Brauer trees of finite reductive groups via Deligne-Lusztig theory

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Local methods in representation theory (1)

Representation theory of a finite group H over a ring Λ.

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Two important cases:

- $\Lambda = K$ field of characteristic zero $\rightsquigarrow$ characters
- $\Lambda = k$ field of prime characteristic $\ell \rightsquigarrow$ modular representation theory
  - simple modules
  - projective modules
  - blocks
  - decomposition matrix...

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Brauer: "understand" the representation theory of H from local subgroups (ℓ-subgroups and their normalizers).

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Assume that S is an abelian Sylow ℓ-subgroup of H

$$\Lambda H\text{-mod} \qquad\qquad \Lambda N_H(S)\text{-mod}$$

where Λ is a finite extension of $\mathbb{Z}_\ell$.

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Broué's conjecture

Assume that S is an abelian Sylow ℓ-subgroup of H

$$D^b(b\Lambda H\text{-mod}) \simeq D^b(b'\Lambda N_H(S)\text{-mod})$$

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Example

$H = \mathfrak{A}_5 = \mathrm{SL}_2(4)$ and $\ell = 5$

$N_H(S) = D_5$

	1	$(12)(34)$	(123)	(12345)	(12354)
1	1	1	1	1	1
χ_3	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3'	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	0	1	0	0

	1	s	r	r^2
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γ_2	2	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
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Geometric setting

Main problem: how to induce the equivalence?

If $H = \mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$ with

- $\mathbf{G}$ a connected reductive algebraic group
- $F : \mathbf{G} \longrightarrow \mathbf{G}$ a Frobenius endomorphism

Then a Sylow ℓ-subgroup is contained in a torus $\mathbf{T}_w^F$ for some $w \in W$

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$$\mathrm{R}\Gamma_c(\mathbf{Y}, \Lambda)$$

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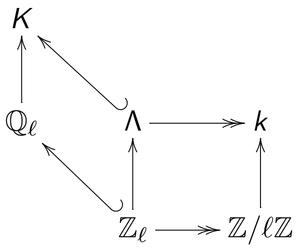

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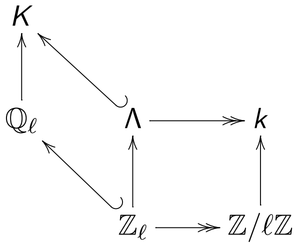

Decomposition into blocks

$$\Lambda H = \bigoplus b_i \Lambda H$$

Principal block: block containing the trivial representation

Geometric framework (1)

$H = \mathrm{SL}_n(q)$ **special linear group** over the finite field $\mathbb{F}_q$

$$|H| = q^{n(n-1)/2}(q^n - 1)(q^{n-1} - 1)\cdots(q^2 - 1)$$

if ℓ does not divide $|H| \rightsquigarrow$ ordinary representations
if $\ell = p \rightsquigarrow$ specific methods
Otherwise ℓ divides $q^d - 1$ for some d

Coxeter case ($d = n$ only)

We assume that q has order n modulo ℓ. The class of q in k is a primitive n-th root of 1.

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$$\mathbf{Y} = \{(x_1, \ldots, x_n) \in \mathbb{A}_n \text{ s.t. } \det\left(\{{x_i}^{q^{j-1}}\}_{1\le i,j\le n}\right) = 1\}$$

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Linear representations

Vector spaces $\mathrm{H}^i_c(\mathbf{Y}, K)$ + linear actions of $\mathrm{SL}_n(q)$ and $\mu_{1+q+\cdots+q^{n-1}}$

Characters in the principal ℓ-block (1)

Representation theory (Fong-Srinivasan)

- $\operatorname{Irr}(\mathrm{SL}_n(q)) \supset \operatorname{Un}(\mathrm{SL}_n(q))$

  $\operatorname{Irr}\mathfrak{S}_n$

  $\lambda \vdash n \rightsquigarrow \chi_\lambda \in \operatorname{Irr}(\mathrm{SL}_n(q))$
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- Principal block: $\chi_{[n-i,1^{(i)}]} \longleftrightarrow \wedge^i V \in \operatorname{Irr}\mathfrak{S}_n$ with V the reflection rep.

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- Irreducible constituants of $\mathrm{H}^i(\mathbf{X}, K)$ are unipotent characters
- $\mathrm{H}^i(\mathbf{X}) = \mathrm{H}^1(\mathbf{X}) \cup \cdots \cup \mathrm{H}^1(\mathbf{X})$
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Cohomology of the Deligne-Lusztig variety $\mathbf{X}$ over K

$H^0(\mathbf{X})$	$H^1(\mathbf{X})$	$\cdots\cdots\cdots$	$H^{n-1}(\mathbf{X})$
$\chi_{[n]}$	$\chi_{[n-1,1]}$	$\cdots\cdots\cdots$	$\chi_{[1^{(n)}]}$

$\rightsquigarrow$ **unipotent** characters in the block

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The **non-unipotent characters** in the block come from $H^{n-1}(\mathbf{Y})$.
We denote by χ_{exc} their sum.

Idea: encode the structure of the block in a graph

Here, if P is a projective indecomposable ΛH-module

$$[P] = \begin{cases} \chi_i + \chi_j & \text{with } i \neq j \\ \chi_i + \chi_{\text{exc}} \end{cases}$$

We define a tree Γ

- vertices: labeled by $\{\chi_{\text{exc}}, \chi_0, \ldots, \chi_{n-1}\}$
- egdes: χ ——— χ' if $\chi + \chi' = [P]$ with P a PIM

$+$ planar embedding encoding extensions between simple modules

The Brauer tree determines the module category over the block up to Morita equivalence

Brauer tree (1)

Idea: encode the structure of the block in a graph

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Cohomology of the Deligne-Lusztig variety $\mathbf{X}$ over K

$\mathrm{H}^0(\mathbf{X})$	$\mathrm{H}^1(\mathbf{X})$	$\cdots\cdots\cdots$	$\mathrm{H}^{n-1}(\mathbf{X})$
$(1_H =)\ \chi_{[n]}$	$\chi_{[n-1,1]}$	$\cdots\cdots\cdots$	$\chi_{[1^{(n)}]}\ (= \mathrm{St}_H)$

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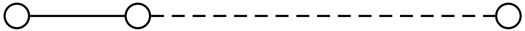

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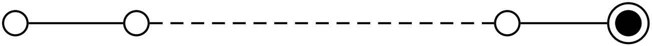

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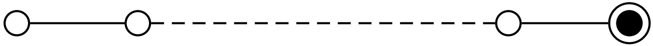

Back to the general case: $(\mathbf{G}, F)$ is any quasi-simple finite reductive group and we assume that the order of q modulo ℓ is the Coxeter number.

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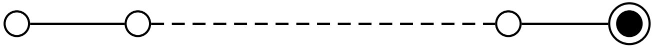

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Conjecture (Hiss-Lübeck-Malle)

The Brauer tree of the principal ℓ-block of $\mathbf{G}^F$ can be deduced, in a very natural way, from the cohomology of the Deligne-Lusztig variety $\mathbf{X}(w)$ associated to a Coxeter element.

Cohomology of **X** for a group of type F_4

i	4	5	6	7	8
$H^i_c(X)$	$(\mathrm{St}, 1)$	$(\phi_{4,13}, q)$	$(\phi''_{6,6}, q^2)$	$(\phi''_{4,1}, q^3)$	(Id, q^4)

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	$(F_4[\theta], \theta q^2)$				
	$(F_4[\theta^2], \theta^2 q^2)$				

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	$(B_{2,\varepsilon}, q^7)$	$(B_{2,r}, q^8)$	$(B_{2,1}, q^9)$		
	$(F_4[\mathrm{i}], q^5)$				
	$(F_4[-\mathrm{i}], q^{11})$				
	$(F_4[\theta], q^6)$				
	$(F_4[\theta^2], q^{10})$				

Where q has order $h = 12$ in $\mathbb{Z}/\ell\mathbb{Z}$

Brauer tree of the principal ℓ-block

Results

In characteristic 0: eigenspaces of $F \rightsquigarrow$ simple modules

In prime characteristic ℓ: generalized eigenspaces of F on the complex $R\Gamma_c(\mathbf{Y}(w), \Lambda) \rightsquigarrow$ projective modules

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Theorem (D)

Assume that p is a good prime number. Then the following are equivalent

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Moreover, in that case, the complex $b\mathrm{R}\Gamma_c(\mathbf{Y}(w), \Lambda)$ induces a derived (and perverse) equivalence between the principal ℓ-blocks of $\mathbf{G}^F$ and $N_{\mathbf{G}^F}(\mathbf{T}_w)$.