Parity Sheaves

Olivier Dudas and Peng Shan

Sheaves in Representation Theory
Isle of Skye

Motivation. Use geometric and homological methods in modular representation theory, namely sheaves with coefficients in a field k

- $\mathrm{char}(k) = 0$: Kazhdan-Lusztig conjecture can be proved using IC complexes and the decomposition theorem
- $\mathrm{char}(k) = p$: IC complexes exist but the decomposition theorem is no longer true

Motivation. Use geometric and homological methods in modular representation theory, namely sheaves with coefficients in a field k

- $\mathrm{char}(k) = 0$: Kazhdan-Lusztig conjecture can be proved using IC complexes and the decomposition theorem
- $\mathrm{char}(k) = p$: IC complexes exist but the decomposition theorem is no longer true

Motivation. Use geometric and homological methods in modular representation theory, namely sheaves with coefficients in a field k

- $\mathrm{char}(k) = 0$: Kazhdan-Lusztig conjecture can be proved using IC complexes and the decomposition theorem
- $\mathrm{char}(k) = p$: IC complexes exist but the decomposition theorem is no longer true

Motivation. Use geometric and homological methods in modular representation theory, namely sheaves with coefficients in a field k

- $\mathrm{char}(k) = 0$: Kazhdan-Lusztig conjecture can be proved using IC complexes and the decomposition theorem
- $\mathrm{char}(k) = p$: IC complexes exist but the decomposition theorem is no longer true

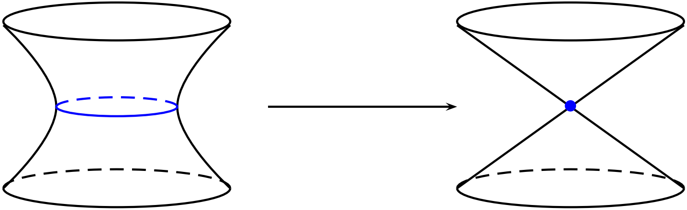
Observation. In the flag variety (as in many other situations)

$$\mathcal{H}^i(\mathrm{IC}(\overline{X}_w, k)) = 0 \text{ unless } i + \ell(w) \text{ is even}$$

when $\mathrm{char}(k) = 0$. But not always true in prime characteristic (torsion).

Failure of the decomposition theorem in $\mathfrak{sl}_2$

Springer resolution of $\mathcal{N} = \{(a, b, c) \in \mathbb{C}^3 \mid a^2 - bc = 0\}$

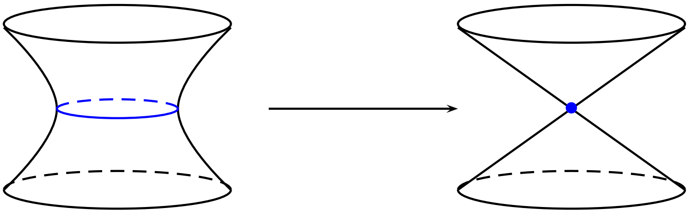


Stalks of $R\pi_*(\underline{k}[2])$

	-2	-1	0
$\mathcal{O}_{\text{reg}}$	k	0	0
$\{0\}$	k	0	k

Failure of the decomposition theorem in $\mathfrak{sl}_2$

Springer resolution of $\mathcal{N} = \{(a, b, c) \in \mathbb{C}^3 \mid a^2 - bc = 0\}$



Stalks of $R\pi_*(\underline{k}[2])$

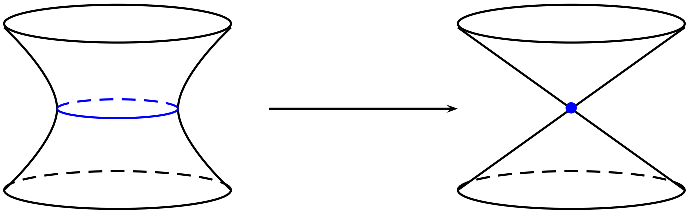
	-2	-1	0
$\mathcal{O}_{\text{reg}}$	k	0	0
$\{0\}$	k	0	k

Stalks of $\mathrm{IC}(\mathcal{N}, k)$ if $\mathrm{char}(k) \neq 2$

	-2	-1	0
$\mathcal{O}_{\text{reg}}$	k	0	0
$\{0\}$	k	0	0

Failure of the decomposition theorem in $\mathfrak{sl}_2$

Springer resolution of $\mathcal{N} = \{(a, b, c) \in \mathbb{C}^3 \mid a^2 - bc = 0\}$



Stalks of $R\pi_*(\underline{k}[2])$

	-2	-1	0
$\mathcal{O}_{reg}$	k	0	0
$\{0\}$	k	0	k

Stalks of $\mathrm{IC}(\mathcal{N}, k)$ if $\mathrm{char}(k) = 2$

	-2	-1	0
$\mathcal{O}_{reg}$	k	0	0
$\{0\}$	k	k	0

Parity complexes

Setting. X complex algebraic variety with Whitney stratification

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

$D_c(X, k) =$ derived category of constructible k-sheaves

Definition

A even sheaf $\mathcal{F}$ is any bounded complex in $D_c(X, k)$ such that

$$\mathcal{H}^i(\mathcal{F}) = \mathcal{H}^i(\mathbb{D}\mathcal{F}) = 0 \ \text{ for odd } i$$

Equivalently, the stalks and costalks are concentrated in even degrees.

$\mathcal{F}$ is parity if it is a direct sum $\mathcal{F}' \oplus \mathcal{F}''[1]$ with $\mathcal{F}'$ and $\mathcal{F}''$ even.

Setting. X complex algebraic G-variety with Whitney stratification

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

$D_c(X, k)$ = derived category of G-equivariant constructible k-sheaves

Definition

A even sheaf $\mathcal{F}$ is any bounded complex in $D_c(X, k)$ such that

$$\mathcal{H}^i(\mathcal{F}) = \mathcal{H}^i(\mathbb{D}\mathcal{F}) = 0 \quad \text{for odd } i$$

Equivalently, the stalks and costalks are concentrated in even degrees.

$\mathcal{F}$ is parity if it is a direct sum $\mathcal{F}' \oplus \mathcal{F}''[1]$ with $\mathcal{F}'$ and $\mathcal{F}''$ even.

Parity complexes

Setting. X complex algebraic variety with Whitney stratification

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

$D_c(X, k) =$ derived category of constructible k-sheaves

Definition

A even sheaf $\mathcal{F}$ is any bounded complex in $D_c(X, k)$ such that

$$\mathcal{H}^i(\mathcal{F}) = \mathcal{H}^i(\mathbb{D}\mathcal{F}) = 0 \quad \text{for odd } i$$

Equivalently, the stalks and costalks are concentrated in even degrees.

$\mathcal{F}$ is parity if it is a direct sum $\mathcal{F}' \oplus \mathcal{F}''[1]$ with $\mathcal{F}'$ and $\mathcal{F}''$ even.

Remark. can adapt the definition when k is a local ring.

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)
- Decomposition theorem in char. 0

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)
- Decomposition theorem in char. 0

Indecomposable parity complexes

Setting. X complex algebraic variety with Whitney stratification

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

satisfying a parity vanishing condition

$$\mathrm{H}^i(X_\lambda, \mathcal{L}_{|X_\lambda}) = 0 \text{ for odd } i \text{ and any local system } \mathcal{L}$$

Indecomposable parity complexes

Setting. X complex algebraic G-variety with Whitney stratification

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

satisfying a parity vanishing condition

$$\mathrm{H}^i_G(X_\lambda, \mathcal{L}_{|X_\lambda}) = 0 \quad \text{for odd } i \text{ and any local system } \mathcal{L}$$

Setting. X complex algebraic variety with Whitney stratification

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

satisfying a parity vanishing condition

$$\mathrm{H}^i(X_\lambda, \mathcal{L}_{|X_\lambda}) = 0 \text{ for odd } i \text{ and any local system } \mathcal{L}$$

Examples

- (Kac-Moody) Flag varieties stratified by the Schubert cells (including the affine Grassmannian)
- G-orbits in the nilpotent cone of $\mathfrak{g}$

Indecomposable parity complexes

Setting. X complex algebraic variety with Whitney stratification

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

satisfying a parity vanishing condition

$$\mathrm{H}^i(X_\lambda, \mathcal{L}_{|X_\lambda}) = 0 \;\text{ for odd } i \text{ and any local system } \mathcal{L}$$

Examples

- (Kac-Moody) Flag varieties stratified by the Schubert cells (including the affine Grassmannian)
- G-orbits in the nilpotent cone of $\mathfrak{g}$

Indecomposable parity complexes (2)

Theorem (Juteau-Mautner-Williamson)

Given an indecomposable local system $\mathcal{L}$ on X_λ there exists, up to isomorphism, at most one indecomposable parity complex $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ such that

- $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ is supported on $\overline{X}_\lambda$
- $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})_{|X_\lambda} \simeq \mathcal{L}[\dim X_\lambda]$

Moreover, any indecomposable parity complex is isomorphic to a shift $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})[m]$ for some indecomposable local system $\mathcal{L}$.

Indecomposable parity complexes (2)

Theorem (Juteau-Mautner-Williamson)

Given an indecomposable local system $\mathcal{L}$ on X_λ there exists, up to isomorphism, at most one indecomposable parity complex $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ such that

- $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ is supported on $\overline{X}_\lambda$
- $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})_{|X_\lambda} \simeq \mathcal{L}[\dim X_\lambda]$

Moreover, any indecomposable parity complex is isomorphic to a shift $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})[m]$ for some indecomposable local system $\mathcal{L}$.

The parity complexes $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ are called parity sheaves

Indecomposable parity complexes (2)

Theorem (Juteau-Mautner-Williamson)

Given an indecomposable local system $\mathcal{L}$ on X_λ there exists, up to isomorphism, at most one indecomposable parity complex $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ such that

- $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ is supported on $\overline{X}_\lambda$
- $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})_{|X_\lambda} \simeq \mathcal{L}[\dim X_\lambda]$

Moreover, any indecomposable parity complex is isomorphic to a shift $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})[m]$ for some indecomposable local system $\mathcal{L}$.

The parity complexes $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ are called parity sheaves

As a consequence $\mathbb{D}\mathcal{E}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathcal{E}(\overline{X}_\lambda, \mathcal{L}^\vee)$

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)
- Decomposition theorem in char. 0

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)
- Decomposition theorem in char. 0

Main features of **parity sheaves**

- Indecomposable parity complexes up to shift are $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathcal{E}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathcal{E}(\overline{X}_\lambda, \mathcal{L}^\vee)$

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)

Main features of **parity sheaves**

- Indecomposable parity complexes up to shift are $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathcal{E}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathcal{E}(\overline{X}_\lambda, \mathcal{L}^\vee)$

Even resolutions

A morphism $\pi : Y = \amalg Y_\mu \longrightarrow X = \amalg X_\lambda$ is a stratified morphism if

(i) each $\pi^{-1}(X_\lambda)$ is a union of strata

(ii) each surjective restriction $Y_\mu \longrightarrow X_\lambda$ is a fibration with smooth fibers

Even resolutions

A morphism $\pi : Y = \amalg Y_\mu \longrightarrow X = \amalg X_\lambda$ is a stratified morphism if

(i) each $\pi^{-1}(X_\lambda)$ is a union of strata

(ii) each surjective restriction $Y_\mu \longrightarrow X_\lambda$ is a fibration with smooth fibers

π is said to be even if moreover

(iii) the fibers of $Y_\mu \longrightarrow X_\lambda$ have cohomology concentrated in even degrees

Even resolutions

A morphism $\pi : Y = \amalg Y_\mu \longrightarrow X = \amalg X_\lambda$ is a stratified morphism if

(i) each $\pi^{-1}(X_\lambda)$ is a union of strata

(ii) each surjective restriction $Y_\mu \longrightarrow X_\lambda$ is a fibration with smooth fibers

π is said to be even if moreover

(iii) the fibers of $Y_\mu \longrightarrow X_\lambda$ have cohomology concentrated in even degrees

Idea

Use pushforward along even proper map/resolution

Even resolutions

A morphism $\pi : Y = \amalg Y_\mu \longrightarrow X = \amalg X_\lambda$ is a stratified morphism if

(i) each $\pi^{-1}(X_\lambda)$ is a union of strata

(ii) each surjective restriction $Y_\mu \longrightarrow X_\lambda$ is a fibration with smooth fibers

π is said to be even if moreover

(iii) the fibers of $Y_\mu \longrightarrow X_\lambda$ have cohomology concentrated in even degrees

Idea

Use pushforward along even proper map/resolution

This gives existence of parity sheaves for

- Flag varieties stratified by Schubert cells
- GL_n-orbits in the nilpotent cone of $\mathfrak{gl}_n$

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)
- Decomposition theorem in char. 0

Main features of **parity sheaves**

- Indecomposable parity complexes up to shift are $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathcal{E}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathcal{E}(\overline{X}_\lambda, \mathcal{L}^\vee)$

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)
- Decomposition theorem in char. 0

Main features of **parity sheaves**

- Indecomposable parity complexes up to shift are $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathcal{E}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathcal{E}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of parity sheaves via even resolution (when such a resolution exists)

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)
- Decomposition theorem in char. 0

Main features of **parity sheaves**

- Indecomposable parity complexes up to shift are $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathcal{E}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathcal{E}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of parity sheaves via even resolution (when such a resolution exists)

Decomposition theorem

For proving the existence, one needs an "analogue" of the decomposition theorem for parity sheaves:

Theorem-Observation (Juteau-Mautner-Williamson)

If $\pi : Y \longrightarrow X$ is a proper even map and $\mathcal{F}$ a parity complex on Y, then $\pi_* \mathcal{F}$ is parity.

Decomposition theorem

For proving the existence, one needs an "analogue" of the decomposition theorem for parity sheaves:

Theorem-Observation (Juteau-Mautner-Williamson)

If $\pi : Y \longrightarrow X$ is a proper even map and $\mathcal{F}$ a parity complex on Y, then $\pi_*\mathcal{F}$ is parity.

Consequences. if Y is smooth

$$\pi_*\underline{k}_Y[\dim Y] \simeq \bigoplus \mathcal{E}(\overline{X}_{\lambda_i}, \mathcal{L}_i)[c_i]$$

In particular, if $\mathrm{char}(k) = 0$, the usual decomposition theorem shows that the IC complexes occurring are parity sheaves.

IC complexes vs parity sheaves

Aim. Find a good substitute for IC complexes in prime characteristic

Main features of **ICs**

- Simple perverse sheaves up to shift are $\mathrm{IC}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathrm{IC}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathrm{IC}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of ICs for any simple local system (via Deligne's construction)
- Decomposition theorem in char. 0

Main features of **parity sheaves**

- Indecomposable parity complexes up to shift are $\mathcal{E}(\overline{X}_\lambda, \mathcal{L})$ which extend irreducible local systems $\mathcal{L}[\dim(X_\lambda)]$
- $\mathbb{D}\mathcal{E}(\overline{X}_\lambda, \mathcal{L}) \simeq \mathcal{E}(\overline{X}_\lambda, \mathcal{L}^\vee)$
- Existence of parity sheaves via even resolution (when such a resolution exists)
- "Decomposition theorem in any characteristic"