Reduced fluid models for self-propelled populations, interacting through alignement

Mihai BOSTAN [*], José Antonio CARRILLO [†]

(April 4, 2017)

Abstract

We perform the asymptotic analysis of kinetic models which describe the behavior of populations interacting through alignement. The asymptotic regime we are interested in, corresponds to a large alignement frequency and assumes that the alignement effects are dominated by the self propulsion and friction forces. The former hypothesis leads to a macroscopic model, while the second one impose a fixed speed in the limit, and thus a reduction of the dynamics to a sphere, in the velocity space. The analysis relies on the averaging techniques, which have been successfully used in the magnetic confinement of charged particles. Since at the limit, the particle distribution is supported on a sphere, we need to work with measures in the velocity space. As for the Euler equations, the fluid model comes by integrating the kinetic equation against the collision invariants, in the velocity space. The main difficulty is the identification of the collision invariants for the averaged alignement kernel.

Keywords: Vlasov-like equations, Measure solutions, Swarming, Cucker-Smale model, Vicsek model, Laplace-Beltrami operator, collision invariants.

AMS classification: 92D50, 82C40, 92C10.

[*]Institut de Mathématiques de Marseille, Centre de Mathématiques et Informatique, UMR 7373, 39 rue Frédéric Joliot Curie, 13453 Marseille Cedex 13 France. E-mail : `mihai.bostan@univ-amu.fr`

[†]Department of Mathematics, Imperial College London, London SW7 2AZ UK. E-mail : `carrillo@imperial.ac.uk`.

1 Introduction

?⟨Intro⟩?

The subject matter of this paper concerns the behavior of living organisms such as flocks of birds, school of fish, swarms of insects, myxobacteria ... These models include short-range repulsion, long-range attraction, self-propelling and friction forces, reorientation or alignment see [44, 34, 25, 39, 38, 2, 3, 40]. We consider self-propelled particles with Rayleigh friction [23, 22, 18, 21], and alignement, introduced through the Cucker-Smale reorientation procedure [26], see also [37, 35, 19, 20, 41] for further details. If we denote by $f = f(t, x, v) \geq 0$ the particle density in the phase space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, with $d \in \{2, 3\}$, the self-propulsion/friction mechanism is given by the term $\mathrm{div}_v\{f(\alpha - \beta|v|^2)v\}$. Notice that the balance between the self-propulsion and friction forces occurs on the velocity sphere $|v| = r := \sqrt{\alpha/\beta}$. We fixe the speed r, meaning that α and β are anytime related by the equality $\alpha = \beta r^2$. The coefficients $\alpha, \beta > 0$ can be interpreted as follows. In the absence of friction, the particles accelerate with αv, leading to a exponential growth of velocity, with frequency α. In the absence of self-propulsion, the inverse of the relative kinetic energy grows linearly, with the frequency $2\beta|v|^2$, where v is the initial velocity of the particle

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{|v|^2}{|V(s)|^2} = -\frac{|v|^2}{|V(s)|^4}2(V(s)\cdot V'(s)) = 2\beta|v|^2.$$

Each individual in the group relaxes his velocity toward the mean velocity of the neighbors, leading to the term $\nu\ \mathrm{div}_v\{f(u[f] - v)\}$, where ν is the reorientation frequency and $u[f]$ is the mean velocity

$$u[f(t)](x) = \frac{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t, x', v')h(x - x')v'\ \mathrm{d}v'\mathrm{d}x'}{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t, x', v')h(x - x')\ \mathrm{d}v'\mathrm{d}x'}.$$

The weight application h is a decreasing, radial, non negative given function that determines the interaction neighborhood around any position. Including also noise in the above kinetic model, leads to the Fokker-Planck like equation

$$\partial_t f + \mathrm{div}_x(fv) + \mathrm{div}_v\{f(\alpha - \beta|v|^2)v\} = \nu\ \mathrm{div}_v\{f(v - u[f])\} + \tau\Delta_v f = \nu\ \mathrm{div}_v\{f(v - u[f]) + \sigma\nabla_v f\} \tag{1}$$ Equ1

where $\sigma = \tau/\nu$ represents the diffusion coefficient in the velocity space. We investigate the large time and space scale regime of (1) that is, we fixe large time and space units. In this case, the equation (1) should be replaced by

$$\delta\{\partial_t f + \mathrm{div}_x(fv)\} + \mathrm{div}_v\{f(\alpha - \beta|v|^2)v\} = \nu\ \mathrm{div}_v\{f(v - u[f]) + \sigma\nabla_v f\},\ \ (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2}$$ Equ2

The choice of a large length unit leads to a local reorientation mechanism : the mean velocity $u[f]$ in (2) is now given by

$$u[f(t)](x) = \frac{\int_{\mathbb{R}^d} f(t,x,v')v' \, \mathrm{d}v'}{\int_{\mathbb{R}^d} f(t,x,v') \, \mathrm{d}v'}.$$

Notice that if $f(t,x,\cdot) = 0$, then the Fokker-Planck collision operator vanishes for any u. In this case we can define $u[f(t)] = 0$, without loss of generality. We assume that the frequencies δ, α, ν scale like

$$\frac{\alpha}{\delta} \approx \frac{1}{\varepsilon_1}, \quad \frac{\nu}{\delta} \approx \frac{1}{\varepsilon_2}$$

for some small parameters $\varepsilon_1, \varepsilon_2 > 0$ and thus the equation (2) becomes

$$\partial_t f^{\varepsilon_1,\varepsilon_2} + \mathrm{div}_x(f^{\varepsilon_1,\varepsilon_2} v) + \frac{1}{\varepsilon_1}\mathrm{div}_v\{f^{\varepsilon_1,\varepsilon_2}(\alpha - \beta|v|^2)v\} = \frac{\nu}{\varepsilon_2}\mathrm{div}_v\{f^{\varepsilon_1,\varepsilon_2}(v - u[f^{\varepsilon_1,\varepsilon_2}]) + \sigma\nabla_v f^{\varepsilon_1,\varepsilon_2}\}. \tag{3}$$

Equ3

Assume for the moment that $\varepsilon_1 \searrow 0$ and ε_2 is fixed. In this situation, the leading order term in the Fokker-Planck equation (3) corresponds to the self-propulsion/friction mechanism, and we expect that the limit density $f^{\varepsilon_2} = \lim_{\varepsilon_1 \searrow 0} f^{\varepsilon_1,\varepsilon_2}$ satisfies

$$\mathrm{div}_v\{f^{\varepsilon_2}(\alpha - \beta|v|^2)v\} = 0.$$

The previous constraint exactly says that at any time t and any position x, the velocity distribution $f^{\varepsilon_2}(t,x,\cdot)$ is a measure supported in $\{0\} \cup r\mathbb{S}^{d-1}$ cf. [10]. The particles will tend to move with asymptotic speed r. These models have been shown to produce complicated dynamics and patterns such as mills, double mills, flocks and clumps, see [31]. Assuming that all individuals move with constant speed also leads to spatial aggregation, patterns, and collective motion [27, 32, 43]. More exactly, it was shown in [10] that, by taking the limit $\varepsilon_1 \searrow 0$, the solutions $f^{\varepsilon_1,\varepsilon_2}$ of (3) converge toward the solution f^{ε_2} of

$$\partial_t f^{\varepsilon_2} + \mathrm{div}_x(f^{\varepsilon_2}\omega) + \frac{\nu}{\varepsilon_2}\mathrm{div}_\omega\left\{f^{\varepsilon_2}\left(I_d - \frac{\omega\otimes\omega}{r^2}\right)u[f^{\varepsilon_2}]\right\} = \frac{\tau}{\varepsilon_2}\Delta_\omega f^{\varepsilon_2}, \quad (t,x,\omega) \in \mathbb{R}_+ \times \mathbb{R}^d \times r\mathbb{S}^{d-1} \tag{4}$$

Equ4

with

$$u[f^{\varepsilon_2}(t)](x) = \frac{\int_{r\mathbb{S}^{d-1}} f^{\varepsilon_2}(t,x,\omega)\omega \, \mathrm{d}\omega}{\int_{r\mathbb{S}^{d-1}} f^{\varepsilon_2}(t,x,\omega) \, \mathrm{d}\omega}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

The above result states that in the limit $\varepsilon_1 \searrow 0$, the Cucker-Smale model with diffusion is reduced to a Vicsek like model. The evolution problem (4) on the phase space $\mathbb{R}^d \times r\mathbb{S}^{d-1}$, with normalized field $u[f^{\varepsilon_2}]$ *i.e.*,

$$\partial_t f + \mathrm{div}_x(f\omega) + \nu\,\mathrm{div}_\omega\left\{f\left(I_d - \frac{\omega\otimes\omega}{r^2}\right)\Omega[f]\right\} = \tau\Delta_\omega f, \quad (t,x,\omega) \in \mathbb{R}_+ \times \mathbb{R}^d \times r\mathbb{S}^{d-1} \tag{5}$$

Equ5

$$\Omega[f(t)](x) = \frac{\int_{r\mathbb{S}^{d-1}} f(t,x,\omega)\omega \, d\omega}{|\int_{r\mathbb{S}^{d-1}} f(t,x,\omega)\omega \, d\omega|}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

was also proposed in the literature as continuum version [29] of the Vicsek model [44, 25]. Our case, based on the relaxation toward the mean velocity $u[f]$, is not included in the model (5), whose alignement mechanism relies only on the direction of the mean velocity $\Omega[f] = u[f]/|u[f]|$. Nevertheless, our method still applies and allows us to handle the model with normalization as well.

The original model in [44, 24] was derived as the mean-field limit of some stochastic particle systems in [5]. In fact, previous particle systems have also been studied with noise in [4] for the mean-field limit (see also [42, 16, 30, 17], in [36] for studying some properties of the Cucker-Smale model with noise, and in [28, 33] for analyzing the phase transition in the Vicsek model.

We assume now that both $\varepsilon_1, \varepsilon_2$ become small. The idea is to justify a macroscopic model for (4), resulting from the balance between two opposite phenomena

1. The reorientation, which tends to align the particle velocities with respect to the mean velocity ;
2. The diffusion, which tends to spread the particle velocities isotropically on the sphere $r\mathbb{S}^{d-1}$.

Such hydrodynamic model was obtained in [29], by letting $\varepsilon_2 \searrow 0$ in the normalized alignement version of (4). Notice that this macroscopic model was obtained by passing to the limit successively in (3) with respect to $\varepsilon_1, \varepsilon_2$. After letting $\varepsilon_1 \searrow 0$, the dynamics was reduced to the phase space $(x,v) \in \mathbb{R}^d \times r\mathbb{S}^{d-1}$, but still captures microscopic behavior in the tangent directions to the sphere $r\mathbb{S}^{d-1}$. The second limit procedure, $\varepsilon_2 \searrow 0$, leads to the macroscopic equations for the density $\int_{r\mathbb{S}^{d-1}} f \, d\omega$ and the direction of the flux $\int_{r\mathbb{S}^{d-1}} \omega f \, d\omega$.

We intend to obtain a macroscopic model, by passing to the limit in (3), simultaneously with respect to $(\varepsilon_1, \varepsilon_2)$. Motivated by the above discussion, we assume that $\varepsilon_1 = \varepsilon^2$ and $\varepsilon_2 = \varepsilon$, where $\varepsilon > 0$ is a small parameter, that is, the self-propulsion/friction mechanism dominates the alignement. Therefore (3) becomes

$$\partial_t f^\varepsilon + \mathrm{div}_x(f^\varepsilon v) + \frac{\mathrm{div}_v\{f^\varepsilon(\alpha - \beta|v|^2)v\}}{\varepsilon^2} = \frac{\nu}{\varepsilon}\mathrm{div}_v\left\{f^\varepsilon(v - u[f^\varepsilon]) + \sigma\nabla_v f^\varepsilon\right\}, (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^{2d} \quad (6)$$

Equ6

supplemented by the initial condition

$$f^\varepsilon(0,x,v) = f^{\mathrm{in}}(x,v), \quad (x,v) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Very recently, by a similar scaling, fluid models have been obtained for the transport of charged particles, under the action of strong magnetic fields, which dominate the collision effects. The resulting macroscopic model is a gyrokinetic version of the Euler equations, in the parallel direction with respect to the magnetic field [13, 15].

The behavior of the family $(f^\varepsilon)_{\varepsilon>0}$, as the parameter ε becomes small, follows by analyzing the formal expansion

$$f^\varepsilon = f + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \ldots \tag{7}$$

FormalExpansio

Plugging the above Ansatz into (6), leads to the constraints

$$\mathrm{div}_v\{f(\alpha - \beta|v|^2)v\} = 0 \tag{8}$$

Equ8

$$\mathrm{div}_v\{f^{(1)}(\alpha - \beta|v|^2)v\} = \nu\ \mathrm{div}_v\{f(v - u[f]) + \sigma\nabla_v f\} \tag{9}$$

Equ9

and to the time evolution equations

$$\begin{aligned}\partial_t f + \mathrm{div}_x(fv) + \mathrm{div}_v\{f^{(2)}(\alpha - \beta|v|^2)v\} &= \nu\ \mathrm{div}_v\{f^{(1)}(v - u[f]) + \sigma\nabla_v f^{(1)}\} \\ &\quad - \nu\ \mathrm{div}_v\left\{ f\ \frac{\int_{\mathbb{R}^d} f^{(1)}(v' - u[f])\ \mathrm{d}v'}{\int_{\mathbb{R}^d} f\ \mathrm{d}v'}\right\}\end{aligned} \tag{10}$$

Equ10

$$\vdots$$

We expect a macroscopic model for the moments of f, similar to that obtained in [29]. The main advantage for considering (6) instead of (4) with $\varepsilon_2 = \varepsilon$ is that the resolution of (6) for small ε will provide a solution supported near $\mathbb{R}^d \times r\mathbb{S}^{d-1}$, which fits much better the behavior of living organism systems, than the solution of (4) on $\mathbb{R}^d \times r\mathbb{S}^{d-1}$. But the prize to be paid is to deal with two Lagrange multipliers, appearing in (10), which have to be eliminated, thanks to the constraints (8) and (9). The first constraint was analyzed in detail in [10]. It exactly says that f is a measure supported in $\mathbb{R}^d \times (\{0\} \cup r\mathbb{S}^{d-1})$. We denote by $\mathcal{M}_b^+(\mathbb{R}^d)$ the set of non negative bounded Radon measure on $\mathbb{R}^d$.

⟨FirstConstraint⟩

Proposition 1.1 *Assume that* $(1+|v|^2)F \in \mathcal{M}_b^+(\mathbb{R}^d)$. *Then* F *solves* $\mathrm{div}_v\{F(\alpha - \beta|v|^2)v\} = 0$ *in* $\mathcal{D}'(\mathbb{R}^d)$ *i.e.,*

$$\int_{\mathbb{R}^d} (\alpha - \beta|v|^2)v \cdot \nabla_v\varphi\ \mathrm{d}F(v) = 0, \ \ \textit{for any } \varphi \in C_c^1(\mathbb{R}^d)$$

if and only if $\mathrm{supp}\ F \subset \{0\} \cup r\mathbb{S}^{d-1}$.

The proof of Proposition 1.1 is based on the resolution of the adjoint problem

$$-(\alpha - \beta|v|^2)v \cdot \nabla_v\varphi = \psi(v), \ \ v \in \mathbb{R}^d$$

for any smooth function ψ with compact support in $\complement(\{0\} \cup r\mathbb{S}^{d-1})$, cf. Lemma 3.1 of [10].

⟨AdjointProblem⟩

Lemma 1.1 *For any C^1 function $\psi = \psi(v)$ with compact support in $\complement(\{0\} \cup r\mathbb{S}^{d-1})$, there is a bounded C^1 function $\varphi = \varphi(v)$ such that $\varphi(0) = 0$ and*

$$-(\alpha - \beta|v|^2)v \cdot \nabla_v \varphi = \psi(v), \quad v \in \mathbb{R}^d.$$

In the sequel, we introduce a projection operator onto the subspace of the constraints in (8). This construction follows closely the gyro-average method in gyro-kinetic theory [6, 7, 8, 9, 11, 12, 14]. An average operator serves to separate between two scales. For example, in gyro-kinetic theory, two time scales exist : a fast time variable, related to the rapid cyclotronic motion, and a slow time variable, related to the parallel motion with respect to the magnetic field. The gyro-average operator represents the average of the fast dynamics over a cyclotronic period, provided that the slow time variable is frozen. Following this technique, we obtain an accurate enough but simpler model, from the numerical approximation point of view. All the fluctuations have been removed and replaced by averaged effects.

Our model (6) presents not two, but three time variables : $t, t/\varepsilon$ and t/ε^2. The dynamics is dominated by the self-propulsion/friction mechanism, introducing the fast time variable $s = t/\varepsilon^2$. The average operator is related to the characteristic flow of the field $\frac{1}{\varepsilon^2}(\alpha - \beta|v|^2)v \cdot \nabla_v$. This characteristic flow $\mathcal{V} = \mathcal{V}(s; v)$, written with respect to $s = t/\varepsilon^2$

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}s} = (\alpha - \beta\,|\mathcal{V}(s; v)|^2)\mathcal{V}(s; v), \quad \mathcal{V}(0; v) = v$$

conserves the direction $\frac{v}{|v|}$ and has as equilibria the elements of $\{0\} \cup r\mathbb{S}^{d-1}$. The Jacobian matrix

$$\partial_v\{(\alpha - \beta|v|^2)v\} = (\alpha - \beta|v|^2)I_d - 2\beta v \otimes v$$

being negative on $r\mathbb{S}^{d-1}$ and definite positive at 0, we deduce that the points of $r\mathbb{S}^{d-1}$ are stable equilibria, and 0 is an unstable equilibrium. For simplicity we neglect the measure of the unstable point 0 in the velocity space. More generally, some results of this work have been rigorously justified, but some other have been discussed only at the formal level. The complete mathematical analysis of these models is out of scope of this paper. We are mainly interested in the two or three dimensional setting, but the same arguments apply for any dimension $d \geq 2$. For the sake of generality, we state and prove all the results in any dimension $d \geq 2$, and we distinguish, if necessary, between the cases $d = 2$ and $d \geq 3$.

Motivated by the previous observations, we define the average of a non negative bounded measure cf. [10]. We will denote by $f(x, v)\ \mathrm{d}v\mathrm{d}x$ the integration against the measure f.

This is done independently of being the measure f absolutely continuous with respect to the Lebesgue measure or not.

Definition 1.1

1. Let $F \in \mathcal{M}_b^+(\mathbb{R}^d)$ be a non negative bounded measure on $\mathbb{R}^d$. We denote by $\langle F \rangle$ the measure corresponding to the linear application

$$\psi \to \int_{\mathbb{R}^d} \psi(v)\, \mathbf{1}_{v=0} F(v)\,\mathrm{d}v + \int_{\mathbb{R}^d} \psi\left(r\frac{v}{|v|}\right) \mathbf{1}_{v\neq 0} F(v)\,\mathrm{d}v\,,$$

for all $\psi \in C_c^0(\mathbb{R}^d)$, i.e.,

$$\int_{\mathbb{R}^d} \psi(v)\,\langle F \rangle(v)\,\mathrm{d}v = \int_{v=0} \psi(v) F(v)\,\mathrm{d}v + \int_{v\neq 0} \psi\left(r\frac{v}{|v|}\right) F(v)\,\mathrm{d}v\,,$$

for all $\psi \in C_c^0(\mathbb{R}^d)$.

2. Let $f \in \mathcal{M}_b^+(\mathbb{R}^d \times \mathbb{R}^d)$ be a non negative bounded measure on $\mathbb{R}^d \times \mathbb{R}^d$. We denote by $\langle f \rangle$ the measure corresponding to the linear application

$$\psi \to \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x,v)\, \mathbf{1}_{v=0} f(x,v)\,\mathrm{d}v\mathrm{d}x + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi\left(x, r\frac{v}{|v|}\right) \mathbf{1}_{v\neq 0} f(x,v)\,\mathrm{d}v\mathrm{d}x\,,$$

for all $\psi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^d)$, i.e.,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x,v)\,\langle f \rangle(x,v)\,\mathrm{d}v\mathrm{d}x = \int_{v=0} \psi(x,v) f(x,v)\,\mathrm{d}v\mathrm{d}x + \int_{v\neq 0} \psi\left(x, r\frac{v}{|v|}\right) f(x,v)\,\mathrm{d}v\mathrm{d}x,$$

for all $\psi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^d)$.

It is easily seen that the average of a non negative bounded measure is a non negative bounded measure, with the same mass, but supported in $\{0\} \cup r\mathbb{S}^{d-1}$, $\mathbb{R}^d \times (\{0\} \cup r\mathbb{S}^{d-1})$ respectively. We have the following characterization (see Proposition 5.1 [10]).

⟨VarChar⟩

Proposition 1.2 *Assume that f is a non negative bounded measure on $\mathbb{R}^d \times \mathbb{R}^d$. Then $\langle f \rangle$ is the unique measure $\tilde{f}$ satisfying* $\operatorname{supp} \tilde{f} \subset \mathbb{R}^d \times (\{0\} \cup r\mathbb{S}^{d-1})$,

$$\int_{v\neq 0} \psi\left(x, r\frac{v}{|v|}\right) \tilde{f}(x,v)\,\mathrm{d}v\mathrm{d}x = \int_{v\neq 0} \psi\left(x, r\frac{v}{|v|}\right) f(x,v)\,\mathrm{d}v\mathrm{d}x, \quad \psi \in C_c^0(\mathbb{R}^d \times \mathbb{R}^d)$$

and $\tilde{f} = f$ on $\mathbb{R}^d \times \{0\}$.

A direct consequence of Proposition 1.2 is that any bounded, non negative measure, supported in $\mathbb{R}^d \times (\{0\} \cup r\mathbb{S}^{d-1})$ is left unchanged by the average operator. Another property of the average operator is that it removes any measure of the form $\mathrm{div}_v\{f(\alpha - \beta|v|^2)v\}$, cf. Proposition 5.2 [10].

⟨Elimination⟩ **Proposition 1.3** *For any* $f \in \mathcal{M}_b^+(\mathbb{R}^d \times \mathbb{R}^d)$ *such that* $\mathrm{div}_v\{f(\alpha - \beta|v|^2)v\} \in \mathcal{M}_b(\mathbb{R}^d \times \mathbb{R}^d)$, *we have* $\left\langle \mathrm{div}_v\{f(\alpha - \beta|v|^2)v\}\right\rangle = 0$.

The above proposition plays a crucial role when eliminating the Lagrange multiplier $f^{(2)}$ in (10). Indeed, for doing that, it is enough to average both hand sides in (10). By the constraint (8), we know that f is supported in $\mathbb{R}^d \times (\{0\} \cup r\mathbb{S}^{d-1})$, and thus is left invariant by the average. We check that $\langle \partial_t f\rangle = \partial_t \langle f\rangle = \partial_t f$, and thus, averaging (10) still leads to a evolution problem for f

$$\partial_t f + \langle \mathrm{div}_x(fv)\rangle = \left\langle \nu \mathrm{div}_v \left\{ f^{(1)}(v - u[f]) + \sigma \nabla_v f^{(1)}\right\}\right\rangle - \left\langle \nu \mathrm{div}_v \left\{ f \frac{\int_{\mathbb{R}^d} f^{(1)}(v' - u[f])\,\mathrm{d}v'}{\int_{\mathbb{R}^d} f\,\mathrm{d}v'}\right\}\right\rangle. \tag{11}$$ Equ12

Certainly, a much more difficult task is to eliminate the Lagrange multiplier $f^{(1)}$. We expect that this can be done thanks to the constraint in (9). The solvability of (9), with respect to $f^{(1)}$, depends on a compatibility condition, to be satisfied by the right hand side. Indeed, by Proposition 1.3, we should have

$$\langle \nu\, \mathrm{div}_v\{f(v - u[f]) + \sigma \nabla_v f\}\rangle = \left\langle \mathrm{div}_v\{f^{(1)}(\alpha - \beta|v|^2)v\}\right\rangle = 0 \tag{12}$$ Equ14

saying that f is a equilibrium for the average collision kernel

$$\langle Q(f)\rangle = 0,\;\; Q(f) = \nu\, \mathrm{div}_v\{f(v - u[f]) + \sigma \nabla_v f\}.$$

The equilibria of the average collision kernel form a d-dimensional manifold, that is one dimension less than the equilibria manifold of the Fokker-Planck operatot Q (see also [29, 33]). For any $l \in \mathbb{R}_+, \Omega \in \mathbb{S}^{d-1}$, we introduce the von Mises-Fisher distribution

$$M_{l\Omega}(\omega)\,\mathrm{d}\omega = \frac{\exp\left(l\Omega \cdot \frac{\omega}{r}\right)}{\int_{r\mathbb{S}^{d-1}} \exp\left(l\Omega \cdot \frac{\omega'}{r}\right)\,\mathrm{d}\omega'}\,\mathrm{d}\omega,\;\; \omega \in r\mathbb{S}^{d-1}.$$

⟨AveEqui⟩ **Proposition 1.4** *Let* $F \in \mathcal{M}_b^+(\mathbb{R}^d)$ *be a non negative bounded measure on* $\mathbb{R}^d$*, supported in* $r\mathbb{S}^{d-1}$*. The following statements are equivalent.*

1. $\langle Q(F)\rangle = 0$*, that is*

$$\int_{v\neq 0} \left\{ -(v - u[F]) \cdot \nabla_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] + \sigma \Delta_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right\} F\,\mathrm{d}v = 0 \;\; \textit{for any } \widetilde{\psi} \in C^2(r\mathbb{S}^{d-1}).$$

2. There are $\rho \in \mathbb{R}_+, \Omega \in \mathbb{S}^{d-1}$ *such that* $F = \rho M_{l\Omega}\mathrm{d}\omega$ *where* $l \in \mathbb{R}_+$ *satisfies*

$$\frac{\int_0^\pi \cos\theta\, e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta} = \frac{\sigma}{r^2}l.$$

The modulus of the mean velocity is not a coordinate on the equilibria manifold, but it is determined by the condition

$$|u| = \frac{\sigma l}{r}, \quad \frac{\int_0^\pi \cos\theta \, e^{l\cos\theta} \sin^{d-2}\theta \, \mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta} \sin^{d-2}\theta \, \mathrm{d}\theta} = \frac{\sigma}{r^2} l.$$

Clearly $l = 0$ is a solution, which corresponds to the isotropic equilibrium

$$F = \rho M_{0\Omega} \, \mathrm{d}\omega = \rho \frac{\mathrm{d}\omega}{\omega_d r^{d-1}}$$

where ω_d represents the area of the unit sphere in $\mathbb{R}^d$. We prove that, if the diffusion is large enough, then the only equilibrium is the isotropic one.

tropicEquilibrium⟩

Proposition 1.5 *Let $\lambda : \mathbb{R}_+ \to \mathbb{R}$ be the function given by*

$$\lambda(l) = \frac{\int_0^\pi \cos\theta e^{l\cos\theta} \sin^{d-2}\theta \, \mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta} \sin^{d-2}\theta \, \mathrm{d}\theta}, \quad l \in \mathbb{R}_+, \quad d \geq 2.$$

The function λ is strictly increasing, strictly concave and verifies

$$\lambda(0) = 0, \quad \lambda'(0) = \frac{1}{d}, \quad \lim_{l \to +\infty} \lambda(l) = 1.$$

If $\frac{\sigma}{r^2} \geq \frac{1}{d}$, then the only solution of $\lambda(l) = \frac{\sigma}{r^2} l$ is $l = 0$. If $\frac{\sigma}{r^2} \in]0, \frac{1}{d}[$, then there is a unique $l = l\left(\frac{\sigma}{r^2}\right) > 0$ such that $\lambda(l) = \frac{\sigma}{r^2} l$.

Our main result establishes the macroscopic equations satisfied by the density ρ and orientation Ω, which parametrize the von Mises-Fisher equilibrium, obtained when passing to the limit for $\varepsilon \searrow 0$ in (6). We retrieve the limit model in [29], written for any space dimension $d \geq 2$.

⟨MainResult1⟩

Theorem 1.1 *For any σ, r such that $\frac{\sigma}{r^2} \in]0, \frac{1}{d}[$, we denote by $l = l\left(\frac{\sigma}{r^2}\right)$ the unique positive solution of $\lambda(l) = \frac{\sigma}{r^2} l$. Let $f^{\text{in}} \in \mathcal{M}_b^+(\mathbb{R}^d \times \mathbb{R}^d)$ be a non negative bounded measure on $\mathbb{R}^d \times \mathbb{R}^d, d \geq 2$. For any $\varepsilon > 0$ we consider the problem*

$$\partial_t f^\varepsilon + \mathrm{div}_x(f^\varepsilon v) + \frac{1}{\varepsilon^2}\mathrm{div}_v(f^\varepsilon(\alpha - \beta|v|^2)v) = \frac{\nu}{\varepsilon}\mathrm{div}_v[f^\varepsilon(v - u[f^\varepsilon]) + \sigma \nabla_v f^\varepsilon], \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \tag{13}$$

Equ71

$$f^\varepsilon(0) = f^{\text{in}}, \quad (x,v) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Therefore the limit distribution $f = \lim_{\varepsilon \searrow 0} f^\varepsilon$, is a von Mises-Fisher equilibrium $f = \rho M_{l\Omega}(\omega) \, \mathrm{d}\omega$ on $r\mathbb{S}^{d-1}$, where the density $\rho(t,x)$ and the orientation $\Omega(t,x)$ satisfy the macroscopic equations

$$\partial_t \rho + \mathrm{div}_x\left(\rho \frac{l\sigma}{r}\Omega\right) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \tag{14}$$

Equ73

$$\partial_t \Omega + k_d\, r(\Omega\cdot\nabla_x)\Omega + \frac{r}{l}(I_d - \Omega\otimes\Omega)\frac{\nabla_x \rho}{\rho} = 0 \tag{15}$$ Equ74

with the initial conditions

$$\rho(0,x) = \int_{\mathbb{R}^d} f^{\mathrm{in}}(x)\,\mathrm{d}v,\ \ \Omega(0,x) = \frac{\int_{\mathbb{R}^d} v f^{\mathrm{in}}(x)\,\mathrm{d}v}{\left|\int_{\mathbb{R}^d} v f^{\mathrm{in}}(x)\,\mathrm{d}v\right|},\ \ x\in\mathbb{R}^d$$

where

$$k_d = \frac{\int_0^\pi e^{l\cos\theta}\chi(\cos\theta)\cos\theta\sin^{d-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}$$

and χ *solves*

$$-\frac{\sigma}{r^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{e^{lc}\chi'(c)(1-c^2)^{\frac{1}{2}}\right\} = re^{lc},\ \ c\in]-1,1[,\ \ \chi(-1)=\chi(1)=0\ \ \textit{if } d=2$$

and

$$-\frac{\sigma}{r^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{e^{lc}\chi'(c)(1-c^2)^{\frac{d-1}{2}}\right\}+(d-2)\frac{\sigma}{r^2}e^{lc}\chi(c)(1-c^2)^{\frac{d-5}{2}} = re^{lc}(1-c^2)^{\frac{d-2}{2}},\ c\in]-1,1[,\ d\geq 3.$$

Our article is organized as follows. In Section 2 we study the equilibria of the average collision operator. This analysis can be carried out by introducing some Bessel functions. In the next section we investigate the notion of collision invariant. We determine the structure of these invariants and present theirs symmetries. Section 4 is devoted to the derivation of the fluid model for the macroscopic quantities, parametrizing the limit von Mises-Fisher equilibrium. The proofs of some technical results can be found in Appendix.

2 The equilibria of the average collision operator

⟨AveColOpe⟩ We consider the collision operator $Q(F) = \nu\mathrm{div}_v\{F(v-u[F]) + \sigma\nabla_v F\}$ where $u[F] = \int_{\mathbb{R}^d} vF\,\mathrm{d}v / \int_{\mathbb{R}^d} F\,\mathrm{d}v$ is the mean velocity. The above operator should be understood in the duality sense between non negative bounded measures on $\mathbb{R}^d$ and smooth functions, compactly supported in $\mathbb{R}^d$

$$\int_{\mathbb{R}^d}\psi(v)Q(F)\,\mathrm{d}v = \nu\int_{\mathbb{R}^d}[-(v-u[F])\cdot\nabla_v\psi(v) + \sigma\Delta_v\psi(v)]F\,\mathrm{d}v$$

for any $F\in\mathcal{M}_b^+(\mathbb{R}^d)$ and $\psi\in C_c^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}|v|F\,\mathrm{d}v < +\infty$. As suggested by the formal expansion (7), we focus on measures satisfying (see (8), (9))

$$\mathrm{div}_v\{F(\alpha-\beta|v|^2)v\} = 0,\ \ Q(F) = \mathrm{div}_v\{F^{(1)}(\alpha-\beta|v|^2)v\}.$$

Thanks to Propositions 1.3, 1.1, we deduce that $\mathrm{supp}\,F\subset\{0\}\cup r\mathbb{S}^{d-1}$ and

$$\langle Q(F)\rangle = \left\langle \mathrm{div}_v\{F^{(1)}(\alpha-\beta|v|^2)v\}\right\rangle = 0.$$

We discuss the case of non negative bounded measures supported on the sphere $r\mathbb{S}^{d-1}$, that is, we discard all difficulties related to the mass of the points at rest. For such measures, the equality $\langle Q(F)\rangle = 0$ can be interpreted in the following sense (see Proposition 1.2)

$$\int_{v\neq 0}\left\{-(v-u[F])\cdot\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]+\sigma\Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right\}=0,\ \text{ for any } \widetilde{\psi}\in C^2(r\mathbb{S}^{d-1}).$$

The complete description of the above equilibria of the average collision operator Q, called the von Mises-Fisher distributions, is given by Proposition 1.4, whose proof is detailed below. We start with the following easy integration by parts formula on spheres. The proof is postponed to Appendix A.

⟨IntByPartsSphere⟩

Lemma 2.1 *Assume that $A = A(v)$ is a C^1 vector field in $\mathcal{O} = \{v \in \mathbb{R}^d\ :\ r_1 < |v| < r_2\}$. Then for any $t \in]r_1, r_2[$ we have*

$$\int_{|\omega|=t}(\mathrm{div}_v A)(\omega)\,\mathrm{d}\omega = \int_{|\omega|=t}\left\{\frac{\omega\otimes\omega}{t^2} : \partial_v A(\omega) + \frac{(d-1)\omega}{t^2}\cdot A(\omega)\right\}\,\mathrm{d}\omega. \tag{16}$$ Ident0

In particular, if $A(v)\cdot v = 0, v \in \mathcal{O}$, then

$$\int_{|\omega|=t}(\mathrm{div}_v A)(\omega)\,\mathrm{d}\omega = 0,\quad t\in]r_1, r_2[\tag{17}$$ Ident1

and for any function $\chi \in C^1(\mathcal{O})$ we have

$$\int_{|\omega|=t}\nabla_v\chi(\omega)\cdot A(\omega)\,\mathrm{d}\omega + \int_{|\omega|=t}\chi(\omega)(\mathrm{div}_v A)(\omega)\,\mathrm{d}\omega = 0,\quad t\in]r_1, r_2[. \tag{18}$$ Ident2

It is very convenient to express the differential operators $\nabla_\omega, \mathrm{div}_\omega$ of functions and vector fields on the sphere $r\mathbb{S}^{d-1}$ in terms of the differential operators ∇_v, div_v applied to extensions of functions and vector fields on a neighborhood of $r\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. The notation $\widetilde{\cdot}$ stands for the restriction on the sphere $r\mathbb{S}^{d-1}$. The proof of the following lemma is detalied in Appendix B.

⟨Extension⟩

Lemma 2.2

1. Let $\psi = \psi(v)$ be a C^1 function in a open set of $\mathbb{R}^d$, containing $r\mathbb{S}^{d-1}$. Then, for any $\omega \in r\mathbb{S}^{d-1}$ we have

$$\nabla_\omega\widetilde{\psi}(\omega) = \left(I_d - \frac{\omega\otimes\omega}{r^2}\right)\widetilde{\nabla_v\psi}(\omega).$$

2. Let $\widetilde{\psi} = \widetilde{\psi}(\omega)$ be a C^1 function on $r\mathbb{S}^{d-1}$ and $\psi : \mathcal{O} = \{v \in \mathbb{R}^d : r_1 < |v| < r_2\} \to \mathbb{R}$ be the function defined by $\psi(v) = \widetilde{\psi}\left(r\frac{v}{|v|}\right), v \in \mathcal{O}$, with $0 < r_1 < r < r_2 < +\infty$. Then, for any $t \in]r_1, r_2[$, we have

$$(\nabla_v\psi)(\omega_t) = (\nabla_{\omega_t}\psi)(\omega_t) = \frac{r}{t}(\nabla_\omega\widetilde{\psi})\left(r\frac{\omega_t}{t}\right),\quad |\omega_t| = t.$$

3. Let $\widetilde{\xi} = \widetilde{\xi}(\omega)$ be a C^1 vector field on $r\mathbb{S}^{d-1}$ and $\xi = \xi(v)$ a C^1 extension of $\widetilde{\xi}$ in the set $\mathcal{O} = \{v \in \mathbb{R}^d : r_1 < |v| < r_2\}$ such that $\xi(v) \cdot v = 0$ for any $v \in \mathcal{O}$. Then we have

$$(\mathrm{div}_\omega \widetilde{\xi})(\omega) = (\widetilde{\mathrm{div}_v \xi})(\omega), \;\; \omega \in r\mathbb{S}^{d-1}.$$

⟨Complement⟩

Remark 2.1 *Consider $\widetilde{\xi} = \widetilde{\xi}(\omega)$ a C^1 vector field on $r\mathbb{S}^{d-1}$ and $\xi(v) = \widetilde{\xi}\left(r\frac{v}{|v|}\right), v \in \mathbb{R}^d \setminus \{0\}$. We have $\xi(v) \cdot v = 0, v \in \mathbb{R}^d \setminus \{0\}$, and for any $t > 0$*

$$(\mathrm{div}_v \xi)(\omega_t) = (\mathrm{div}_{\omega_t} \xi)(\omega_t) = \frac{r}{t}(\mathrm{div}_\omega \widetilde{\xi})\left(r\frac{\omega_t}{t}\right), \;\; \omega_t \in t\mathbb{S}^{d-1}.$$

The first equality comes by the third statement of Lemma 2.2. In oder to check the second equality, pick a C^1 function ψ_t on $t\mathbb{S}^{d-1}$ and consider the function $\widetilde{\psi}(\omega) = \psi_t(t\omega/r), \omega \in r\mathbb{S}^{d-1}$. We have

$$\nabla_\omega \widetilde{\psi}(\omega) = \frac{t}{r}(\nabla_{\omega_t}\psi_t)\left(t\frac{\omega}{r}\right)$$

and thus

$$\begin{aligned}
-\int_{|\omega_t|=t} (\mathrm{div}_{\omega_t}\xi)(\omega_t)\psi_t(\omega_t)\,\mathrm{d}\omega_t &= \int_{|\omega_t|=t} \xi(\omega_t)\cdot\nabla_{\omega_t}\psi_t(\omega_t)\,\mathrm{d}\omega_t \\
&= \int_{|\omega|=r} \xi\left(t\frac{\omega}{r}\right)\cdot(\nabla_{\omega_t}\psi_t)\left(t\frac{\omega}{r}\right)\left(\frac{t}{r}\right)^{d-1}\,\mathrm{d}\omega \\
&= \int_{|\omega|=r} \widetilde{\xi}(\omega)\cdot\nabla_\omega\widetilde{\psi}(\omega)\left(\frac{t}{r}\right)^{d-2}\,\mathrm{d}\omega \\
&= -\int_{|\omega|=r} (\mathrm{div}_\omega\widetilde{\xi})(\omega)\widetilde{\psi}(\omega)\left(\frac{t}{r}\right)^{d-2}\,\mathrm{d}\omega \\
&= -\int_{|\omega_t|=t} \frac{r}{t}(\mathrm{div}_\omega\widetilde{\xi})\left(r\frac{\omega_t}{t}\right)\psi_t(\omega_t)\,\mathrm{d}\omega_t.
\end{aligned}$$

We deduce that $(\mathrm{div}_{\omega_t}\xi)(\omega_t) = \frac{r}{t}(\mathrm{div}_\omega\widetilde{\xi})(r\omega_t/t)$ for any $\omega_t \in t\mathbb{S}^{d-1}$.

Before giving the proof of Proposition 1.4, we indicate a formula which will be used several times in our computations. For any continuous function $G : [-r, r] \to \mathbb{R}$, $d \geq 2, \Omega \in \mathbb{S}^{d-1}$, we have

$$\int_{r\mathbb{S}^{d-1}} G(\omega\cdot\Omega)\,\mathrm{d}\omega = \int_0^\pi G(r\cos\theta)\sin^{d-2}\theta\,\mathrm{d}\theta\; r^{d-1}\omega_{d-1}$$

with $\omega_1 = 2$. In particular, for any continuous function $g : [-r, r] \to \mathbb{R}$, we have

$$\int_{r\mathbb{S}^{d-1}} g(\omega\cdot\Omega)M_{l\Omega}(\omega)\,\mathrm{d}\omega = \frac{\int_{r\mathbb{S}^{d-1}} g(\omega\cdot\Omega)\exp\left(l\Omega\cdot\frac{\omega}{r}\right)\,\mathrm{d}\omega}{\int_{r\mathbb{S}^{d-1}}\exp\left(l\Omega\cdot\frac{\omega}{r}\right)\,\mathrm{d}\omega} = \frac{\int_0^\pi g(r\cos\theta)e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}. \tag{19}$$

ForMag

Proof. (of Proposition 1.4)

1. $\Longrightarrow$ 2. We assume that F is a equilibrium for the average collision kernel. We claim

that $\int_{\mathbb{R}^d}\varphi(v)F\ \mathrm{d}v = 0$ for any smooth function φ satisfying $\int_{r\mathbb{S}^{d-1}}\varphi(\omega)M(\omega)\ \mathrm{d}\omega = 0$, with $M(v) = \exp\left(-\frac{|v-u[F]\ |^2}{2\sigma}\right), v \in \mathbb{R}^d$. The idea is to solve the problem

$$-\mathrm{div}_\omega(M(\omega)\nabla_\omega\widetilde{\psi}) = M(\omega)\widetilde{\varphi}(\omega),\ \ \omega \in r\mathbb{S}^{d-1} \tag{20}$$ Equ14

where $\widetilde{\varphi}$ is the restriction on $r\mathbb{S}^{d-1}$ of φ and the notations $\mathrm{div}_\omega, \nabla_\omega$ stand for the divergence and gradient operators along the sphere $r\mathbb{S}^{d-1}$. Notice that we have

$$\int_{r\mathbb{S}^{d-1}}\widetilde{\varphi}(\omega)M(\omega)\ \mathrm{d}\omega = \int_{r\mathbb{S}^{d-1}}\varphi(\omega)M(\omega)\ \mathrm{d}\omega = 0.$$

We introduce the Hilbert spaces

$$L^2(r\mathbb{S}^{d-1}) = \{\chi : r\mathbb{S}^{d-1} \to \mathbb{R},\ \int_{r\mathbb{S}^{d-1}}(\chi(\omega))^2M(\omega)\ \mathrm{d}\omega < +\infty\}$$

$$H^1(r\mathbb{S}^{d-1}) = \{\chi : r\mathbb{S}^{d-1} \to \mathbb{R},\ \int_{r\mathbb{S}^{d-1}}\{(\chi(\omega))^2 + |\nabla_\omega\chi|^2\}M(\omega)\ \mathrm{d}\omega < +\infty\}$$

endowed with the scalar products

$$(\chi,\theta)_r = \int_{r\mathbb{S}^{d-1}}\chi(\omega)\theta(\omega)M(\omega)\ \mathrm{d}\omega,\ \ \chi,\theta \in L^2(r\mathbb{S}^{d-1})$$

$$((\chi,\theta))_r = \int_{r\mathbb{S}^{d-1}}[\chi(\omega)\theta(\omega) + \nabla_\omega\chi\cdot\nabla_\omega\theta]\ M(\omega)\ \mathrm{d}\omega,\ \ \chi,\theta \in H^1(r\mathbb{S}^{d-1}).$$

We denote by $|\cdot|_r, \|\cdot\|_r$ the norm induced by the above scalar products. There is a constant C_r such that the following Poincaré inequality holds true

$$|\chi|_r^2 = \int_{r\mathbb{S}^{d-1}}(\chi(\omega))^2M(\omega)\ \mathrm{d}\omega \le C_r\int_{r\mathbb{S}^{d-1}}|\nabla_\omega\chi|^2M(\omega)\ \mathrm{d}\omega = C_r|\nabla_\omega\chi|_r^2$$

for any $\chi \in H^1(r\mathbb{S}^{d-1})$ satisfying $\int_{r\mathbb{S}^{d-1}}\chi(\omega)M(\omega)\ \mathrm{d}\omega = 0$. The previous inequality guarantees that the application $\chi \to |\nabla_\omega\chi|_r$ is a norm equivalent to $\|\cdot\|_r$ on

$$\tilde{H}^1(r\mathbb{S}^{d-1}) := H^1(r\mathbb{S}^{d-1}) \cap \{\theta \in L^2(r\mathbb{S}^{d-1})\ :\ \int_{r\mathbb{S}^{d-1}}\theta(\omega)M(\omega)\ \mathrm{d}\omega = 0\}.$$

Therefore, the bilinear form

$$(\chi,\theta) \in \tilde{H}^1(r\mathbb{S}^{d-1}) \times \tilde{H}^1(r\mathbb{S}^{d-1}) \to \int_{r\mathbb{S}^{d-1}}\nabla_\omega\chi\cdot\nabla_\omega\theta\ M(\omega)\ \mathrm{d}\omega$$

is symmetric, bounded and coercive. By the Lax-Milgram lemma, there is a unique solution $\widetilde{\psi} \in \tilde{H}^1(r\mathbb{S}^{d-1})$ for the variational problem (20)

$$\int_{r\mathbb{S}^{d-1}}\nabla_\omega\widetilde{\psi}\cdot\nabla_\omega\chi\ M(\omega)\ \mathrm{d}\omega = \int_{r\mathbb{S}^{d-1}}\widetilde{\varphi}(\omega)\chi(\omega)M(\omega)\ \mathrm{d}\omega \tag{21}$$ Equ15

for any $\chi \in \tilde{H}^1(r\mathbb{S}^{d-1})$. Observe that (21) still holds true for any constant function on $r\mathbb{S}^{d-1}$, thanks to the compatibility condition $\int_{r\mathbb{S}^{d-1}} \widetilde{\varphi}(\omega) M(\omega)\ \mathrm{d}\omega = 0$. Therefore the variational formulation is valid for any function $\chi \in H^1(r\mathbb{S}^{d-1})$, implying that

$$-\mathrm{div}_\omega (M(\omega)\nabla_\omega \widetilde{\psi}) = M(\omega)\widetilde{\varphi}(\omega), \ \ \omega \in r\mathbb{S}^{d-1}.$$

We consider the homogeneous function ψ of degree 0, which coincides with $\widetilde{\psi}$ on $r\mathbb{S}^{d-1}$

$$\psi(v) = \widetilde{\psi}\left(r\frac{v}{|v|}\right), \ \ v \in \mathbb{R}^d \setminus \{0\}.$$

By Lemma 2.2, statements 2 and 3, we check that for any $v \in r\mathbb{S}^{d-1}$ we have

$$M(v)\left\{\frac{v - u[F]}{\sigma} \cdot \nabla_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] - \Delta_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right\} = -\mathrm{div}_\omega (M\nabla_\omega \widetilde{\psi}) = M(v)\widetilde{\varphi}(v)$$

and therefore we obtain

$$\int_{\mathbb{R}^d} \varphi(v) F\ \mathrm{d}v = \int_{\mathbb{R}^d} \left\{\frac{v - u[F]}{\sigma} \cdot \nabla_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] - \Delta_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right\} F\ \mathrm{d}v = 0.$$

We deduce that the linear forms $\varphi \to \int_{r\mathbb{S}^{d-1}} \varphi(\omega)M(\omega)\ \mathrm{d}\omega$ and $\varphi \to \int_{\mathbb{R}^d} \varphi(v) F\ \mathrm{d}v$ are proportional, and thus there is $\tilde{C}$ such that for any $\varphi \in C(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \varphi(v) F\ \mathrm{d}v = \tilde{C}\int_{r\mathbb{S}^{d-1}} \varphi(\omega)M(\omega)\ \mathrm{d}\omega = \rho\frac{\int_{r\mathbb{S}^{d-1}} \varphi(\omega)\exp\left(\frac{\omega\cdot u[F]}{\sigma}\right)\ \mathrm{d}\omega}{\int_{r\mathbb{S}^{d-1}} \exp\left(\frac{\omega\cdot u[F]}{\sigma}\right)\ \mathrm{d}\omega}, \ \ \rho = \tilde{C}\int_{r\mathbb{S}^{d-1}} M(\omega)\ \mathrm{d}\omega.$$

Therefore the measure F has a positive density with respect to $\mathrm{d}\omega$ on $r\mathbb{S}^{d-1}$

$$F = \rho\frac{\exp\left(\frac{\omega\cdot u[F]}{\sigma}\right)\ \mathrm{d}\omega}{\int_{r\mathbb{S}^{d-1}} \exp\left(\frac{\omega'\cdot u[F]}{\sigma}\right)\ \mathrm{d}\omega'}.$$

If $\rho = 0$, we obtain $F = 0$, and we can take $l = 0$ and any $\Omega \in \mathbb{S}^{d-1}$. Assume now that $\rho > 0$. If $u[F] = 0$, we obtain $F = \rho\frac{\mathrm{d}\omega}{\omega_d r^{d-1}}$ which corresponds to $l = 0$ and any $\Omega \in \mathbb{S}^{d-1}$. If $u[F] \neq 0$, we introduce $\Omega[F] = \frac{u[F]}{|u[F]|}$. By the definition of $u[F]$, we have

$$u[F] = \frac{\int_{r\mathbb{S}^{d-1}} \exp\left(\frac{\omega\cdot u[F]}{\sigma}\right)\omega\ \mathrm{d}\omega}{\int_{r\mathbb{S}^{d-1}} \exp\left(\frac{\omega\cdot u[F]}{\sigma}\right)\ \mathrm{d}\omega} = \frac{\int_0^\pi r\cos\theta\exp\left(\frac{r|u[F]|}{\sigma}\cos\theta\right)\sin^{d-2}\theta\ \mathrm{d}\theta}{\int_0^\pi \exp\left(\frac{r|u[F]|}{\sigma}\cos\theta\right)\sin^{d-2}\theta\ \mathrm{d}\theta}\Omega[F]. \tag{22}$$ Equ20

For the last equality use the fact that

$$\int_{r\mathbb{S}^{d-1}} \exp\left(\frac{\omega\cdot u[F]}{\sigma}\right)\omega\ \mathrm{d}\omega = \int_{r\mathbb{S}^{d-1}} \exp\left(\frac{\omega\cdot u[F]}{\sigma}\right)(\omega\cdot\Omega)\ \mathrm{d}\omega\ \Omega$$

and formula (19). The equality (22) reduces to the condition

$$\frac{|u[F]|}{r} = \frac{\int_0^\pi \cos\theta\exp\left(\frac{r|u[F]|}{\sigma}\cos\theta\right)\sin^{d-2}\theta\ \mathrm{d}\theta}{\int_0^\pi \exp\left(\frac{r|u[F]|}{\sigma}\cos\theta\right)\sin^{d-2}\theta\ \mathrm{d}\theta}.$$

We introduce the function $\lambda : \mathbb{R}_+ \to \mathbb{R}$

$$\lambda(l) = \frac{\int_0^\pi \cos\theta e^{l\cos\theta} \sin^{d-2}\theta \, \mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta} \sin^{d-2}\theta \, \mathrm{d}\theta}, \quad l \in \mathbb{R}_+.$$

Therefore the non negative number $\frac{r|u[F]|}{\sigma}$ satisfies $\lambda(l) = \frac{\sigma}{r^2} l$, and thus the measure F is given by

$$F = \rho \frac{\exp\left(\frac{r|u[F]|}{\sigma}\frac{\omega}{r}\cdot\Omega\right)\,\mathrm{d}\omega}{\int_{r\mathbb{S}^{d-1}} \exp\left(\frac{r|u[F]|}{\sigma}\frac{\omega\,'}{r}\cdot\Omega\right)\,\mathrm{d}\omega\,'} = \rho M_{l\Omega}\,\mathrm{d}\omega$$

with $\rho \in \mathbb{R}_+$, $\Omega = \frac{u[F]}{|u[F]|} \in \mathbb{S}^{d-1}$, $l \in \mathbb{R}_+$ satisfying $\lambda(l) = \frac{\sigma}{r^2} l$.

2. $\implies$ 1. Conversely, let F be a measure given by $F = \rho M_{l\Omega} \mathrm{d}\omega$ for some $\rho \in \mathbb{R}_+, \Omega \in \mathbb{S}^{d-1}, l \in \mathbb{R}_+$ such that $\lambda(l) = \frac{\sigma}{r^2} l$. If $\rho = 0$, F is the trivial equilibrium (with $u[F] = 0$). If $\rho > 0$, the mean velocity writes

$$\begin{aligned} u[F] &= \frac{\int_{\mathbb{R}^d} vF\,\mathrm{d}v}{\int_{\mathbb{R}^d} F\,\mathrm{d}v} = \frac{\int_{r\mathbb{S}^{d-1}} (\omega\cdot\Omega)\exp\left(l\frac{\omega}{r}\cdot\Omega\right)\,\mathrm{d}\omega}{\int_{r\mathbb{S}^{d-1}} \exp\left(l\frac{\omega}{r}\cdot\Omega\right)\,\mathrm{d}\omega}\Omega \\ &= \frac{r\int_0^\pi \cos\theta e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\Omega = r\lambda(l)\Omega = \frac{\sigma}{r} l\Omega \end{aligned}$$

saying that $\frac{u[F]}{|u[F]|} = \Omega$ and $|u[F]| = \frac{\sigma l}{r}$. For any test function $\widetilde{\psi} \in C^2(r\mathbb{S}^{d-1})$ we have

$$M(v)\left[(v - u[F])\cdot\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] - \sigma\Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right] = -\sigma \mathrm{div}_\omega(M\nabla_\omega\widetilde{\psi}), \quad v \in r\mathbb{S}^{d-1}$$

where $M(v) = \exp\left(-\frac{|v-u[F]|^2}{2\sigma}\right), v \in \mathbb{R}^d$. Notice that for any $v \in r\mathbb{S}^{d-1}$ we have

$$M(v) = \exp\left(-\frac{r^2 + \frac{\sigma^2 l^2}{r^2}}{2\sigma}\right)\int_{r\mathbb{S}^{d-1}} \exp\left(l\Omega\cdot\frac{\omega}{r}\right)\,\mathrm{d}\omega\; M_{l\Omega}(\omega)$$

and thus, the above equality becomes

$$M_{l\Omega}(v)\left\{(v - u[F])\cdot\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] - \sigma\Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right\} = -\sigma \mathrm{div}_\omega(M_{l\Omega}\nabla_\omega\widetilde{\psi}), \quad v \in r\mathbb{S}^{d-1}.$$

Therefore we obtain

$$\begin{aligned} &\int_{v\neq 0}\left\{(v - u[F])\cdot\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] - \sigma\Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right\}F\,\mathrm{d}v \\ &\quad = \int_{|v|=r}\left\{(v - u[F])\cdot\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] - \sigma\Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right\}\rho M_{l\Omega}(v)\,\mathrm{d}v \\ &\quad = -\rho\sigma\int_{r\mathbb{S}^{d-1}} \mathrm{div}_\omega(M_{l\Omega}(\omega)\nabla_\omega\widetilde{\psi})\,\mathrm{d}\omega = 0. \end{aligned}$$

□

The properties of the function λ are summarized in Proposition 1.5, whose proof is detalied below.

Proof. (of Proposition 1.5)

We introduce the function

$$\beta_0(l) = \frac{1}{\pi}\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta \ \mathrm{d}\theta, \ \ l \in \mathbb{R}.$$

It is a Bessel like function [1]. Indeed, it verifies the linear second order differential equation

$$l^2\beta_0''(l) + (d-1)l\beta_0'(l) = l^2\beta_0(l), \ \ l \in \mathbb{R}. \tag{23}$$ EquBessel

We recall that the standard modified Bessel function $I_n(l) = \frac{1}{\pi}\int_0^\pi e^{l\cos\theta}\cos(n\theta)\ \mathrm{d}\theta, n \in \mathbb{N}$, satisfy

$$l^2 I_n''(l) + lI_n'(l) = (l^2+n^2)I_n(l), \ \ l \in \mathbb{R}.$$

Clearly $\beta_0'(l) = \frac{1}{\pi}\int_0^\pi \cos\theta e^{l\cos\theta}\sin^{d-2}\theta\ \mathrm{d}\theta$ and thus the function λ writes

$$\lambda(l) = \frac{\beta_0'(l)}{\beta_0(l)}.$$

It is easily seen that $\beta_0'(0) = 0$, implying that $\lambda(0) = 0$. Indeed, we have

$$\pi\beta_0'(0) = \int_0^\pi \cos\theta\sin^{d-2}\theta\ \mathrm{d}\theta = \int_0^\pi \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\sin^{d-1}\theta}{d-1}\ \mathrm{d}\theta = 0, \ \ d \geq 2.$$

Moreover, λ is strictly increasing. This comes by the formula

$$\lambda'(l) = \frac{\beta_0''(l)\beta_0(l) - (\beta_0'(l))^2}{\beta_0^2(l)} \tag{24}$$ EquDerLam

and by observing that the Cauchy inequality implies

$$\begin{aligned}(\beta_0'(l))^2 &= \left(\frac{1}{\pi}\int_0^\pi \cos\theta e^{l\cos\theta}\sin^{d-2}\theta\ \mathrm{d}\theta\right)^2 \\ &< \frac{1}{\pi}\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\ \mathrm{d}\theta\ \frac{1}{\pi}\int_0^\pi \cos^2\theta e^{l\cos\theta}\sin^{d-2}\theta\ \mathrm{d}\theta = \beta_0(l)\beta_0''(l).\end{aligned}$$

The derivative of λ at $l = 0$ is

$$\lambda'(0) = \frac{\beta_0''(0)}{\beta_0(0)} = \frac{\int_0^\pi \cos^2\theta\sin^{d-2}\theta\ \mathrm{d}\theta}{\int_0^\pi \sin^{d-2}\theta\ \mathrm{d}\theta} = \frac{\int_0^\pi \cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\sin^{d-1}\theta}{d-1}\ \mathrm{d}\theta}{\int_0^\pi \sin^{d-2}\theta\ \mathrm{d}\theta} = \frac{\int_0^\pi \sin^d\theta\ \mathrm{d}\theta}{(d-1)\int_0^\pi \sin^{d-2}\theta\ \mathrm{d}\theta}.$$

But we also have

$$\lambda'(0) = 1 - \frac{\int_0^\pi \sin^d\theta\ \mathrm{d}\theta}{\int_0^\pi \sin^{d-2}\theta\ \mathrm{d}\theta}.$$

We deduce that

$$\frac{\int_0^\pi \sin^d\theta\ \mathrm{d}\theta}{\int_0^\pi \sin^{d-2}\theta\ \mathrm{d}\theta} = 1 - \lambda'(0) = (d-1)\lambda'(0)$$

which yields $\lambda'(0) = 1/d$. We claim that λ is strictly concave. Combining (24), (23), we obtain for any $l > 0$

$$\lambda'(l) = \frac{\left(\beta_0(l) - \frac{d-1}{l}\beta_0'(l)\right)\beta_0(l)}{\beta_0^2(l)} - \left(\frac{\beta_0'(l)}{\beta_0(l)}\right)^2 = 1 - \frac{d-1}{l}\lambda(l) - \lambda^2(l). \tag{25}$$ Equ22

As λ is positive and strictly increasing, we deduce that λ is strictly concave on $\mathbb{R}_+$. Clearly the function λ is bounded on $\mathbb{R}_+$

$$0 = \lambda(0) < \lambda(l) = \frac{\int_0^\pi \cos\theta e^{l\cos\theta} \sin^{d-2}\theta \, \mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta} \sin^{d-2}\theta \, \mathrm{d}\theta} < 1$$

and $\frac{1}{d} = \lambda'(0) > \lambda'(l) > 0, l > 0$. Let us denote by Λ_0, Λ_1 the limits

$$\Lambda_0 = \lim_{l\to+\infty} \lambda(l) \in]0,1], \;\; \Lambda_1 = \lim_{l\to+\infty} \lambda'(l) \in [0, \frac{1}{d}[.$$

If $\Lambda_1 > 0$ then the inequality $\lambda'(l) > \Lambda_1, l > 0$, implies

$$\lim_{l\to+\infty} \lambda(l) = \lim_{l\to+\infty} \{\lambda(l) - \lambda(0)\} \geq \lim_{l\to+\infty} l\Lambda_1 = +\infty$$

which contradicts the boundedness of λ. Therefore $\Lambda_1 = 0$ and thus $\lambda'([0,+\infty[) =]0, \lambda'(0)] =]0, 1/d]$. Passing to the limit, when $l \to +\infty$, in (25), yields $\Lambda_0 = \lim_{l\to+\infty} \lambda(l) = 1$.
If $\frac{\sigma}{r^2} \geq \frac{1}{d}$, the function $l \to \lambda(l) - \frac{\sigma}{r^2} l$ is strictly decreasing on $\mathbb{R}_+$, and vanishes at $l = 0$

$$\lambda'(l) - \frac{\sigma}{r^2} < \lambda'(0) - \frac{\sigma}{r^2} = \frac{1}{d} - \frac{\sigma}{r^2} \leq 0, \;\; l > 0$$

implying that the only solution of $\lambda(l) = \frac{\sigma}{r^2} l$ on $\mathbb{R}_+$ is $l = 0$. If $\frac{\sigma}{r^2} \in]0, \frac{1}{d}[$, there is a unique $\tilde{l}(\frac{\sigma}{r^2}) > 0$ such that $\lambda'(\tilde{l}) = \frac{\sigma}{r^2}$ and the function $l \to \lambda'(l) - \frac{\sigma}{r^2}$ is positive on $]0, \tilde{l}(\frac{\sigma}{r^2})[$ and negative on $]\tilde{l}(\frac{\sigma}{r^2}), +\infty[$. Therefore the function $l \to \lambda(l) - \frac{\sigma}{r^2} l$ is strictly increasing on $[0, \tilde{l}(\frac{\sigma}{r^2})]$, strictly decreasing on $[\tilde{l}(\frac{\sigma}{r^2}), +\infty[$

$$\left\{\lambda(l) - \frac{\sigma}{r^2} l\right\}|_{l=0} = 0, \;\; \lim_{l\to+\infty} \left\{\lambda(l) - \frac{\sigma}{r^2} l\right\} = -\infty.$$

We deduce that there is a unique solution $l(\frac{\sigma}{r^2}) > 0$ such that $\lambda(l(\frac{\sigma}{r^2})) = \frac{\sigma}{r^2} l(\frac{\sigma}{r^2})$. □

Remark 2.2 *The value $l = 0$ corresponds to the isotropic equilibrium $M_{0\Omega} \, \mathrm{d}\omega = \frac{\mathrm{d}\omega}{\omega_d r^{d-1}}$. The limit when $l \to +\infty$ leads to the Dirac measure on $r\mathbb{S}^{d-1}$, concentrated at $r\Omega$, that is, for any function $\widetilde{\psi} \in C(r\mathbb{S}^{d-1})$ we have*

$$\lim_{l\to+\infty} \int_{r\mathbb{S}^{d-1}} \widetilde{\psi}(\omega) M_{l\Omega}(\omega) \, \mathrm{d}\omega = \widetilde{\psi}(r\Omega).$$

The function λ can be computed explicitly, at least for $d = 3$. Nevertheless, very good explicite approximations are available in any dimension d.

⟨Approximation⟩

Lemma 2.3

1. Consider the function

$$\mu : \mathbb{R}_+ \to \mathbb{R}_+, \ \ \mu(l) = \frac{\sqrt{d^2+4l^2}-d}{2l} = \frac{2l}{\sqrt{d^2+4l^2}+d}, \ \ l \in \mathbb{R}_+.$$

The function μ is strictly increasing, strictly concave and we have

$$\mu(0) = \lambda(0) = 0, \ \ \mu'(0) = \lambda'(0) = \frac{1}{d}, \ \ \lim_{l\to+\infty} \mu(l) = 1$$

$$\mu'(l) < 1 - \frac{d-1}{l}\mu(l) - \mu^2(l), \ \ \mu(l) < \lambda(l), \ \ l > 0.$$

2. If $d = 3$, the function λ is given by $\lambda(l) = \frac{\cosh(l)}{\sinh(l)} - \frac{1}{l}, l > 0$.

Proof.

1. By direct computations we obtain

$$\mu'(l) = \frac{2d}{\sqrt{d^2+4l^2}(\sqrt{d^2+4l^2}+d)} > 0, \ \ l \in \mathbb{R}_+$$

and

$$1 - \frac{d-1}{l}\mu(l) - \mu^2(l) = \frac{2}{\sqrt{d^2+4l^2}+d}.$$

Therefore μ satisfies the first order differential inequation

$$\mu'(l) = \frac{2d}{\sqrt{d^2+4l^2}(\sqrt{d^2+4l^2}+d)} < \frac{2}{\sqrt{d^2+4l^2}+d} = 1 - \frac{d-1}{l}\mu(l) - \mu^2(l), \ \ l > 0$$

and the initial condition $\mu(0) = 0$. Recall that λ satisfies the first order differential equation (cf. (25))

$$\lambda'(l) = 1 - \frac{d-1}{l}\lambda(l) - \lambda^2(l), \ \ l > 0$$

with the initial condition $\lambda(0) = 0$. By comparison principle, it follows that $\mu(l) < \lambda(l)$ for any $l > 0$. Clearly $\mu'(0) = \frac{1}{d} = \lambda'(0), \lim_{l\to+\infty} \mu(l) = 1$, $\mu'(l) > 0, l \in \mathbb{R}_+$, and μ' is strictly decreasing, saying that μ is strictly increasing and strictly concave on $\mathbb{R}_+$.

2. In the case $d = 3$ we obtain

$$\pi\beta_0(l) = \int_0^\pi e^{l\cos\theta}\sin\theta \,\mathrm{d}\theta = \frac{e^l - e^{-l}}{l}, \ \ l > 0$$

$$\pi\beta_0'(l) = \int_0^\pi e^{l\cos\theta}\cos\theta\sin\theta \,\mathrm{d}\theta = \frac{e^l + e^{-l}}{l} - \frac{e^l - e^{-l}}{l^2}, \ \ l > 0$$

implying that

$$\lambda(l) = \frac{\beta_0'(l)}{\beta_0(l)} = \frac{\cosh(l)}{\sinh(l)} - \frac{1}{l}, l > 0.$$

□

In order to exploit the constraint (9) we will need to compute $Q(F)$, where F is a von Mises-Fisher equilibrium, let us say $F = M_{l\Omega}(\omega)\mathrm{d}\omega$. This computation is detailed in the following lemma. The notation $(\cdot,\cdot)$ stands for the pairing between distributions and smooth functions.

Lemma 2.4 *Let $F = M_{l\Omega}(\omega)\mathrm{d}\omega$ be a von Mises-Fisher equilibrium. Then we have, for any function $\varphi \in C_c^2(\mathbb{R}^d)$*

$$(Q(F), \varphi) = \nu\sigma \frac{M_{l\Omega}}{M} \frac{\mathrm{d}}{\mathrm{d}t}|_{t=r} \int_{|\omega_t|=t} M(\omega_t)(\nabla_v \varphi)(\omega_t) \cdot \frac{\omega_t}{t} \,\mathrm{d}\omega_t$$

where $M(v) = \exp\left(-\frac{|v-u[F]|^2}{2\sigma}\right), v \in \mathbb{R}^d$.

Proof. Pick a test function $\varphi \in C_c^2(\mathbb{R}^d)$ and notice that

$$\begin{aligned}
(Q(F), \varphi) &= \nu(F, \sigma\Delta_v\varphi - (v - u[F]) \cdot \nabla_v\varphi) \\
&= \nu\left(F, \sigma\frac{\operatorname{div}_v(M\nabla_v\varphi)}{M(v)}\right) \\
&= \nu\sigma \int_{r\mathbb{S}^{d-1}} \operatorname{div}_v(M\nabla_v\varphi)(\omega)\frac{M_{l\Omega}(\omega)}{M(\omega)} \,\mathrm{d}\omega.
\end{aligned}$$

It is easily seen that the function $\frac{M_{l\Omega}}{M}$ is constant on the sphere $r\mathbb{S}^{d-1}$

$$\frac{M_{l\Omega}(\omega)}{M(\omega)} = \frac{\exp\left(\frac{r^2+|u[F]|^2}{2\sigma}\right)}{\int_{r\mathbb{S}^{d-1}} \exp\left(l\Omega \cdot \frac{\omega'}{r}\right) \,\mathrm{d}\omega\,'}, \quad \omega \in r\mathbb{S}^{d-1}$$

and therefore we have

$$\begin{aligned}
(Q(F), \varphi) &= \nu\sigma\frac{M_{l\Omega}}{M}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=r} \int_{|v|<t} \operatorname{div}_v(M\nabla_v\varphi) \,\mathrm{d}v \\
&= \nu\sigma\frac{M_{l\Omega}}{M}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=r} \int_{|\omega_t|=t} M(\omega_t)\nabla_v\varphi(\omega_t) \cdot \frac{\omega_t}{t} \,\mathrm{d}\omega_t.
\end{aligned}$$

□

Thanks to the above result, we can determine $F^{(1)} - \left\langle F^{(1)} \right\rangle$ in terms of F. More exactly we prove

⟨F0neF⟩
Lemma 2.5 *Let $F = M_{l\Omega}(\omega)\mathrm{d}\omega$ be a von Mises-Fisher equilibrium and $F^{(1)}$ a bounded measure such that*

$$\operatorname{div}_v\{F^{(1)}(\alpha - \beta|v|^2)v\} = Q(F).$$

Then for any function $\chi \in C_c^1(\mathbb{R}^d)$, such that $\chi|_{r\mathbb{S}^{d-1}} = 0$ we have

$$\int_{\mathbb{R}^d} \chi(v)\left(F^{(1)} - \left\langle F^{(1)} \right\rangle\right) \,\mathrm{d}v = \int_{v\neq 0} \chi(v)F^{(1)} \,\mathrm{d}v = \nu\sigma\frac{M_{l\Omega}}{M}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=r} \int_{|\omega_t|=t} \frac{M(\omega_t)\chi(\omega_t)}{t\beta(t^2 - r^2)} \,\mathrm{d}\omega_t.$$

Proof. For any function $\varphi \in C_c^1(\mathbb{R}^d)$, we know that

$$-\int_{\mathbb{R}^d} (\alpha - \beta|v|^2) v \cdot \nabla_v \varphi \, F^{(1)} \, \mathrm{d}v = (Q(F), \varphi) = \nu\sigma \frac{M_{l\Omega}}{M} \frac{\mathrm{d}}{\mathrm{d}t}|_{t=r} \int_{|\omega_t|=t} M(\omega_t) \nabla_v \varphi(\omega_t) \cdot \frac{\omega_t}{t} \, \mathrm{d}\omega_t.$$

The idea is to solve the adjoint problem (cf. Lemma 1.1)

$$-(\alpha - \beta|v|^2) v \cdot \nabla_v \varphi = \chi(v)$$

and to express the normal derivative of φ in terms of χ. Indeed, for any $\omega_t \in t\mathbb{S}^{d-1}$, we have

$$\nabla_v \varphi(\omega_t) \cdot \frac{\omega_t}{t} = \frac{\chi(\omega_t)}{t(\beta t^2 - \alpha)} = \frac{\chi(\omega_t)}{t\beta(t^2 - r^2)}.$$

Finally we obtain the formula

$$\int_{v \neq 0} \chi(v) F^{(1)} \, \mathrm{d}v = (Q(F), \varphi) = \nu\sigma \frac{M_{l\Omega}}{M} \frac{\mathrm{d}}{\mathrm{d}t}|_{t=r} \int_{|\omega_t|=t} \frac{M(\omega_t)\chi(\omega_t)}{t\beta(t^2 - r^2)} \, \mathrm{d}\omega_t.$$

□

Once we have determined the form of the dominant distribution $f(t,x) = \rho(t,x) M_{l\Omega(t,x)} \mathrm{d}\omega$, we search for macroscopic equations characterizing $\rho(t,x)$ and $\Omega(t,x)$. For doing that, we use the moments of (11) with respect to the velocity. The key point is how to eliminate $f^{(1)}$ in the right hand side of (11). Notice that this right hand side is the linearization around f, with $\int_{\mathbb{R}^d} f \, \mathrm{d}v > 0$, computed in the direction $f^{(1)}$, of the average collision kernel Q

$$\begin{aligned}
\lim_{\varepsilon \searrow 0} \frac{\langle Q(f + \varepsilon f^{(1)}) \rangle - \langle Q(f) \rangle}{\varepsilon} &= \left\langle \nu \mathrm{div}_v \left[f^{(1)}(v - u[f]) + \sigma \nabla_v f^{(1)} \right] \right\rangle \\
&\quad - \left\langle \nu \mathrm{div}_v \left[f \frac{\int_{\mathbb{R}^d} f^{(1)}(v' - u[f]) \, \mathrm{d}v'}{\int_{\mathbb{R}^d} f \, \mathrm{d}v'} \right] \right\rangle \\
&= \left\langle \mathrm{div}_v A_f(f^{(1)}) \right\rangle
\end{aligned}$$

where

$$A_f(f^{(1)}) = \nu \left[f^{(1)}(v - u[f]) + \sigma \nabla_v f^{(1)} \right] - \nu f \frac{\int_{\mathbb{R}^d} f^{(1)}(v' - u[f]) \, \mathrm{d}v'}{\int_{\mathbb{R}^d} f \, \mathrm{d}v'}.$$

We are looking for functions such that

$$\int_{\mathbb{R}^d} \psi(v) \left\langle \mathrm{div}_v A_f(f^{(1)}) \right\rangle \, \mathrm{d}v \tag{26}$$

Equ23

can be expressed in terms of the velocity moments of f, in order to get a closure for the macroscopic quantities $\rho(t,x), \Omega(t,x)$. For example $\psi(v) = 1$ leads to the continuity equation

$$\partial_t \int_{\mathbb{R}^d} f \, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} v f \, \mathrm{d}v = 0$$

which also writes

$$\partial_t \rho + \mathrm{div}_x \left(\rho \frac{\sigma}{r} l \Omega \right) = 0.$$

Naturally, we need to find other functions ψ, which will allow us to characterize the time evolution of the orientation Ω. Recall that the constraint (9) determines $f^{(1)} - \left\langle f^{(1)} \right\rangle$ (in terms of f), but not $\left\langle f^{(1)} \right\rangle$. Motivated by this, we are looking for functions ψ such that

$$\int_{\mathbb{R}^d} \psi(v) \left\langle \mathrm{div}_v A_f(g^{(1)}) \right\rangle \; \mathrm{d}v = 0$$

for any measures $f, g^{(1)}$ supported in $\mathbb{R}^d \times r\mathbb{S}^{d-1}$. Indeed, in that case the expression in (26) can be computed in terms of f, provided that we neglect the mass of $f^{(1)}$ at $\mathbb{R}^d \times \{0\}$

$$\begin{aligned}\int_{\mathbb{R}^d} \psi(v) \left\langle \mathrm{div}_v A_f(f^{(1)}) \right\rangle \; \mathrm{d}v &= \int_{\mathbb{R}^d} \psi \left\langle \mathrm{div}_v A_f \left\langle f^{(1)} \right\rangle \right\rangle \; \mathrm{d}v + \int_{\mathbb{R}^d} \psi \left\langle \mathrm{div}_v A_f \left[f^{(1)} - \left\langle f^{(1)} \right\rangle \right] \right\rangle \; \mathrm{d}v \\ &= \int_{\mathbb{R}^d} \psi(v) \left\langle \mathrm{div}_v A_f \left[f^{(1)} - \left\langle f^{(1)} \right\rangle \right] \right\rangle \; \mathrm{d}v.\end{aligned}$$

Let us concentrate now on the collision invariants of the average collision operator. Recall that the linearized of $\langle Q \rangle$, around a measure F such that $\int_{\mathbb{R}^d} F \; \mathrm{d}v > 0$, writes

$$\lim_{\varepsilon \searrow 0} \frac{\langle Q(F + \varepsilon F^{(1)}) \rangle - \langle Q(F) \rangle}{\varepsilon} = \left\langle \mathrm{div}_v A_F(F^{(1)}) \right\rangle$$

where

$$A_F(F^{(1)}) = \nu \left[F^{(1)}(v - u[F]) + \sigma \nabla_v F^{(1)} \right] - \nu F \frac{\int_{\mathbb{R}^d} F^{(1)}(v' - u[F]) \; \mathrm{d}v'}{\int_{\mathbb{R}^d} F \; \mathrm{d}v'}.$$

We search for functions $\psi = \psi(v)$ such that

$$\int_{\mathbb{R}^d} \psi(v) \left\langle \mathrm{div}_v A_F(G^{(1)}) \right\rangle \; \mathrm{d}v = 0 \tag{27}$$ Equ41

for any bounded measures $F, G^{(1)}$ supported in $r\mathbb{S}^{d-1}$. Actually, since we already know that the dominant term is a von Mises-Fisher distribution, it is enough to impose (27) only for $F = M_{l\Omega} \mathrm{d}\omega$, with $\lambda(l) = \frac{\sigma}{r^2} l$, for some given $\Omega \in \mathbb{S}^{d-1}$. Doing that, to any orientation Ω, we associate a family of suitable pseudo-collision invariants, allowing us to determine the macroscopic equations satisfied by the moments ρ, Ω, see also [29] for a similar construction. Nevertheless the approach is not exactly the same. Actually, once we have determined ψ such that (27) is verified for any bounded measure $G^{(1)}$ supported in $r\mathbb{S}^{d-1}$, we need to check that (27) still holds true for any bounded measure, not necessarily supported in $r\mathbb{S}^{d-1}$, satisfying the constraint (9) (see Proposition 3.4 and C). The condition (27) should be understood in the following sense

$$\int_{v \neq 0} \widetilde{\psi} \left(r \frac{v}{|v|} \right) \mathrm{div}_v \{ A_F(G^{(1)}) \} \; \mathrm{d}v = 0, \quad F = M_{l\Omega} \; \mathrm{d}\omega$$

for any $G^{(1)} \in \mathcal{M}_b(\mathbb{R}^d)$, $\operatorname{supp} G^{(1)} \subset r\mathbb{S}^{d-1}$, that is

$$\int_{v\neq 0} \left\{ -(v-u[F])\cdot\nabla_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] + \sigma\Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right\} G^{(1)}\,\mathrm{d}v + \int_{v\neq 0} \frac{\int_{v'\neq 0}(v'-u[F])G^{(1)}\,\mathrm{d}v'}{\int_{\mathbb{R}^d} F\,\mathrm{d}v'}\cdot\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] F\,\mathrm{d}v = 0 \quad (28)$$ Equ42

for $F = M_{l\Omega}\mathrm{d}\omega$ and any $G^{(1)} \in \mathcal{M}_b(\mathbb{R}^d)$, $\operatorname{supp} G^{(1)} \subset r\mathbb{S}^{d-1}$. Taking into account the equalities

$$\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] = \nabla_\omega\widetilde{\psi},\ \ \Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] = \Delta_\omega\widetilde{\psi},\ \ |v| = r$$

the condition (28) becomes

$$\left[-(\omega - u[M_{l\Omega}])\cdot\nabla_\omega\widetilde{\psi} + \sigma\operatorname{div}_\omega(\nabla_\omega\widetilde{\psi})\right] + (\omega - u[M_{l\Omega}])\cdot\frac{\int_{r\mathbb{S}^{d-1}}\nabla_{\omega'}\widetilde{\psi}M_{l\Omega}\,\mathrm{d}\omega'}{\int_{r\mathbb{S}^{d-1}}M_{l\Omega}\,\mathrm{d}\omega'} = 0,\ \ \omega\in r\mathbb{S}^{d-1}. \quad (29)$$ Equ43

3 The collision invariants

?⟨ColInv⟩?

In the sequel we concentrate on the resolution of the linear equation (29). If we introduce the vector

$$W[\widetilde{\psi}] = \frac{\int_{r\mathbb{S}^{d-1}}\nabla_\omega\widetilde{\psi}M_{l\Omega}(\omega)\,\mathrm{d}\omega}{\int_{r\mathbb{S}^{d-1}}M_{l\Omega}(\omega)\,\mathrm{d}\omega} = \int_{r\mathbb{S}^{d-1}}\nabla_\omega\widetilde{\psi}M_{l\Omega}(\omega)\,\mathrm{d}\omega$$

the equation (29) becomes elliptic on $r\mathbb{S}^{d-1}$ and writes

$$-\sigma\operatorname{div}_\omega(M_{l\Omega}\nabla_\omega\widetilde{\psi}) = M_{l\Omega}(\omega)(\omega - u[M_{l\Omega}])\cdot W[\widetilde{\psi}]. \quad (30)$$ Equ62

The solvability of (30) requires that the integral of the right hand side over $r\mathbb{S}^{d-1}$ vanishes, which is true, by the definition of the mean velocity. But there is another compatibility condition to be fullfiled. Take any vector $W' \in \mathbb{R}^d$ and multiply the equation (30) by the scalar function $\omega \to W'\cdot\omega$, whose gradient along $r\mathbb{S}^{d-1}$ is $\left(I_d - \frac{\omega\otimes\omega}{r^2}\right)W'$. Integrating by parts yields

$$\sigma\int_{r\mathbb{S}^{d-1}}M_{l\Omega}(\omega)\nabla_\omega\widetilde{\psi}\,\mathrm{d}\omega\cdot W' = \int_{r\mathbb{S}^{d-1}}M_{l\Omega}(\omega)(\omega - u[M_{l\Omega}])\otimes(\omega - u[M_{l\Omega}])\,\mathrm{d}\omega : W[\widetilde{\psi}]\otimes W'$$

saying that $W[\widetilde{\psi}]$ is an eigenvector of the matrix

$$\mathcal{M}_{l\Omega} := \int_{r\mathbb{S}^{d-1}}M_{l\Omega}(\omega)(\omega - u[M_{l\Omega}])\otimes(\omega - u[M_{l\Omega}])\,\mathrm{d}\omega$$

corresponding to the eigenvalue σ. The following lemma details the spectral properties of the matrix $\mathcal{M}_{l\Omega}$.

⟨SpecProp⟩
Lemma 3.1 *For any* $l \in \mathbb{R}_+$ *such that* $\lambda(l) = \frac{\sigma}{r^2} l$*, and* $\Omega \in r\mathbb{S}^{d-1}$*, the matrix* $\mathcal{M}_{l\Omega}$ *is symmetric, definite positive and*

$$\mathcal{M}_{0\Omega} = \frac{r^2}{d} I_d, \;\; \mathcal{M}_{l\Omega} = (r^2 - (d-1)\sigma - |u|^2)\Omega \otimes \Omega + \sigma(I_d - \Omega \otimes \Omega), \;\; l > 0, \;\; 0 < \frac{\sigma}{r^2} < \frac{1}{d}.$$

If $0 < \frac{\sigma}{r^2} < \frac{1}{d}$*, we have* $r^2 - (d-1)\sigma - |u|^2 < \sigma$ *and, in particular* $\ker(\mathcal{M}_{l\Omega} - \sigma I_d) = (\mathbb{R}\Omega)^\perp$.

Proof. Clearly $\mathcal{M}_{l\Omega}$ is symmetric and definite positive. The case $l = 0$ is trivial, and we have $\mathcal{M}_{0\Omega} = \frac{r^2}{d} I_d$. Assume now that $l > 0$ and thus necessarily $\frac{\sigma}{r^2} \in]0, \frac{1}{d}[$ cf. Proposition 1.5. We consider a orthonormal basis $\{E_1, ..., E_{d-1}, \Omega\}$. It is easily seen that

$$\begin{aligned}
\mathcal{M}_{l\Omega} &= \int_{r\mathbb{S}^{d-1}} (\omega - u) \otimes \omega M_{l\Omega} \,\mathrm{d}\omega \\
&= \int_{r\mathbb{S}^{d-1}} [((\omega \cdot \Omega) - |u|)\Omega + \sum_{i=1}^{d-1} (\omega \cdot E_i) E_i] \otimes [(\omega \cdot \Omega)\Omega + \sum_{i=1}^{d-1} (\omega \cdot E_i) E_i] M_{l\Omega} \,\mathrm{d}\omega \\
&= \int_{r\mathbb{S}^{d-1}} ((\omega \cdot \Omega) - |u|)(\omega \cdot \Omega) M_{l\Omega} \,\mathrm{d}\omega \; \Omega \otimes \Omega + \sum_{i=1}^{d-1} \int_{r\mathbb{S}^{d-1}} (\omega \cdot E_i)^2 M_{l\Omega} \,\mathrm{d}\omega \; E_i \otimes E_i \\
&= \int_{r\mathbb{S}^{d-1}} ((\omega \cdot \Omega)^2 - |u|^2) M_{l\Omega} \,\mathrm{d}\omega \; \Omega \otimes \Omega + \frac{1}{d-1} \int_{r\mathbb{S}^{d-1}} (r^2 - (\omega \cdot \Omega)^2) M_{l\Omega} \,\mathrm{d}\omega (I_d - \Omega \otimes \Omega).
\end{aligned}$$

We show that

$$\int_{r\mathbb{S}^{d-1}} (\omega \cdot \Omega)^2 M_{l\Omega} \,\mathrm{d}\omega = r^2 - (d-1)\sigma.$$

This comes by the condition $\lambda(l) = \frac{\sigma}{r^2} l$ and integrations by parts

$$\begin{aligned}
r^2 - \int_{r\mathbb{S}^{d-1}} (\omega \cdot \Omega)^2 M_{l\Omega} \,\mathrm{d}\omega &= \frac{r^2 \int_0^\pi \sin\theta e^{l\cos\theta} \sin^{d-1}\theta \,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta} \sin^{d-2}\theta \,\mathrm{d}\theta} \\
&= -\frac{r^2}{l} \frac{\int_0^\pi \frac{\mathrm{d}}{\mathrm{d}\theta} e^{l\cos\theta} \sin^{d-1}\theta \,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta} \sin^{d-2}\theta \,\mathrm{d}\theta} \\
&= (d-1) \frac{r^2}{l} \frac{\int_0^\pi \cos\theta e^{l\cos\theta} \sin^{d-2}\theta \,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta} \sin^{d-2}\theta \,\mathrm{d}\theta} \\
&= (d-1) \frac{r^2}{l} \lambda(l) \\
&= (d-1) \frac{r^2}{l} \frac{\sigma}{r^2} l = (d-1)\sigma.
\end{aligned}$$

We deduce also that

$$\int_{r\mathbb{S}^{d-1}} ((\omega \cdot \Omega)^2 - |u|^2) M_{l\Omega} \,\mathrm{d}\omega = r^2 - (d-1)\sigma - |u|^2$$

and therefore

$$\mathcal{M}_{l\Omega} = (r^2 - (d-1)\sigma - |u|^2)\Omega \otimes \Omega + \sigma(I_d - \Omega \otimes \Omega).$$

We claim that the biggest eigenvalue is σ, that is $r^2 - (d-1)\sigma - |u|^2 < \sigma$, or equivalently $r^2 < d\sigma + |u|^2$. This is a consequence of Lemma 2.3. Indeed, since $l > 0$, we know that

$$\mu(l) = \frac{2l}{\sqrt{d^2 + 4l^2} + d} < \lambda(l) = \frac{\sigma}{r^2} l$$

implying that

$$\sqrt{d^2 + 4l^2} > \frac{2r^2}{\sigma} - d > 0, \text{ since } r^2 > d\sigma$$

or equivalently

$$4l^2 > 4\frac{r^4}{\sigma^2} - 4d\frac{r^2}{\sigma}.$$

Replacing $l = \frac{|u|r}{\sigma}$ in the above inequality, yields $r^2 < d\sigma + |u|^2$. □

The resolution of (29) follows immediately, thanks to Lemma 3.1. As (29) is linear and admits any constant function on $r\mathbb{S}^{d-1}$ as solution, we will work with zero mean solutions on $r\mathbb{S}^{d-1}$, that is $\int_{r\mathbb{S}^{d-1}} \widetilde{\psi}(\omega)\, \mathrm{d}\omega = 0$.

⟨CollInvBis⟩ **Proposition 3.1** *Let $M_{l\Omega}$ be a von Mises-Fisher distribution i.e., $\Omega \in \mathbb{S}^{d-1}, l \in \mathbb{R}_+, \lambda(l) = \frac{\sigma}{r^2} l$, and $E_1, ..., E_{d-1}$ be a orthonormal basis of $(\mathbb{R}\Omega)^\perp$.*

1. *If $l = 0$ and $\frac{\sigma}{r^2} \neq \frac{1}{d}$, then the only (zero mean) solution of (29) is the trivial one.*

2. *If $l = 0$ and $\frac{\sigma}{r^2} = \frac{1}{d}$, then the family of zero mean solutions for (29) is a linear space of dimension d. A basis is given by the functions $\widetilde{\psi}_1, ..., \widetilde{\psi}_d$ satisfying*

$$-\sigma \mathrm{div}_\omega (M_{0\Omega} \nabla_\omega \widetilde{\psi}_i) = M_{0\Omega}(\omega)(\omega \cdot E_i), \quad \int_{r\mathbb{S}^{d-1}} \widetilde{\psi}_i(\omega)\, \mathrm{d}\omega = 0, \quad i \in \{1, ..., d\}, \ E_d = \Omega. \tag{31}$$

Equ63

3. *If $0 < \frac{\sigma}{r^2} < \frac{1}{d}, l > 0, \lambda(l) = \frac{\sigma}{r^2} l$, then the family of zero mean solutions for (29) is a linear space of dimension $d-1$. A basis is given by the functions $\widetilde{\psi}_1, ..., \widetilde{\psi}_{d-1}$ satisfying*

$$-\sigma \mathrm{div}_\omega (M_{l\Omega} \nabla_\omega \widetilde{\psi}_i) = M_{l\Omega}(\omega)(\omega \cdot E_i), \quad \int_{r\mathbb{S}^{d-1}} \widetilde{\psi}_i(\omega)\, \mathrm{d}\omega = 0, \quad i \in \{1, ..., d-1\}. \tag{32}$$

Equ65

Proof.

1. Let $\widetilde{\psi}$ be a zero mean solution of (29). Multiplying by $(\omega \cdot W')$, with $W' \in \mathbb{R}^d$, and integrating by parts over $r\mathbb{S}^{d-1}$ yield

$$\sigma W[\widetilde{\psi}] \cdot W' = \sigma \int_{r\mathbb{S}^{d-1}} M_{0\Omega} \nabla_\omega \widetilde{\psi} \cdot W'\, \mathrm{d}\omega = \int_{r\mathbb{S}^{d-1}} M_{0\Omega}(\omega - 0) \cdot W[\widetilde{\psi}](\omega \cdot W')\, \mathrm{d}\omega$$
$$= \mathcal{M}_{0\Omega} W[\widetilde{\psi}] \cdot W' = \frac{r^2}{d} W[\widetilde{\psi}] \cdot W'.$$

Therefore $\left(\sigma - \frac{r^2}{d}\right) W[\widetilde{\psi}] = 0$, implying that $W[\widetilde{\psi}] = 0$ and

$$-\mathrm{div}_\omega(M_{0\Omega}(\omega)\nabla_\omega \widetilde{\psi}) = 0.$$

We deduce that $\widetilde{\psi}$ is a constant, zero mean function on $r\mathbb{S}^{d-1}$, and thus $\widetilde{\psi} = 0$.
2. As $l = 0$, then $\int_{r\mathbb{S}^{d-1}} \omega M_{0\Omega}(\omega)\,\mathrm{d}\omega = u = 0$. Therefore the right hand sides in (31) are zero mean functions on $r\mathbb{S}^{d-1}$, and by Lax-Milgram lemma, the zero mean functions $(\widetilde{\psi}_i)_{1\le i\le d}$ are well defined. Notice that these functions also solve (29). Indeed, after multiplication by $(\omega \cdot W')$, with $W' \in \mathbb{R}^d$, and integration by parts we obtain, for any $i \in \{1, ..., d\}$

$$\sigma \int_{r\mathbb{S}^{d-1}} \nabla_\omega \widetilde{\psi}_i \cdot W' M_{0\Omega}\,\mathrm{d}\omega = \int_{r\mathbb{S}^{d-1}} (\omega \cdot E_i)(\omega \cdot W') M_{0\Omega}\,\mathrm{d}\omega = \mathcal{M}_{0\Omega} E_i \cdot W'.$$

We deduce that

$$\sigma \int_{r\mathbb{S}^{d-1}} M_{0\Omega}(\omega)\nabla_\omega \widetilde{\psi}_i\,\mathrm{d}\omega = \mathcal{M}_{0\Omega} E_i = \frac{r^2}{d} E_i = \sigma E_i,\ \ i \in \{1, .., d\} \tag{33}$$ Equ66

which eactly says that $(\widetilde{\psi}_i)_{1\le i\le d}$ solve (29). It is easily seen that the family $(\widetilde{\psi}_i)_{1\le i\le d}$ is linearly independent : if $\sum_{i=1}^d c_i \widetilde{\psi}_i = 0$, then by (33) one gets

$$\sum_{i=1}^d c_i E_i = \sum_{i=1}^d c_i \int_{r\mathbb{S}^{d-1}} M_{0\Omega}(\omega)\nabla_\omega \widetilde{\psi}_i\,\mathrm{d}\omega = 0$$

implying that $c_i = 0, i \in \{1, ..., d\}$. We show now that any zero mean solution $\widetilde{\psi}$ for (29) is a linear combination of $(\widetilde{\psi}_i)_{1\le i\le d}$. Let $(c_i)_{1\le i\le d}$ be the coordinates of the vector $W[\widetilde{\psi}]$ with respect to the basis $(E_i)_{1\le i\le d}$

$$W[\widetilde{\psi}] = \int_{r\mathbb{S}^{d-1}} M_{0\Omega}(\omega)\nabla_\omega \widetilde{\psi}\,\mathrm{d}\omega = \sum_{i=1}^d c_i E_i.$$

We claim that $\widetilde{\psi} = \sum_{i=1}^d c_i \widetilde{\psi}_i$. Indeed, since $\widetilde{\psi}$ and $\sum_{i=1}^d c_i \widetilde{\psi}_i$ have zero mean, thanks to the uniqueness of zero mean solution, it is enough to check that $\sum_{i=1}^d c_i \widetilde{\psi}_i$ solves (30), with the right hand side $M_{0\Omega}\omega \cdot W[\widetilde{\psi}]$. Indeed, we have

$$-\sigma \mathrm{div}_\omega(M_{0\Omega}\nabla_\omega \sum_{i=1}^d c_i \widetilde{\psi}_i) = \sum_{i=1}^d c_i M_{0\Omega}(\omega \cdot E_i) = M_{0\Omega}(\omega - 0) \cdot W[\widetilde{\psi}]$$

implying that $\widetilde{\psi} = \sum_{i=1}^d c_i \widetilde{\psi}_i$.
3. The arguments are similar. The solutions $(\widetilde{\psi}_i)_{1\le i\le d-1}$ in (32) also solve (29), and are linearly independent. But for any solution $\widetilde{\psi}$ of (29), we have for any $W' \in \mathbb{R}^d$

$$\sigma W[\widetilde{\psi}]\cdot W' = \sigma \int_{r\mathbb{S}^{d-1}} M_{l\Omega}\nabla_\omega \widetilde{\psi} \cdot W'\,\mathrm{d}\omega = \int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega - u[M_{l\Omega}])\cdot W[\widetilde{\psi}](\omega \cdot W')\,\mathrm{d}\omega = \mathcal{M}_{l\Omega} W[\widetilde{\psi}]\cdot W'.$$

Therefore $W[\widetilde{\psi}] \in \ker(\mathcal{M}_{l\Omega} - \sigma I_d) = (\mathbb{R}\Omega)^\perp = \mathrm{span}\{E_1, ..., E_{d-1}\}$ and we deduce that $\widetilde{\psi} = \sum_{i=1}^{d-1} c_i \widetilde{\psi}_i$, with $W[\widetilde{\psi}] = \sum_{i=1}^{d-1} c_i E_i$. □

We focus now on the structure of the solutions of (29). This is a consequence of the symmetry of $M_{l\Omega}$, by rotations leaving invariant the orientation Ω. We concentrate on the case $0 < \frac{\sigma}{r^2} < \frac{1}{d}, \lambda(l) = \frac{\sigma}{r^2} l, l > 0$.

⟨Struct1⟩

Proposition 3.2 *For any $W \in \mathbb{R}^d, W \cdot \Omega = 0$, let us denote by $\widetilde{\psi}_W$ the unique solution of the problem*

$$-\sigma \mathrm{div}_\omega (M_{l\Omega} \nabla_\omega \widetilde{\psi}_W) = M_{l\Omega}\ (\omega - u) \cdot W = M_{l\Omega}\ (\omega \cdot W), \ \int_{r\mathbb{S}^{d-1}} \widetilde{\psi}_W \ \mathrm{d}\omega = 0.$$

For any orthogonal transformation $\mathcal{O}$ of $\mathbb{R}^d$, leaving invariant the orientation Ω, that is $\mathcal{O}\Omega = \Omega$, we have

$$\widetilde{\psi}_W(\mathcal{O}\omega) = \widetilde{\psi}_{{}^t\mathcal{O}W}(\omega), \ \ \omega \in r\mathbb{S}^{d-1}.$$

Proof. We know that $\widetilde{\psi}_W$ is the minimum point of the functional

$$J_W(z) = \frac{\sigma}{2} \int_{r\mathbb{S}^{d-1}} M_{l\Omega} |\nabla_\omega z|^2 \ \mathrm{d}\omega - \int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega \cdot W) z(\omega) \ \mathrm{d}\omega, z \in H^1(r\mathbb{S}^{d-1}), \int_{r\mathbb{S}^{d-1}} z(\omega) \ \mathrm{d}\omega = 0.$$

It is easily seen that, for any orthogonal transformation $\mathcal{O}$ of $\mathbb{R}^d$, and any function $z \in H^1(r\mathbb{S}^{d-1})$, $\int_{r\mathbb{S}^{d-1}} z(\omega) \ \mathrm{d}\omega = 0$, we have

$$z_{\mathcal{O}} := z \circ \mathcal{O} \in H^1(r\mathbb{S}^{d-1}), \ \ \int_{r\mathbb{S}^{d-1}} z_{\mathcal{O}}(\omega) \ \mathrm{d}\omega = 0$$

and

$$(\nabla_\omega z_{\mathcal{O}})(\omega) = {}^t\mathcal{O}(\nabla_\omega z)(\mathcal{O}\omega), \ \ \omega \in r\mathbb{S}^{d-1}.$$

Moreover, for any $z \in H^1(r\mathbb{S}^{d-1}), \int_{r\mathbb{S}^{d-1}} z(\omega) \ \mathrm{d}\omega = 0$, and any orthogonal transformation leaving invariant the orientation Ω we obtain

$$\begin{aligned} J_{{}^t\mathcal{O}W}(z_{\mathcal{O}}) &= \frac{\sigma}{2} \int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega) |{}^t\mathcal{O}(\nabla_\omega z)(\mathcal{O}\omega)|^2 \ \mathrm{d}\omega - \int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega)(\omega \cdot {}^t\mathcal{O}W) z(\mathcal{O}\omega) \ \mathrm{d}\omega \\ &= \frac{\sigma}{2} \int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\mathcal{O}\omega) |(\nabla_\omega z)(\mathcal{O}\omega)|^2 \ \mathrm{d}\omega - \int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\mathcal{O}\omega)(\mathcal{O}\omega \cdot W) z(\mathcal{O}\omega) \ \mathrm{d}\omega \\ &= \frac{\sigma}{2} \int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega) |\nabla_\omega z(\omega)|^2 \ \mathrm{d}\omega - \int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega)(\omega \cdot W) z(\omega) \ \mathrm{d}\omega \\ &= J_W(z). \end{aligned}$$

Finally, one gets for any $z \in H^1(r\mathbb{S}^{d-1}), \int_{r\mathbb{S}^{d-1}} z(\omega) \ \mathrm{d}\omega = 0$

$$J_{{}^t\mathcal{O}W}(\widetilde{\psi}_W \circ \mathcal{O}) = J_W(\widetilde{\psi}_W) \leq J_W(z \circ {}^t\mathcal{O}) = J_{{}^t\mathcal{O}W}(z)$$

saying that $\widetilde{\psi}_W \circ \mathcal{O} = \widetilde{\psi}_{{}^t\mathcal{O}W}$. □

We claim that there is a function χ such that, for any $i \in \{1, ..., d-1\}$, the solution $\widetilde{\psi}_i$ writes

$$\widetilde{\psi}_i(\omega) = \chi\left(\Omega \cdot \frac{\omega}{r}\right) c_i(\omega), \;\; c_i(\omega) = \frac{\omega \cdot E_i}{\sqrt{r^2 - (\omega \cdot \Omega)^2}}, \;\; \omega \in r\mathbb{S}^{d-1} \setminus \{\pm r\Omega\}.$$

⟨InvField⟩ **Lemma 3.2** *We consider the vector field F given by*

$$F(\omega) = \sum_{i=1}^{d-1} \widetilde{\psi}_i(\omega) E_i, \;\; \omega \in r\mathbb{S}^{d-1}.$$

Then the vector field F do not depend on the orthonormal basis $\{E_1, ..., E_{d-1}\}$ of $(\mathbb{R}\Omega)^\perp$ and for any orthogonal transformation $\mathcal{O}$ of $\mathbb{R}^d$, preserving Ω, we have

$$F(\mathcal{O}\omega) = \mathcal{O}F(\omega), \;\; \omega \in r\mathbb{S}^{d-1}.$$

There is a function χ such that

$$F(\omega) = \chi\left(\Omega \cdot \frac{\omega}{r}\right) \frac{(I_d - \Omega \otimes \Omega)(\omega)}{\sqrt{r^2 - (\Omega \cdot \omega)^2}}, \;\; \omega \in r\mathbb{S}^{d-1} \setminus \{\pm r\Omega\}$$

and thus, for any $i \in \{1, ..., d-1\}$, we have

$$\widetilde{\psi}_i(\omega) = \chi\left(\Omega \cdot \frac{\omega}{r}\right) \frac{\omega \cdot E_i}{\sqrt{r^2 - (\Omega \cdot \omega)^2}}, \;\; \omega \in r\mathbb{S}^{d-1} \setminus \{\pm r\Omega\}.$$

Proof. Consider any other orthonormal basis $\{F_1, ..., F_{d-1}\}$ of $(\mathbb{R}\Omega)^\perp$. Thanks to the identities

$$E_1 \otimes E_1 + ... + E_{d-1} \otimes E_{d-1} + \Omega \otimes \Omega = I_d, \;\; F_1 \otimes F_1 + ... + F_{d-1} \otimes F_{d-1} + \Omega \otimes \Omega = I_d$$

we obtain

$$\begin{aligned}\sum_{i=1}^{d-1} \widetilde{\psi}_i E_i &= \sum_{i=1}^{d-1} \widetilde{\psi}_{E_i} E_i = \sum_{i=1}^{d-1} \widetilde{\psi}_{\sum_{j=1}^{d-1}(E_i \cdot F_j)F_j} E_i = \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} (E_i \cdot F_j) \widetilde{\psi}_{F_j} E_i \\ &= \sum_{j=1}^{d-1} \widetilde{\psi}_{F_j} \sum_{i=1}^{d-1} (E_i \cdot F_j) E_i = \sum_{j=1}^{d-1} \widetilde{\psi}_{F_j} F_j.\end{aligned}$$

Pick $\mathcal{O}$ any orthogonal transformation of $\mathbb{R}^d$, leaving invariant Ω. For any $\omega \in r\mathbb{S}^{d-1}$, we can write, by Proposition 3.2

$$F(\mathcal{O}\omega) = \sum_{i=1}^{d-1} \widetilde{\psi}_{E_i}(\mathcal{O}\omega) E_i = \sum_{i=1}^{d-1} \widetilde{\psi}_{^t\mathcal{O}E_i}(\omega) E_i = \mathcal{O} \sum_{i=1}^{d-1} \widetilde{\psi}_{^t\mathcal{O}E_i}(\omega)\, {}^t\mathcal{O}E_i = \mathcal{O}F(\omega)$$

where, in the last equality, we have used the independence of F with respect to the orthonormal basis of $(\mathbb{R}\Omega)^\perp$. Take now $\omega \in r\mathbb{S}^{d-1} \setminus \{\pm r\Omega\}$ and

$$E = \frac{(Id - \Omega \otimes \Omega)\omega}{\sqrt{r^2 - (\Omega \cdot \omega)^2}}.$$

Clearly $E \cdot \Omega = 0, |E| = 1$.

If $d = 2$, as we know that $F(\omega) \cdot \Omega = 0$, there is $\Lambda = \Lambda(\omega)$ such that

$$F(\omega) = \Lambda(\omega) E = \Lambda(\omega) \frac{(I_2 - \Omega \otimes \Omega)\omega}{\sqrt{r^2 - (\Omega \cdot \omega)^2}}.$$

If $d \geq 3$, take any unitary vector ${}^{\perp}E$, orthogonal to E and Ω, and consider the symmetry

$$\mathcal{O} = I_d - 2\, {}^{\perp}E \otimes {}^{\perp}E.$$

The above orthogonal transformation leaves invariant Ω, and thus, by the hypothesis, we know that

$$F(\mathcal{O}\omega') = \mathcal{O}F(\omega'),\ \ \omega' \in r\mathbb{S}^{d-1}.$$

Observe that

$$0 = {}^{\perp}E \cdot E = {}^{\perp}E \cdot \frac{\omega - (\omega \cdot \Omega)\Omega}{\sqrt{r^2 - (\Omega \cdot \omega)^2}} = \frac{{}^{\perp}E \cdot \omega}{\sqrt{r^2 - (\Omega \cdot \omega)^2}},\ \ \mathcal{O}\omega = \omega$$

and thus

$$F(\omega) = F(\mathcal{O}\omega) = (I_d - 2\, {}^{\perp}E \otimes {}^{\perp}E)F(\omega) = F(\omega) - 2(F(\omega) \cdot {}^{\perp}E)\, {}^{\perp}E.$$

We deduce that $F(\omega) \cdot {}^{\perp}E = 0$ for any vector ${}^{\perp}E$, orthogonal to E and Ω. As $F(\Omega) \cdot \Omega = 0$, we deduce that $F(\omega)$ is orthogonal to any vector orthogonal to E, anf thus there is $\Lambda = \Lambda(\omega)$ such that

$$F(\omega) = \Lambda(\omega) E = \Lambda(\omega) \frac{(I_d - \Omega \otimes \Omega)\omega}{\sqrt{r^2 - (\Omega \cdot \omega)^2}},\ \ \omega \in r\mathbb{S}^{d-1} \setminus \{\pm r\Omega\}.$$

We claim that $\Lambda(\omega)$ depends only on $\Omega \cdot \frac{\omega}{r}$. Indeed, for any $d \geq 2$, and any orthogonal transformation $\mathcal{O}$, such that $\mathcal{O}\Omega = \Omega$, we have $F(\mathcal{O}\omega) = \mathcal{O}F(\omega)$

$$(I_d - \Omega \otimes \Omega)\mathcal{O}\omega = \mathcal{O}\omega - (\Omega \cdot \mathcal{O}\omega)\Omega = \mathcal{O}\omega - (\Omega \cdot \omega)\mathcal{O}\Omega = \mathcal{O}(I_d - \Omega \otimes \Omega)\omega,\ \ \omega \in r\mathbb{S}^{d-1} \setminus \{\pm r\Omega\}$$

$$\sqrt{r^2 - (\Omega \cdot \mathcal{O}\omega)^2} = |(I_d - \Omega \otimes \Omega)\mathcal{O}\omega| = |\mathcal{O}(I_d - \Omega \otimes \Omega)\omega| = |(I_d - \Omega \otimes \Omega)\omega| = \sqrt{r^2 - (\Omega \cdot \omega)^2}$$

implying that $\Lambda(\mathcal{O}\omega) = \Lambda(\omega), \omega \in r\mathbb{S}^{d-1} \setminus \{\pm r\Omega\}$. Actually, the previous equality holds true for any $\omega \in r\mathbb{S}^{d-1}$, since $\mathcal{O}\Omega = \Omega$. We are done if we prove that $\Lambda(\omega) = \Lambda(\omega')$ for any $\omega, \omega' \in r\mathbb{S}^{d-1} \setminus \{\pm r\Omega\}$ such that $\Omega \cdot \omega = \Omega \cdot \omega', \omega \neq \omega'$. Consider the rotation $\mathcal{O}$ such that

$$\mathcal{O}E = E',\ \ (\mathcal{O} - I_d)|_{\text{span}\{E,E'\}^{\perp}} = 0,\ \ E = \frac{(I_d - \Omega \otimes \Omega)\omega}{\sqrt{r^2 - (\Omega \cdot \omega)^2}},\ \ E' = \frac{(I_d - \Omega \otimes \Omega)\omega'}{\sqrt{r^2 - (\Omega \cdot \omega')^2}}.$$

Notice that the condition $\mathcal{O}E = E'$ exactly says that $\mathcal{O}\omega = \omega'$ and thus $\Lambda(\omega') = \Lambda(\mathcal{O}\omega) = \Lambda(\omega)$. We deduce that there is a function χ such that $\Lambda(\omega) = \chi\left(\Omega \cdot \frac{\omega}{r}\right)$ and therefore

$$\sum_{i=1}^{d-1} \widetilde{\psi}_i(\omega) E_i = F(\omega) = \chi\left(\Omega \cdot \frac{\omega}{r}\right) \frac{(I_d - \Omega \otimes \Omega)\omega}{\sqrt{r^2 - (\Omega \cdot \omega)^2}} = \sum_{i=1}^{d-1} \chi\left(\Omega \cdot \frac{\omega}{r}\right) \frac{\omega \cdot E_i}{\sqrt{r^2 - (\Omega \cdot \omega)^2}} E_i$$

implying that

$$\widetilde{\psi}_i(\omega) = \chi\left(\Omega \cdot \frac{\omega}{r}\right) \frac{\omega \cdot E_i}{\sqrt{r^2 - (\Omega\cdot\omega)^2}}, \;\; i \in \{1, ..., d-1\}, \;\; \omega \in r\mathbb{S}^{d-1} \setminus \{\pm r\Omega\}.$$

□

?⟨BiDim⟩?

Remark 3.1 *In the case $d = 2$, we take $E_1 = {}^\perp\Omega, \omega = r(\cos\theta\Omega + \sin\theta\, {}^\perp\Omega)$ and therefore $\widetilde{\psi}_1$ writes*

$$\widetilde{\psi}_1(r(\cos\theta\Omega + \sin\theta\, {}^\perp\Omega)) = \chi(\cos\theta)\text{sign}(\sin\theta), \;\; \theta \in]-\pi, 0[\, \cup\,]0, \pi[.$$

Clearly, the function $\overline{\psi}_1(\theta) := \widetilde{\psi}_1(r(\cos\theta\Omega + \sin\theta\, {}^\perp\Omega))$ is odd (in particular $\int_{r\mathbb{S}^1} \widetilde{\psi}_1(\omega)\mathrm{d}\omega = \int_{-\pi}^{\pi} \overline{\psi}_1(\theta) r\mathrm{d}\theta = 0$) and the condition

$$\int_{r\mathbb{S}^1} |\nabla_\omega \widetilde{\psi}_1|^2 M_{l\Omega}(\omega)\mathrm{d}\omega < +\infty$$

implies that $\int_{-\pi}^{\pi} |\partial_\theta \overline{\psi}_1|^2 \mathrm{d}\theta < +\infty$. Therefore $\overline{\psi}_1$ is continuous on $]-\pi, \pi[$, and thus $\chi(1) = 0$. Notice that $\chi(-1) = 0$ as well, since $\lim_{\theta\nearrow\pi} \overline{\psi}_1(\theta) = \widetilde{\psi}_1(-r\Omega) = \lim_{\theta\searrow-\pi} \overline{\psi}_1(\theta)$.

Thanks to Lemma 3.2, in order to determine $\widetilde{\psi}_i, i \in \{1, ..., d-1\}$, we only need to solve for χ. The idea is to analyse the behavior of the functionals J_{E_i} on the set of functions $\Psi_{i,h}(\omega) = h\left(\Omega \cdot \frac{\omega}{r}\right) c_i(\omega), \omega \in r\mathbb{S}^{d-1}$. The notation P_ω stands for the orthogonal projection on the tangent space to $r\mathbb{S}^{d-1}$ at ω, that is, $P_\omega = I_d - \frac{\omega\otimes\omega}{r^2}$.

Proposition 3.3 *The function χ constructed in Lemma 3.2 solves the problem*

$$-\frac{\sigma}{r^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{e^{lc}\chi'(c)(1-c^2)^{\frac{1}{2}}\right\} = re^{lc}, \;\; c \in]-1, 1[, \;\; \chi(-1) = \chi(1) = 0 \;\; \textit{if } d = 2 \qquad (34)$$ Prob2D

and

$$-\frac{\sigma}{r^2}\frac{\mathrm{d}}{\mathrm{d}c}\left\{e^{lc}\chi'(c)(1-c^2)^{\frac{d-1}{2}}\right\} + (d-2)\frac{\sigma}{r^2}e^{lc}\chi(c)(1-c^2)^{\frac{d-5}{2}} = re^{lc}(1-c^2)^{\frac{d-2}{2}}, \; c \in]-1, 1[, \; \textit{if } d \geq 3. \qquad (35)$$ Prob3D

Proof. For any $i \in \{1, ..., d-1\}$, the gradient of $\Psi_{i,h}$ writes

$$\nabla_\omega \Psi_{i,h} = h'\left(\Omega \cdot \frac{\omega}{r}\right) c_i(\omega) \frac{P_\omega\Omega}{r} + h\left(\Omega \cdot \frac{\omega}{r}\right) \nabla_\omega c_i$$

where

$$\nabla_\omega c_i = \frac{P_\omega E_i}{\sqrt{r^2 - (\omega\cdot\Omega)^2}} + \frac{(\omega \cdot E_i)(\omega\cdot\Omega)}{(r^2 - (\omega\cdot\Omega)^2)^{3/2}} P_\omega \Omega.$$

Therefore we obtain

$$\nabla_\omega \psi_{i,h} = h'\left(\Omega\cdot\frac{\omega}{r}\right)\frac{\omega\cdot E_i}{\sqrt{r^2-(\omega\cdot\Omega)^2}}\frac{P_\omega\Omega}{r} + \frac{h\left(\Omega\cdot\frac{\omega}{r}\right)}{\sqrt{r^2-(\omega\cdot\Omega)^2}}\left[P_\omega E_i + \frac{(\omega\cdot E_i)(\omega\cdot\Omega)}{r^2-(\omega\cdot\Omega)^2}P_\omega\Omega\right].$$

Notice that $P_\omega\Omega$ and $\nabla_\omega c_i$ are orthogonal, thanks to the equality $|P_\omega\Omega|^2 = 1-\frac{(\omega\cdot\Omega)^2}{r^2}$. Indeed, we have

$$P_\omega\Omega\cdot\left[P_\omega E_i + \frac{(\omega\cdot E_i)(\omega\cdot\Omega)}{r^2-(\omega\cdot\Omega)^2}P_\omega\Omega\right] = -\frac{(\omega\cdot E_i)(\omega\cdot\Omega)}{r^2} + \frac{(\omega\cdot E_i)(\omega\cdot\Omega)}{r^2-(\omega\cdot\Omega)^2}|P_\omega\Omega|^2 = 0.$$

Observe also that

$$|\nabla_\omega c_i|^2 = \frac{1}{r^2-(\omega\cdot\Omega)^2}\left[1-\frac{(\omega\cdot E_i)^2}{r^2-(\omega\cdot\Omega)^2}\right]$$

implying that

$$\begin{aligned}|\nabla_\omega \Psi_{i,h}|^2 &= \left(h'\left(\Omega\cdot\frac{\omega}{r}\right)c_i(\omega)\right)^2\frac{|P_\omega\Omega|^2}{r^2} + \left(h\left(\Omega\cdot\frac{\omega}{r}\right)\right)^2|\nabla_\omega c_i|^2\\ &= \frac{\left(h'\left(\Omega\cdot\frac{\omega}{r}\right)\right)^2(\omega\cdot E_i)^2}{r^4} + \frac{\left(h\left(\Omega\cdot\frac{\omega}{r}\right)\right)^2}{r^2-(\omega\cdot\Omega)^2}\left[1-\frac{(\omega\cdot E_i)^2}{r^2-(\omega\cdot\Omega)^2}\right].\end{aligned}$$

Performing orthogonal changes of coordinates, which preserve Ω, we deduce that the integrals $\int_{r\mathbb{S}^{d-1}}|\nabla_\omega\Psi_{i,h}|^2 M_{l\Omega}\,\mathrm{d}\omega$ do not depend on $i\in\{1,...,d-1\}$, and thus

$$\begin{aligned}\int_{r\mathbb{S}^{d-1}}|\nabla_\omega\Psi_{i,h}|^2 M_{l\Omega}\,\mathrm{d}\omega &= \frac{1}{d-1}\int_{r\mathbb{S}^{d-1}}\frac{\left(h'\left(\Omega\cdot\frac{\omega}{r}\right)\right)^2}{r^4}[r^2-(\omega\cdot\Omega)^2]M_{l\Omega}\,\mathrm{d}\omega \\ &\quad + \frac{d-2}{d-1}\int_{r\mathbb{S}^{d-1}}\frac{\left(h\left(\Omega\cdot\frac{\omega}{r}\right)\right)^2}{r^2-(\omega\cdot\Omega)^2}M_{l\Omega}\,\mathrm{d}\omega.\end{aligned} \tag{36}$$

EquFunct1

We also need to compute the linear part of the functional J_{E_i}

$$\int_{r\mathbb{S}^{d-1}} M_{l\Omega}\,(\omega\cdot E_i)h\left(\Omega\cdot\frac{\omega}{r}\right)c_i(\omega)\,\mathrm{d}\omega = \frac{1}{d-1}\int_{r\mathbb{S}^{d-1}} M_{l\Omega}\,h\left(\Omega\cdot\frac{\omega}{r}\right)\sqrt{r^2-(\omega\cdot\Omega)^2}\,\mathrm{d}\omega. \tag{37}$$

EquFunct2

The expression of $J_{E_i}(\psi_{i,h})$ follows by (36), (37)

$$\begin{aligned}J_{E_i}(\psi_{i,h}) &= \frac{\sigma}{2(d-1)}\int_{r\mathbb{S}^{d-1}} M_{l\Omega}\left(h'\left(\Omega\cdot\frac{\omega}{r}\right)\right)^2\frac{r^2-(\Omega\cdot\omega)^2}{r^4}\,\mathrm{d}\omega\\ &\quad + \frac{\sigma}{2}\frac{d-2}{d-1}\int_{r\mathbb{S}^{d-1}} M_{l\Omega}\frac{\left(h\left(\Omega\cdot\frac{\omega}{r}\right)\right)^2}{r^2-(\omega\cdot\Omega)^2}\,\mathrm{d}\omega\\ &\quad - \frac{1}{d-1}\int_{r\mathbb{S}^{d-1}} M_{l\Omega}h\left(\Omega\cdot\frac{\omega}{r}\right)\sqrt{r^2-(\omega\cdot\Omega)^2}\,\mathrm{d}\omega\\ &= \frac{\sigma}{2(d-1)r^2}\frac{\int_0^\pi e^{l\cos\theta}(h'(\cos\theta))^2\sin^d\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\\ &\quad + \frac{\sigma}{2}\frac{d-2}{d-1}\frac{\int_0^\pi e^{l\cos\theta}\left(\frac{h(\cos\theta)}{r\sin\theta}\right)^2\sin^{d-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\\ &\quad - \frac{1}{d-1}\frac{\int_0^\pi e^{l\cos\theta}h(\cos\theta)r\sin\theta\sin^{d-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\\ &= \frac{J(h)}{(d-1)\pi\beta_0(l)}\end{aligned}$$

where $\pi\beta_0(l) = \int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta \,\mathrm{d}\theta$ and

$$J(h) = \frac{\sigma}{2r^2}\int_{-1}^{1} e^{lc}(h'(c))^2(1-c^2)^{\frac{d-1}{2}}\,\mathrm{d}c + \frac{\sigma}{2}\frac{d-2}{r^2}\int_{-1}^{1} e^{lc}(h(c))^2(1-c^2)^{\frac{d-5}{2}}\,\mathrm{d}c$$
$$- r\int_{-1}^{1} e^{lc}h(c)(1-c^2)^{\frac{d-2}{2}}\,\mathrm{d}c.$$

We consider the Hilbert spaces

$$H_2 = \{h :]-1,1[\to\mathbb{R},\ \ (1-c^2)^{1/4}h' \in L^2(]-1,1[),\ \ h(-1) = h(1) = 0\}$$

and

$$H_d = \{h :]-1,1[\to\mathbb{R},\ \ (1-c^2)^{\frac{d-1}{4}}h' \in L^2(]-1,1[),\ \ (1-c^2)^{\frac{d-5}{4}}h \in L^2(]-1,1[)\},\ \ d \geq 3$$

endowed with the scalar products

$$(g,h)_2 = \int_{-1}^{1} g'(c)h'(c)\sqrt{1-c^2}\,\mathrm{d}c,\ \ g,h\in H_2$$

and

$$(g,h)_d = \int_{-1}^{1} g'(c)h'(c)(1-c^2)^{\frac{d-1}{2}}\,\mathrm{d}c + \int_{-1}^{1} g(c)h(c)(1-c^2)^{\frac{d-5}{2}}\,\mathrm{d}c,\ \ g,h\in H_d,\ \ \text{if } d\geq 3.$$

By Lemma 3.2, there is a function χ such that $\widetilde{\psi}_i = \chi\left(\Omega\cdot\frac{\omega}{r}\right)c_i(\omega), i\in\{1,...,d-1\}$. We know that $\widetilde{\psi}_i, i\in\{1,...,d-1\}$, minimize the functionals $J_{E_i}(z)$, with $z\in H^1(r\mathbb{S}^{d-1})$, $\int_{r\mathbb{S}^{d-1}} z(\omega)\,\mathrm{d}\omega = 0$. In particular, for any $h\in H_d, d\geq 2$, we have

$$J_{E_i}(\Psi_{i,h}) \geq J_{E_i}(\widetilde{\psi}_i),\ \ \Psi_{i,h}(\omega) = h\left(\Omega\cdot\frac{\omega}{r}\right)c_i(\omega)$$

implying that χ, which belongs to H_d, is the solution of the minimization problem

$$J(h) \geq J(\chi),\ \ h\in H_d.$$

Thanks to the Lax-Milgram lemma, we deduce that χ is the solution of the problem (34) if $d=2$, and (35) if $d\geq 3$. □

Up to now, for a given equilibrium $F = M_{l\Omega}\,\mathrm{d}\omega$, we have determined the functions ψ such that

$$\int_{\mathbb{R}^d}\psi(v)\lim_{\varepsilon\searrow 0}\frac{\langle Q(F+\varepsilon G^{(1)})\rangle - \langle Q(F)\rangle}{\varepsilon}\,\mathrm{d}v = 0$$

for any bounded measure $G^{(1)}$, supported in $r\mathbb{S}^{d-1}$. But we need to control the linearization of $\langle Q\rangle$ around the equilibrium F in the direction $F^{(1)}$, which is not necessarily supported in $r\mathbb{S}^{d-1}$. It happens that the constraint $\mathrm{div}_v\{F^{(1)}(\alpha-\beta|v|^2)v\} = Q(F)$, see (9), will guarantee that

$$\int_{\mathbb{R}^d}\psi(v)\lim_{\varepsilon\searrow 0}\frac{\langle Q(F+\varepsilon F^{(1)})\rangle - \langle Q(F)\rangle}{\varepsilon}\,\mathrm{d}v = \int_{\mathbb{R}^d}\psi(v)\left\langle\mathrm{div}_v A_F(F^{(1)})\right\rangle\,\mathrm{d}v = 0.$$

These computations are a little bit tedious and can be found in Appendix C.

⟨ZeroAvePart⟩

Proposition 3.4 *Let $F = M_{l\Omega}\mathrm{d}\omega$ be a von Mises-Fisher distribution with $l > 0$, and $F^{(1)}$ be a bounded measure (not charging a small neighborhood of 0, for simplifying), satisfying $\mathrm{div}_v\{F^{(1)}(\alpha - \beta|v|^2)v\} = Q(F)$. Then the linearized of $\langle Q\rangle$ around F in the direction $F^{(1)}$ verifies*

$$\int_{\mathbb{R}^d} \widetilde{\psi}(v) \left\langle \mathrm{div}_v A_F(F^{(1)})\right\rangle \mathrm{d}v = 0, \text{ for any collision invariant } \widetilde{\psi} \text{ of } \langle Q\rangle .$$

4 The limit model

⟨LimMod⟩

We identify the model satisfied by the limit distribution $f = \lim_{\varepsilon\searrow 0} f^{\varepsilon}$. We already know that f is a von Mises-Fisher distribution $f = \rho(t,x)M_{l\Omega(t,x)}(\omega)\mathrm{d}\omega$ with $\rho \geq 0, \Omega \in \mathbb{S}^{d-1}, l \geq 0, \lambda(l) = \frac{\sigma}{r^2}l$. If $\frac{\sigma}{r^2} \geq \frac{1}{d}$, then $l = 0$ and $M_{l\Omega}\mathrm{d}\omega$ reduces to the isotropic measure on $r\mathbb{S}^{d-1}$, that is $f = \rho(t,x)\frac{\mathrm{d}\omega}{r^{d-1}\omega_d}$, with zero mean velocity $u[f] = \int_{r\mathbb{S}^{d-1}} \omega\rho M_{l\Omega}\ \mathrm{d}\omega = 0$. In this case, the continuity equation reduces to the trivial limit model $\partial_t \rho = 0, t \in \mathbb{R}_+$. From now on, we assume that $\frac{\sigma}{r^2} \in]0, \frac{1}{d}[$, and we consider $l > 0$ the unique solution for $\lambda(l) = \frac{\sigma}{r^2}l$ cf. Proposition 1.5. We are ready to justify the main result in Theorem 1.1.

Proof. (of Theorem 1.1)

The continuity equation (14) comes from the continuity equation of (13)

$$\partial_t \int_{\mathbb{R}^d} f\ \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} fv\ \mathrm{d}v = \lim_{\varepsilon\searrow 0}\left\{\partial_t \int_{\mathbb{R}^d} f^{\varepsilon}\ \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} f^{\varepsilon} v\ \mathrm{d}v\right\} = 0$$

and the formula for the mean velocity of a von Mises-Fisher equilibrium

$$u[f] = \int_{r\mathbb{S}^{d-1}} \omega\rho M_{l\Omega}\ \mathrm{d}\omega = \rho\frac{l\sigma}{r}\Omega = \rho\lambda(l)r\Omega.$$

Equivalently, (14) is obtained by using the collision invariant $\widetilde{\psi} = 1$. The equation (15) will follow, by using the $(d-1)$ dimensional linear space of collision invariants studied in Proposition 3.1. Revisiting the expansion (7), we obtain

$$\partial_t f + \mathrm{div}_x(fv) + \mathrm{div}_v\{f^{(2)}(\alpha - \beta|v|^2)v\} = \nu\mathrm{div}_v(A_f(f^{(1)})) \tag{38}$$ Equ75

together with the constarints

$$\mathrm{div}_v\{f(\alpha - \beta|v|^2)v\} = 0 \tag{39}$$ Equ76

$$\mathrm{div}_v\{f^{(1)}(\alpha - \beta|v|^2)v\} = Q(f). \tag{40}$$ Equ77

The first constraint (39) says that, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\mathrm{supp}\, f(t,x) \subset \{0\} \cup r\mathbb{S}^{d-1}$. Averaging the second constraint (40) leads to

$$\langle Q(f)\rangle = \left\langle \mathrm{div}_v\{f^{(1)}(\alpha - \beta|v|^2)v\}\right\rangle = 0$$

and thus $f(t,x) = \rho(t,x) M_{l\Omega(t,x)}(\omega) \mathrm{d}\omega, \omega \in r\mathbb{S}^{d-1}$. Averaging (38) allows us to get rid of $f^{(2)}$

$$\partial_t \langle f \rangle + \mathrm{div}_x \langle v f \rangle = \nu \left\langle \mathrm{div}_v A_f(f^{(1)}) \right\rangle. \tag{41}$$ Equ78

In order to eliminate $f^{(1)}$ as well, we test (41) against the functions $\psi_i(v) = \widetilde{\psi}_i \left(r\frac{v}{|v|} \right)$, where $(\widetilde{\psi}_i)_{1 \le i \le d-1}$ are the collision invariants constructed in Proposition 3.1. Indeed, by Proposition 3.4, we know that

$$\int_{v \neq 0} \widetilde{\psi} \left(r\frac{v}{|v|} \right) \left\langle \mathrm{div}_v A_f(f^{(1)}) \right\rangle \, \mathrm{d}v = \int_{v \neq 0} \widetilde{\psi} \left(r\frac{v}{|v|} \right) \mathrm{div}_v A_f(f^{(1)}) \, \mathrm{d}v = I[\widetilde{\psi}_i] = 0, \;\; i \in \{1, ..., d-1\}$$

and therefore

$$\int_{r\mathbb{S}^{d-1}} \partial_t (\rho M_{l\Omega}) \widetilde{\psi}_i \, \mathrm{d}\omega + \int_{r\mathbb{S}^{d-1}} \mathrm{div}_x (\rho M_{l\Omega} \omega) \widetilde{\psi}_i(\omega) \, \mathrm{d}\omega = 0, \;\; i \in \{1, ..., d-1\}. \tag{42}$$ Equ79

Let $\{E_1, ..., E_{d-1}, \Omega\}$ be a orthonormal basis and $\widetilde{\psi}_1, ..., \widetilde{\psi}_{d-1}$ be the solutions of the problems (32). We recall that

$$\sum_{i=1}^{d-1} \widetilde{\psi}_i E_i = F(\omega) = \chi \left(\Omega \cdot \frac{\omega}{r} \right) \frac{(I_d - \Omega \otimes \Omega) \frac{\omega}{r}}{\sqrt{1 - \left(\Omega \cdot \frac{\omega}{r} \right)^2}}.$$

The equation (42), written for $i \in \{1, ..., d-1\}$, says that

$$(I_d - \Omega \otimes \Omega) \int_{r\mathbb{S}^{d-1}} [\partial_t (\rho M_{l\Omega}) + \mathrm{div}_x (\rho M_{l\Omega} \omega)] \frac{\chi \left(\frac{\omega}{r} \cdot \Omega \right)}{\sqrt{1 - \left(\Omega \cdot \frac{\omega}{r} \right)^2}} \frac{\omega}{r} \, \mathrm{d}\omega = 0.$$

We need to compute the vectors

$$U_1 = \int_{r\mathbb{S}^{d-1}} \partial_t \rho M_{l\Omega}(\omega) \frac{\chi \left(\frac{\omega}{r} \cdot \Omega \right)}{\sqrt{1 - \left(\Omega \cdot \frac{\omega}{r} \right)^2}} \frac{\omega}{r} \, \mathrm{d}\omega$$

$$U_2 = \int_{r\mathbb{S}^{d-1}} \rho M_{l\Omega}(\omega) \; l \; \partial_t \Omega \cdot \frac{\omega}{r} \frac{\chi \left(\frac{\omega}{r} \cdot \Omega \right)}{\sqrt{1 - \left(\Omega \cdot \frac{\omega}{r} \right)^2}} \frac{\omega}{r} \, \mathrm{d}\omega$$

$$U_3 = \int_{r\mathbb{S}^{d-1}} \omega \cdot \nabla_x \rho M_{l\Omega}(\omega) \frac{\chi \left(\frac{\omega}{r} \cdot \Omega \right)}{\sqrt{1 - \left(\Omega \cdot \frac{\omega}{r} \right)^2}} \frac{\omega}{r} \, \mathrm{d}\omega$$

$$U_4 = \int_{r\mathbb{S}^{d-1}} l \rho \omega \cdot {}^t \partial_x \Omega \frac{\omega}{r} \; M_{l\Omega}(\omega) \frac{\chi \left(\frac{\omega}{r} \cdot \Omega \right)}{\sqrt{1 - \left(\Omega \cdot \frac{\omega}{r} \right)^2}} \frac{\omega}{r} \, \mathrm{d}\omega$$

and to impose

$$\sum_{i=1}^{4} (I_d - \Omega \otimes \Omega) U_i = 0. \tag{43}$$ Equ85

Clearly, the first vector U_1 is parallel to Ω, and thus

$$(I_d - \Omega \otimes \Omega) U_1 = 0. \tag{44}$$ Equ86

The treatment of the second and third vectors requires to compute

$$\mathcal{A} := \int_{r\mathbb{S}^{d-1}} \frac{\omega}{r} \otimes \frac{\omega}{r} M_{l\Omega}(\omega) \frac{\chi\left(\frac{\omega}{r}\cdot\Omega\right)}{\sqrt{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}}\,\mathrm{d}\omega$$

$$= \sum_{i=1}^{d-1} \int_{r\mathbb{S}^{d-1}} \frac{(\omega\cdot E_i)^2}{r^2} M_{l\Omega}(\omega) \frac{\chi\left(\frac{\omega}{r}\cdot\Omega\right)}{\sqrt{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}}\,\mathrm{d}\omega\; E_i\otimes E_i$$

$$+ \int_{r\mathbb{S}^{d-1}} \frac{(\omega\cdot\Omega)^2}{r^2} M_{l\Omega}(\omega) \frac{\chi\left(\frac{\omega}{r}\cdot\Omega\right)}{\sqrt{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}}\,\mathrm{d}\omega\; \Omega\otimes\Omega$$

$$= \frac{1}{d-1}\int_{r\mathbb{S}^{d-1}} \left[1-\left(\Omega\cdot\frac{\omega}{r}\right)^2\right] M_{l\Omega}(\omega) \frac{\chi\left(\frac{\omega}{r}\cdot\Omega\right)}{\sqrt{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}}\,\mathrm{d}\omega \sum_{i=1}^{d-1} E_i\otimes E_i$$

$$+ \int_{r\mathbb{S}^{d-1}} \frac{(\omega\cdot\Omega)^2}{r^2} M_{l\Omega}(\omega) \frac{\chi\left(\frac{\omega}{r}\cdot\Omega\right)}{\sqrt{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}}\,\mathrm{d}\omega\; \Omega\otimes\Omega$$

$$= \frac{\int_0^\pi \sin^2\theta e^{l\cos\theta}\,\frac{\chi(\cos\theta)}{\sin\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\;\frac{I_d-\Omega\otimes\Omega}{d-1}$$

$$+ \frac{\int_0^\pi \cos^2\theta e^{l\cos\theta}\,\frac{\chi(\cos\theta)}{\sin\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\Omega\otimes\Omega.$$

We obtain, thanks to the identity $\partial_t\Omega\cdot\Omega = \frac{1}{2}\partial_t|\Omega|^2 = 0$

$$(I_d-\Omega\otimes\Omega)U_2 = (I_d-\Omega\otimes\Omega)\rho l\mathcal{A}\partial_t\Omega = \frac{\rho l}{d-1}\frac{\int_0^\pi e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\partial_t\Omega \tag{45}$$

Equ87

and

$$(I_d-\Omega\otimes\Omega)U_3 = r(I_d-\Omega\otimes\Omega)\mathcal{A}\nabla_x\rho = \frac{r}{d-1}\frac{\int_0^\pi e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}(I_d-\Omega\otimes\Omega)\nabla_x\rho. \tag{46}$$

Equ88

We concentrate now on the last vector U_4. Observe that

$$(I_d-\Omega\otimes\Omega)U_4 = r\rho l\int_{r\mathbb{S}^{d-1}} \frac{\omega}{r}\otimes\frac{\omega}{r} : \partial_x\Omega\; M_{l\Omega}(\omega)\frac{\chi\left(\frac{\omega}{r}\cdot\Omega\right)}{\sqrt{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}}\sum_{i=1}^{d-1}\left(E_i\cdot\frac{\omega}{r}\right)E_i\,\mathrm{d}\omega$$

and for any $i\in\{1,...,d-1\}$

$$\int_{r\mathbb{S}^{d-1}} \frac{\omega}{r}\otimes\frac{\omega}{r} : \partial_x\Omega\; M_{l\Omega}(\omega)\frac{\chi\left(\frac{\omega}{r}\cdot\Omega\right)}{\sqrt{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}}\left(E_i\cdot\frac{\omega}{r}\right)\,\mathrm{d}\omega$$

$$= \int_{r\mathbb{S}^{d-1}} \frac{(\omega\cdot E_i)^2\,(\omega\cdot\Omega)}{r^3} M_{l\Omega}(\omega)\frac{\chi\left(\frac{\omega}{r}\cdot\Omega\right)}{\sqrt{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}}\,\mathrm{d}\omega\,[E_i\otimes\Omega:\partial_x\Omega + \Omega\otimes E_i:\partial_x\Omega]$$

$$= \int_{r\mathbb{S}^{d-1}} \frac{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}{d-1}\;\frac{(\omega\cdot\Omega)}{r} M_{l\Omega}(\omega)\frac{\chi\left(\frac{\omega}{r}\cdot\Omega\right)}{\sqrt{1-\left(\Omega\cdot\frac{\omega}{r}\right)^2}}\,\mathrm{d}\omega\,[E_i\otimes\Omega:\partial_x\Omega + \Omega\otimes E_i:\partial_x\Omega]$$

$$= \frac{1}{d-1}\frac{\int_0^\pi \sin^2\theta\cos\theta e^{l\cos\theta}\,\frac{\chi(\cos\theta)}{\sin\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}(\partial_x\Omega\Omega\cdot E_i + {}^t\partial_x\Omega\Omega\cdot E_i).$$

Thanks to the formula ${}^t\partial_x\Omega\Omega = \frac{1}{2}\nabla_x|\Omega|^2 = 0$, we obtain

$$(I_d - \Omega\otimes\Omega)U_4 = \frac{r\rho l}{d-1}\frac{\int_0^\pi \cos\theta e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\sum_{i=1}^{d-1}(\partial_x\Omega\Omega\cdot E_i)E_i \quad (47)\ \text{Equ89}$$
$$= \frac{r\rho l}{d-1}\frac{\int_0^\pi \cos\theta e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}(I_d - \Omega\otimes\Omega)\partial_x\Omega\Omega$$
$$= \frac{r\rho l}{d-1}\frac{\int_0^\pi \cos\theta e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\partial_x\Omega\Omega.$$

The evolution equation for the orientation Ω comes now by (43), (44), (45), (46), (47)

$$\frac{\rho l\partial_t\Omega + r(I_d-\Omega\otimes\Omega)\nabla_x\rho}{d-1}\frac{\int_0^\pi e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}$$
$$+\frac{r\rho l}{d-1}\frac{\int_0^\pi \cos\theta e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta\,\mathrm{d}\theta}\partial_x\Omega\Omega = 0$$

which also writes

$$\partial_t\Omega + r\frac{\int_0^\pi \cos\theta e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\chi(\cos\theta)\sin^{d-1}\theta\,\mathrm{d}\theta}(\Omega\cdot\nabla_x)\Omega + \frac{r}{l}(I_d-\Omega\otimes\Omega)\frac{\nabla_x\rho}{\rho} = 0.$$

□

Remark 4.1 *Taking the scalar product of the equation* (15) *with* Ω*, we obtain*

$$\frac{1}{2}\partial_t|\Omega|^2 + \frac{k_d r}{2}(\Omega\cdot\nabla_x)|\Omega|^2 = 0,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d$$

implying that $|\Omega(t,x)| = 1, (t,x)\in\mathbb{R}_+\times\mathbb{R}^d$*, provided that* $|\Omega(0,x)| = 1, x\in\mathbb{R}^d$.

A Integration by parts on spheres

⟨A⟩

Proof. (of Lemma 2.1)

We pick a function $\eta\in C_c^1(]r_1,r_2[)$ and observe that

$$\mathrm{div}_v\{\eta(|v|)A(v)\} = \eta'(|v|)\frac{v}{|v|}\cdot A(v) + \eta(|v|)(\mathrm{div}_v A)(v),\quad v\in\mathcal{O}.$$

Integrating with respect to v over $\mathcal{O}$ leads to

$$0 = \int_{\mathcal{O}}\mathrm{div}_v\{\eta(|v|)A(v)\}\,\mathrm{d}v = \int_{\mathcal{O}}\eta'(|v|)\frac{v}{|v|}\cdot A(v)\,\mathrm{d}v + \int_{\mathcal{O}}\eta(|v|)(\mathrm{div}_v A)(v)\,\mathrm{d}v$$
$$= \int_{r_1}^{r_2}\eta'(t)\int_{|\omega|=1}\omega\cdot A(t\omega)t^{d-1}\,\mathrm{d}\omega\mathrm{d}t + \int_{r_1}^{r_2}\eta(t)\int_{|\omega|=1}(\mathrm{div}_v A)(t\omega)t^{d-1}\,\mathrm{d}\omega\mathrm{d}t$$
$$= \int_{r_1}^{r_2}\eta(t)\left[-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|\omega|=1}\omega\cdot A(t\omega)t^{d-1}\,\mathrm{d}\omega + \int_{|\omega|=1}(\mathrm{div}_v A)(t\omega)t^{d-1}\,\mathrm{d}\omega\right]\,\mathrm{d}t.$$

We deduce that

$$\begin{aligned}\int_{|\omega|=t} (\mathrm{div}_v A)(\omega)\ \mathrm{d}\omega &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{|\omega|=1} \omega\cdot A(t\omega)t^{d-1}\ \mathrm{d}\omega\\ &= \int_{|\omega|=1} \{\omega\cdot\partial_v A(t\omega)\omega t^{d-1} + \omega\cdot A(t\omega)(d-1)t^{d-2}\}\ \mathrm{d}\omega\\ &= \int_{|\omega|=t}\left\{\frac{\omega\otimes\omega}{t^2} : \partial_v A(\omega) + \frac{(d-1)\omega}{t^2}\cdot A(\omega)\right\}\ \mathrm{d}\omega.\end{aligned}$$

Assume now that $A(v)\cdot v = 0, v\in\mathcal{O}$. Taking the gradient with respect to v yields ${}^t\partial_v A(v)v + A(v) = 0$ implying $\partial_v A(v) : v\otimes v = -A(v)\cdot v = 0, v\in\mathcal{O}$. In this case (16) reduces to (17). The formula in (18) follows easily by applying (17) with the field $v\to\chi(v)A(v)$. $\square$

B Differential operators on spheres

⟨B⟩

Proof. (of Lemma 2.2)

1. Pick a point $\omega\in r\mathbb{S}^{d-1}$ and a tangent vector $X\in T_\omega(r\mathbb{S}^{d-1})$. Let $\gamma :]-\varepsilon,\varepsilon[\to r\mathbb{S}^{d-1}$ be a smooth curve such that $\gamma(0)=\omega$, $\gamma\,'(0) = X$. Then we have

$$\begin{aligned}\nabla_\omega\widetilde{\psi}\cdot X = \mathrm{d}\widetilde{\psi}_\omega(X) &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\widetilde{\psi}(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\psi(\gamma(t))\\ &= \widetilde{\nabla_v\psi}(\omega)\cdot X = \left(I_d - \frac{\omega\otimes\omega}{r^2}\right)\widetilde{\nabla_v\psi}(\omega)\cdot X\end{aligned}$$

saying that

$$\nabla_\omega\widetilde{\psi} - \left(I_d - \frac{\omega\otimes\omega}{r^2}\right)\widetilde{\nabla_v\psi}\in T_\omega(r\mathbb{S}^{d-1})\cap(T_\omega(r\mathbb{S}^{d-1}))^\perp = \{0\}.$$

Therefore we deduce that $\nabla_\omega\widetilde{\psi} = \left(I_d - \frac{\omega\otimes\omega}{r^2}\right)\widetilde{\nabla_v\psi}$.

2. For any $\omega_t\in t\mathbb{S}^{d-1}$ and $X\in T_{\omega_t}(t\mathbb{S}^{d-1})$, pick a smooth curve $\gamma :]-\varepsilon,\varepsilon[\to t\mathbb{S}^{d-1}$ such that $\gamma(0)=\omega_t, \gamma\,'(0) = X$. Therefore we have

$$\nabla_{\omega_t}\psi(\omega_t)\cdot X = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\psi(\gamma(s)) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\widetilde{\psi}\left(r\frac{\gamma(s)}{t}\right) = \nabla_\omega\widetilde{\psi}\left(r\frac{\omega_t}{t}\right)\cdot\frac{r}{t}X$$

saying that $(\nabla_{\omega_t}\psi)(\omega_t) = \frac{r}{t}(\nabla_\omega\widetilde{\psi})\left(r\frac{\omega_t}{t}\right)$. Actually the function ψ has only tangent gradient (to the spheres), and thus

$$(\nabla_v\psi)(\omega_t) = (\nabla_{\omega_t}\psi)(\omega_t) = \frac{r}{t}(\nabla_\omega\widetilde{\psi})\left(r\frac{\omega_t}{t}\right),\ \ |\omega_t| = t.$$

3. Consider a C^1 function $\widetilde{\psi}$ on $r\mathbb{S}^{d-1}$ and ψ a C^1 extension of $\widetilde{\psi}$ on $\mathcal{O}$. By Lemma 2.1, we know that

$$\int_{|\omega|=r}\widetilde{\nabla_v\psi}(\omega)\cdot\widetilde{\xi}(\omega)\ \mathrm{d}\omega + \int_{|\omega|=r}\widetilde{\psi}(\omega)\widetilde{\mathrm{div}_v\xi}(\omega)\ \mathrm{d}\omega = 0. \qquad (48)\ \texttt{EquStep1}$$

But, by the previous statement, we can write

$$\widetilde{\nabla_v \psi}(\omega)\cdot\widetilde{\xi}(\omega) = \widetilde{\nabla_v \psi}(\omega)\cdot\left(I_d - \frac{\omega\otimes\omega}{r^2}\right)\widetilde{\xi}(\omega) = \left(I_d - \frac{\omega\otimes\omega}{r^2}\right)\widetilde{\nabla_v \psi}(\omega)\cdot\widetilde{\xi}(\omega) = \nabla_\omega \widetilde{\psi}(\omega)\cdot\widetilde{\xi}(\omega). \tag{49}$$ EquStep2

Combining (48), (49) yields

$$\int_{|\omega|=r} \widetilde{\psi}(\omega)\mathrm{div}_\omega \widetilde{\xi}(\omega)\ \mathrm{d}\omega = -\int_{|\omega|=r} \nabla_\omega \widetilde{\psi}(\omega)\cdot\widetilde{\xi}(\omega)\ \mathrm{d}\omega = \int_{|\omega|=r} \widetilde{\psi}(\omega)\widetilde{\mathrm{div}_v \xi}(\omega)\ \mathrm{d}\omega,\ \ \widetilde{\psi}\in C^1(r\mathbb{S}^{d-1})$$

implying that $\mathrm{div}_\omega \widetilde{\xi} = \widetilde{\mathrm{div}_v \xi}$.

□

C Collision invariants and linearization of $\langle Q\rangle$

⟨C⟩

Proof. (of Proposition 3.4)

Consider a collision invariant $\widetilde{\psi}$, and let us compute

$$I[\widetilde{\psi}] := \int_{v\neq 0} \widetilde{\psi}\left(r\frac{v}{|v|}\right)\mathrm{div}_v A_F(F^{(1)})\ \mathrm{d}v$$

that is

$$\begin{aligned} I[\widetilde{\psi}] &= \nu \int_{v\neq 0}\left\{-(v-u[F])\cdot\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] + \sigma\Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\right\}F^{(1)}\ \mathrm{d}v \\ &+ \nu\int_{v\neq 0}(v-u[F])\cdot\frac{\int_{v'\neq 0}\nabla_{v'}\left[\widetilde{\psi}\left(r\frac{v'}{|v'|}\right)\right]F\ \mathrm{d}v'}{\int_{\mathbb{R}^d}F\ \mathrm{d}v'}F^{(1)}\ \mathrm{d}v. \end{aligned}$$

We consider the application

$$\begin{aligned} \chi(v) &= -(v-u[F])\cdot\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] + \sigma\Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] \\ &+ (v-u[F])\cdot\frac{\int_{v'\neq 0}\nabla_{v'}\left[\widetilde{\psi}\left(r\frac{v'}{|v'|}\right)\right]F\ \mathrm{d}v'}{\int_{\mathbb{R}^d}F\ \mathrm{d}v'} \\ &= u[F]\cdot\nabla_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] + \sigma\Delta_v\left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] \\ &+ (v-u[F])\cdot\int_{r\mathbb{S}^{d-1}}(\nabla_{\omega'}\widetilde{\psi})(\omega')M_{l\Omega}(\omega')\ \mathrm{d}\omega\,',\ \ v\neq 0. \end{aligned}$$

As $\widetilde{\psi}$ is a collision invariant, we have $\chi(\omega)=0$, for any $\omega\in r\mathbb{S}^{d-1}$ cf. (29). Thanks to Lemma 2.5, the integral $I[\widetilde{\psi}]$ can be written

$$\begin{aligned} I[\widetilde{\psi}] &= \nu\int_{v\neq 0}\chi(v)F^{(1)}\ \mathrm{d}v = \frac{\nu^2\sigma}{\beta}\frac{M_{l\Omega}}{M}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=r}\int_{|\omega_t|=t}M(\omega_t)\frac{\chi(\omega_t)}{t(t^2-r^2)}\ \mathrm{d}\omega_t \\ &= \frac{\nu^2\sigma}{\beta}\frac{M_{l\Omega}}{M}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=r}\int_{|\omega|=r}M\left(t\frac{\omega}{r}\right)\frac{\chi\left(t\frac{\omega}{r}\right)}{t(t^2-r^2)}\left(\frac{t}{r}\right)^{d-1}\ \mathrm{d}\omega. \end{aligned}$$

Thanks to the second statement in Lemma 2.2, we can write

$$\nabla_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\left(t\frac{\omega}{r}\right) = \frac{r}{t}(\nabla_\omega \widetilde{\psi})(\omega)$$

and by Remark 2.1, we have

$$\Delta_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right]\left(t\frac{\omega}{r}\right) = \left(\frac{r}{t}\right)^2 (\Delta_\omega \widetilde{\psi})(\omega).$$

Therefore, the function $t \to \chi\left(t\frac{\omega}{r}\right)$ is given by

$$\chi\left(t\frac{\omega}{r}\right) = \frac{r}{t}u[F]\cdot(\nabla_\omega\widetilde{\psi})(\omega) + \sigma\frac{r^2}{t^2}(\Delta_\omega\widetilde{\psi})(\omega) + \left(t\frac{\omega}{r} - u[F]\right)\cdot W[\widetilde{\psi}]$$

with $W[\widetilde{\psi}] = \int_{r\mathbb{S}^{d-1}} \nabla_\omega\widetilde{\psi} M_{l\Omega}(\omega)\ \mathrm{d}\omega$. As $\chi(\omega) = 0, \omega \in r\mathbb{S}^{d-1}$, because $\widetilde{\psi}$ is a collision invariant, we obtain

$$\begin{aligned}
M\left(t\frac{\omega}{r}\right)\frac{\chi\left(t\frac{\omega}{r}\right)}{t(t^2-r^2)} &= M\left(t\frac{\omega}{r}\right)\frac{\chi\left(t\frac{\omega}{r}\right)-\chi(\omega)}{t(t^2-r^2)}\\
&= M\left(t\frac{\omega}{r}\right)\frac{\frac{r-t}{t}u[F]\cdot(\nabla_\omega\widetilde{\psi})(\omega) + \sigma\frac{r^2-t^2}{t^2}(\Delta_\omega\widetilde{\psi})(\omega) + \frac{t-r}{r}\omega\cdot W[\widetilde{\psi}]}{t(t-r)(t+r)}\\
&= M\left(t\frac{\omega}{r}\right)\frac{\omega\cdot W[\widetilde{\psi}]}{rt(t+r)} - M\left(t\frac{\omega}{r}\right)\frac{\sigma}{t^3}(\Delta_\omega\widetilde{\psi})(\omega) - M\left(t\frac{\omega}{r}\right)\frac{u[F]\cdot(\nabla_\omega\widetilde{\psi})(\omega)}{t^2(t+r)}\\
&= \frac{M\left(t\frac{\omega}{r}\right)}{rt(t+r)}[\omega\cdot W[\widetilde{\psi}] + u[F]\cdot(\nabla_\omega\widetilde{\psi})(\omega)] - \frac{\sigma}{t^3}\mathrm{div}_\omega\left(M\left(t\frac{\omega}{r}\right)\nabla_\omega\widetilde{\psi}\right).
\end{aligned}$$

It is easily seen that $\int_{r\mathbb{S}^{d-1}} M\left(t\frac{\omega}{r}\right)\omega\ \mathrm{d}\omega \in \mathbb{R}\Omega$ and, as we know that $W[\widetilde{\psi}] \in (\mathbb{R}\Omega)^\perp$, we deduce that

$$\int_{r\mathbb{S}^{d-1}} M\left(t\frac{\omega}{r}\right)\omega\cdot W[\widetilde{\psi}]\ \mathrm{d}\omega = 0.$$

Taking into account that

$$\int_{r\mathbb{S}^{d-1}} \mathrm{div}_\omega\left\{M\left(t\frac{\omega}{r}\right)\nabla_\omega\widetilde{\psi}\right\}\ \mathrm{d}\omega = 0$$

we deduce that

$$\begin{aligned}
I[\widetilde{\psi}] &= \frac{\nu^2\sigma}{\beta}\frac{M_{l\Omega}}{M}\frac{\mathrm{d}}{\mathrm{dt}}|_{t=r}\left[\left(\frac{t}{r}\right)^{d-1}\int_{r\mathbb{S}^{d-1}}\frac{M\left(t\frac{\omega}{r}\right)\nabla_\omega\widetilde{\psi}\cdot u[F]}{rt(t+r)}\ \mathrm{d}\omega\right]\\
&= \frac{\nu^2\sigma}{\beta}\frac{M_{l\Omega}}{M}\frac{\mathrm{d}}{\mathrm{dt}}|_{t=r}\left[\left(\frac{t}{r}\right)^{d-1}\frac{1}{rt(t+r)}\right]\int_{r\mathbb{S}^{d-1}} M(\omega)\nabla_\omega\widetilde{\psi}\cdot u[F]\,\mathrm{d}\omega\\
&+ \frac{\nu^2\sigma}{2r^3\beta}\frac{M_{l\Omega}}{M}\frac{\mathrm{d}}{\mathrm{dt}}|_{t=r}\int_{r\mathbb{S}^{d-1}} M\left(t\frac{\omega}{r}\right)\nabla_\omega\widetilde{\psi}\cdot u[F]\,\mathrm{d}\omega.
\end{aligned}$$

As before

$$\frac{M_{l\Omega}}{M}\int_{r\mathbb{S}^{d-1}} M(\omega)\nabla_\omega\widetilde{\psi}\ \mathrm{d}\omega\cdot u[F] = \int_{r\mathbb{S}^{d-1}} M_{l\Omega}\nabla_\omega\widetilde{\psi}\ \mathrm{d}\omega\cdot u[F] = W[\widetilde{\psi}]\cdot u[F] = 0$$

implying that

$$
\begin{aligned}
I[\widetilde{\psi}] &= \frac{\nu^2\sigma}{2r^3\beta}\frac{M_{l\Omega}}{M}\frac{\mathrm{d}}{\mathrm{d}t}|_{t=r}\int_{r\mathbb{S}^{d-1}} M\left(t\frac{\omega}{r}\right)\nabla_\omega\widetilde{\psi}\cdot u[F]\,\mathrm{d}\omega\\
&= \frac{\nu^2\sigma}{2r^3\beta}\frac{M_{l\Omega}}{M}\int_{r\mathbb{S}^{d-1}} M(\omega)\left(\frac{u[F]-\omega}{\sigma}\cdot\frac{\omega}{r}\right)\left(\nabla_\omega\widetilde{\psi}\cdot u[F]\right)\,\mathrm{d}\omega\\
&= \frac{\nu^2}{2r^4\beta}\int_{r\mathbb{S}^{d-1}} M_{l\Omega}(u[F]\cdot\omega - r^2)(\nabla_\omega\widetilde{\psi}\cdot u[F])\,\mathrm{d}\omega\\
&= \frac{\nu^2}{2r^4\beta}\int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\nabla_\omega\widetilde{\psi}\cdot u[F])(\omega\cdot u[F])\,\mathrm{d}\omega.
\end{aligned}
$$

In the last equality we have used one more time that $W[\widetilde{\psi}]\cdot u[F]=0$. We claim that the last integral vanishes. Indeed, multiplying by $(\omega\cdot u[F])^2$ the equation (30) satisfied by the collision invariant $\widetilde{\psi}$ one gets

$$
\begin{aligned}
2\sigma\int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\nabla_\omega\widetilde{\psi}\cdot u[F])(\omega\cdot u[F])\,\mathrm{d}\omega &= W[\widetilde{\psi}]\cdot\int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega\cdot u[F])^2(\omega-u[F])\,\mathrm{d}\omega\\
&= W[\widetilde{\psi}]\cdot\int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega\cdot u[F])^2\omega\,\mathrm{d}\omega.
\end{aligned}
$$

It is easily seen that $\int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega\cdot u[F])^2\omega\,\mathrm{d}\omega\in\mathbb{R}\Omega$ and therefore

$$
W[\widetilde{\psi}]\cdot\int_{r\mathbb{S}^{d-1}} M_{l\Omega}(\omega\cdot u[F])^2\omega\,\mathrm{d}\omega = 0
$$

saying that $I[\widetilde{\psi}]=0$. □

Acknowledgments

References

AbraSteg [1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions : with formulas, graphs and mathematical tables (Dover Books on Mathematics) 1965.

BTTYB [2] A. Barbaro, K. Taylor, P.F. Trethewey, L. Youseff, B. Birnir, Discrete and continuous models of the dynamics of pelagic fish: application to the capelin, Mathematics and Computers in Simulation 79(2009) 3397-3414.

BEBSVPSS [3] A. Barbaro, B. Einarsson, B. Birnir, S. Sigurthsson, H. Valdimarsson, O.K. Palsson, S. Sveinbjornsson, T. Sigurthsson, Modelling and simulations of the migration of pelagic fish, ICES J. Mar. Sci. 66(2009) 826-838.

BCC11 [4] F. Bolley, J.A. Cañizo, J.A. Carrillo, Stochastic Mean-Field Limit: Non-Lipschitz Forces & Swarming, Math. Mod. Meth. Appl. Sci. 21(2011) 2179-2210.

BCC12 [5] F. Bolley, J.A. Cañizo, J.A. Carrillo, Mean-field limit for the stochastic Vicsek model, Appl. Math. Letters 25(2012) 339-343.

BosAsyAna [6] M. Bostan, The Vlasov-Poisson system with strong external magnetic field. Finite Larmor radius regime, Asymptot. Anal. 61(2009) 91-123.

BosTraEquSin [7] M. Bostan, Transport equations with disparate advection fields. Application to the gyrokinetic models in plasma physics, J. Differential Equations 249(2010) 1620-1663.

BosGuiCen3D [8] M. Bostan, Gyro-kinetic Vlasov equation in three dimensional setting. Second order approximation, SIAM J. Multiscale Model. Simul. 8(2010) 1923-1957.

Bos12 [9] M. Bostan, Transport of charged particles under fast oscillating magnetic fields, SIAM J. Math. Anal. 44(2012) 1415-1447.

BosCar13 [10] M. Bostan, J.-A. Carrillo, Asymptotic fixed-speed reduced dynamics for kinetic equations in swarming, Math. Mod. Meth. Appl. Sci. 23(2013) 2353-2393.

BosCalQAM1 [11] M. Bostan, C. Caldini-Queiros, Finite Larmor radius approximation for collisional magnetic confinement. Part I: the linear Boltzmann equation, Quart. Appl. Math. Vol. LXXII(2014) 323-345.

BosCalQAM2 [12] M. Bostan, C. Caldini-Queiros, Finite Larmor radius approximation for collisional magnetic confinement. Part II: the Fokker-Planck-Landau equation, Quart. Appl. Math. Vol. LXXII(2014) 513-548.

BosDCDS15 [13] M. Bostan, On the Boltzmann equation for charged particle beams under the effect of strong magnetic fields, Discrete Contin. Dyn. Syst. Ser. B 20(2015) 339-371.

BosAnisoDiff [14] M. Bostan, Strongly anisotropic diffusion problems; asymptotic analysis, J. Differential Equations 256(2016) 1043-1092.

BosIHP15 [15] M. Bostan, High magnetic field equilibria for the Fokker-Planck-Landau equation, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire.

BraHep77 [16] W. Braun, K. Hepp, The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles, Commun. Math. Phys. 56(1977) 101-113.

CCR10 [17] J.A. Cañizo, J.A. Carrillo, J. Rosado, *A well-posedness theory in measures for some kinetic models of collective motion*, Math. Mod. Meth. Appl. Sci. 21(2011) 515-539.

CarDorPan09 [18] J.A. Carrillo, M.R. D'Orsogna, V. Panferov, Double milling in a self-propelled swarms from kinetic theory, Kinet. Relat. Models 2(2009) 363-378.

CFRT10 [19] J.A. Carrillo, M. Fornasier, J. Rosado, G. Toscani, Asymptotic flocking dynamics for the kinetic Cucker-Smale model, SIAM J. Math. Anal. 42(2010) 218-236.

review [20] Carrillo, J.A., Fornasier, M., Toscani, G., Vecil, F., *Particle, Kinetic, and Hydrodynamic Models of Swarming.* Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, Series: Modelling and Simulation in Science and Technology, Birkhauser (2010), 297-336.

UAB25 [21] J.A. Carrillo, A. Klar, S. Martin, S. Tiwari, Self-propelled interacting particle systems with roosting force, Math. Mod. Meth. Appl. Sci. 20(2010) 1533-1552.

ChuDorMarBerCha07 [22] Y.-L. Chuang, M.R. D'Orsogna, D. Marthaler, A.L. Bertozzi, L. Chayes, State transitions and the continuum limit for a 2D interacting, self-propelled particle system, Physica D 232(2007) 33-47.

ChuHuaDorBer07 [23] Y.-L. Chuang, Y.R. Huang, M.R. D'Orsogna, A.L. Bertozzi, Multi-vehicle flocking : scalability of cooperative control algorithms using pairwise potentials, IEEE International Conference on Robotics and Automation (2007) 2292-2299.

CouKraFraLev02 [24] I. D. Couzin, J. Krause, R. James, G. D. Ruxton and N. R. Franks, Collective Memory and Spatial Sorting in Animal Groups, J. theor. Biol. 218(2002) 1-11.

CouKraFraLev05 [25] I.D. Couzin, J. Krause, N.R. Franks, S.A. Levin, Effective leadership and decision making in animal groups on the move, Nature 433(2005) 513-516.

CS2 [26] F. Cucker and S. Smale. Emergent behavior in flocks. *IEEE Trans. Automat. Control* 52(2007) 852-862.

CziStaVic97 [27] A. Czirok, H.E. Stanley, T. Vicsek, Spontaneously ordered motion of self-propelled particles, J. Phys. A: Math. Gen. 30(1997) 1375-1385.

DFL10 [28] P. Degond, A. Frouvelle, J.-G. Liu, Macroscopic limits and phase transition in a system of self-propelled particles, J. Nonlinear Sci. 23 (2013) 427-456.

DM08 [29] P. Degond, S. Motsch, Continuum limit of self-driven particles with orientation interaction, Math. Mod. Meth. Appl. Sci. 18(2008) 1193-1215.

Dob79 [30] R. Dobrushin, Vlasov equations, Funct. Anal. Appl. 13(1979) 115-123.

DorChuBerCha06 [31] M.R. D'Orsogna, Y.-L. Chuang, A.L. Bertozzi, L. Chayes, Self-propelled particles with soft-core interactions : Patterns, stability and collapse, Phys. Rev. Lett. 96(2006) 104302-1/4.

EbeErd03 [32] W. Ebeling, U. Erdmann, Nonequilibrium statistical mechanics of swarms of driven particles, Complexity 8(2003) 23-30.

FL11 [33] A. Frouvelle, J.-G. Liu, Dynamics in a kinetic model of oriented particles with phase transition, SIAM J. Math. Anal. 44 (2012) 791-826.

GreCha04 [34] G. Grégoire, H. Chaté, On set of collective and cohesive motion, Phys. Rev. Lett. 92(2004) 025702-1/4.

HL08 [35] S.-Y. Ha, J.-G. Liu, A simple proof of the Cucker-Smale flocking dynamics and mean-field limit, Comm. Math. Sci. 7(2009) 297-325.

HLL09 [36] S.-Y. Ha, K. Lee, D. Levy, Emergence of Time-Asymptotic Flocking in a Stochastic Cucker-Smale System, Comm. Math. Sci. 7(2009) 453-469.

HT08 [37] S.-Y. Ha, E. Tadmor, From particle to kinetic and hydrodynamic descriptions of flocking, Kinetic and Related Models 1(2008) 415-435.

HCH09 [38] H. Hildenbrandt, C. Carere, C.K. Hemelrijk, Self-organised complex aerial displays of thousands of starlings: a model, Behav. Ecol. 21(2010) 1349-1359.

HW [39] A. Huth, C. Wissel, The Simulation of the Movement of Fish Schools, Journal of Theoretical Biology 156(1992) 365-385.

LevRapCoh00 [40] H. Levine, W.-J. Rappel, I. Cohen, Self-organization in systems of self-propelled particles, Phys. Rev. E 63(2000) 017101-1/4.

MT11 [41] S. Motsch, E. Tadmor, A new model for self-organized dynamics and its flocking behavior, J. Stat. Phys. 144(2011) 923-947.

Neu77 [42] H. Neunzert, The Vlasov equation as a limit of Hamiltonian classical mechanical systems of interacting particles, Trans. Fluid Dynamics, 18(1977) 663-678.

ParEde99 [43] J. Parrish, L. Edelstein-Keshet, Complexity, pattern and evolutionary trade-offs in animal aggregation, Science 294(1999) 99-101.

VicCziBenCohSho95 [44] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen, O. Shochet, Novel type of phase transition in a system of self-driven particles, Phys. Rev. Lett. 75(1995) 1226-1229.