$\mathfrak{m}(u,v) = \int_{\mathbb{R}^m}\left\{\langle D\rangle(y)\nabla u + \lim_{S\to+\infty}\frac{1}{S}\int_0^S (G(s)D - \langle D\rangle)\nabla\zeta(2s)u\;\mathrm{d}s\right\}\cdot\nabla v\;\mathrm{d}y$$

for any $u,v\in H^1_P$, where $\langle D\rangle$ is the average of D along the C^0-group $(G(s))_{s\in\mathbb{R}}$ cf. Theorem 3.2

$$\langle D\rangle = \lim_{S\to+\infty}\frac{1}{S}\int_0^S G(s)D\;\mathrm{d}s \quad \text{in } H_Q.$$

The construction of the corrector requires a second bilinear form, cf. Proposition 7.1. We establish the following convergence result, see Theorem 8.1 for all the details, under suitable hypotheses : smoothness hypotheses on u^{in}, b and D, existence of a matrix field P which satisfies (18), (19) and structural assumptions associated to the fields b and D, see Sections 5.2, 7.1 and 7.2.

Theorem
Assume that u^{in}, b, D are smooth enough. Moreover, we assume that the hypotheses (18), (19) are satisfied, as well as the structural hypotheses given in Sections 5.2, 7.1 and 7.2. For any $\varepsilon\in]0,1]$ let us denote by $u^\varepsilon \in C_b(\mathbb{R}_+;L^2(\mathbb{R}^m))\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^1_P)$ the unique variational solution of (1), (2)

$$u^\varepsilon(0) = u^{\mathrm{in}}, \;\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} u^\varepsilon(t,y)\varphi(y)\;\mathrm{d}y + \mathfrak{a}^\varepsilon(u^\varepsilon(t),\varphi) = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}_+), \;\varphi\in H^1_P$$

and by $v \in C_b(\mathbb{R}_+;L^2(\mathbb{R}^m))\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^1_P)$ the unique variational solution

$$v(0) = u^{\mathrm{in}}, \;\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} v(t,y)\varphi(y)\;\mathrm{d}y + \mathfrak{m}(v(t),\varphi) = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}_+), \;\varphi\in H^1_P.$$

For any $T\in\mathbb{R}_+$ there is a constant C_T such that

$$\left|u^\varepsilon - e^{-\frac{t}{\varepsilon}\mathcal{B}}v\right|_{L^\infty([0,T];L^2(\mathbb{R}^m))} + \left|\nabla u^\varepsilon - \nabla e^{-\frac{t}{\varepsilon}\mathcal{B}}v\right|_{L^2([0,T];X_P)} \le C_T\varepsilon, \;\; 0<\varepsilon\le 1.$$

When the initial condition is well prepared, that is $\mathcal{T}u^{\mathrm{in}} = 0$, there is no boundary layer at $t = 0$ and the limit model is given by the parabolic equation associated to the average matrix field $\langle D \rangle$, see Remark 8.1

$$v(0) = u^{\mathrm{in}}, \;\; \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} v(t,y)\varphi(y) \;\mathrm{d}y + \int_{\mathbb{R}^m} \langle D \rangle (y) \nabla v(t) \cdot \nabla \varphi \;\mathrm{d}y = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}_+), \; \varphi \in H^1_P.$$

Our paper is organized as follows. The main lines of the asymptotic analysis are presented first in the finite dimensional case, cf. Section 2. The infinite dimensional case requires several tools and hypotheses. We define average operators for functions and matrix fields, see Section 3. The spectral properties of the operator $\mathcal{B}$, as well as its semi-group, are studied in Section 4. The eigen-spaces of the operator $\mathcal{B}$ will play a crucial role ; a characterization of these eigen-spaces is shown and a description of the associated projections is given, in terms of ergodic averages. The bilinear form $\mathfrak{m}$ is constructed in Section 5 and we study its main properties. The well posedness of the problems associated to the bilinear forms $\mathfrak{a}^\varepsilon$ and $\mathfrak{m}$ is established in Section 6, and uniform estimates for the solutions are highlighted. A second bilinear form $\mathfrak{n}$ is emphasized in Section 7, which will allow us to construct a corrector term. Finally, in Section 8, we establish the asymptotic behavior of the problem (1), (2) cf. Theorem 8.1.

2 The finite dimensional case

We intend to investigate the behavior of the family $(u^\varepsilon)_\varepsilon$ of solutions for the parabolic problems (1), (2). It is very instructive to consider first the case of linear operators on finite dimensional spaces. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be two real matrices and for any $\varepsilon > 0$ consider the problem

$$\frac{\mathrm{d}}{\mathrm{d}t} u^\varepsilon + Au^\varepsilon(t) + \frac{1}{\varepsilon} Bu^\varepsilon(t) = 0, \;\; t \in \mathbb{R}_+ \tag{4}$$

$$u^\varepsilon(0) = u^{\mathrm{in}} \in \mathbb{R}^n. \tag{5}$$

In the case when A and B are commuting, *i.e.*, $BA - AB = 0$, it is easily seen that $e^{-\tau B}A = Ae^{-\tau B}, \tau \in \mathbb{R}$, and a direct computation shows that $t \to e^{\frac{t}{\varepsilon}B} u^\varepsilon(t)$ satisfies the problem

$$\frac{\mathrm{d}}{\mathrm{d}t} v + Av(t) = 0, \;\; t \in \mathbb{R}_+ \tag{6}$$

$$v(0) = u^{\mathrm{in}} \in \mathbb{R}^n. \tag{7}$$

We obtain the well-known commutation formula between the matrices $e^{-tA}, e^{-\tau B}$

$$e^{-t(A + \frac{B}{\varepsilon})} u^{\mathrm{in}} = u^\varepsilon(t) = e^{-\frac{t}{\varepsilon}B} v(t) = e^{-\frac{t}{\varepsilon}B} e^{-tA} u^{\mathrm{in}}, \;\; t \in \mathbb{R}_+, \;\; \varepsilon > 0$$

which allows us to describe the behavior of the family $(u^\varepsilon)_\varepsilon$ in terms of the solution of problem (6), (7), and the semi-group $(e^{-\tau B})_{\tau \in \mathbb{R}_+}$. For studying the general case, we need a decomposition formula for the matrix A. Assume for example that B is symmetric, and let us denote by $E_1, ..., E_r$ the eigen-spaces of B, corresponding to the eigen-values $\lambda_1, ..., \lambda_r$

$$E_i = \ker(B - \lambda_i I_n), \;\; \lambda_i \in \mathbb{R}, \;\; 1 \le i \le r, \;\; E_1 \oplus ... \oplus E_r = \mathbb{R}^n.$$

For any $i \in \{1, ..., r\}$, the notation $(B - \lambda_i I_n)^{-1}$ stands for the reciprocal application of the isomorphism $(B - \lambda_i I_n)|_{E_i^\perp} : E_i^\perp \to \text{Range } (B - \lambda_i I_n) = E_i^\perp$. We consider the linear applications

$$m_i(u) = \mathrm{Proj}_{E_i} Au, \;\; n_i(u) = (B - \lambda_i I_n)^{-1}(Au - \mathrm{Proj}_{E_i} Au), \;\; u \in E_i, \;\; i \in \{1, ..., r\} \tag{8}$$

and we denote by M, N the matrices of the linear applications

$$m = m_1 \oplus ... \oplus m_r, \;\; n = n_1 \oplus ... \oplus n_r$$

that is

$$Mu = m(u) = m_i(u), \;\; Nu = n(u) = n_i(u), \;\; u \in E_i, \;\; i \in \{1, ..., r\}.$$

We claim that the following decomposition holds true

$$A = M + BN - NB, \;\; BM - MB = O_n. \tag{9}$$

Indeed, for any $i \in \{1, ..., r\}$ and $u \in E_i$ we have

$$\begin{aligned}(BN - NB)u &= BNu - \lambda_i Nu = (B - \lambda_i I_n)|_{E_i^\perp} n_i(u) \\ &= Au - \mathrm{Proj}_{E_i} Au = Au - m_i(u) = Au - Mu\end{aligned}$$

and $(BM - MB)u = BMu - \lambda_i Mu = 0$, since $Mu = m_i(u) = \mathrm{Proj}_{E_i} Au \in E_i$. Based on the decomposition (9), we obtain the asymptotic behavior for the solution of (4), (5), when ε becomes small.

Proposition 2.1
Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be two real matrices and $u^{\mathrm{in}} \in \mathbb{R}^n$. We assume that B is symmetric, positive and consider the matrices M, N verifying (9). For any $T \in \mathbb{R}_+$, there is a constant C_T such that for any $\varepsilon > 0$ we have

$$|u^\varepsilon(t) - e^{-\frac{t}{\varepsilon}B} e^{-tM} u^{\mathrm{in}}| \le C_T \varepsilon, \;\; t \in [0, T].$$

Proof. The idea is to introduce a corrector. Let us consider the function $u^1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^n$ given by

$$u^1(t, \tau) = e^{-\tau B} N e^{-tM} u^{\mathrm{in}} - N e^{-\tau B} e^{-tM} u^{\mathrm{in}}, \;\; (t, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{10}$$

Notice that we have $u^1(t, 0) = 0, t \in \mathbb{R}_+$ and

$$\begin{aligned}\partial_\tau u^1 &= -Be^{-\tau B} N e^{-tM} u^{\mathrm{in}} + NBe^{-\tau B} e^{-tM} u^{\mathrm{in}} \\ &= -B\left(e^{-\tau B} N e^{-tM} u^{\mathrm{in}} - N e^{-\tau B} e^{-tM} u^{\mathrm{in}}\right) - (BN - NB) e^{-\tau B} e^{-tM} u^{\mathrm{in}} \\ &= -Bu^1(t, \tau) - (BN - NB) e^{-\tau B} e^{-tM} u^{\mathrm{in}}.\end{aligned}$$

Therefore, using the notation $\tilde{u}^\varepsilon = e^{-\frac{t}{\varepsilon}B} e^{-tM} u^{\mathrm{in}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\varepsilon u^1(t, t/\varepsilon)\} + (BN - NB)\tilde{u}^\varepsilon(t) + \frac{B}{\varepsilon}\{\varepsilon u^1(t, t/\varepsilon)\} = \varepsilon \partial_t u^1(t, t/\varepsilon). \tag{11}$$

Taking into account that B and M are commuting, observe that

$$\frac{\mathrm{d}\tilde{u}^\varepsilon}{\mathrm{d}t} + M\tilde{u}^\varepsilon(t) + \frac{B}{\varepsilon}\tilde{u}^\varepsilon(t) = 0$$

which combined with (11) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\tilde{u}^\varepsilon(t) + \varepsilon u^1(t, t/\varepsilon)\} + \left(A + \frac{B}{\varepsilon}\right)\{\tilde{u}^\varepsilon(t) + \varepsilon u^1(t, t/\varepsilon)\} = \varepsilon \partial_t u^1(t, t/\varepsilon) + \varepsilon A u^1(t, t/\varepsilon).$$

Finally, the function $t \to r^\varepsilon(t) := u^\varepsilon(t) - \tilde{u}^\varepsilon(t) - \varepsilon u^1(t, t/\varepsilon)$ satisfies the problem

$$\frac{\mathrm{d}r^\varepsilon}{\mathrm{d}t} + Ar^\varepsilon(t) + \frac{B}{\varepsilon} r^\varepsilon(t) = -\varepsilon(\partial_t u^1 + Au^1)(t, t/\varepsilon), \;\; t \in \mathbb{R}_+$$

$$r^\varepsilon(0) = u^\varepsilon(0) - \tilde{u}^\varepsilon(0) - \varepsilon u^1(0, 0) = u^{\mathrm{in}} - u^{\mathrm{in}} = 0.$$

Taking the scalar product with $r^\varepsilon(t)$ and using the positivity of B imply

$$|r^\varepsilon(t)| \le \varepsilon \int_0^T \{|\partial_t u^1(t', t'/\varepsilon)| + |A|\, |u^1(t', t'/\varepsilon)|\}\, \mathrm{d}t' + |A| \int_0^t |r^\varepsilon(t')|\, \mathrm{d}t', \;\; t \in [0, T], \;\; \varepsilon > 0.$$

Here, for any matrix C, the notation $|C|$ stands for the norm subordonated to the Euclidean norm

$$|C| = \sup_{\xi \ne 0} \frac{|C\xi|}{|\xi|} \le (C : C)^{1/2}.$$

By Gronwall's lemma we deduce that

$$|r^\varepsilon(t)| \leq \varepsilon \int_0^T \{|\partial_t u^1(t', t'/\varepsilon)| + |A|\ |u^1(t', t'/\varepsilon)|\}\ \mathrm{d}t'\ e^{T|A|},\ \ t \in [0,T],\ \ \varepsilon > 0$$

and we are done provided that there is a constant $\tilde{C}_T$ such that

$$|u^1(t,\tau)| + |\partial_t u^1(t,\tau)| \leq \tilde{C}_T,\ \ t \in [0,T],\ \ \tau \in \mathbb{R}_+.$$

But thanks to the positivity of B, it is easily seen that

$$|u^1(t,\tau)| \leq 2|N|\ |e^{-tM} u^{\mathrm{in}}| \leq 2|N|\ |u^{\mathrm{in}}|\ e^{T|M|},\ \ t \in [0,T],\ \ \tau \in \mathbb{R}_+$$

and

$$|\partial_t u^1(t,\tau)| \leq 2|N|\ |M|\ |u^{\mathrm{in}}|\ e^{T|M|},\ \ t \in [0,T],\ \ \tau \in \mathbb{R}_+.$$

□

Remark 2.1
The key point of the above proof is the choice of the corrector u^1. We retrieve formally the expression of u^1 in (10) *by appealing to the usual two scale Ansatz*

$$u^\varepsilon(t) = u(t, t/\varepsilon) + \varepsilon u^1(t, t/\varepsilon) + \ldots .$$

Indeed, plugging the previous Ansatz in (4)*, leads to*

$$\partial_\tau u(t,\tau) + Bu(t,\tau) = 0 \tag{12}$$

$$\partial_t u(t,\tau) + Au(t,\tau) + \partial_\tau u^1(t,\tau) + Bu^1(t,\tau) = 0 \tag{13}$$

$$\vdots$$

The equation (12) *says that for any $t \in \mathbb{R}_+$ there is a function $v(t) = u(t,0)$ such that $u(t,\tau) = e^{-\tau B} v(t)$. The time evolution for v comes from* (13)*, and we take as initial condition $v(0) = u(0,0) = u^{\mathrm{in}}$, which is obtained by letting formally $\varepsilon \searrow 0$ in the equality $u^{\mathrm{in}} = u^\varepsilon(0) = u(0,0) + \varepsilon u^1(0,0) + \ldots$. We appeal to the decomposition* (9)*. Notice that we have*

$$\partial_t u(t,\tau) + Mu(t,\tau) = \partial_t e^{-\tau B} v(t) + M e^{-\tau B} v(t) = e^{-\tau B}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + Mv(t)\right)$$

and

$$(BN - NB)u(t,\tau) + \partial_\tau u^1(t,\tau) + Bu^1(t,\tau) = e^{-\tau B}\partial_\tau\{e^{\tau B} N e^{-\tau B} v(t) + e^{\tau B} u^1(t,\tau)\}.$$

Therefore the equation (13) *becomes*

$$e^{-\tau \mathcal{B}}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + Mv(t) + \partial_\tau\{e^{\tau B} N e^{-\tau B} v(t) + e^{\tau B} u^1(t,\tau)\}\right) = 0 \tag{14}$$

or equivalently

$$\frac{\mathrm{d}v}{\mathrm{d}t} + Mv(t) + \partial_\tau\{e^{\tau B} N e^{-\tau B} v(t) + e^{\tau B} u^1(t,\tau)\} = 0. \tag{15}$$

Here we have used that $(e^{-\tau \mathcal{B}})_{\tau \in \mathbb{R}}$ is a group. Notice that (14) *still implies* (15) *when $(e^{-\tau \mathcal{B}})_{\tau \in \mathbb{R}_+}$ is only a semi-group, satisfying the backward uniqueness (as for the heat equation, for example). Averaging with respect to the fast time variable suggests to consider*

$$\frac{\mathrm{d}v}{\mathrm{d}t} + Mv(t) = 0 \quad \textit{and} \quad e^{\tau B} N e^{-\tau B} v(t) + e^{\tau B} u^1(t,\tau) = Nv(t) + u^1(t,0).$$

The solution satisfying the condition $u^1(t,0) = 0, t \in \mathbb{R}_+$ corresponds to the choice in (10)*. Notice that the corrector in* (10) *is defined only in terms of the semi-groups $(e^{-\tau B})_{\tau \in \mathbb{R}_+}, (e^{-tM})_{t \in \mathbb{R}_+}$ and not of the groups $(e^{-\tau B})_{\tau \in \mathbb{R}}, (e^{-tM})_{t \in \mathbb{R}}$. Therefore it will be possible to use it when analyzing* (1), (2)*, in which case only semi-groups will be available.*

Remark 2.2

1. *The decomposition in* (9)*, with* B *symmetric, is unique. More exactly, if*

$$A = \tilde{M} + B\tilde{N} - \tilde{N}B, \;\; B\tilde{M} - \tilde{M}B = 0, \;\; \tilde{N}E_i \subset E_i^\perp, \;\; i \in \{1, ..., r\}$$

then $\tilde{M} = M$ *and* $\tilde{N} = N$*. Indeed, for any* $i \in \{1, ..., r\}$ *and any* $u \in E_i$ *we have*

$$Au = \tilde{M}u + (B - \lambda_i I_n)\tilde{N}u, \;\; B\tilde{M}u = \tilde{M}Bu = \lambda_i \tilde{M}u$$

saying that $Au - \tilde{M}u \in \text{Range}\,(B - \lambda_i I_n) = E_i^\perp, \tilde{M}u \in E_i$*. Therefore we obtain*

$$\tilde{M}u = \text{Proj}_{E_i} Au = Mu, \;\; i \in \{1, ..., r\}, \;\; u \in E_i$$

and

$$(B - \lambda_i I_n)\tilde{N}u = Au - \tilde{M}u = Au - Mu = (B - \lambda_i I_n)Nu.$$

As we know that $\tilde{N}u, Nu \in E_i^\perp$*, one gets* $\tilde{N}u = Nu$ *for any* $u \in E_i$*,* $i \in \{1, ..., r\}$*.*

2. *In particular, if* A *and* B *are symmetric, the matrix* M *is symmetric and the matrix* N *is skew-symmetric.*

Before ending this section, let us observe that the convergence of $(u^\varepsilon)_{\varepsilon>0}$ when ε becomes small is not uniform on $[0, T], T \in \mathbb{R}_+$, except for well prepared initial conditions $u^{\text{in}} \in \ker B$. Indeed, if $u^{\text{in}} \in \ker B$, then the commutation property between B and M allows us to write

$$e^{-\frac{t}{\varepsilon}B} e^{-tM} u^{\text{in}} = e^{-tM} e^{-\frac{t}{\varepsilon}B} u^{\text{in}} = e^{-tM} u^{\text{in}}$$

and therefore $(u^\varepsilon)_\varepsilon$ converges uniformly on $[0, T]$ toward $e^{-tM}u^{\text{in}}$, when $\varepsilon \searrow 0$. If the initial condition is not well prepared, that is, if $u^{\text{in}} \notin \ker B$, the limit function $\lim_{\varepsilon \searrow 0} u^\varepsilon$ is not continuous in $t = 0$, and thus the convergence is not uniform on $[0, T], T \in \mathbb{R}_+$. In order to check that, we appeal to the long time behavior of $(e^{-\tau B})_{\tau \in \mathbb{R}_+}$

$$\left|e^{-\tau B}v - \text{Proj}_{\ker B} v\right| \le e^{-\tau c} \left|v - \text{Proj}_{\ker B} v\right|, \;\; v \in \mathbb{R}^n, \;\; \tau \in \mathbb{R}_+$$

with $c := \inf_{|v|=1, v \perp \ker B} Bv \cdot v > 0$. Thanks to Proposition 2.1 we obtain the pointwise convergence

$$\lim_{\varepsilon \searrow 0} u^\varepsilon(t) = \lim_{\varepsilon \searrow 0} e^{-tM} e^{-\frac{t}{\varepsilon}B} u^{\text{in}} = e^{-tM} \lim_{\varepsilon \searrow 0} e^{-\frac{t}{\varepsilon}B} u^{\text{in}} = \begin{cases} u^{\text{in}} & , t = 0 \\ e^{-tM}\text{Proj}_{\ker B} u^{\text{in}} & , t > 0 \end{cases}$$

which is discontinuous at $t = 0$ when $u^{\text{in}} \notin \ker B$. A time boundary layer $[0, T_\varepsilon]$, of size $\mathcal{O}(\varepsilon)$ occurs at $t = 0$, during which any curve u^ε connects the initial condition u^{in} to $e^{-T_\varepsilon M} e^{-\frac{T_\varepsilon}{\varepsilon}B} u^{\text{in}} \approx \text{Proj}_{\ker B} u^{\text{in}}$.

3 Average operators

We intend to generalize Proposition 2.1 for the parabolic problems (1), (2). In this section, we specify the definition and the properties of the average operators along a characteristic flow, for matrix fields and functions. The construction of the average operator for matrix fields relies on the existence of a matrix field P satisfying (18), (19). We introduce the transport operator $\mathcal{T} = \text{div}_y(\cdot\, b)$, defined on

$$\text{dom}\mathcal{T} = \{u \in L^2(\mathbb{R}^m) \; : \; \text{div}_y(ub) \in L^2(\mathbb{R}^m)\}.$$

We make the following standard assumptions on the vector field b

$$b \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^m), \;\; \text{div}_y b = 0 \tag{16}$$

and

$$\exists\, C > 0 \text{ such that } |b(y)| \le C(1 + |y|), \;\; y \in \mathbb{R}^m. \tag{17}$$

Sometimes we will also write $\mathcal{T} = b(y) \cdot \nabla_y$, motivated by the fact that b is divergence free. Under the above hypotheses, the vector field b possesses a global smooth characteristic flow $Y \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \times \mathbb{R}^m)$

$$\frac{\text{d}Y}{\text{d}s} = b(Y(s; y)), \;\; (s, y) \in \mathbb{R} \times \mathbb{R}^m, \;\; Y(0; y) = y, \;\; y \in \mathbb{R}^m.$$

Since the field b is divergence free, the transformation $y \in \mathbb{R}^m \to Y(s;y) \in \mathbb{R}^m$ is measure preserving for any $s \in \mathbb{R}$. We introduce the C^0-group of unitary operators $(\zeta(s))_{s\in\mathbb{R}}$ given by

$$\zeta(s)u = u \circ Y(s;\cdot), \;\; u \in L^2(\mathbb{R}^m), \;\; s \in \mathbb{R}.$$

The transport operator $\mathcal{T}$ appears as the infinitesimal generator of the C^0-group $(\zeta(s))_{s\in\mathbb{R}}$. Sometimes we will use the notation $f_s(z) = f(Y(s;z))$, given a function $f = f(y)$.
Any time a C^0-group of unitary operators acts on a Hilbert space, the orthogonal projection on the kernel of its infinitesimal generator coincides with the ergodic mean of the group [14].

Theorem 3.1 *(von Neumann's ergodic mean theorem)*
Let $(\mathcal{G}(s))_{s\in\mathbb{R}}$ be a C^0-group of unitary operators on a Hilbert space $(\mathcal{H}, (\cdot,\cdot))$ and $\mathcal{L}$ be its infinitesimal generator. Then for any $x \in \mathcal{H}$, we have the strong convergence in $\mathcal{H}$

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} \mathcal{G}(s)x \, \mathrm{d}s = \mathrm{Proj}_{\ker \mathcal{L}} x, \quad \text{uniformly with respect to } r \in \mathbb{R}.$$

As a direct consequence of Theorem 3.1 we obtain the following representation for the orthogonal projection on $\ker \mathcal{T} = \{u \in L^2(\mathbb{R}^m) : u(Y(s;\cdot)) = u, \; \forall\, s \in \mathbb{R}\}$.

Proposition 3.1 (Average of $L^2(\mathbb{R}^m)$ functions)
Assume that (16), (17) hold true. Then for any $u \in L^2(\mathbb{R}^m)$ we have the strong convergence in $L^2(\mathbb{R}^m)$

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} u(Y(s;\cdot)) \, \mathrm{d}s = \mathrm{Proj}_{\ker \mathcal{T}} u \quad \text{uniformly with respect to } r \in \mathbb{R}.$$

We introduce the average operator $\langle u \rangle = \lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} u(Y(s;\cdot))\mathrm{d}s, u \in L^2(\mathbb{R}^m)$. The previous result says that the average operator coincides with the orthogonal projection on $\ker \mathcal{T}$. In order to handle parabolic operators, we will also need to average matrix fields of a L^2 weighted space and L^∞ weighted space. We assume that there is a matrix field P such that

$$^tP = P, \;\; P(y)\xi\cdot\xi > 0, \;\; \xi \in \mathbb{R}^m \setminus \{0\}, \;\; y \in \mathbb{R}^m, \;\; P^{-1}, P \in L^2_{\mathrm{loc}}(\mathbb{R}^m) \tag{18}$$

$$[b, P] := (b\cdot\nabla_y)P - \partial_y b P - P\, {}^t\partial_y b = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}_+). \tag{19}$$

We refer to Proposition 3.8 [3]

Proposition 3.2
Consider $b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m)$ (not necessarily divergence free) with at most linear growth at infinity and $A(y) \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Then $[b, A] = 0$ in $\mathcal{D}'(\mathbb{R}_+)$ iff

$$A(Y(s;y)) = \partial Y(s;y)A(y)\, {}^t\partial Y(s;y), \;\; s \in \mathbb{R}, \;\; y \in \mathbb{R}^m.$$

Let us consider some useful spaces.

Definition 3.1 *We introduce the linear space*

$$H_Q = \left\{ A : \mathbb{R}^m \to \mathcal{M}_m(\mathbb{R}) \text{ measurable } : \; Q^{1/2}AQ^{1/2} \in L^2 \right\},$$

where $Q = P^{-1}$, which is a Hilbert space for the natural scalar product

$$(A,\, B)_{H_Q} = \int_{\mathbb{R}^m} Q^{1/2}\, A\, Q^{1/2} : Q^{1/2}\, B\, Q^{1/2} \, \mathrm{d}y = \int_{\mathbb{R}^m} QA : BQ \, \mathrm{d}y, \quad \forall A,\, B \in H_Q.$$

The associated norm is denoted by $|A|_{H_Q}$.
Similarly we introduce the Banach space

$$H_Q^\infty = \left\{ A : \mathbb{R}^m \to \mathcal{M}_m(\mathbb{R}) \text{ measurable } : \; Q^{1/2}AQ^{1/2} \in L^\infty \right\},$$

endowed with the norm

$$|A|_{H_Q^\infty} := |Q^{1/2}AQ^{1/2}|_{L^\infty}.$$

Assume that there is a continuous function ψ, which is left invariant by the flow of b, and goes to infinity when $|y|$ goes to infinity

$$\psi \in C(\mathbb{R}^m), \;\; \psi \circ Y(s;\cdot) = \psi \;\text{ for any } s \in \mathbb{R}, \; \lim_{|y|\to+\infty} \psi(y) = +\infty. \tag{20}$$

Since the compact sets $\{\psi \le k\}$, for $k \in \mathbb{N}$, are left invariant by the flow of b, we will be able to perform our analysis in the local spaces

$$H_{Q,\mathrm{loc}} = \left\{A : \mathbb{R}^m \to \mathcal{M}_m(\mathbb{R}) \text{ measurable } : \mathbf{1}_{\{\psi \le k\}} A \in H_Q \text{ for any } k \in \mathbb{N}\right\}.$$

We say that a family $(A_i)_i \subset H_{Q,\mathrm{loc}}$ converges in $H_{Q,\mathrm{loc}}$ toward some $A \in H_{Q,\mathrm{loc}}$ iff for any $k \in \mathbb{N}$, the family $(\mathbf{1}_{\{\psi\le k\}} A_i)_i$ converges in H_Q toward $\mathbf{1}_{\{\psi \le k\}} A$. Notice that we have the continuous inclusion $H_Q \subset H_{Q,\mathrm{loc}}$. As suggested by the characterization in Proposition 3.2, we introduce the family of linear transformations $(G(s))_{s\in\mathbb{R}}$, acting on H_Q (see Proposition 4.1 [4] for more details). Moreover, under the assumption (20), the group $(G(s))_{s\in\mathbb{R}}$ also acts on $H_{Q,\mathrm{loc}}$.

Proposition 3.3
Assume that the hypotheses (16), (17), (18), (19) *hold true.*

1. *The family of applications*

$$A \to G(s)A := \partial Y^{-1}(s;\cdot) A_s \, {}^t\partial Y^{-1}(s;\cdot) = \partial Y(-s; Y(s;\cdot)) A_s \, {}^t\partial Y(-s;Y(s;\cdot))$$

   is a C^0-group of unitary operators on H_Q.

2. *If A is a field of symmetric matrices, then so is $G(s)A$, for any $s \in \mathbb{R}$.*

3. *If A is a field of positive semi-definite matrices, then so is $G(s)A$, for any $s \in \mathbb{R}$.*

4. *Let $\mathcal{S} \subset \mathbb{R}^m$ be an invariant set of the flow of b, that is $Y(s;\mathcal{S}) = \mathcal{S}$, for any $s \in \mathbb{R}$. If there is $d > 0$ such that $Q^{1/2}(y)A(y)Q^{1/2}(y) \ge dI_m, y \in \mathcal{S}$, then for any $s \in \mathbb{R}$ we have $Q^{1/2}(y)(G(s)A)(y)Q^{1/2}(y) \ge dI_m, y \in \mathcal{S}$.*

5. *Moreover, if* (20) *holds true, then the family of applications $(G(s))_{s\in\mathbb{R}}$ acts on $H_{Q,\mathrm{loc}}$, that is, if $A \in H_{Q,\mathrm{loc}}$, then $G(s)A \in H_{Q,\mathrm{loc}}$ for any $s \in \mathbb{R}$. We have*

$$\mathbf{1}_{\{\psi\le k\}} G(s)A = G(s)(\mathbf{1}_{\{\psi \le k\}} A), \;\; A \in H_{Q,\mathrm{loc}}, \;\; s \in \mathbb{R}, \;\; k \in \mathbb{N}.$$

Proof.
1. Thanks to the characterization in Proposition 3.2 we know that

$$P_s = \partial Y(s;\cdot) P \, {}^t\partial Y(s;\cdot), \;\; s \in \mathbb{R}. \tag{21}$$

For any $s \in \mathbb{R}$ we consider the matrix field $\mathcal{O}(s;\cdot) = Q_s^{1/2} \partial Y(s;\cdot) Q^{-1/2}$. Observe that $\mathcal{O}(s;\cdot)$ is a field of orthogonal matrices, for any $s \in \mathbb{R}$. Indeed we have, thanks to (21)

$$\begin{aligned}
{}^t\mathcal{O}(s;\cdot)\mathcal{O}(s;\cdot) &= Q^{-1/2} \, {}^t\partial Y(s;\cdot) Q_s^{1/2} Q_s^{1/2} \partial Y(s;\cdot) Q^{-1/2} \\
&= Q^{-1/2} \left(\partial Y^{-1}(s;\cdot) P_s \, {}^t\partial Y^{-1}(s;\cdot)\right)^{-1} Q^{-1/2} \\
&= Q^{-1/2} P^{-1} Q^{-1/2} \\
&= I_m
\end{aligned}$$

implying that for any matrix field A we have

$$Q^{1/2} G(s) A Q^{1/2} = Q^{1/2} \partial Y^{-1}(s;\cdot) A_s \, {}^t\partial Y^{-1}(s;\cdot) Q^{1/2} = {}^t\mathcal{O}(s;\cdot) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s;\cdot). \tag{22}$$

It is easily seen that if $A \in H_Q$, then for any $s \in \mathbb{R}$

$$\begin{aligned}
|G(s)A|_Q^2 &= \int_{\mathbb{R}^m} Q^{1/2} G(s) A Q^{1/2} : Q^{1/2} G(s) A Q^{1/2} \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} {}^t\mathcal{O}(s;\cdot) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s;\cdot) : \, {}^t\mathcal{O}(s;\cdot) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s;\cdot) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} Q_s^{1/2} A_s Q_s^{1/2} : Q_s^{1/2} A_s Q_s^{1/2} \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} Q^{1/2} A Q^{1/2} : Q^{1/2} A Q^{1/2} \, \mathrm{d}y = |A|_{H_Q}^2
\end{aligned}$$

proving that $G(s)$ is a unitary transformation for any $s \in \mathbb{R}$. The group property of the family $(G(s))_{s\in\mathbb{R}}$ follows easily from the group property of the flow $(Y(s;\cdot))_{s\in\mathbb{R}}$

$$\begin{aligned} G(s)G(t)A &= \partial Y^{-1}(s;\cdot)(G(t)A)_s {}^t\partial Y^{-1}(s;\cdot) \\ &= \partial Y^{-1}(s;\cdot)\partial Y^{-1}(t;Y(s;\cdot))(A_t)_s {}^t\partial Y^{-1}(t;Y(s;\cdot)) {}^t\partial Y^{-1}(s;\cdot) \\ &= \partial Y^{-1}(t+s;\cdot)A_{t+s} {}^t\partial Y^{-1}(t+s;\cdot) = G(t+s)A, \quad A \in H_Q. \end{aligned}$$

The continuity of the group, *i.e.*, $\lim_{s\to 0} G(s)A = A$ strongly in H_Q, is left to the reader.
2. Notice that $G(s)$ commutes with transposition

$$\begin{aligned} {}^t(G(s)A) &= {}^t\left(\partial Y^{-1}(s;\cdot)A_s {}^t\partial Y^{-1}(s;\cdot)\right) \\ &= \partial Y^{-1}(s;\cdot) {}^tA_s {}^t\partial Y^{-1}(s;\cdot) \\ &= G(s) {}^tA. \end{aligned}$$

In particular, if ${}^tA = A$, then ${}^t(G(s)A) = G(s)A$.
3. We use the formula (22). For any $\xi \in \mathbb{R}^m$ we have

$$\begin{aligned} G(s)A : Q^{1/2}\xi \otimes Q^{1/2}\xi = Q^{1/2}G(s)AQ^{1/2} : \xi\otimes\xi \\ &= {}^t\mathcal{O}(s;\cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;\cdot) : \xi\otimes\xi \\ &= Q_s^{1/2}A_sQ_s^{1/2} : \mathcal{O}(s;\cdot)(\xi\otimes\xi) {}^t\mathcal{O}(s;\cdot) \\ &= Q_s^{1/2}A_sQ_s^{1/2} : (\mathcal{O}(s;\cdot)\xi)\otimes(\mathcal{O}(s;\cdot)\xi) \\ &= A_s : (Q_s^{1/2}\mathcal{O}(s;\cdot)\xi)\otimes(Q_s^{1/2}\mathcal{O}(s;\cdot)\xi). \end{aligned}$$

As A is a field of positive semi-definite matrices, therefore $G(s)A$ is a field of positive semi-definite matrices as well.
4. Assume that there is $\alpha > 0$ such that $Q^{1/2}AQ^{1/2} \geq \alpha I_m$ on $\mathcal{S}$. As before we write for any $\xi \in \mathbb{R}^m, y \in \mathcal{S}$

$$Q^{1/2}G(s)AQ^{1/2} : \xi\otimes\xi = (Q^{1/2}AQ^{1/2})_s : (\mathcal{O}(s;\cdot)\xi)\otimes(\mathcal{O}(s;\cdot)\xi) \geq \alpha|\mathcal{O}(s;\cdot)\xi|^2 = \alpha|\xi|^2$$

saying that $Q^{1/2}G(s)AQ^{1/2} \geq \alpha I_m$ on $\mathcal{S}$.
5. Here $G(s)$ stands for the application $A \to \partial Y(-s;Y(s;\cdot))A(Y(s;\cdot)) {}^t\partial Y(-s;Y(s;\cdot))$ independently of A being in H_Q or in $H_{Q,\mathrm{loc}}$. As ψ is left invariant by the flow of b, so is $\mathbf{1}_{\{\psi\leq k\}}$, for any $k \in \mathbb{N}$. If A belongs to $H_{Q,\mathrm{loc}}$, we have

$$\mathbf{1}_{\{\psi\leq k\}}G(s)A = G(s)(\mathbf{1}_{\{\psi\leq k\}}A) \in H_Q, \quad k\in\mathbb{N}, \quad s\in\mathbb{R}$$

saying that $G(s)A \in H_{Q,\mathrm{loc}}, s \in \mathbb{R}$. Moreover, the applications $(G(s))_{s\in\mathbb{R}}$ preserve locally the norm of H_Q

$$\left|\mathbf{1}_{\{\psi\leq k\}}G(s)A\right|_{H_Q} = \left|G(s)(\mathbf{1}_{\{\psi\leq k\}}A)\right|_{H_Q} = \left|\mathbf{1}_{\{\psi\leq k\}}A\right|_{H_Q}, \quad k\in\mathbb{N}, \quad s\in\mathbb{R}.$$

□

We introduce the infinitesimal generator of the group G

$$L : \mathrm{dom}L \subset H_Q \to H_Q, \quad \mathrm{dom}L = \{A \in H_Q \ : \ \exists \lim_{s\to 0}\frac{G(s)A - A}{s} \ \text{ in } \ H_Q\}$$

and $LA = \lim_{s\to 0}\frac{G(s)A-A}{s}$ for any $A \in \mathrm{dom}L$. Notice that $C_c^1(\mathbb{R}^m) \subset \mathrm{dom}L$ and $LA = (b\cdot\nabla_y)A - \partial_y bA - A {}^t\partial_y b$, $A \in C_c^1(\mathbb{R}^m)$ (use the hypothesis $Q \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ and the dominated convergence theorem). The main properties of the operator L are summarized below (see [3] Proposition 3.13 for details).

Proposition 3.4
Assume that the hypotheses (16), (17), (18), (19) *hold true.*

1. *The domain of L is dense in H_Q and L is closed.*

2. *The matrix field $A \in H_Q$ belongs to* dom$L$ *iff there is a constant $C > 0$ such that*

$$|G(s)A - A|_{H_Q} \le C|s|, \quad s \in \mathbb{R}.$$

3. *The operator L is skew-adjoint and we have the orthogonal decomposition $H_Q = \ker L \overset{\perp}{\oplus} \overline{\text{Range}\, L}$.*

Remark 3.1 *When working on $H_{Q,\text{loc}}$, the generator of $(G(s))_{s\in\mathbb{R}}$, which is still denoted by L, is defined by*

$$A \in \text{dom}(L) \ \textit{iff} \ \exists \lim_{s\to 0} \frac{G(s)(\mathbf{1}_{\{\psi\le k\}}A) - \mathbf{1}_{\{\psi\le k\}}A}{s} \in H_Q, \ \ k \in \mathbb{N}$$

and

$$\mathbf{1}_{\{\psi\le k\}}L(A) = \lim_{s\to 0} \frac{G(s)(\mathbf{1}_{\{\psi\le k\}}A) - \mathbf{1}_{\{\psi\le k\}}A}{s}, \ \ k \in \mathbb{N}.$$

Clearly, the generator in $H_{Q,\text{loc}}$ extends the generator in H_Q.

The transformations $(G(s))_{s\in\mathbb{R}}$ also behave nicely in the weighted L^∞ space H_Q^∞. More precisely, for any $s \in \mathbb{R}$, and any $A \in H_Q^\infty$, we have $G(s)A \in H_Q^\infty$ and $|G(s)A|_{H_Q^\infty} = |A|_{H_Q^\infty}$. Indeed, thanks to (22) and to the orthogonality of $\mathcal{O}(s;\cdot)$, observe that

$$\begin{aligned} Q^{1/2}G(s)AQ^{1/2} : Q^{1/2}G(s)AQ^{1/2} &= {}^t\mathcal{O}(s;\cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;\cdot) : {}^t\mathcal{O}(s;\cdot)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;\cdot) \\ &= (Q^{1/2}\,A\,Q^{1/2} : Q^{1/2}\,A\,Q^{1/2})_s, \quad s \in \mathbb{R} \end{aligned}$$

and our claim follows immediately. Applying Theorem 3.1 to the group $(G(s))_{s\in\mathbb{R}}$, we deduce that the average of a matrix field $\langle A\rangle := \lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} G(s)A \,\mathrm{d}s$ is well defined and coincides with the orthogonal projection on $\ker L$. Moreover, by Proposition 3.3, $(G(s))_{s\in\mathbb{R}}$ also acts on $H_{Q,\text{loc}}$, and any matrix field of $H_Q^\infty \subset H_{Q,\text{loc}}$ possesses an average in $H_{Q,\text{loc}}$, still denoted by $\langle\cdot\rangle$ as for the matrix fields in H_Q.

Theorem 3.2 (Average of $H_{Q,\text{loc}}$ matrix fields)
Assume that (16), (17), (18), (19) *hold true.*

1. *For any matrix field $A \in H_Q$ we have the strong convergence in H_Q*

$$\langle A\rangle := \lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} \partial Y(-s;Y(s;\cdot))A(Y(s;\cdot))\,{}^t\partial Y(-s;Y(s;\cdot))\,\mathrm{d}s = \text{Proj}_{\ker L}A$$

uniformly with respect to $r \in \mathbb{R}$.

2. *If $A \in H_Q$ is a field of symmetric positive semi-definite matrices, then so is $\langle A\rangle$.*

3. *Let $\mathcal{S} \subset \mathbb{R}^m$ be an invariant set of the flow of b, that is $Y(s;\mathcal{S}) = \mathcal{S}$ for any $s \in \mathbb{R}$. If $A \in H_Q$ and there is $d > 0$ such that*

$$Q^{1/2}(y)A(y)Q^{1/2}(y) \ge dI_m, \ \ y \in \mathcal{S}$$

therefore we have

$$Q^{1/2}(y)\,\langle A\rangle\,(y)Q^{1/2}(y) \ge dI_m, \ \ y \in \mathcal{S}$$

and in particular, $\langle A\rangle(y)$ is definite positive, $y \in \mathcal{S}$.

4. *If $A \in H_Q \cap H_Q^\infty$, then $\langle A\rangle \in H_Q \cap H_Q^\infty$ and*

$$|\langle A\rangle|_{H_Q} \le |A|_{H_Q}, \ \ |\langle A\rangle|_{H_Q^\infty} \le |A|_{H_Q^\infty}.$$

5. *Moreover, assume that* (20) *holds true. For any matrix field $A \in H_{Q,\text{loc}}$, the family*

$$\left(\frac{1}{S}\int_r^{r+S} \partial Y(-s;Y(s;\cdot))A(Y(s;\cdot))\,{}^t\partial Y(-s;Y(s;\cdot))\,\mathrm{d}s\right)_{S>0}$$

converges in $H_{Q,\mathrm{loc}}$, when S goes to infinity, uniformly with respect to $r \in \mathbb{R}$, for any fixed $k \in \mathbb{N}$. Its limit, denoted by $\langle A \rangle$, satisfies

$$\mathbf{1}_{\{\psi \le k\}} \langle A \rangle = \left\langle \mathbf{1}_{\{\psi \le k\}} A \right\rangle, \text{ for any } k \in \mathbb{N}$$

where the symbol $\langle \cdot \rangle$ in the right hand side stands for the average operator on H_Q. In particular, any matrix field $A \in H_Q^\infty$ has an average in $H_{Q,\mathrm{loc}}$ and $|\langle A \rangle|_{H_Q^\infty} \le |A|_{H_Q^\infty}$. If $A \in H_{Q,\mathrm{loc}}$ is such that

$$Q^{1/2}(y) A(y) Q^{1/2}(y) \ge \alpha I_m, \ \ y \in \mathbb{R}^m,$$

for some $\alpha > 0$, then we have

$$Q^{1/2}(y) \langle A \rangle (y) Q^{1/2}(y) \ge \alpha I_m, \ \ y \in \mathbb{R}^m.$$

Proof. We only sketch the arguments. For more details we refer to Theorem 2.1 [4]. The first and second statements are obvious.
3. For any $\xi \in \mathbb{R}^m, \psi \in C_c^0(\mathcal{S}), \psi \ge 0$ we have $\psi(\cdot) P^{1/2}\xi \otimes P^{1/2}\xi \in H_Q$ and we can write, thanks to (22)

$$\begin{aligned}
(G(s)A, \psi(\cdot) P^{1/2}\xi \otimes P^{1/2}\xi)_Q &= \int_{\mathbb{R}^m} \psi(y) Q^{1/2} G(s) A Q^{1/2} : \xi \otimes \xi \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} \psi(y)\, {}^t\mathcal{O}(s;y) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s;y)\xi \cdot \xi \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} \psi(y) Q_s^{1/2} A_s Q_s^{1/2} : \mathcal{O}(s;y)\xi \otimes \mathcal{O}(s;y)\xi \,\mathrm{d}y \\
&\ge \alpha \int_{\mathbb{R}^m} |\mathcal{O}(s;y)\xi|^2 \psi(y) \,\mathrm{d}y \\
&= \alpha |\xi|^2 \int_{\mathbb{R}^m} \psi(y) \,\mathrm{d}y.
\end{aligned}$$

Taking the average over $[0, S]$ and letting $S \to +\infty$ yield

$$\int_{\mathbb{R}^m} \psi(y) Q^{1/2} \langle A \rangle Q^{1/2} : \xi \otimes \xi \,\mathrm{d}y = (\langle A \rangle, \psi P^{1/2}\xi \otimes P^{1/2}\xi)_Q \ge \int_{\mathbb{R}^m} \alpha |\xi|^2 \psi(y) \,\mathrm{d}y$$

implying that

$$Q^{1/2}(y) \langle A \rangle (y) Q^{1/2}(y) \ge \alpha I_m, \ \ y \in \mathcal{S}.$$

4. Obviously, for any $A \in H_Q$, we have by the properties of the orthogonal projection on $\ker L$ that $|\langle A \rangle|_{H_Q} = |\mathrm{Proj}_{\ker L} A|_{H_Q} \le |A|_{H_Q}$. For the last inequality, consider $M \in \mathcal{M}_m(\mathbb{R})$ a fixed matrix, $\psi \in C_c^0(\mathbb{R}^m), \psi \ge 0$ and, as before, observe that $\psi P^{1/2} M P^{1/2} \in H_Q$, which allows us to write

$$\begin{aligned}
(G(s)A, \psi P^{1/2} M P^{1/2})_Q &= \int_{\mathbb{R}^m} Q^{1/2} G(s) A Q^{1/2} : \psi M \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} {}^t\mathcal{O}(s;y) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s;y) : \psi M \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} Q_s^{1/2} A_s Q_s^{1/2} : \mathcal{O}(s;y) M\, {}^t\mathcal{O}(s;y) \psi \,\mathrm{d}y \\
&\le \int_{\mathbb{R}^m} \sqrt{Q_s^{1/2} A_s Q_s^{1/2} : Q_s^{1/2} A_s Q_s^{1/2}} \sqrt{\mathcal{O}(s;y) M\, {}^t\mathcal{O}(s;y) : \mathcal{O}(s;y) M\, {}^t\mathcal{O}(s;y)}\, \psi \,\mathrm{d}y \\
&\le |A|_{H_Q^\infty} (M : M)^{1/2} \int_{\mathbb{R}^m} \psi(y) \,\mathrm{d}y.
\end{aligned}$$

Taking the average over $[0, S]$ and letting $S \to +\infty$, lead to

$$\int_{\mathbb{R}^m} Q^{1/2} \langle A \rangle Q^{1/2} : M \psi(y) \,\mathrm{d}y = (\langle A \rangle, \psi P^{1/2} M P^{1/2})_Q \le |A|_{H_Q^\infty} (M : M)^{1/2} \int_{\mathbb{R}^m} \psi(y) \,\mathrm{d}y.$$

We deduce that

$$Q^{1/2}(y)\langle A\rangle(y)Q^{1/2}(y) : M \le |A|_{H_Q^\infty}(M:M)^{1/2},\ \ y\in\mathbb{R}^m,\ \ M\in\mathcal{M}_m(\mathbb{R})$$

saying that

$$|\langle A\rangle|_{H_Q^\infty} = \text{ess sup}_{y\in\mathbb{R}^m}\sqrt{Q^{1/2}(y)\langle A\rangle(y)Q^{1/2}(y) : Q^{1/2}(y)\langle A\rangle(y)Q^{1/2}(y)} \le |A|_{H_Q^\infty}.$$

5. Let A be a matrix field in $H_{Q,\text{loc}}$. For any $k\in\mathbb{N}$, $\mathbf{1}_{\{\psi\le k\}}A$ belongs to H_Q, and by the first statement we know that

$$\lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S} G(s)(\mathbf{1}_{\{\psi\le k\}}A)\,\mathrm{d}s = \left\langle \mathbf{1}_{\{\psi\le k\}}A\right\rangle \in H_Q$$

uniformly with respect to $r\in\mathbb{R}$, for any fixed $k\in\mathbb{N}$. It is easily seen that for any $k,l\in\mathbb{N}$ we have

$$\lim_{S\to+\infty}\frac{1}{S}\int_0^{S} G(s)(\mathbf{1}_{\{\psi\le k\}}A)\,\mathrm{d}s = \lim_{S\to+\infty}\frac{1}{S}\int_0^{S} G(s)(\mathbf{1}_{\{\psi\le l\}}A)\,\mathrm{d}s$$

almost everywhere on $\{\psi\le\min(k,l)\}$, and thus, there is a matrix field denoted by $\langle A\rangle$, whose restriction on $\{\psi\le k\}$ coincides with $\left\langle \mathbf{1}_{\{\psi\le k\}}A\right\rangle$ for any $k\in\mathbb{N}$. Notice also that for any $k\in\mathbb{N}$ we have $\left\langle \mathbf{1}_{\{\psi\le k\}}A\right\rangle = 0$ almost everywhere on $\{\psi>k\}$ and thus we obtain

$$\mathbf{1}_{\{\psi\le k\}}\langle A\rangle = \left\langle \mathbf{1}_{\{\psi\le k\}}A\right\rangle,\ \ k\in\mathbb{N}.$$

Observe that for any $k\in\mathbb{N}$, we have the uniform, with respect to $r\in\mathbb{R}$, convergence in H_Q

$$\lim_{S\to+\infty}\mathbf{1}_{\{\psi\le k\}}\frac{1}{S}\int_r^{r+S} G(s)(A)\,\mathrm{d}s = \lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S} G(s)(\mathbf{1}_{\{\psi\le k\}}A)\,\mathrm{d}s = \left\langle \mathbf{1}_{\{\psi\le k\}}A\right\rangle = \mathbf{1}_{\{\psi\le k\}}\langle A\rangle$$

saying that $\lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S} G(s)A\,\mathrm{d}s = \langle A\rangle$ in $H_{Q,\text{loc}}$ (uniformly with respect to $r\in\mathbb{R}$, for any fixed $k\in\mathbb{N}$). The inclusion $H_Q^\infty\subset H_{Q,\text{loc}}$ follows by the compactness of $\{\psi\le k\}, k\in\mathbb{N}$. By the fourth statement we have

$$|\langle A\rangle|_{H_Q^\infty} = \sup_{k\in\mathbb{N}}|\mathbf{1}_{\{\psi\le k\}}\langle A\rangle|_{H_Q^\infty} = \sup_{k\in\mathbb{N}}|\left\langle \mathbf{1}_{\{\psi\le k\}}A\right\rangle|_{H_Q^\infty} \le \sup_{k\in\mathbb{N}}|\mathbf{1}_{\{\psi\le k\}}A|_{H_Q^\infty} = |A|_{H_Q^\infty}.$$

Let A be a matrix field of $H_{Q,\text{loc}}$, such that $Q^{1/2}(y)A(y)Q^{1/2}(y)\ge\alpha I_m, y\in\mathbb{R}^m$, for some $\alpha>0$. For any $k\in\mathbb{N}$ we have $\mathbf{1}_{\{\psi\le k\}}A\in H_Q$ and

$$Q^{1/2}(y)\mathbf{1}_{\{\psi\le k\}}A(y)Q^{1/2}(y)\ge\alpha I_m,\ \ y\in\{\psi\le k\}.$$

By the third statement we deduce that for any $k\in\mathbb{N}$

$$Q^{1/2}(y)\mathbf{1}_{\{\psi\le k\}}\langle A\rangle(y)Q^{1/2}(y) = Q^{1/2}(y)\left\langle \mathbf{1}_{\{\psi\le k\}}A\right\rangle(y)Q^{1/2}(y)\ge\alpha I_m,\ \ y\in\{\psi\le k\}$$

saying that $Q^{1/2}(y)\langle A\rangle(y)Q^{1/2}(y)\ge\alpha I_m,\ \ y\in\mathbb{R}^m$. □

Remark 3.2

1. *We have the following variational characterization of the average operator on $H_{Q,\text{loc}}$: for any matrix field $A\in H_{Q,\text{loc}}$, the average matrix field $\langle A\rangle$ is the unique matrix field in $H_{Q,\text{loc}}$ satisfying*

$$(\mathbf{1}_{\{\psi\le k\}}(A-\langle A\rangle), M)_{H_Q} = 0,\ \ \textit{for any } M\in H_Q.$$

2. *It is easily seen that the average operator on $H_{Q,\text{loc}}$ extends the average operator on H_Q.*

3. *Let A be a matrix field in $H_{Q,\text{loc}}$. For any $k\in\mathbb{N}$ we have*

$$\frac{G(s)(\mathbf{1}_{\{\psi\le k\}}\langle A\rangle)-\mathbf{1}_{\{\psi\le k\}}\langle A\rangle}{s} = \frac{G(s)\left\langle \mathbf{1}_{\{\psi\le k\}}A\right\rangle-\left\langle \mathbf{1}_{\{\psi\le k\}}A\right\rangle}{s} = 0$$

saying that $\langle A\rangle\in\text{dom}(L)$ and $\mathbf{1}_{\{\psi\le k\}}L\langle A\rangle = 0,\ \ k\in\mathbb{N}$, see Remark 3.1. Therefore $L\langle A\rangle = 0$, for any $A\in H_{Q,\text{loc}}$.

We also introduce the linear spaces

$$X_Q = \{c : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable } : \int_{\mathbb{R}^m} Q(y) : c(y) \otimes c(y) \, \mathrm{d}y < +\infty\}$$

$$X_Q^\infty = \{c : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable } : |Q^{1/2}c| \in L^\infty(\mathbb{R}^m)\}.$$

The linear space X_Q, endowed with the scalar product

$$(\cdot,\cdot)_{X_Q} : X_Q \times X_Q \to \mathbb{R}, \ (c,d)_{X_Q} = \int_{\mathbb{R}^m} Q(y) : c(y) \otimes d(y) \, \mathrm{d}y, \ \ c,d \in X_Q$$

becomes a Hilbert space, whose norm is denoted by $|c|_{X_Q} = (c,c)^{1/2}_{X_Q}, c \in X_Q$.
The linear space X_Q^∞ is a Banach space with respect to the norm

$$|c|_{X_Q^\infty} = \text{ess sup}_{y\in\mathbb{R}^m} |Q^{1/2}(y)c(y)|, \ \ c \in X_Q^\infty.$$

Notice that for any $c \in X_Q \cap X_Q^\infty$, we have $c \otimes c \in H_Q \cap H_Q^\infty$ and

$$|c \otimes c|_{H_Q^\infty} = \text{ess sup}_{y\in\mathbb{R}^m} |Q^{1/2}(y)c(y)|^2 = |c|^2_{X_Q^\infty}$$

$$|c \otimes c|_{H_Q} = \left(\int_{\mathbb{R}^m} |Q^{1/2}(y)c(y)|^4 \, \mathrm{d}y \right)^{1/2} \leq |c|_{X_Q} |c|_{X_Q^\infty}.$$

Replacing the matrix field Q by the matrix field P, we obtain the linear spaces X_P, X_P^∞.
For solving the parabolic problems (1), (2), we appeal to variational methods. We consider the following linear subspace of $L^2(\mathbb{R}^m)$

$$H_P^1 = \{u \in L^2(\mathbb{R}^m) \ : \ \nabla_y u \in X_P\}. \tag{23}$$

It becomes a Hilbert space, when endowed with the scalar product

$$(u,v)_{H_P^1} = \int_{\mathbb{R}^m} u(y)v(y) \, \mathrm{d}y + \int_{\mathbb{R}^m} P(y) : \nabla_y u \otimes \nabla_y v \, \mathrm{d}y, \ \ u,v \in H_P^1.$$

The choice of the above weighted H^1 space is motivated by the fact that the C^0-group $(\zeta(s))_{s\in\mathbb{R}}$ acts on H_P^1.

Proposition 3.5 (Average of H_P^1 functions)
Assume that the hypotheses (16), (17), (18), (19) *hold true. For any $s \in \mathbb{R}$ and $u \in H_P^1$ we have $u_s \in H_P^1$ and $|u_s|_{H_P^1} = |u|_{H_P^1}$. The family of applications $u \in H_P^1 \to \zeta^1(s)u = u \circ Y(s;\cdot) \in H_P^1$ is a C^0-group of unitary operators on H_P^1. In particular, for any $u \in H_P^1$ we have $\langle u \rangle \in H_P^1$*

$$\nabla_y \langle u \rangle = \lim_{S\to+\infty} \frac{1}{S} \int_r^{r+S} \nabla_y u_s \, \mathrm{d}s, \ \ \textit{strongly in } X_P, \ \ \textit{uniformly with respect to } r \in \mathbb{R}$$

$$u - \langle u \rangle \perp \ker \mathcal{T} \cap H_P^1 \ \ \textit{in } H_P^1, \ \ |\nabla_y \langle u \rangle|_{X_P} \leq |\nabla_y u|_{X_P}.$$

Proof. Let $u = u(y)$ be a function in H_P^1. As the flow satisfies $Y \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \times \mathbb{R}^m)$, we have $\nabla u_s = {}^t\partial Y(s;\cdot)(\nabla u)_s$. By Proposition 3.2 we know that $P_s = \partial Y(s;\cdot)P \, {}^t\partial Y(s;\cdot)$, and therefore we can write

$$\begin{aligned} |u_s|^2_{H_P^1} &= \int_{\mathbb{R}^m} (u_s(y))^2 \, \mathrm{d}y + \int_{\mathbb{R}^m} P(y) \nabla u_s \cdot \nabla u_s \, \mathrm{d}y \\ &= \int_{\mathbb{R}^m} (u(y))^2 \, \mathrm{d}y + \int_{\mathbb{R}^m} \underbrace{\partial Y(s;y)P(y) \, {}^t\partial Y(s;y)}_{P_s} : (\nabla u)_s \otimes (\nabla u)_s \, \mathrm{d}y \\ &= |u|^2_{L^2(\mathbb{R}^m)} + |\nabla u|^2_{X_P} = |u|^2_{H_P^1}. \end{aligned}$$

The group property of $(\zeta^1(s))_{s\in\mathbb{R}}$ comes by the group property of $(\zeta(s))_{s\in\mathbb{R}}$. In order to check the continuity of $(\zeta^1(s))_{s\in\mathbb{R}}$, observe that for any $u \in H^1_P$, we can write

$$\begin{aligned}
|\zeta^1(s)u - u|^2_{H^1_P} - |\zeta(s)u - u|^2_{L^2(\mathbb{R}^m)} &= |\nabla u_s - \nabla u|^2_{X_P} \\
&= 2|\nabla u|^2_{X_P} - 2(\nabla u_s, \nabla u)_{X_P} \\
&= 2|\nabla u|^2_{X_P} - 2\int_{\mathbb{R}^m} \underbrace{P^{1/2}(y)\ {}^t\partial Y(s;y)}_{{}^t\mathcal{O}(s;y)P_s^{1/2}}(\nabla u)_s \cdot P^{1/2}(y)\nabla u \,\mathrm{d}y \\
&= 2|\nabla u|^2_{X_P} - 2\int_{\mathbb{R}^m} {}^t\mathcal{O}(s;y)(P^{1/2}\nabla u)_s \cdot P^{1/2}\nabla u \,\mathrm{d}y \\
&= |(P^{1/2}\nabla u)_s - P^{1/2}\nabla u|^2_{L^2(\mathbb{R}^m)} - 2\int_{\mathbb{R}^m}(P^{1/2}\nabla u)_s \cdot (\mathcal{O} - I_m)P^{1/2}\nabla u \,\mathrm{d}y.
\end{aligned}$$

Thanks to the continuity of $(\zeta(s))_{s\in\mathbb{R}}$, we are done provided that the last integral terms converges to 0, as $s \to 0$. The convergence $\lim_{s\to 0}\partial Y(s;y) = I_m, y \in \mathbb{R}^m$, implies the convergences

$$\lim_{s\to 0} P(Y(s;y)) = \lim_{s\to 0}\partial Y(s;y)P(y)\ {}^t\partial Y(s;y) = P(y),\ \ \lim_{s\to 0} P^{1/2}(Y(s;y)) = P^{1/2}(y)$$

$$\lim_{s\to 0} Q(Y(s;y)) = \lim_{s\to 0}\ {}^t\partial Y^{-1}(s;y)Q(y)\ \partial Y^{-1}(s;y) = Q(y),\ \ \lim_{s\to 0} Q^{1/2}(Y(s;y)) = Q^{1/2}(y)$$

$$\lim_{s\to 0}\mathcal{O}(s;y) = \lim_{s\to 0} Q^{1/2}(Y(s;y))\partial Y(s;y)Q^{-1/2}(y) = I_m,\ \ y \in \mathbb{R}^m.$$

Since $\mathcal{O}(s;y)$ is orthogonal, we have $|\mathcal{O}(s;y)| = 1$ for any $s \in \mathbb{R}, y \in \mathbb{R}^m$, and by the dominated convergence theorem we obtain

$$\lim_{s\to 0}\int_{\mathbb{R}^m} P^{1/2}\nabla u \cdot (\mathcal{O}(s;y) - I_m)P^{1/2}\nabla u \,\mathrm{d}y = 0$$

implying that

$$\begin{aligned}
&\lim_{s\to 0}\int_{\mathbb{R}^m}(P^{1/2}\nabla u)_s \cdot (\mathcal{O}(s;y) - I_m)P^{1/2}\nabla u \,\mathrm{d}y \\
&= \lim_{s\to 0}\int_{\mathbb{R}^m}\Big[(P^{1/2}\nabla u)_s - P^{1/2}\nabla u\Big]\cdot(\mathcal{O}(s;y) - I_m)P^{1/2}\nabla u \,\mathrm{d}y = 0.
\end{aligned}$$

For the last limit we have used the convergence $\lim_{s\to 0}(P^{1/2}\nabla u)_s = P^{1/2}\nabla u$ in $L^2(\mathbb{R}^m)$, and the upper bound $|\mathcal{O}(s;y) - I_m| \le 2$, $s \in \mathbb{R}, y \in \mathbb{R}^m$. Finally, by Theorem 3.1 we deduce the strong convergence in H^1_P

$$\lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S}\zeta^1(s)u \,\mathrm{d}s = \mathrm{Proj}_{\ker\mathcal{T}\cap H^1_P} u$$

implying that $\langle u\rangle = \mathrm{Proj}_{\ker\mathcal{T}} u = \mathrm{Proj}_{\ker\mathcal{T}\cap H^1_P} u \in H^1_P$, $(\nabla u - \nabla\langle u\rangle, \nabla v)_{X_P} = 0$ for any $v \in \ker\mathcal{T}\cap H^1_P$, and the strong convergence in X_P, uniformly with respect to $r \in \mathbb{R}$

$$\lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S}\nabla u_s \,\mathrm{d}s = \nabla\langle u\rangle.$$

For the last statement use $|\nabla u_s|_{X_P} = |\nabla u|_{X_P}, s \in \mathbb{R}$ and the above convergence. □

4 Properties of the operator $\mathcal{B} = -\mathcal{T}^2$

We introduce the operator $\mathcal{B} = -\mathcal{T}^2 = -\mathrm{div}_y(\mathrm{div}_y(\cdot\ b)b)$ defined for any function in the domain

$$\mathrm{dom}\mathcal{B} = \{u \in \mathrm{dom}\mathcal{T}\ :\ \mathrm{div}_y(ub) \in \mathrm{dom}\mathcal{T}\} \subset L^2(\mathbb{R}^m).$$

Clearly, this operator will play a crucial role when analyzing the asymptotic behavior for the solutions of (1), (2) with small $\varepsilon > 0$. In this section we study the semi-group generated by the operator $-\mathcal{B}$, together with the spectral properties of $\mathcal{B}$. More precisely, we indicate a characterization of the eigen-spaces of $\mathcal{B}$ and give a description, in terms of ergodic averages, of the orthogonal projections on these eigen-spaces.

4.1 Semi-group generated by the operator $-\mathcal{B}$

For any $\theta > 0$, the notation M_θ stands for the one dimension Maxwellian, of temperature θ

$$M_\theta(s) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{s^2}{2\theta}}, \;\; s \in \mathbb{R}.$$

The semi-group $(e^{-\tau\mathcal{B}})_{\tau\in\mathbb{R}_+}$ is given by

Proposition 4.1 (Semi-group generated by $-\mathcal{B}$)
Let us consider the family of applications

$$\varphi_\tau u = \int_{\mathbb{R}} u_s M_{2\tau}(s) \, \mathrm{d}s = \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} u(Y(s;\cdot)) e^{-\frac{s^2}{4\tau}} \, \mathrm{d}s, \;\; u \in L^2(\mathbb{R}^m), \;\; \tau > 0$$

and $\varphi_0 u = u, u \in L^2(\mathbb{R}^m)$. The family $(\varphi_\tau)_{\tau\in\mathbb{R}_+}$ is a C^0 semi-group of contractions on $L^2(\mathbb{R}^m)$, whose infinitesimal generator is $-\mathcal{B}$, i.e. $\varphi_\tau = e^{-\tau\mathcal{B}}, \tau \in \mathbb{R}_+$.

Proof. Clearly, for any $u \in L^2(\mathbb{R}^m), \tau > 0$, we have

$$\int_{\mathbb{R}^m} (\varphi_\tau u)^2 \, \mathrm{d}y \le \int_{\mathbb{R}^m}\int_{\mathbb{R}} u_s^2 M_{2\tau}(s) \, \mathrm{d}s \, \mathrm{d}y = \int_{\mathbb{R}} M_{2\tau}(s) \int_{\mathbb{R}^m} u_s^2 \, \mathrm{d}y \, \mathrm{d}s = \int_{\mathbb{R}^m} u^2(y) \, \mathrm{d}y$$

saying that φ_τ is a contraction of $L^2(\mathbb{R}^m)$. The semi-group property follows immediately, thanks to the formula $M_{\theta_1} * M_{\theta_2} = M_{\theta_1+\theta_2}, \theta_1, \theta_2 > 0$. Indeed, for any $\tau, h > 0, u \in L^2(\mathbb{R}^m)$ we have

$$\begin{aligned}
\varphi_\tau\varphi_h u &= \int_{\mathbb{R}} M_{2\tau}(s)(\varphi_h u)_s \, \mathrm{d}s = \int_{\mathbb{R}} M_{2\tau}(s) \int_{\mathbb{R}} M_{2h}(s') u_{s+s'} \, \mathrm{d}s' \, \mathrm{d}s \\
&= \int_{\mathbb{R}} u_s (M_{2\tau} * M_{2h})(s) \, \mathrm{d}s = \int_{\mathbb{R}} u_s M_{2(\tau+h)}(s) \, \mathrm{d}s = \varphi_{\tau+h} u.
\end{aligned}$$

The continuity of the semi-group comes by the density of $C_c(\mathbb{R}^m)$ in $L^2(\mathbb{R}^m)$ and the contraction property, noticing that $\varphi_\tau u = \int_{\mathbb{R}} M_1(r) u_{\sqrt{2\tau}\,r} \, \mathrm{d}r$. It remains to check that the infinitesimal generator of $(\varphi_\tau)_{\tau\in\mathbb{R}_+}$ is $-\mathcal{B}$. Consider $u \in \mathrm{dom}\mathcal{B}$, that is $u, \mathcal{T}u = \mathrm{div}_y(ub), \mathcal{T}^2 u = \mathrm{div}_y(\mathrm{div}_y(ub)b) \in L^2(\mathbb{R}^m)$ and let us establish that $\frac{\mathrm{d}}{\mathrm{d}\tau}|_{\tau=0}\varphi_\tau u = -\mathcal{B}u$ in $L^2(\mathbb{R}^m)$. Thanks to the equality in $L^2(\mathbb{R}^m)$

$$u_h = u + h\mathrm{div}_y(ub) + h^2 \int_0^1 (1-s)(\mathcal{T}^2 u)_{hs} \, \mathrm{d}s$$

we can write for any $\tau > 0$

$$\begin{aligned}
\frac{\varphi_\tau u - u}{\tau} &= \int_{\mathbb{R}} M_1(r) \frac{u_{\sqrt{2\tau}r} - u}{\tau} \, \mathrm{d}r \\
&= \int_{\mathbb{R}} M_1(r) \frac{u_{\sqrt{2\tau}r} - u - \sqrt{2\tau}r\mathrm{div}_y(ub)}{\tau} \, \mathrm{d}r \\
&= \int_{\mathbb{R}} M_1(r) 2r^2 \int_0^1 (1-s)(\mathcal{T}^2 u)_{\sqrt{2\tau}rs} \, \mathrm{d}s\mathrm{d}r \\
&\underset{\tau\searrow 0}{\to} \int_{\mathbb{R}} M_1(r) r^2 \mathcal{T}^2 u \, \mathrm{d}r = \mathcal{T}^2 u = -\mathcal{B}u \;\text{ in } L^2(\mathbb{R}^m).
\end{aligned}$$

Conversely, assume that $u \in L^2(\mathbb{R}^m)$ such that the following limit exists in $L^2(\mathbb{R}^m)$

$$\lim_{\tau\searrow 0} \frac{\varphi_\tau u - u}{\tau} = w \in L^2(\mathbb{R}^m).$$

A straightforward computation shows that for any $\tau > 0$ we have

$$|(\varphi_\tau u)_h - \varphi_\tau u|_{L^2(\mathbb{R}^m)} \le \frac{|h||u|_{L^2(\mathbb{R}^m)}}{\sqrt{\pi\tau}}, \;\; h \in \mathbb{R}$$

$$|(\varphi_\tau u)_h + (\varphi_\tau u)_{-h} - 2\varphi_\tau u|_{L^2(\mathbb{R}^m)} \le \frac{2h^2|u|_{L^2(\mathbb{R}^m)}}{\tau}, \;\; h \in \mathbb{R}$$

saying that $\varphi_\tau u \in \mathrm{dom}\mathcal{B}$ for any $\tau > 0$ and

$$|\mathcal{T}\varphi_\tau u|_{L^2(\mathbb{R}^m)} \leq \frac{|u|_{L^2(\mathbb{R}^m)}}{\sqrt{\pi\tau}}, \;\; |\mathcal{T}^2\varphi_\tau u|_{L^2(\mathbb{R}^m)} \leq \frac{2|u|_{L^2(\mathbb{R}^m)}}{\tau}, \;\; \tau > 0.$$

The semi-group property guarantees that

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi_\tau u\right|_{L^2(\mathbb{R}^m)} = |\varphi_\tau w|_{L^2(\mathbb{R}^m)} \leq |w|_{L^2(\mathbb{R}^m)}, \;\; \tau > 0$$

implying that

$$\begin{aligned}
|\mathcal{T}\varphi_\tau u|^2_{L^2(\mathbb{R}^m)} &= -\int_{\mathbb{R}^m} \varphi_\tau u \, \mathcal{T}^2\varphi_\tau u \, \mathrm{d}y \\
&= -\int_{\mathbb{R}^m} \varphi_\tau u \frac{\mathrm{d}}{\mathrm{d}\tau}\varphi_\tau u \, \mathrm{d}y \\
&\leq |\varphi_\tau u|_{L^2(\mathbb{R}^m)} \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi_\tau u\right|_{L^2(\mathbb{R}^m)} \\
&\leq |u|_{L^2(\mathbb{R}^m)}|w|_{L^2(\mathbb{R}^m)}, \;\; \tau > 0.
\end{aligned}$$

For any $h \in \mathbb{R}$ we can write

$$|\varphi_\tau(u_h - u)|_{L^2(\mathbb{R}^m)} = |(\varphi_\tau u)_h - \varphi_\tau u|_{L^2(\mathbb{R}^m)} \leq |h||\mathcal{T}\varphi_\tau u|_{L^2(\mathbb{R}^m)} \leq |h||u|^{1/2}_{L^2(\mathbb{R}^m)}|w|^{1/2}_{L^2(\mathbb{R}^m)}, \;\; \tau > 0$$

and thanks to the continuity of the semi-group, we deduce

$$|u_h - u|_{L^2(\mathbb{R}^m)} \leq |h||u|^{1/2}_{L^2(\mathbb{R}^m)}|w|^{1/2}_{L^2(\mathbb{R}^m)}, \;\; h \in \mathbb{R}$$

saying that $u \in \mathrm{dom}\mathcal{T}$. For any smooth function $v \in C_c^\infty(\mathbb{R}^m)$ we have (using the symmetry of the Maxwellians $M_{2\tau}(-s) = M_{2\tau}(s), s \in \mathbb{R}, \tau > 0$)

$$\int_{\mathbb{R}^m} \varphi_\tau u \, v \, \mathrm{d}y = \int_{\mathbb{R}^m} u \, \varphi_\tau v \, \mathrm{d}y, \;\; \tau \in \mathbb{R}_+$$

and therefore we obtain

$$\begin{aligned}
\int_{\mathbb{R}^m} wv \, \mathrm{d}y &= \int_{\mathbb{R}^m} \frac{\mathrm{d}}{\mathrm{d}\tau}|_{\tau=0}\varphi_\tau u \, v \, \mathrm{d}y = \int_{\mathbb{R}^m} u \, \frac{\mathrm{d}}{\mathrm{d}\tau}|_{\tau=0}\varphi_\tau v \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} u\mathcal{T}^2 v \, \mathrm{d}y = -\int_{\mathbb{R}^m} \mathcal{T}u\mathcal{T}v \, \mathrm{d}y.
\end{aligned}$$

We deduce that $\mathcal{T}u \in \mathrm{dom}\mathcal{T}$ and $\mathcal{T}^2 u = w \in L^2(\mathbb{R}^m)$. □

Properties of the semi-group $(\varphi_\tau)_{\tau\in\mathbb{R}_+}$

We inquire about the regularity propagation along the semi-group $(\varphi_\tau)_{\tau\in\mathbb{R}_+}$. These properties will be useful when justifying the regularity of the solution for the effective problem, and of the corrector, see Theorem 8.1.

1. The semi-group $(\varphi_\tau)_{\tau\in\mathbb{R}_+}$ also acts on H^1_P. Indeed, for any function $u \in H^1_P$ and any $s \in \mathbb{R}$ we have
$$P^{1/2}(y)\nabla u_s = P^{1/2}(y) \, {}^t\partial Y(s;y)(\nabla u)_s = {}^t\mathcal{O}(s;y)P_s^{1/2}(\nabla u)_s.$$
As the matrices $\mathcal{O}(s;y)$ are orthogonal, we obtain
$$P^{1/2}(y)\nabla(\varphi_\tau u) = \int_{\mathbb{R}} P^{1/2}(y)\nabla u_s M_{2\tau}(s) \, \mathrm{d}s = \int_{\mathbb{R}} {}^t\mathcal{O}(s;y)(P^{1/2}\nabla u)_s M_{2\tau}(s) \, \mathrm{d}s$$
implying that
$$|\nabla(\varphi_\tau u)|_{X_P} = |P^{1/2}\nabla(\varphi_\tau u)|_{L^2(\mathbb{R}^m)} \leq \int_{\mathbb{R}} |(P^{1/2}\nabla u)_s|_{L^2(\mathbb{R}^m)} M_{2\tau}(s) \, \mathrm{d}s = |\nabla u|_{X_P}.$$
Therefore $(\varphi_\tau)_{\tau\in\mathbb{R}_+}$ are also contractions on H^1_P
$$|\varphi_\tau u|^2_{H^1_P} = |\varphi_\tau u|^2_{L^2(\mathbb{R}^m)} + |\nabla(\varphi_\tau u)|^2_{X_P} \leq |u|^2_{L^2(\mathbb{R}^m)} + |\nabla u|^2_{X_P} = |u|^2_{H^1_P}, \;\; u \in H^1_P, \; \tau \in \mathbb{R}_+.$$

2. If $u \in H^1_P$ such that $\mathrm{div}_y(P\nabla u) \in L^2(\mathbb{R}^m)$, then $\mathrm{div}_y(P\nabla \varphi_\tau u) \in L^2(\mathbb{R}^m)$ for any $\tau \in \mathbb{R}_+$, and

$$|\mathrm{div}_y(P\nabla \varphi_\tau u)|_{L^2(\mathbb{R}^m)} \leq |\mathrm{div}_y(P\nabla u)|_{L^2(\mathbb{R}^m)}, \;\; \tau \in \mathbb{R}_+.$$

Indeed, for any $\psi \in C^1_c(\mathbb{R}^m)$ we have

$$\begin{aligned}
\int_{\mathbb{R}^m} P\nabla \varphi_\tau u \cdot \nabla \psi \;\mathrm{d}y &= \int_{\mathbb{R}}\int_{\mathbb{R}^m} P\nabla u_s \cdot \nabla \psi \;\mathrm{d}y M_{2\tau}(s)\;\mathrm{d}s \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^m} P\nabla u \cdot \nabla \psi_{-s} \;\mathrm{d}y M_{2\tau}(s)\;\mathrm{d}s \\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}^m} \mathrm{div}_y(P\nabla u)\psi_{-s}(y)\;\mathrm{d}y M_{2\tau}(s)\;\mathrm{d}s \\
&\leq \int_{\mathbb{R}} |\mathrm{div}_y(P\nabla u)|_{L^2(\mathbb{R}^m)}|\psi_{-s}|_{L^2(\mathbb{R}^m)} M_{2\tau(s)}\;\mathrm{d}s \\
&= |\mathrm{div}_y(P\nabla u)|_{L^2(\mathbb{R}^m)}|\psi|_{L^2(\mathbb{R}^m)}
\end{aligned}$$

saying that $\mathrm{div}_y(P\nabla \varphi_\tau u) \in L^2(\mathbb{R}^m)$, and $|\mathrm{div}_y(P\nabla \varphi_\tau u)|_{L^2(\mathbb{R}^m)} \leq |\mathrm{div}_y(P\nabla u)|_{L^2(\mathbb{R}^m)}$ for any $\tau \in \mathbb{R}_+$.

3. More generally, the semi-group $(\varphi_\tau)_{\tau\in\mathbb{R}_+}$ preserves all derivations $c \cdot \nabla_y$ along vector fields $c : \mathbb{R}^m \to \mathbb{R}^m$ in involution with respect to b, *i.e.* $[b, c] = 0$. More exactly, let c be a smooth field in involution with b, with growth at most linear and bounded divergence

$$c \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m), \;\; \sup_{y\in\mathbb{R}^m} \frac{|c(y)|}{1+|y|} < +\infty, \;\; [b, c] = 0, \;\; \mathrm{div}_y c \in L^\infty(\mathbb{R}^m)$$

and let us denote by $Z(h; y)$ the characteristic flow of c. For any function $u \in \mathrm{dom}(c \cdot \nabla_y)$ we have

$$\begin{aligned}
|(\varphi_\tau u) \circ Z(h; \cdot) - \varphi_\tau u|_{L^2(\mathbb{R}^m)} &= |\varphi_\tau(u \circ Z(h; \cdot) - u)|_{L^2(\mathbb{R}^m)} \\
&\leq |u \circ Z(h; \cdot) - u|_{L^2(\mathbb{R}^m)} \leq |h|\; e^{\frac{|h|}{2}|\mathrm{div}_y c|_{L^\infty}} |c \cdot \nabla u|_{L^2(\mathbb{R}^m)}, \;\; \tau \in \mathbb{R}_+
\end{aligned}$$

saying that $\varphi_\tau u \in \mathrm{dom}(c \cdot \nabla_y)$ and $|c \cdot \nabla(\varphi_\tau u)|_{L^2(\mathbb{R}^m)} \leq |c \cdot \nabla u|_{L^2(\mathbb{R}^m)}, \tau \in \mathbb{R}_+$. Letting $h \to 0$ in the equality

$$\frac{(\varphi_\tau u) \circ Z(h; \cdot) - \varphi_\tau u}{h} = \varphi_\tau \frac{u \circ Z(h; \cdot) - u}{h}$$

gives the commutation of φ_τ and $c \cdot \nabla_y$, that is

$$c \cdot \nabla(\varphi_\tau u) = \varphi_\tau(c \cdot \nabla u), \;\; \tau \in \mathbb{R}_+.$$

Moreover, if c_1, c_2 are two smooth fields in involution with respect to b and $c_1 \cdot \nabla_y(c_2 \cdot \nabla_y u) \in L^2(\mathbb{R}^m)$, then $c_1 \cdot \nabla_y(c_2 \cdot \nabla_y \varphi_\tau u) \in L^2(\mathbb{R}^m)$ and

$$|c_1 \cdot \nabla_y(c_2 \cdot \nabla_y \varphi_\tau u)|_{L^2(\mathbb{R}^m)} \leq |c_1 \cdot \nabla_y(c_2 \cdot \nabla_y u)|_{L^2(\mathbb{R}^m)}, \;\; \tau \in \mathbb{R}_+.$$

The above arguments allow us to propagate derivations along fields in involution with respect to b, of any order, uniformly with respect to $\tau \in \mathbb{R}_+$.

4.2 Spectral properties of the operator $\mathcal{B}$

We concentrate now on the spectral properties of $\mathcal{B}$.

Proposition 4.2
The operator $\mathcal{B}$ is self-adjoint and positive. In particular the eigen-spaces are orthogonal, and for any λ we have $\ker(\mathcal{B} - \lambda Id)^\perp = \overline{\mathrm{Range}\;(\mathcal{B} - \lambda Id)}$.

Proof. For any $u, v \in \mathrm{dom}\mathcal{B}$ and $\tau \in \mathbb{R}_+$ we have

$$\int_{\mathbb{R}^m} \varphi_\tau u\, v \;\mathrm{d}y = \int_{\mathbb{R}^m} u\, \varphi_\tau v \;\mathrm{d}y$$

implying that $\int_{\mathbb{R}^m}\mathcal{B}u\, v\ \mathrm{d}y = \int_{\mathbb{R}^m}u\, \mathcal{B}v\ \mathrm{d}y$. Therefore we have $\mathrm{dom}\mathcal{B} \subset \mathrm{dom}\mathcal{B}^\star$ and $\mathcal{B}^\star v = \mathcal{B}v$ for any $v \in \mathrm{dom}\mathcal{B}$. Conversely, assume that $v \in \mathrm{dom}\mathcal{B}^\star$, that is there is a constant C such that

$$\int_{\mathbb{R}^m} \mathcal{B}u\, v\ \mathrm{d}y \leq C|u|_{L^2(\mathbb{R}^m)},\quad \text{for any } u \in \mathrm{dom}\mathcal{B}.$$

For any $u \in \mathrm{dom}\mathcal{B}$ and $h \in \mathbb{R}$ we have

$$u_h = u + h\mathcal{T}u + h^2 \int_0^1 (1-s)(\mathcal{T}^2u)_{hs}\ \mathrm{d}s,\ \ u_{-h} = u - h\mathcal{T}u + h^2 \int_0^1 (1-s)(\mathcal{T}^2u)_{-hs}\ \mathrm{d}s$$

and thus we obtain

$$\begin{aligned}\int_{\mathbb{R}^m}(v_h + v_{-h} - 2v)u\ \mathrm{d}y &= \int_{\mathbb{R}^m} v(u_h + u_{-h} - 2u)\ \mathrm{d}y \\ &= h^2 \int_{\mathbb{R}^m} v\left[\int_0^1 (1-s)(\mathcal{T}^2u)_{hs}\ \mathrm{d}s + \int_0^1 (1-s)(\mathcal{T}^2u)_{-hs}\ \mathrm{d}s\right]\ \mathrm{d}y \\ &= h^2 \int_0^1 (1-s) \int_{\mathbb{R}^m} v(\mathcal{T}^2u_{hs} + \mathcal{T}^2u_{-hs})\ \mathrm{d}y\mathrm{d}s \leq C|h|^2|u|_{L^2(\mathbb{R}^m)}.\end{aligned}$$

As the domain of $\mathcal{B}$ is dense in $L^2(\mathbb{R}^m)$, it comes that

$$\int_{\mathbb{R}^m}(v_h + v_{-h} - 2v)u\ \mathrm{d}y \leq Ch^2|u|_{L^2(\mathbb{R}^m)},\ \ h \in \mathbb{R},\ \ u \in L^2(\mathbb{R}^m) \tag{24}$$

implying that

$$|v_h + v_{-h} - 2v|_{L^2(\mathbb{R}^m)} \leq Ch^2,\ \ h \in \mathbb{R}. \tag{25}$$

In particular, taking $u = -v \in L^2(\mathbb{R}^m)$ in (24), one gets

$$|v_h - v|^2_{L^2(\mathbb{R}^m)} = -\int_{\mathbb{R}^m}(v_h + v_{-h} - 2v)v\ \mathrm{d}y \leq Ch^2|v|_{L^2(\mathbb{R}^m)}. \tag{26}$$

The estimates (26), (25) guarantee that $v \in \mathrm{dom}\mathcal{B}$ and thus $\mathcal{B}^\star = \mathcal{B}$. Clearly, for any $u \in \mathrm{dom}\mathcal{B}$ we have $\int_{\mathbb{R}^m}\mathcal{B}u\, u\ \mathrm{d}y = \int_{\mathbb{R}^m}(\mathcal{T}u)^2\ \mathrm{d}y \geq 0$, and therefore all the eigen-values belong to $\mathbb{R}_+$. □

Description of the eigen-spaces and of the associated projections

For any $\lambda \in \mathbb{R}_+$ we denote by E_λ the subspace $E_\lambda = \ker(\mathcal{B} - \lambda Id)$. Thanks to the equality $\int_{\mathbb{R}^m}\mathcal{B}u\, u\ \mathrm{d}y = \int_{\mathbb{R}^m}(\mathcal{T}u)^2\ \mathrm{d}y, u \in \mathrm{dom}\mathcal{B}$, it is easily seen that

$$E_0 = \ker\mathcal{B} = \ker\mathcal{T} = \{u \in L^2(\mathbb{R}^m)\ :\ u_s = u, s \in \mathbb{R}\}.$$

By Proposition 3.1 we know that

$$\mathrm{Proj}_{E_0}u = \lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S} u_s\ \mathrm{d}s,\quad \text{strongly in } L^2(\mathbb{R}^m),\ \text{ uniformly with respect to } r \in \mathbb{R}.$$

We will see that the orthogonal projections on the subspaces E_λ are also given by average operators. For any $\lambda > 0$ we introduce the family of transformations of $L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$

$$\zeta_\lambda(s)(u,v) = {}^t\left(\mathcal{R}(\sqrt{\lambda}s)\ {}^t(u_s, v_s)\right) = (u_s, v_s)\mathcal{R}(-\sqrt{\lambda}s),\ \ (u,v) \in L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m),\ \ s \in \mathbb{R}$$

where $\mathcal{R}$ stands for the rotation of angle $\theta \in \mathbb{R}$.

Proposition 4.3
For any $\lambda > 0$ the family $(\zeta_\lambda(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary transformations of $L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$. The subspace E_λ writes

$$\begin{aligned}E_\lambda &= \left\{u \in \mathrm{dom}\mathcal{T}\ :\ u_s = \cos(\sqrt{\lambda}s)u + \sin(\sqrt{\lambda}s)\frac{\mathcal{T}}{\sqrt{\lambda}}u,\ \ \textit{for any } s \in \mathbb{R}\right\} \\ &= \left\{u \in \mathrm{dom}\mathcal{T}\ :\ \left(u + i\frac{\mathcal{T}}{\sqrt{\lambda}}u\right)_s = e^{-i\sqrt{\lambda}s}\left(u + i\frac{\mathcal{T}}{\sqrt{\lambda}}u\right),\ \ \textit{for any } s \in \mathbb{R}\right\}\end{aligned}$$

and for any $u \in L^2(\mathbb{R}^m)$ *we have*

$$\mathrm{Proj}_{E_\lambda} u = \lim_{S\to+\infty} \frac{2}{S}\int_r^{r+S} \cos(\sqrt{\lambda}s) u_s \,\mathrm{d}s, \quad \frac{\mathcal{T}}{\sqrt{\lambda}}\mathrm{Proj}_{E_\lambda} u = \lim_{S\to+\infty} \frac{2}{S}\int_r^{r+S} \sin(\sqrt{\lambda}s) u_s \,\mathrm{d}s$$

in $L^2(\mathbb{R}^m)$*, uniformly with respect to* $r \in \mathbb{R}$*. If* $u \in \mathrm{dom}\mathcal{T}$*, the orthogonal projection on* E_λ *also writes*

$$\mathrm{Proj}_{E_\lambda} u = -\lim_{S\to+\infty} \frac{2}{S}\int_r^{r+S} \sin(\sqrt{\lambda}s)\frac{\mathcal{T}}{\sqrt{\lambda}} u_s \,\mathrm{d}s$$

in $L^2(\mathbb{R}^m)$*, uniformly with respect to* $r \in \mathbb{R}$*.*

Proof. Clearly we have for any $(u, v) \in L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$, $s, h \in \mathbb{R}$

$$\begin{aligned}\zeta_\lambda(s)\zeta_\lambda(h)(u,v) &= (\zeta_\lambda(h)(u,v))_s \mathcal{R}(-\sqrt{\lambda}s)\\ &= (u_h, v_h)_s \mathcal{R}(-\sqrt{\lambda}h)\mathcal{R}(-\sqrt{\lambda}s)\\ &= \zeta_\lambda(s+h)(u,v)\end{aligned}$$

and

$$|\zeta_\lambda(s)(u,v)|^2_{L^2\times L^2} = \int_{\mathbb{R}^m} \{(u_s)^2 + (v_s)^2\}\,\mathrm{d}y = \int_{\mathbb{R}^m}\{u^2+v^2\}\,\mathrm{d}y.$$

The continuity of the group $(\zeta(s))_{s\in\mathbb{R}}$ guarantees the continuity of the group $(\zeta_\lambda(s))_{s\in\mathbb{R}}$. We denote by T_λ the infinitesimal generator of $(\zeta_\lambda(s))_{s\in\mathbb{R}}$. Its domain is given by the pairs $(u, v) \in L^2(\mathbb{R}^m)\times L^2(\mathbb{R}^m)$ such that it exists

$$T_\lambda(u,v) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\zeta_\lambda(s)(u,v) \in L^2(\mathbb{R}^m)\times L^2(\mathbb{R}^m).$$

It coincides with the set of the pairs $(u, v) \in L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$ such that it exists

$$\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\{\zeta_\lambda(s)(u,v)\mathcal{R}(\sqrt{\lambda}s)\} = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}(u_s, v_s) \in L^2(\mathbb{R}^m)\times L^2(\mathbb{R}^m).$$

Therefore $\mathrm{dom}T_\lambda = \mathrm{dom}\mathcal{T}\times\mathrm{dom}\mathcal{T}$, and for any $(u, v) \in \mathrm{dom}T_\lambda$ we have

$$T_\lambda(u,v) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\zeta_\lambda(s)(u,v) = \sqrt{\lambda}(u,v)\mathcal{R}(-\pi/2) + (\mathcal{T}u, \mathcal{T}v) = (\mathcal{T}u - \sqrt{\lambda}v, \mathcal{T}v+\sqrt{\lambda}u).$$

The kernel of T_λ is $F_\lambda = \{(u, \frac{\mathcal{T}}{\sqrt{\lambda}}u), u \in E_\lambda\}$. Notice that $u \in E_\lambda$ iff $(u, \frac{\mathcal{T}}{\sqrt{\lambda}}u) \in F_\lambda$, or equivalently iff

$$u \in \mathrm{dom}\mathcal{T}, \quad \left(u_s, \frac{\mathcal{T}}{\sqrt{\lambda}}u_s\right)\mathcal{R}(-\sqrt{\lambda}s) = \left(u, \frac{\mathcal{T}}{\sqrt{\lambda}}u\right), \quad \text{for any } s \in \mathbb{R}.$$

We deduce that $u \in E_\lambda$ iff $u \in \mathrm{dom}\mathcal{T}$ and

$$u_s = \cos(\sqrt{\lambda}s)u + \sin(\sqrt{\lambda}s)\frac{\mathcal{T}}{\sqrt{\lambda}}u, \quad \text{for any } s \in \mathbb{R} \tag{27}$$

$$\frac{\mathcal{T}}{\sqrt{\lambda}}u_s = -\sin(\sqrt{\lambda}s)u + \cos(\sqrt{\lambda}s)\frac{\mathcal{T}}{\sqrt{\lambda}}u, \quad \text{for any } s \in \mathbb{R}. \tag{28}$$

Observe that (28) comes from (27), by taking the derivative with respect to s and therefore we obtain the following characterization for the subspace E_λ

$$\begin{aligned}E_\lambda &= \left\{u \in \mathrm{dom}\mathcal{T} \,:\, u_s = \cos(\sqrt{\lambda}s)u + \sin(\sqrt{\lambda}s)\frac{\mathcal{T}}{\sqrt{\lambda}}u \text{ for any } s\in\mathbb{R}\right\}\\ &= \left\{u \in \mathrm{dom}\mathcal{T} \,:\, \left(u + i\frac{\mathcal{T}}{\sqrt{\lambda}}u\right)_s = e^{-i\sqrt{\lambda}s}\left(u + i\frac{\mathcal{T}}{\sqrt{\lambda}}u\right) \text{ for any } s\in\mathbb{R}\right\}.\end{aligned}$$

By Theorem 3.1, we know that for any $(u, v) \in L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$ we have

$$\mathrm{Proj}_{F_\lambda}(u,v) = \lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S}\zeta_\lambda(s)(u,v)\,\mathrm{d}s$$

in $L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$, uniformly with respect to $r \in \mathbb{R}$. In particular, we deduce that

$$\text{Proj}_{F_\lambda}(u,0) = \lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} \zeta_\lambda(s)(u,0)\;\mathrm{d}s = \lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S} (\cos(\sqrt{\lambda}s)u_s, \sin(\sqrt{\lambda}s)u_s)\;\mathrm{d}s$$

in $L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$, uniformly with respect to $r \in \mathbb{R}$. But $\text{Proj}_{F_\lambda}(u,0) = (U, \frac{\mathcal{T}}{\sqrt{\lambda}}U)$ for some $U \in E_\lambda$ satisfying

$$\int_{\mathbb{R}^m}\left\{(u-U)V - \frac{\mathcal{T}}{\sqrt{\lambda}}U\frac{\mathcal{T}}{\sqrt{\lambda}}V\right\}\;\mathrm{d}y = 0,\;\; V \in E_\lambda$$

which also writes

$$\int_{\mathbb{R}^m}(u-2U)V\;\mathrm{d}y = 0,\;\; V \in E_\lambda.$$

This exactly means that

$$\text{Proj}_{E_\lambda}u = 2U = \lim_{S\to+\infty}\frac{2}{S}\int_r^{r+S}\cos(\sqrt{\lambda}s)u_s\;\mathrm{d}s$$

in $L^2(\mathbb{R}^m)$, uniformly with respect to $r \in \mathbb{R}$. Notice that we also have

$$\frac{\mathcal{T}}{\sqrt{\lambda}}\text{Proj}_{E_\lambda}u = 2\frac{\mathcal{T}}{\sqrt{\lambda}}U = \lim_{S\to+\infty}\frac{2}{S}\int_r^{r+S}\sin(\sqrt{\lambda}s)u_s\;\mathrm{d}s$$

in $L^2(\mathbb{R}^m)$, uniformly with respect to $r \in \mathbb{R}$. If $u \in \text{dom}\mathcal{T}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{\sin(\sqrt{\lambda}s)\frac{u_s}{\sqrt{\lambda}}\right\} = \cos(\sqrt{\lambda}s)u_s + \sin(\sqrt{\lambda}s)\frac{\mathcal{T}}{\sqrt{\lambda}}u_s \text{ in } L^2(\mathbb{R}^m)$$

and thus we deduce

$$\begin{aligned}\text{Proj}_{E_\lambda}u &= \lim_{S\to+\infty}\frac{2}{S}\int_r^{r+S}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[\sin(\sqrt{\lambda}s)\frac{u_s}{\sqrt{\lambda}}\right] - \sin(\sqrt{\lambda}s)\frac{\mathcal{T}}{\sqrt{\lambda}}u_s\right\}\;\mathrm{d}s \\ &= -\lim_{S\to+\infty}\frac{2}{S}\int_r^{r+S}\sin(\sqrt{\lambda}s)\frac{\mathcal{T}}{\sqrt{\lambda}}u_s\;\mathrm{d}s\end{aligned}$$

in $L^2(\mathbb{R}^m)$, uniformly with respect to $r \in \mathbb{R}$. □

Remark 4.1

1. *It is easily seen that* $E_0 = \ker\mathcal{T} = \ker\mathcal{B}$ *is left invariant by the group* $(\zeta(s))_{s\in\mathbb{R}}$ *and that* $\text{Proj}_{E_0} = \langle\cdot\rangle$ *is commuting with* $(\zeta(s))_{s\in\mathbb{R}}$ *and* $\mathcal{T}$

$$\text{Proj}_{E_0}u_s = \text{Proj}_{E_0}u = (\text{Proj}_{E_0}u)_s,\;\; u \in L^2(\mathbb{R}^m),\;\; s \in \mathbb{R}$$

$$\text{Proj}_{E_0}\mathcal{T}u = 0 = \mathcal{T}\text{Proj}_{E_0}u,\;\; u \in \text{dom}\mathcal{T},\;\; s \in \mathbb{R}.$$

2. *The subspaces* $E_\lambda = \ker(\mathcal{B} - \lambda Id), \lambda > 0$ *are left invariant by the group* $(\zeta(s))_{s\in\mathbb{R}}$. *Indeed, for any* $u \in \text{dom}\mathcal{T}$ *such that*

$$\left(u + i\frac{\mathcal{T}}{\sqrt{\lambda}}u\right)_s = e^{-i\sqrt{\lambda}s}\left(u + i\frac{\mathcal{T}}{\sqrt{\lambda}}u\right),\;\; s \in \mathbb{R}$$

we have

$$u_h \in \text{dom}\mathcal{T},\;\; \left(u_h + i\frac{\mathcal{T}}{\sqrt{\lambda}}u_h\right)_s = e^{-i\sqrt{\lambda}s}\left(u_h + i\frac{\mathcal{T}}{\sqrt{\lambda}}u_h\right),\;\; s \in \mathbb{R}$$

saying that $u_h \in E_\lambda, h \in \mathbb{R}, \lambda > 0$. *In particular the subspaces* $E_\lambda \cap H_P^1, \lambda > 0$ *are left invariant by the group* $(\zeta^1(s))_{s\in\mathbb{R}}$.

3. *The application* $u \in E_\lambda \cap H^1_P \to \frac{\mathcal{T}}{\sqrt{\lambda}} u$ *is a isometry with respect to the* H^1_P *norm. Indeed, for any* $u \in E_\lambda \cap H^1_P$ *we have*

$$u_s = \cos(\sqrt{\lambda} s) u + \sin(\sqrt{\lambda} s) \frac{\mathcal{T}}{\sqrt{\lambda}} u, \ \ s \in \mathbb{R}.$$

As we already know that $u, u_s \in E_\lambda \cap H^1_P$, *we deduce that* $\frac{\mathcal{T}}{\sqrt{\lambda}} u \in E_\lambda \cap H^1_P$ *and thus we can write*

$$\begin{aligned}\int_{\mathbb{R}^m} P\nabla u \cdot \nabla u \ \mathrm{d}y &= \int_{\mathbb{R}^m} P\nabla u_s \cdot \nabla u_s \ \mathrm{d}y \\ &= \int_{\mathbb{R}^m} \left| \cos(\sqrt{\lambda} s) P^{1/2} \nabla u + \sin(\sqrt{\lambda} s) P^{1/2} \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|^2 \ \mathrm{d}y \\ &= \cos(\sqrt{\lambda} s)^2 \int_{\mathbb{R}^m} P\nabla u \cdot \nabla u \ \mathrm{d}y + \sin(\sqrt{\lambda} s)^2 \int_{\mathbb{R}^m} P\nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \cdot \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \ \mathrm{d}y \\ &+ \sin(2\sqrt{\lambda} s) \int_{\mathbb{R}^m} P\nabla u \cdot \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \ \mathrm{d}y.\end{aligned}$$

Taking the derivatives with respect to s *at* $s = 0$, *we deduce that*

$$\int_{\mathbb{R}^m} P\nabla u \cdot \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \ \mathrm{d}y = 0$$

which implies that

$$\int_{\mathbb{R}^m} P\nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \cdot \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \ \mathrm{d}y = \int_{\mathbb{R}^m} P\nabla u \cdot \nabla u \ \mathrm{d}y.$$

Notice also that

$$\int_{\mathbb{R}^m} \left(\frac{\mathcal{T}}{\sqrt{\lambda}} u \right)^2 \ \mathrm{d}y = -\int_{\mathbb{R}^m} u \frac{\mathcal{T}^2}{\lambda} u \ \mathrm{d}y = \int_{\mathbb{R}^m} u^2 \ \mathrm{d}y$$

and thus we have $|\frac{\mathcal{T}}{\sqrt{\lambda}} u|_{H^1_P} = |u|_{H^1_P}$ *for any* $u \in E_\lambda \cap H^1_P$ *and also* $(\frac{\mathcal{T}}{\sqrt{\lambda}} u, \frac{\mathcal{T}}{\sqrt{\lambda}} v)_{H^1_P} = (u, v)_{H^1_P}$ *for any* $u, v \in E_\lambda \cap H^1_P$. *The reciprocal application of* $\frac{\mathcal{T}}{\sqrt{\lambda}}|_{E_\lambda \cap H^1_P}$ *is* $-\frac{\mathcal{T}}{\sqrt{\lambda}}|_{E_\lambda \cap H^1_P}$.

4. *The orthogonal projection on* $E_\lambda, \lambda > 0$ *are commuting with the group* $(\zeta(s))_{s \in \mathbb{R}}$

$$\mathrm{Proj}_{E_\lambda} u_s = (\mathrm{Proj}_{E_\lambda} u)_s, \ \ s \in \mathbb{R}.$$

In particular $\mathrm{Proj}_{E_\lambda}, \lambda > 0$ *are commuting with* $\mathcal{T}$, *that is, for any* $u \in \mathrm{dom}\mathcal{T}, \lambda > 0$ *we have the equalities in* $L^2(\mathbb{R}^m)$

$$\begin{aligned}\mathrm{Proj}_{E_\lambda} \mathcal{T} u = \mathrm{Proj}_{E_\lambda} \lim_{s \to 0} \frac{u_s - u}{s} &= \lim_{s \to 0} \frac{\mathrm{Proj}_{E_\lambda} u_s - \mathrm{Proj}_{E_\lambda} u}{s} \\ &= \lim_{s \to 0} \frac{(\mathrm{Proj}_{E_\lambda} u)_s - \mathrm{Proj}_{E_\lambda} u}{s} = \mathcal{T} \mathrm{Proj}_{E_\lambda} u.\end{aligned}$$

5. *Many other commutations hold true, for example between* $(\zeta_\lambda(s))_{s \in \mathbb{R}}, (\zeta_\mu(s))_{s \in \mathbb{R}}$, *thanks to the equalities*

$$\mathcal{R}(\sqrt{\lambda} s) \mathcal{R}(\sqrt{\mu} h) = \mathcal{R}(\sqrt{\mu} h) \mathcal{R}(\sqrt{\lambda} s), \ \ \lambda, \mu > 0, \ \ s, h \in \mathbb{R}.$$

We study now the action of $(\zeta_\lambda(s))_{s \in \mathbb{R}}$ on $H^1_P \times H^1_P$, for any $\lambda > 0$. As in Proposition 3.5 we prove

Proposition 4.4
Assume that the hypotheses (16), (17), (18), (19) *hold true. For any* $(u, v) \in H^1_P \times H^1_P$, *we have* $\zeta_\lambda(s)(u, v) \in H^1_P \times H^1_P$ *and* $|\zeta_\lambda(s)(u, v)|_{H^1_P \times H^1_P} = |(u, v)|_{H^1_P \times H^1_P}$. *The family of applications* $(\zeta^1_\lambda(s))_{s \in \mathbb{R}} = (\zeta_\lambda(s)|_{H^1_P \times H^1_P})_{s \in \mathbb{R}}$ *is a* C^0*-group of unitary operators on* $H^1_P \times H^1_P$. *In particular for any* $u \in H^1_P$ *we have* $\mathrm{Proj}_{E_\lambda} u \in H^1_P$

$$\nabla \mathrm{Proj}_{E_\lambda} u = \lim_{S \to +\infty} \frac{2}{S} \int_r^{r+S} \cos(\sqrt{\lambda} s) \nabla u_s \ \mathrm{d}s, \ \ \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} \mathrm{Proj}_{E_\lambda} u = \lim_{S \to +\infty} \frac{2}{S} \int_r^{r+S} \sin(\sqrt{\lambda} s) \nabla u_s \ \mathrm{d}s \qquad (29)$$

strongly in X_P, *uniformly with respect to* $r \in \mathbb{R}$ *and*

$$u - \mathrm{Proj}_{E_\lambda} u \perp E_\lambda \cap H^1_P \ \text{ in } H^1_P, \ \ |\nabla \mathrm{Proj}_{E_\lambda} u|^2_{X_P} \leq 2|\nabla u|^2_{X_P}.$$

Proof. Let (u,v) be an element of $H^1_P \times H^1_P$. By Proposition 3.5 we know that

$$(u_s, v_s) \in H^1_P \times H^1_P, \;\; |u_s|_{L^2(\mathbb{R}^m)} = |u|_{L^2(\mathbb{R}^m)}, \; |v_s|_{L^2(\mathbb{R}^m)} = |v|_{L^2(\mathbb{R}^m)}$$
$$|\nabla u_s|_{X_P} = |\nabla u|_{X_P}, \; |\nabla v_s|_{X_P} = |\nabla v|_{X_P}$$

and therefore we deduce

$$\begin{aligned}|\zeta^1_\lambda(s)(u,v)|^2_{H^1_P\times H^1_P} &= |\zeta_\lambda(s)(u,v)|^2_{L^2(\mathbb{R}^m)\times L^2(\mathbb{R}^m)} + |\cos(\sqrt{\lambda}s)\nabla u_s - \sin(\sqrt{\lambda}s)\nabla v_s|^2_{X_P}\\ &+ |\sin(\sqrt{\lambda}s)\nabla u_s + \cos(\sqrt{\lambda}s)\nabla v_s|^2_{X_P}\\ &= |(u,v)|^2_{L^2(\mathbb{R}^m)\times L^2(\mathbb{R}^m)} + |\nabla u_s|^2_{X_P} + |\nabla v_s|^2_{X_P}\\ &= |(u,v)|^2_{H^1_P\times H^1_P}.\end{aligned}$$

The group property of $(\zeta^1_\lambda(s))_{s\in\mathbb{R}}$ comes by the group property of $(\zeta_\lambda(s))_{s\in\mathbb{R}}$, cf. Proposition 4.3. Notice that

$$\begin{aligned}|\zeta^1_\lambda(s)(u,v) - (u,v)|^2_{H^1_P\times H^1_P} &= |\zeta_\lambda(s)(u,v) - (u,v)|^2_{L^2(\mathbb{R}^m)\times L^2(\mathbb{R}^m)}\\ &+ |\cos(\sqrt{\lambda}s)\nabla u_s - \sin(\sqrt{\lambda}s)\nabla v_s - \nabla u|^2_{X_P}\\ &+ |\sin(\sqrt{\lambda}s)\nabla u_s + \cos(\sqrt{\lambda}s)\nabla v_s - \nabla v|^2_{X_P}\end{aligned}$$

and therefore, the continuity of $(\zeta^1_\lambda(s))_{s\in\mathbb{R}}$ on $H^1_P \times H^1_P$ follows by the continuity of $(\zeta_\lambda(s))_{s\in\mathbb{R}}$ on $L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$ and by the continuity of $(\zeta^1(s))_{s\in\mathbb{R}}$ on H^1_P, thanks to the inequalities

$$\begin{aligned}|\cos(\sqrt{\lambda}s)\nabla u_s - \sin(\sqrt{\lambda}s)\nabla v_s - \nabla u|_{X_P} &\leq (1-\cos(\sqrt{\lambda}s))|\nabla u_s|_{X_P} + |\nabla u_s - \nabla u|_{X_P}\\ &+ |\sin(\sqrt{\lambda}s)|\,|\nabla v_s|_{X_P}\end{aligned}$$

and

$$\begin{aligned}|\sin(\sqrt{\lambda}s)\nabla u_s + \cos(\sqrt{\lambda}s)\nabla v_s - \nabla v|_{X_P} &\leq |\sin(\sqrt{\lambda}s)|\,|\nabla u_s|_{X_P} + |\nabla v_s - \nabla v|_{X_P}\\ &+(1-\cos(\sqrt{\lambda}s))\,|\nabla v_s|_{X_P}\end{aligned}$$

and to the equalities $|\nabla u_s|_{X_P} = |\nabla u|_{X_P}, |\nabla v_s|_{X_P} = |\nabla v|_{X_P}$. Notice that the set of elements in $H^1_P \times H^1_P$ which are left invariant by the group $(\zeta^1_\lambda(s))_{s\in\mathbb{R}}$ is given by

$$\begin{aligned}&\{(u,v) \in H^1_P \times H^1_P \;:\; \zeta^1_\lambda(s)(u,v) = (u,v), \; s \in \mathbb{R}\}\\ &= \{(u,v) \in L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m) \;:\; \zeta_\lambda(s)(u,v) = (u,v), \; s \in \mathbb{R}\} \cap (H^1_P \times H^1_P)\\ &= F_\lambda \cap (H^1_P \times H^1_P) = \left\{\left(u, \frac{\mathcal{T}}{\sqrt{\lambda}}u\right) \;:\; u \in E_\lambda\right\} \cap (H^1_P \times H^1_P)\\ &= \left\{\left(u, \frac{\mathcal{T}}{\sqrt{\lambda}}u\right) \;:\; u \in E_\lambda \cap H^1_P\right\}.\end{aligned}$$

For the last point we have used the third point of Remark 4.1.
Applying Theorem 3.1, we deduce for any $u \in H^1_P$, the strong convergence in $H^1_P \times H^1_P$, uniformly with respect to $r \in \mathbb{R}$

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} \zeta^1_\lambda(s)(u,0)\,\mathrm{d}s = \mathrm{Proj}_{F_\lambda\cap(H^1_P\times H^1_P)}(u,0) = \left(U, \frac{\mathcal{T}}{\sqrt{\lambda}}U\right)$$

for some $U \in E_\lambda \cap H^1_P$. This implies the following strong convergences in $L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^m)$, uniformly with respect to $r \in \mathbb{R}$

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} (\cos(\sqrt{\lambda}s), \sin(\sqrt{\lambda}s))u_s\,\mathrm{d}s = \left(U, \frac{\mathcal{T}}{\sqrt{\lambda}}U\right)$$

and in $X_P \times X_P$, uniformly with respect to $r \in \mathbb{R}$

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} (\cos(\sqrt{\lambda}s), \sin(\sqrt{\lambda}s))\nabla u_s\,\mathrm{d}s = \left(\nabla U, \nabla\frac{\mathcal{T}}{\sqrt{\lambda}}U\right). \tag{30}$$

By Proposition 4.3 we have the strong convergence in $L^2 \times L^2$

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} (\cos(\sqrt{\lambda}s), \sin(\sqrt{\lambda}s))u_s \,\mathrm{d}s = \lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} \zeta_\lambda(s)(u,0)\,\mathrm{d}s$$
$$= \frac{1}{2}\left(\mathrm{Proj}_{E_\lambda} u, \frac{\mathcal{T}}{\sqrt{\lambda}}\mathrm{Proj}_{E_\lambda} u\right)$$

implying that $\mathrm{Proj}_{E_\lambda} u = 2U \in E_\lambda \cap H^1_P$ and the statements in (29) follow by (30). As $(u,0) - \frac{1}{2}(\mathrm{Proj}_{E_\lambda} u, \frac{\mathcal{T}}{\sqrt{\lambda}}\mathrm{Proj}_{E_\lambda} u)$ is orthogonal on F_λ with respect to the scalar product of $L^2(\mathbb{R}^m)\times L^2(\mathbb{R}^m)$, and also on $F_\lambda \cap (H^1_P \times H^1_P)$ with respect to the scalar product of $H^1_P \times H^1_P$, we deduce that for any $V \in E_\lambda \cap H^1_P$ we have

$$\left(\nabla u - \frac{1}{2}\nabla \mathrm{Proj}_{E_\lambda} u, \nabla V\right)_{X_P} + \left(0 - \frac{1}{2}\nabla\frac{\mathcal{T}}{\sqrt{\lambda}}\mathrm{Proj}_{E_\lambda} u, \nabla\frac{\mathcal{T}}{\sqrt{\lambda}}V\right)_{X_P} = 0.$$

By the third statement of Remark 4.1, we deduce that

$$(\nabla u - \nabla \mathrm{Proj}_{E_\lambda} u, \nabla V)_{X_P} = 0,\ \ V \in E_\lambda \cap H^1_P$$

which together with the orthogonality of $u - \mathrm{Proj}_{E_\lambda} u$ on E_λ in $L^2(\mathbb{R}^m)$, gives the orthogonality of $u - \mathrm{Proj}_{E_\lambda} u$ on $E_\lambda \cap H^1_P$ in H^1_P. The last conclusion follows thanks to Proposition 3.5, noticing that

$$|\nabla \mathrm{Proj}_{E_\lambda} u|_{X_P} \le \left(\liminf_{S\to+\infty} \frac{2}{S}\int_r^{r+S} |\cos(\sqrt{\lambda}s)|\,\mathrm{d}s\right)|\nabla u|_{X_P}$$
$$\le \liminf_{S\to+\infty} 2\left(\frac{1}{S}\int_r^{r+S} (\cos(\sqrt{\lambda}s))^2\,\mathrm{d}s\right)^{1/2} |\nabla u|_{X_P}$$
$$= \sqrt{2}|\nabla u|_{X_P}.$$

□

Remark 4.2

The convergence in Proposition 4.4 being uniform with respect to $r \in \mathbb{R}$, it allows us to obtain, by changing s to $-s$

$$\lim_{S\to+\infty} \frac{2}{S}\int_r^{r+S} (\cos(\sqrt{\lambda}s)\nabla u_{-s}, -\sin(\sqrt{\lambda}s)\nabla u_{-s})\,\mathrm{d}s = \left(\nabla \mathrm{Proj}_{E_\lambda} u, \nabla\frac{\mathcal{T}}{\sqrt{\lambda}}\mathrm{Proj}_{E_\lambda} u\right)$$

strongly in $X_P \times X_P$, uniformly with respect to $r \in \mathbb{R}$.

5 The effective problem

The goal of this section is to introduce the effective bilinear form $\mathfrak{m}$ and to justify its well definition, see Proposition 5.5. In order to achieve this, in Section 5.1 we prove some technical lemmas, which will provide the existence of the limit

$$\lim_{S\to+\infty} \frac{1}{S}\int_0^S G(s)D\nabla\{u(Y(2s;\cdot))\}\,\mathrm{d}s \tag{31}$$

strongly in X_Q, for any $u \in E_\lambda \cap H^1_P$, $\lambda > 0$ and $D \in H^\infty_Q$. Moreover, the limit in (31) is explicited through a new family of projections associated to the eigen-spaces of the operator $-L^2$, where L is the infinitesimal generator of the group $(G(s))_{s\in\mathbb{R}}$. These projections are studied in Proposition 5.1. In Section 5.2 we indicate a structural hypothesis which allows us to justify the existence of the limit (31) for any $u \in H^1_P$.

5.1 Technical tools

For further developements, we need the following lemma.

Lemma 5.1

1. *For any matrix field $D \in H_Q \cap H_Q^\infty$ and any vector field $c \in X_P$ we have the convergence*

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} G(s)Dc \,\mathrm{d}s = \langle D\rangle\, c, \ \text{strongly in } X_Q, \ \text{uniformly with respect to } r\in\mathbb{R}.$$

2. *The above convergence still holds true for any matrix field $D \in H_Q^\infty$, and any vector field $c \in X_P$, where the average of D is considered in $H_{Q,\mathrm{loc}}$ cf. Theorem 3.2.*

Proof.
1. We know by Theorem 3.2 that

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} G(s)D \,\mathrm{d}s = \langle D\rangle \text{ strongly in } H_Q, \text{ uniformly with respect to } r\in\mathbb{R}.$$

We define the sequence $c_k = \mathbf{1}_{\{\{|P^{1/2}c|\le k\}\}}c,\ k\in\mathbb{N}$. Any vector field c_k belongs to X_P^∞ and we have the convergence $\lim_{k\to+\infty} c_k = c$ in X_P. For any $k\in\mathbb{N}$ we have the convergence

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} G(s)Dc_k \,\mathrm{d}s = \langle D\rangle\, c_k, \ \text{ strongly in } X_Q, \text{ uniformly with respect to } r\in\mathbb{R}$$

thanks to the inequality

$$\left|\frac{1}{S}\int_r^{r+S} G(s)Dc_k \,\mathrm{d}s - \langle D\rangle\, c_k\right|_{X_Q} \le \left|\frac{1}{S}\int_r^{r+S} G(s)D \,\mathrm{d}s - \langle D\rangle\right|_{H_Q} |c_k|_{X_P^\infty}$$

Observe that

$$\begin{aligned}
\left|\frac{1}{S}\int_r^{r+S} G(s)Dc \,\mathrm{d}s - \langle D\rangle\, c\right|_{X_Q} &\le \left|\frac{1}{S}\int_r^{r+S} G(s)D(c-c_k) \,\mathrm{d}s\right|_{X_Q} \\
&\quad + \left|\frac{1}{S}\int_r^{r+S} G(s)Dc_k \,\mathrm{d}s - \langle D\rangle\, c_k\right|_{X_Q} + |\langle D\rangle\,(c_k - c)|_{X_Q} \\
&\le \left|\frac{1}{S}\int_r^{r+S} G(s)D \,\mathrm{d}s\right|_{H_Q^\infty} |c-c_k|_{X_P} \\
&\quad + \left|\frac{1}{S}\int_r^{r+S} G(s)Dc_k \,\mathrm{d}s - \langle D\rangle\, c_k\right|_{X_Q} + |\langle D\rangle\,|_{H_Q^\infty}|c_k - c|_{X_P} \\
&\le 2|D|_{H_Q^\infty}|c_k - c|_{X_P} + \left|\frac{1}{S}\int_r^{r+S} G(s)Dc_k \,\mathrm{d}s - \langle D\rangle\, c_k\right|_{X_Q}
\end{aligned}$$

which implies that for any $k\in\mathbb{N}$

$$\limsup_{S\to+\infty}\sup_{r\in\mathbb{R}} \left|\frac{1}{S}\int_r^{r+S} G(s)Dc \,\mathrm{d}s - \langle D\rangle\, c\right|_{X_Q} \le 2|D|_{H_Q^\infty}|c_k - c|_{X_P}.$$

Our conclusion follows by letting $k\to+\infty$.
2. For any $k\in\mathbb{N}$ we consider $D_k = \mathbf{1}_{\{\psi\le k\}}D$. Since $D\in H_Q^\infty \subset H_{Q,\mathrm{loc}}$, we deduce that $D_k \in H_Q\cap H_Q^\infty$, and by the previous statement, we have for any $k\in\mathbb{N}$

$$\lim_{S\to+\infty}\sup_{r\in\mathbb{R}} \left|\frac{1}{S}\int_r^{r+S} G(s)D_k c \,\mathrm{d}s - \langle D_k\rangle\, c\right|_{X_Q} = 0.$$

Notice that

$$\left|\frac{1}{S}\int_r^{r+S} G(s)Dc\,\mathrm{d}s - \langle D\rangle c\right|_{X_Q} - \left|\frac{1}{S}\int_r^{r+S} G(s)D_k c\,\mathrm{d}s - \langle D_k\rangle c\right|_{X_Q} \leq \left|\frac{1}{S}\int_r^{r+S} G(s)(D-D_k)c\,\mathrm{d}s\right|_{X_Q}$$
$$+ |\langle D_k - D\rangle c|_{X_Q}$$
$$= \left|\frac{1}{S}\int_r^{r+S} G(s)D\mathbf{1}_{\{\psi>k\}}c\,\mathrm{d}s\right|_{X_Q} + \left|\langle D\rangle \mathbf{1}_{\{\psi>k\}}c\right|_{X_Q}$$
$$\leq \frac{1}{S}\int_r^{r+S} |G(s)D|_{H_Q^\infty} |\mathbf{1}_{\{\psi>k\}}c|_{X_P}\,\mathrm{d}s + |\langle D\rangle|_{H_Q^\infty} |\mathbf{1}_{\{\psi>k\}}c|_{X_P}$$
$$\leq 2|\langle D\rangle|_{H_Q^\infty} |\mathbf{1}_{\{\psi>k\}}c|_{X_P}$$

which implies that

$$\limsup_{S\to+\infty}\sup_{r\in\mathbb{R}}\left|\frac{1}{S}\int_r^{r+S} G(s)Dc\,\mathrm{d}s - \langle D\rangle c\right|_{X_Q} \leq 2|\langle D\rangle|_{H_Q^\infty} |\mathbf{1}_{\{\psi>k\}}c|_{X_P}.$$

Our conclusion follows by letting $k\to+\infty$. □

The purpose of the following proposition is to introduce the orthogonal projections on the eigen-spaces of $-L^2$, by appealing to the von Neumann ergodic mean theorem, in respect with a new family of unitary C^0-groups. These new projections will allow us to justify the existence of the limit in (31). As suggested in Proposition 4.3, for any $\lambda>0$, we introduce the family of transformations of $H_Q\times H_Q$

$$G_\lambda(s)(A,B) = (\cos(\sqrt{\lambda}s)G(s)A - \sin(\sqrt{\lambda}s)G(s)B, \sin(\sqrt{\lambda}s)G(s)A + \cos(\sqrt{\lambda}s)G(s)B)$$

for any $(A,B)\in H_Q\times H_Q, s\in\mathbb{R}$.

Proposition 5.1
For any $\lambda>0$, the family $(G_\lambda(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary transformations of $H_Q\times H_Q$, whose infinitesimal generator L_λ is given by

$$\mathrm{dom}L_\lambda = \mathrm{dom}L\times\mathrm{dom}L,\;\; L_\lambda(A,B) = (LA-\sqrt{\lambda}B, LB+\sqrt{\lambda}A),\;\; (A,B)\in\mathrm{dom}L\times\mathrm{dom}L$$

where L is the infinitesimal generator of the group $(G(s))_{s\in\mathbb{R}}$. For any $A\in H_Q$ we have, with the notation $\mathcal{E}_\lambda = \ker(-L^2-\lambda Id)$

$$\left(\mathrm{Proj}_{\mathcal{E}_\lambda}A, \frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda}A\right) = \lim_{S\to+\infty}\frac{2}{S}\int_r^{r+S}(\cos(\sqrt{\lambda}s)G(s)A, \sin(\sqrt{\lambda}s)G(s)A)\,\mathrm{d}s$$

strongly in $H_Q\times H_Q$, uniformly with respect to $r\in\mathbb{R}$, and

$$|\mathrm{Proj}_{\mathcal{E}_\lambda}A|_{H_Q} = \left|\frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda}A\right|_{H_Q} \leq |A|_{H_Q}.$$

Moreover, if $A\in H_Q\cap H_Q^\infty$, then $\mathrm{Proj}_{\mathcal{E}_\lambda}A, \frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda}A\in H_Q^\infty$ and

$$|\mathrm{Proj}_{\mathcal{E}_\lambda}A|_{H_Q^\infty} \leq 2|A|_{H_Q^\infty},\quad \left|\frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda}A\right|_{H_Q^\infty} \leq 2|A|_{H_Q^\infty}.$$

Proof. It is easily seen that $(G_\lambda(s))_{s\in\mathbb{R}}$ is a C^0-group. For any $(A,B)\in H_Q\times H_Q$ we obtain by direct computations

$$|G_\lambda(s)(A,B)|^2_{H_Q\times H_Q} = |(A,B)|^2_{H_Q\times H_Q}$$

saying that $(G_\lambda(s))_{s\in\mathbb{R}}$ are unitary transformations of $H_Q\times H_Q$. As before we check that $\mathrm{dom}L_\lambda = \mathrm{dom}L\times\mathrm{dom}L$ and

$$L_\lambda(A,B) = (LA-\sqrt{\lambda}B, LB+\sqrt{\lambda}A),\;\; (A,B)\in\mathrm{dom}L\times\mathrm{dom}L.$$

The kernel of L_λ is given by

$$\ker L_\lambda = \left\{ \left(A, \frac{L}{\sqrt{\lambda}} A \right) \ : \ A \in \mathcal{E}_\lambda \right\}.$$

Notice that $A \in \mathcal{E}_\lambda$ iff $\left(A, \frac{L}{\sqrt{\lambda}}A\right) \in \ker L_\lambda$, or equivalently iff $A \in \mathrm{dom}L$ and $G_\lambda(s)\left(A, \frac{L}{\sqrt{\lambda}}A\right) = \left(A, \frac{L}{\sqrt{\lambda}}A\right)$ for any $s \in \mathbb{R}$, that is

$$G(s)A = \cos(\sqrt{\lambda}s)A + \sin(\sqrt{\lambda}s)\frac{L}{\sqrt{\lambda}}A, \ \ s \in \mathbb{R} \tag{32}$$

$$\frac{L}{\sqrt{\lambda}}G(s)A = -\sin(\sqrt{\lambda}s)A + \cos(\sqrt{\lambda}s)\frac{L}{\sqrt{\lambda}}A, \ \ s \in \mathbb{R}. \tag{33}$$

Observe that (33) comes from (32), by taking the derivative with respect to s and therefore we obtain the characterization

$$\begin{aligned}
\mathcal{E}_\lambda &= \left\{ A \in \mathrm{dom}L \ : \ G(s)A = \cos(\sqrt{\lambda}s)A + \sin(\sqrt{\lambda}s)\frac{L}{\sqrt{\lambda}}A, \ \ s \in \mathbb{R} \right\} \\
&= \left\{ A \in \mathrm{dom}L \ : \ G(s)\left(A + i\frac{L}{\sqrt{\lambda}}A\right) = e^{-i\sqrt{\lambda}s}\left(A + i\frac{L}{\sqrt{\lambda}}A\right), \ \ s \in \mathbb{R} \right\}.
\end{aligned}$$

Applying Theorem 3.1, we know that for any $(A, B) \in H_Q \times H_Q$ we have

$$\mathrm{Proj}_{\ker L_\lambda}(A, B) = \lim_{S \to +\infty} \frac{1}{S} \int_r^{r+S} G_\lambda(s)(A, B) \, \mathrm{d}s$$

in $H_Q \times H_Q$, uniformly with respect to $r \in \mathbb{R}$. In particular we have the uniform convergence in $H_Q \times H_Q$, with respect to $r \in \mathbb{R}$

$$\mathrm{Proj}_{\ker L_\lambda}(A, 0) = \lim_{S \to +\infty} \frac{1}{S} \int_r^{r+S} (\cos(\sqrt{\lambda}s)G(s)A, \sin(\sqrt{\lambda}s)G(s)A) \, \mathrm{d}s = \left(Z, \frac{L}{\sqrt{\lambda}}Z\right) \tag{34}$$

for some $Z \in \mathcal{E}_\lambda$. Therefore, for any $W \in \mathcal{E}_\lambda$ we have

$$(A - Z, W)_{H_Q} + \left(0 - \frac{L}{\sqrt{\lambda}}Z, \frac{L}{\sqrt{\lambda}}W\right)_{H_Q} = 0.$$

As $(G(s))_{s \in \mathbb{R}}$ is a unitary group, its infinitesimal generator L is skew-adjoint, and thus

$$(A - 2Z, W)_{H_Q} = 0, \ \ W \in \mathcal{E}_\lambda$$

saying that $\mathrm{Proj}_{\mathcal{E}_\lambda} A = 2Z$. Thanks to (34) we obtain the uniform convergence in $H_Q \times H_Q$, with respect to $r \in \mathbb{R}$

$$\left(\mathrm{Proj}_{\mathcal{E}_\lambda} A, \frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda} A\right) = \lim_{S \to +\infty} \frac{2}{S} \int_r^{r+S} (\cos(\sqrt{\lambda}s)G(s)A, \sin(\sqrt{\lambda}s)G(s)A) \, \mathrm{d}s.$$

Assume now that $A \in H_Q \cap H_Q^\infty$. The above convergence in $H_Q \times H_Q$ guarantees the existence of a sequence $(S_n)_n$ such that $\lim_{n \to +\infty} S_n = +\infty$ and

$$\lim_{n \to +\infty} \frac{2}{S_n} \int_0^{S_n} \cos(\sqrt{\lambda}s)G(s)A \, \mathrm{d}s = \mathrm{Proj}_{\mathcal{E}_\lambda} A, \ \ \text{for a.a. } y \in \mathbb{R}^m$$

$$\lim_{n \to +\infty} \frac{2}{S_n} \int_0^{S_n} \sin(\sqrt{\lambda}s)G(s)A \, \mathrm{d}s = \frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda} A, \ \ \text{for a.a. } y \in \mathbb{R}^m$$

But the sequence $\left(\frac{2}{S_n}\int_0^{S_n}(\cos(\sqrt{\lambda}s)G(s)A, \sin(\sqrt{\lambda}s)G(s)A) \, \mathrm{d}s\right)_n$ is bounded in $H_Q^\infty \times H_Q^\infty$

$$\left| \frac{2}{S_n} \int_0^{S_n} \cos(\sqrt{\lambda}s)G(s)A \, \mathrm{d}s \right|_{H_Q^\infty} \leq 2|A|_{H_Q^\infty}, \ \ \left| \frac{2}{S_n} \int_0^{S_n} \sin(\sqrt{\lambda}s)G(s)A \, \mathrm{d}s \right|_{H_Q^\infty} \leq 2|A|_{H_Q^\infty}.$$

We deduce that $\text{Proj}_{\mathcal{E}_\lambda} A, \frac{L}{\sqrt{\lambda}} \text{Proj}_{\mathcal{E}_\lambda} A \in H_Q^\infty$ and

$$|\text{Proj}_{\mathcal{E}_\lambda} A|_{H_Q^\infty} \le 2|A|_{H_Q^\infty}, \quad \left| \frac{L}{\sqrt{\lambda}} \text{Proj}_{\mathcal{E}_\lambda} A \right|_{H_Q^\infty} \le 2|A|_{H_Q^\infty}.$$

□

Remark 5.1
It is easily seen, thanks to the skew-symmetry of L that $\mathcal{E}_0 := \ker(-L^2) = \ker L$ and thus $\text{Proj}_{\mathcal{E}_0} A = \text{Proj}_{\ker L} A = \langle A \rangle$, for any matrix field $A \in H_Q$.

Remark 5.2

1. *As in the last statement of Theorem 3.2, the operator $\text{Proj}_{\mathcal{E}_\lambda}$ extends from H_Q to $H_{Q,\text{loc}}$. Indeed, let A be a matrix field in $H_{Q,\text{loc}}$. For any $k \in \mathbb{N}$, $A_k := \mathbf{1}_{\{\psi \le k\}} A$ belongs to H_Q, and by Proposition 5.1 we know that*

$$\lim_{S \to +\infty} \frac{2}{S} \int_r^{r+S} (\cos(\sqrt{\lambda} s) G(s) A_k, \sin(\sqrt{\lambda} s) G(s) A_k) \, ds = \left(\text{Proj}_{\mathcal{E}_\lambda} A_k, \frac{L}{\sqrt{\lambda}} \text{Proj}_{\mathcal{E}_\lambda} A_k \right)$$

strongly in $H_Q \times H_Q$, uniformly with respect to $r \in \mathbb{R}$. We have

$$\lim_{S \to +\infty} \frac{2}{S} \int_r^{r+S} (\cos(\sqrt{\lambda} s), \sin(\sqrt{\lambda} s)) G(s) A_k \, ds = \lim_{S \to +\infty} \frac{2}{S} \int_r^{r+S} (\cos(\sqrt{\lambda} s), \sin(\sqrt{\lambda} s)) G(s) A_l \, ds$$

almost everywhere on $\{\psi \le \min(k,l)\}$, and thus there are two matrix fields $B, C \in H_{Q,\text{loc}}$ such that

$$\begin{aligned} \lim_{S \to +\infty} \frac{2}{S} \int_r^{r+S} (\cos(\sqrt{\lambda} s) G(s) A_k, \sin(\sqrt{\lambda} s) G(s) A_k) \, ds &= \mathbf{1}_{\{\psi \le k\}} (B, C) \\ &= \left(\text{Proj}_{\mathcal{E}_\lambda} A_k, \frac{L}{\sqrt{\lambda}} \text{Proj}_{\mathcal{E}_\lambda} A_k \right) \end{aligned}$$

strongly in $H_Q \times H_Q$, uniformly with respect to $r \in \mathbb{R}$, for any fixed $k \in \mathbb{N}$. We claim that $B \in \text{dom}(L^2)$, and $L^2 B + \lambda B = 0$, that is $B \in \ker(-L^2 - \lambda Id)$, where L is considered in $H_{Q,\text{loc}}$. Indeed, we have for any $k \in \mathbb{N}$

$$\mathbf{1}_{\{\psi \le k\}} B = \text{Proj}_{\mathcal{E}_\lambda} A_k \in \text{dom}(L|_{H_Q})$$

saying that $B \in \text{dom}(L)$. Moreover

$$\mathbf{1}_{\{\psi \le k\}} LB = L(\mathbf{1}_{\{\psi \le k\}} B) = L(\text{Proj}_{\mathcal{E}_\lambda} A_k) \in \text{dom}(L|_{H_Q})$$

implying that $LB \in \text{dom}(L)$ and

$$\begin{aligned} \mathbf{1}_{\{\psi \le k\}} L^2 B = \mathbf{1}_{\{\psi \le k\}} L(LB) = L(\mathbf{1}_{\{\psi \le k\}} LB) &= L(L(\text{Proj}_{\mathcal{E}_\lambda} A_k)) \\ &= -\lambda \text{Proj}_{\mathcal{E}_\lambda} A_k = -\mathbf{1}_{\{\psi \le k\}} \lambda B, \quad k \in \mathbb{N}. \end{aligned}$$

We deduce that $B \in \text{dom}(L^2)$ and $L^2 B + \lambda B = 0$. Notice that for any $k \in \mathbb{N}$ we have

$$\mathbf{1}_{\{\psi \le k\}} C = \frac{L}{\sqrt{\lambda}} \text{Proj}_{\mathcal{E}_\lambda} A_k = \frac{L}{\sqrt{\lambda}} (\mathbf{1}_{\{\psi \le k\}} B) = \mathbf{1}_{\{\psi \le k\}} \frac{L}{\sqrt{\lambda}} B \in \text{dom}(L|_{H_Q})$$

saying that $C = \frac{L}{\sqrt{\lambda}} B \in \text{dom}(L)$. It is easily seen that the matrix field $B \in H_{Q,\text{loc}}$ satisfies

$$B \in \ker(-L^2 - \lambda Id), \quad (\mathbf{1}_{\{\psi \le k\}} (A - B), M)_{H_Q} = 0 \quad \textit{for any } M \in \mathcal{E}_\lambda, k \in \mathbb{N}$$

and that B is uniquely determined by the above variational characterization. Moreover, if $A \in H_Q$, then B coincides with $\text{Proj}_{\mathcal{E}_\lambda} A \subset \mathcal{E}_\lambda \subset H_Q$. Therefore, for any $A \in H_{Q,\text{loc}}$, the family

$$\left(\frac{2}{S} \int_r^{r+S} (\cos(\sqrt{\lambda} s) G(s) A, \sin(\sqrt{\lambda} s) G(s) A) \, ds \right)_{S > 0}$$

converges in $H_{Q,\text{loc}} \times H_{Q,\text{loc}}$, as $S \to +\infty$, toward $(B, \frac{L}{\sqrt{\lambda}} B)$, where the application $A \in H_{Q,\text{loc}} \to B \in H_{Q,\text{loc}}$ extends the projection $\text{Proj}_{\mathcal{E}_\lambda} : H_Q \to H_Q$. We use the same notation $B = \text{Proj}_{\mathcal{E}_\lambda} A$ independently of A being in H_Q or in $H_{Q,\text{loc}}$.

2. *For any matrix field $A \in H_Q^\infty \subset H_{Q,\mathrm{loc}}$ and any $k \in \mathbb{N}$, the matrix field $A_k = \mathbf{1}_{\{\psi \le k\}} A$ belongs to $H_Q \cap H_Q^\infty$, and by Theorem 5.1 we have*

$$|\mathrm{Proj}_{\mathcal{E}_\lambda} A_k|_{H_Q^\infty} \le 2|A_k|_{H_Q^\infty} \le 2|A|_{H_Q^\infty}$$

$$\left|\frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda} A_k\right|_{H_Q^\infty} \le 2|A_k|_{H_Q^\infty} \le 2|A|_{H_Q^\infty}.$$

We deduce that

$$|\mathrm{Proj}_{\mathcal{E}_\lambda} A|_{H_Q^\infty} = \sup_{k\in\mathbb{N}} |\mathbf{1}_{\{\psi\le k\}}\mathrm{Proj}_{\mathcal{E}_\lambda} A|_{H_Q^\infty} = \sup_{k\in\mathbb{N}} |\mathrm{Proj}_{\mathcal{E}_\lambda} A_k|_{H_Q^\infty} \le 2|A|_{H_Q^\infty}$$

and

$$\left|\frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda} A\right|_{H_Q^\infty} = \sup_{k\in\mathbb{N}} \left|\mathbf{1}_{\{\psi\le k\}}\frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda} A\right|_{H_Q^\infty} = \sup_{k\in\mathbb{N}} \left|\frac{L}{\sqrt{\lambda}}\mathrm{Proj}_{\mathcal{E}_\lambda} A_k\right|_{H_Q^\infty} \le 2|A|_{H_Q^\infty}.$$

The unitary C^0-groups $(G_\lambda(s))_{s\in\mathbb{R}}, \lambda > 0$ emphasized in Proposition 5.1 allow us to establish the following convergences.

Lemma 5.2

1. *For any matrix field $D \in H_Q \cap H_Q^\infty$ and any function $u \in E_\lambda \cap H_P^1, \lambda > 0$, we have the convergence*

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} G(s) D\nabla u_{2s}\, \mathrm{d}s = \frac{1}{2}\mathrm{Proj}_{\mathcal{E}_{4\lambda}} D\nabla u + \frac{1}{2}\frac{L}{\sqrt{4\lambda}}\mathrm{Proj}_{\mathcal{E}_{4\lambda}} D\nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u$$

strongly in X_Q, uniformly with respect to $r \in \mathbb{R}$.

2. *The above convergence still holds true for any matrix field $D \in H_Q^\infty$ and any function $u \in E_\lambda \cap H_P^1, \lambda > 0$, where the operators $\mathrm{Proj}_{\mathcal{E}_{4\lambda}}, L$ are considered in $H_{Q,\mathrm{loc}}$ cf. Remarks 5.2, 3.1.*

Proof.
1. For any $s \in \mathbb{R}$ we have $u_{2s} = \cos(\sqrt{4\lambda}s)u + \sin(\sqrt{4\lambda}s)\frac{\mathcal{T}}{\sqrt{\lambda}}u$. By the third statement of Remark 4.1, we know that $\frac{\mathcal{T}}{\sqrt{\lambda}}u \in E_\lambda \cap H_P^1$ and therefore we have the following equality in X_P

$$\nabla u_{2s} = \cos(\sqrt{4\lambda}s)\nabla u + \sin(\sqrt{4\lambda}s)\nabla\frac{\mathcal{T}}{\sqrt{\lambda}}u. \tag{35}$$

We claim that

$$\lim_{S\to+\infty} \frac{2}{S}\int_r^{r+S} \cos(\sqrt{4\lambda}s)G(s)D\nabla u\, \mathrm{d}s = \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D\nabla u \tag{36}$$

$$\lim_{S\to+\infty} \frac{2}{S}\int_r^{r+S} \sin(\sqrt{4\lambda}s)G(s)D\nabla\frac{\mathcal{T}}{\sqrt{\lambda}}u\, \mathrm{d}s = \frac{L}{\sqrt{4\lambda}}\mathrm{Proj}_{\mathcal{E}_{4\lambda}} D\nabla\frac{\mathcal{T}}{\sqrt{\lambda}}u \tag{37}$$

strongly in X_Q, uniformly with respect to $r \in \mathbb{R}$. We introduce the sequence $c_k = \mathbf{1}_{\{\{|P^{1/2}\nabla u| \le k\}\}}\nabla u$. Any vector field c_k belongs to X_P^∞ and we have the convergence $\lim_{k\to+\infty} c_k = \nabla u$ in X_P. By Proposition 5.1 we have

$$\lim_{S\to+\infty} \frac{2}{S}\int_r^{r+S} \cos(\sqrt{4\lambda}s)G(s)D\, \mathrm{d}s = \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D$$

strongly in H_Q, uniformly with respect to $r \in \mathbb{R}$. As in the proof of Lemma 5.1 we have

$$\left|\frac{2}{S}\int_r^{r+S} \cos(\sqrt{4\lambda}s)G(s)D\nabla u\, \mathrm{d}s - \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D\nabla u\right|_{X_Q} \le 4|D|_{H_Q^\infty}|\nabla u - c_k|_{X_P}$$

$$+ \left|\frac{2}{S}\int_r^{r+S} \cos(\sqrt{4\lambda}s)G(s)D\, \mathrm{d}s - \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D\right|_{H_Q} |c_k|_{X_P^\infty}$$

which implies that for any $k \in \mathbb{N}$

$$\limsup_{S \to +\infty} \sup_{r \in \mathbb{R}} \left| \frac{2}{S} \int_r^{r+S} \cos(\sqrt{4\lambda} s) G(s) D \nabla u \, \mathrm{d}s - \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D \nabla u \right|_{X_Q} \leq 4 |D|_{H_Q^\infty} |\nabla u - c_k|_{X_P}.$$

The formula (36) follows by letting $k \to +\infty$. For the formula (37) use the field $\nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \in X_P$ and the convergence

$$\lim_{S \to +\infty} \frac{2}{S} \int_r^{r+S} \sin(\sqrt{4\lambda} s) G(s) D \, \mathrm{d}s = \frac{L}{\sqrt{4\lambda}} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D$$

strongly in H_Q, uniformly with respect to $r \in \mathbb{R}$. Combining (36), (37), (35) yields

$$\lim_{S \to +\infty} \frac{1}{S} \int_r^{r+S} G(s) D \nabla u_{2s} \, \mathrm{d}s = \frac{1}{2} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D \nabla u + \frac{1}{2} \frac{L}{\sqrt{4\lambda}} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u$$

strongly in X_Q, uniformly with respect to $r \in \mathbb{R}$.
2. For any $k \in \mathbb{N}$, let us consider $D_k = \mathbf{1}_{\{\psi \leq k\}} D \in H_Q \cap H_Q^\infty$. By the previous statement we have

$$\lim_{S \to +\infty} \sup_{r \in \mathbb{R}} \left| \frac{1}{S} \int_r^{r+S} G(s) D_k \nabla u_{2s} \, \mathrm{d}s - \frac{1}{2} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D_k \nabla u - \frac{1}{2} \frac{L}{\sqrt{4\lambda}} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D_k \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_Q} = 0. \quad (38)$$

Notice also that we have

$$\begin{aligned} & \left| \frac{1}{S} \int_r^{r+S} G(s) D \nabla u_{2s} \, \mathrm{d}s - \frac{1}{2} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D \nabla u - \frac{1}{2} \frac{L}{\sqrt{4\lambda}} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_Q} \\ & - \left| \frac{1}{S} \int_r^{r+S} G(s) D_k \nabla u_{2s} \, \mathrm{d}s - \frac{1}{2} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D_k \nabla u - \frac{1}{2} \frac{L}{\sqrt{4\lambda}} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D_k \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_Q} \\ & \leq \left| \frac{1}{S} \int_r^{r+S} G(s) (D - D_k) \nabla u_{2s} \, \mathrm{d}s \right|_{X_Q} + \frac{1}{2} \left| \mathrm{Proj}_{\mathcal{E}_{4\lambda}} (D_k - D) \nabla u \right|_{X_Q} \\ & + \frac{1}{2} \left| \frac{L}{\sqrt{4\lambda}} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} (D_k - D) \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_Q}. \end{aligned}$$

It is easily seen that

$$\begin{aligned} \left| \frac{1}{S} \int_r^{r+S} G(s) (D - D_k) \nabla u_{2s} \, \mathrm{d}s \right|_{X_Q} & = \left| \frac{1}{S} \int_r^{r+S} G(s) D \mathbf{1}_{\{\psi > k\}} \left(\cos(\sqrt{4\lambda} s) \nabla u + \sin(\sqrt{4\lambda} s) \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right) \right|_{X_Q} \\ & \leq |D|_{H_Q^\infty} \left(|\mathbf{1}_{\{\psi > k\}} \nabla u|_{X_P} + \left| \mathbf{1}_{\{\psi > k\}} \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_P} \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \left| \mathrm{Proj}_{\mathcal{E}_{4\lambda}} (D_k - D) \nabla u \right|_{X_Q} + \frac{1}{2} \left| \frac{L}{\sqrt{4\lambda}} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} (D_k - D) \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_Q} \\ & \leq \frac{1}{2} |\mathrm{Proj}_{\mathcal{E}_{4\lambda}} D|_{H_Q^\infty} \; |\mathbf{1}_{\{\psi > k\}} \nabla u|_{X_P} + \frac{1}{2} \left| \frac{L}{\sqrt{4\lambda}} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D \right|_{H_Q^\infty} \left| \mathbf{1}_{\{\psi > k\}} \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_P} \\ & \leq |D|_{H_Q^\infty} \left(|\mathbf{1}_{\{\psi > k\}} \nabla u|_{X_P} + \left| \mathbf{1}_{\{\psi > k\}} \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_P} \right). \end{aligned}$$

Combining the convergence (38) with the previous estimates yields

$$\begin{aligned} & \limsup_{S \to +\infty} \sup_{r \in \mathbb{R}} \left| \frac{1}{S} \int_r^{r+S} G(s) D \nabla u_{2s} \, \mathrm{d}s - \frac{1}{2} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D \nabla u - \frac{1}{2} \frac{L}{\sqrt{4\lambda}} \mathrm{Proj}_{\mathcal{E}_{4\lambda}} D \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_Q} \\ & \leq 2 |D|_{H_Q^\infty} \left(|\mathbf{1}_{\{\psi > k\}} \nabla u|_{X_P} + \left| \mathbf{1}_{\{\psi > k\}} \nabla \frac{\mathcal{T}}{\sqrt{\lambda}} u \right|_{X_P} \right). \end{aligned}$$

Our conclusion follows by letting $k \to +\infty$. □

5.2 Structural hypotheses associated to $\mathcal{B}$

Notice that any E_λ is closed and $E_\lambda \perp E_\mu$ for any $\lambda \neq \mu$, thanks to the symmetry of $\mathcal{B}$. In order to extend the existence of the limit (31) to any function $u \in H^1_P$, we need to decompose the space H^1_P through the spaces $(E_\lambda)_{\lambda \geq 0}$. We assume that $L^2(\mathbb{R}^m)$ is the Hilbertian sum of a countable family of subspaces E_λ *i.e.*, $\text{span}(\cup_{n\in\mathbb{N}} E_{\lambda_n})$ is dense in $L^2(\mathbb{R}^m)$

$$L^2(\mathbb{R}^m) = \underset{n\in\mathbb{N}}{\oplus} E_{\lambda_n}, \;\; E_{\lambda_n} = \ker(\mathcal{B} - \lambda_n Id), \;\; \lambda_n \geq 0, \;\; n \in \mathbb{N}. \tag{39}$$

Without loss of generality, we assume that $\lambda_0 = 0$ (independently with respect to 0 being an eigen-value of $\mathcal{B}$ or not) and $\lambda_n > 0, E_{\lambda_n} = \ker(\mathcal{B} - \lambda_n Id) \neq \{0\}, n \in \mathbb{N}^\star$.

Example 5.1 *(Periodic case)*
Assume that the characteristic flow $Y(s;y)$ is S_0-periodic, that is

$$\exists\, S_0 > 0 \;\; \textit{such that} \;\; Y(s + S_0; y) = Y(s;y), \;\; s \in \mathbb{R}, \;\; y \in \mathbb{R}^m.$$

We claim that any eigen-value of $\mathcal{B} = -\mathcal{T}^2$ writes $\sqrt{\lambda_n} = n\omega_0, n \in \mathbb{N}, \omega_0 = 2\pi/S_0$. Indeed, if $\lambda_0 = 0$ is an eigen-value of $\mathcal{B}$, it corresponds to $n = 0$. Let $\lambda > 0$ be a positive eigen-value of $\mathcal{B}$. This means that there is $u \in \text{dom}\mathcal{T}, u \neq 0$ such that

$$u_s + i\frac{\mathcal{T}}{\sqrt{\lambda}} u_s = e^{-i\sqrt{\lambda}s}\left(u + i\frac{\mathcal{T}}{\sqrt{\lambda}}u\right), \;\; s \in \mathbb{R}.$$

Taking $s = S_0$, one gets

$$\left(u + i\frac{\mathcal{T}}{\sqrt{\lambda}}u\right)\left(1 - e^{-i\sqrt{\lambda}S_0}\right) = 0$$

implying that $\sqrt{\lambda}S_0 = 2\pi n, n \in \mathbb{N}^\star$. In this case the hypothesis (39) holds true. Indeed, if $u \in L^2(\mathbb{R}^m)$ is such that $\text{Proj}_{E_{\lambda_n}} = 0$ for any eigen-value λ_n, then we have

$$\text{Proj}_{E_0} = \frac{1}{S_0}\int_0^{S_0} u_s \,\mathrm{d}s = 0$$

and for any $n \in \mathbb{N}^\star$

$$\text{Proj}_{E_{n^2\omega_0^2}} u = \frac{2}{S_0}\int_0^{S_0} \cos(n\omega_0 s) u_s \,\mathrm{d}s = 0, \;\; \frac{\mathcal{T}}{n\omega_0}\text{Proj}_{E_{n^2\omega_0^2}} u = \frac{2}{S_0}\int_0^{S_0} \sin(n\omega_0 s) u_s \,\mathrm{d}s = 0.$$

Therefore, all the Fourier coefficients of the S_0-periodic function $s \to u(Y(s;\cdot)) \in L^2(\mathbb{R}^m)$ vanish, and thus $u(Y(s;\cdot)) = 0$ for any $s \in \mathbb{R}$, saying that $u = 0$.

Example 5.2 *(Almost periodic case)*
We investigate now a very important particular case: that when the C^0-group $(\zeta(s))_{s\in\mathbb{R}}$ is almost periodic. We assume that for any $u \in L^2(\mathbb{R}^m)$, the function $s \in \mathbb{R} \to \zeta(s)u \in L^2(\mathbb{R}^m)$ is almost periodic, that is, the trajectory $s \to \zeta(s)u \in L^2(\mathbb{R}^m)$ is the limit in $C(\mathbb{R}; L^2(\mathbb{R}^m))$ of a sequence of trigonometric polynomials with coefficients in $L^2(\mathbb{R}^m)$ (see [6] for a detalied study of almost periodic functions with values in Banach spaces).

Proposition 5.2
Assume that the hypotheses (16), (17) hold true and that the C^0-group $(\zeta(s))_{s\in\mathbb{R}}$ is almost periodic. Then the family of non trivial subspaces $E_\lambda = \ker(\mathcal{B} - \lambda Id), \lambda \in \mathbb{R}_+$ is countable and $L^2(\mathbb{R}^m) = \underset{n}{\oplus} E_{\lambda_n}, E_{\lambda_n} = \ker(\mathcal{B} - \lambda_n Id) \neq \{0\}$.

Proof. Let $u \in L^2(\mathbb{R}^m)$ be a function orthogonal to $E_\lambda = \ker(\mathcal{B} - \lambda Id)$ for any eigen-value λ. Therefore u is orthogonal to $E_\lambda = \ker(\mathcal{B} - \lambda Id)$ for any $\lambda \in \mathbb{R}_+$

$$\text{Proj}_{E_0} u = \lim_{S\to+\infty} \frac{1}{S}\int_0^S \zeta(s) u \,\mathrm{d}s = 0, \;\; \text{Proj}_{E_\lambda} u = \lim_{S\to+\infty} \frac{2}{S}\int_0^S \cos(\sqrt{\lambda}s)\zeta(s) u \,\mathrm{d}s = 0, \;\; \lambda > 0.$$

Notice that we also have

$$\frac{\mathcal{T}}{\sqrt{\lambda}}\mathrm{Proj}_{E_\lambda} u = \lim_{S\to+\infty}\frac{2}{S}\int_0^S \sin(\sqrt{\lambda}s)\zeta(s)u \,\mathrm{d}s = 0,\ \lambda > 0.$$

Therefore all the Fourier coefficients of $s \to \zeta(s)u$ vanish, implying that $u = 0$ in $L^2(\mathbb{R}^m)$ and thus $\overline{\mathrm{span}(\cup_{\lambda\in\mathbb{R}_+} E_\lambda)} = L^2(\mathbb{R}^m)$. As $L^2(\mathbb{R}^m)$ is separable and the subspaces $(E_\lambda)_{\lambda\in\mathbb{R}_+}$ are orthogonal, we deduce that $E_\lambda \neq \{0\}$ only for a countable set $\{\lambda_n\}$, saying that (39) holds true. □

A direct consequence of (39) is given by

Proposition 5.3
The space H^1_P is the Hilbertian sum of the spaces $(E_{\lambda_n} \cap H^1_P)_{n\in\mathbb{N}}$.

Proof. The spaces $(E_{\lambda_n} \cap H^1_P)_{n\in\mathbb{N}}$ are closed in H^1_P, since $(E_{\lambda_n})_{n\in\mathbb{N}}$ are closed in $L^2(\mathbb{R}^m)$. By Propositions 3.5, 4.4 we have

$$u - \mathrm{Proj}_{E_{\lambda_n}} u \perp E_{\lambda_n} \cap H^1_P \ \text{ in } H^1_P,\ u \in H^1_P,\ n \in \mathbb{N}.$$

Therefore, for any $u \in E_{\lambda_k} \cap H^1_P, k \neq n$ we have

$$u - 0 \perp E_{\lambda_n} \cap H^1_P \ \text{ in } H^1_P$$

saying that $E_{\lambda_k} \cap H^1_P \perp E_{\lambda_n} \cap H^1_P$ in H^1_P, for any $k \neq n$. Let u be an element of H^1_P. As $L^2(\mathbb{R}^m)$ is the Hilbertian sum of $(E_{\lambda_n})_{n\in\mathbb{N}}$, we have $u = \sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}} u$ in $L^2(\mathbb{R}^m)$. For any $n \in \mathbb{N}$ we have $\mathrm{Proj}_{E_{\lambda_n}} u = \mathrm{Proj}_{E_{\lambda_n}\cap H^1_P} u$ and therefore the Bessel inequality

$$\sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}} u|^2_{H^1_P} = \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}\cap H^1_P} u|^2_{H^1_P} \le |u|^2_{H^1_P}$$

guarantees that $\sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}} u$ converges also in H^1_P. Its sum in $L^2(\mathbb{R}^m)$ being u, we deduce that $u = \sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}} u$ in H^1_P, saying that $H^1_P = \overline{\mathrm{span}(\cup_{n\in\mathbb{N}}(E_{\lambda_n} \cap H^1_P))}$. □

By Lemmas 5.1, 5.2 we deduce that $\left(\frac{1}{S}\int_r^{r+S} G(s)D\nabla u_{2s}\,\mathrm{d}s\right)_{S>0}$ converges strongly in X_Q, when $S \to +\infty$, uniformly with respect to $r \in \mathbb{R}$, toward some limit not depending on r, for any $u \in \mathrm{span}\left\{\cup_{n\in\mathbb{N}}(E_{\lambda_n} \cap H^1_P)\right\}$. Thanks to the inequality

$$\left|\frac{1}{S}\int_r^{r+S} G(s)D\nabla u_{2s}\,\mathrm{d}s\right|_{X_Q} \le |D|_{H^\infty_Q}|\nabla u|_{X_P},$$

we deduce that the above convergence holds true strongly in X_Q, uniformly with respect to r, toward some limit not depending on r, for any $u \in \oplus_{n\in\mathbb{N}}(E_{\lambda_n} \cap H^1_P) = H^1_P$. We are led to the following result.

Proposition 5.4
For any matrix field $D \in H^\infty_Q$ and any function $u \in H^1_P$, the quantity

$$\frac{1}{S}\int_r^{r+S} G(s)D\nabla u_{2s}\,\mathrm{d}s$$

converges strongly in X_Q, when $S \to +\infty$, uniformly with respect to $r \in \mathbb{R}$, toward some limit not depending on r.

5.3 Definition and properties of the bilinear form $\mathfrak{m}$

We intend to apply variational methods for solving (1), (2). We need to construct the bilinear form corresponding to the limit problem. We perform this construction for any field D of symmetric positive matrices, satisfying

$$Q^{1/2}(y)(D(y) + b(y)\otimes b(y))Q^{1/2}(y) \ge dI_m,\ \ y \in \mathbb{R}^m \tag{40}$$

for some constant $d > 0$. We suppose also that

$$D \in H^\infty_Q,\ \ b \in X^\infty_Q. \tag{41}$$

The above hypotheses have to be considered together with the previous assumptions in (16), (17), (18), (19), (20), (39). We introduce the following bilinear applications.

Proposition 5.5

1. *For any $\varepsilon > 0$, let us consider the application* $\mathfrak{a}^\varepsilon : H^1_P \times H^1_P \to \mathbb{R}$

$$\mathfrak{a}^\varepsilon(u,v) = \underbrace{\int_{\mathbb{R}^m} D(y)\nabla u \cdot \nabla v \;\mathrm{d}y}_{\mathfrak{a}(u,v)} + \frac{1}{\varepsilon}\int_{\mathbb{R}^m} (b\cdot\nabla u)(b\cdot\nabla v)\;\mathrm{d}y, \;\; u,v \in H^1_P.$$

The bilinear form $\mathfrak{a}^\varepsilon$ is well defined, continuous, symmetric, positive. For any $\varepsilon \in]0,1]$ it is coercive on H^1_P with respect to $L^2(\mathbb{R}^m)$.

2. *For any $r \in \mathbb{R}$, let us consider the application* $\mathfrak{m} : H^1_P \times H^1_P \to \mathbb{R}$

$$\mathfrak{m}(u,v) = \int_{\mathbb{R}^m}\left\{\langle D\rangle(y)\nabla u + \lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S}(G(s)D - \langle D\rangle)\nabla u_{2s}\;\mathrm{d}s\right\}\cdot\nabla v\;\mathrm{d}y$$

The bilinear form $\mathfrak{m}$ is well defined, not depending on $r \in \mathbb{R}$, continuous, symmetric, positive and also writes

$$\mathfrak{m}(u,v) = \int_{\mathbb{R}^m}\langle D\rangle(y)\nabla u\cdot\nabla v\;\mathrm{d}y + \lim_{S\to+\infty}\frac{1}{S}\int_{-S/2}^{S/2}\int_{\mathbb{R}^m}(D-\langle D\rangle)\nabla u_s\cdot\nabla v_{-s}\;\mathrm{d}y\mathrm{d}s, u,v\in H^1_P. \quad (42)$$

The following equality is satisfied

$$\mathfrak{m}(u,v) = \int_{\mathbb{R}^m} D\nabla u\cdot\nabla \mathrm{Proj}_{E_{\lambda_n}} v\;\mathrm{d}y,\;\; u\in E_{\lambda_n}\cap H^1_P,\; n\in\mathbb{N},\; v\in H^1_P. \quad (43)$$

3. *The bilinear form $\mathfrak{m}$ satifies the following commutation property with the operator $\mathcal{B}$*

$$\mathfrak{m}(u,\mathcal{B}v) = \mathfrak{m}(\mathcal{B}u,v) \;\; \text{for any } u,v\in H^1_P \text{ such that } \mathcal{T}u, \mathcal{T}v, \mathcal{B}u, \mathcal{B}v \in H^1_P.$$

Moreover, the bilinear form $(u,v) \to \mathfrak{m}(u,v) + \int_{\mathbb{R}^m}(b\cdot\nabla u)(b\cdot\nabla v)\;\mathrm{d}y$ is coercive on H^1_P with respect to $L^2(\mathbb{R}^m)$.

Proof.
1. For any $u,v\in H^1_P$ we have

$$|D\nabla u\cdot\nabla v| = |Q^{1/2}DQ^{1/2} : (P^{1/2}\nabla v)\otimes(P^{1/2}\nabla u)| \le |D|_{H^\infty_Q}|P^{1/2}\nabla v|\;|P^{1/2}\nabla u|.$$

and

$$\begin{aligned}|(b\cdot\nabla u)(b\cdot\nabla v)| &= |Q^{1/2}b\otimes bQ^{1/2} : (P^{1/2}\nabla v)\otimes(P^{1/2}\nabla u)|\\ &\le |b\otimes b|_{H^\infty_Q}|P^{1/2}\nabla v|\;|P^{1/2}\nabla u| = |b|^2_{X^\infty_Q}|P^{1/2}\nabla v|\;|P^{1/2}\nabla u|.\end{aligned}$$

We deduce that

$$|\mathfrak{a}^\varepsilon(u,v)| \le \left(|D|_{H^\infty_Q} + \frac{|b|^2_{X^\infty_Q}}{\varepsilon}\right)|u|_{H^1_P}|v|_{H^1_P}$$

saying that $\mathfrak{a}^\varepsilon$ is well defined, and continuous on H^1_P. It is also symmetric and positive, thanks to the symmetry and positivity of $D(y), y\in\mathbb{R}^m$. The coercivity comes by (40), observing that for any $u\in H^1_P, 0<\varepsilon\le 1$ we have

$$\begin{aligned}\mathfrak{a}^\varepsilon(u,u) + d|u|^2_{L^2(\mathbb{R}^m)} &= \int_{\mathbb{R}^m} Q^{1/2}\left(D + \frac{b\otimes b}{\varepsilon}\right)Q^{1/2} : (P^{1/2}\nabla u)\otimes(P^{1/2}\nabla u)\;\mathrm{d}y + d|u|^2_{L^2(\mathbb{R}^m)}\\ &\ge d|u|^2_{H^1_P}.\end{aligned}$$

2. We justify that $\mathfrak{m}$ is well defined and not depending on $r\in\mathbb{R}$. By Proposition 5.4, see also Lemmas 5.1, 5.2, we know that for any $u\in H^1_P$, the family

$$\left(\frac{1}{S}\int_r^{r+S}(G(s)D-\langle D\rangle)\nabla u_{2s}\;\mathrm{d}s\right)_{S>0} = \left(\frac{1}{S}\int_r^{r+S}(G(s)(D-\langle D\rangle))\nabla u_{2s}\;\mathrm{d}s\right)_{S>0}$$

converges strongly in X_Q, when $S \to +\infty$, uniformly with respect to $r \in \mathbb{R}$, toward some limit not depending on $r \in \mathbb{R}$. Therefore $\mathfrak{m}(u,v)$ is well defined for any $u, v \in H^1_P$. Obviously $\mathfrak{m}$ is bilinear. In order to establish the symmetry observe that

$$\begin{aligned}&G(s)(D - \langle D\rangle)\nabla u_{2s} \cdot \nabla v\\ &\qquad = G(s)(D - \langle D\rangle)\, {}^t\partial Y(s;y)(\nabla u_s)(Y(s;y)) \cdot {}^t\partial Y(s;y)(\nabla v_{-s})(Y(s;y))\\ &\qquad = ((D - \langle D\rangle)\nabla u_s \cdot \nabla v_{-s})_s\end{aligned}$$

implying that

$$\mathfrak{m}(u,v) = \int_{\mathbb{R}^m} \langle D\rangle\, \nabla u \cdot \nabla v \,\mathrm{d}y + \lim_{S\to+\infty} \frac{1}{S}\int_{-S/2}^{S/2}\int_{\mathbb{R}^m} (D - \langle D\rangle)\nabla u_s \cdot \nabla v_{-s} \,\mathrm{d}y\mathrm{d}s.$$

Obviously $\mathfrak{m}$ is continuous on $H^1_P \times H^1_P$

$$|\mathfrak{m}(u,v)| \le 3|D|_{H^\infty_Q} |\nabla u|_{X_P} |\nabla v|_{X_P} \le 3|D|_{H^\infty_Q} |u|_{H^1_P} |v|_{H^1_P}, \;\; u, v \in H^1_P.$$

The symmetry of $\mathfrak{m}$ comes by the symmetry of $D(y), y \in \mathbb{R}^m$, after performing the change of variable $s \to -s$. Another useful formula for $\mathfrak{m}$ comes by observing that for any $u \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}$, we have

$$u_{2s} + u = 2\cos(\sqrt{\lambda_n} s) u_s, \;\; s \in \mathbb{R}. \tag{44}$$

Indeed, the above formula is trivial when $n = 0$. When $n \in \mathbb{N}^\star$, notice that

$$u_{2s} + i\frac{\mathcal{T}}{\sqrt{\lambda_n}} u_{2s} = e^{-i\sqrt{\lambda_n}s}\left(u_s + i\frac{\mathcal{T}}{\sqrt{\lambda_n}}u_s\right), \;\; u + i\frac{\mathcal{T}}{\sqrt{\lambda_n}}u = e^{i\sqrt{\lambda_n}s}\left(u_s + i\frac{\mathcal{T}}{\sqrt{\lambda_n}}u_s\right)$$

implying that $u_{2s} + u = \Re\{2\cos(\sqrt{\lambda_n}s)(u_s + i\frac{\mathcal{T}}{\sqrt{\lambda_n}}u_s)\} = 2\cos(\sqrt{\lambda_n}s)u_s, s \in \mathbb{R}$. Thanks to (44), the average term of $\mathfrak{m}$ writes, for any $u \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}^\star, v \in H^1_P$

$$\begin{aligned}&\int_{\mathbb{R}^m} \frac{1}{S}\int_r^{r+S} (G(s)D - \langle D\rangle)\nabla u_{2s} \,\mathrm{d}s \cdot \nabla v \,\mathrm{d}y\\ &\qquad = \int_{\mathbb{R}^m} \frac{2}{S}\int_r^{r+S} \cos(\sqrt{\lambda_n}s) G(s)D\nabla u_s \,\mathrm{d}s \cdot \nabla v \,\mathrm{d}y\\ &\qquad - \int_{\mathbb{R}^m} \frac{1}{S}\int_r^{r+S} G(s)D\nabla u \,\mathrm{d}s \cdot \nabla v \,\mathrm{d}y - \int_{\mathbb{R}^m} \frac{1}{S}\int_r^{r+S} \nabla u_{2s} \,\mathrm{d}s \cdot \langle D\rangle\, \nabla v \,\mathrm{d}y.\end{aligned} \tag{45}$$

By Lemma 5.1 we have

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} G(s)D\nabla u = \langle D\rangle\, \nabla u$$

strongly in X_Q, uniformly with respect to $r \in \mathbb{R}$ and thus

$$\lim_{S\to+\infty} \int_{\mathbb{R}^m} \frac{1}{S}\int_r^{r+S} G(s)D\nabla u \,\mathrm{d}s \cdot \nabla v \,\mathrm{d}y = \int_{\mathbb{R}^m} \langle D\rangle\, \nabla u \cdot \nabla v \,\mathrm{d}y. \tag{46}$$

By Proposition 3.5, we deduce thanks to the orthogonality $E_{\lambda_n} \perp E_0, n \in \mathbb{N}^\star$

$$\lim_{S\to+\infty} \frac{1}{S}\int_r^{r+S} \nabla u_{2s} \,\mathrm{d}s = \nabla \mathrm{Proj}_{E_0} u = 0$$

strongly in X_P, uniformly with respect to $r \in \mathbb{R}$ and thus

$$\lim_{S\to+\infty} \int_{\mathbb{R}^m} \frac{1}{S}\int_r^{r+S} \nabla u_{2s} \,\mathrm{d}s \cdot \langle D\rangle\, \nabla v \,\mathrm{d}y = 0. \tag{47}$$

For the remaining term in the right hand side of equation (45), notice that

$$G(s)D\nabla u_s \cdot \nabla v = \partial Y(s;y) G(s) D\, {}^t\partial Y(s;y)(\nabla u)_s \cdot (\nabla v_{-s})_s = (D\nabla u \cdot \nabla v_{-s})_s$$

and therefore we obtain

$$\int_{\mathbb{R}^m} \frac{2}{S}\int_r^{r+S} \cos(\sqrt{\lambda_n} s) G(s) D\nabla u_s \;\mathrm{d}s \cdot \nabla v \;\mathrm{d}y = \int_{\mathbb{R}^m} D\nabla u \cdot \frac{2}{S}\int_r^{r+S} \cos(\sqrt{\lambda_n} s)\nabla v_{-s}\;\mathrm{d}s\;\mathrm{d}y.$$

By Remark 4.2 we know that

$$\lim_{S\to+\infty} \frac{2}{S}\int_r^{r+S} \cos(\sqrt{\lambda_n} s)\nabla v_{-s}\;\mathrm{d}s = \nabla \mathrm{Proj}_{E_{\lambda_n}} v$$

strongly in X_P, uniformly with respect to $r \in \mathbb{R}$, implying that

$$\lim_{S\to+\infty} \int_{\mathbb{R}^m} \frac{2}{S}\int_r^{r+S} \cos(\sqrt{\lambda_n} s) G(s) D\nabla u_s \;\mathrm{d}s \cdot \nabla v \;\mathrm{d}y = \int_{\mathbb{R}^m} D\nabla u \cdot \nabla \mathrm{Proj}_{E_{\lambda_n}} v \;\mathrm{d}y \tag{48}$$

uniformly with respect to $r \in \mathbb{R}$. Combining (45), (46), (47), (48) leads to the following expression for the average term of $\mathfrak{m}$

$$\lim_{S\to+\infty} \int_{\mathbb{R}^m} \frac{1}{S}\int_r^{r+S} (G(s)D - \langle D\rangle)\nabla u_{2s}\;\mathrm{d}s \cdot \nabla v\;\mathrm{d}y = \int_{\mathbb{R}^m} D\nabla u \cdot \nabla \mathrm{Proj}_{E_{\lambda_n}} v\;\mathrm{d}y$$
$$- \int_{\mathbb{R}^m} \langle D\rangle \nabla u \cdot \nabla v \;\mathrm{d}y, \;\; u \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}^\star, v \in H^1_P$$

and therefore

$$\mathfrak{m}(u,v) = \int_{\mathbb{R}^m} D\nabla u \cdot \nabla \mathrm{Proj}_{E_{\lambda_n}} v \;\mathrm{d}y, \;\; u \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}^\star, v \in H^1_P. \tag{49}$$

We claim that the above formula also holds true for $u \in E_0 \cap H^1_P, v \in H^1_P$. Indeed, taking into account that $u = u_{2s} = u_s(Y(s;\cdot))$, $v = v_{-s}(Y(s;\cdot))$ we obtain

$$G(s)D\nabla u_{2s}\cdot \nabla v = G(s) D\, {}^t\partial Y(s;\cdot)(\nabla u_s)_s \cdot {}^t\partial Y(s;\cdot)(\nabla v_{-s})_s$$
$$= (D\nabla u_s \cdot \nabla v_{-s})_s = (D\nabla u \cdot \nabla v_{-s})_s.$$

By Proposition 3.5 we deduce

$$\mathfrak{m}(u,v) = \lim_{S\to+\infty} \int_{\mathbb{R}^m} D\nabla u \cdot \frac{1}{S}\int_r^{r+S} \nabla v_{-s}\;\mathrm{d}s\;\mathrm{d}y = \int_{\mathbb{R}^m} D\nabla u \cdot \nabla \mathrm{Proj}_{E_0} v\;\mathrm{d}y.$$

By (49) and the hypothesis $D \geq 0$ we have

$$\mathfrak{m}(u,v) = 0, \; u \in E_{\lambda_n} \cap H^1_P, v \in E_{\lambda_k} \cap H^1_P, n \neq k$$

and

$$\mathfrak{m}(u,u) = \int_{\mathbb{R}^m} D(y)\nabla u \cdot \nabla u \;\mathrm{d}y \geq 0, \; u \in E_{\lambda_n} \cap H^1_P, \; n \in \mathbb{N}.$$

As $\mathfrak{m}$ is bounded on $H^1_P \times H^1_P$, it is easily seen that for any $u \in H^1_P$ we have

$$\mathfrak{m}(u,u) = \mathfrak{m}\left(\sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}} u, \sum_{k\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_k}} u\right) = \sum_{n\in\mathbb{N}} \mathfrak{m}(\mathrm{Proj}_{E_{\lambda_n}} u, \mathrm{Proj}_{E_{\lambda_n}} u) \geq 0$$

saying that the quadratic form $u \to \mathfrak{m}(u,u)$ is positive on H^1_P. Notice also that for any $u, v \in H^1_P$ we have

$$\mathfrak{m}(u,v) = \sum_{n\in\mathbb{N}} \mathfrak{m}(\mathrm{Proj}_{E_{\lambda_n}} u, \mathrm{Proj}_{E_{\lambda_n}} v) = \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^m} D\nabla \mathrm{Proj}_{E_{\lambda_n}} u \cdot \nabla \mathrm{Proj}_{E_{\lambda_n}} v \;\mathrm{d}y$$
$$\leq \sum_{n\in\mathbb{N}} |D|_{H^\infty_Q} |\nabla \mathrm{Proj}_{E_{\lambda_n}} u|_{X_P} |\nabla \mathrm{Proj}_{E_{\lambda_n}} v|_{X_P}$$
$$\leq |D|_{H^\infty_Q} |\nabla u|_{X_P} |\nabla v|_{X_P} \leq |D|_{H^\infty_Q} |u|_{H^1_P} |v|_{H^1_P}.$$

3. We focus on the equality $\mathfrak{m}(u, \mathcal{B}v) = \mathfrak{m}(\mathcal{B}u, v)$, with $u, v \in H^1_P$, such that $\mathcal{T}u, \mathcal{T}v, \mathcal{B}u, \mathcal{B}v \in H^1_P$. Observe that

$$\begin{aligned}\int_{\mathbb{R}^m} \langle D\rangle \nabla u \cdot \nabla \mathcal{T} v \;\mathrm{d}y &= \lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}^m} \langle D\rangle \nabla u \cdot \nabla (v_h - v)\;\mathrm{d}y \\ &= \lim_{h\to 0} \frac{1}{h} \left\{ \int_{\mathbb{R}^m} G(h) \langle D\rangle \nabla u \cdot \nabla v_h \;\mathrm{d}y - \int_{\mathbb{R}^m} \langle D\rangle \nabla u \cdot \nabla v \;\mathrm{d}y \right\} \\ &= \lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}^m} (\langle D\rangle \nabla u_{-h} - \langle D\rangle \nabla u) \cdot \nabla v \;\mathrm{d}y \\ &= - \int_{\mathbb{R}^m} \langle D\rangle \nabla \mathcal{T} u \cdot \nabla v \;\mathrm{d}y\end{aligned}$$

and thus

$$\int_{\mathbb{R}^m} \langle D\rangle \nabla u \cdot \nabla \mathcal{B} v \;\mathrm{d}y = \int_{\mathbb{R}^m} \langle D\rangle \nabla \mathcal{T} u \cdot \nabla \mathcal{T} v \;\mathrm{d}y = \int_{\mathbb{R}^m} \langle D\rangle \nabla \mathcal{B} u \cdot \nabla v \;\mathrm{d}y. \tag{50}$$

For the second term in the right hand side of (42) we notice that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s} \left\{ \int_{\mathbb{R}^m} (D - \langle D\rangle) \nabla u_s \cdot \nabla \mathcal{T} v_{-s} \;\mathrm{d}y + \int_{\mathbb{R}^m} (D - \langle D\rangle) \nabla \mathcal{T} u_s \cdot \nabla v_{-s} \;\mathrm{d}y \right\} \\ &\qquad = \int_{\mathbb{R}^m} (D - \langle D\rangle) \nabla u_s \cdot \nabla \mathcal{B} v_{-s} \;\mathrm{d}y - \int_{\mathbb{R}^m} (D - \langle D\rangle) \nabla \mathcal{B} u_s \cdot \nabla v_{-s} \;\mathrm{d}y\end{aligned}$$

and therefore

$$\begin{aligned}&\lim_{S\to+\infty} \frac{1}{S} \int_{-S/2}^{S/2} \left\{ \int_{\mathbb{R}^m} (D - \langle D\rangle) \nabla u_s \cdot \nabla \mathcal{B} v_{-s} \;\mathrm{d}y - \int_{\mathbb{R}^m} (D - \langle D\rangle) \nabla \mathcal{B} u_s \cdot \nabla v_{-s} \;\mathrm{d}y \right\} \mathrm{d}s \\ &\quad = \lim_{S\to+\infty} \frac{1}{S} \left[\int_{\mathbb{R}^m} (D - \langle D\rangle) \nabla u_s \cdot \nabla \mathcal{T} v_{-s} \;\mathrm{d}y + \int_{\mathbb{R}^m} (D - \langle D\rangle) \nabla \mathcal{T} u_s \cdot \nabla v_{-s} \;\mathrm{d}y \right]_{-S/2}^{S/2} \\ &\quad = 0.\end{aligned} \tag{51}$$

Combining (50), (51), we deduce that $\mathfrak{m}(u, \mathcal{B}v) = \mathfrak{m}(\mathcal{B}u, v)$.
It remains to justify the coercivity. For any $u \in E_{\lambda_n} \cap H^1_P, v \in E_{\lambda_k} \cap H^1_P$, we have

$$(u, v)_{H^1_P} = \delta_{nk} (u, v)_{H^1_P}, \;\; \mathfrak{m}(u, v) = \delta_{nk} \int_{\mathbb{R}^m} D(y) \nabla u \cdot \nabla v \;\mathrm{d}y$$

$$\begin{aligned}\int_{\mathbb{R}^m} (b \cdot \nabla u)(b \cdot \nabla v) \;\mathrm{d}y &= - \int_{\mathbb{R}^m} \mathcal{T}^2 u \; v \;\mathrm{d}y = \lambda_n \int_{\mathbb{R}^m} u(y) v(y) \;\mathrm{d}y \\ &= \lambda_n \delta_{nk} \int_{\mathbb{R}^m} u(y) v(y) \;\mathrm{d}y = \delta_{nk} \int_{\mathbb{R}^m} (b \cdot \nabla u)(b \cdot \nabla v) \;\mathrm{d}y.\end{aligned}$$

It is easily seen, thanks to (40), that for any $u \in \operatorname{span} \cup_{n\in\mathbb{N}} (E_{\lambda_n} \cap H^1_P)$ we have

$$\mathfrak{m}(u, u) + \int_{\mathbb{R}^m} (b \cdot \nabla u)^2 \;\mathrm{d}y + d \int_{\mathbb{R}^m} u^2(y) \;\mathrm{d}y \geq d |u|^2_{H^1_P}.$$

Since the bilinear forms $(u, v) \to \mathfrak{m}(u, v), (u, v) \to \int_{\mathbb{R}^m} (b \cdot \nabla u)(b \cdot \nabla v) \;\mathrm{d}y$ are bounded on $H^1_P \times H^1_P$ (use the hypotheses $b \in X^\infty_Q$ for the second form), the above inequality still holds true for any $u \in \oplus_{n\in\mathbb{N}} (E_{\lambda_n} \cap H^1_P) = H^1_P$, saying that $(u, v) \to \mathfrak{m}(u, v) + \int_{\mathbb{R}^m} (b \cdot \nabla u)(b \cdot \nabla v) \;\mathrm{d}y$ is coercive on H^1_P with respect to $L^2(\mathbb{R}^m)$. □

Remark 5.3
When $u \in E_{\lambda_n} \cap H^1_P$ such that $\operatorname{div}_y (D(y) \nabla u) \in L^2(\mathbb{R}^m)$, we deduce by (43)

$$\mathfrak{m}(u, v) = - \int_{\mathbb{R}^m} \operatorname{div}_y (D \nabla u) \operatorname{Proj}_{E_{\lambda_n}} v \;\mathrm{d}y = - \int_{\mathbb{R}^m} \operatorname{Proj}_{E_{\lambda_n}} \operatorname{div}_y (D \nabla u) \, v(y) \;\mathrm{d}y, \;\; v \in H^1_P$$

saying that the restriction on $E_{\lambda_n} \cap H^1_P$ of the linear operator associated to the bilinear form $\mathfrak{m}$ is $\operatorname{Proj}_{E_{\lambda_n}} (-\operatorname{div}_y (D(y) \nabla u))$, for any $u \in E_{\lambda_n} \cap H^1_P$ such that $\operatorname{div}_y (D(y) \nabla u) \in L^2(\mathbb{R}^m)$, see also (8).

6 Uniform estimates

In this section, we justify the well posedness of the problem (1), (2) and of the effective problem associated to the bilinear form $\mathfrak{m}$. We indicate uniform estimates for the solutions of these problems. We consider the continuous embedding $H^1_P \hookrightarrow L^2(\mathbb{R}^m)$, with dense image (since $C^1_c(\mathbb{R}^m) \subset H^1_P$). In the following propositions, we are looking for variationnal solutions of the above problems.

Proposition 6.1
Let u^{in} be a function in $L^2(\mathbb{R}^m)$. For any $\varepsilon \in]0,1]$ there is a unique variational solution of (1), (2). Moreover we have

$$|u^\varepsilon|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le |u^{\text{in}}|_{L^2(\mathbb{R}^m)}, \;\; |\nabla u^\varepsilon|_{L^2(\mathbb{R}_+;X_P)} \le \frac{|u^{\text{in}}|_{L^2(\mathbb{R}^m)}}{\sqrt{2d}}, \;\; 0 < \varepsilon \le 1.$$

Proof. This is a direct consequence of Theorems 1, 2 [7] p.513, see also [11]. By Proposition 5.5 we know that, for any $\varepsilon \in]0,1]$, the bilinear form $\mathfrak{a}^\varepsilon$ is coercive on H^1_P with respect to $L^2(\mathbb{R}^m)$. We deduce that, for any $u^{\text{in}} \in L^2(\mathbb{R}^m)$, there is a unique variational solution u^ε for (1), (2), that is $u^\varepsilon \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1_P)$ and

$$u^\varepsilon(0) = u^{\text{in}}, \;\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} u^\varepsilon(t,y)\varphi(y)\;\mathrm{d}y + \mathfrak{a}^\varepsilon(u^\varepsilon(t),\varphi) = 0 \;\; \text{in } \mathcal{D}'(\mathbb{R}_+) \;\; \text{for any } \varphi \in H^1_P.$$

By the energy balance we obtain for any $t \in \mathbb{R}_+, \varepsilon \in]0,1]$

$$\frac{1}{2}|u^\varepsilon(t)|^2_{L^2(\mathbb{R}^m)} + d\int_0^t |\nabla u^\varepsilon(s)|^2_{X_P}\;\mathrm{d}s \le \frac{1}{2}|u^\varepsilon(t)|^2_{L^2(\mathbb{R}^m)} + \int_0^t \mathfrak{a}^\varepsilon(u^\varepsilon(s),u^\varepsilon(s))\;\mathrm{d}s = \frac{1}{2}|u^{\text{in}}|^2_{L^2(\mathbb{R}^m)}$$

implying that

$$|u^\varepsilon(t)|_{L^2(\mathbb{R}^m)} \le |u^{\text{in}}|_{L^2(\mathbb{R}^m)}, \;\; \int_0^{+\infty} |\nabla u^\varepsilon(s)|^2_{X_P}\;\mathrm{d}s \le \frac{|u^{\text{in}}|^2_{L^2(\mathbb{R}^m)}}{2d}.$$

□

We intend to proceed similarly for solving the variational problem associated to the bilinear form $\mathfrak{m}$. As shown in Proposition 5.5, we only know that $(u,v) \to \mathfrak{m}(u,v) + \int_{\mathbb{R}^m}(b\cdot\nabla u)(b\cdot\nabla v)\;\mathrm{d}y$ is coercive on H^1_P with respect to $L^2(\mathbb{R}^m)$. Nevertheless $\mathfrak{m}$ is coercive on $E_{\lambda_n} \cap H^1_P$ with respect to E_{λ_n}, for any $n \in \mathbb{N}$. Indeed, for any $n \in \mathbb{N}, u \in E_{\lambda_n} \cap H^1_P$, we have, thanks to (40)

$$\begin{aligned}\mathfrak{m}(u,u) + (\lambda_n + d)|u|^2_{L^2(\mathbb{R}^m)} &= \int_{\mathbb{R}^m} D\nabla u\cdot\nabla u\;\mathrm{d}y + \int_{\mathbb{R}^m}(b\cdot\nabla u)^2\;\mathrm{d}y + d\int_{\mathbb{R}^m} u^2(y)\;\mathrm{d}y \\ &= \int_{\mathbb{R}^m} Q^{1/2}(D + b\otimes b)Q^{1/2} : (P^{1/2}\nabla u)\otimes(P^{1/2}\nabla u)\;\mathrm{d}y + d\int_{\mathbb{R}^m} u^2(y)\;\mathrm{d}y \ge d|u|^2_{H^1_P}.\end{aligned}$$

Proposition 6.2
For any $n \in \mathbb{N}$, let u^{in}_n be an element of E_{λ_n}. There is a unique function $v_n \in C_b(\mathbb{R}_+; E_{\lambda_n}) \cap L^2_{\text{loc}}(\mathbb{R}_+; E_{\lambda_n} \cap H^1_P)$ such that

$$v_n(0) = u^{\text{in}}_n, \;\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} v_n(t,y)\varphi(y)\;\mathrm{d}y + \mathfrak{m}(v_n(t),\varphi) = 0 \;\; \textit{in } \mathcal{D}'(\mathbb{R}_+), \;\; \textit{for any } \varphi \in E_{\lambda_n} \cap H^1_P.$$

Moreover we have

$$|v_n|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le |u^{\text{in}}_n|_{L^2(\mathbb{R}^m)}, \;\; |\mathcal{T} v_n|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le |\mathcal{T} u^{\text{in}}_n|_{L^2(\mathbb{R}^m)}$$

$$|\nabla v_n|_{L^2([0,t];X_P)} \le \frac{|u^{\text{in}}_n|_{L^2(\mathbb{R}^m)}}{\sqrt{2d}} + \sqrt{\frac{t}{d}}\;|\mathcal{T} u^{\text{in}}_n|_{L^2(\mathbb{R}^m)}, \; t \in \mathbb{R}_+, \; n \in \mathbb{N}.$$

If $u^{\text{in}}_n \in E_{\lambda_n} \cap H^1_P$ and there is a function $f_n \in E_{\lambda_n}$ (the function f_n will be denoted by $\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla u^{\text{in}}_n))$) such that

$$\int_{\mathbb{R}^m} D(y)\nabla u^{\text{in}}_n\cdot\nabla\varphi\;\mathrm{d}y = \int_{\mathbb{R}^m} f_n(y)\varphi(y)\;\mathrm{d}y, \;\; \textit{for any } \varphi \in E_{\lambda_n} \cap H^1_P$$

then

$$|\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla v_n(t)))|_{L^2(\mathbb{R}^m)} \le |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla u^{\text{in}}_n))|_{L^2(\mathbb{R}^m)}, \;\; t \in \mathbb{R}_+$$

and

$$\partial_t v_n + \mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla v_n(t))) = 0, \;\; t \in \mathbb{R}_+.$$

Proof. We use the inclusion $E_{\lambda_n} \cap H^1_P \hookrightarrow E_{\lambda_n}, n \in \mathbb{N}$. The existence and uniqueness of the variational solutions $(v_n)_n$ come by Theorems 1,2 [7]. The energy balance gives

$$\frac{1}{2}|v_n(t)|^2_{L^2(\mathbb{R}^m)} + \int_0^t \int_{\mathbb{R}^m} D(y)\nabla v_n \cdot \nabla v_n \, \mathrm{d}y\mathrm{d}s = \frac{1}{2}|u^{\mathrm{in}}_n|^2_{L^2(\mathbb{R}^m)}$$

implying

$$|v_n(t)|_{L^2(\mathbb{R}^m)} \leq |u^{\mathrm{in}}_n|_{L^2(\mathbb{R}^m)}, \;\; |\mathcal{T} v_n(t)|_{L^2(\mathbb{R}^m)} \leq |\mathcal{T} u^{\mathrm{in}}_n|_{L^2(\mathbb{R}^m)}, \;\; t \in \mathbb{R}_+, n \in \mathbb{N}.$$

For the last estimate we have used the equality $\int_{\mathbb{R}^m} (\mathcal{T} v)^2 \, \mathrm{d}y = \lambda_n \int_{\mathbb{R}^m} v^2(y) \, \mathrm{d}y, v \in E_{\lambda_n}$. Observe also that for any $t \in \mathbb{R}_+, n \in \mathbb{N}$ we have

$$\begin{aligned} d\int_0^t |\nabla v_n(s)|^2_{X_P} \, \mathrm{d}s &\leq \int_0^t \int_{\mathbb{R}^m} \{D\nabla v_n(s) \cdot \nabla v_n(s) + (b \cdot \nabla v_n(s))^2\} \, \mathrm{d}y\mathrm{d}s \\ &\leq \frac{1}{2}|u^{\mathrm{in}}_n|^2_{L^2(\mathbb{R}^m)} + t|\mathcal{T} u^{\mathrm{in}}_n|^2_{L^2(\mathbb{R}^m)}. \end{aligned}$$

Assume now that $u^{\mathrm{in}}_n \in E_{\lambda_n} \cap H^1_P$ such that $\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla u^{\mathrm{in}}_n))$ exists. For any $h \in \mathbb{R}^\star_+$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} (v_n(t+h,y) - v_n(t,y))\varphi(y) \, \mathrm{d}y + \mathfrak{m}(v_n(t+h) - v_n(t), \varphi) = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}_+), \;\; \varphi \in E_{\lambda_n} \cap H^1_P$$

implying that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} (v_n(t+h,y) - v_n(t,y))^2 \, \mathrm{d}y = -\mathfrak{m}(v_n(t+h) - v_n(t), v_n(t+h) - v_n(t)) \leq 0.$$

We deduce that

$$|v_n(t+h) - v_n(t)|_{L^2(\mathbb{R}^m)} \leq |v_n(h) - v_n(0)|_{L^2(\mathbb{R}^m)}, \;\; t, h \in \mathbb{R}_+. \tag{52}$$

Notice also that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}h}|v_n(h) - v_n(0)|^2_{L^2(\mathbb{R}^m)} &+ \int_{\mathbb{R}^m} D(y)\nabla(v_n(h) - v_n(0)) \cdot \nabla(v_n(h) - v_n(0)) \, \mathrm{d}y \\ &= -\int_{\mathbb{R}^m} D(y)\nabla v_n(0) \cdot \nabla(v_n(h) - v_n(0)) \, \mathrm{d}y \end{aligned}$$

and therefore

$$\begin{aligned} \frac{1}{2}|v_n(h) - v_n(0)|^2_{L^2(\mathbb{R}^m)} &\leq -\int_0^h \int_{\mathbb{R}^m} \mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla u^{\mathrm{in}}_n))(v_n(s,y) - v_n(0,y)) \, \mathrm{d}y\mathrm{d}s \\ &\leq |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla u^{\mathrm{in}}_n))|_{L^2(\mathbb{R}^m)} \int_0^h |v_n(s) - v_n(0)|_{L^2(\mathbb{R}^m)} \, \mathrm{d}s. \end{aligned}$$

Thanks to Bellman's lemma, one gets

$$|v_n(h) - v_n(0)|_{L^2(\mathbb{R}^m)} \leq h|\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla u^{\mathrm{in}}_n))|_{L^2(\mathbb{R}^m)}. \tag{53}$$

Combining (52), (53) we deduce

$$|v_n(t+h) - v_n(t)|_{L^2(\mathbb{R}^m)} \leq |v_n(h) - v_n(0)|_{L^2(\mathbb{R}^m)} \leq h|\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla u^{\mathrm{in}}_n))|_{L^2(\mathbb{R}^m)}$$

saying that

$$|\partial_t v_n|_{L^2(\mathbb{R}^m)} \leq |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla u^{\mathrm{in}}_n))|_{L^2(\mathbb{R}^m)}, \;\; t \in \mathbb{R}_+.$$

By the variational formulation we know that

$$\int_{\mathbb{R}^m} \partial_t v_n \varphi \, \mathrm{d}y + \int_{\mathbb{R}^m} D(y)\nabla v_n(t) \cdot \nabla\varphi \, \mathrm{d}y = 0, \;\; \varphi \in E_{\lambda_n} \cap H^1_P$$

implying that

$$\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla v_n(t))) = -\partial_t v_n \in E_{\lambda_n}, \;\; t \in \mathbb{R}_+$$

and thus

$$|\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla v_n(t)))|_{L^2(\mathbb{R}^m)} = |\partial_t v_n(t)|_{L^2(\mathbb{R}^m)} \leq |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla u^{\mathrm{in}}_n))|_{L^2}.$$

□

Corollary 6.1
Assume that $(\lambda_n)_n$ *is increasing and* $\lim_{n\to+\infty}\lambda_n = +\infty$. *Under the hypotheses of Proposition 6.2, for any* $u^{\rm in} \in \mathrm{dom}\mathcal{T}$, *there is a unique* $v \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2_{\rm loc}(\mathbb{R}_+; H^1_P)$, $\mathcal{T}v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ *such that*

$$v(0) = u^{\rm in}, \;\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} v(t,y)\varphi(y)\;\mathrm{d}y + \mathfrak{m}(v(t),\varphi) = 0 \;\; \textit{in } \mathcal{D}'(\mathbb{R}_+), \; \textit{for any } \varphi \in H^1_P. \tag{54}$$

Moreover we have

$$|v|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le |u^{\rm in}|_{L^2(\mathbb{R}^m)}, \;\; |\mathcal{T}v|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le |\mathcal{T}u^{\rm in}|_{L^2(\mathbb{R}^m)}$$

$$|\nabla v|_{L^2([0,t];X_P)} \le \frac{|u^{\rm in}|_{L^2(\mathbb{R}^m)}}{\sqrt{2d}} + \sqrt{\frac{t}{d}}\;|\mathcal{T}u^{\rm in}|_{L^2(\mathbb{R}^m)}.$$

If $u^{\rm in} \in H^1_P$ *and* $\sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla \mathrm{Proj}_{E_{\lambda_n}} u^{\rm in}))|^2_{L^2(\mathbb{R}^m)} < +\infty$, *then*

$$\sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla \mathrm{Proj}_{E_{\lambda_n}} v(t)))|^2_{L^2(\mathbb{R}^m)} < +\infty$$

and

$$\partial_t v + \sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla \mathrm{Proj}_{E_{\lambda_n}} v(t))) = 0, \;\; t \in \mathbb{R}_+.$$

Proof. For any $n \in \mathbb{N}$, we denote by v_n the solution given by Proposition 6.2, corresponding to $u^{\rm in}_n = \mathrm{Proj}_{E_{\lambda_n}} u^{\rm in}$. By Remark 4.1 we know that $\mathcal{T}\mathrm{Proj}_{E_{\lambda_n}} u^{\rm in} = \mathrm{Proj}_{E_{\lambda_n}} \mathcal{T}u^{\rm in}, n \in \mathbb{N}$, implying that

$$\sum_{n\in\mathbb{N}} |v_n(t)|^2_{L^2(\mathbb{R}^m)} \le \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}} u^{\rm in}|^2_{L^2(\mathbb{R}^m)} = |u^{\rm in}|^2_{L^2(\mathbb{R}^m)}, \;\; t \in \mathbb{R}_+$$

$$\sum_{n\in\mathbb{N}} |\mathcal{T}v_n(t)|^2_{L^2(\mathbb{R}^m)} \le \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}} \mathcal{T}u^{\rm in}|^2_{L^2(\mathbb{R}^m)} = |\mathcal{T}u^{\rm in}|^2_{L^2(\mathbb{R}^m)}, \; t \in \mathbb{R}_+.$$

Therefore $\sum_{n\in\mathbb{N}} v_n(t), \sum_{n\in\mathbb{N}} \mathcal{T}v_n(t)$ converge in $L^2(\mathbb{R}^m)$, for any $t \in \mathbb{R}_+$. Let us introduce $v(t) = \sum_{n\in\mathbb{N}} v_n(t)$, $w(t) = \sum_{n\in\mathbb{N}} \mathcal{T}v_n(t), t \in \mathbb{R}_+$. For any $\varphi \in \mathrm{dom}\mathcal{T}$ and any $N \in \mathbb{N}$ we can write

$$\int_{\mathbb{R}^m} \sum_{n=0}^N v_n(t,y)\mathcal{T}\varphi\;\mathrm{d}y + \int_{\mathbb{R}^m} \sum_{n=0}^N \mathcal{T}v_n(t)\varphi(y)\;\mathrm{d}y = 0$$

which implies, by letting $N \to +\infty$

$$\int_{\mathbb{R}^m} v(t,y)\mathcal{T}\varphi\;\mathrm{d}y + \int_{\mathbb{R}^m} w(t,y)\varphi(y)\;\mathrm{d}y = 0.$$

Therefore $v(t) \in \mathrm{dom}\mathcal{T}$ and $w(t) = \mathcal{T}v(t)$. In particular

$$|\mathcal{T}v(t)|^2_{L^2(\mathbb{R}^m)} = \sum_{n\in\mathbb{N}} |\mathcal{T}v_n|^2_{L^2(\mathbb{R}^m)} \le \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}} \mathcal{T}u^{\rm in}|^2_{L^2(\mathbb{R}^m)} = |\mathcal{T}u^{\rm in}|^2_{L^2(\mathbb{R}^m)}.$$

Actually the convergence $v = \sum_{n\in\mathbb{N}} v_n$ is uniform with respect to $t \in \mathbb{R}_+$, thanks to

$$|v(t) - \sum_{n=0}^N v_n(t)|^2_{L^2(\mathbb{R}^m)} \le \frac{1}{\lambda_N}\sum_{n>N} \lambda_n |v_n(t)|^2_{L^2(\mathbb{R}^m)} = \frac{1}{\lambda_N}\sum_{n>N} |\mathcal{T}v_n(t)|^2_{L^2(\mathbb{R}^m)} \le \frac{1}{\lambda_N}|\mathcal{T}u^{\rm in}|^2_{L^2(\mathbb{R}^m)}$$

and therefore $v \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m))$. By Proposition 5.3, we know that the spaces $(E_{\lambda_n} \cap H^1_P)_{n\in\mathbb{N}}$ are orthogonal in H^1_P and thus

$$\int_0^t (v_n(s), v_k(s))_{H^1_P}\;\mathrm{d}s = 0, \;\; t \in \mathbb{R}_+, \; n \neq k.$$

Moreover we have for any $t \in \mathbb{R}_+$

$$
\begin{aligned}
\sum_{n\in\mathbb{N}} |v_n|^2_{L^2([0,t];H^1_P)} &= \sum_{n\in\mathbb{N}} \{|v_n|^2_{L^2([0,t];L^2(\mathbb{R}^m))} + |\nabla v_n|^2_{L^2([0,t];X_P)}\} \\
&\le t \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}} u^{\mathrm{in}}|^2_{L^2(\mathbb{R}^m)} + \frac{1}{2d} \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}} u^{\mathrm{in}}|^2_{L^2(\mathbb{R}^m)} + \frac{t}{d} \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}} \mathcal{T} u^{\mathrm{in}}|^2_{L^2(\mathbb{R}^m)} \\
&= \left(t + \frac{1}{2d} \right) |u^{\mathrm{in}}|^2_{L^2(\mathbb{R}^m)} + \frac{t}{d} |\mathcal{T} u^{\mathrm{in}}|^2_{L^2(\mathbb{R}^m)} < +\infty
\end{aligned}
$$

implying that there is $z \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1_P)$ such that $\sum_{n\in\mathbb{N}} v_n = z$ in $L^2([0,t]; H^1_P)$ for any $t \in \mathbb{R}_+$. In particular $\sum_{n\in\mathbb{N}} v_n = z$ in $L^2([0,t]; L^2(\mathbb{R}^m))$ for any $t \in \mathbb{R}_+$ and therefore $\sum_{n\in\mathbb{N}} v_n(t) = z(t)$ in $L^2(\mathbb{R}^m)$ for a.a. $t \in \mathbb{R}_+$. We deduce that $v = z \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1_P)$, $v = \sum_{n\in\mathbb{N}} v_n$ in $L^2([0,t]; H^1_P)$ for any $t \in \mathbb{R}_+$ and

$$
|\nabla v|_{L^2([0,t];X_P)} \le \frac{|u^{\mathrm{in}}|_{L^2(\mathbb{R}^m)}}{\sqrt{2d}} + \sqrt{\frac{t}{d}}\, |\mathcal{T} u^{\mathrm{in}}|_{L^2(\mathbb{R}^m)}, \; t \in \mathbb{R}_+.
$$

For any $\eta \in C^1_c(\mathbb{R}_+), \varphi \in H^1_P$ and $N \in \mathbb{N}$ we have

$$
\begin{aligned}
-\eta(0) \int_{\mathbb{R}^m} \sum_{n=0}^{N} \mathrm{Proj}_{E_{\lambda_n}} u^{\mathrm{in}} \varphi(y) \,\mathrm{d}y - \int_0^{+\infty} \eta'(t) \int_{\mathbb{R}^m} \sum_{n=0}^{N} v_n(t,y)\varphi(y) \,\mathrm{d}y\mathrm{d}t \\
+ \int_0^{+\infty} \eta(t) \mathfrak{m}(\sum_{n=0}^{N} v_n(t), \varphi) \,\mathrm{d}t = 0.
\end{aligned}
$$

Letting $N \to +\infty$, it is easily seen, thanks to the boundedness of the bilinear form $\mathfrak{m}$ on H^1_P, that

$$
-\eta(0) \int_{\mathbb{R}^m} u^{\mathrm{in}}(y)\varphi(y) \,\mathrm{d}y - \int_0^{+\infty} \eta'(t) \int_{\mathbb{R}^m} v(t,y)\varphi(y) \,\mathrm{d}y\mathrm{d}t + \int_0^{+\infty} \eta(t) \mathfrak{m}(v(t), \varphi) \,\mathrm{d}t = 0
$$

saying that

$$
v(0) = u^{\mathrm{in}}, \;\; \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} v(t,y)\varphi(y) \,\mathrm{d}y + \mathfrak{m}(v(t), \varphi) = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}_+) \;\text{ for any } \varphi \in H^1_P.
$$

The uniqueness follows by the energy balance and the positivity of the quadratic form $u \in H^1_P \to \mathfrak{m}(u,u)$.
Assume now that $u^{\mathrm{in}} \in H^1_P, \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla u^{\mathrm{in}}_n))|^2_{L^2(\mathbb{R}^m)} < +\infty$ with $u^{\mathrm{in}}_n = \mathrm{Proj}_{E_{\lambda_n}} u^{\mathrm{in}} \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}$. By Proposition 6.2 we know that

$$
\partial_t v_n + \mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla v_n(t))) = 0, \;\; t \in \mathbb{R}_+
$$

$$
|\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla v_n(t)))|_{L^2(\mathbb{R}^m)} \le |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla u^{\mathrm{in}}_n))|_{L^2(\mathbb{R}^m)}, \; t \in \mathbb{R}_+.
$$

Therefore we obtain for any $t \in \mathbb{R}_+$

$$
\sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla v_n(t)))|^2_{L^2(\mathbb{R}^m)} \le \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla u^{\mathrm{in}}_n))|^2_{L^2(\mathbb{R}^m)} < +\infty.
$$

We claim that $t \to v(t)$ is differentiable in $L^2(\mathbb{R}^m)$ and that $\partial_t v = \sum_{n\in\mathbb{N}} \partial_t v_n$. Indeed we have

$$
\begin{aligned}
|v(t+h) - v(t)|^2_{L^2(\mathbb{R}^m)} &= \sum_{n\in\mathbb{N}} |v_n(t+h) - v_n(t)|^2_{L^2(\mathbb{R}^m)} \\
&\le \sum_{n\in\mathbb{N}} |v_n(h) - v_n(0)|^2_{L^2(\mathbb{R}^m)} \le h^2 \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}}(-D(y)\nabla u^{\mathrm{in}}_n)|^2_{L^2(\mathbb{R}^m)}
\end{aligned}
$$

saying that $\partial_t v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$. For any $\eta \in C_c^1(\mathbb{R}_+), \varphi \in L^2(\mathbb{R}^m)$ we have

$$\begin{aligned}\int_0^{+\infty} \eta(t) \int_{\mathbb{R}^m} \partial_t v \varphi(y) \;\mathrm{d}y\mathrm{d}t &= -\eta(0) \int_{\mathbb{R}^m} u^{\mathrm{in}}(y)\varphi(y) \;\mathrm{d}y - \int_0^{+\infty} \eta'(t) \int_{\mathbb{R}^m} v(t,y)\varphi(y) \;\mathrm{d}y\mathrm{d}t \\ &= -\sum_{n\in\mathbb{N}} \eta(0) \int_{\mathbb{R}^m} u_n^{\mathrm{in}}(y)\varphi(y) \;\mathrm{d}y - \sum_{n\in\mathbb{N}} \int_0^{+\infty} \eta'(t) \int_{\mathbb{R}^m} v_n(t,y)\varphi(y) \;\mathrm{d}y\mathrm{d}t \\ &= \sum_{n\in\mathbb{N}} \int_0^{+\infty} \eta(t) \int_{\mathbb{R}^m} \partial_t v_n \varphi(y) \;\mathrm{d}y\mathrm{d}t.\end{aligned}$$

But for any $t \in \mathbb{R}_+$ we have

$$\sum_{n\in\mathbb{N}} |\partial_t v_n(t)|_{L^2}^2 = \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla v_n(t)))|_{L^2}^2 \le \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D\nabla u_n^{\mathrm{in}}))|_{L^2}^2$$

saying that $\sum_{n\in\mathbb{N}} \partial_t v_n(t)$ converges in $L^2(\mathbb{R}^m)$, for any $t \in \mathbb{R}_+$. We deduce, thanks to the dominated convergence theorem

$$\int_0^{+\infty} \eta(t) \int_{\mathbb{R}^m} \partial_t v \varphi(y) \;\mathrm{d}y\mathrm{d}t = \int_0^{+\infty} \eta(t) \int_{\mathbb{R}^m} \sum_{n\in\mathbb{N}} \partial_t v_n \varphi(y) \;\mathrm{d}y\mathrm{d}t$$

implying that

$$\begin{aligned}\partial_t v = \sum_{n\in\mathbb{N}} \partial_t v_n &= -\sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}}(-\mathrm{div}_y(D(y)\nabla v_n(t))) \\ &= -\sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}}\left(-\mathrm{div}_y(D(y)\nabla \mathrm{Proj}_{E_{\lambda_n}} v(t))\right), \quad t \in \mathbb{R}_+.\end{aligned}$$

□

Remark 6.1 *The following conditions are equivalent*

1. $u \in H_P^1$ *and* $\sum_{n\in\mathbb{N}} \left|\mathrm{Proj}_{E_{\lambda_n}}\left(-\mathrm{div}_y(D(y)\nabla \mathrm{Proj}_{E_{\lambda_n}} u)\right)\right|^2_{L^2(\mathbb{R}^m)} < +\infty$.
2. $u \in H_P^1$ *and there is a constant* $C \in \mathbb{R}_+$ *such that*

$$\mathfrak{m}(u,\varphi) \le C|\varphi|_{L^2(\mathbb{R}^m)}, \quad \varphi \in H_P^1.$$

For any u *satisfying 1. or 2. we have*

$$\mathfrak{m}(u,\varphi) = -\int_{\mathbb{R}^m} \varphi(y) \sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}} \mathrm{div}_y(D(y)\nabla \mathrm{Proj}_{E_{\lambda_n}} u) \;\mathrm{d}y, \quad \varphi \in H_P^1.$$

Indeed, if 1. holds true, then for any $u \in H_P^1$ *we have* $u = \sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}} u$ *in* H_P^1*, and by the continuity of* $\mathfrak{m}$ *we deduce, thanks to* (43)

$$\begin{aligned}\mathfrak{m}(u,\varphi) &= \sum_{n\in\mathbb{N}} \mathfrak{m}(\mathrm{Proj}_{E_{\lambda_n}} u, \varphi) = \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^m} D(y)\nabla \mathrm{Proj}_{E_{\lambda_n}} u \cdot \nabla \mathrm{Proj}_{E_{\lambda_n}} \varphi \;\mathrm{d}y \\ &= \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^m} \mathrm{Proj}_{E_{\lambda_n}}\left(-\mathrm{div}_y(D(y)\nabla \mathrm{Proj}_{E_{\lambda_n}} u)\right) \varphi(y) \;\mathrm{d}y \\ &\le \left|\sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}}\left(-\mathrm{div}_y(D(y)\nabla \mathrm{Proj}_{E_{\lambda_n}} u)\right)\right|_{L^2(\mathbb{R}^m)} |\varphi|_{L^2(\mathbb{R}^m)} \\ &= \left(\sum_{n\in\mathbb{N}} \left|\mathrm{Proj}_{E_{\lambda_n}}\left(-\mathrm{div}_y(D(y)\nabla \mathrm{Proj}_{E_{\lambda_n}} u)\right)\right|^2_{L^2(\mathbb{R}^m)}\right)^{1/2} |\varphi|_{L^2(\mathbb{R}^m)}, \; \varphi \in H_P^1.\end{aligned}$$

Conversely if 2. holds true, there is a function $f \in L^2(\mathbb{R}^m)$ *such that*

$$\mathfrak{m}(u,\varphi) = \int_{\mathbb{R}^m} f(y)\varphi(y)\,\mathrm{d}y, \varphi \in H^1_P.$$

In particular, for any $\varphi \in E_{\lambda_n} \cap H^1_P$ *we obtain*

$$\int_{\mathbb{R}^m} \mathrm{Proj}_{E_{\lambda_n}} f\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^m} f(y)\varphi(y)\,\mathrm{d}y = \mathfrak{m}(u,\varphi) = \mathfrak{m}(\varphi,u) = \int_{\mathbb{R}^m} D\nabla\varphi\cdot\nabla\mathrm{Proj}_{E_{\lambda_n}} u\,\mathrm{d}y$$

and therefore

$$\sum_{n\in\mathbb{N}} \left|\mathrm{Proj}_{E_{\lambda_n}}\left(-\mathrm{div}_y(D(y)\nabla\mathrm{Proj}_{E_{\lambda_n}} u)\right)\right|^2_{L^2(\mathbb{R}^m)} = \sum_{n\in\mathbb{N}} |\mathrm{Proj}_{E_{\lambda_n}} f|^2_{L^2(\mathbb{R}^m)} = |f|^2_{L^2(\mathbb{R}^m)} < +\infty.$$

The linear transformation

$$\mathcal{M}u = \sum_{n\in\mathbb{N}} \mathrm{Proj}_{E_{\lambda_n}}\left(-\mathrm{div}_y(D(y)\nabla\mathrm{Proj}_{E_{\lambda_n}} u)\right)$$

defined on

$$\mathrm{dom}\mathcal{M} = \left\{u \in H^1_P,\ \sum_{n\in\mathbb{N}}\left|\mathrm{Proj}_{E_{\lambda_n}}\left(-\mathrm{div}_y(D(y)\nabla\mathrm{Proj}_{E_{\lambda_n}} u)\right)\right|^2_{L^2(\mathbb{R}^m)} < +\infty\right\}$$

is the operator associated to the bilinear form $\mathfrak{m} : H^1_P \times H^1_P$ *i,e.,*

$$\mathfrak{m}(u,\varphi) = \int_{\mathbb{R}^m} \mathcal{M}u\varphi(y)\,\mathrm{d}y,\ \ u \in \mathrm{dom}\mathcal{M},\ \ \varphi \in H^1_P.$$

Therefore the last statement in Corrolary 6.1 says that if $u^{\mathrm{in}} \in \mathrm{dom}\mathcal{M}$*, then*

$$v(t) \in \mathrm{dom}\mathcal{M}\ \ \textit{and}\ \partial_t v + \mathcal{M}v(t) = 0,\ \ t \in \mathbb{R}_+.$$

Remark 6.2

1. *The Corollary 6.1 defines a* C^0 *semi-group of contractions with respect to the* L^2 *norm*

   $$\psi_t u^{\mathrm{in}} = v(t),\ \ t \in \mathbb{R}_+,\ \ u^{\mathrm{in}} \in \mathrm{dom}\mathcal{T}$$

   where v *is the unique solution of* (54)*. This semi-group extends by continuity to a* C^0 *semi-group of contractions on whole* $L^2(\mathbb{R}^m)$*, still denoted by* $(\psi_t)_{t\in\mathbb{R}_+}$ *(use the density of* $\mathrm{dom}\mathcal{T}$ *in* $L^2(\mathbb{R}^m)$*).*

2. *We claim that the* C^0 *semi-groups* $(e^{-\tau\mathcal{B}})_{\tau\in\mathbb{R}_+}$ *and* $(\psi_t)_{t\in\mathbb{R}_+}$ *are commuting. Indeed, for any* $u \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}$*, there is, cf. Proposition 6.2 a unique function* $v \in C_b(\mathbb{R}_+; E_{\lambda_n}) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; E_{\lambda_n} \cap H^1_P)$ *such that*

   $$v(0) = u,\ \ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} v(t,y)\varphi(y)\,\mathrm{d}y + \mathfrak{m}(v(t),\varphi) = 0\ \ \textit{in}\ \mathcal{D}'(\mathbb{R}_+),\ \ \varphi \in E_{\lambda_n} \cap H^1_P. \tag{55}$$

   Notice that for any $\varphi \in E_{\lambda_k} \cap H^1_P, k \neq n$ *we have*

   $$\int_{\mathbb{R}^m} v(t,y)\varphi(y)\,\mathrm{d}y = 0,\ \ \mathfrak{m}(v(t),\varphi) = 0,\ \ t \in \mathbb{R}_+$$

   and thus (55) *holds true for any* $\varphi \in \overline{\mathrm{span}\{\cup_{k\in\mathbb{N}}(E_{\lambda_k} \cap H^1_P)\}} = H^1_P$*. Therefore we have* $\psi_t u = v(t) \in E_{\lambda_n} \cap H^1_P$*, for any* $t \in \mathbb{R}_+, n \in \mathbb{N}$*. We obtain*

   $$e^{-\tau\mathcal{B}}\psi_t u = e^{-\tau\lambda_n}\psi_t u = \psi_t(e^{-\tau\lambda_n}u) = \psi_t e^{-\tau\mathcal{B}}u,\ \ t,\tau \in \mathbb{R}_+, u \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}$$

   which extends by density to any $u \in L^2(\mathbb{R}^m)$*.*

3. For any $\varepsilon \in]0,1]$, we know by the second statement of Proposition 5.5 that $(u,v) \to \mathfrak{m}(u,v) + \frac{1}{\varepsilon}\int_{\mathbb{R}^m}(b\cdot\nabla u)(b\cdot\nabla v)\,\mathrm{d}y$ is coercive on H^1_P with respect to $L^2(\mathbb{R}^m)$. Therefore, for any $u \in L^2(\mathbb{R}^m)$ there is a unique function $\tilde{u}^\varepsilon \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1_P)$ such that $\tilde{u}^\varepsilon(0) = u^{\mathrm{in}}$ and for any $\varphi \in H^1_P$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} \tilde{u}^\varepsilon(t,y)\varphi(y)\,\mathrm{d}y + \mathfrak{m}(\tilde{u}^\varepsilon(t),\varphi) + \frac{1}{\varepsilon}\int_{\mathbb{R}^m}(b\cdot\nabla\tilde{u}^\varepsilon(t))(b\cdot\nabla\varphi)\,\mathrm{d}y = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_+).$$

We claim that $\tilde{u}^\varepsilon(t) = e^{-\frac{t}{\varepsilon}\mathcal{B}}\psi_t u^{\mathrm{in}} = \psi_t e^{-\frac{t}{\varepsilon}\mathcal{B}} u^{\mathrm{in}}, t \in \mathbb{R}_+$. We are done if we prove it for $u^{\mathrm{in}} \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}$, that is

$$\tilde{u}^\varepsilon(t) = e^{-\frac{t}{\varepsilon}\lambda_n}\psi_t u^{\mathrm{in}}, \quad t \in \mathbb{R}_+.$$

Indeed, for any $\varphi \in H^1_P$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} e^{-\frac{t}{\varepsilon}\lambda_n}\psi_t u^{\mathrm{in}}\varphi(y)\,\mathrm{d}y + \mathfrak{m}(e^{-\frac{t}{\varepsilon}\lambda_n}\psi_t u^{\mathrm{in}},\varphi) + \frac{1}{\varepsilon}\int_{\mathbb{R}^m} b\cdot\nabla(e^{-\frac{t}{\varepsilon}\lambda_n}\psi_t u^{\mathrm{in}})\, b\cdot\nabla\varphi\,\mathrm{d}y \\ = e^{-\frac{t}{\varepsilon}\lambda_n}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m}\psi_t u^{\mathrm{in}}\varphi(y)\,\mathrm{d}y + \mathfrak{m}(\psi_t u^{\mathrm{in}},\varphi)\right\} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_+).\end{aligned}$$

Therefore $\tilde{u}^\varepsilon, e^{-\frac{t}{\varepsilon}\lambda_n}\psi_t u^{\mathrm{in}}$ satisfy the same variational formulation, with the same initial condition u^{in}. By the uniqueness of the solution, we deduce that $\tilde{u}^\varepsilon = e^{-\frac{t}{\varepsilon}\lambda_n}\psi_t u^{\mathrm{in}}, t \in \mathbb{R}_+$.

7 The operator $\mathcal{N}$ and the associated bilinear form $\mathfrak{n}$

As suggested by Proposition 2.1, we intend to establish $u^\varepsilon(t) = e^{-\frac{t}{\varepsilon}\mathcal{B}}v(t) + \mathcal{O}(\varepsilon)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$, as $\varepsilon \searrow 0$, where $v(t) = \psi_t u^{\mathrm{in}}, t \in \mathbb{R}_+$. The key point is to emphasize a corrector like in (10), which requires the construction of a second operator $\mathcal{N}$, which enters a decomposition of $\mathcal{A} = -\mathrm{div}_y(D(y)\nabla_y)$ with respect to $\mathcal{B} = -\mathcal{T}^2$ similar to (9). More exactly we are interested in solving for $(\mathcal{B} - \lambda_n Id)u = \mathcal{A}u - \mathcal{M}u = \mathcal{A}u - \mathrm{Proj}_{E_{\lambda_n}}\mathcal{A}u, n \in \mathbb{N}$, see (8). Obviously, this is not always possible, since $\mathcal{A}u - \mathcal{M}u$ belongs to $E^\perp_{\lambda_n} = \overline{\mathrm{Range}\,(\mathcal{B} - \lambda_n Id)}$ which is larger than $\mathrm{Range}\,(\mathcal{B} - \lambda_n Id)$, when the range of $\mathcal{B} - \lambda_n Id$ is not closed. In order to define the bilinear form associated to the operator $\mathcal{N}$ we introduce new structural hypotheses for the matrix field D.

7.1 Structural hypotheses for the matrix field D

Recall that the infinitesimal generator L of the group $(G(s))_{s\in\mathbb{R}}$ is skew-adjoint on H_Q and thus $\overline{\mathrm{Range}\,L} = (\ker L)^\perp$, implying that $D - \langle D\rangle \in \overline{\mathrm{Range}\,L}$. We assume that D is a matrix field in H^∞_Q such that $D - \langle D\rangle \in \mathrm{Range}\,L$, that is

$$\exists\, C \in \mathrm{dom}L \cap H^\infty_Q \ \text{ such that } D = \langle D\rangle + LC, \tag{56}$$

where the operators $\langle\cdot\rangle, L$ are considered in $H_{Q,\mathrm{loc}}$, see Proposition 3.2 and Remark 3.1. Replacing C by $C - \langle C\rangle$ we can suppose that $C \in \ker\langle\cdot\rangle \cap H^\infty_Q$. Thanks to the symmetry of $D, \langle D\rangle$, we have

$$L(C - {}^tC) = LC - {}^t(LC) = D - \langle D\rangle - {}^t(D - \langle D\rangle) = 0$$

implying that $C - {}^tC \in \ker L \cap \ker\langle\cdot\rangle = \{0\}$ and thus C is also symmetric. Moreover, we will require that

$$\exists\, C_0 \in \ker\langle\cdot\rangle \cap H^\infty_Q \ \text{ such that } C_0 = {}^tC_0,\ LC_0 = -C. \tag{57}$$

Notice that (56), (57) say that there is $C_0 \in \ker\langle\cdot\rangle \cap H^\infty_Q$ such that $C_0 = {}^tC_0$, $LC_0 \in \mathrm{dom}L \cap H^\infty_Q$ and $D - \langle D\rangle = -L^2C_0$. We also make the following hypotheses for any $n \in \mathbb{N}^\star$

$$C - \mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C = (-L^2 - 4\lambda_n Id)C_n,\ \ C_n \in \ker\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} \cap H^\infty_Q,\ \ LC_n \in H^\infty_Q,\ \ C_n = {}^tC_n \tag{58}$$

where the operators $\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}, n \in \mathbb{N}^\star$ are considered in $H_{Q,\mathrm{loc}}$ cf. Remark 5.2. In the sequel we work under the hypotheses (16), (17), (18), (19), (40), (41), (39), supplemented by (56), (57), (58).

7.2 Definition and properties of the bilinear form $\mathfrak{n}$

We introduce a second bounded bilinear form on $H^1_P \times H^1_P$ and a corrector is constructed in terms of the operator associated to this form.

Proposition 7.1
Assume that the following conditions hold true

$$\sum_{n\geq 1} \frac{1}{\lambda_n} < +\infty, \;\; \sum_{n\geq 1} \lambda_n \left(|C_n|_{H^\infty_Q} + \left| \frac{L}{\sqrt{4\lambda_n}} C_n \right|_{H^\infty_Q} \right)^2 < +\infty. \tag{59}$$

We consider the application $\mathfrak{n} : H^1_P \times H^1_P \to \mathbb{R}$

$$\mathfrak{n}(u,v) = \int_{\mathbb{R}^m} \lim_{S\to+\infty} \frac{1}{S} \int_0^S (S-s) \left[G(s) C \nabla u_{2s} - \lim_{S'\to+\infty} \frac{1}{S'} \int_0^{S'} G(s') C \nabla u_{2s'} \mathrm{d}s' \right] \mathrm{d}s \cdot \nabla v \,\mathrm{d}y$$

for any $u, v \in H^1_P$. *The bilinear form* $\mathfrak{n}$ *is well defined, bounded on* $H^1_P \times H^1_P$, *skew-symmetric and verifies*

$$\mathfrak{a}(u,v) = \mathfrak{m}(u,v) + \mathfrak{n}(u, \mathcal{B}v) - \mathfrak{n}(\mathcal{B}u, v), \tag{60}$$

for any $u, v \in H^1_P$ *such that* $\mathcal{T}u, \mathcal{T}v, \mathcal{B}u, \mathcal{B}v \in H^1_P$. *Here, by* $\mathcal{T}u \in H^1_P$ *we understand that* u *belongs to the domain of the infinitesimal generator of the* C^0*-group* $(\zeta^1(s))_{s\in\mathbb{R}}$ *and* $\mathfrak{a}(u,v) = \int_{\mathbb{R}^m} D(y)\nabla u \cdot \nabla v \,\mathrm{d}y, u, v \in H^1_P$. *Moreover we have* $\mathfrak{n}(u,v) = 0$ *for any* $u, v \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}$.

The proof of Proposition 7.1 is very technical and it is postponed to Appendix A.

Remark 7.1
We denote by $\mathcal{N}$ *the operator associated to the bilinear form* $\mathfrak{n}$, *that is*

$$\int_{\mathbb{R}^m} \lim_{S\to+\infty} \frac{1}{S} \int_0^S (S-s) \left[G(s) C \nabla u_{2s} - \lim_{S'\to+\infty} \frac{1}{S'} \int_0^{S'} G(s') C \nabla u_{2s'} \,\mathrm{d}s' \right] \mathrm{d}s \cdot \nabla v \,\mathrm{d}y$$
$$= \mathfrak{n}(u,v) = \int_{\mathbb{R}^m} \mathcal{N}u \, v \,\mathrm{d}y, \;\; u \in \mathrm{dom}\mathcal{N}, \; v \in H^1_P.$$

We deduce that

$$\begin{aligned}
\mathcal{N}u &= -\mathrm{div}_y \left\{ \lim_{S\to+\infty} \frac{1}{S} \int_0^S (S-s) \left[G(s) C \nabla u_{2s} - \lim_{S'\to+\infty} \frac{1}{S'} \int_0^{S'} G(s') C \nabla u_{2s'} \,\mathrm{d}s' \right] \mathrm{d}s \right\} \\
&= -\mathrm{div}_y \left\{ C_0 \nabla \mathrm{Proj}_{E_0} u + \sum_{n\geq 1} \left[\frac{1}{8\sqrt{\lambda_n}} \frac{L}{\sqrt{4\lambda_n}} \mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C + L C_n \right] \nabla \mathrm{Proj}_{E_{\lambda_n}} u \right. \\
&\quad \left. + \sum_{n\geq 1} \left[\frac{1}{8\sqrt{\lambda_n}} \mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C - \sqrt{4\lambda_n} C_n \right] \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} \mathrm{Proj}_{E_{\lambda_n}} u \right\}
\end{aligned}$$

provided that b, D, u *are smooth enough.*

8 Asymptotic behavior

We are ready to establish the asymptotic behavior of the variational solutions $(u^\varepsilon)_{\varepsilon>0}$ for (1), (2). We follow the arguments in the proof of Proposition 2.1.

Theorem 8.1
Let u^{in} *be an element in the domain of* $\mathcal{T}$. *We assume that the vector field* b *and the matrix field* D *satisfy the following hypotheses* (16), (17), (18), (19), (20), (40), (41) *and that the structural hypotheses* (39), (56), (57), (58), (59) *hold true. For any* $\varepsilon \in]0,1]$ *let us denote by* $u^\varepsilon \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1_P)$ *the unique variational solution of* (1), (2)

$$u^\varepsilon(0) = u^{\mathrm{in}}, \;\; \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} u^\varepsilon(t,y)\varphi(y)\,\mathrm{d}y + \underbrace{\int_{\mathbb{R}^m} D(y)\nabla u^\varepsilon \cdot \nabla\varphi \,\mathrm{d}y}_{\mathfrak{a}(u^\varepsilon,\varphi)} + \frac{1}{\varepsilon} \int_{\mathbb{R}^m} (b\cdot\nabla u^\varepsilon)(b\cdot\nabla\varphi)\,\mathrm{d}y = 0$$

in $\mathcal{D}'(\mathbb{R}_+)$, for any $\varphi \in H^1_P$ cf. Proposition 6.1, and by $v \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1_P)$ the unique variational solution (see Proposition 5.5 for the definition of the bilinear form $\mathfrak{m}$)

$$v(0) = u^{\text{in}}, \ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} v(t,y)\varphi(y)\,\mathrm{d}y + \mathfrak{m}(v(t),\varphi) = 0 \ \ \textit{in } \mathcal{D}'(\mathbb{R}_+), \ \varphi \in H^1_P$$

cf. Corollary 6.1. Then, provided that $u^{\text{in}}, v, b, D, P$ are smooth enough, for any $T \in \mathbb{R}_+$, there is a constant C_T such that

$$\left| u^\varepsilon - e^{-\frac{t}{\varepsilon}\mathcal{B}} v \right|_{L^\infty([0,T];L^2(\mathbb{R}^m))} + \left| \nabla u^\varepsilon - \nabla e^{-\frac{t}{\varepsilon}\mathcal{B}} v \right|_{L^2([0,T];X_P)} \le C_T \varepsilon, \ \ 0 < \varepsilon \le 1.$$

Proof. By Proposition 6.1 and Corollary 6.1 we know that

$$|u^\varepsilon|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le |u^{\text{in}}|_{L^2(\mathbb{R}^m)}, \ \ |\nabla u^\varepsilon|_{L^2(\mathbb{R}_+;X_P)} \le \frac{|u^{\text{in}}|_{L^2(\mathbb{R}^m)}}{\sqrt{2d}}, \ \ 0 < \varepsilon \le 1$$

and

$$|v|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le |u^{\text{in}}|_{L^2(\mathbb{R}^m)}, \ \ |\mathcal{T} v|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le |\mathcal{T} u^{\text{in}}|_{L^2(\mathbb{R}^m)}$$

$$|\nabla v|_{L^2([0,t];X_P)} \le \frac{|u^{\text{in}}|_{L^2(\mathbb{R}^m)}}{\sqrt{2d}} + \sqrt{\frac{t}{d}}\, |\mathcal{T} u^{\text{in}}|_{L^2(\mathbb{R}^m)}.$$

We consider the function

$$u^1(t,\tau,\cdot) = e^{-\tau\mathcal{B}} \mathcal{N} v(t,\cdot) - \mathcal{N} e^{-\tau\mathcal{B}} v(t,\cdot), \ \ (t,\tau) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

We assume that v is regular enough, such that $\mathcal{N} v(t,\cdot)$ is well defined, see Remark 7.1. Moreover, as the semi-group $(e^{-\tau\mathcal{B}})_{\tau\in\mathbb{R}_+}$ preserves the regularity, see Section 4.1. We deduce that $\mathcal{N} e^{-\tau\mathcal{B}} v(t,\cdot)$ is also well defined, uniformly with respect to $\tau \in \mathbb{R}_+$, implying that

$$|u^1(t,\tau,\cdot)|_{L^2(\mathbb{R}^m)} \le \tilde{C}_T, \ \ t \in [0,T], \ \ \tau \in \mathbb{R}_+$$

for some constant $\tilde{C}_T$. We also ask for the estimates

$$\int_0^T \sup_{\tau\in\mathbb{R}_+} |\partial_t u^1(t,\tau,\cdot)|_{L^2(\mathbb{R}^m)}\,\mathrm{d}t + \int_0^T \sup_{\tau\in\mathbb{R}_+} |\mathrm{div}_y(D\nabla u^1(t,\tau,\cdot))|_{L^2(\mathbb{R}^m)}\,\mathrm{d}t$$
$$+ \left(\int_0^T \sup_{\tau\in\mathbb{R}_+} |\nabla u^1(t,\tau,\cdot)|^2_{X_P}\,\mathrm{d}t \right)^{1/2} \le \tilde{\tilde{C}}_T$$

which can be achieved provided that $u^{\text{in}}, v, b, D, P$ are smooth enough. We also assume the existence of smooth fields in involution with respect to b, in order to guarantee the propagation of the regularity along the semi-group $(e^{-\tau\mathcal{B}})_{\tau\in\mathbb{R}_+}$. The derivative of u^1 with respect to the variable τ writes (assuming that v is regular enough)

$$\begin{aligned}
\partial_\tau u^1 &= -\mathcal{B} e^{-\tau\mathcal{B}} \mathcal{N} v(t) + \mathcal{N}\mathcal{B} e^{-\tau\mathcal{B}} v(t) \\
&= -\mathcal{B}\left(e^{-\tau\mathcal{B}}\mathcal{N} v(t) - \mathcal{N} e^{-\tau\mathcal{B}} v(t)\right) + \mathcal{N}\mathcal{B} e^{-\tau\mathcal{B}} v(t) - \mathcal{B}\mathcal{N} e^{-\tau\mathcal{B}} v(t) \\
&= -\mathcal{B} u^1 - (\mathcal{B}\mathcal{N} - \mathcal{N}\mathcal{B}) e^{-\tau\mathcal{B}} v(t)
\end{aligned}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\varepsilon u^1(t,t/\varepsilon)\right\} + (\mathcal{B}\mathcal{N} - \mathcal{N}\mathcal{B}) e^{-\frac{t}{\varepsilon}\mathcal{B}} v(t) + \frac{\mathcal{B}}{\varepsilon}\{\varepsilon u^1(t,t/\varepsilon)\} = \varepsilon \partial_t u^1(t,t/\varepsilon). \tag{61}$$

By the third statement of Remark 6.2 we know that $\tilde{u}^\varepsilon(t) = e^{-\frac{t}{\varepsilon}\mathcal{B}} v(t) \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1_P)$ satisfies

$$\tilde{u}^\varepsilon(0) = u^{\text{in}}, \ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} \tilde{u}^\varepsilon(t,y)\varphi(y)\,\mathrm{d}y + \mathfrak{m}(\tilde{u}^\varepsilon(t),\varphi) + \frac{1}{\varepsilon}\int_{\mathbb{R}^m} (b\cdot\nabla\tilde{u}^\varepsilon(t))(b\cdot\nabla\varphi)\,\mathrm{d}y = 0 \tag{62}$$

in $\mathcal{D}'(\mathbb{R}_+)$, for any $\varphi \in H^1_P$. Combining (61), (62) we obtain $\tilde{u}^\varepsilon(0) + \varepsilon u^1(0,0) = v(0) = u^{\rm in}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m}\{\tilde{u}^\varepsilon(t,y) + \varepsilon u^1(t,t/\varepsilon,y)\}\varphi(y)\;\mathrm{d}y + \mathfrak{m}(\tilde{u}^\varepsilon(t),\varphi) + \mathfrak{n}(\tilde{u}^\varepsilon(t),\mathcal{B}\varphi) - \mathfrak{n}(\mathcal{B}\tilde{u}^\varepsilon(t),\varphi)$$
$$+\frac{1}{\varepsilon}\int_{\mathbb{R}^m}(b\cdot\nabla[\tilde{u}^\varepsilon(t) + \varepsilon u^1(t,t/\varepsilon)])(b\cdot\nabla\varphi)\;\mathrm{d}y = \varepsilon\int_{\mathbb{R}^m}\partial_t u^1(t,t/\varepsilon,y)\varphi(y)\;\mathrm{d}y \qquad (63)$$

in $\mathcal{D}'(\mathbb{R}_+)$, for any $\varphi \in H^1_P$ such that $\mathcal{T}\varphi, \mathcal{B}\varphi \in H^1_P$. By Proposition 7.1 we know that

$$\mathfrak{m}(\tilde{u}^\varepsilon(t),\varphi) + \mathfrak{n}(\tilde{u}^\varepsilon(t),\mathcal{B}\varphi) - \mathfrak{n}(\mathcal{B}\tilde{u}^\varepsilon(t),\varphi) = \mathfrak{a}(\tilde{u}^\varepsilon(t),\varphi)$$

and therefore (63) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m}\{\tilde{u}^\varepsilon(t,y) + \varepsilon u^1(t,t/\varepsilon,y)\}\varphi\;\mathrm{d}y + \mathfrak{a}(\tilde{u}^\varepsilon(t),\varphi) + \frac{1}{\varepsilon}\int_{\mathbb{R}^m}(b\cdot\nabla[\tilde{u}^\varepsilon(t) + \varepsilon u^1(t,t/\varepsilon)])(b\cdot\nabla\varphi)\;\mathrm{d}y$$
$$= \varepsilon\int_{\mathbb{R}^m}\partial_t u^1(t,t/\varepsilon,y)\varphi(y)\;\mathrm{d}y$$

in $\mathcal{D}'(\mathbb{R}_+)$, for any $\varphi \in H^1_P$. Finally the functions $r^\varepsilon(t,y) = u^\varepsilon(t,y) - \tilde{u}^\varepsilon(t,y) - \varepsilon u^1(t,t/\varepsilon,y)$ satisfy the variational problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} r^\varepsilon(t,y)\varphi\;\mathrm{d}y + \mathfrak{a}(r^\varepsilon(t),\varphi) + \frac{1}{\varepsilon}\int_{\mathbb{R}^m}(b\cdot\nabla r^\varepsilon(t))(b\cdot\nabla\varphi)\;\mathrm{d}y$$
$$= -\varepsilon\int_{\mathbb{R}^m}[\partial_t u^1(t,t/\varepsilon,y) - \mathrm{div}_y(D\nabla u^1(t,t/\varepsilon,y)\,)]\varphi\;\mathrm{d}y$$

in $\mathcal{D}'(\mathbb{R}_+)$, for any $\varphi \in H^1_P$ and the initial condition $r^\varepsilon(0) = 0$. Thanks to the coercivity condition (40) we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|r^\varepsilon(t)|^2_{L^2(\mathbb{R}^m)} + d|\nabla r^\varepsilon(t)|^2_{X_P} \leq \varepsilon|r^\varepsilon(t)|_{L^2(\mathbb{R}^m)}$$
$$\times\left[\sup_{\tau\in\mathbb{R}_+}|\partial_t u^1(t,\tau)|_{L^2(\mathbb{R}^m)} + \sup_{\tau\in\mathbb{R}_+}|\mathrm{div}_y(D\nabla u^1(t,\tau))|_{L^2(\mathbb{R}^m)}\right],\;\; 0<\varepsilon\leq 1.$$

We obtain the estimates

$$|r^\varepsilon|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \leq \varepsilon\int_0^T\sup_{\tau\in\mathbb{R}_+}|\partial_t u^1(t,\tau)|_{L^2(\mathbb{R}^m)}\,\mathrm{d}t$$
$$+\,\varepsilon\int_0^T\sup_{\tau\in\mathbb{R}_+}|\mathrm{div}_y(D\nabla u^1(t,\tau))|_{L^2(\mathbb{R}^m)}\,\mathrm{d}t \leq \varepsilon\tilde{\tilde{C}}_T,\; 0<\varepsilon\leq 1$$

and

$$|\nabla r^\varepsilon|_{L^2([0,T];X_P)} \leq \varepsilon\frac{\tilde{\tilde{C}}_T}{\sqrt{d}},\; 0<\varepsilon\leq 1$$

which implies immediately that

$$\left|u^\varepsilon(t) - e^{-\frac{t}{\varepsilon}\mathcal{B}}v(t)\right|_{L^2(\mathbb{R}^m)} \leq \varepsilon\sup_{(t',\tau)\in[0,T]\times\mathbb{R}_+}|u^1(t',\tau)|_{L^2(\mathbb{R}^m)} + \varepsilon\tilde{\tilde{C}}_T \leq \varepsilon(\tilde{C}_T + \tilde{\tilde{C}}_T),\; t\in[0,T]$$

$$\left|\nabla u^\varepsilon(t) - \nabla e^{-\frac{t}{\varepsilon}\mathcal{B}}v(t)\right|_{L^2([0,T];X_P)} \leq \varepsilon\tilde{\tilde{C}}_T\left(1 + \frac{1}{\sqrt{d}}\right)$$

for any $0<\varepsilon\leq 1$. □

Remark 8.1

1. *For any $\varphi \in H^1_P$ we have by Proposition 5.5 and Lemma 5.1*

$$\begin{aligned}
&\mathfrak{m}(v(t), \langle\varphi\rangle) = \mathfrak{m}(\langle v(t)\rangle, \langle\varphi\rangle) = \mathfrak{m}(\langle v(t)\rangle, \varphi) \\
&= \int_{\mathbb{R}^m} \langle D\rangle \nabla \langle v(t)\rangle \cdot \nabla\varphi \,\mathrm{d}y + \int_{\mathbb{R}^m} \lim_{S\to+\infty} \frac{1}{S}\int_0^S G(s)(D - \langle D\rangle)\langle v(t)\rangle \,\mathrm{d}s \cdot \nabla\varphi \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} \langle D\rangle \nabla \langle v(t)\rangle \cdot \nabla\varphi \,\mathrm{d}y
\end{aligned}$$

and therefore we have the equalities in $\mathcal{D}'(\mathbb{R}_+)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} \langle v(t)\rangle(y)\varphi(y)\,\mathrm{d}y &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} v(t,y)\langle\varphi\rangle(y)\,\mathrm{d}y = -\mathfrak{m}(v(t), \langle\varphi\rangle) \\
&= -\int_{\mathbb{R}^m} \langle D\rangle(y)\nabla\langle v(t)\rangle \cdot \nabla\varphi\,\mathrm{d}y.
\end{aligned}$$

The function $\langle v\rangle \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^m)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1_P)$ satisfies the variational problem

$$\langle v(0)\rangle = \langle u^{\mathrm{in}}\rangle, \ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} \langle v(t)\rangle(y)\varphi(y)\,\mathrm{d}y + \int_{\mathbb{R}^m} \langle D\rangle(y)\nabla\langle v(t)\rangle\cdot\nabla\varphi\,\mathrm{d}y = 0 \ \text{in } \mathcal{D}'(\mathbb{R}_+)$$

for any $\varphi \in H^1_P$.

2. *If the initial condition is well prepared, i.e., $\mathcal{T}u^{\mathrm{in}} = 0$, we deduce thanks to the inequality $|\mathcal{T}v|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^m))} \le |\mathcal{T}u^{\mathrm{in}}|_{L^2(\mathbb{R}^m)} = 0$, that $\mathcal{T}v = 0$ and in this case $v = \langle v\rangle$ satisfies the parabolic problem associated to the average matrix field $\langle D\rangle$*

$$v(0) = u^{\mathrm{in}} \in \ker\mathcal{T}, \ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} v(t,y)\varphi(y)\,\mathrm{d}y + \int_{\mathbb{R}^m}\langle D\rangle(y)\nabla v(t)\cdot\nabla\varphi\,\mathrm{d}y = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}_+)$$

for any $\varphi \in H^1_P$.

A Proofs of Proposition 7.1

Proof. (of Proposition 7.1)
Boundedness of $\mathfrak{n}$
We show that $\lim_{S\to+\infty}\frac{1}{S}\int_0^S (S-s)\left[G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'} G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right]\mathrm{d}s$ converges, as $S\to+\infty$, strongly in X_Q, for any $u\in H^1_P$, which will imply that $\mathfrak{n}(u,v)$ is well defined for any $(u,v)\in H^1_P\times H^1_P$. We appeal to the Hilbertian sum $H^1_P = \oplus_{n\in\mathbb{N}}(E_{\lambda_n}\cap H^1_P)$. If $u\in E_0\cap H^1_P$, we know by Lemma 5.1 that

$$\lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'} G(s')C\nabla u_{2s'}\,\mathrm{d}s' = \lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'} G(s')C\nabla u\,\mathrm{d}s' = \langle C\rangle\nabla u = 0$$

strongly in X_Q and therefore, by (57), we obtain

$$\begin{aligned}
&\lim_{S\to+\infty}\frac{1}{S}\int_0^S (S-s)\left[G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'} G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right]\mathrm{d}s \qquad (64)\\
&= -\frac{1}{S}\int_0^S (S-s)\left\{\frac{\mathrm{d}}{\mathrm{d}s}G(s)C_0\right\}\nabla u\,\mathrm{d}s \\
&= -\frac{1}{S}[(S-s)G(s)C_0\nabla u]_0^S - \frac{1}{S}\int_0^S G(s)C_0\nabla u\,\mathrm{d}s \underset{S\to+\infty}{\longrightarrow} C_0\nabla u - \langle C_0\rangle\nabla u = C_0\nabla u
\end{aligned}$$

strongly in X_Q. Assume now that $u\in E_{\lambda_n}\cap H^1_P, n\in\mathbb{N}^\star$. We know by Lemma 5.2 that

$$\lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'} G(s')C\nabla u_{2s'}\,\mathrm{d}s' = \frac{1}{2}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C\nabla u + \frac{1}{2}\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u$$

strongly in X_Q and therefore

$$f_u(s) := G(s)C\nabla u_{2s} - \lim_{S'\to+\infty} \frac{1}{S'} \int_0^{S'} G(s')C\nabla u_{2s'} \,\mathrm{d}s' = U(s)\nabla u + V(s)\nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} u$$

with

$$U(s) = \cos(\sqrt{4\lambda_n} s)G(s)C - \frac{1}{2}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C, \;\; V(s) = \sin(\sqrt{4\lambda_n} s)G(s)C - \frac{1}{2}\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C.$$

Notice that the hypothesis (58) writes

$$-L_{4\lambda_n}\left(C_n, \frac{L}{\sqrt{4\lambda_n}} C_n\right) = \left(0, \frac{C - \mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C}{\sqrt{4\lambda_n}}\right),$$

where the operators $L, L_{4\lambda_n}$ are considered on $H_{Q,\mathrm{loc}}$, $H_{Q,\mathrm{loc}} \times H_{Q,\mathrm{loc}}$ respectively, see Remark 3.1. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} G_{4\lambda_n}(s)\left(C_n, \frac{L}{\sqrt{4\lambda_n}} C_n\right) &= -G_{4\lambda_n}(s)\left(0, \frac{C - \mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C}{\sqrt{4\lambda_n}}\right) \\ &= (\sin(\sqrt{4\lambda_n} s), -\cos(\sqrt{4\lambda_n} s))\frac{G(s)C - G(s)\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C}{\sqrt{4\lambda_n}}.\end{aligned}$$

We obtain

$$\begin{aligned}\sin(\sqrt{4\lambda_n} s)G(s)C = \sin(\sqrt{4\lambda_n} s)&\underbrace{\left[\cos(\sqrt{4\lambda_n} s)\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C + \sin(\sqrt{4\lambda_n} s)\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C\right]}_{G(s)\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C} \\ &+ \sqrt{4\lambda_n}\frac{\mathrm{d}}{\mathrm{d}s}\left[\cos(\sqrt{4\lambda_n} s)G(s)C_n - \sin(\sqrt{4\lambda_n} s)G(s)\frac{L}{\sqrt{4\lambda_n}} C_n\right]\end{aligned}$$

and

$$\begin{aligned}\cos(\sqrt{4\lambda_n} s)G(s)C = \cos(\sqrt{4\lambda_n} s)&\underbrace{\left[\cos(\sqrt{4\lambda_n} s)\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C + \sin(\sqrt{4\lambda_n} s)\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C\right]}_{G(s)\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C} \\ &- \sqrt{4\lambda_n}\frac{\mathrm{d}}{\mathrm{d}s}\left[\sin(\sqrt{4\lambda_n} s)G(s)C_n + \cos(\sqrt{4\lambda_n} s)G(s)\frac{L}{\sqrt{4\lambda_n}} C_n\right]\end{aligned}$$

and the matrix fields $U(s), V(s)$ write

$$\begin{aligned}U(s) = &\frac{1}{2}\cos(4\sqrt{\lambda_n} s)\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C + \frac{1}{2}\sin(4\sqrt{\lambda_n} s)\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C \\ &- \sqrt{4\lambda_n}\frac{\mathrm{d}}{\mathrm{d}s}\left[\sin(\sqrt{4\lambda_n} s)G(s)C_n + \cos(\sqrt{4\lambda_n} s)G(s)\frac{L}{\sqrt{4\lambda_n}} C_n\right]\end{aligned} \tag{65}$$

$$\begin{aligned}V(s) = &\frac{1}{2}\sin(4\sqrt{\lambda_n} s)\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C - \frac{1}{2}\cos(4\sqrt{\lambda_n} s)\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}} C \\ &+ \sqrt{4\lambda_n}\frac{\mathrm{d}}{\mathrm{d}s}\left[\cos(\sqrt{4\lambda_n} s)G(s)C_n - \sin(\sqrt{4\lambda_n} s)G(s)\frac{L}{\sqrt{4\lambda_n}} C_n\right].\end{aligned} \tag{66}$$

Recall that we intend to establish the convergence, as $S \to +\infty$, of

$$\frac{1}{S}\int_0^S (S-s)f_u(s)\,\mathrm{d}s = \frac{1}{S}\int_0^S (S-s)\{U(s)\nabla u + V(s)\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}} u\}\,\mathrm{d}s$$

in X_Q. Observe that

$$\lim_{S\to+\infty}\frac{1}{S}\int_0^S (S-s)\cos(4\sqrt{\lambda_n} s)\,\mathrm{d}s = 0, \;\; \lim_{S\to+\infty}\frac{1}{S}\int_0^S (S-s)\sin(4\sqrt{\lambda_n} s)\,\mathrm{d}s = \frac{1}{4\sqrt{\lambda_n}}. \tag{67}$$

After integration by parts one gets

$$
\begin{aligned}
&-\frac{1}{S}\int_0^S (S-s)\sqrt{4\lambda_n}\frac{\mathrm{d}}{\mathrm{d}s}\left[\sin(\sqrt{4\lambda_n}s)G(s)C_n+\cos(\sqrt{4\lambda_n}s)G(s)\frac{L}{\sqrt{4\lambda_n}}C_n\right]\,\mathrm{d}s\nabla u\\
&+\frac{1}{S}\int_0^S (S-s)\sqrt{4\lambda_n}\frac{\mathrm{d}}{\mathrm{d}s}\left[\cos(\sqrt{4\lambda_n}s)G(s)C_n-\sin(\sqrt{4\lambda_n}s)G(s)\frac{L}{\sqrt{4\lambda_n}}C_n\right]\,\mathrm{d}s\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u\\
&=LC_n\nabla u-\sqrt{4\lambda_n}C_n\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u+\frac{\sqrt{4\lambda_n}}{S}\int_0^S G_{4\lambda_n}(s)\left(C_n,\frac{L}{\sqrt{4\lambda_n}}C_n\right)\,\mathrm{d}s\left(\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u,-\nabla u\right). \quad (68)
\end{aligned}
$$

As $\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C_n=0$, we know, cf. Remark 5.2, that

$$
\lim_{S\to+\infty}\frac{2}{S}\int_r^{r+S}(\cos(\sqrt{4\lambda_n}s)G(s)C_n,\sin(\sqrt{4\lambda_n}s)G(s)C_n)\,\mathrm{d}s=(0,0)
$$

in $H_{Q,\mathrm{loc}}\times H_{Q,\mathrm{loc}}$, implying that

$$
\begin{aligned}
\mathrm{Proj}_{\ker L_{4\lambda_n}}\left(C_n,\frac{L}{\sqrt{4\lambda_n}}C_n\right)&=\lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S}G_{4\lambda_n}(s)\left(C_n,\frac{L}{\sqrt{4\lambda_n}}C_n\right)\,\mathrm{d}s\\
&=\lim_{S\to+\infty}\frac{1}{S}\int_r^{r+S}\Big(\cos(\sqrt{4\lambda_n}s)G(s)C_n-\sin(\sqrt{4\lambda_n}s)G(s)\frac{L}{\sqrt{4\lambda_n}}C_n,\\
&\qquad\sin(\sqrt{4\lambda_n}s)G(s)C_n+\cos(\sqrt{4\lambda_n}s)G(s)\frac{L}{\sqrt{4\lambda_n}}C_n\Big)\,\mathrm{d}s\\
&\qquad=(0,0)
\end{aligned}
$$

in $H_{Q,\mathrm{loc}}\times H_{Q,\mathrm{loc}}$. As $C_n,\frac{L}{\sqrt{4\lambda_n}}C_n$ belong to H_Q^∞ and $\nabla u,\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u$ belong to X_P, we prove, by adapting the arguments in Lemma 5.1, the strong convergence in X_Q

$$
\lim_{S\to+\infty}\frac{1}{S}\int_0^S G_{4\lambda_n}(s)\left(C_n,\frac{L}{\sqrt{4\lambda_n}}C_n\right)\,\mathrm{d}s\left(\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u,-\nabla u\right)=0. \quad (69)
$$

Finally (65), (66), (67), (68), (69) lead to the convergence in X_Q

$$
\begin{aligned}
\lim_{S\to+\infty}\frac{1}{S}\int_0^S(S-s)f_u(s)\,\mathrm{d}s&=\left[\frac{1}{8\sqrt{\lambda_n}}\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C+LC_n\right]\nabla u\\
&+\left[\frac{1}{8\sqrt{\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C-\sqrt{4\lambda_n}C_n\right]\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u. \quad (70)
\end{aligned}
$$

Up to now, we know that $\left(\frac{1}{S}\int_0^S(S-s)f_u(s)\,\mathrm{d}s\right)_{S>0}$ converges in X_Q, as $S\to+\infty$, for any $u\in\operatorname{span}\cup_{n\in\mathbb{N}}(E_{\lambda_n}\cap H_P^1)$. In order to justify the existence of the previous limit for any $u\in H_P^1$, it is enough to bound $\left(\frac{1}{S}\int_0^S(S-s)f_u(s)\,\mathrm{d}s\right)_{S>0}$ in X_Q uniformly with respect to $S>0$ and $u\in\operatorname{span}\cup_{n\in\mathbb{N}}(E_{\lambda_n}\cap H_P^1)$, $|u|_{H_P^1}\le 1$. By (64) we have

$$
\left|\frac{1}{S}\int_0^S(S-s)f_u(s)\,\mathrm{d}s\right|_{X_Q}\le 2|C_0|_{H_Q^\infty}|\nabla u|_{X_P},\;\; u\in E_0\cap H_P^1
$$

and by the previous computations, Remark 5.2 and the fourth statement of Remark 4.1 we obtain for any $u\in E_{\lambda_n}\cap H_P^1, n\in\mathbb{N}^\star$

$$
\begin{aligned}
\left|\frac{1}{S}\int_0^S(S-s)f_u(s)\,\mathrm{d}s\right|_{X_Q}&\le\frac{1}{2\sqrt{\lambda_n}}\left[\left|\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C\right|_{H_Q^\infty}+|\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C|_{H_Q^\infty}\right]|\nabla u|_{X_P}\\
&+3\sqrt{4\lambda_n}\left[\left|\frac{L}{\sqrt{4\lambda_n}}C_n\right|_{H_Q^\infty}+|C_n|_{H_Q^\infty}\right]|\nabla u|_{X_P}\\
&\le\frac{2}{\sqrt{\lambda_n}}|C|_{H_Q^\infty}\;|\nabla u|_{X_P}+6\sqrt{\lambda_n}\left[|C_n|_{H_Q^\infty}+\left|\frac{L}{\sqrt{4\lambda_n}}C_n\right|_{H_Q^\infty}\right]|\nabla u|_{X_P}.
\end{aligned}
$$

Pick $u \in \text{span} \cup_{n\in\mathbb{N}} (E_{\lambda_n} \cap H^1_P)$, that is $u = \sum_{n=0}^N u_n$, $u_n \in E_{\lambda_n} \cap H^1_P$ for any $n \in \{0, ..., N\}$ and let us introduce the notation $c_n = |C_n|_{H^\infty_Q} + \left|\frac{L}{\sqrt{4\lambda_n}} C_n\right|_{H^\infty_Q}$, $n \in \mathbb{N}^\star$. Using the orthogonality of $(\nabla u_n)_{0\le n\le N}$ in X_P, we deduce

$$
\begin{aligned}
\frac{1}{S}\left|\int_0^S (S-s) f_u(s)\,\mathrm{d}s\right|_{X_Q} &\le 2|C_0|_{H^\infty_Q} |\nabla u_0|_{X_P} + \sum_{n=1}^N \left[\frac{2}{\sqrt{\lambda_n}}|C|_{H^\infty_Q} + 6\sqrt{\lambda_n} c_n\right] |\nabla u_n|_{X_P} \\
&\le 2|C_0|_{H^\infty_Q}|\nabla u|_{X_P} + 2|C|_{H^\infty_Q} \left(\sum_{n=1}^N \frac{1}{\lambda_n}\right)^{1/2} |\nabla u|_{X_P} + 6\left(\sum_{n=1}^N \lambda_n c_n^2\right)^{1/2} |\nabla u|_{X_P} \\
&\le \left[2|C_0|_{H^\infty_Q} + 2|C|_{H^\infty_Q}\left(\sum_{n\ge1}\frac{1}{\lambda_n}\right)^{1/2} + 6\left(\sum_{n\ge1}\lambda_n c_n^2\right)^{1/2}\right] |u|_{H^1_P}
\end{aligned}
$$

saying that $\mathfrak{n}$ is well defined and bounded on $H^1_P \times H^1_P$.

Skew-symmetry of $\mathfrak{n}$

Let us focus now on the skew-symmetry of $\mathfrak{n}$. We are done if we show that for any $u \in H^1_P$ we have the convergence in X_Q

$$
\begin{aligned}
&\lim_{S\to+\infty} \frac{1}{S}\int_0^S (S-s)\left\{G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'} G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right\}\mathrm{d}s \\
&\quad = -\lim_{S\to+\infty}\frac{1}{S}\int_{-S}^0 (S+s)\left\{G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_{-S'}^0 G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right\}\mathrm{d}s. \qquad (71)
\end{aligned}
$$

Indeed, let us assume for the moment that (71) holds true. As the field C of symmetric matrices belongs to $\ker\langle\cdot\rangle \cap H^\infty_Q$, we know by Proposition 5.5 that

$$
\mathfrak{m}_C(u,v) := \int_{\mathbb{R}^m} \lim_{S'\to+\infty}\frac{1}{S'}\int_{-S'}^0 G(s')C\nabla u_{2s'}\,\mathrm{d}s' \cdot \nabla v\,\mathrm{d}y
$$

defines a symmetric bounded bilinear form on $H^1_P \times H^1_P$. We obtain, thanks to the symmetries of C and $\mathfrak{m}_C$

$$
\begin{aligned}
\mathfrak{n}(u,v) &= \int_{\mathbb{R}^m}\lim_{S\to+\infty}\frac{1}{S}\int_0^S (S-s)\left\{G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'} G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right\}\mathrm{d}s\cdot\nabla v\,\mathrm{d}y \\
&= -\int_{\mathbb{R}^m}\lim_{S\to+\infty}\frac{1}{S}\int_{-S}^0 (S+s)\left\{G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_{-S'}^0 G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right\}\mathrm{d}s\cdot\nabla v\,\mathrm{d}y \\
&= -\lim_{S\to+\infty}\frac{1}{S}\int_{-S}^0 (S+s)\left[\int_{\mathbb{R}^m} C\nabla u_s\cdot\nabla v_{-s}\,\mathrm{d}y - \mathfrak{m}_C(u,v)\right]\mathrm{d}s \\
&= -\lim_{S\to+\infty}\frac{1}{S}\int_0^S (S-s)\left[\int_{\mathbb{R}^m} C\nabla u_{-s}\cdot\nabla v_s\,\mathrm{d}y - \mathfrak{m}_C(u,v)\right]\mathrm{d}s \\
&= -\lim_{S\to+\infty}\frac{1}{S}\int_0^S (S-s)\left[\int_{\mathbb{R}^m} C\nabla v_s\cdot\nabla u_{-s}\,\mathrm{d}y - \mathfrak{m}_C(v,u)\right]\mathrm{d}s \\
&= -\int_{\mathbb{R}^m}\lim_{S\to+\infty}\frac{1}{S}\int_0^S (S-s)\left\{G(s)C\nabla v_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'} G(s')C\nabla v_{2s'}\,\mathrm{d}s'\right\}\mathrm{d}s\cdot\nabla u\,\mathrm{d}y \\
&= -\mathfrak{n}(v,u).
\end{aligned}
$$

The key point when justifying (71) is that $(G(s))_{s\in\mathbb{R}}, (G_{4\lambda_n}(s))_{s\in\mathbb{R}}, n \in \mathbb{N}^\star$ are groups, and thus the previous arguments work also with $s \in \mathbb{R}_-$. It is enough to check (71) for $u \in \text{span} \cup_{n\in\mathbb{N}} (E_{\lambda_n} \cap H^1_P)$. If $u \in E_0 \cap H^1_P$, we know by Lemma 5.1 that

$$
\lim_{S'\to+\infty}\frac{1}{S'}\int_{-S'}^0 G(s')C\nabla u_{2s'}\,\mathrm{d}s' = \lim_{S'\to+\infty}\frac{1}{S'}\int_{-S'}^0 G(s')C\nabla u\,\mathrm{d}s' = \langle C\rangle \nabla u = 0
$$

strongly in X_Q, and by (57) we obtain

$$-\frac{1}{S}\int_{-S}^{0}(S+s)G(s)C\nabla u_{2s}\,\mathrm{d}s = \frac{1}{S}\int_{-S}^{0}(S+s)\frac{\mathrm{d}}{\mathrm{d}s}G(s)C_0\nabla u\,\mathrm{d}s$$
$$= \frac{1}{S}\left[(S+s)G(s)C_0\nabla u\right]_{-S}^{0} - \frac{1}{S}\int_{-S}^{0}G(s)C_0\nabla u\,\mathrm{d}s$$
$$\underset{S\to+\infty}{\longrightarrow} C_0\nabla u - \langle C_0\rangle\nabla u = C_0\nabla u$$
$$= \lim_{S\to+\infty}\frac{1}{S}\int_0^S(S-s)\left\{G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'}G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right\}\mathrm{d}s$$

cf. (64). Assume now that $u\in E_{\lambda_n}\cap H_P^1, n\in\mathbb{N}^\star$. By Lemma 5.2 we know that

$$\lim_{S'\to+\infty}\frac{1}{S'}\int_{-S'}^{0}G(s')C\nabla u_{2s'}\,\mathrm{d}s' = \frac{1}{2}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C\nabla u + \frac{1}{2}\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u$$

strongly in X_Q and therefore

$$-\frac{1}{S}\int_{-S}^{0}(S+s)\left\{G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_{-S'}^{0}G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right\}\,\mathrm{d}s$$
$$= -\frac{1}{S}\int_{-S}^{0}(S+s)\left\{U(s)\nabla u + V(s)\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u\right\}\,\mathrm{d}s$$

where the matrix fields $U(s), V(s)$ were defined in (65), (66). Following the same arguments as before we deduce

$$\lim_{S\to+\infty}\frac{1}{S}\int_{-S}^{0}(S+s)\cos(4\sqrt{\lambda_n}s)\,\mathrm{d}s = 0,\quad \lim_{S\to+\infty}\frac{1}{S}\int_{-S}^{0}(S+s)\sin(4\sqrt{\lambda_n}s)\,\mathrm{d}s = -\frac{1}{4\sqrt{\lambda_n}}$$

$$\lim_{S\to+\infty}\frac{1}{S}\int_{-S}^{0}(S+s)\frac{\mathrm{d}}{\mathrm{d}s}\left[\sin(\sqrt{4\lambda_n}s)G(s)C_n + \cos(\sqrt{4\lambda_n}s)G(s)\frac{L}{\sqrt{4\lambda_n}}C_n\right]\,\mathrm{d}s\nabla u$$
$$= \frac{L}{\sqrt{4\lambda_n}}C_n\nabla u \text{ in } X_Q$$

and

$$\lim_{S\to+\infty}\frac{1}{S}\int_{-S}^{0}(S+s)\frac{\mathrm{d}}{\mathrm{d}s}\left[\cos(\sqrt{4\lambda_n}s)G(s)C_n - \sin(\sqrt{4\lambda_n}s)G(s)\frac{L}{\sqrt{4\lambda_n}}C_n\right]\,\mathrm{d}s\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u$$
$$= C_n\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u \text{ in } X_Q.$$

Finally we obtain cf. (70)

$$\lim_{S\to+\infty} -\frac{1}{S}\int_{-S}^{0}(S+s)\left\{G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_{-S'}^{0}G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right\}\mathrm{d}s$$
$$= \left[\frac{1}{8\sqrt{\lambda_n}}\frac{L}{\sqrt{4\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C + LC_n\right]\nabla u + \left[\frac{1}{8\sqrt{\lambda_n}}\mathrm{Proj}_{\mathcal{E}_{4\lambda_n}}C - \sqrt{4\lambda_n}C_n\right]\nabla\frac{\mathcal{T}}{\sqrt{\lambda_n}}u$$
$$= \lim_{S\to+\infty}\frac{1}{S}\int_0^S(S-s)\left\{G(s)C\nabla u_{2s} - \lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'}G(s')C\nabla u_{2s'}\,\mathrm{d}s'\right\}\,\mathrm{d}s.$$

Decomposition formula

Let us check (60). Assume that $u, v, \mathcal{T}u, \mathcal{T}v, \mathcal{B}u, \mathcal{B}v\in H_P^1$. Therefore we have

$$\mathfrak{n}(u,\mathcal{B}v) - \mathfrak{n}(\mathcal{B}u, v)$$
$$= \lim_{S\to+\infty}\frac{1}{S}\int_0^S(S-s)\left\{\int_{\mathbb{R}^m}G(s)C\nabla u_{2s}\cdot\nabla\mathcal{B}v - G(s)C\nabla\mathcal{B}u_{2s}\cdot\nabla v\,\mathrm{d}y\right.$$
$$\left.-\lim_{S'\to+\infty}\frac{1}{S'}\int_0^{S'}\int_{\mathbb{R}^m}G(s')C\nabla u_{2s'}\cdot\nabla\mathcal{B}v - G(s')C\nabla\mathcal{B}u_{2s'}\cdot\nabla v\,\mathrm{d}y\,\mathrm{d}s'\right\}\mathrm{d}s.$$

It is easily seen that for any $h \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^m} G(s)C\nabla u_{2s} \cdot \nabla v_h \, \mathrm{d}y = \int_{\mathbb{R}^m} G(h)G(s-h)C\nabla (u_{2s-h})_h \cdot \nabla v_h \, \mathrm{d}y$$
$$= \int_{\mathbb{R}^m} G(s-h)C\nabla u_{2s-h} \cdot \nabla v \, \mathrm{d}y$$

implying, thanks to the hypotheses $u, \mathcal{T}u, v, \mathcal{T}v \in H^1_P, C \in H^\infty_Q$ that

$$\int_{\mathbb{R}^m} G(s)C\nabla u_{2s} \cdot \nabla \mathcal{T} v \, \mathrm{d}y = \lim_{h\to 0} \int_{\mathbb{R}^m} G(s)C\nabla u_{2s} \cdot \nabla \frac{v_h - v}{h} \, \mathrm{d}y$$
$$= \lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}^m} [G(s-h)C\nabla u_{2s-h} - G(s)C\nabla u_{2s}] \cdot \nabla v \, \mathrm{d}y$$
$$= -\int_{\mathbb{R}^m} [G(s)(D - \langle D\rangle)\nabla u_{2s} + G(s)C\nabla \mathcal{T} u_{2s}] \cdot \nabla v \, \mathrm{d}y.$$

Applying twice the above formula, by taking into account that $u, \mathcal{T}u, \mathcal{B}u, v, \mathcal{T}v, \mathcal{B}v \in H^1_P$, we obtain

$$\int_{\mathbb{R}^m} G(s')C\nabla u_{2s'} \cdot \nabla \mathcal{B} v - G(s')C\nabla \mathcal{B} u_{2s'} \cdot \nabla v \, \mathrm{d}y = -\int_{\mathbb{R}^m} G(s')C\nabla \mathcal{B} u_{2s'} \cdot \nabla v \, \mathrm{d}y$$
$$+ \int_{\mathbb{R}^m} [G(s')(D - \langle D\rangle)\nabla u_{2s'} + G(s')C\nabla \mathcal{T} u_{2s'}] \cdot \nabla \mathcal{T} v \, \mathrm{d}y$$
$$= \int_{\mathbb{R}^m} G(s')(D - \langle D\rangle)\nabla u_{2s'} \cdot \nabla \mathcal{T} v \, \mathrm{d}y - \int_{\mathbb{R}^m} G(s')(D - \langle D\rangle)\nabla \mathcal{T} u_{2s'} \cdot \nabla v \, \mathrm{d}y$$
$$= \int_{\mathbb{R}^m} (D - \langle D\rangle)\nabla u_{s'} \cdot \nabla \mathcal{T} v_{-s'} \, \mathrm{d}y - \int_{\mathbb{R}^m} (D - \langle D\rangle)\nabla \mathcal{T} u_{s'} \cdot \nabla v_{-s'} \, \mathrm{d}y$$
$$= -\frac{\mathrm{d}}{\mathrm{d}s'} \int_{\mathbb{R}^m} (D - \langle D\rangle)\nabla u_{s'} \cdot \nabla v_{-s'} \, \mathrm{d}y$$

and therefore

$$\lim_{S'\to +\infty} \frac{1}{S'} \int_0^{S'} \int_{\mathbb{R}^m} G(s')C\nabla u_{2s'} \cdot \nabla \mathcal{B} v - G(s')C\nabla \mathcal{B} u_{2s'} \cdot \nabla v \, \mathrm{d}y\mathrm{d}s'$$
$$= \lim_{S'\to +\infty} \frac{1}{S'} \int_{\mathbb{R}^m} (D - \langle D\rangle) : [\nabla v \otimes \nabla u - \nabla v_{-S'} \otimes \nabla u_{S'}] \, \mathrm{d}y = 0.$$

Similarly we have

$$\frac{1}{S} \int_0^S (S-s) \int_{\mathbb{R}^m} G(s)C\nabla u_{2s} \cdot \nabla \mathcal{B} v - G(s)C\nabla \mathcal{B} u_{2s} \cdot \nabla v \, \mathrm{d}y\mathrm{d}s$$
$$= -\frac{1}{S} \int_0^S (S-s) \frac{\mathrm{d}}{\mathrm{d}s} \int_{\mathbb{R}^m} (D - \langle D\rangle)\nabla u_s \cdot \nabla v_{-s} \, \mathrm{d}y\mathrm{d}s$$
$$= \int_{\mathbb{R}^m} (D - \langle D\rangle)\nabla u \cdot \nabla v \, \mathrm{d}y - \frac{1}{S} \int_0^S \int_{\mathbb{R}^m} (D - \langle D\rangle)\nabla u_s \cdot \nabla v_{-s} \, \mathrm{d}y\mathrm{d}s$$
$$\underset{S\to +\infty}{\longrightarrow} \mathfrak{a}(u, v) - \mathfrak{m}(u, v).$$

Orthogonality condition

Let us check that $\mathfrak{n}(u, v) = 0$ for any $u, v \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}$. Notice that for any $u, v \in H^1_P$ we have

$$G(s)[P(\nabla v \otimes \nabla u)P] = \partial Y^{-1}(s;\cdot)P_s[(\nabla v)_s \otimes (\nabla u)_s]P_s \, {}^t\partial Y^{-1}(s;\cdot)$$
$$= \underbrace{\partial Y^{-1}(s;\cdot)P_s \, {}^t\partial Y^{-1}(s;\cdot)}_{G(s)P=P} \, {}^t\partial Y(s;\cdot)[(\nabla v)_s \otimes (\nabla u)_s]\partial Y(s;\cdot) \underbrace{\partial Y^{-1}(s;\cdot)P_s \, {}^t\partial Y^{-1}(s;\cdot)}_{{}^tG(s)P={}^tP=P}$$
$$= P(\nabla v_s \otimes \nabla u_s)P. \tag{72}$$

In particular, when $u, v \in E_0 \cap H^1_P$, the matrix field $P(\nabla v \otimes \nabla u)P$ is left invariant by $(G(s))_{s\in\mathbb{R}}$ and thus, by Lemma 5.1, we obtain cf. (64)

$$\begin{aligned}
\mathfrak{n}(u,v) &= \int_{\mathbb{R}^m} C_0 \nabla u \cdot \nabla v \,\mathrm{d}y = \int_{\mathbb{R}^m} QC_0 : P(\nabla v \otimes \nabla u)PQ \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} QG(s)C_0 : P(\nabla v \otimes \nabla u)PQ \,\mathrm{d}y = \int_{\mathbb{R}^m} G(s)C_0 \nabla u \cdot \nabla v \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} \frac{1}{S} \int_0^S G(s)C_0 \nabla u \,\mathrm{d}s \cdot \nabla v \,\mathrm{d}y \underset{S\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Consider now $u, v \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}^\star$. We claim that the matrix field

$$P\left(\nabla v \otimes \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} u - \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} v \otimes \nabla u\right) P$$

is left invariant by $(G(s))_{s\in\mathbb{R}}$. Indeed, by formula (72) we have

$$\begin{aligned}
G(s)P&\left(\nabla v \otimes \nabla \frac{\mathcal{T}u}{\sqrt{\lambda_n}} - \nabla \frac{\mathcal{T}v}{\sqrt{\lambda_n}} \otimes \nabla u\right) P = P\left(\nabla v_s \otimes \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} u_s - \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} v_s \otimes \nabla u_s\right) P \\
&= P\left[\cos(\sqrt{\lambda_n}s)\nabla v + \sin(\sqrt{\lambda_n}s)\nabla \frac{\mathcal{T}v}{\sqrt{\lambda_n}}\right] \otimes \left[\cos(\sqrt{\lambda_n}s)\nabla \frac{\mathcal{T}u}{\sqrt{\lambda_n}} - \sin(\sqrt{\lambda_n}s)\nabla u\right] P \\
&- P\left[\cos(\sqrt{\lambda_n}s)\nabla \frac{\mathcal{T}v}{\sqrt{\lambda_n}} - \sin(\sqrt{\lambda_n}s)\nabla v\right] \otimes \left[\cos(\sqrt{\lambda_n}s)\nabla u + \sin(\sqrt{\lambda_n}s)\nabla \frac{\mathcal{T}u}{\sqrt{\lambda_n}}\right] P \\
&= P\left(\nabla v \otimes \nabla \frac{\mathcal{T}u}{\sqrt{\lambda_n}} - \nabla \frac{\mathcal{T}v}{\sqrt{\lambda_n}} \otimes \nabla u\right) P.
\end{aligned}$$

We deduce, thanks to Lemma 5.1, that

$$\begin{aligned}
&\int_{\mathbb{R}^m} C(y) : \left(\nabla v \otimes \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} u - \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} v \otimes \nabla u\right) \mathrm{d}y \\
&= \lim_{S\to+\infty} \int_{\mathbb{R}^m} \frac{1}{S} \int_0^S G(s)C \,\mathrm{d}s : \left(\nabla v \otimes \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} u - \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} v \otimes \nabla u\right) \mathrm{d}y = 0.
\end{aligned}$$

Now we are ready to check that $\mathfrak{n}(u,v) = 0, u, v \in E_{\lambda_n} \cap H^1_P, n \in \mathbb{N}^\star$. For any $s \in \mathbb{R}$ we obtain, thanks to the equalities $u_{2s'} = u_{s'}(Y(s';\cdot)), v = v_{-s'}(Y(s';\cdot))$

$$\begin{aligned}
&\int_{\mathbb{R}^m} G(s')C\nabla u_{2s'} \cdot \nabla v \,\mathrm{d}y = \int_{\mathbb{R}^m} C(y)\nabla u_{s'} \cdot \nabla v_{-s'} \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} C\left[\cos(\sqrt{\lambda_n}s')\nabla u + \sin(\sqrt{\lambda_n}s')\nabla \frac{\mathcal{T}u}{\sqrt{\lambda_n}}\right] \cdot \left[\cos(\sqrt{\lambda_n}s')\nabla v - \sin(\sqrt{\lambda_n}s')\nabla \frac{\mathcal{T}v}{\sqrt{\lambda_n}}\right] \mathrm{d}y \\
&= \cos^2(\sqrt{\lambda_n}s') \int_{\mathbb{R}^m} C(y) : \nabla v \otimes \nabla u \,\mathrm{d}y - \sin^2(\sqrt{\lambda_n}s') \int_{\mathbb{R}^m} C(y) : \nabla \frac{\mathcal{T}v}{\sqrt{\lambda_n}} \otimes \nabla \frac{\mathcal{T}u}{\sqrt{\lambda_n}} \,\mathrm{d}y \\
&+ \sin(\sqrt{\lambda_n}s')\cos(\sqrt{\lambda_n}s') \int_{\mathbb{R}^m} C(y) : \left(\nabla v \otimes \nabla \frac{\mathcal{T}u}{\sqrt{\lambda_n}} - \nabla \frac{\mathcal{T}v}{\sqrt{\lambda_n}} \otimes \nabla u\right) \mathrm{d}y \\
&= \cos^2(\sqrt{\lambda_n}s') \int_{\mathbb{R}^m} C(y) : \nabla v \otimes \nabla u \,\mathrm{d}y - \sin^2(\sqrt{\lambda_n}s') \int_{\mathbb{R}^m} C(y) : \nabla \frac{\mathcal{T}v}{\sqrt{\lambda_n}} \otimes \nabla \frac{\mathcal{T}u}{\sqrt{\lambda_n}} \,\mathrm{d}y.
\end{aligned}$$

Averaging over $[0, S']$ and letting $S' \to +\infty$ yield

$$\int_{\mathbb{R}^m} \lim_{S'\to+\infty} \frac{1}{S'} \int_0^{S'} G(s')C\nabla u_{2s'} \,\mathrm{d}s' \cdot \nabla v \,\mathrm{d}y = \frac{1}{2} \int_{\mathbb{R}^m} C : \left(\nabla v \otimes \nabla u - \nabla \frac{\mathcal{T}v}{\sqrt{\lambda_n}} \otimes \frac{\mathcal{T}u}{\sqrt{\lambda_n}}\right) \mathrm{d}y$$

and finally, thanks to (67), we obtain

$$\begin{aligned}
\mathfrak{n}(u,v) = \lim_{S\to+\infty} \frac{1}{S} \int_0^S (S-s) &\left\{ \left[\cos^2(\sqrt{\lambda_n}s) - \frac{1}{2}\right] \int_{\mathbb{R}^m} C : \nabla v \otimes \nabla u \,\mathrm{d}y \right. \\
&\left. - \left[\sin^2(\sqrt{\lambda_n}s) - \frac{1}{2}\right] \int_{\mathbb{R}^m} C : \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} v \otimes \nabla \frac{\mathcal{T}}{\sqrt{\lambda_n}} u \,\mathrm{d}y \right\} \mathrm{d}s = 0.
\end{aligned}$$

□

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