ASYMPTOTIC BEHAVIOR OF WEAK SOLUTIONS FOR THE RELATIVISTIC VLASOV-MAXWELL EQUATIONS WITH LARGE LIGHT SPEED. CONVERGENCE TOWARD WEAK SOLUTIONS FOR THE CLASSICAL VLASOV-POISSON EQUATIONS

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Abstract. *We study here the behavior of time periodic weak solutions for the relativistic Vlasov-Maxwell boundary value problem in a three dimensional bounded domain with strictly star-shaped boundary when the light speed becomes infinite. We prove the convergence toward a time periodic weak solution for the classical Vlasov-Poisson equations.*

Key words. Vlasov-Maxwell equations, Vlasov-Poisson equations, weak formulation, permanent regimes.

AMS subject classifications. 35F30, 35L40.

1. Introduction.

In this paper we analyze the behavior of weak solutions for the relativistic Vlasov-Maxwell equations with boundary conditions, when the light speed goes to infinite. We prove the convergence toward a weak solution for the classical Vlasov-Poisson equations. Our main interests focus on permanent regimes, *i.e.,* stationary or time periodic solutions.

The Vlasov equation describes the kinetic of charged particles of a plasma. This equation is coupled to evolution equations for the electro-magnetic field. If the magnetic field is neglected, we end up with the Poisson equation for an electrostatic potential ; this leads to the Vlasov-Poisson system. Otherwise, if the magnetic field is not small, the full Maxwell equations must be considered ; this gives the Vlasov-Maxwell system.

Consider Ω an open bounded subset of $\mathbb{R}^3_x$, with boundary $\partial\Omega$ regular. We introduce the notations $\Sigma = \partial\Omega \times \mathbb{R}^3_p$ and :

$$\Sigma^{\pm} = \{(x,p) \in \partial\Omega \times \mathbb{R}^3_p \mid \pm (v(p) \cdot n(x)) > 0\}, \tag{1.1}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at x and $v(p)$ is the velocity associated to some energy function $\mathcal{E}(p)$ by $v(p) = \nabla_p \mathcal{E}(p),\ p \in \mathbb{R}^3_p$. The functions to be considered are :

$$\mathcal{E}(p) = \frac{|p|^2}{2m}, \ \ v(p) = \frac{p}{m}, \tag{1.2}$$

for the classical case and :

$$\mathcal{E}_c(p) = mc^2 \left(\left(1 + \frac{|p|^2}{m^2c^2}\right)^{1/2} - 1 \right), \ \ v_c(p) = \frac{p}{m}\left(1 + \frac{|p|^2}{m^2c^2}\right)^{-1/2}, \tag{1.3}$$

for the relativistic case, where m is the mass of particles, c is the light speed in the vacuum. We denote by $f(t,x,p)$ the particles distribution depending on the time t, the position $x \in \Omega$ and the momentum $p \in \mathbb{R}^3_p$ and by $(E(t,x), B(t,x))$ the electro-magnetic field depending on t and x. If we note by $F(t,x,p) = q \cdot (E(t,x) + v(p) \wedge B(t,x))$ the electro-magnetic force, the Vlasov problem is given by :

$$\partial_t f + v(p) \cdot \nabla_x f + q \cdot (E(t,x) + v(p) \wedge B(t,x)) \cdot \nabla_p f = 0, \ \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p, \tag{1.4}$$

$$f(t,x,p) = g(t,x,p), \ \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-, \tag{1.5}$$

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where q is the charge of particles and g is a given T periodic function representing the distribution of the incoming particles. The problem (1.4), (1.5) is coupled with the Maxwell equations :

$$\partial_t E - c^2 \cdot \text{rot } B = -\frac{j}{\varepsilon_0}, \;\; \partial_t B + \text{rot } E = 0, \;\; \text{div } E = \frac{\rho}{\varepsilon_0}, \;\; \text{div } B = 0, \;\; (t,x) \in \mathbb{R}_t \times \Omega, \tag{1.6}$$

with the boundary condition :

$$n(x) \wedge E(t,x) + c \cdot n(x) \wedge (n(x) \wedge B(t,x)) = h(t,x), \;\; (t,x) \in \mathbb{R}_t \times \partial\Omega, \tag{1.7}$$

where ε_0 is the permittivity of the vacuum, $\rho(t,x) = q \int_{\mathbb{R}^3_p} f(t,x,p)dp$ is the charge density, $j(t,x) = q \int_{\mathbb{R}^3_p} f(t,x,p)v(p)dp$ is the current density and h is a given T periodic function on the boundary $\mathbb{R}_t \times \partial\Omega$ such that $(n \cdot h)|_{\mathbb{R}_t \times \partial\Omega} = 0$. We suppose that the boundary data have finite energy $\int_0^T\int_{\Sigma^-} |(v(p) \cdot n(x))|\mathcal{E}(p) g(t,x,p) \; dt d\sigma dp + \int_0^T\int_{\partial\Omega} |h(t,x)|^2 \; dt d\sigma < +\infty$ and $0 \le g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$.

Various results were obtained for the free space system of Vlasov-Poisson. Weak solutions were constructed by Arseneev [1], Horst and Hunze [21]. The existence of classical solutions has been studied by Ukai and Okabe [30], Horst [20], Batt [2], Pfaffelmoser [25]. The existence of global classical solutions for the Vlasov-Poisson equations with small initial data is a result of Bardos and Degond [3], see also Schaeffer [28], [29]. The propagation of the moments for the three dimensional Vlasov-Poisson system was studied by Lions and Perthame in [23]. The existence of global weak solution for the Vlasov-Maxwell system in three dimensions was obtained by DiPerna and Lions [12], one of the key points being the compactness result of velocity averages (see also [16]). Results for the relativistic case were obtained by Glassey and Schaeffer [14], Glassey and Strauss [15].

Results for the initial-boundary value problem were obtained by Ben Abdallah [4] for the Vlasov-Poisson system in three dimensions and Guo [18] for the Vlasov-Maxwell system. The stationary problem for the Vlasov-Poisson equations was studied by Greengard and Raviart [17] in one dimension and by Poupaud [26] in three dimensions for the Vlasov-Maxwell system. An asymptotic analysis of the Vlasov-Poisson system was done by Degond and Raviart [11] in the case of the plane diode. The regularity of the solutions for the Vlasov-Maxwell system in a half line has been studied by Guo [19]. The convergence of smooth solutions for the Vlasov-Maxwell equations toward a solution for the Vlasov-Poisson equations when the light velocity goes to infinity was proved by Degond [10], Schaeffer [27]. Results for the time periodic case can be found in [5], [6], [7], [8].

We start by constructing T periodic weak solutions for the relativistic Vlasov-Maxwell system when the light speed c is fixed. The main ingredient are the a priori estimates, which derive from the conservation laws of the mass, momentum and total energy. As usual we multiply the Vlasov equation by $\mathcal{E}_c(p)$ and the Maxwell equations by $(E, c^2 \cdot B)$ to obtain formally :

$$\begin{aligned} \frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3_p} & \mathcal{E}_c(p) f(t,x,p) \; dxdp + \frac{\varepsilon_0}{2} \cdot \frac{d}{dt}\int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2) \; dx \\ & + \int_{\Sigma^+} (v_c(p) \cdot n(x)) \mathcal{E}_c(p) \gamma^+ f(t,x,p) \; d\sigma dp + \frac{\varepsilon_0 c}{2}\int_{\partial\Omega} (|n \wedge E(t,x)|^2 + c^2 \cdot |n \wedge B(t,x)|^2) \; d\sigma \\ & = \int_{\Sigma^-} |(v_c(p) \cdot n(x))| \mathcal{E}_c(p) g(t,x,p) \; d\sigma dp + \frac{\varepsilon_0 c}{2}\int_{\partial\Omega} |h(t,x)|^2 \; d\sigma, \;\; t \in \mathbb{R}_t, \end{aligned}$$

where $\gamma^+ f$ represents the trace of f on $\mathbb{R}_t \times \Sigma^+$. Note that in the time periodic case the above formula doesn't provide bounds for the total (kinetic and electro-magnetic) energy, since we don't dispose of initial conditions. Nevertheless, after integration over one period we obtain :

$$\begin{aligned} \int_0^T\int_{\Sigma^+} & (v_c(p) \cdot n(x)) \mathcal{E}_c(p) \gamma^+ f(t,x,p) \; dt d\sigma dp + \frac{\varepsilon_0 c}{2}\int_0^T\int_{\partial\Omega} (|n \wedge E(t,x)|^2 + c^2 \cdot |n \wedge B(t,x)|^2) \; dt d\sigma \\ & = \int_0^T\int_{\Sigma^-} |(v_c(p) \cdot n(x))| \mathcal{E}_c(p) g(t,x,p) \; dt d\sigma dp + \frac{\varepsilon_0 c}{2}\int_0^T\int_{\partial\Omega} |h(t,x)|^2 \; dt d\sigma. \end{aligned} \tag{1.8}$$

Of coarse, the estimate (1.8) is not sufficient, but other a priori estimates can be obtained by using the momentum conservation law (cf. [6]). For this we need to impose a geometrical hypothesis on the boundary : we assume that $\partial\Omega$ is strictly star-shaped (see also [22]).

Once we have constructed T periodic solutions for every $c > 0$, in order to study the behavior of these solutions when $c \to +\infty$ we are looking for uniform estimates with respect to c. Remark that for $c \geq 1$, the inequality (1.8) gives uniform estimates for the tangential traces of the electro-magnetic field :

$$\frac{\varepsilon_0}{2}\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E(t,x)|^2 + c^2\cdot|n\wedge B(t,x)|^2)\ dtd\sigma \leq \int_0^T\!\!\int_{\Sigma^-}|(v_c(p)\cdot n(x))|\mathcal{E}_c(p)g(t,x,p)\ dtd\sigma dp + \frac{\varepsilon_0}{2}\int_0^T\!\!\int_{\partial\Omega}|h(t,x)|^2\ dtd\sigma. \quad (1.9)$$

In particular, the inequality (1.9) implies that $\lim_{c\to+\infty}\|n\wedge B\|_{L^2(]0,T[\times\partial\Omega)^3} = 0$. Similarly, we need to estimate the total electro-magnetic energy $\frac{\varepsilon_0}{2}\int_0^T\!\int_\Omega(|E(t,x)|^2 + c^2\cdot|B(t,x)|^2)\ dtdx$ and the normal traces $\frac{\varepsilon_0}{2}\int_0^T\!\int_{\partial\Omega}(|(n\cdot E(t,x))|^2 + c^2\cdot|(n\cdot B(t,x))|^2)\ dtd\sigma$ in order to conclude that in the limit model (when $c \to +\infty$) the magnetic field vanishes and thus (f,E) verify the Vlasov-Poisson model. Indeed, following the ideas of [6] the total energy can be estimated in term of the tangential traces of the electro-magnetic field and the outgoing kinetic energy $K_c^+ := \int_0^T\!\int_{\Sigma^+}(v_c(p)\cdot n(x))\mathcal{E}_c(p)\gamma^+f(t,x,p)\ dtd\sigma dp$, but the problem is that the inequality (1.8) doesn't guarantee uniform estimate of K_c^+ with respect to c (unless $h = 0$). One of the main difficulties is to remove this dependence on c.

For example in the stationary case we write $E = -\nabla_x\Phi$ and by multiplying the Vlasov equation by $\mathcal{E}_c(p) + q(\Phi(x) + a)$, $a \in \mathbb{R}$ we find :

$$\int_\Sigma (v_c(p)\cdot n(x))(\mathcal{E}_c(p) + q(\gamma\Phi(x) + a))\gamma f\ d\sigma dp = 0. \quad (1.10)$$

By using Sobolev and interpolation inequalities we have :

$$\begin{aligned}\int_\Sigma (v_c(p)\cdot n(x))\mathcal{E}_c(p)\gamma f\ d\sigma dp &= -\int_\Sigma (v_c(p)\cdot n(x))q(\gamma\Phi + a)\gamma f\ d\sigma dp\\ &\leq |q|\inf_{a\in\mathbb{R}}\|\gamma\Phi + a\|_{L^5(\partial\Omega)}\cdot\left\|\int_{\mathbb{R}^3_p}(v_c(p)\cdot n(\cdot))\gamma f(\cdot,p)\ dp\right\|_{L^{\frac{5}{4}}(\partial\Omega)}\\ &\leq C\cdot\inf_{a\in\mathbb{R}}\|\gamma\Phi + a\|_{H^1(\partial\Omega)}\left(\int_\Sigma|(v_c\cdot n)|(1+\mathcal{E}_c(p))\gamma f\ d\sigma dp\right)^{\frac{4}{5}}\cdot\|g\|_{L^\infty}^{\frac{1}{5}}\\ &\leq C\cdot\|n\wedge E\|_{L^2(\partial\Omega)^3}\cdot(K_c^+ + K_c^- + 2M_c^-)^{\frac{4}{5}}, \quad (1.11)\end{aligned}$$

where $M_c^- := \int_{\Sigma^-}|(v_c(p)\cdot n(x))|g(x,p)\ d\sigma dp$, $K_c^- := \int_{\Sigma^-}|(v_c(p)\cdot n(x))|\mathcal{E}_c(p)g(x,p)\ d\sigma dp$. The inequalities (1.9), (1.11) imply uniform bounds for K_c^+. The time periodic case is more complicated ; we need to assume more regularity with respect to t for h, for example $\partial_t h \in L^2(]0,T[\times\partial\Omega)^3$. After establishing uniform estimates with respect to c, we conclude by weak stability results (cf. [12]).

The paper is organized as follows: first we establish the a priori estimates for T periodic solutions for the Vlasov-Maxwell system (classical or relativistic case) when the light speed c is fixed. In section 3 we show that, in fact, the above estimates are uniform with respect to the light speed. In section 4 we justify the weak convergence toward a T periodic weak solution for the classical Vlasov-Poisson equations. We end this paper with some remarks concerning other systems.

2. The existence of weak solution for the Vlasov-Maxwell equations.

In this section we justify the existence of weak solution for the Vlasov-Maxwell equations. We analyze the permanent regimes (time periodic or stationary solutions). The same method applies for both relativistic and classical cases. The arguments are standard. First we analyze a regularized system (the existence of solution for such a system can be obtained by using a fixed point method). Secondly we deduce a priori estimates for the regularized solutions. We conclude by weak stability under uniform estimates. We only recall here how to obtain a priori estimates for smooth solutions, compactly supported in momentum. For the other details the reader can refer to [6], [8], [7]. We suppose that (f, E, B) is a smooth (C^1) T periodic solution for the Vlasov-Maxwell equations in the relativistic or classical case (we denote by $\mathcal{E}(p), v(p)$ the energy and velocity functions in both cases). For reasons which we will justify later on, it is convenient to start with the analysis of the perturbed Vlasov-Maxwell system :

$$\alpha f + \partial_t f + v(p) \cdot \nabla_x f + q \cdot (E(t,x) + v(p) \wedge B(t,x)) \cdot \nabla_p f = 0, \ \ (t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p, \quad (2.1)$$

$$\alpha \cdot E(t,x) + \partial_t E - c_0^2 \cdot \text{rot } B = -\frac{j(t,x)}{\varepsilon_0}, \ \ \alpha \cdot B(t,x) + \partial_t B + \text{rot } E = 0, \ \ (t,x) \in \mathbb{R}_t \times \Omega, \quad (2.2)$$

$$f(t,x,p) = g(t,x,p), \ \ (t,x,p) \in \mathbb{R}_t \times \Sigma^-, \quad (2.3)$$

$$n(x) \wedge E(t,x) + c \cdot n(x) \wedge (n(x) \wedge B(t,x)) = h(t,x), \ \ (t,x) \in \mathbb{R}_t \times \partial\Omega, \quad (2.4)$$

where $\alpha > 0$ is a small parameter, $0 \le g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$ and h are given T periodic functions verifying :

$$\int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))|(1 + \mathcal{E}(p)) g(t,x,p) \, dt d\sigma dp + \int_0^T \int_{\partial\Omega} |h(t,x)|^2 \, dt d\sigma < +\infty.$$

First of all remark that since $e^{\alpha t} f$ is constant along characteristics (*i. e.*, solutions of $\frac{dX}{ds} = v(P(s))$, $\frac{dP}{ds} = q \cdot (E(s, X(s)) + v(P(s)) \wedge B(s, X(s)))$) we have :

$$\|f\|_{L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)} \le \|g\|_{L^\infty(\mathbb{R}_t \times \Sigma^-)}, \quad (2.5)$$

and also $f \ge 0$. If we denote by $\gamma^+ f$ the trace of f on $\mathbb{R}_t \times \Sigma^+$ we have also :

$$\|\gamma^+ f\|_{L^\infty(\mathbb{R}_t \times \Sigma^+)} \le \|g\|_{L^\infty(\mathbb{R}_t \times \Sigma^-)}, \quad (2.6)$$

and $\gamma^+ f \ge 0$. In order to simplify our computations we suppose also that f is uniformly compacted supported in momentum, $\exists R > 0$ such that for all $(t,x,p) \in \mathbb{R}_t \times \Omega \times \mathbb{R}^3_p$ with $|p| > R$ we have $f(t,x,p) = 0$. After integration of the Vlasov equations with respect to $p \in \mathbb{R}^3_p$ we deduce the continuity equation :

$$\alpha \cdot \rho + \partial_t \rho + \text{div } j = 0, \ \ (t,x) \in \mathbb{R}_t \times \Omega. \quad (2.7)$$

By taking the divergence of the Maxwell equations we deduce as usual that :

$$\alpha \cdot \text{div } E + \partial_t \text{div } E = -\frac{\text{div } j}{\varepsilon_0} = \alpha \cdot \frac{\rho}{\varepsilon_0} + \partial_t \frac{\rho}{\varepsilon_0},$$

which implies :

$$\alpha \cdot \left(\text{div } E - \frac{\rho}{\varepsilon_0} \right) + \partial_t \left(\text{div } E - \frac{\rho}{\varepsilon_0} \right) = 0, \ (t,x) \in \mathbb{R}_t \times \Omega.$$

By time periodicity we conclude that $\text{div } E = \frac{\rho}{\varepsilon_0}$. Similarly one gets that $\alpha \cdot \text{div } B + \partial_t \text{div } B = 0$ which implies by periodicity that $\text{div } B = 0$, $(t,x) \in \mathbb{R}_t \times \Omega$. Notice that the above argument fails if $\alpha = 0$. This is why we introduce the small perturbations $\alpha(f, E, B)$ in the Vlasov-Maxwell equations. We introduce the notations :

$$M^- := \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))| g(t,x,p)\ dtd\sigma dp,\ K^- := \int_0^T\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|\mathcal{E}(p) g(t,x,p)\ dtd\sigma dp,$$

$$M^+ := \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f(t,x,p)\ dtd\sigma dp,\ K^+ := \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f(t,x,p)\ dtd\sigma dp,$$

and $H := \int_0^T\!\int_{\partial\Omega} |h(t,x)|^2\ dtd\sigma$. We integrate the Vlasov equation with respect to $(x,p) \in \Omega \times \mathbb{R}^3_p$ and we deduce the mass conservation law :

$$\alpha \cdot \int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\ dxdp + \frac{d}{dt}\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\ dxdp + \int_\Sigma (v(p)\cdot n(x))\gamma f(t,x,p)\ d\sigma dp = 0,\ \ t \in \mathbb{R}_t, \tag{2.8}$$

which implies that :

$$\alpha \cdot \int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\ dtdxdp + M^+ = M^-. \tag{2.9}$$

Notice that we obtained an estimate of the outgoing mass M^+ uniformly with respect to $\alpha > 0, c > 0$. We multiply now the Vlasov equation by $\mathcal{E}(p)$ and we integrate with respect to $(x,p) \in \Omega \times \mathbb{R}^3_p$:

$$\alpha \cdot \int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f(t,x,p)\ dxdp + \frac{d}{dt}\int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f(t,x,p)\ dxdp + \int_\Sigma (v(p)\cdot n(x))\mathcal{E}(p)\gamma f(t,x,p)\ d\sigma dp$$
$$= \int_\Omega j(t,x)\cdot E(t,x)\ dx. \tag{2.10}$$

We multiply now the Maxwell equations by $(E, c^2 \cdot B)$ and after integration with respect to $x \in \Omega$ we deduce that :

$$\alpha \cdot \int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2)\ dx + \frac{1}{2}\frac{d}{dt}\int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2)\ dx$$
$$- c^2 \cdot \int_{\partial\Omega} (n \wedge B)\cdot E\ d\sigma = -\frac{1}{\varepsilon_0}\int_\Omega j(t,x)\cdot E(t,x)\ dx.$$

A direct computation shows that :

$$-c(n\wedge B)\cdot E = \frac{1}{2}(|n \wedge E|^2 + c^2\cdot|n\wedge B|^2) - \frac{1}{2}|h|^2.$$

Finally one gets :

$$\alpha\varepsilon_0 \cdot \int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2)\ dx + \frac{\varepsilon_0}{2}\frac{d}{dt}\int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2)\ dx \tag{2.11}$$
$$+\frac{\varepsilon_0 c}{2}\int_{\partial\Omega} (|n\wedge E|^2 + c^2 \cdot |n \wedge B|^2)\ d\sigma = -\int_\Omega j(t,x)\cdot E(t,x)\ dx + \frac{\varepsilon_0 c}{2}\int_{\partial\Omega} |h(t,x)|^2\ d\sigma.$$

By adding (2.10), (2.11) we deduce the energy conservation law :

$$\alpha \int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f(t,x,p)\ dxdp + \alpha\varepsilon_0 \int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2)\ dx$$
$$+\frac{d}{dt}\left(\int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f(t,x,p)\ dxdp + \frac{\varepsilon_0}{2}\int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2)\ dx\right)$$
$$+\int_\Sigma (v(p)\cdot n(x))\mathcal{E}(p)\gamma f(t,x,p)\ d\sigma dp + \frac{\varepsilon_0 c}{2}\int_{\partial\Omega} (|n\wedge E|^2 + c^2\cdot|n\wedge B|^2)\ d\sigma$$
$$= \frac{\varepsilon_0 c}{2}\int_{\partial\Omega} |h(t,x)|^2\ d\sigma,\ \ t \in \mathbb{R}_t. \tag{2.12}$$

In particular, after integration on $]0,T[$ we deduce that :

$$\alpha\cdot\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f(t,x,p)\;dtdxdp+\alpha\varepsilon_0\int_0^T\!\!\int_\Omega(|E(t,x)|^2+c^2\cdot|B(t,x)|^2)\;dtdx$$
$$+\int_0^T\!\!\int_{\Sigma^+}(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+f(t,x,p)\;dtd\sigma dp+\frac{\varepsilon_0 c}{2}\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E|^2+c^2\cdot|n\wedge B|^2)\;dtd\sigma$$
$$=\int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|\mathcal{E}(p)g(t,x,p)\;dtd\sigma dp+\frac{\varepsilon_0 c}{2}\int_0^T\!\!\int_{\partial\Omega}|h(t,x)|^2\;dtd\sigma,\;\;t\in\mathbb{R}_t,\quad(2.13)$$

which provides uniform estimates in $\alpha>0$ for the outgoing kinetic energy K^+ and the tangential traces of the electro-magnetic field. Remark that the previous inequality allows us to obtain the following estimates for the tangential traces of the electro-magnetic field :

$$\frac{\varepsilon_0}{2}\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E|^2+c^2\cdot|n\wedge B|^2)\;dtd\sigma\le K^-+\frac{\varepsilon_0}{2}H,\;\;\alpha>0,\;\;c\ge 1.\qquad(2.14)$$

In order to establish a priori estimates for the total energy and the normal traces of the electro-magnetic field we also use the momentum conservation law. We suppose that $\partial\Omega$ is strictly star-shaped with respect to some point $x_0\in\Omega$ (i.e., $\exists r>0$ such that $((x-x_0)\cdot n(x))\ge r,\forall x\in\partial\Omega$). After translation we can assume that $x_0=0\in\Omega$ and thus $(x\cdot n(x))\ge r,\ \forall x\in\partial\Omega$. This hypothesis was used in order to estimate the solutions of the Maxwell equations by using the multiplier method (see [22]). We multiply the Vlasov equation by $(p\cdot x)$ and integrate with respect to $(x,p)\in\Omega\times\mathbb{R}^3_p$:

$$\alpha\cdot\int_\Omega\int_{\mathbb{R}^3_p}(p\cdot x)f(t,x,p)\;dxdp+\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3_p}(p\cdot x)f(t,x,p)\;dxdp+\int_\Sigma(v(p)\cdot n(x))(p\cdot x)f(t,x,p)\;d\sigma dp$$
$$=\int_\Omega\int_{\mathbb{R}^3_p}(v(p)\cdot p)f(t,x,p)\;dxdp+\int_\Omega(\rho E+j\wedge B)(t,x)\cdot x\;dx.$$

By using the perturbed Maxwell equations we check by direct computation that :

$$\rho E+j\wedge B=\varepsilon_0(E\mathrm{div}\;E-E\wedge\mathrm{rot}\;E)+\varepsilon_0c^2(B\mathrm{div}\;B-B\wedge\mathrm{rot}\;B)-\varepsilon_0\partial_t(E\wedge B)-2\alpha\varepsilon_0(E\wedge B),$$

and therefore we obtain that :

$$\alpha\cdot\int_\Omega\int_{\mathbb{R}^3_p}(p\cdot x)f(t,x,p)\;dxdp+2\alpha\varepsilon_0\int_\Omega(E\wedge B)\cdot x\;dx+\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^3_p}(p\cdot x)f(t,x,p)\;dxdp$$
$$+\varepsilon_0\frac{d}{dt}\int_\Omega(E\wedge B)\cdot x\;dx+\int_\Sigma(v(p)\cdot n(x))(p\cdot x)f(t,x,p)\;d\sigma dp\qquad(2.15)$$
$$=\int_\Omega\int_{\mathbb{R}^3_p}(v(p)\cdot p)f\;dxdp+\varepsilon_0\int_\Omega\{(E\mathrm{div}\;E-E\wedge\mathrm{rot}\;E)+c^2(B\mathrm{div}\;B-B\wedge\mathrm{rot}\;B)\}\cdot x\;dx.$$

Remark also that we have the identity :

$$u_i\mathrm{div}\;u-(u\wedge\mathrm{rot}\;u)_i=\sum_{j=1}^3\frac{\partial}{\partial x_j}(u_iu_j)-\frac{1}{2}\frac{\partial}{\partial x_i}|u|^2,\;\;\forall 1\le i\le 3,\qquad(2.16)$$

where $u=(u_i)_{1\le i\le 3}$ is a smooth function (in fact this identity still holds in $\mathcal{D}'(\Omega)$ for $u\in H(\mathrm{div}\;;\Omega)\cap H(\mathrm{rot}\;;\Omega)$). After integration by parts we deduce that :

$$\int_\Omega[u\mathrm{div}\;u-(u\wedge\mathrm{rot}\;u)]\cdot x\;dx=\sum_{i=1}^3\int_\Omega\left(\sum_{j=1}^3\frac{\partial}{\partial x_j}(u_iu_j)-\frac{1}{2}\frac{\partial}{\partial x_i}|u|^2\right)x_i\;dx$$
$$=\int_{\partial\Omega}(x\cdot u)(n\cdot u)\;d\sigma-\frac{1}{2}\int_{\partial\Omega}(n\cdot x)|u|^2\;d\sigma+\frac{1}{2}\int_\Omega|u|^2\;dx.\qquad(2.17)$$

We use the decomposition $u = (n \cdot u)n - n \wedge (n \wedge u)$ and we can write :

$$(x \cdot u)(n \cdot u) = (n \cdot x)|(n \cdot u)|^2 - ((n \wedge (n \wedge u)) \cdot x)(n \cdot u). \tag{2.18}$$

Since Ω is bounded with boundary strictly star-shaped, there are $0 < r \leq R$ such that $r \leq (n(x) \cdot x) \leq R$, $\forall x \in \partial\Omega$. By combining (2.17), (2.18) we deduce that :

$$\begin{aligned}\int_\Omega (u\text{div } u - u \wedge \text{rot } u) \cdot x \, dx =& \frac{1}{2}\int_{\partial\Omega} (n \cdot x)|(n \cdot u)|^2 \, d\sigma - \frac{1}{2}\int_{\partial\Omega} (n \cdot x)|n \wedge u|^2 \, d\sigma \\ & - \int_{\partial\Omega} ((n \wedge (n \wedge u)) \cdot x)(n \cdot u) \, d\sigma + \frac{1}{2}\int_\Omega |u|^2 \, dx \\ \geq & \frac{r}{2}\int_{\partial\Omega} (n \cdot u)^2 \, d\sigma + \frac{1}{2}\int_\Omega |u|^2 \, dx - \frac{R}{2}\int_{\partial\Omega} |n \wedge u|^2 \, d\sigma - R\int_{\partial\Omega} |n \wedge u| \cdot |(n \cdot u)| \, d\sigma \\ \geq & \frac{r}{2}\int_{\partial\Omega} (n \cdot u)^2 \, d\sigma + \frac{1}{2}\int_\Omega |u|^2 \, dx - \frac{R}{2}\int_{\partial\Omega} |n \wedge u|^2 \, d\sigma - \frac{r}{4}\int_{\partial\Omega} (n \cdot u)^2 \, d\sigma - \frac{R^2}{r}\int_{\partial\Omega} |n \wedge u|^2 \, d\sigma \\ = & \frac{r}{4}\int_{\partial\Omega} (n \cdot u)^2 \, d\sigma + \frac{1}{2}\int_\Omega |u|^2 \, dx - \left(\frac{R}{2} + \frac{R^2}{r}\right)\int_{\partial\Omega} |n \wedge u|^2 \, d\sigma. \end{aligned} \tag{2.19}$$

By taking now $u = E$ and $u = B$ and by observing that $(v(p) \cdot p) \geq \mathcal{E}(p)$, $\forall p \in \mathbb{R}^3_p$ (relativistic or classical case), (2.15), (2.19) yield :

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f \, dxdp &+ \frac{\varepsilon_0}{2}\int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2) \, dx + \frac{\varepsilon_0 r}{4}\int_{\partial\Omega} ((n \cdot E)^2 + c^2 \cdot (n \cdot B)^2) \, d\sigma \\ \leq & \alpha \cdot \int_\Omega \int_{\mathbb{R}^3_p} (p \cdot x) f \, dxdp + 2\alpha \cdot \varepsilon_0 \int_\Omega (E \wedge B) \cdot x \, dx + \frac{d}{dt}\int_\Omega \int_{\mathbb{R}^3_p} (p \cdot x) f \, dxdp \\ & + \frac{d}{dt}\int_\Omega \varepsilon_0 (E \wedge B) \cdot x \, dx + \int_\Sigma (v(p) \cdot n(x))(p \cdot x)\gamma f \, d\sigma dp \\ & + \varepsilon_0 \left(\frac{R}{2} + \frac{R^2}{r}\right)\int_{\partial\Omega} (|n \wedge E|^2 + c^2 \cdot |n \wedge B|^2) \, d\sigma, \quad t \in \mathbb{R}_t. \end{aligned} \tag{2.20}$$

Note also that there is a constant $C = C(m) > 0$ not depending on c such that for $c \geq 1$ we have $|p| \leq C(m)(1 + \mathcal{E}_c(p))$, $\forall p \in \mathbb{R}^3_p$. Therefore, by using (2.9), (2.13) we obtain the estimate :

$$\begin{aligned}\left|\int_0^T \int_\Sigma (v(p) \cdot n(x))(p \cdot x) f \, dt d\sigma dp\right| \leq & \int_0^T \int_\Sigma |(v(p) \cdot n(x))| \cdot R \cdot C(m)(1 + \mathcal{E}(p)) f \, dt d\sigma dp \\ \leq & 2 \cdot R \cdot C(m)(M^- + K^-) + \frac{\varepsilon_0 c}{2} \cdot R \cdot C(m) \cdot H, \end{aligned} \tag{2.21}$$

and :

$$\varepsilon_0 \int_0^T \int_{\partial\Omega} (|n \wedge E|^2 + c^2 \cdot |n \wedge B|^2) \, dt d\sigma \leq \frac{2}{c}K^- + \varepsilon_0 H. \tag{2.22}$$

After integration of (2.20) with respect to $t \in]0, T[$ and by using the time periodicity finally we obtain that :

$$\begin{aligned}\int_0^T \int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f \, dt dx dp &+ \frac{\varepsilon_0}{2}\int_0^T \int_\Omega (|E|^2 + c^2 |B|^2) \, dt dx + \frac{\varepsilon_0 r}{4}\int_0^T \int_{\partial\Omega} ((n \cdot E)^2 + c^2 \cdot (n \cdot B)^2) \, dt d\sigma \\ \leq & R \cdot C(m)\left(M^- + K^- + \int_0^T \int_{\Sigma^+} (v(p) \cdot n(x))(1 + \mathcal{E}(p)) f \, dt d\sigma dp\right) \\ & + \left(\frac{R}{2} + \frac{R^2}{r}\right) \cdot \left(\frac{2}{c}K^- + \varepsilon_0 H\right) + \alpha \cdot \int_0^T \int_\Omega \int_{\mathbb{R}^3_p} C(m) \cdot R(1 + \mathcal{E}(p)) f \, dt dx dp \\ & + \frac{\alpha \varepsilon_0 R}{c}\int_0^T \int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2) \, dt dx. \end{aligned} \tag{2.23}$$

By using (2.9) we have also $\alpha \int_0^T\int_\Omega \int_{\mathbb{R}^3_p} f\ dtdxdp \le M^-$ and by taking $\alpha C(m)R < \frac{1}{2}$, $\frac{\alpha R}{c} < \frac{1}{4}$, $c \ge 1$, the inequality (2.23) implies :

$$\frac{1}{2}\int_0^T\int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f\ dtdxdp + \frac{\varepsilon_0}{4}\int_0^T\int_\Omega (|E|^2 + c^2|B|^2)\ dtdx + \frac{\varepsilon_0 r}{4}\int_0^T\int_{\partial\Omega} ((n\cdot E)^2 + c^2\cdot(n\cdot B)^2)\ dtd\sigma$$
$$\le \left(2R\cdot C(m) + R + \frac{2R^2}{r}\right)(M^- + K^-) + R\cdot C(m)(M^+ + K^+)$$
$$+\varepsilon_0\left(\frac{R}{2} + \frac{R^2}{r}\right)H. \tag{2.24}$$

The inequalities (2.9), (2.13) assure that :

$$M^+ \le M^-, \quad \text{and } K^+ \le K^- + \frac{\varepsilon_0 c}{2}H. \tag{2.25}$$

Therefore it is possible to obtain uniform estimates in $\alpha > 0$ (when $c \ge 1$ is fixed) :

$$\int_0^T\int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f\ dtdxdp + \frac{\varepsilon_0}{2}\int_0^T\int_\Omega (|E|^2 + c^2|B|^2)\ dtdx + \int_0^T\int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p)) f\ dtd\sigma dp$$
$$+\frac{\varepsilon_0 r}{2}\int_0^T\int_{\partial\Omega} (|n\wedge E|^2 + c^2\cdot|n\wedge B|^2)\ dtd\sigma + \frac{\varepsilon_0 r}{2}\int_0^T\int_{\partial\Omega} ((n\cdot E)^2 + c^2\cdot(n\cdot B)^2)\ dtd\sigma$$
$$\le C_1\cdot M^- + C_2\cdot K^- + C_3\cdot H, \tag{2.26}$$

where $C_1 = 1 + 6R\cdot C(m) + 2R + \frac{4R^2}{r}$, $C_2 = 1 + 6R\cdot C(m) + 2R + \frac{4R^2}{r} + r$, $C_3 = \varepsilon_0(R + \frac{2R^2}{r} + \frac{r}{2} + \frac{c}{2} + cR\cdot C(m))$. The total mass can be estimated by using the equation div $E = \frac{\rho}{\varepsilon_0}$:

$$\int_0^T\int_\Omega \int_{\mathbb{R}^3_p} f\ dtdxdp = \int_0^T\int_\Omega \rho\ dtdx = \varepsilon_0\int_0^T\int_\Omega \text{div } E\ dtdx = \varepsilon_0\int_0^T\int_{\partial\Omega} (n\cdot E)\ dtd\sigma$$
$$\le \varepsilon_0\left(\int_0^T\int_{\partial\Omega} |(n\cdot E)|^2\ dtd\sigma\right)^{1/2}\left(T\cdot\int_{\partial\Omega} d\sigma\right)^{1/2}. \tag{2.27}$$

Note that the only dependence in c in the estimate (2.26) comes from the estimate of K^+ by $K^- + \frac{\varepsilon_0 c}{2}H$. In fact, later on we will see that it is possible to estimate the outgoing kinetic energy K^+ uniformly in c and therefore the inequality (2.26) will provide uniform estimates with respect to c. For the moment assume that c is fixed and thus (2.26) allows us to prove the existence of T periodic weak solution for the Vlasov-Maxwell equations :

THEOREM 2.1. *Assume that Ω is bounded with $\partial\Omega$ smooth and strictly star-shaped, $g \in L^\infty(\mathbb{R}_t\times\Sigma^-)$ and h are T periodic such that $g \ge 0$, $(n\cdot h)|_{\mathbb{R}_t\times\partial\Omega} = 0$ and $\int_0^T\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p))g\ dtd\sigma dp < +\infty$, $\int_0^T\int_{\partial\Omega} |h|^2\ dtd\sigma < +\infty$. Then there is a T periodic weak solution $(f,E,B) \in L^\infty(\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p)\times L^2_{loc}(\mathbb{R}_t; L^2(\Omega)^3)^2$ for the Vlasov-Maxwell system (classical or relativistic case) :*

$$\partial_t f + v(p)\cdot\nabla_x f + q\cdot(E(t,x) + v(p)\wedge B(t,x))\cdot\nabla_p f = 0,\ \ (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p,$$

$$\partial_t E - c^2\cdot rot\ B = -\frac{j(t,x)}{\varepsilon_0},\ \ \partial_t B + rot\ E = 0,\ \ div\ E = \frac{\rho}{\varepsilon_0},\ \ div\ B = 0,\ \ (t,x)\in\mathbb{R}_t\times\Omega,$$

$$f(t,x,p) = g(t,x,p),\ \ (t,x,p)\in\mathbb{R}_t\times\Sigma^-,\ \ n\wedge E + c\cdot n\wedge(n\wedge B) = h(t,x),\ \ (t,x)\in\mathbb{R}_t\times\partial\Omega.$$

Moreover, the continuity equation $\partial_t \rho + div\ j = 0$ is satisfied, there is trace functions $\gamma^+ f = f|_{\mathbb{R}_t \times \Sigma^+}$, $\|\gamma^+ f\|_{L^\infty} \leq \|g\|_{L^\infty}$, normal and tangential traces $(n \cdot E, n \cdot B)$, $(n \wedge E, n \wedge B)$ verifying :

$$\int_0^T \int_{\Sigma^+} (v(p) \cdot n(x)) \gamma^+ f\ dt d\sigma dp = \int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))| g\ dt d\sigma dp = M^-, \tag{2.28}$$

$$\int_0^T \int_{\Sigma^+} (v(p) \cdot n(x)) \mathcal{E}(p) \gamma^+ f\ dt d\sigma dp + \frac{\varepsilon_0 c}{2} \int_0^T \int_{\partial\Omega} (|n \wedge E|^2 + c^2 \cdot |n \wedge B|^2)\ dt d\sigma$$
$$\leq \int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x)) \mathcal{E}(p) g\ dt d\sigma dp + \frac{\varepsilon_0 c}{2} \int_0^T \int_{\partial\Omega} |h|^2\ dt d\sigma = K^- + \frac{\varepsilon_0 c}{2} H, \tag{2.29}$$

and for some constant $C(m, \varepsilon_0, c, \Omega)$ we have :

$$\int_0^T \int_\Omega \int_{\mathbb{R}^3_p} \mathcal{E}(p) f\ dt dx dp + \frac{\varepsilon_0}{2} \int_0^T \int_\Omega (|E|^2 + c^2 |B|^2)\ dt dx + \frac{\varepsilon_0 r}{2} \int_0^T \int_{\partial\Omega} ((n \cdot E)^2 + c^2 \cdot (n \cdot B)^2)\ dt d\sigma$$
$$\leq C(m, \varepsilon_0, c, \Omega) \cdot (M^- + K^- + H). \tag{2.30}$$

Proof. The proof follows by standard arguments. We construct T periodic solutions for the perturbed Vlasov-Maxwell system. When $c \geq 1$ is fixed, the estimates (2.5), (2.6), (2.26) allow us to extract subsequences such that $f_k \rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$, $\gamma^+ f_k \rightharpoonup \gamma^+ f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Sigma^+)$, $(E_k, B_k) \rightharpoonup (E, B)$ weakly in $L^2(]0, T[; L^2(\Omega)^3)^2$, $(n \cdot E_k, n \cdot B_k) \rightharpoonup (n \cdot E, n \cdot B)$ weakly in $L^2(]0, T[; L^2(\partial\Omega))^2$, $(n \wedge E_k, n \wedge B_k) \rightharpoonup (n \wedge E, n \wedge B)$ weakly in $L^2(]0, T[; L^2(\partial\Omega)^3)^2$ when $k \to +\infty$. One of the key points consists of applying the velocity average result of DiPerna and Lions (cf. [12]) in order to pass to the limit the non linear term of the Vlasov equation :

$$\lim_{k \to +\infty} \int_0^T \int_\Omega \int_{\mathbb{R}^3_p} f_k q (E_k + v(p) \wedge B_k) \cdot \nabla_p \varphi\ dt dx dp = \int_0^T \int_\Omega \int_{\mathbb{R}^3_p} f q (E + v(p) \wedge B) \cdot \nabla_p \varphi\ dt dx dp,$$

for all $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p)$ T periodic and compactly supported in momentum. The equality (2.28) and the inequalities (2.29), (2.30) for the solution (f, E, B) follow as usual by weak limit. $\square$

3. Uniform estimates with respect to c.

As said before, the only dependence in c in (2.30) comes from the estimate of the outgoing kinetic energy K^+. In order to remove this dependence in c we will prove that K^+ can be estimated uniformly with respect to c. Before analyzing the general time periodic case, let us start by studying the stationary case, which is much simpler.

3.1. Stationary case.

We need the following lemmas :

LEMMA 3.1. *Assume that $\Omega \subset \mathbb{R}^3_x$ is a smooth open bounded set simply connected and consider $E \in L^2(\Omega)^3$ verifying rot $E = 0$ in $\mathcal{D}'(\Omega)$ and $n \wedge E \in L^2(\partial\Omega)^3$. Then there is $\Phi \in H^1(\Omega)$ such that $E = -\nabla_x \Phi$, $\varphi := \gamma\Phi \in H^1(\partial\Omega)$ and $\|\gamma\Phi\|_{H^1(\partial\Omega)} \leq C(\Omega) \cdot \|n \wedge E\|_{L^2(\partial\Omega)^3}$.*

Proof. Since rot $E = 0$ and Ω is simply connected there is $\Phi \in H^1(\Omega)$ such that $E = -\nabla_x \Phi$. We can suppose that $\int_{\partial\Omega} \gamma\Phi\ d\sigma = 0$. We take $\tilde{\Phi}_k \in C^2(\overline{\Omega})$ such that $\tilde{\Phi}_k \to \Phi$ in $H^1(\Omega)$ (in particular $\tilde{E}_k := -\nabla_x \tilde{\Phi}_k \to -\nabla_x \Phi =: E$ in $L^2(\Omega)^3$) and $-n \wedge \nabla_x \tilde{\Phi}_k = n \wedge \tilde{E}_k \to n \wedge E$ in

$L^2(\partial\Omega)^3$. We consider also $\Phi_k = \tilde{\Phi}_k - \frac{1}{\int_{\partial\Omega} 1 \, d\sigma} \int_{\partial\Omega} \tilde{\Phi}_k \, d\sigma$. Since $\tilde{\Phi}_k \to \gamma\Phi$ in $L^2(\partial\Omega)$ we have $\lim_{k\to+\infty} \int_{\partial\Omega} \tilde{\Phi}_k \, d\sigma = \int_{\partial\Omega} \gamma\Phi \, d\sigma = 0$ and thus $(\Phi_k)_k$ converges to Φ in $H^1(\Omega)$. Finally we have $\Phi_k \to \Phi$ in $H^1(\Omega)$, $E_k := -\nabla_x\Phi_k \to -\nabla_x\Phi =: E$ in $L^2(\Omega)^3$, $n \wedge E_k = -n \wedge \nabla_x\Phi_k \to n \wedge E$ in $L^2(\partial\Omega)^3$ and $\int_{\partial\Omega} \Phi_k \, d\sigma = 0$, $\forall k \geq 1$. By construction we have rot $E_k = 0$ and thus after multiplication by $\nabla_x\chi$ with $\chi \in C^2(\overline{\Omega})$ one gets :

$$\int_{\partial\Omega} (n \wedge E_k) \cdot \nabla_x\chi \, d\sigma = \int_\Omega \text{rot } E_k \cdot \nabla_x\chi \, dx - \int_\Omega E_k \cdot \text{rot } \nabla_x\chi \, dx = 0.$$

We denote by ∇_τ the tangential gradient on $\partial\Omega$. Since for smooth functions we have $\nabla_x\chi = \nabla_\tau\chi + \frac{\partial\chi}{\partial n}n$, finally we deduce that $\int_{\partial\Omega}(n \wedge E_k) \cdot \nabla_\tau\chi \, d\sigma = 0$ or $n \wedge E_k \in L^2(\partial\Omega)^3$ is a divergence free tangential field on $\partial\Omega$. Therefore there is $\varphi_k \in H^1(\partial\Omega)$ such that $n \wedge E_k = -n \wedge \nabla_\tau\varphi_k$. This is a consequence of an orthogonal decomposition result for tangential fields of $L^2(\partial\Omega)^3$ (see the Appendix for details). Moreover we can suppose that $\int_{\partial\Omega} \varphi_k \, d\sigma = 0$ and therefore we have $\|\varphi_k\|_{H^1(\partial\Omega)} \leq C(\Omega) \cdot \|n \wedge E_k\|_{L^2(\partial\Omega)^3}$, $\forall k \geq 1$. Since $(n \wedge E_k)_k$ converges in $L^2(\partial\Omega)^3$, $(\varphi_k)_k$ is a Cauchy sequence in $H^1(\partial\Omega)$ and thus converges to some $\varphi \in H^1(\partial\Omega)$ with $\|\varphi\|_{H^1(\partial\Omega)} \leq C(\Omega) \cdot \|n \wedge E\|_{L^2(\partial\Omega)^3}$. By writing $E_k = -\nabla_\tau\Phi_k - \frac{\partial\Phi_k}{\partial n}n$ when $x \in \partial\Omega$ we deduce that $n \wedge E_k = -n \wedge \nabla_\tau\Phi_k$ and thus $\nabla_\tau(\Phi_k - \varphi_k) = 0$ on $\partial\Omega$, which implies that there is $c_k \in \mathbb{R}$ such that $\Phi_k - \varphi_k = c_k$ on $\partial\Omega$. Hence, as $\int_{\partial\Omega} \Phi_k \, d\sigma = \int_{\partial\Omega} \varphi_k \, d\sigma = 0$ we deduce that $c_k = 0$, or $\Phi_k = \varphi_k$ on $\partial\Omega$. We have $\gamma\Phi = \lim_{k\to+\infty} \gamma\Phi_k$ in $H^{1/2}(\partial\Omega)$ and therefore in $L^2(\partial\Omega)$. On the other hand $\lim_{k\to+\infty} \gamma\Phi_k = \lim_{k\to+\infty} \varphi_k = \varphi$ in $H^1(\partial\Omega)$. It follows that $\gamma\Phi = \varphi \in H^1(\partial\Omega)$ and $\|\gamma\Phi\|_{H^1(\partial\Omega)} \leq C(\Omega) \cdot \|n \wedge E\|_{L^2(\partial\Omega)^3}$.
□

Lemma 3.2. *Assume that* $0 \leq f \in L^\infty(\Omega \times \mathbb{R}^3_p)$ *such that* $\int_\Omega \int_{\mathbb{R}^3_p} (1 + \mathcal{E}(p))f \, dxdp < +\infty$ *(classical or relativistic case). Then we have the interpolation inequality :*

$$\left\| \int_{\mathbb{R}^3_p} \frac{f(\cdot,p)}{|p|} \, dp \right\|_{L^2(\Omega)} \leq C \cdot \|f\|^{\frac{1}{2}}_{L^\infty} \cdot \left(\int_\Omega \int_{\mathbb{R}^3_p} (1 + \mathcal{E}(p))f(x,p) \, dxdp \right)^{\frac{1}{2}}.$$

Proof. As usual we write for $R > 0$:

$$\int_{\mathbb{R}^3_p} \frac{f(x,p)}{|p|} \, dp = \int_{|p|\leq R} \frac{f(x,p)}{|p|} \, dp + \int_{|p|>R} \frac{f(x,p)}{|p|} \, dp \leq 2\pi R^2 \|f\|_{L^\infty} + \frac{C}{R^2} \int_{\mathbb{R}^3_p} (1 + \mathcal{E}(p))f \, dp.$$

The conclusion follows by taking the optimal value for R and by integrating with respect to $x \in \Omega$.
□

Lemma 3.3. *Assume that* $\Omega \subset \mathbb{R}^3_x$ *is a smooth open bounded set and consider* $0 \leq f \in L^\infty(\Omega \times \mathbb{R}^3_p)$ *a stationary weak solution for the Vlasov problem (classical or relativistic case) :*

$$v(p) \cdot \nabla_x f + q(-\nabla_x\Phi + v(p) \wedge B(x)) \cdot \nabla_p f = 0, \ \ (x,p) \in \Omega \times \mathbb{R}^3_p, \ \ f(x,p) = g(x,p), \ \ (x,p) \in \Sigma^-,$$

with finite mass and kinetic energy $\int_\Omega \int_{\mathbb{R}^3_p} (1 + \mathcal{E}(p))f(x,p) \, dxdp < +\infty$ *and trace* $0 \leq \gamma^+ f \in L^\infty(\Sigma^+)$*, where* $0 \leq g \in L^\infty(\Sigma^-)$, $\Phi \in H^1(\Omega)$, $B \in L^2(\Omega)^3$ *are given functions verifying* $\int_{\Sigma^-} |(v(p) \cdot n(x))|g(x,p) \, d\sigma dp < +\infty$. *Then, for all function* $F \in C^1_b(\mathbb{R})$ *(i.e.,* $F \in C^1(\mathbb{R})$ *with* F, F' *bounded), we have :*

$$\int_\Sigma (v(p) \cdot n(x)) \, \gamma f(x,p) \, F(\mathcal{E}(p) + q\gamma\Phi(x)) \, d\sigma dp = 0.$$

Proof. Since f is a weak solution for the Vlasov problem, we have for all test function $\theta \in C^1(\overline{\Omega} \times \mathbb{R}^3_p)$, compactly supported in momentum :

$$-\int_\Omega \int_{\mathbb{R}^3_p} (v(p) \cdot \nabla_x\theta + q(E(x) + v(p) \wedge B(x)) \cdot \nabla_p\theta)f \, dxdp + \int_\Sigma (v(p) \cdot n(x))\theta(x,p)\gamma f \, d\sigma dp = 0,$$

where $E = -\nabla_x \Phi$. Since $E, B \in L^2(\Omega)^3$, $f \in L^\infty(\Omega \times \mathbb{R}^3_p)$, $\gamma^+ f \in L^\infty(\Sigma^+)$, by an easy density argument we deduce that the above formulation holds also for $\theta \in H^1(\Omega \times \mathbb{R}^3_p)$, compactly supported in momentum :

$$-\int_\Omega \int_{\mathbb{R}^3_p} (v(p) \cdot \nabla_x \theta + q(E(x) + v(p) \wedge B(x)) \cdot \nabla_p \theta) f \, dxdp + \int_\Sigma (v(p) \cdot n(x)) \gamma\theta \, \gamma f \, d\sigma dp = 0,$$

where $\gamma\theta \in H^{\frac{1}{2}}(\Sigma)$ is the trace of $\theta \in H^1(\Omega \times \mathbb{R}^3_p)$. Consider now $F \in C^1_b(\mathbb{R})$ and $\chi \in C^1_c(\mathbb{R})$ such that $\chi(u) = 1$ if $|u| \leq 1$, $\chi(u) = 0$ if $|u| \geq 2$, $0 \leq \chi \leq 1$ and denote by χ_R the function $\chi_R(u) = \chi(\frac{u}{R})$, $\forall u \in \mathbb{R}$, $R > 0$. The function $\theta_R(x,p) = F(\mathcal{E}(p) + q\Phi(x)) \cdot \chi_R(|p|)$, $(x,p) \in \Omega \times \mathbb{R}^3_p$ belongs to $H^1(\Omega \times \mathbb{R}^3_p)$ and has compact support in momentum. Remark also that :

$$v(p) \cdot \nabla_x \theta_R + q(E(x) + v(p) \wedge B(x)) \cdot \nabla_p \theta_R = qE(x) \; F(\mathcal{E}(p) + q\Phi(x)) \cdot \frac{1}{R} \cdot \chi' \left(\frac{|p|}{R} \right) \cdot \frac{p}{|p|},$$

and $\gamma\theta_R(x,p) = F(\mathcal{E}(p) + q\gamma\Phi(x))\chi_R(|p|)$, $(x,p) \in \Sigma$. By applying the weak formulation with the test function θ_R we find for every $R > 0$:

$$\int_\Omega \int_{\mathbb{R}^3_p} qE \; F(\mathcal{E}(p) + q\Phi(x)) \frac{1}{R} \chi' \left(\frac{|p|}{R} \right) \cdot \frac{p}{|p|} f \, dxdp = \int_\Sigma (v(p) \cdot n(x)) F(\mathcal{E}(p) + q\gamma\Phi(x)) \chi_R(|p|) \gamma f \, d\sigma dp. \tag{3.1}$$

Consider for the moment the function $F = 1$. We deduce that :

$$-\int_\Omega \int_{\mathbb{R}^3_p} qE \cdot \frac{1}{R} \cdot \chi' \left(\frac{|p|}{R} \right) \cdot \frac{p}{|p|} f \, dxdp + \int_{\Sigma^+} (v(p) \cdot n(x)) \chi_R(|p|) \gamma^+ f \, d\sigma dp = \int_{\Sigma^-} |(v(p) \cdot n(x))| \chi_R(|p|) g \, d\sigma dp. \tag{3.2}$$

Since $E \in L^2(\Omega)^3$ and $\int_{\mathbb{R}^3_p} \frac{f(\cdot,p)}{|p|} \, dp \in L^2(\Omega)$ (see *Lemma* 3.2) we deduce that :

$$\left| qE \cdot \frac{1}{R} \cdot \chi' \left(\frac{|p|}{R} \right) \cdot \frac{p}{|p|} f \right| \leq C \cdot |q| \cdot |E(x)| \cdot \frac{f(x,p)}{|p|} \in L^1(\Omega \times \mathbb{R}^3_p).$$

By using the dominated convergence theorem we deduce that :

$$\lim_{R \to +\infty} \int_\Omega \int_{\mathbb{R}^3_p} qE \cdot \frac{1}{R} \cdot \chi' \left(\frac{|p|}{R} \right) \cdot \frac{p}{|p|} f \, dxdp = 0.$$

Finally, by letting $R \to +\infty$ in (3.2) and by applying the monotone convergence theorem we find that :

$$\int_{\Sigma^+} (v(p) \cdot n(x)) \gamma^+ f(x,p) \, d\sigma dp = \int_{\Sigma^-} |(v(p) \cdot n(x))| g(x,p) \, d\sigma dp. \tag{3.3}$$

Suppose now that $F \in C^1_b(\mathbb{R})$. By using that $E \in L^2(\Omega)^3$, $\int_{\mathbb{R}^3_p} \frac{f(\cdot,p)}{|p|} \, dp \in L^2(\Omega)$, $(v(p) \cdot n(x))\gamma^+ f \in L^1(\Sigma^+)$, $(v(p) \cdot n(x)) g \in L^1(\Sigma^-)$, by passing to the limit for $R \to +\infty$ in (3.1) we find :

$$\int_{\Sigma^+} (v(p) \cdot n(x)) \gamma^+ f(x,p) \; F(\mathcal{E}(p) + q\gamma\Phi(x)) \, d\sigma dp = \int_{\Sigma^-} |(v(p) \cdot n(x))| g(x,p) \; F(\mathcal{E}(p) + q\gamma\Phi(x)) \, d\sigma dp.$$

□

LEMMA 3.4. *Assume that $F \in L^\infty(\Sigma_1)$ is a non negative function such that $\int_{\Sigma_1} |(v(p) \cdot n(x))| (1 + \mathcal{E}(p)) F(x,p) \, d\sigma dp < +\infty$ (classical or relativistic case), where $\Sigma_1 \subset \Sigma$. For $x \in \partial\Omega$ denote by $P(x)$ the set $\{p \in \mathbb{R}^3_p \mid (x,p) \in \Sigma_1\}$. Then $\int_{\mathbb{R}^3_p} |(v(p) \cdot n(x))| \mathbf{1}_{P(x)} F(x,p) \, dp \in L^{5/4}(\partial\Omega)$ and we have the inequality :*

$$\left\| \int_{\mathbb{R}^3_p} |(v(p) \cdot n(\cdot))| \mathbf{1}_{P(\cdot)} F(\cdot,p) \, dp \right\|_{L^{\frac{5}{4}}(\partial\Omega)} \leq C \cdot \|F\|^{\frac{1}{5}}_{L^\infty} \cdot \left(\int_{\Sigma_1} |(v(p) \cdot n(x))| (1 + \mathcal{E}(p)) F(x,p) \, d\sigma dp \right)^{\frac{4}{5}},$$

for some constant C not depending on the light speed if $c \geq 1$.

Proof. Observe that for all $0 < c \leq +\infty$ we have $|v(p)| \leq \frac{|p|}{m}$. There is a constant $C = C(m)$ such that $|p| \leq C(m) \cdot (1 + \mathcal{E}(p))$, $p \in \mathbb{R}^3_p$, $c \geq 1$. For $R > 0$ we write :

$$\begin{aligned}\int_{\mathbb{R}^3_p} |(v(p)\cdot n(x))| \mathbf{1}_{P(x)}(p) F(x,p)\, dp =& \int_{|p|>R} |(v(p)\cdot n(x))| \mathbf{1}_{P(x)}(p) F(x,p)\, dp \\ &+ \int_{|p|\leq R} |(v(p)\cdot n(x))| \mathbf{1}_{P(x)}(p) F(x,p)\, dp \\ \leq& \frac{C}{R}\int_{\mathbb{R}^3_p} |(v(p)\cdot n(x))| \cdot C(m)\cdot (1+\mathcal{E}(p)) F(x,p)\, dp \\ &+ C\cdot R^4 \cdot \|F\|_{L^\infty}.\end{aligned}$$

Take the optimal value for R and integrate with respect to $x \in \Omega$. □

Proposition 3.5. *Assume that $\Omega \subset \mathbb{R}^3_x$ is a smooth open bounded simply connected set and consider $0 \leq f(x,p) \in L^\infty(\Omega \times \mathbb{R}^3_p)$ a stationary weak solution for the Vlasov problem (classical or relativistic case) :*

$$v(p)\cdot\nabla_x f + q(E(x) + v(p)\wedge B(x))\cdot\nabla_p f = 0,\ \ (x,p)\in\Omega\times\mathbb{R}^3_p,\ \ f(x,p) = g(x,p),\ \ (x,p)\in\Sigma^-,$$

with finite mass and kinetic energy and bounded trace $0 \leq \gamma^+ f$ where $0 \leq g \in L^\infty(\Sigma^-)$, $E, B \in L^2(\Omega)^3$ are given functions verifying rot $E = 0$, $n \wedge E \in L^2(\partial\Omega)^3$ and $\int_{\Sigma^-} |(v(p)\cdot n(x))|(1 + \mathcal{E}(p))g(x,p)\, d\sigma dp < +\infty$. Then the outgoing energy is uniformly bounded with respect to the light speed and we have for some constant C :

$$\int_{\Sigma^+} (v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f\, d\sigma dp \leq C\cdot (1 + \|g\|^{\frac{1}{5}}_{L^\infty} + \|\gamma^+ f\|^{\frac{1}{5}}_{L^\infty})\cdot (M^- + K^- + \|n\wedge E\|^5_{L^2(\partial\Omega)^3}).$$

Proof. By *Lemma* 3.1 we can write $E = -\nabla_x \Phi$ with $\Phi \in H^1(\Omega)$, $\gamma\Phi \in H^1(\partial\Omega)$, such that $\|\gamma\Phi\|_{H^1(\partial\Omega)} \leq C(\Omega)\cdot\|n\wedge E\|_{L^2(\partial\Omega)^3}$. By using Sobolev inequalities we deduce that $\varphi = \gamma\Phi \in L^r(\partial\Omega)$, $\forall 1 \leq r < \infty$ and $\|\varphi\|_{L^r(\partial\Omega)} \leq C_r(\Omega)\cdot\|\varphi\|_{H^1(\partial\Omega)}$. We apply *Lemma* 3.3 with the function $F_R(u) = u\cdot\chi_R(u)$ and we obtain :

$$\int_{\Sigma^+} (v(p)\cdot n(x))(\mathcal{E}(p) + q\gamma\Phi(x))\chi_R(W)\gamma^+ f\, d\sigma dp = \int_{\Sigma^-} |(v(p)\cdot n(x))|(\mathcal{E}(p) + q\gamma\Phi(x))\chi_R(W) g\, d\sigma dp, \tag{3.4}$$

where $W(x,p) = \mathcal{E}(p) + q\varphi(x)$, $\forall (x,p) \in \Sigma$. By using *Lemma* 3.4 we have $\int_{(v(p)\cdot n(x))<0} |(v(p)\cdot n(x))| g\, dp \in L^{5/4}(\partial\Omega)$ and therefore we have :

$$\begin{aligned}\int_{\Sigma^-} |(v(p)\cdot n(x))|\cdot|q\varphi(x)|\chi_R(W) g\, d\sigma dp &\leq |q|\cdot\int_{\partial\Omega} |\varphi(x)| \left(\int_{(v(p)\cdot n(x))<0} |(v(p)\cdot n(x))| g\, dp\right) d\sigma \\ &\leq |q|\cdot\|\varphi\|_{L^5(\partial\Omega)}\cdot\left\|\int_{(v(p)\cdot n(x))<0} |(v(p)\cdot n(x))| g\, dp\right\|_{L^{\frac{5}{4}}(\partial\Omega)} \\ &\leq C\cdot|q|\cdot\|\varphi\|_{H^1(\partial\Omega)}\cdot\|g\|^{\frac{1}{5}}_{L^\infty}\cdot\left(\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p)) g\, d\sigma dp\right)^{\frac{4}{5}}.\end{aligned} \tag{3.5}$$

Remark that for a.e. $x \in \partial\Omega$ the function $p \to (v(p)\cdot n(x))\gamma^+ f(x,p)\mathcal{E}(p)\chi_R(W(x,p))$ has compact support in momentum and therefore is integrable on $p \in \mathbb{R}^3_p$ such that $(v(p)\cdot n(x)) > 0$:

$$\int_{(v(p)\cdot n(x))>0} (v(p)\cdot n(x))\gamma^+ f(x,p)\mathcal{E}(p)\chi_R(W(x,p))\, dp < +\infty,\ \ \text{a.e. } x \in \partial\Omega.$$

As before we can write :

$$
\begin{aligned}
|\int_{(v\cdot n)>0} (v(p)\cdot n(x))\gamma^+ f\, q\varphi(x)\chi_R(W)\,dp\,| &\leq |q|\cdot|\varphi(x)| \int_{(v\cdot n)>0} (v(p)\cdot n(x))\gamma^+ f \chi_R(W)\,dp \qquad (3.6)\\
&\leq |q|\cdot|\varphi(x)|\cdot C\cdot \|\gamma^+ f\|_{L^\infty}^{\frac15}\cdot\left(\int_{(v\cdot n)>0}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\chi_R(W)\,dp\right)^{\frac45}\\
&\leq C\cdot|q|\cdot\|\gamma^+ f\|_{L^\infty}^{\frac15}\cdot\left(\frac{|\varphi(x)|^5}{5\delta^4}+\frac{4\delta}{5}\int_{(v\cdot n)>0}(v(p)\cdot n(x))\gamma^+ f\chi_R(W)(1+\mathcal{E}(p))\,dp\right).
\end{aligned}
$$

Finally one gets :

$$
\begin{aligned}
\int_{\Sigma^+}(v\cdot n)\gamma^+ f(1+\mathcal{E}(p)+q\varphi(x))\chi_R(W)\,d\sigma dp \geq &\left(1-\frac45\delta C_1\right)\int_{\Sigma^+}(v\cdot n)\gamma^+ f(1+\mathcal{E}(p))\chi_R(W)\,d\sigma dp\\
&-\frac{C_1}{5\delta^4}\int_{\partial\Omega}|\varphi(x)|^5\,d\sigma, \qquad (3.7)
\end{aligned}
$$

where $C_1 = |q|\cdot C\cdot\|\gamma^+ f\|_{L^\infty}^{\frac15}$. By taking into account that $\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\chi_R(W)\,d\sigma dp \leq \int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\,d\sigma dp = \int_{\Sigma^-}|(v(p)\cdot n(x))|g\,d\sigma dp$, the inequalities (3.4), (3.5), (3.7) imply :

$$
\begin{aligned}
\left(1-\frac45\delta C_1\right)\int_{\Sigma^+}(v\cdot n)\gamma^+ f(1+\mathcal{E}(p))\chi_R(W)\,d\sigma dp \leq &\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g\,d\sigma dp\\
+\frac{C_1}{5\delta^4}\int_{\partial\Omega}|\varphi(x)|^5\,d\sigma &+ C\cdot|q|\cdot\|\varphi\|_{H^1(\partial\Omega)}\cdot\|g\|_{L^\infty}^{\frac15}\cdot(M^-+K^-)^{\frac45}.
\end{aligned}
$$

By taking δ small enough we deduce that $\int_{\Sigma^+}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\chi_R(W)\,d\sigma dp$ is bounded uniformly with respect to $R>0$ and by the Fatou lemma we find that :

$$
\begin{aligned}
\left(1-\frac45\delta C_1\right)&\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(1+\mathcal{E}(p))\,d\sigma dp \leq M^- + K^- + \frac{C_1}{5\delta^4}\|\varphi\|_{L^5(\partial\Omega)}^5\\
&+C\cdot|q|\cdot\|\varphi\|_{H^1(\partial\Omega)}\cdot\|g\|_{L^\infty}^{\frac15}\cdot(M^-+K^-)^{\frac45}\\
&\leq M^-+K^-+C\cdot\|\gamma^+ f\|_{L^\infty}^{\frac15}\cdot\|\varphi\|_{H^1(\partial\Omega)}^5 + C\cdot\|g\|_{L^\infty}^{\frac15}\cdot(\|\varphi\|_{H^1(\partial\Omega)}^5+M^-+K^-),
\end{aligned}
$$

and the conclusion follows.
□

Remark 3.6. *By using Proposition* 3.5 *we can now estimate uniformly with respect to* c *the solutions constructed in Theorem* 2.1 *in the stationary case :*

$$
\begin{aligned}
\int_\Omega\int_{\mathbb{R}^3_p}(1+\mathcal{E}(p))f\,dxdp &+ \frac{\varepsilon_0}{2}\int_\Omega(|E|^2+c^2|B|^2)\,dx + \int_{\Sigma^+}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\,d\sigma dp\\
&+\frac{\varepsilon_0}{2}\int_{\partial\Omega}(|n\wedge E|^2+(n\cdot E)^2)\,d\sigma + \frac{\varepsilon_0 c^2}{2}\int_{\partial\Omega}(|n\wedge B|^2+(n\cdot B)^2)\,d\sigma \leq C, \qquad (3.8)
\end{aligned}
$$

where C *depends on* $m,\varepsilon_0,\Omega,M^-,K^-,H$ *and* $\|g\|_{L^\infty(\Sigma^-)}$.

3.2. The time periodic case.

In this paragraph we deduce uniform estimates with respect to c for time periodic solutions of the Vlasov-Maxwell equations. Notice that the same estimates hold for T periodic solutions of the perturbed Vlasov-Maxwell system (2.1), (2.2), (2.3), (2.4) uniformly with respect to the regularization parameter $\alpha>0$; for this just replace the derivative ∂_t by $\alpha+\partial_t$. More precisely we need to estimate uniformly the outgoing kinetic energy. This is a direct consequence of (2.13) in the case $h = 0$:

$$\int_0^T\!\!\int_{\Sigma^+} (v\cdot n)\mathcal{E}(p) f \; dtd\sigma dp + \frac{\varepsilon_0 c}{2}\int_0^T\!\!\int_{\partial\Omega} (|n\wedge E|^2 + c^2\cdot |n\wedge B|^2)\; dtd\sigma \leq \int_0^T\!\!\int_{\Sigma^-} |(v\cdot n)|\mathcal{E}(p) g \; dtd\sigma dp.$$

For the general case we decompose the electro-magnetic field into the self-consistent field (E_s, B_s) and the exterior field (E_0, B_0) :

$$\partial_t E_s - c^2\cdot \text{rot } B_s = -\frac{j(t,x)}{\varepsilon_0},\;\; \partial_t B_s + \text{rot } E_s = 0,\;\; \text{div } E_s = \frac{\rho(t,x)}{\varepsilon_0},\;\; \text{div } B_s = 0,\;\; (t,x)\in \mathbb{R}_t\times\Omega, \tag{3.9}$$

$$n(x)\wedge E_s(t,x) + c\cdot n(x)\wedge(n(x)\wedge B_s(t,x)) = 0,\;\; (t,x)\in\mathbb{R}_t\times\partial\Omega, \tag{3.10}$$

and :

$$\partial_t E_0 - c^2\cdot\text{rot } B_0 = 0,\;\; \partial_t B_0 + \text{rot } E_0 = 0,\;\; \text{div } E_0 = 0,\;\; \text{div } B_0 = 0,\;\; (t,x)\in\mathbb{R}_t\times\Omega, \tag{3.11}$$

$$n(x)\wedge E_0(t,x) + c\cdot n(x)\wedge(n(x)\wedge B_0(t,x)) = h(t,x),\;\; (t,x)\in\mathbb{R}_t\times\partial\Omega. \tag{3.12}$$

PROPOSITION 3.7. *Assume that Ω is bounded, regular (C^1) and strictly star-shaped (with respect to $0\in\Omega$), $h\in L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega)^3)$ is T periodic verifying $(n\cdot h) = 0$, $(t,x)\in\mathbb{R}_t\times\partial\Omega$. Then there is a unique T periodic weak solution $(E_0, B_0)\in L^2_{loc}(\mathbb{R}_t; L^2(\Omega)^3)^2$ for the problem* (3.11), (3.12). *Moreover the solution (E_0,B_0) has tangential and normal traces $(n\wedge E_0, n\wedge B_0)\in L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega)^3)^2$, $(n\cdot E_0, n\cdot B_0)\in L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega))^2$ and verifies the estimates :*

$$\begin{aligned}\int_0^T\!\!\int_\Omega (|E_0|^2 + c^2\cdot|B_0|^2)\; dtdx + r\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E_0|^2 + (n\cdot E_0)^2 + c^2|n\wedge B_0|^2 + c^2(n\cdot B_0)^2)\; dtd\sigma \\ \leq C(\Omega)\int_0^T\!\!\int_{\partial\Omega}|h|^2\; dtd\sigma.\end{aligned} \tag{3.13}$$

Proof. The existence part is similar to the existence of time periodic solution for the Vlasov-Maxwell equations (take $g = 0$ and thus $f = 0$). We only sketch the proof. First we regularize h and for $\alpha > 0$ we consider the unique T periodic smooth solution for :

$$\alpha E_\varepsilon + \partial_t E_\varepsilon - c^2\cdot\text{rot } B_\varepsilon = 0,\;\; \alpha B_\varepsilon + \partial_t B_\varepsilon + \text{rot } E_\varepsilon = 0,\;\; (t,x)\in\mathbb{R}_t\times\Omega, \tag{3.14}$$

$$n(x)\wedge E_\varepsilon(t,x) + c\cdot n(x)\wedge(n(x)\wedge B_\varepsilon(t,x)) = h_\varepsilon(t,x),\;\; (t,x)\in\mathbb{R}_t\times\partial\Omega. \tag{3.15}$$

As before, by taking the divergence in the perturbed Maxwell equations we find that :

$$(\alpha+\partial_t)\text{div } E_\varepsilon = 0,\;\; (\alpha+\partial_t)\text{div } B_\varepsilon = 0\;\; (t,x)\in\mathbb{R}_t\times\Omega, \tag{3.16}$$

and by periodicity we deduce also that $\text{div } E_\varepsilon = 0$, $\text{div } B_\varepsilon = 0$, $(t,x)\in\mathbb{R}_t\times\Omega$. After multiplication of the perturbed Maxwell equations by $(E_\varepsilon, c^2 B_\varepsilon)$ we deduce :

$$\alpha\int_0^T\!\!\int_\Omega(|E_\varepsilon|^2 + c^2\cdot|B_\varepsilon|^2)\; dtdx + \frac{c}{2}\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E_\varepsilon|^2 + c^2\cdot|n\wedge B_\varepsilon|^2)\; dtd\sigma = \frac{c}{2}\int_0^T\!\!\int_{\partial\Omega}|h_\varepsilon|^2\; dtd\sigma. \tag{3.17}$$

The Maxwell equations (3.14) and $\text{div } E_\varepsilon = 0$, $\text{div } B_\varepsilon = 0$ imply :

$$E_\varepsilon\text{div } E_\varepsilon - E_\varepsilon\wedge\text{rot } E_\varepsilon + c^2(B_\varepsilon\text{div } B_\varepsilon - B_\varepsilon\wedge\text{rot } B_\varepsilon) = \partial_t(E_\varepsilon\wedge B_\varepsilon) + 2\alpha(E_\varepsilon\wedge B_\varepsilon). \tag{3.18}$$

By using the identity (2.16), after multiplication of (3.18) by x and integration by parts one gets as before that :

$$\begin{aligned}\frac{1}{2}\int_0^T\!\!\int_\Omega (|E_\varepsilon|^2 + c^2\cdot|B_\varepsilon|^2)\,dtdx + \frac{r}{4}\int_0^T\!\!\int_{\partial\Omega}((n\cdot E_\varepsilon)^2 + c^2\cdot(n\cdot B_\varepsilon)^2)\,dtd\sigma \\ \le 2\alpha\int_0^T\!\!\int_\Omega (E_\varepsilon\wedge B_\varepsilon)\cdot x\,dtdx + \left(\frac{R}{2}+\frac{R^2}{r}\right)\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E_\varepsilon|^2 + c^2\cdot|n\wedge B_\varepsilon|^2)\,dtd\sigma,\end{aligned}\tag{3.19}$$

where $0 < r \le R$ such that $r \le (n\cdot x) \le R$, $\forall x \in \partial\Omega$. Remark also that for $\frac{\alpha R}{c} \le \frac{1}{4}$ we have :

$$2\alpha\int_0^T\!\!\int_\Omega (E_\varepsilon\wedge B_\varepsilon)\cdot x\,dtdx \le \frac{\alpha R}{c}\int_0^T\!\!\int_\Omega(|E_\varepsilon|^2 + c^2\cdot|B_\varepsilon|^2)\,dtdx \le \frac{1}{4}\int_0^T\!\!\int_\Omega(|E_\varepsilon|^2 + c^2\cdot|B_\varepsilon|^2)\,dtdx,\tag{3.20}$$

and finally (3.19) and (3.17) imply that :

$$\begin{aligned}\frac{1}{4}\int_0^T\!\!\int_\Omega(|E_\varepsilon|^2 + c^2\cdot|B_\varepsilon|^2)\,dtdx + \frac{r}{4}\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E_\varepsilon|^2 + (n\cdot E_\varepsilon)^2 + c^2(|n\wedge B_\varepsilon|^2 + (n\cdot B_\varepsilon)^2))\,dtd\sigma \\ \le \left(\frac{R}{2}+\frac{R^2}{r}+\frac{r}{4}\right)\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E_\varepsilon|^2 + c^2\cdot|n\wedge B_\varepsilon|^2)\,dtd\sigma \\ \le C_1(\Omega)\int_0^T\!\!\int_{\partial\Omega}|h_\varepsilon|^2\,dtd\sigma,\end{aligned}\tag{3.21}$$

where $C_1(\Omega) = \frac{R}{2} + \frac{R^2}{r} + \frac{r}{4}$. Therefore the solution $(E_\varepsilon, B_\varepsilon)$ verify the estimate (3.13) with $C(\Omega) = 4C_1(\Omega)$, $\forall \varepsilon > 0, 0 < \alpha \le \frac{c}{4R}$. Now by taking $h_\varepsilon \to h$ in $L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega)^3)$ and $\alpha = \varepsilon \searrow 0$, it is clear that $(E_\varepsilon, B_\varepsilon)$ converges strongly in $L^2_{loc}(\mathbb{R}_t; L^2(\Omega)^3)^2$ to a T periodic weak solution (E, B) of (3.14) with tangential traces $(n\wedge E, n\wedge B) = \lim_{\varepsilon\searrow 0}(n\wedge E_\varepsilon, n\wedge B_\varepsilon)$ strongly in $L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega)^3)^2$ and normal traces $(n\cdot E, n\cdot B) = \lim_{\varepsilon\searrow 0}(n\cdot E_\varepsilon, n\cdot B_\varepsilon)$ strongly in $L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega))^2$. Moreover the solution verifies :

$$\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E|^2 + c^2\cdot|n\wedge B|^2)\,dtd\sigma = \int_0^T\!\!\int_{\partial\Omega}|h|^2\,dtd\sigma,$$

and the estimate (3.13). In order to prove the uniqueness it is sufficient to show that all T periodic weak solution verifies the estimate (3.13). This can be done by regularization.
□

PROPOSITION 3.8. *Under the hypotheses of Proposition* 3.7 *assume that the time derivative* $\partial_t h$ *belongs to* $L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega)^3)$. *Then the T periodic weak solution of the problem* (3.11), (3.12) *verifies* $(\partial_t E, \partial_t B) \in L^2(]0,T[; L^2(\Omega)^3)^2$, $(\partial_t(n\wedge E), \partial_t(n\wedge B)) \in L^2(]0,T[; L^2(\partial\Omega)^3)^2$, $(\partial_t(n\cdot E), \partial_t(n\cdot B)) \in L^2(]0,T[; L^2(\partial\Omega))^2$ *and the estimate :*

$$\begin{aligned}\int_0^T\!\!\int_\Omega(|\partial_t E|^2 + c^2|\partial_t B|^2)\,dtdx + r\int_0^T\!\!\int_{\partial\Omega}(|\partial_t(n\wedge E)|^2 + c^2|\partial_t(n\wedge B)|^2)\,dtd\sigma \\ + r\int_0^T\!\!\int_{\partial\Omega}(|\partial_t(n\cdot E)|^2 + c^2|\partial_t(n\cdot B)|^2)\,dtd\sigma \\ \le C(\Omega)\int_0^T\!\!\int_{\partial\Omega}|\partial_t h|^2\,dtd\sigma.\end{aligned}$$

Proof. For every real number η and function u we denote by $D_\eta u$ the function $D_\eta u(t,x) = u(t+\eta, x) - u(t,x)$. If (E,B) is the T periodic weak solution for the problem (3.11), (3.12), therefore $(D_\eta E, D_\eta B)$ is T periodic weak solution for :

$$\partial_t D_\eta E - c^2\cdot\mathrm{rot}\ D_\eta B = 0,\ \ \partial_t D_\eta B + \mathrm{rot}\ D_\eta E = 0,\ \ \mathrm{div}\ D_\eta E = 0,\ \ \mathrm{div}\ D_\eta B = 0,\ \ (t,x)\in\mathbb{R}_t\times\Omega,$$

$$n(x) \wedge D_\eta E(t,x) + c \cdot n(x) \wedge (n(x) \wedge D_\eta B(t,x)) = D_\eta h(t,x), \ \ (t,x) \in \mathbb{R}_t \times \partial\Omega.$$

By *Proposition* 3.7 we have the estimate :

$$\begin{aligned}\int_0^T\!\!\int_\Omega (|D_\eta E|^2 + c^2|D_\eta B|^2)\, dtdx + r\int_0^T\!\!\int_{\partial\Omega} (|n \wedge D_\eta E|^2 + c^2|n \wedge D_\eta B|^2)\, dtd\sigma \\ + r\int_0^T\!\!\int_{\partial\Omega} (|(n \cdot D_\eta E)|^2 + c^2|(n \cdot D_\eta B)|^2)\, dtd\sigma \\ \leq C(\Omega)\int_0^T\!\!\int_{\partial\Omega} |D_\eta h|^2\, dtd\sigma, \\ \leq C(\Omega)\cdot|\eta|^2\cdot\int_0^T\!\!\int_{\partial\Omega} |\partial_t h|^2\, dtd\sigma, \quad (3.22)\end{aligned}$$

and our conclusion follows.
□

In the following we establish the divergence equations verified on the boundary $\mathbb{R}_t \times \partial\Omega$ by T periodic weak solutions for the Maxwell equations. We denote by $\nabla_{(t,\tau)}$, $\mathrm{div}_{(t,\tau)}$ the gradient and divergence operator on $\mathbb{R}_t \times \partial\Omega$ (see the *Appendix* for a brief presentation of these operators).

Proposition 3.9. *Assume that Ω is regular and consider $(E,B) \in L^2_{loc}(\mathbb{R}_t; L^2(\Omega)^3)^2$ a T periodic weak solution for the Maxwell equations :*

$$\partial_t E - c^2 \cdot rot\ B = -\frac{j(t,x)}{\varepsilon_0},\ \ \partial_t B + rot\ E = 0,\ \ div\ E = \frac{\rho(t,x)}{\varepsilon_0},\ \ div\ B = 0,\ \ (t,x) \in \mathbb{R}_t \times \Omega, \quad (3.23)$$

with tangential and normal traces $(n \wedge E, n \wedge B) \in L^2_{loc}(\mathbb{R}_t; L^2(\Omega)^3)^2$, respectively $((n \cdot E), (n \cdot B)) \in L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega))^2$. We assume also that the charge density ρ belongs to $L^1_{loc}(\mathbb{R}_t; L^1(\Omega))$, the current density belongs to $L^1_{loc}(\mathbb{R}_t; L^1(\Omega)^3)$ and that the continuity equation $\partial_t\rho + div\ j = 0$ holds true in $\mathcal{D}'_{per}(\mathbb{R}_t \times \overline{\Omega})$ (i.e., $\int_0^T\!\int_\Omega \rho\partial_t\varphi\ dtdx + \int_0^T\!\int_\Omega j \cdot \nabla_x\varphi\ dtdx = \int_0^T\!\int_{\partial\Omega} (n \cdot j)\varphi\ dtd\sigma$, $\forall \varphi \in C^1(\mathbb{R}_t \times \overline{\Omega})$, T periodic, for some function $(n \cdot j) \in L^1_{loc}(\mathbb{R}_t; L^1(\partial\Omega)^3)$). Then the traces of the electro-magnetic field verify the following divergence equations in $\mathcal{D}'_{per}(\mathbb{R}_t \times \partial\Omega)$:

$$div_{(t,\tau)}\ ((n \cdot E), c^2(n \wedge B)) = -\frac{(n \cdot j)}{\varepsilon_0},\ \ div_{(t,\tau)}\ ((n \cdot B), -(n \wedge E)) = 0,$$

i.e.,

$$-\int_0^T\!\!\int_{\partial\Omega} (n \cdot E)\partial_t\psi\ dtd\sigma - c^2\int_0^T\!\!\int_{\partial\Omega} (n \wedge B) \cdot \nabla_\tau\psi\ dtd\sigma = -\frac{1}{\varepsilon_0}\int_0^T\!\!\int_{\partial\Omega} (n \cdot j)\psi\ dtd\sigma,$$

and

$$-\int_0^T\!\!\int_{\partial\Omega} (n \cdot B)\partial_t\psi\ dtd\sigma + \int_0^T\!\!\int_{\partial\Omega} (n \wedge E) \cdot \nabla_\tau\psi\ dtd\sigma = 0,$$

for all function $\psi \in C^1(\mathbb{R}_t \times \partial\Omega)$, T periodic.

Proof. Consider the test function $\eta(t)\nabla_x\varphi$, where $\eta \in C^1(\mathbb{R}_t)$ is T periodic and $\varphi \in C^1(\overline{\Omega})$. By using the first equation of (3.23) with this test function, we deduce :

$$-\int_0^T\!\!\int_\Omega E(t,x)\eta'(t)\nabla_x\varphi\ dtdx - c^2\int_0^T\!\!\int_{\partial\Omega} (n \wedge B)\eta(t)\nabla_x\varphi\ dtd\sigma = -\frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega \eta(t)j(t,x) \cdot \nabla_x\varphi\ dtdx. \quad (3.24)$$

By using now the third equation of (3.23) with the test function $-\eta'(t)\varphi(t,x)$ we deduce that :

$$-\int_0^T\int_{\partial\Omega}(n\cdot E)\eta'(t)\varphi(t,x)\;dtd\sigma+\int_0^T\int_\Omega \eta'(t)E(t,x)\nabla_x\varphi\;dtdx=-\frac{1}{\varepsilon_0}\int_0^T\int_\Omega \rho(t,x)\eta'(t)\varphi(t,x)\;dtdx. \tag{3.25}$$

By adding the equations (3.24), (3.25), by observing that $(n\wedge B)\cdot\nabla_x\varphi=(n\wedge B)\cdot\nabla_\tau\varphi$ and by using the continuity equation finally we obtain that :

$$-\int_0^T\int_{\partial\Omega}(n\cdot E)\partial_t\psi\;dtd\sigma-c^2\int_0^T\int_{\partial\Omega}(n\wedge B)\cdot\nabla_\tau\psi\;dtd\sigma=-\frac{1}{\varepsilon_0}\int_0^T\int_{\partial\Omega}(n\cdot j)\psi\;dtd\sigma,$$

for all $\psi(t,x)=\eta(t)\varphi(x)$. By density we deduce that the previous equality holds for all test function $\psi\in C^1(\mathbb{R}_t\times\partial\Omega)$, T periodic, or $\mathrm{div}_{(t,\tau)}\;((n\cdot E),c^2(n\wedge B))=-\frac{(n\cdot j)}{\varepsilon_0}$ in $\mathcal{D}'_{per}(\mathbb{R}_t\times\partial\Omega)$. In order to establish the second divergence equation on the boundary we use the second equation of (3.23) with the test function $\eta(t)\nabla_x\varphi$ which gives :

$$-\int_0^T\int_\Omega \eta'(t)B(t,x)\cdot\nabla_x\varphi\;dtdx+\int_0^T\int_{\partial\Omega}\eta(t)(n\wedge E)\cdot\nabla_x\varphi\;dtd\sigma=0.$$

By using also the fourth equation of (3.23) one gets finally :

$$-\int_0^T\int_{\partial\Omega}(n\cdot B)\partial_t\psi\;dtd\sigma+\int_0^T\int_{\partial\Omega}(n\wedge E)\cdot\nabla_\tau\psi\;dtd\sigma=0,$$

or $\mathrm{div}_{(t,\tau)}\;((n\cdot B),-(n\wedge E))=0$ in $\mathcal{D}'_{per}(\mathbb{R}_t\times\partial\Omega)$.
□

We give now an estimate for the outgoing kinetic energy in terms of the total electro-magnetic field and the exterior electro-magnetic field :

PROPOSITION 3.10. *Assume that the hypotheses of Theorem* 2.1 *hold and consider* (f,E,B) *the* T *periodic weak solution constructed in Theorem* 2.1. *We suppose also that* $\partial_t h$ *belongs to* $L^2_{loc}(\mathbb{R}_t;L^2(\partial\Omega)^3)$ *and denote by* (E_0,B_0) *the* T *periodic solution for the problem* (3.11), (3.12) *(cf. Proposition* 3.7*) and by* $(E_s,B_s)=(E-E_0,B-B_0)$ *the self-consistent electro-magnetic field. Then we have the inequality :*

$$K^++\frac{\varepsilon_0 c}{2}\int_0^T\int_{\partial\Omega}(|n\wedge E_s|^2+c^2|n\wedge B_s|^2)\;dtd\sigma\le K^-+\varepsilon_0c^2\int_0^T\int_{\partial\Omega}(n\wedge B)\cdot E_0\;dtd\sigma$$
$$+\,\varepsilon_0\int_0^T\int_\Omega(\partial_tE_0\cdot E(t,x)-c^2\partial_tB_0\cdot B(t,x))\;dtdx.$$

Proof. By using the boundary condition $n\wedge E+c\cdot n\wedge(n\wedge B)=h$ and the inequality (2.29) we have

$$K^+-\varepsilon_0c^2\int_0^T\int_{\partial\Omega}(n\wedge B)\cdot E\;dtd\sigma\le K^-. \tag{3.26}$$

By using also the boundary condition $n\wedge E_s+cn\wedge(n\wedge B_s)=0$ we can write :

$$c^2(n\wedge B)\cdot E=-\frac{c}{2}[|n\wedge E_s|^2+c^2|n\wedge B_s|^2]+c^2(n\wedge B_s)\cdot E_0+c^2(n\wedge B_0)\cdot E. \tag{3.27}$$

Since $\partial_t h$ belongs to $L^2(]0,T[\times\partial\Omega)^3$, from *Proposition* 3.8 we have $\partial_tE_0,\mathrm{rot}\;B_0\in L^2(]0,T[\times\Omega)^3$ and therefore, after multiplication of $\partial_tE_0-c^2\mathrm{rot}\;B_0=0$ by E we have :

$$\int_0^T\int_\Omega\partial_tE_0\cdot E(t,x)\;dtdx-c^2\int_0^T\int_\Omega \mathrm{rot}\;B_0\cdot E(t,x)\;dtdx=0. \tag{3.28}$$

We use also the equation $\partial_t B + \text{rot } E = 0$ with the test function B_0 (which is possible since $\partial_t B_0, \text{rot } B_0 \in L^2(]0,T[\times\Omega)^3)$:

$$-\int_0^T\int_\Omega B(t,x)\cdot\partial_t B_0\ dtdx - \int_0^T\int_{\partial\Omega}(n\wedge B_0)\cdot E(t,x)\ dtd\sigma + \int_0^T\int_\Omega E(t,x)\cdot\text{rot } B_0\ dtdx = 0. \quad (3.29)$$

Finally from (3.27), (3.28), (3.29) we obtain :

$$\varepsilon_0 c^2\int_0^T\int_{\partial\Omega}(n\wedge B)\cdot E\ dtd\sigma + \frac{\varepsilon_0 c}{2}\int_0^T\int_{\partial\Omega}(|n\wedge E_s|^2 + c^2|n\wedge B_s|^2)\ dtd\sigma = \varepsilon_0 c^2\int_0^T\int_{\partial\Omega}(n\wedge B_s)\cdot E_0\ dtd\sigma$$
$$+\varepsilon_0\int_0^T\int_\Omega(\partial_t E_0\cdot E - c^2\partial_t B_0\cdot B)\ dtdx.$$

The conclusion follows by combining with (3.26) and by taking into account that $\int_0^T\int_{\partial\Omega}(n\wedge B_0)\cdot E_0\ dtd\sigma = \frac{1}{c^2}\int_0^T\frac{1}{2}\frac{d}{dt}\int_\Omega(|E_0(t,x)|^2 + c^2\cdot|B_0(t,x)|^2)\ dx\ dt = 0.$ □

In order to estimate the term $\varepsilon_0 c^2\int_0^T\int_{\partial\Omega}(n\wedge B)\cdot E_0(t,x)\ dtd\sigma$ we need the following representation for free divergence fields on $\mathbb{R}_t\times\partial\Omega$ (see the *Appendix*, *Corollary* 6.11 for more details). We denote by $H^1([0,T]\times\partial\Omega)$ the closure of $\{\varphi\in C^1(\mathbb{R}_t\times\partial\Omega)\mid\varphi \text{ is } T \text{ periodic}\}$ in the H^1 norm.

LEMMA 3.11. *Assume that $\Omega\subset\mathbb{R}^3_x$ is smooth, open, bounded with $\partial\Omega$ simply connected and consider $f = (f_0,\vec{f}) := (f_0,f_1,f_2,f_3)\in L^2_{loc}(\mathbb{R}_t;L^2(\partial\Omega)^4)$ a T periodic field such that $\int_0^T\int_{\partial\Omega}f_0(t,x)\ dtd\sigma = 0$, $(n\cdot\vec{f})|_{\mathbb{R}_t\times\partial\Omega} = 0$ and $div_{(t,\tau)}\ f = 0$ in $\mathcal{D}'_{per}(\mathbb{R}_t\times\partial\Omega)$. Then there is $A = (A_0,\vec{A})\in H^1([0,T]\times\partial\Omega)^4$ verifying $\int_0^T\int_{\partial\Omega}A_0(t,x)\ dtd\sigma = 0$, $(n\cdot\vec{A})|_{\mathbb{R}_t\times\partial\Omega} = 0$, $div_{(t,\tau)}\ A = 0$ in $\mathcal{D}'_{per}(\mathbb{R}_t\times\partial\Omega)$ such that :*

$$f_0 = -div_\tau\ (n\wedge\vec{A}),\quad \vec{f} = n\wedge(\partial_t\vec{A} - \nabla_\tau\ A_0).$$

Moreover we have the estimate :

$$\|\nabla_{(t,\tau)}\ A_0\|^2_{L^2(]0,T[\times\partial\Omega)^4} + \|\vec{A}\|^2_{H^1(]0,T[\times\partial\Omega)^3}\le C(\Omega)\cdot\|f\|^2_{L^2(]0,T[\times\partial\Omega)^4},$$

where the constant $C(\Omega)$ depends on Ω but not on T.

By a straightforward scaling argument we obtain :

PROPOSITION 3.12. *Assume that the hypotheses of Proposition 3.7 are verified and denote by (E_0,B_0) the T periodic weak solution of the problem (3.11), (3.12). Then there is $A = (A_0,\vec{A})\in H^1([0,T]\times\partial\Omega)^4$ with $\int_0^T\int_{\partial\Omega}A_0(t,x)\ dtd\sigma = 0$, $(n\cdot\vec{A})|_{\mathbb{R}_t\times\partial\Omega} = 0$, such that :*

$$\frac{1}{c}\partial_t A_0 + div_\tau\ \vec{A} = 0,\quad (n\cdot B_0) = div_\tau\left(n\wedge\frac{\vec{A}}{c}\right),\quad n\wedge E_0 = n\wedge\left(\frac{1}{c}\partial_t\vec{A} - \nabla_\tau\ A_0\right).$$

Moreover we have the estimates :

$$\|\partial_t A_0\|^2_{L^2(]0,T[\times\partial\Omega)}\le C(\Omega)\cdot c^2\cdot H,\quad \|\nabla_\tau\ A_0\|^2_{L^2(]0,T[\times\partial\Omega)^3}\le C(\Omega)\cdot H,$$

$$\|\partial_t\vec{A}\|^2_{L^2(]0,T[\times\partial\Omega)^3}\le C(\Omega)\cdot c^2\cdot H,\quad \|\vec{A}\|^2_{L^2(]0,T[\times\partial\Omega)^3} + \|\nabla_\tau\ \vec{A}\|^2_{L^2(]0,T[\times\partial\Omega)^9}\le C(\Omega)\cdot H.$$

Proof. From *Proposition* 3.9 we know that $\text{div}_{(t,\tau)}\ ((n\cdot B_0), -n\wedge E_0) = 0$. We introduce $\tilde{t} = c\cdot t$, $\tilde{T} = c\cdot T$ and if $u(t)$ is an arbitrary T periodic function of t we denote by $\tilde{u}(t)$ the $\tilde{T}$ periodic function given by $\tilde{u}(\tilde{t}) = u(\tilde{t}/c)$, $\tilde{t}\in\mathbb{R}$. We obtain $\text{div}\ _{(\tilde{t},\tau)}(c\cdot(n\cdot\tilde{B}_0), -n\wedge\tilde{E}_0) = 0$. Remark that since div ${}_x B_0 = 0$ we have div ${}_x \tilde{B}_0 = 0$ and thus $\int_0^{\tilde{T}} \int_{\partial\Omega} (n \cdot \tilde{B}_0)\ d\tilde{t} d\sigma = 0$. Obviously $(n \wedge \tilde{E}_0)$ is a tangential field and therefore the previous lemma applies for the field $(-c(n{\cdot}\tilde{B}_0), n{\wedge}\tilde{E}_0)$. We deduce that there is $\tilde{A} = (\tilde{A}_0, \vec{\tilde{A}}) \in H^1([0,\tilde{T}] \times \partial\Omega)^4$ with $\int_0^{\tilde{T}} \int_{\partial\Omega} \tilde{A}_0(\tilde{t}, x)\ d\tilde{t} d\sigma = 0$, $(n \cdot \vec{\tilde{A}})|_{\mathbb{R}_t \times \partial\Omega} = 0$, div ${}_{(\tilde{t},\tau)} \tilde{A} = 0$ such that :

$$c(n \cdot \tilde{B}_0) = \mathrm{div}_\tau\ (n \wedge \vec{\tilde{A}}),\ \ n \wedge \tilde{E}_0 = n \wedge (\partial_{\tilde{t}} \vec{\tilde{A}} - \nabla_\tau\ \tilde{A}_0),$$

and :

$$\|\nabla_{(\tilde{t},\tau)} \tilde{A}_0\|^2_{L^2(]0,\tilde{T}[\times\partial\Omega)^4} + \|\vec{\tilde{A}}\|^2_{H^1(]0,\tilde{T}[\times\partial\Omega)^3} \le C(\Omega) \cdot [\|c(n{\cdot}\tilde{B}_0)\|^2_{L^2(]0,\tilde{T}[\times\partial\Omega)} + \|n{\wedge}\tilde{E}_0\|^2_{L^2(]0,\tilde{T}[\times\partial\Omega)^3}].$$

The conclusion follows by taking $A(t,x) = (A_0(t,x), \vec{A}(t,x)) = (\tilde{A}_0(c \cdot t, x), \vec{\tilde{A}}(c \cdot t, x))$.
□

Proposition 3.13. *Assume that the hypotheses of Proposition 3.8 are verified and denote by* (E_0, B_0) *the* T *periodic solution of the problem* 3.11, 3.12. *Then we have the estimates :*

$$\|\partial_t A_0\|^2_{L^2(]0,T[\times\partial\Omega)} + \|\nabla_\tau\ \partial_t A_0\|^2_{L^2(]0,T[\times\partial\Omega)^3} + \|\nabla_\tau\ A_0\|^2_{L^2(]0,T[\times\partial\Omega)^3} \le C(\Omega) \cdot (H + H_1),$$

$$\|\partial_t \vec{A}\|^2_{L^2(]0,T[\times\partial\Omega)^3} + \|\nabla_\tau\ \partial_t \vec{A}\|^2_{L^2(]0,T[\times\partial\Omega)^9} \le C(\Omega) \cdot H_1,$$

where $H_1 := \int_0^T\int_{\partial\Omega} |\partial_t h|^2\ dt d\sigma$.

Proof. By *Proposition* 3.12 we have the estimate :

$$\|\nabla_\tau\ \partial_t A_0\|^2_{L^2(]0,T[\times\partial\Omega)^3} \le C(\Omega) \cdot H_1,\ \ \|\nabla_\tau\ A_0\|^2_{L^2(]0,T[\times\partial\Omega)^3} \le C(\Omega) \cdot H.$$

Since $\frac{1}{c}\partial_t A_0 + \mathrm{div}_\tau\ \vec{A} = 0$ we have $\int_{\partial\Omega} \partial_t A_0\ d\sigma = 0$, a.e. $t \in \mathbb{R}_t$. By using the Poincaré inequality we have :

$$\|\partial_t A_0\|^2_{L^2(]0,T[\times\partial\Omega)} \le C_P \cdot \int_0^T \left\{ \int_{\partial\Omega} |\nabla_\tau \partial_t A_0(t,x)|^2\ d\sigma + \left| \int_{\partial\Omega} \partial_t A_0(t,x)\ d\sigma \right|^2 \right\}\ dt \le C_1(\Omega) \cdot H_1.$$

The second estimate of our proposition follows directly from *Proposition* 3.12.
□

Proposition 3.14. *Assume that the hypotheses of Theorem* 2.1 *and Proposition* 3.8 *hold and consider* (f, E, B) *the* T *periodic weak solution constructed in Theorem* 2.1. *Then we have the estimates :*

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \int_{\mathbb{R}^3_p} (1+\mathcal{E}(p)) f\ dtdxdp + \frac{\varepsilon_0}{2} \int_0^T\!\!\int_\Omega (|E|^2 + c^2|B|^2)\ dtdx + \int_0^T\!\!\int_{\Sigma^+} (v(p) \cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\ dtd\sigma dp \\
&\quad + \frac{\varepsilon_0 r}{2} \int_0^T\!\!\int_{\partial\Omega} (|n \wedge E|^2 + c^2 \cdot |n \wedge B|^2)\ dtd\sigma + \frac{\varepsilon_0 r}{2} \int_0^T\!\!\int_{\partial\Omega} ((n \cdot E)^2 + c^2 \cdot (n \cdot B)^2)\ dtd\sigma \\
&\quad \le C(m, \varepsilon_0, T, \Omega, M^-, K^-, H, H_1, \|g\|_{L^\infty}).
\end{aligned} \tag{3.30}$$

Proof. We need to estimate the outgoing kinetic energy K^+. By *Proposition* 3.10 we have :

$$\begin{aligned}
K^+ + \frac{\varepsilon_0 c}{2} \int_0^T\!\!\int_{\partial\Omega} (|n \wedge (E - E_0)|^2 + c^2|n \wedge (B - B_0)|^2)\ dtd\sigma \le K^- + \varepsilon_0 c^2 \int_0^T\!\!\int_{\partial\Omega} (n \wedge B) \cdot E_0\ dtd\sigma \\
+ \varepsilon_0 \left(\int_0^T\!\!\int_\Omega (|\partial_t E_0|^2 + c^2 |\partial_t B_0|^2)\ dtdx \right)^{\frac{1}{2}} \cdot \left(\int_0^T\!\!\int_\Omega (|E|^2 + c^2|B|^2)\ dtdx \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.31}$$

In order to estimate the term $\varepsilon_0 c^2 \int_0^T\int_{\partial\Omega}(n\wedge B)\cdot E_0\ dtd\sigma = \varepsilon_0 c^2 \int_0^T\int_{\partial\Omega}(n\wedge(n\wedge B))\cdot(n\wedge E_0)\ dtd\sigma$ we use the decomposition $n\wedge E_0 = n\wedge(\frac{1}{c}\partial_t\vec{A} - \nabla_\tau\ A_0)$ of *Proposition* 3.12. By using *Proposition* 3.13 and the inequality (2.29) we have for $c\geq 1$:

$$\left|c^2\int_0^T\int_{\partial\Omega}(n\wedge(n\wedge B))\cdot(n\wedge\frac{1}{c}\partial_t\vec{A})\ dtd\sigma\right| \leq \|c(n\wedge B)\|_{L^2(]0,T[\times\partial\Omega)^3}\cdot\|\partial_t\vec{A}\|_{L^2(]0,T[\times\partial\Omega)^3}$$

$$\leq C(\Omega)\cdot\left(\frac{2}{\varepsilon_0}K^- + H\right)^{\frac{1}{2}}\cdot H_1^{\frac{1}{2}}. \tag{3.32}$$

We want now to estimate the term $c^2\int_0^T\int_{\partial\Omega}(n\wedge(n\wedge B))\cdot(n\wedge\nabla_\tau\ A_0)\ dtd\sigma$. For this we use the first divergence equation proved in *Proposition* 3.9 :

$$\mathrm{div}_{(t,\tau)}\ ((n\cdot E),\ c^2(n\wedge B)) = -\frac{(n\cdot j)}{\varepsilon_0}. \tag{3.33}$$

By using the test function A_0 we deduce that :

$$\int_0^T\int_{\partial\Omega}(n\cdot E)\partial_t A_0\ dtd\sigma + c^2\int_0^T\int_{\partial\Omega}(n\wedge(n\wedge B))\cdot(n\wedge\nabla_\tau\ A_0)\ dtd\sigma = \frac{1}{\varepsilon_0}\int_0^T\int_{\partial\Omega}(n\cdot j)A_0\ dtd\sigma. \tag{3.34}$$

From *Proposition* 3.13 we obtain :

$$\left|\int_0^T\int_{\partial\Omega}(n\cdot E)\partial_t A_0\ dtd\sigma\right| \leq \|(n\cdot E)\|_{L^2(]0,T[\times\partial\Omega)}\cdot C(\Omega)\cdot(H+H_1)^{\frac{1}{2}}. \tag{3.35}$$

In order to estimate the term $\int_0^T\int_{\partial\Omega}(n\cdot j)A_0\ dtd\sigma$ we can use Sobolev inequalities. By using the condition $\int_0^T\int_{\partial\Omega}A_0(t,x)\ dtd\sigma = 0$, the Poincaré inequality and *Proposition* 3.13 we deduce that $\|A_0\|^2_{H^1(]0,T[\times\partial\Omega)} \leq C(T,\Omega)\cdot(H+H_1)$. By Sobolev inequalities we have $\|A_0\|^2_{L^5(]0,T[\times\partial\Omega)} \leq C\cdot\|A_0\|^2_{H^1(]0,T[\times\partial\Omega)} \leq C(T,\Omega)\cdot(H+H_1)$. Now by adapting *Lemma* 3.4 for the time periodic case we obtain :

$$\left|\int_0^T\int_{\partial\Omega}(n\cdot j)A_0\ dtd\sigma\right| \leq \|(n\cdot j)\|_{L^{\frac{5}{4}}(]0,T[\times\partial\Omega)}\cdot\|A_0\|_{L^5(]0,T[\times\partial\Omega)} \tag{3.36}$$

$$\leq C\|g\|^{\frac{1}{5}}_{L^\infty(\mathbb{R}_t\times\Sigma^-)}\left(\int_0^T\int_\Sigma|(v(p)\cdot n(x))|(1+\mathcal{E}(p))\gamma f\ dtd\sigma dp\right)^{\frac{4}{5}}\cdot(H+H_1)^{\frac{1}{2}}.$$

Finally, by using (3.31), (3.32), (3.34), (3.35), (3.36) we obtain :

$$K^+ + \frac{\varepsilon_0 c}{2}\int_0^T\int_{\partial\Omega}(|n\wedge(E-E_0)|^2 + c^2|n\wedge(B-B_0)|^2)\ dtd\sigma \leq \varepsilon_0 C H_1^{\frac{1}{2}}\cdot\left(\int_0^T\int_\Omega(|E|^2+c^2|B|^2)\ dtdx\right)^{\frac{1}{2}}$$

$$+K^- + C\varepsilon_0 H_1^{\frac{1}{2}}\cdot\left(\frac{2}{\varepsilon_0}K^- + H\right)^{\frac{1}{2}} + C\varepsilon_0(H+H_1)^{\frac{1}{2}}\cdot\|(n\cdot E)\|_{L^2(]0,T[\times\partial\Omega)}$$

$$+C\cdot\|g\|^{\frac{1}{5}}_{L^\infty(\mathbb{R}_t\times\Sigma^-)}\cdot(H+H_1)^{\frac{1}{2}}\cdot(M^- + K^- + M^+ + K^+)^{\frac{4}{5}}. \tag{3.37}$$

Remember that the solution (f,E,B) verifies the estimate (see (2.24)) :

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p}\mathcal{E}(p)f\ dtdxdp + \frac{\varepsilon_0}{2}\int_0^T\int_\Omega(|E|^2+c^2|B|^2)\ dtdx + \int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\ dtd\sigma dp$$

$$+\frac{\varepsilon_0 r}{2}\int_0^T\int_{\partial\Omega}[(|n\wedge E|^2 + c^2\cdot|n\wedge B|^2) + ((n\cdot E)^2 + c^2\cdot(n\cdot B)^2)]\ dtd\sigma$$

$$\leq C(m,\varepsilon_0,\Omega)\cdot(M^- + K^- + K^+ + H). \tag{3.38}$$

The conclusion follows easily by combining (3.37), (3.38) (the estimate for the total mass can be obtained as in (2.27)). Remark also that the previous computations give the estimate :

$$c \cdot \int_0^T\!\!\int_{\partial\Omega} (|n \wedge (E - E_0)|^2 + c^2 |n \wedge (B - B_0)|^2)\, dt d\sigma \le C, \tag{3.39}$$

where C depends on $m, \varepsilon_0, T, \Omega, M^-, K^-, H, H_1, \|g\|_{L^\infty}$, but not on c. □

Now we can prove that the total energy is uniformly bounded with respect to $t \in \mathbb{R}_t$ and $c \ge 1$. We check this (only) for regular T periodic solutions of the Vlasov-Maxwell system. By using the energy conservation law (see 2.12) we have for all $t \in [T, 2T]$ and $s \in [0, T]$:

$$\begin{aligned}\int_\Omega\int_{\mathbb{R}^3_p} \mathcal{E}(p) f(t,x,p)\, dxdp &+ \frac{\varepsilon_0}{2}\int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2)\, dx + \int_s^t\!\!\int_{\Sigma^+} (v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f d\tau\, d\sigma dp \\ &- \varepsilon_0 c^2 \int_s^t\!\!\int_{\partial\Omega} (n \wedge B)\cdot E d\tau\, d\sigma = \int_\Omega\int_{\mathbb{R}^3_p} \mathcal{E}(p) f(s,x,p)\, dxdp \\ &+ \frac{\varepsilon_0}{2}\int_\Omega (|E(s,x)|^2 + c^2 |B(s,x)|^2)\, dx + \int_s^t\!\!\int_{\Sigma^-} |(v(p)\cdot n(x))|\mathcal{E}(p) g d\tau\, d\sigma dp.\end{aligned} \tag{3.40}$$

We introduce the notations $M(t) = \int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)\ dxdp$, $K(t) = \int_\Omega\int_{\mathbb{R}^3_p} \mathcal{E}(p) f(t,x,p)\ dxdp$, $W^{em}(t) = \frac{\varepsilon_0}{2}\int_\Omega (|E(t,x)|^2 + c^2 \cdot |B(t,x)|^2)\ dx$, $W_0^{em}(t) = \frac{\varepsilon_0}{2}\int_\Omega (|E_0|^2 + c^2 \cdot |B_0|^2)\ dx$. By performing similar computations as in the proof of *Proposition* 3.10 we obtain :

$$\begin{aligned}-\varepsilon_0 c^2 \int_s^t\!\!\int_{\partial\Omega} (n\wedge B)\cdot E d\tau\, d\sigma = &\frac{\varepsilon_0 c}{2}\int_s^t\!\!\int_{\partial\Omega} (|n\wedge E_s|^2 + c^2|n\wedge B_s|^2) d\tau\, d\sigma - \varepsilon_0 c^2 \int_s^t\!\!\int_{\partial\Omega} (n\wedge B)\cdot E_0 d\tau\, d\sigma \\ &+ W_0^{em}(t) - W_0^{em}(s) - \varepsilon_0 \int_s^t\int_\Omega (\partial_t E_0 \cdot E - c^2 \partial_t B_0 \cdot B) d\tau\, dx \\ &- \varepsilon_0 c^2 \int_\Omega B(t,x)\cdot B_0(t,x)\, dx + \varepsilon_0 c^2 \int_\Omega B(s,x)\cdot B_0(s,x)\, dx.\end{aligned} \tag{3.41}$$

By combining (3.40), (3.41) one gets :

$$\begin{aligned}K(t) + W^{em}(t) &+ \frac{\varepsilon_0 c}{2}\int_s^t\int_{\partial\Omega} (|n\wedge E_s|^2 + c^2|n\wedge B_s|^2)\, d\sigma d\tau + \int_s^t\int_{\Sigma^+} (v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f\, d\sigma dp d\tau \\ &= K(s) + W^{em}(s) + \int_{\Sigma^-} |(v(p)\cdot n(x))|\mathcal{E}(p) g\, d\sigma dp d\tau + \varepsilon_0 c^2 \int_s^t\int_{\partial\Omega} (n\wedge B)\cdot E_0\, d\sigma d\tau + W_0^{em}(s) \\ &- \varepsilon_0 c^2 \int_\Omega B(s,x)\cdot B_0(s,x)\, dx - W_0^{em}(t) + \varepsilon_0 c^2 \int_\Omega B(t,x)\cdot B_0(t,x)\, dx \\ &+ \varepsilon_0 \int_s^t\int_\Omega (\partial_t E_0\cdot E - c^2\partial_t B_0 \cdot B)\, dx d\tau.\end{aligned} \tag{3.42}$$

By the *Propositions* 3.7, 3.8 we deduce easily that $(E_0, B_0) \in L^\infty(\mathbb{R}_t; L^2(\Omega))^6$, $(n\wedge E_0, n\wedge B_0) \in L^\infty(\mathbb{R}_t; L^2(\partial\Omega))^6$, $(n\cdot E_0, n\cdot B_0) \in L^\infty(\mathbb{R}_t; L^2(\partial\Omega))^2$ and we have the estimate :

$$\begin{aligned}\int_\Omega (|E_0|^2 + c^2\cdot|B_0|^2)\, dx + r\cdot \int_{\partial\Omega} (|n\wedge E_0|^2 + c^2 \cdot |n\wedge B_0|^2)\, d\sigma + r\cdot\int_{\partial\Omega} ((n\cdot E_0)^2 + c^2\cdot(n\cdot B_0)^2)\, d\sigma \\ \le C\cdot(H + H_1),\ \ \forall t \in \mathbb{R}_t.\end{aligned} \tag{3.43}$$

Now, after integration of (3.42) with respect to $s \in [0, T]$ and by using the estimates of *Proposition* 3.14 finally one gets :

$$\begin{aligned}K(t) + W^{em}(t) &\le C(m, \varepsilon_0, T, \Omega, M^-, K^-, H, H_1, \|g\|_{L^\infty}) + C(\varepsilon_0, \Omega, T, H, H_1)\cdot W^{em}(t)^{\frac{1}{2}} \\ &+ \frac{\varepsilon_0 c^2}{T}\int_0^T\int_s^t\int_{\partial\Omega} (n\wedge B)\cdot E_0\, ds d\tau d\sigma,\ \ \forall t\in\mathbb{R}_t.\end{aligned} \tag{3.44}$$

We need to estimate the last term of the above inequality. As in the proof of *Proposition* 3.14 we write :

$$I(s) = \varepsilon_0 c^2 \int_s^t \int_{\partial\Omega} (n \wedge B) \cdot E_0 \; d\tau d\sigma = \varepsilon_0 c^2 \int_s^t \int_{\partial\Omega} (n \wedge (n \wedge B)) \cdot (n \wedge (\frac{1}{c}\partial_t \vec{A} - \nabla_\tau \; A_0)) \; d\tau d\sigma$$
$$= I_1(s) + I_2(s). \tag{3.45}$$

The first term can be estimated uniformly with respect to $s \in [0,T]$:

$$\begin{aligned} |I_1(s)| &\leq \int_0^{2T} \int_{\partial\Omega} \varepsilon_0 c |n \wedge B| \cdot |\partial_t \vec{A}| dt \; d\sigma \\ &\leq 2\sqrt{\varepsilon_0} \left(\int_0^T \int_{\partial\Omega} \varepsilon_0 c^2 |n \wedge B|^2 \; dt d\sigma \right)^{\frac{1}{2}} \cdot \left(\int_0^T \int_{\partial\Omega} |\partial_t \vec{A}|^2 \; dt d\sigma \right)^{\frac{1}{2}} \\ &\leq C(m, \varepsilon_0, \Omega, M^-, K^-, H, H_1). \end{aligned} \tag{3.46}$$

For the second term we use (3.33) with the test function A_0 on $[s,t] \times \partial\Omega$:

$$\begin{aligned} - \int_s^t \int_{\partial\Omega} (n \cdot E) \partial_t A_0(u,x) \; du \; d\sigma + \int_{\partial\Omega} (n \cdot E) A_0(t,x) \; d\sigma - \int_{\partial\Omega} (n \cdot E) A_0(s,x) \; d\sigma \\ - \int_s^t \int_{\partial\Omega} c^2 (n \wedge B) \cdot \nabla_\tau \; A_0(u,x) \; du \; d\sigma \\ = - \frac{1}{\varepsilon_0} \int_s^t \int_{\partial\Omega} (n \cdot j) A_0(u,x) \; du \; d\sigma. \end{aligned} \tag{3.47}$$

By using the *Propositions* 3.13, 3.14 we find that :

$$\left| \frac{1}{T} \int_0^T I_2(s) \; ds \right| \leq C(m, \varepsilon_0, T, \Omega, M^-, K^-, H, H_1, \|g\|_{L^\infty}) + \left| \int_{\partial\Omega} (n \cdot E) A_0(t,x) \; d\sigma \right|. \tag{3.48}$$

Since $\|\nabla_\tau \; A_0\|^2_{L^2(]0,T[\times\partial\Omega)^3} + \|\partial_t \nabla_\tau \; A_0\|^2_{L^2(]0,T[\times\partial\Omega)^3} \leq C(\Omega) \cdot (H + H_1)$, we deduce by periodicity that $\|\nabla_\tau \; A_0\|^2_{L^\infty(\mathbb{R}_t; L^2(\partial\Omega)^3)} \leq C(\Omega, T) \cdot (H + H_1)$. Since $\int_0^T \int_{\partial\Omega} A_0(t,x) \; dt d\sigma = 0$ we have $\|A_0\|^2_{L^2(]0,T[\times\partial\Omega)} \leq C \cdot \|\nabla_{(t,\tau)} A_0\|^2_{L^2(]0,T[\times\partial\Omega)^4} \leq C \cdot (H + H_1)$ and from $\|A_0\|^2_{L^2(]0,T[\times\partial\Omega)} + \|\partial_t A_0\|^2_{L^2(]0,T[\times\partial\Omega)} \leq C(\Omega) \cdot (H + H_1)$ we deduce that $\|A_0\|^2_{L^\infty(\mathbb{R}_t; L^2(\partial\Omega))} \leq C(\Omega, T) \cdot (H + H_1)$. Finally we obtain that $\|A_0\|^2_{L^\infty(\mathbb{R}_t; H^1(\partial\Omega))} \leq C \cdot (H + H_1)$. Take now $F_0 \in H^1(\Omega)$ such that $\gamma(F_0) = A_0(t)$ and $\|F_0\|_{H^1(\Omega)} \leq C(\Omega) \cdot \|A_0(t)\|_{H^{1/2}(\partial\Omega)} \leq C(\Omega) \|A_0(t)\|_{H^1(\partial\Omega)}$. By using the equation div $E = \frac{\rho}{\varepsilon_0}$ we have :

$$\int_{\partial\Omega} (n \cdot E) A_0(t,x) \; d\sigma = \int_\Omega E(t,x) \cdot \nabla_x F_0 \; dx + \int_\Omega \text{div } E \cdot F_0(x) \; dx = J_1(t) + J_2(t).$$

For the first term we can write :

$$|J_1(t)| \leq \|E(t)\|_{L^2(\Omega)^3} \cdot \|F_0\|_{H^1(\Omega)} \leq C(\Omega, T, H, H_1) \cdot W^{em}(t)^{\frac{1}{2}}. \tag{3.49}$$

For the second term we can use interpolation and Sobolev inequalities :

$$\begin{aligned} |J_2(t)| &\leq \|\frac{\rho}{\varepsilon_0}\|_{L^{\frac{4}{3}}} \cdot \|F_0\|_{L^4(\Omega)} \leq C(m, \varepsilon_0, \Omega) \cdot \|g\|^{\frac{1}{4}}_{L^\infty} \cdot \left(\int_\Omega \int_{\mathbb{R}^3_p} (1 + \mathcal{E}(p)) f(t,x,p) \; dx dp \right)^{\frac{3}{4}} \cdot \|F_0\|_{H^1(\Omega)} \\ &\leq C(m, \varepsilon_0, \Omega, T, H, H_1, \|g\|_{L^\infty}) \cdot (M(t) + K(t))^{\frac{3}{4}}. \end{aligned} \tag{3.50}$$

Finally by using (3.44), (3.45), (3.46), (3.48), (3.49), (3.50) we deduce for all $t \in \mathbb{R}_t$:

$$K(t) + W^{em}(t) \leq C(m, \varepsilon_0, \Omega, T, M^-, K^-, H, H_1, \|g\|_{L^\infty}) \cdot \left(1 + W^{em}(t)^{\frac{1}{2}} + (M(t) + K(t))^{\frac{3}{4}} \right). \tag{3.51}$$

The conclusion follows easily by observing that $M(t)$ is bounded since we have for $s, t \in [0,T]$:

$$M(t) = M(s) - \int_s^t \int_\Sigma (v(p)\cdot n(x))\gamma f(u,x,p)\ du\ d\sigma dp,$$

which implies after integration on $s \in [0,T]$ that :

$$\begin{aligned} M(t) &\leq \frac{1}{T}\int_0^T M(s)ds + \frac{1}{T}\int_0^T ds\left|\int_s^t \int_\Sigma (v(p)\cdot n(x))\gamma f(u,x,p)\ du\ d\sigma dp\right| \\ &\leq \frac{1}{T}\int_0^T M(s)\ ds + 2M^-, \quad \forall t\in[0,T]. \end{aligned} \tag{3.52}$$

4. Asymptotic behavior when $c \to +\infty$.

In this section we study the behavior of the T periodic weak solutions for the relativistic Vlasov-Maxwell system (cf. *Theorem* 2.1) when c becomes large. We denote by $\mathcal{E}_c(p), v_c(p)$ the relativistic energy and velocity functions corresponding to the light speed $c > 0$. The classical energy and velocity functions will be denoted by $\mathcal{E}(p), v(p)$ respectively. Observe that we have the convergence :

$$\lim_{c\to+\infty} \mathcal{E}_c(p) = \mathcal{E}(p), \quad \lim_{c\to+\infty} v_c(p) = v(p), \text{ uniformly on compact sets of } \mathbb{R}^3_p.$$

We denote by (f_c, E_c, B_c) the T periodic weak solutions for the relativistic Vlasov-Maxwell system constructed in *Theorem* 2.1 and we introduce the notations :

$$K_c(t) := \int_\Omega\int_{\mathbb{R}^3_p} \mathcal{E}_c(p) f_c\ dxdp, \quad W_c^{em}(t) := \frac{\varepsilon_0}{2}\int_\Omega (|E_c|^2 + c^2|B_c|^2)\ dx, \quad W_c(t) = K_c(t) + W_c^{em}(t),$$

$$K_c := \int_0^T K_c(t)\ dt, \quad W_c^{em} := \int_0^T W_c^{em}(t)\ dt, \quad W_c := K_c + W_c^{em},$$

$$M_c(t) := \int_\Omega\int_{\mathbb{R}^3_p} f_c\ dxdp, \quad M_c := \int_0^T M_c(t)\ dt,$$

$$M_c^\pm := \int_0^T\!\!\int_{\Sigma^\pm} |(v_c(p)\cdot n(x))|\gamma^\pm f_c\ dtd\sigma dp, \quad K_c^\pm := \int_0^T\!\!\int_{\Sigma^\pm} |(v_c(p)\cdot n(x))|\mathcal{E}_c(p)\gamma^\pm f_c\ dtd\sigma dp,$$

$$W_{c,\tau}^{em} := \frac{\varepsilon_0}{2}\int_0^T\!\!\int_{\partial\Omega} (|n\wedge E_c|^2 + c^2|n\wedge B_c|^2)\ dtd\sigma, \quad W_{c,n}^{em} := \frac{\varepsilon_0}{2}\int_0^T\!\!\int_{\partial\Omega} ((n\cdot E_c)^2 + c^2(n\cdot B_c)^2)\ dtd\sigma.$$

THEOREM 4.1. *Assume that $\Omega \subset \mathbb{R}^3_x$ is open, bounded, with boundary $\partial\Omega$ smooth, strictly star-shaped and consider g and h T periodic functions verifying $0 \leq g \in L^\infty(\mathbb{R}_t\times\Sigma^-)$, $M^- + K^- = \int_0^T\!\int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\ dtd\sigma dp < +\infty$, $(n\cdot h)|_{\mathbb{R}_t\times\partial\Omega} = 0$, $\int_0^T\!\int_{\partial\Omega}(|h|^2 + |\partial_t h|^2)\ dtd\sigma < +\infty$. Then for all sequence $(c_r)_r$ with $\lim_{r\to+\infty} c_r = +\infty$ there is a subsequence $(c_{r_k})_k$ such that $f_{c_{r_k}} \rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p)$, $E_{c_{r_k}} \rightharpoonup E$ weakly $\star$ in $L^\infty(\mathbb{R}_t; L^2(\Omega)^3)$ where (f,E) is a T periodic weak solution for the classical Vlasov-Poisson system :*

$$\partial_t f + v(p)\cdot\nabla_x f + qE(t,x)\cdot\nabla_p f = 0, \quad (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p, \quad f(t,x,p) = g(t,x,p), \ (t,x,p)\in\mathbb{R}_t\times\Sigma^-,$$

$$rot\ E = 0, \ \ div\ E = \frac{\rho}{\varepsilon_0}, \ \ (t,x) \in \mathbb{R}_t \times \Omega.$$

Moreover the solution (f,E) *has traces* $\gamma^+ f \in L^\infty(\mathbb{R}_t \times \Sigma^+)$, $n \wedge E = n \wedge \nabla_\tau\ h_2 \in L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega)^3)$ *(where* $h = \nabla_\tau\ h_1 + n \wedge \nabla_\tau\ h_2$, $h_1, h_2 \in L^2_{loc}(\mathbb{R}_t; H^1(\partial\Omega))$ *is the orthogonal decomposition in* $L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega)^3)$ *of* h *into irrotational/rotational parts, cf. Appendix, Proposition* 6.5*),* $(n \cdot E) \in L^2_{loc}(\mathbb{R}_t; L^2(\partial\Omega))$ *and the following estimates hold :*

$$\begin{aligned} ess \sup_{t\in\mathbb{R}_t} & \left\{ \int_\Omega \int_{\mathbb{R}^3_p} (1+\mathcal{E}(p)) f(t,x,p)\ dxdp + \frac{\varepsilon_0}{2} \int_\Omega |E(t,x)|^2\ dx \right\} \\ & + \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\ dtd\sigma dp + \frac{\varepsilon_0 r}{2} \int_0^T\!\!\int_{\partial\Omega} (|n\wedge E|^2 + (n\cdot E)^2)\ dtd\sigma \\ & \leq C(m, \varepsilon_0, \Omega, T, M^-, K^-, H, H_1, \|g\|_{L^\infty}). \end{aligned} \tag{4.1}$$

Proof. We have $\|f_{c_r}\|_{L^\infty(\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p)} \leq \|g\|_{L^\infty(\mathbb{R}_t\times\Sigma^-)}\ \forall r$. By observing that $K^-_c \leq K^-$ and $M^-_c \leq M^-$, $\forall c > 0$ we deduce also that :

$$\begin{aligned} \|E_{c_r}\|_{L^\infty(\mathbb{R}_t;L^2(\Omega)^3)} + c_r \cdot \|B_{c_r}\|_{L^\infty(\mathbb{R}_t;L^2(\Omega)^3)} + \|M_{c_r}(\cdot)\|_{L^\infty(\mathbb{R}_t)} + \|K_{c_r}(\cdot)\|_{L^\infty(\mathbb{R}_t)} \\ + M^+_{c_r} + K^+_{c_r} + W^{em}_{c_r,\tau} + W^{em}_{c_r,n} \\ \leq C(m, \varepsilon_0, \Omega, T, M^-, K^-, H, H_1, \|g\|_{L^\infty}), \ \ \forall\ r. \end{aligned} \tag{4.2}$$

Therefore there is a subsequence (c_{r_k}) such that $f_k := f_{c_{r_k}} \rightharpoonup f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}^3_p)$, $\gamma^+ f_k := \gamma^+ f_{c_{r_k}} \rightharpoonup \gamma^+ f$ weakly $\star$ in $L^\infty(\mathbb{R}_t \times \Sigma^+)$, $E_k := E_{c_{r_k}} \rightharpoonup E$ weakly $\star$ in $L^\infty(\mathbb{R}_t; L^2(\Omega)^3)$, $n \wedge E_k := n \wedge E_{c_{r_k}} \rightharpoonup n \wedge E$ weakly in $L^2(]0,T[\times\partial\Omega)^3$, $n \cdot E_k := n \cdot E_{c_{r_k}} \rightharpoonup n \cdot E$ weakly in $L^2(]0,T[\times\partial\Omega)$, $c_k B_k := c_{r_k} B_{c_{r_k}} \rightharpoonup A$ weakly $\star$ in $L^\infty(\mathbb{R}_t; L^2(\Omega)^3)$, $c_k(n \wedge B_k) := c_{r_k}(n \wedge B_{c_{r_k}}) \rightharpoonup n \wedge A$ weakly in $L^2(]0,T[\times\partial\Omega)^3$, $c_k(n \cdot B_k) := c_{r_k}(n \cdot B_{c_{r_k}}) \rightharpoonup (n \cdot A)$ weakly in $L^2(]0,T[\times\partial\Omega)$. In particular we have the convergences :

$$\lim_{k\to+\infty} B_k = 0, \ \ \lim_{k\to+\infty} (n \wedge B_k) = 0, \ \ \lim_{k\to+\infty} (n \cdot B_k) = 0, \tag{4.3}$$

in the spaces $L^\infty(\mathbb{R}_t; L^2(\Omega)^3)$, $L^2(]0,T[; L^2(\partial\Omega)^3)$, $L^2(]0,T[; L^2(\partial\Omega))$ respectively. By weak limits we deduce that $\|f\|_{L^\infty} \leq \|g\|_{L^\infty}$, $\|\gamma^+ f\|_{L^\infty} \leq \|g\|_{L^\infty}$ and :

$$ess \sup_{t\in\mathbb{R}_t} \left\{ \int_\Omega \int_{\mathbb{R}^3_p} (1+\mathcal{E}(p)) f(t,x,p)\ dxdp \right\} \leq C, \tag{4.4}$$

$$\|E\|_{L^\infty(\mathbb{R}_t;L^2(\Omega)^3)} + \|A\|_{L^\infty(\mathbb{R}_t;L^2(\Omega)^3)} \leq C, \tag{4.5}$$

$$M^+ + K^+ := \int_0^T\!\!\int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f\ dtd\sigma dp \leq C, \tag{4.6}$$

$$\|n\wedge E\|^2_{L^2(]0,T[\times\partial\Omega)^3} + \|(n\cdot E)\|^2_{L^2(]0,T[\times\partial\Omega)} + \|n\wedge A\|^2_{L^2(]0,T[\times\partial\Omega)^3} + \|(n\cdot A)\|^2_{L^2(]0,T[\times\partial\Omega)} \leq C, \tag{4.7}$$

where C depends on $m, \varepsilon_0, \Omega, T, M^-, K^-, H, H_1, \|g\|_{L^\infty}$. By using the velocity average lemma (see [12]) we can pass to the limit the non linear term of the Vlasov equation and we deduce that f is a T periodic weak solution for :

$$\partial_t f + v(p)\cdot\nabla_x f + qE(t,x)\cdot\nabla_p f = 0, \ \ (t,x,p) \in \mathbb{R}_t\times\Omega\times\mathbb{R}^3_p, \ \ f(t,x,p) = g(t,x,p), \ (t,x,p) \in \mathbb{R}_t\times\Sigma^-.$$

Take now a test function $\psi(t,x) = \eta(t) \cdot \Phi(x)$, where $\eta \in C^1(\mathbb{R}_t)$, T periodic and $\Phi \in C^1(\overline{\Omega})^3$. We have :

$$-\int_0^T\!\!\int_\Omega \eta'(t)\Phi(x)B_k(t,x)\;dtdx + \int_0^T\!\!\int_{\partial\Omega}(n\wedge E_k)\eta(t)\Phi(x)\;dtd\sigma + \int_0^T\!\!\int_\Omega \eta(t)\mathrm{rot}\;\Phi E_k(t,x)\;dtdx = 0, k \geq 1.$$

By passing to the limit for $k \to +\infty$ and by using (4.3) we deduce that :

$$\int_0^T \eta(t)\left(\int_{\partial\Omega}(n\wedge E)\cdot\Phi(x)\;d\sigma + \int_\Omega E(t,x)\cdot \mathrm{rot}\;\Phi\;dx\right)\;dt = 0,\;\;\forall \eta \in C^1(\mathbb{R}_t)\;T\text{ periodic},$$

and therefore we obtain that the field E verifies rot $E = 0$ and has tangential trace $n \wedge E \in L^2(]0,T[\times\partial\Omega)^3$. By using the equation div $E_k = \frac{\rho_k}{\varepsilon_0}$ with the test function $\eta(t)\varphi(x)$, $\varphi \in C^1(\overline{\Omega})$ we have :

$$\int_0^T\!\!\int_{\partial\Omega}\eta(t)(n\cdot E_k)\varphi(x)\;dtd\sigma - \int_0^T\!\!\int_\Omega \eta(t)E_k(t,x)\nabla_x\varphi\;dtdx = \frac{1}{\varepsilon_0}\int_0^T\!\!\int_\Omega \eta(t)\rho_k(t,x)\varphi(x)\;dtdx,$$

and after passing to the limit for $k \to +\infty$ one gets that the field $E(t)$ verifies div $E = \frac{\rho}{\varepsilon_0}$ and has normal trace $(n\cdot E) \in L^2(]0,T[\times\partial\Omega)$, where $\rho(t,x) = q\int_{\mathbb{R}^3_p} f(t,x,p)\;dp$. In order to identify the tangential trace of the electric field we use the divergence equations on the boundary (see *Proposition* 3.9). For all test function $\eta(t)\theta(x)$ where $\eta \in C^1(\mathbb{R}_t)$ periodic and $\theta \in C^1(\partial\Omega)$ we have :

$$-\int_0^T\!\!\int_{\partial\Omega}(n\cdot B_k)\eta'(t)\theta(x)\;dtd\sigma + \int_0^T\!\!\int_{\partial\Omega}\eta(t)(n\wedge E_k)\cdot\nabla_\tau\;\theta\;dtd\sigma = 0,\;\;k\geq 1.$$

After passing to the limit for $k \to +\infty$ and by using (4.3) we deduce that $\int_0^T\!\!\int_{\partial\Omega}(n\wedge E)\cdot\nabla_\tau\;(\eta\theta)\;dtd\sigma = 0$. By density we obtain that $\int_0^T\!\!\int_{\partial\Omega}(n\wedge E)\cdot\nabla_\tau\;\varphi\;dtd\sigma = 0$, $\varphi \in L^2(]0,T[;H^1(\partial\Omega))$. By taking into account that $\int_0^T\!\!\int_{\partial\Omega}(n\wedge\nabla_\tau\;h_2)\cdot\nabla_\tau\;\varphi\;dtd\sigma = 0$, $\forall\varphi \in L^2(]0,T[;H^1(\partial\Omega))$ we can write :

$$\int_0^T\!\!\int_{\partial\Omega}(n\wedge E - n\wedge\nabla_\tau\;h_2)\cdot\nabla_\tau\;\varphi\;dtd\sigma = 0,\;\;\forall\varphi\in L^2(]0,T[;H^1(\partial\Omega)). \tag{4.8}$$

We have also for all $k \geq 1$:

$$-\frac{1}{c_k}\int_0^T\!\!\int_{\partial\Omega}(n\cdot E_k)\eta'(t)\theta(x)\;dtd\sigma - \int_0^T\!\!\int_{\partial\Omega}\eta(t)c_k(n\wedge B_k)\cdot\nabla_\tau\;\theta\;dtd\sigma = -\frac{1}{\varepsilon_0 c_k}\int_0^T\!\!\int_{\partial\Omega}(n\cdot j_k)\eta(t)\theta(x)\;dtd\sigma. \tag{4.9}$$

Remark that :

$$\begin{aligned}\int_0^T\!\!\int_{\partial\Omega}\eta(t)c_k(n\wedge B_k)\cdot\nabla_\tau\;\theta\;dtd\sigma &= \int_0^T\!\!\int_{\partial\Omega}\eta(t)c_k(n\wedge(n\wedge B_k))\cdot(n\wedge\nabla_\tau\;\theta)\;dtd\sigma\\ &= \int_0^T\!\!\int_{\partial\Omega}\eta(t)(h(t,x) - n\wedge E_k(t,x))\cdot(n\wedge\nabla_\tau\;\theta))\;dtd\sigma.\end{aligned}$$

By passing to the limit for $k \to +\infty$ in (4.9) we deduce :

$$\int_0^T\!\!\int_{\partial\Omega}(h(t,x) - (n\wedge E(t,x)))\cdot(n\wedge\nabla_\tau\;(\eta\theta))\;dtd\sigma = 0. \tag{4.10}$$

By taking into account that $\int_0^T\!\!\int_{\partial\Omega}\nabla_\tau\;h_1\cdot(n\wedge\nabla_\tau\;(\eta\theta))\;dtd\sigma = 0$ we have also $\int_0^T\!\!\int_{\partial\Omega}(n\wedge E - n\wedge\nabla_\tau\;h_2)\cdot(n\wedge\nabla_\tau\;(\eta\theta))\;dtd\sigma = 0$ and by density one gets :

$$\int_0^T\!\!\int_{\partial\Omega}(n\wedge E - n\wedge\nabla_\tau\;h_2)\cdot(n\wedge\nabla_\tau\;\psi)\;dtd\sigma = 0,\;\;\forall\psi\in L^2(]0,T[;H^1(\partial\Omega)). \tag{4.11}$$

By using (4.8), (4.11) and the orthogonal decomposition of tangential fields of $L^2(]0,T[;L^2(\partial\Omega)^3)$ into irrotational part $\nabla_\tau\ \varphi$ and rotational part $n\wedge\nabla_\tau\ \psi$ we deduce that $n\wedge E = n\wedge\nabla_\tau\ h_2$. Note also that the field A (the weak limit of $(c_kB_k)_k$) verifies div $A=0$ and has normal trace $(n\cdot A)\in L^2(]0,T[\times\partial\Omega)$. By using the equation $\partial_t E_k - c_k^2 \text{rot } B_k = -\frac{j_k}{\varepsilon_0}$ we have :

$$-\frac{1}{c_k}\int_0^T\!\!\int_\Omega \eta'(t)\varphi(x)E_k(t,x)\ dtdx - \int_0^T\!\!\int_{\partial\Omega} c_k(n\wedge B_k)\eta(t)\varphi(x)\ dtd\sigma - \int_0^T\!\!\int_\Omega c_kB_k\eta(t)\text{rot }\varphi\ dtdx$$
$$= -\frac{1}{\varepsilon_0 c_k}\int_0^T\!\!\int_\Omega \eta(t)\varphi(x)j_k(t,x)\ dtdx,\quad k\geq 1. \tag{4.12}$$

After passing to the limit for $k\to+\infty$ we obtain that the field A verifies rot $A=0$ and has tangential trace $n\wedge A\in L^2(]0,T[\times\partial\Omega)^3$. In fact, by using the boundary condition $n\wedge E_k + c_k n\wedge(n\wedge B_k) = h$, $k\geq 1$ we deduce easily that $n\wedge A = -n\wedge\nabla_\tau\ h_1$.
□
In fact it is possible to show that the tangential traces converge strongly which is equivalent to :

$$\lim_{c\to+\infty}(n\wedge E_c, cn\wedge(n\wedge B_c)) = (n\wedge\nabla_\tau\ h_2, \nabla_\tau\ h_1),\ \text{strongly in }\ L^2(]0,T[\times\partial\Omega)^6.$$

This follows from the inequality (see (2.29)) :

$$\frac{\varepsilon_0 c}{2}\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E_c|^2 + c^2|n\wedge B_c|^2)\ dtd\sigma \leq K^- + \frac{\varepsilon_0 c}{2}\int_0^T\!\!\int_{\partial\Omega}|h|^2\ dtd\sigma,$$

and the following easy lemma :

Lemma 4.2. *Consider two sequences $(x_k)_k, (y_k)_k$ in a Hilbert space $(H,\langle\cdot,\cdot\rangle)$ verifying :*
(i) $\lim_{k\to+\infty} x_k = x,\ \lim_{k\to+\infty} y_k = y,$ *weakly in* H ;
(ii) there is $z\in H$ such that $x_k+y_k=z, \forall k$;
(iii) $\langle x,y\rangle = 0$;
(iv) $\limsup_{k\to+\infty}\{|x_k|^2+|y_k|^2\}\leq |z|^2$.
Then we have $\lim_{k\to+\infty} x_k = x,\ \lim_{k\to+\infty} y_k = y,$ *strongly in* H.

Remark also that in the case $h=0$ we have :

$$\frac{\varepsilon_0}{2}\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E_c|^2 + c^2|n\wedge B_c|^2)\ dtd\sigma \leq \frac{K^-}{c} = \mathcal{O}\left(\frac{1}{c}\right).$$

5. Other systems.

The previous analysis applies for other kinetic models. It is possible to treat systems with several species of charged particles. We can also replace the boundary condition of the Vlasov problem by the condition :

$$f(t,x,p) = g(t,x,p) + a(t,x,p)f(t,x,p-2(n(x)\cdot p)n(x)),\ \ (t,x,p)\in\mathbb{R}_t\times\Sigma^-, \tag{5.1}$$

where $0\leq a(t,x,p)\leq a_0<1$, $\forall(t,x,p)\in\mathbb{R}_t\times\Sigma^-$ and $0\leq g\in L^\infty(\mathbb{R}_t\times\Sigma^-)$ verifies :

$$M^- + K^- = \int_0^T\!\!\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\ dtd\sigma dp < +\infty.$$

In this case we obtain the estimates :

$$\int_0^T\!\!\int_{\Sigma^\pm}|(v_c(p)\cdot n(x))|\gamma^\pm f_c(t,x,p)\ dtd\sigma dp \leq \frac{1}{1-a_0}M^-, \tag{5.2}$$

$$(1-a_0)\int_0^T\!\!\int_{\Sigma^+}(v_c(p)\cdot n(x))\mathcal{E}_c(p)\gamma^+ f_c(t,x,p)\ dtd\sigma dp+\frac{\varepsilon_0 c}{2}\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E_c|^2+c^2|n\wedge B_c|^2)\ dtd\sigma \leq K^- + \frac{\varepsilon_0 c}{2}H, \tag{5.3}$$

and :

$$(1-a_0)\int_0^T\!\!\int_{\Sigma^-}|(v_c(p)\cdot n(x))|\mathcal{E}_c(p)\gamma^- f_c(t,x,p)\ dtd\sigma dp+a_0\frac{\varepsilon_0 c}{2}\int_0^T\!\!\int_{\partial\Omega}(|n\wedge E_c|^2+c^2|n\wedge B_c|^2)\ dtd\sigma \leq K^- + a_0\frac{\varepsilon_0 c}{2}H. \tag{5.4}$$

Notice that the inequality (5.3) still gives uniform estimates for the tangential traces of the electro-magnetic field and thus the other computations (estimates for the total energy, outgoing kinetic energy, normal traces of the electro-magnetic field) follow in similar way. Note also that the Vlasov-Maxwell system with initial-boundary conditions can be analyzed by using the same method.

6. Appendix.

For the sake of presentation we give in this section some details concerning the orthogonal decomposition of tangential fields of $L^2(\partial\Omega)^3$ (or $L^2(]0,T[\times\partial\Omega)^4$). The results are analogous to the well-known orthogonal decomposition result for fields of $L^2(\Omega)^3$ (see [13], p. 22). We assume that $\partial\Omega$ is bounded and smooth (generally C^1). We denote by $(x_{r_1},x_{r_2},x_{r_3})=(x_r',x_{r_3})$, with $1\leq r\leq M$, a system of local coordinates *i.e.*, there is $\alpha,\beta>0$ such that $\{(x_r',x_{r_3})\mid a_r(x_r')-\beta<x_{r_3}<a_r(x_r'),\ x_r'\in\Delta_r\}\subset\mathbb{R}^3_x-\overline{\Omega}$, $\{(x_r',x_{r_3})\mid x_{r_3}=a_r(x_r'),\ x_r'\in\Delta_r\}\subset\partial\Omega$, $\{(x_r',x_{r_3})\mid a_r(x_r')<x_{r_3}<a_r(x_r')+\beta,\ x_r'\in\Delta_r\}\subset\Omega$, where $\Delta_r=\{x_r'\mid |x_{r_1}|<\alpha,\ |x_{r_2}|<\alpha\}$ and $a_r\in C^1(\overline{\Delta}_r)$. The exterior unit normal is given locally by $n(x)=(\frac{\partial a_r}{\partial x_{r_1}},\frac{\partial a_r}{\partial x_{r_2}},-1)\cdot(1+|\nabla_{x_r'}a_r|^2)^{-\frac{1}{2}}$. If f belongs to $C^1(\partial\Omega)$ the tangential gradient of f is given locally by $\nabla_\tau\ f(x)=A(x)\cdot\nabla_{x_r'}f_r$, where $f_r(x_r')=f(x_r',a_r(x_r'))$, $x_r'\in\Delta_r$ and $A=(a_{ij})\in\mathcal{M}_{3,2}$, $a_{ij}=\delta_{ij}-n_i(x)n_j(x)$, $1\leq i\leq 3, 1\leq j\leq 2$ (the tangential gradient doesn't depend on the system of local coordinates). Notice that we have $n\cdot\nabla_\tau\ f=0$. We also define $\mathrm{rot}\ _\tau f=n\wedge\nabla_\tau\ f$ for $f\in C^1(\partial\Omega)$. Obviously we have $n\cdot\mathrm{rot}\ _\tau f=0$. A direct computation shows that ∇_τ and $\mathrm{rot}\ _\tau$ are orthogonal in $L^2(\partial\Omega)^3$:

$$\int_{\partial\Omega}\nabla_\tau\ f\cdot(n\wedge\nabla_\tau\ g)\ d\sigma=0,\ \ \forall\ f,g\in C^1(\partial\Omega).$$

Moreover, by density we have also :

$$\int_{\partial\Omega}\nabla_\tau\ \varphi\cdot(n\wedge\nabla_\tau\ \psi)\ d\sigma=0,\ \ \forall\ \varphi,\psi\in H^1(\partial\Omega). \tag{6.1}$$

For the definition of Sobolev spaces on $\partial\Omega$ the reader can refer to [24]. Consider now a tangential field $f\in C^1(\partial\Omega)^3$, $n\cdot f=0,\ x\in\partial\Omega$ and assume that $\partial\Omega\in C^2$. The divergence of f is given locally by $\mathrm{div}_\tau\ f=n_3\mathrm{div}\ _{x_r'}\left(\frac{f_r}{n_3}\right)$, $x_r'\in\Delta_r$. By direct computations we check that for $f\in C^1(\partial\Omega)^3$, $n\cdot f=0$, $\varphi\in C^1(\partial\Omega)$ we have :

$$\int_{\partial\Omega}\varphi\ \mathrm{div}_\tau\ f\ d\sigma+\int_{\partial\Omega}f\cdot\nabla_\tau\ \varphi\ d\sigma=0.$$

In particular we have $\int_{\partial\Omega}\mathrm{div}_\tau\ f\ d\sigma=0$, $\forall f\in C^1(\partial\Omega)^3$, $n\cdot f=0$. The above identities hold also for $f\in H^1(\partial\Omega)^3,\ n\cdot f=0$, $\varphi\in H^1(\partial\Omega)$. We can prove the *Poincaré* inequality :

Lemma 6.1. *Assume that $\partial\Omega$ is bounded, connected and regular (C^1). Then there is a constant $C_P(\Omega)>0$ such that :*

$$\int_{\partial\Omega}|\varphi(x)|^2\ d\sigma\leq C_P(\Omega)\left\{\left|\int_{\partial\Omega}\varphi(x)\ d\sigma\right|^2+\int_{\partial\Omega}|\nabla_\tau\ \varphi|^2\ d\sigma\right\},\ \ \forall\ \varphi\in H^1(\partial\Omega).$$

We use the notations : $\|u\|^2_{0,\partial\Omega} = \int_{\partial\Omega} |u(x)|^2 \; d\sigma$, $|u|^2_{1,\partial\Omega} = \int_{\partial\Omega} |\nabla_\tau \; u|^2 \; d\sigma$, $\|u\|^2_{1,\partial\Omega} = \|u\|^2_{0,\partial\Omega} + |u|^2_{1,\partial\Omega}$. As a consequence of the *Poincaré* inequality we obtain the classical result :

LEMMA 6.2. *Assume that $\partial\Omega$ is bounded, connected and regular (C^1). Denote by K the subspace of constant functions and consider the quotient space $H^1(\partial\Omega)/K$, endowed with the quotient norm $\|\hat{u}\|_{H^1(\partial\Omega)/K} = \inf_{u\in\hat{u}} \|u\|_{1,\partial\Omega}$. Then $|\cdot|_{1,\partial\Omega}$ is a norm on $H^1(\partial\Omega)/K$ equivalent to the quotient norm and we have :*

$$|u|_{1,\partial\Omega} \leq \|\hat{u}\|_{H^1(\partial\Omega)/K} \leq (1 + C_P(\Omega))^{\frac{1}{2}} \cdot |u|_{1,\partial\Omega}, \;\; \forall u \in H^1(\partial\Omega).$$

By direct computations we check that :

$$\int_S \mathrm{div}_\tau \; f \; d\sigma = \int_{\partial S} (n \wedge f) \; d\tau, \tag{6.2}$$

where $f \in C^1(\partial\Omega)^3$, $n \cdot f = 0$ and S is a region of $\partial\Omega$ such that ∂S is a smooth closed path (for details about integration of differential forms on manifolds and Stokes formulae the reader can refer to [9]). The following result is classical :

PROPOSITION 6.3. *Assume that $\partial\Omega$ is bounded, simply connected and regular (C^1) and consider $f \in L^2(\partial\Omega)^3$, $n \cdot f = 0$. Then the following statements are equivalent :*
(i) $div_\tau \; f = 0$ in $\mathcal{D}'(\partial\Omega)$ (i.e., $\int_{\partial\Omega} f \cdot \nabla_\tau \; \varphi \, d\sigma = 0$, $\forall \varphi \in H^1(\partial\Omega)$) ;
(ii) $\exists \psi \in H^1(\partial\Omega)$ such that $f = n \wedge \nabla_\tau \; \psi$.

Proof. The implication $(ii) \to (i)$ follows by formula (6.1). For the implication $(i) \to (ii)$ consider first smooth fields f and use the formula (6.2). The general case follows by density.
□
Similarly we have :

PROPOSITION 6.4. *Assume that $\partial\Omega$ is bounded, simply connected and regular (C^1) and consider $f \in L^2(\partial\Omega)^3$, $n \cdot f = 0$. Then the following statements are equivalent :*
(i) div $_\tau(n \wedge f) = 0$ in $\mathcal{D}'(\partial\Omega)$ (i.e., $\int_{\partial\Omega} (n \wedge f) \cdot \nabla_\tau \; \psi \, d\sigma = 0$, $\forall \psi \in H^1(\partial\Omega)$) ;
(ii) $\exists \varphi \in H^1(\partial\Omega)$ such that $f = \nabla_\tau \; \varphi$.

We introduce the notations : $X = \{f \in L^2(\partial\Omega)^3 \mid n \cdot f(x) = 0 \text{ a.e. } x \in \partial\Omega\}$, $Y = \{\nabla_\tau \; \varphi \mid \varphi \in H^1(\partial\Omega)\}$, $Z = \{n \wedge \nabla_\tau \; \psi \mid \psi \in H^1(\partial\Omega)\}$.

PROPOSITION 6.5. *Assume that $\partial\Omega$ is bounded, simply connected and regular (C^1). Then Y and Z are closed orthogonal subspaces of X and we have the decomposition :*

$$X = Y + Z. \tag{6.3}$$

Proof. By using formula (6.1) we deduce that $Y \perp Z$. By the *Lemma* 6.2 we check easily that Y, Z are closed subspaces of X. Let us prove now that Y is dense in $Z^\perp$: take $f \in Z^\perp$ such that $f \perp Y$. By *Proposition* 6.4 the condition $f \in Z^\perp$ implies that $f = \nabla_\tau \; \varphi$, $\varphi \in H^1(\partial\Omega)$. Since $f \perp Y$ we deduce that $\int_{\partial\Omega} |f|^2 \; d\sigma = \int_{\partial\Omega} f \cdot \nabla_\tau \; \varphi \; d\sigma = 0$, or $f = 0$. Therefore we have :

$$Y + Z = \overline{Y} + \overline{Z} = Z^\perp + (Z^\perp)^\perp = X.$$

□

The previous proposition has the following direct consequences :

COROLLARY 6.6. *Consider $f \in X$. Under the hypotheses of Proposition 6.5 the following statements are equivalent ;*
(i) $div_\tau\ f = 0,\ div\ _\tau(n \wedge f) = 0$ in $\mathcal{D}'(\partial\Omega)$;
(ii) $f = 0$.

We denote by $H^1_\tau(\partial\Omega) \subset H^1(\partial\Omega)^3$ the closed subspace of tangential fields $f \in H^1(\partial\Omega)^3$, $n \cdot f = 0$ and let $L^2_0(\partial\Omega) = \{u \in L^2(\partial\Omega) \mid \int_{\partial\Omega} u(x)\ d\sigma = 0\}$. By using the orthogonal decomposition (6.3) we can deduce the following representation of $(H^1_\tau(\partial\Omega))'$:

PROPOSITION 6.7. *Assume that $\partial\Omega$ is bounded, simply connected and regular (C^2). Then we have :*

$$(H^1_\tau(\partial\Omega))' = \{\nabla_\tau\ \varphi + n \wedge \nabla_\tau\ \psi \mid \varphi, \psi \in L^2_0(\partial\Omega)\}.$$

Moreover, for all $l \in (H^1_\tau(\partial\Omega))'$, the representation $l = \nabla_\tau\ \varphi + n \wedge \nabla_\tau\ \psi$, with $\varphi, \psi \in L^2_0(\partial\Omega)$ is unique and there is a constant $C(\Omega) > 0$ such that $\|\varphi\|_{0,\partial\Omega} + \|\psi\|_{0,\partial\Omega} \le C(\Omega) \cdot \|l\|_{-1,\partial\Omega}$.

Proof. Denote by W the set $W = \{\nabla_\tau\ \varphi + n \wedge \nabla_\tau\ \psi \mid \varphi, \psi \in L^2_0(\partial\Omega)\} \subset (H^1_\tau(\partial\Omega))'$. We will prove that W is closed and dense in $(H^1_\tau(\partial\Omega))'$. Consider $F : (H^1_\tau(\partial\Omega))' \to \mathbb{R}$ a linear continuous form on $(H^1_\tau(\partial\Omega))'$, vanishing on W. There is $u \in H^1_\tau(\partial\Omega)$ such that $F(l) = l(u)$, $\forall l \in (H^1_\tau(\partial\Omega))'$ and therefore we have :

$$-\int_{\partial\Omega} \varphi \mathrm{div}_\tau\ u\ d\sigma + \int_{\partial\Omega} \psi \mathrm{div}_\tau\ (n \wedge u)\ d\sigma = 0,\ \ \forall \varphi, \psi \in L^2_0(\partial\Omega),$$

which implies that $\mathrm{div}_\tau\ u = \mathrm{div}_\tau\ (n \wedge u) = 0$ (for this observe that $\mathrm{div}_\tau\ u, \mathrm{div}_\tau\ (n \wedge u) \in L^2_0(\partial\Omega)$ and thus is possible to take $(\varphi, \psi) = (\mathrm{div}_\tau\ u, 0)$ and $(\varphi, \psi) = (0, \mathrm{div}_\tau\ (n \wedge u))$). By *Corollary* 6.6 we deduce that $u = 0$ and thus $\overline{W} = (H^1_\tau(\partial\Omega))'$. In order to show that W is closed we will prove that for all $\varphi, \psi \in L^2_0(\partial\Omega)$ we have $\|\varphi\|_{0,\partial\Omega} + \|\psi\|_{0,\partial\Omega} \le C(\Omega) \cdot \|\nabla_\tau\ \varphi + n \wedge \nabla_\tau\ \psi\|_{-1,\partial\Omega}$ for some constant $C(\Omega)$. Denote by l the form $\nabla_\tau\ \varphi + n \wedge \nabla_\tau\ \psi$:

$$l(v) = -\int_{\partial\Omega} \varphi \mathrm{div}_\tau\ v\ d\sigma + \int_{\partial\Omega} \psi \mathrm{div}_\tau\ (n \wedge v)\ d\sigma,\ \ \forall v \in H^1_\tau(\partial\Omega). \tag{6.4}$$

Take $\hat{\theta} \in H^1(\partial\Omega)/K$ the unique solution for the variational problem :

$$\int_{\partial\Omega} \nabla_\tau\ \theta \cdot \nabla_\tau\ \chi\ d\sigma = \int_{\partial\Omega} \varphi \chi\ d\sigma,\ \ \forall \hat{\chi} \in H^1(\partial\Omega)/K.$$

Note that the application $\hat{\chi} \to \int_{\partial\Omega} \varphi\chi\ d\sigma$ is well defined since $\varphi \in L^2_0(\partial\Omega)$. We have $\|\nabla_\tau\ \theta\|_{0,\partial\Omega} \le C \cdot \|\varphi\|_{0,\partial\Omega}$. Moreover by elliptic regularity results we have $u = \nabla_\tau\ \theta \in H^1_\tau(\partial\Omega)$ with $\|u\|_{1,\partial\Omega} \le C \cdot \|\varphi\|_{0,\partial\Omega}$. By taking $v = u$ in (6.4) we obtain that :

$$\int_{\partial\Omega} |\varphi|^2\ d\sigma = -\int_{\partial\Omega} \varphi \mathrm{div}_\tau\ u\ d\sigma = l(u) \le \|l\|_{-1,\partial\Omega} \cdot \|u\|_{1,\partial\Omega} \le C \cdot \|l\|_{-1,\partial\Omega} \cdot \|\varphi\|_{0,\partial\Omega},$$

which implies that $\|\varphi\|_{0,\partial\Omega} \le C \cdot \|l\|_{-1,\partial\Omega}$. The analogous estimate for ψ follows in the same manner by observing that :

$$l(n \wedge v) = -\int_{\partial\Omega} \varphi \mathrm{div}_\tau\ (n \wedge v)\ d\sigma - \int_{\partial\Omega} \psi \mathrm{div}_\tau\ v\ d\sigma,\ \ \forall v \in H^1_\tau(\partial\Omega).$$

□

As a consequence of *Proposition* 6.7 we obtain :

PROPOSITION 6.8. *Assume that $\partial\Omega$ is bounded, simply connected and regular (C^2). Then we have :*

$$H^1_\tau(\partial\Omega) = \{f \in X \mid div_\tau\ f \in L^2(\partial\Omega),\ div_\tau\ (n \wedge f) \in L^2(\partial\Omega)\}.$$

Moreover there is a constant $C(\Omega) > 0$ such that :

$$\|f\|^2_{1,\partial\Omega} \leq C(\Omega) \cdot \{\|div_\tau\ f\|^2_{0,\partial\Omega} + \|div_\tau\ (n \wedge f)\|^2_{0,\partial\Omega}\},\ \ \forall f \in H^1_\tau(\partial\Omega). \tag{6.5}$$

Proof. Observe that for all $l \in (H^1_\tau(\partial\Omega))'$ and $f \in H^1(\partial\Omega)$ we have :

$$\begin{aligned} l(f) =& \langle \nabla_\tau\ \varphi + n \wedge \nabla_\tau\ \psi, f\rangle = -\int_{\partial\Omega} \varphi \mathrm{div}_\tau\ f\ d\sigma + \int_{\partial\Omega} \psi \mathrm{div}_\tau\ (n \wedge f)\ d\sigma \\ \leq& (\|\varphi\|^2_{0,\partial\Omega} + \|\psi\|^2_{0,\partial\Omega})^{\frac{1}{2}} \cdot (\|\mathrm{div}_\tau\ f\|^2_{0,\partial\Omega} + \|\mathrm{div}_\tau\ (n \wedge f)\|^2_{0,\partial\Omega})^{\frac{1}{2}} \\ \leq& C(\Omega)\|l\|_{-1,\partial\Omega} \cdot (\|\mathrm{div}_\tau\ f\|^2_{0,\partial\Omega} + \|\mathrm{div}_\tau\ (n \wedge f)\|^2_{0,\partial\Omega})^{\frac{1}{2}}, \end{aligned}$$

which implies that $\|f\|_{1,\partial\Omega} \leq C(\Omega) \cdot (\|\mathrm{div}_\tau\ f\|^2_{0,\partial\Omega} + \|\mathrm{div}_\tau\ (n \wedge f)\|^2_{0,\partial\Omega})^{\frac{1}{2}}$. The conclusion follows easily by regularization.
□

We introduce also the differential operators :

$$\mathrm{rot}_\tau\ A = -n\ \mathrm{div}_\tau\ (n \wedge A) - n \wedge \nabla_\tau\ (n \cdot A),\ \ \forall A \in C^1(\partial\Omega)^3,$$

$$\nabla_{(t,\tau)}\ f = (\partial_t f, \nabla_\tau\ f),\ \ \forall f \in C^1(\mathbb{R}_t \times \partial\Omega),$$

$$\mathrm{rot}_{(t,\tau)}\ A = (n \cdot \mathrm{rot}_\tau\ \vec{A}, n \wedge (\partial_t \vec{A} - \nabla_\tau\ A_0)),\ \ \forall A = (A_0, \vec{A}) \in C^1(\mathbb{R}_t \times \partial\Omega)^4,$$

$$\mathrm{div}_{(t,\tau)}\ A = \partial_t A_0 + \mathrm{div}_\tau\ \vec{A},\ \ \forall A = (A_0, \vec{A}) \in C^1(\mathbb{R}_t \times \partial\Omega)^4.$$

Note that for tangential fields $A \in C^1(\partial\Omega)^3$ we have $\mathrm{rot}_\tau\ A = -n\ \mathrm{div}_\tau\ (n \wedge A)$. The following identities follow by direct computations :

$$\int_{\partial\Omega} \mathrm{rot}_\tau\ A \cdot \nabla_\tau\ \varphi\ d\sigma = 0,\ \ \forall A \in C^1(\partial\Omega)^3,\ \ \forall \varphi \in C^1(\partial\Omega),$$

$$\int_0^T\!\!\int_{\partial\Omega} \mathrm{rot}_{(t,\tau)}\ A \cdot \nabla_{(t,\tau)}\ \varphi\ dtd\sigma = 0,\ \ \forall A \in C^1(\mathbb{R}_t \times \partial\Omega)^4,\ \ \forall \varphi \in C^1(\mathbb{R}_t \times \partial\Omega),\ T \text{ periodic},$$

$$\int_{\partial\Omega} \mathrm{rot}_\tau\ A \cdot B\ d\sigma - \int_{\partial\Omega} A \cdot \mathrm{rot}_\tau\ B\ d\sigma = 0,\ \ \forall A, B \in C^1(\partial\Omega)^3,$$

$$\int_0^T\!\!\int_{\partial\Omega} \mathrm{rot}_{(t,\tau)}\ A \cdot B\ dtd\sigma - \int_0^T\!\!\int_{\partial\Omega} A \cdot \mathrm{rot}_{(t,\tau)}\ B\ dtd\sigma = 0,\ \ \forall A, B \in C^1(\mathbb{R}_t \times \partial\Omega)^4,\ T \text{ periodic}.$$

Obviously, the above identities hold for functions/fields in the corresponding time periodic Sobolev spaces H^1. We introduce also the notations :

$$X_T = \{f = (f_0, \vec{f}) \in L^2(]0, T[\times\partial\Omega)^4 \mid n \cdot \vec{f} = 0,\ \int_0^T\!\!\int_{\partial\Omega} f_0(t,x)\ dtd\sigma = 0\},$$

$$Y_T = \{\nabla_{(t,\tau)}\ \varphi \mid \varphi \in H^1([0,T] \times \partial\Omega),\ \int_0^T\!\!\int_{\partial\Omega} \varphi(t,x)\ dtd\sigma = 0\},$$

$$Z_T = \{\mathrm{rot}_{(t,\tau)}\ \psi \mid \psi = (\psi_0, \vec{\psi}) \in H^1([0,T] \times \partial\Omega)^4\ , \mathrm{div}_{(t,\tau)}\ \psi = 0,\ n \cdot \vec{\psi} = 0,\ \int_0^T \int_{\partial\Omega} \psi_0\ dt d\sigma = 0\}.$$

As for 3-component tangential fields we have the analogous result for 4-component tangential fields :

Proposition 6.9. *Assume that $\partial\Omega$ is bounded, simply connected and regular (C^2) and consider $f \in X_T$ such that $div_{(t,\tau)}\ f \in L^2(]0,T[\times\partial\Omega)$ and $rot_{(t,\tau)}\ f \in L^2(]0,T[\times\partial\Omega)^4$ (in $\mathcal{D}'([0,T] \times \partial\Omega)$ periodic). Then f belongs to $H^1([0,T] \times \partial\Omega)^4$ and there is a constant $C(\Omega) > 0$ (depending on Ω but not on T) such that :*

$$|f_0|^2_{1,[0,T]\times\partial\Omega} + \|\vec{f}\|^2_{1,[0,T]\times\partial\Omega} \le C(\Omega) \cdot \{\|div_{(t,\tau)}\ f\|^2_{0,[0,T]\times\partial\Omega} + \|rot_{(t,\tau)}\ f\|^2_{0,[0,T]\times\partial\Omega}\}. \tag{6.6}$$

Proof. It is sufficient to prove the inequality (6.6) for T periodic smooth fields. We have :

$$\begin{aligned}\|\mathrm{div}_{(t,\tau)}\ f\|^2_{0,[0,T]\times\partial\Omega} + \|\mathrm{rot}_{(t,\tau)}\ f\|^2_{0,[0,T]\times\partial\Omega} &= \|\partial_t f_0 + \mathrm{div}_\tau\ \vec{f}\|_0^2 + \|\mathrm{div}_\tau\ (n \wedge \vec{f})\|_0^2 + \|\partial_t \vec{f} - \nabla_\tau\ f_0\|_0^2 \\ &= \|\partial_t f_0\|_0^2 + \|\nabla_\tau\ f_0\|_0^2 + \|\partial_t \vec{f}\|_0^2 + \|\mathrm{div}_\tau\ \vec{f}\|_0^2 + \|\mathrm{div}_\tau\ (n \wedge \vec{f})\|_0^2.\end{aligned}$$

By using (6.5) we deduce that :

$$\|\mathrm{div}_\tau\ \vec{f}\|_0^2 + \|\mathrm{div}_\tau\ (n \wedge \vec{f})\|_0^2 \ge \frac{1}{C(\Omega)} \left\{\|\vec{f}\|_0^2 + \|\nabla_\tau\ \vec{f}\|_0^2\right\}. \tag{6.7}$$

Finally one gets that :

$$\|\mathrm{div}_{(t,\tau)}\ f\|^2_{0,[0,T]\times\partial\Omega} + \|\mathrm{rot}_{(t,\tau)}\ f\|^2_{0,[0,T]\times\partial\Omega} \ge \min\left(1, \frac{1}{C(\Omega)}\right) \cdot (|f_0|_1^2 + \|\vec{f}\|_1^2).$$

□

Now we can prove the orthogonal decomposition result for 4-component fields of $L^2(]0,T[\times\partial\Omega)^4$.

Proposition 6.10. *Assume that $\partial\Omega$ is bounded, simply connected and regular. Then Y_T and Z_T are closed orthogonal subspaces of X_T and we have the decomposition $X_T = Y_T + Z_T$.*

Proof. By using the *Poincaré* inequality we check easily that Y_T is closed. By *Proposition* 6.9 combined with the *Poincaré* inequality we deduce also that Z_T is closed. An easy computation shows that $Y_T \perp Z_T$. We will prove that Y_T is dense in $Z_T^\perp$ which implies that :

$$X_T = Y_T + Y_T^\perp = Y_T + (Z_T^\perp)^\perp = Y_T + \overline{Z}_T = Y_T + Z_T.$$

Indeed, consider $f \in X_T$ such that $f \perp Y_T$, $f \perp Z_T$. We deduce that $\int_0^T\int_{\partial\Omega} f \cdot \nabla_{(t,\tau)}\ \varphi\ dt d\sigma = 0$, $\forall \varphi \in H^1([0,T] \times \partial\Omega)^4$, or $\mathrm{div}_{(t,\tau)}\ f = 0$ in $\mathcal{D}'([0,T] \times \partial\Omega)$. Consider now $\psi = (\psi_0, \vec{\psi}) \in H^1([0,T] \times \partial\Omega)^4$ with $n \cdot \vec{\psi} = 0$. Take $\varphi \in H^2([0,T] \times \partial\Omega)^4$ such that $-\mathrm{div}_{(t,\tau)}\ \nabla_{(t,\tau)}\ \varphi = \mathrm{div}_{(t,\tau)}\ \psi$ (such a solution exists since $\int_0^T\int_{\partial\Omega} \mathrm{div}_{(t,\tau)}\ \psi\ dt d\sigma = 0$). Consider now the field $\Psi = \psi - (\langle\psi_0\rangle, 0) + \nabla_{(t,\tau)}\ \varphi \in H^1([0,T] \times \partial\Omega)^4$, where $\langle\psi_0\rangle = \frac{\int_0^T\int_{\partial\Omega} \psi_0\ dt d\sigma}{\int_0^T\int_{\partial\Omega} 1\ dt d\sigma}$. By construction we have $\mathrm{div}_{(t,\tau)}\ \Psi = 0$, $n \cdot \vec{\Psi} = 0$ and $\int_0^T\int_{\partial\Omega} \Psi_0\ dt d\sigma = 0$, or $\mathrm{rot}_{(t,\tau)}\ \Psi \in Z_T$. We deduce that $\int_0^T\int_{\partial\Omega} f \cdot \mathrm{rot}_{(t,\tau)}\ \psi\ dt d\sigma = \int_0^T\int_{\partial\Omega} f \cdot \mathrm{rot}_{(t,\tau)}\ \Psi\ dt d\sigma = 0$ or $\mathrm{rot}_{(t,\tau)}\ f = 0$ in $\mathcal{D}'([0,T] \times \partial\Omega)$. By *Proposition* 6.9 and the the condition $\int_0^T\int_{\partial\Omega} f_0(t,x)\ dt d\sigma = 0$ we deduce that $f = 0$. □

Consider now a decomposition $f = \nabla_{(t,\tau)}\ \varphi + \mathrm{rot}_{(t,\tau)}\ \psi$ as in *Proposition* 6.10. We deduce that :

$$\|f\|^2_{0,[0,T]\times\partial\Omega} = \|\nabla_{(t,\tau)}\ \varphi\|^2_{0,[0,T]\times\partial\Omega} + \|\mathrm{rot}_{(t,\tau)}\ \psi\|^2_{0,[0,T]\times\partial\Omega}.$$

By using *Proposition* 6.9 and the condition $\mathrm{div}_{(t,\tau)}\ \psi = 0$ we obtain that :

$$|\varphi|^2_{1,[0,T]\times\partial\Omega} + |\psi_0|^2_{1,[0,T]\times\partial\Omega} + \|\vec{\psi}\|^2_{1,[0,T]\times\partial\Omega} \leq C(\Omega)\cdot\|f\|^2_{0,[0,T]\times\partial\Omega}.$$

By using also the conditions $\int_0^T\!\!\int_{\partial\Omega}\varphi(t,x)\ dtd\sigma = 0$, $\int_0^T\!\!\int_{\partial\Omega}\psi_0(t,x)\ dtd\sigma = 0$ and the *Poincaré* inequality we obtain :

$$\|\varphi\|^2_{1,[0,T]\times\partial\Omega} + \|\psi_0\|^2_{1,[0,T]\times\partial\Omega} + \|\vec{\psi}\|^2_{1,[0,T]\times\partial\Omega} \leq C(T,\Omega)\cdot\|f\|^2_{0,[0,T]\times\partial\Omega}.$$

The previous proposition has the following direct consequences :

COROLLARY 6.11. *Consider $f \in X_T$. Under the hypotheses of Proposition 6.10 we have ;*
(i) $div_{(t,\tau)}\ f = 0$, in $\mathcal{D}'([0,T]\times\partial\Omega)$ (i.e., $\int_0^T\!\!\int_{\partial\Omega} f\cdot\nabla_{(t,\tau)}\ \varphi\ dtd\sigma = 0$, $\forall\varphi\in H^1([0,T]\times\partial\Omega)$) iff $f\in Z_T$;
(ii) $rot_{(t,\tau)}\ f = 0$, in $\mathcal{D}'([0,T]\times\partial\Omega)$ (i.e., $\int_0^T\!\!\int_{\partial\Omega} f\cdot rot_{(t,\tau)}\ \psi\ dtd\sigma = 0$, $\forall\psi\in H^1([0,T]\times\partial\Omega)^4$) iff $f\in Y_T$;
(iii) $div_{(t,\tau)}\ f = 0$ and $rot_{(t,\tau)}\ f = 0$ in $\mathcal{D}'([0,T]\times\partial\Omega)$ iff $f = 0$.

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