Translations on a torus: Minimal complexity

Nicolas Bédaride, Jean françois Bertazzon

Translation on a torus.

$$\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$$

$$\begin{array}{ccc} \mathbb{T}^k & \to & \mathbb{T}^k \\ \mathbf{x} & \mapsto & \mathbf{x}+\mathbf{a} \end{array}$$

Minimal translation: every point has a dense orbit in $\mathbb{T}^k$.

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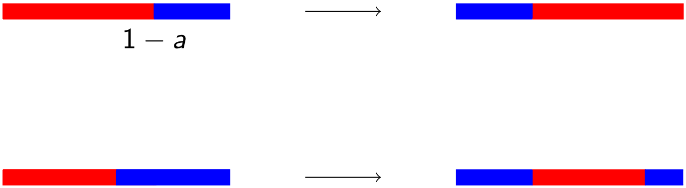
Fundamental domain

Let $\mathcal{D}$ be a subset of $\mathbb{R}^k$ of finite volume which tiles the space by action of $\mathbb{Z}^k$.

$$\begin{array}{ccc} \mathbb{T}^k & \rightarrow & \mathbb{T}^k \\ | & & | \\ \mathcal{D} & \rightarrow & \mathcal{D} \end{array}$$

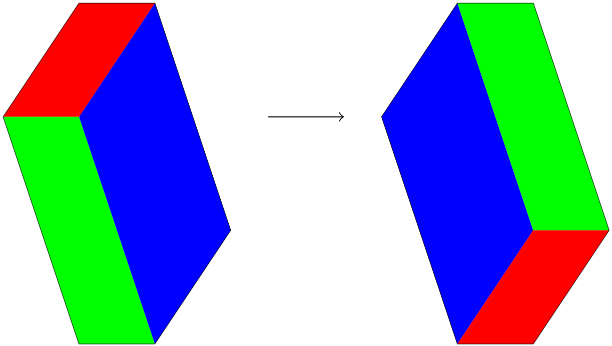
The translation becomes a piecewise translation defined on $\mathcal{D}$.

Examples in dimension one



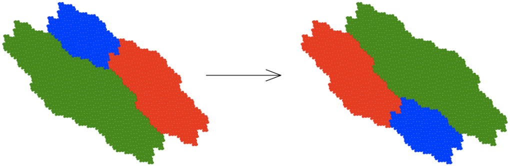
Same value for a, different partitions.

Example in dimension two



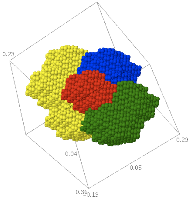
Picture for every value of **a**. The set $\mathcal{D}$ is an hexagon with parallel sides.

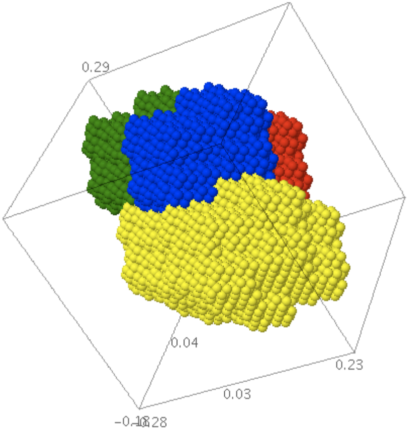
Example in dimension two



Particular value of **a**.

Dimension three





Piecewise translation associated to a translation

Consider $(T, \mathcal{D}_1, \ldots, \mathcal{D}_m)$ as a map defined on $\mathcal{D} = \bigcup_{i=1}^{m} \mathcal{D}_i$ such that for each $\mathbf{x} \in \mathcal{D}$: $T(\mathbf{x}) = \mathbf{x} + \mathbf{a} + \mathbf{n}(\mathbf{x})$ where:

- $\mathbf{n} : \mathcal{D} \mapsto \mathbb{Z}^k$ is a measurable map,
- $\mathcal{D} = \bigcup_{i=1}^{m} \mathcal{D}_i$ is a fundamental domain of the torus,
- for each integer $i \in \{1, \ldots, m\}$, there exists a vector $\mathbf{r}_i \in \mathbb{Q}^k$ such that:

$$\int_{\mathcal{D}_i} \mathbf{n}(\mathbf{x})\, \mathrm{d}\lambda(\mathbf{x}) = \lambda(\mathcal{D}_i)\mathbf{r}_i.$$

- The domain of $\mathcal{D}$ is not assumed to be bounded.
- The dynamical symbolic system is not conjugate to the translation on the torus
- We allow multiple vectors of translation in each subset of the fundamental domain.
- The map $\mathbf{n}$ can take an infinity of values.

Words

- Piecewise translations
- Alphabet
- Orbit of a point
- Word (infinite)

Question

Complexity of one orbit ?

Fundamental domain $\mathcal{D}$ with the lowest complexity ?

Theorem

Let $k \geq 1$ and $m \geq 1$ be two integers, let $\mathbf{a}$ be a vector in $\mathbb{R}^k$ such that the translation by $\mathbf{a}$ on the torus $\mathbb{T}^k$ is minimal. Let $(T, \mathcal{D}_1, \ldots, \mathcal{D}_m)$ be a piecewise translation associated to this translation. Then the complexity function of the piecewise translation fulfills

$$\forall n \geq 1, \quad p_k(n) \geq kn + 1.$$

Remark

Same result in dimension two by Bertazzon.

Substitutions

Morphism of free monoid:

$$\sigma_k = \sigma : \begin{cases} a_1 \mapsto a_1 a_2 \\ a_2 \mapsto a_1 a_3 \\ \vdots \\ a_k \mapsto a_1. \end{cases}$$

k-bonacci substitution

Theorem (Messaoudi)

The fixed point of k-bonacci substitution is an infinite word of complexity

$$p(n) = (k-1)n + 1.$$

The associated subshift is conjugated to a translation on the torus $\mathbb{T}^{k-1}$.

Remark

The bound is sharp in Theorem 1. The vector $\mathbf{a}$ *is an eigenvector of the matrix of the substitution* σ_k.

Theorem (Chevallier)

For a minimal translation on the torus $\mathbb{T}^2$, if $\mathcal{D}$ is a polygon, then there exists two constants a, b such that

$$an^2 \leq p_2(n) \leq bn^2$$

Examples

- In this case for every sturmian word $p_1(n) = n + 1$.
- In this case $p_2(n) = n^2 + n + 1$.
- In this case for the Tribonacci translation $p_2(n) = 2n + 1$.

Steps

Proposition

Let $k \geq 1$ and $m \geq 1$ be two integers, $\mathbf{a}$ a vector in $\mathbb{R}^k$ such that the translation by $\mathbf{a}$ on the torus $\mathbb{T}^k$ is minimal. Let $(T, \mathcal{D}_1, \ldots, \mathcal{D}_m)$ be a piecewise translation associated to this translation. Then we have:

$$m \geq k + 1.$$

Proposition

Let $k \geq 1$ and $m \geq 1$ be two integers, $\mathbf{a}$ a vector of $\mathbb{R}^k$ such that the translation by $\mathbf{a}$ on the torus $\mathbb{T}^k$ is minimal. Let $(T, \mathcal{D}_1, \ldots, \mathcal{D}_m)$ be a piecewise translation associated to this translation. Then the complexity function fulfills for every integer n:

$$p_k(n+1) - p_k(n) \geq k.$$

First proposition

$$T^N(\mathbf{x}) = \mathbf{x} + N\mathbf{a} + \sum_{k=0}^{N-1} \mathbf{n}\left(T^k\mathbf{x}\right)$$

$$= \mathbf{x} + N\mathbf{a} + \sum_{i=1}^{m}\sum_{k=0}^{N-1} \mathbf{n}\left(T^k\mathbf{x}\right) 1_{\mathcal{D}_i}\left(T^k\mathbf{x}\right).$$

$$\frac{T^{N_p}(\mathbf{x})}{N_p} = \frac{\mathbf{x}}{N_p} + \mathbf{a} + \sum_{i=1}^{m}\frac{1}{N_p}\sum_{k=0}^{N_p-1} \mathbf{n}\left(T^k\mathbf{x}\right) 1_{\mathcal{D}_i}\left(T^k\mathbf{x}\right).$$

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Since $\mathbf{x}$ is a recurrent point for T, there exists an integer sequence $(N_p)_{p\in\mathbb{N}}$ such that $T^{N_p}(\mathbf{x})/N_p$ converges to zero.

$$\frac{T^{N_p}(\mathbf{x})}{N_p} = \frac{\mathbf{x}}{N_p} + \mathbf{a} + \sum_{i=1}^{m} \frac{1}{N_p} \sum_{k=0}^{N_p-1} \mathbf{n}\left(T^k\mathbf{x}\right) 1_{\mathcal{D}_i}\left(T^k\mathbf{x}\right).$$

Remember

We deduce

$$0 = \mathbf{a} + A_1\mathbf{r}_1 + \cdots + A_m\mathbf{r}_m.$$

Contradiction with the minimality.

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Lemma

Let $\alpha_1, \ldots \alpha_m$ be m real numbers and let $(\mathbf{n}_i)_{1\leq i\leq m}$ be m vectors of $\mathbb{Q}^k$ such that

$$\mathbf{a} = \alpha_1\mathbf{n}_1 + \cdots \alpha_m\mathbf{n}_m.$$

Assume $m < k$, then there exists k rational numbers $q_1, \ldots, q_k$, non all equal to zero, such that

$$a_1q_1 + \cdots a_kq_k = 0.$$

Second proposition

- Same method.
- Definition of the Rauzy graphs.
- Euler characteristic of the graph.