

Lectures on modular Deligne–Lusztig theory

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Abstract. These notes are based on a series of lectures given by the author at the Centre Bernoulli (EPFL) in July 2016. They aim at illustrating the importance of the mod- ℓ cohomology of Deligne–Lusztig varieties in the modular representation theory of finite reductive groups.

Introduction

In order to construct and study the complex representations of finite reductive groups $G(q)$ (such as $\mathrm{GL}_n(q)$, $\mathrm{Sp}_{2n}(q)$, ...) Deligne and Lusztig introduced in 1976 a family of algebraic varieties acted on by $G(q)$ [20]. The subsequent work of Lusztig on the cohomology of these *Deligne–Lusztig varieties* led to a complete classification of the irreducible characters of finite reductive groups [58].

The purpose of these lectures is to present a generalization of the theory of Deligne–Lusztig to the modular setting, that is, for representations over fields of positive characteristic. This originated in the work of Broué [11] and Bonnafé–Rouquier [5].

In the ordinary case (in characteristic zero), the representation theory is controlled by the simple objects, which are in turn determined by a numerical datum, their characters. The situation is far more complicated for representations in positive characteristic; several classes of indecomposable objects are of particular interest, and more information is needed to understand the representations, namely:

- information of numerical nature: characters of projective modules, multiplicities of simple modules in a given ordinary character, all of which are encoded in the so-called decomposition matrix;
- information of homological nature: extensions between simple modules, Loewy series of projective modules, projective resolutions of simple objects.

The alternating sum of the cohomology groups of Deligne–Lusztig varieties produces a virtual character – an element of the Grothendieck group of the category of representations. In the modular framework, this object does not contain enough information, and one should consider each individual cohomology group, or rather the cohomology complex of the variety. This object now lives in the derived category of representations, and it encodes many aspects of the modular representation theory of the group. One crucial incarnation of this phenomenon is the geometric version of Broué’s abelian defect group conjecture, which predicts that the cohomology complex of a suitably chosen Deligne–Lusztig variety induces a derived

equivalence between the principal block of a finite reductive group and its Brauer correspondent. Not so many cases of this conjecture are known to hold but a lot of numerical evidence and many partial results have been obtained in that direction.

The first part of these lectures aims at introducing the mod- ℓ cohomology of Deligne–Lusztig varieties using the modern language of derived and homotopy categories. Unlike most of the textbooks on étale and ℓ -adic cohomology, we avoid the definition and focus on the properties of the cohomology complexes of varieties acted on by a finite group (such as perfectness), with particular attention on how one can compute such complexes (using decompositions, quotients or fixed points). In the second part we present several recent results obtained using this approach. They include the computation of decomposition numbers in §4 (a joint work with G. Malle) and the determination of Brauer trees in §5 (a joint work with D. Craven and R. Rouquier). This illustrates how powerful the geometric methods are for solving representation theoretic problems for finite reductive groups. There is a converse to that statement, and we explain in a final chapter how to use representation theory to show that the cohomology of a particular Deligne–Lusztig variety is torsion-free.

Acknowledgement: The author thanks Marc Cabanes, Gerhard Hiß and Gunter Malle for many valuable comments on an earlier version of these lecture notes.

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1. Introduction to derived categories

Throughout this chapter, A will denote a ring with unit. The category of left A -modules (resp. finitely generated left A -modules) will be denoted by $A\text{-Mod}$ (resp. $A\text{-mod}$).

The purpose of this first chapter is to introduce two categories, the homotopy category $\mathrm{Ho}(A\text{-Mod})$ and the derived category $\mathrm{D}(A\text{-Mod})$. Here is a non-exhaustive list of reasons why we are going to work in this framework, instead of working with A -modules or complexes of A -modules:

- to get rid of (split) exact sequences;
- to have uniqueness of projective or injective resolutions;
- to have a good notion of duality (e.g. over \mathbb{Z});
- to work with non-exact functors.

Several steps are needed to understand the construction of the homotopy and derived categories of A -modules:

$$\begin{array}{ccccccc}
 A\text{-Mod} & \rightsquigarrow & \underbrace{C(A\text{-Mod})}_{\text{complexes of } A\text{-modules}} & \rightsquigarrow & \mathrm{Ho}(A\text{-Mod}) & \rightsquigarrow & \mathrm{D}(A\text{-Mod}) \\
 & & & & \downarrow \text{getting rid of } \textit{split} \text{ exact sequences} & & \downarrow \text{inverting quasi-isomorphisms}
 \end{array}$$

Note however that this chapter is not intended to provide a detailed account on this construction. For further reading on the subject we recommend for example the excellent textbooks by Gelfand–Manin [47] and Neeman [60].

1.1. Complexes of A -modules. A *complex of A -modules* is a sequence

$$C_{\bullet} = (\cdots \rightarrow C_n \xrightarrow{d_n} C_{n+1} \xrightarrow{d_{n+1}} C_{n+2} \rightarrow \cdots)$$

where, for each $n \in \mathbb{Z}$, C_n is a left A -module and $d_n : C_n \rightarrow C_{n+1}$ is a morphism of A -modules (the *differential*) satisfying $d_{n+1} \circ d_n = 0$.

A morphism between two complexes $f = (C_{\bullet}, d) \rightarrow (D_{\bullet}, \partial)$ is given by a family of morphisms of A -modules $f_n : C_n \rightarrow D_n$ making the following diagram commute

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n & \xrightarrow{d_n} & C_{n+1} & \xrightarrow{d_{n+1}} & C_{n+2} \longrightarrow \cdots \\ & & \downarrow f_n & & \downarrow f_{n+1} & & \downarrow f_{n+2} \\ \cdots & \longrightarrow & D_n & \xrightarrow{\partial_n} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_{n+2} \longrightarrow \cdots \end{array}$$

Here is an example of a morphism between two complexes of \mathbb{Z} -modules (we will see later that it is a quasi-isomorphism)

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{m} & \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/m\mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \end{array}$$

The category of complexes of A -modules will be denoted $\mathbf{C}(A\text{-Mod})$. It is an abelian category. This can be seen by considering the ring $A[X]/X^2 =: A(d) \simeq A \oplus Ad$ with $d^2 = 0$ and d commuting with A . Then the functor

$$\begin{aligned} \mathbf{C}(A\text{-Mod}) &\xrightarrow{\sim} A(d)\text{-Mod} \\ C_{\bullet} &\mapsto \bigoplus_{n \in \mathbb{Z}} C_n \text{ with } d|_{C_n} = d_n \end{aligned}$$

is an equivalence of categories. The abelian structure on $\mathbf{C}(A\text{-Mod})$ is obtained via the equivalence from the abelian structure of $A(d)\text{-Mod}$. As a consequence, we can consider kernels and cokernels of morphisms between complexes, as well as exact sequences of complexes.

We say that a complex C_{\bullet} is *bounded above* (resp. *bounded below*, resp. *bounded*) if $C_n = 0$ for $n \gg 0$ (resp. $n \ll 0$, resp. $|n| \gg 0$). The corresponding full subcategory of $\mathbf{C}(A\text{-Mod})$ will be denoted by $\mathbf{C}^-(A\text{-Mod})$ (resp. $\mathbf{C}^+(A\text{-Mod})$, resp. $\mathbf{C}^b(A\text{-Mod})$).

Given $k \in \mathbb{Z}$ and C_{\bullet} a complex of A -modules, we define the *k -th shift of C_{\bullet}* , denoted by $C_{\bullet}[k]$, to be the complex with terms $(C_{\bullet}[k])_n = C_{n+k}$ and differential

$d_{C_\bullet[k]} = (-1)^k d_{C_\bullet}$. If M is an A -module, the notation $M[k]$ stands for the complex with zero terms outside the degree $-k$ and M in the degree $-k$. The functor

$$\begin{aligned} A\text{-Mod} &\longrightarrow \mathbf{C}(A\text{-Mod}) \\ M &\longmapsto M[0] \end{aligned}$$

is fully faithful. In other words, $A\text{-Mod}$ can be identified with complexes with zero terms outside the degree 0.

Since $d_n \circ d_{n-1} = 0$, $\text{Im } d_{n-1}$ is a submodule of $\text{Ker } d_n$. The quotient $H^n(C_\bullet) = \text{Ker } d_n / \text{Im } d_{n-1}$ is an A -module, called the *degree n cohomology group* of C_\bullet . We write

$$H^\bullet(C_\bullet) := \bigoplus_{n \in \mathbb{Z}} H^n(C_\bullet) = \text{Ker } d / \text{Im } d.$$

We say that a complex C_\bullet is *exact* or *acyclic* if $H^n(C_\bullet) = 0$ for all $n \in \mathbb{Z}$. The compatibility of maps between complexes and the respective differentials ensures that any morphism of complexes $f = (C_\bullet, d) \rightarrow (D_\bullet, \partial)$ induces a family of morphisms of A -modules $H^n(f) : H^n(C_\bullet) \rightarrow H^n(D_\bullet)$.

From now on, we will omit the subscript \bullet in the notation of complexes, as well as the reference to the differentials for morphisms of complexes.

Proposition 1.1. *Let $0 \rightarrow C \xrightarrow{\iota} C' \xrightarrow{\pi} C'' \rightarrow 0$ be a short exact sequence of complexes of A -modules. Then there are boundary maps $\delta_n : H^n(C'') \rightarrow H^{n+1}(C)$ for all $n \in \mathbb{Z}$ yielding a long exact sequence of A -modules*

$$\dots \rightarrow H^n(C) \xrightarrow{H^n(\iota)} H^n(C') \xrightarrow{H^n(\pi)} H^n(C'') \xrightarrow{\delta_n} H^{n+1}(C) \rightarrow \dots$$

Sketch of proof. Let $c'' \in \text{Ker } d_n''$, which we write $c'' = \pi_n(c')$ for some $c' \in C'_n$. Since π is a morphism of complexes, $\pi_{n+1}(d'_n(c')) = d_n''(\pi_n(c')) = 0$, hence $d'_n(c') \in \text{Ker } \pi_{n+1} = \text{Im } \iota_{n+1}$. Now write $d'_n(c') = \iota_{n+1}(c)$ and set $\delta_n(c'') := c$. \square

Exercise 1.2. Check that δ_n is well-defined, and that it induces the long exact sequence stated in the proposition.

In terms of $A(d)$ -modules, the proposition shows the existence of a morphism of $A(d)$ -modules $\delta : H^\bullet(C'') \rightarrow H^\bullet(C)[1]$ which fits in a triangle

$$\begin{array}{ccc} & H^\bullet(C'') & \\ [1] \swarrow & & \searrow \\ H^\bullet(C) & \xrightarrow{\delta} & H^\bullet(C') \end{array}$$

1.2. The homotopy category. A morphism $f : (C, d) \rightarrow (D, \partial)$ between complexes of A -modules is said to be *null-homotopic* if it is of the form $f = s \circ d + \partial \circ s$

for some map $s : C \rightarrow D[-1]$ (not necessarily a morphism of complexes). We illustrate this with the following diagram:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & C_n & \xrightarrow{d_n} & C_{n+1} & \xrightarrow{d_{n+1}} & C_{n+2} & \longrightarrow & \cdots \\
 & & \downarrow & \nearrow s_n & \downarrow & \nearrow s_{n+1} & \downarrow & \nearrow s_{n+2} & \\
 & & + & & + & & + & & \\
 \cdots & \longrightarrow & D_n & \xrightarrow{\partial_n} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_{n+2} & \longrightarrow & \cdots
 \end{array}$$

Each vertical map $f_n : C_n \rightarrow D_n$ satisfies $f_n = s_n \circ d_n + \partial_{n-1} \circ s_{n-1}$ (this is represented by the symbol “+” in the previous diagram).

Given two morphisms of complexes $f, f' : C \rightarrow D$ we write $f \sim f'$ if $f - f'$ is null-homotopic. This is an equivalence relation, compatible with the sum and the composition of morphisms. A complex C is *null-homotopic* if the identity map 1_C is null-homotopic (*i.e.* $1_C \sim 0$). We say that f is a *homotopy equivalence* if there exists a morphism $g : D \rightarrow C$ such that $f \circ g \sim 1_D$ and $g \circ f \sim 1_C$.

Definition 1.3. The *homotopy category of A -modules*, denoted by $\mathrm{Ho}(A\text{-Mod})$, is the category with

- objects: complexes of A -modules (same as $\mathbf{C}(A\text{-Mod})$),
- morphisms: $\mathrm{Hom}_{\mathrm{Ho}(A\text{-Mod})}(C, D) := \mathrm{Hom}_{\mathbf{C}(A\text{-Mod})}(C, D) / \sim$.

It is an additive category (but non-abelian in general). The isomorphisms in the homotopy category are exactly the classes of the homotopy equivalences.

Exercise 1.4. Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of A -modules, and C be the complex associated to this sequence, with L in degree 0. Show that C is null-homotopic if and only if the exact sequence splits.

A complex of the form $(\cdots \rightarrow 0 \rightarrow M \xrightarrow{f} M \rightarrow 0 \rightarrow \cdots)$ is null-homotopic if and only if f is an isomorphism. More generally, a complex C is null-homotopic if and only if it decomposes as a direct sum of complexes of the form

$$(\cdots \rightarrow 0 \rightarrow M \xrightarrow{\sim} M \rightarrow 0 \rightarrow \cdots).$$

See also Exercise 1.4. When working in the homotopy category we will often consider reduced complexes where all the null-homotopic direct summands are removed. This is well-defined whenever we consider bounded complexes of finitely generated A -modules (since $\mathbf{C}^b(A\text{-mod})$ is a Krull-Schmidt category).

When f is null-homotopic, the corresponding morphism on cohomology groups is zero. As a consequence, if $f \sim g$ then $H^\bullet(f) = H^\bullet(g)$. Also, if f is a homotopy equivalence then $H^\bullet(f)$ is an isomorphism. In particular, if $C \simeq D$ in $\mathrm{Ho}(A\text{-Mod})$ then $H^\bullet(C) \simeq H^\bullet(D)$.

We mentioned earlier that projective and injective resolutions are unique in the homotopy and derived categories. Recall that a projective resolution P of an A -module M is a complex of projective A -modules

$$\cdots \rightarrow P_{-n} \rightarrow P_{-n+1} \rightarrow \cdots \rightarrow P_{-1} \rightarrow P_0 \rightarrow 0 \rightarrow \cdots$$

such that $H^\bullet(P) \simeq M[0]$. In other words, there is a surjective map $P_0 \twoheadrightarrow M = H^0(P)$ which fits in the following long exact sequence

$$\cdots \longrightarrow P_{-n} \longrightarrow P_{-n+1} \longrightarrow \cdots \longrightarrow P_{-1} \longrightarrow P_0 \twoheadrightarrow M \longrightarrow 0 \longrightarrow \cdots .$$

The idea of projective resolutions is to replace M by a complex with the same cohomology (M in degree 0) but whose terms are ‘nicer’.

Proposition 1.5. *If P and Q are two projective resolutions of M then $P \simeq Q$ in $\text{Ho}(A\text{-Mod})$.*

Sketch of proof. We only show how to construct the morphisms $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ which will be mutually inverse in the homotopy category.

Let us denote by d (resp. ∂) the differential of the complex P (resp. Q) and by $d_0 : P_0 \twoheadrightarrow M$ (resp. $\partial_0 : Q_0 \twoheadrightarrow M$) the respective projections. Since ∂_0 is surjective and P_0 is projective, the map $d_0 : P_0 \longrightarrow M$ factors through $Q_0 \twoheadrightarrow M$. In other words, there exists $f_0 : P_0 \longrightarrow Q_0$ such that $d_0 = \partial_0 \circ f_0$.

Since $\partial_0 f_0 d_{-1} = d_0 d_{-1} = 0$, we have $\text{Im}(f_0 d_{-1}) \subset \text{Ker } \partial_0 = \text{Im } \partial_{-1}$. Therefore $f_0 d_{-1}$ can be seen as a map from P_{-1} to $\text{Im } \partial_{-1}$. Since P_{-1} is projective, it should factor through the surjective map $\partial_{-1} : Q_{-1} \twoheadrightarrow \text{Im } \partial_{-1}$. In other words, there exists $f_{-1} : P_{-1} \longrightarrow Q_{-1}$ such that $\partial_{-1} f_{-1} = f_0 d_{-1}$. By iterating this construction, we obtain, for all $n < 0$, maps $f_n : P_n \longrightarrow Q_n$ such that $\partial_n f_n = f_{n+1} d_n$. This means that $f = (f_n)_{n \in \mathbb{Z}} : P \longrightarrow Q$ is a morphism of complexes. The construction of g is similar. \square

Exercise 1.6. Show that $g \circ f \sim 1_P$ and $f \circ g \sim 1_Q$.

More generally, if C is a complex of A -modules, a *projective resolution* P of C is a bounded above complex of projective modules together with a morphism $s : P \longrightarrow C$ such that $H^n(s)$ is an isomorphism for all $n \in \mathbb{Z}$ (a *quasi-isomorphism*, see §1.3). We shall see in Proposition 1.15 that P is uniquely determined by C up to homotopy equivalence, which generalizes the previous proposition.

Exercise 1.7. Spell out the case where $C = M[0]$ has only one non-zero term, say M in degree 0.

Definition 1.8. The *mapping cone* of a morphism of complexes $f : (C, d) \longrightarrow (D, \partial)$ is the complex $\text{Cone}(f) = C[1] \oplus D$ with differential

$$d_{\text{Cone}(f)} = \begin{bmatrix} d[1] & 0 \\ f[1] & \partial \end{bmatrix}.$$

If $f = 0$, the mapping cone is just the direct sum of $C[1]$ and D in the category of complexes. However, if f is non-zero, it encodes more information. For example, if $C = M[0]$ and $D = N[0]$ are complexes concentrated in degree 0, then f is induced by a morphism of A -modules $f_0 : M \longrightarrow N$. In this case the complex $\text{Cone}(f)$ has only two non-zero terms (in degree -1 and 0) and its cohomology is

$H^{-1}(\text{Cone}(f)) = \text{Ker } f$ and $H^0(\text{Cone}(f)) = \text{CoKer } f$. We deduce the existence of a long exact sequence of A -modules

$$0 \longrightarrow \underbrace{H^{-1}(\text{Cone}(f))}_{\text{Ker } f} \longrightarrow M \xrightarrow{f} N \longrightarrow \underbrace{H^0(\text{Cone}(f))}_{\text{CoKer } f} \longrightarrow 0.$$

This generalizes to morphisms of complexes as follows.

Proposition 1.9. *Any morphism of complexes $f : C \rightarrow D$ induces a long exact sequence*

$$\dots \longrightarrow H^n(C) \longrightarrow H^n(D) \longrightarrow H^n(\text{Cone}(f)) \longrightarrow H^{n+1}(C) \longrightarrow \dots$$

The maps $H^n(D) \rightarrow H^n(\text{Cone}(f))$ and $H^n(\text{Cone}(f)) \rightarrow H^{n+1}(C)$ are induced by the natural morphisms of complexes $\iota : D \rightarrow \text{Cone}(f)$ and $\pi : \text{Cone}(f) \rightarrow C[1]$. Note that $\pi \circ \iota = 0$. The maps $f \circ \pi$ and $\iota \circ f$ are non-zero in general, and only null-homotopic. This ensures that we have a triangle

$$\begin{array}{ccc} & \text{Cone}(f) & \\ [1] \swarrow & & \nwarrow \\ C & \xrightarrow{f} & D \end{array}$$

in the homotopy category $\text{Ho}(A\text{-Mod})$.

Exercise 1.10. Show that $f \circ \pi$ and $\iota \circ f$ are null-homotopic.

More generally, a *triangle* in $\text{Ho}(A\text{-Mod})$ is $C \xrightarrow{f} D \xrightarrow{g} E \xrightarrow{h} C[1]$ such that $g \circ f$, $h \circ g$ and $f[1] \circ h$ are null-homotopic. We represent it as $C \rightarrow D \rightarrow E \rightsquigarrow$ or as a triangular diagram

$$\begin{array}{ccc} & E & \\ [1] \swarrow & & \nwarrow \\ C & \longrightarrow & D \end{array}$$

A morphism of triangles between $C \rightarrow D \rightarrow E \rightsquigarrow$ and $C' \rightarrow D' \rightarrow E' \rightsquigarrow$ is the data of morphisms of complexes $u : C \rightarrow C'$, $v : D \rightarrow D'$ and $w : E \rightarrow E'$ making the following diagram commute

$$\begin{array}{ccccccc} C & \xrightarrow{f} & D & \xrightarrow{g} & E & \xrightarrow{h} & C[1] \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow u[1] \\ C' & \xrightarrow{f'} & D' & \xrightarrow{g'} & E' & \xrightarrow{h'} & C'[1] \end{array}$$

A triangle is *distinguished* if it is isomorphic to a triangle $C \xrightarrow{f} D \rightarrow \text{Cone}(f) \rightsquigarrow$ for some morphism of complexes $f : C \rightarrow D$. In particular, by Proposition 1.9 any distinguished triangle $C \rightarrow D \rightarrow E \rightsquigarrow$ yields a long exact sequence in cohomology

$$\dots \longrightarrow H^n(C) \longrightarrow H^n(D) \longrightarrow H^n(E) \longrightarrow H^{n+1}(C) \longrightarrow \dots$$

The category $\mathrm{Ho}(A\text{-Mod})$ together with the shift functor [1] (called *suspension functor*) and the collection of distinguished triangles is a *triangulated category* (see for example [60, Chap. 1] for the list of axioms of triangulated categories).

Morphisms in the homotopy category can be expressed in terms of the cohomology of the total Hom complex. Given C and D two complexes of A -modules, the total Hom complex, denoted by $\mathrm{Hom}_A^\bullet(C, D)$, is defined by

$$\mathrm{Hom}_A^n(C, D) = \prod_{j-i=n} \mathrm{Hom}_A(C_i, D_j)$$

with the differential given by

$$\delta(f_{i,j}) = \partial_j \circ f_{i,j} - (-1)^{j-i} f_{i,j} \circ d_{i-1}$$

for every $f_{i,j} \in \mathrm{Hom}_A(C_i, D_j)$. One can readily check that $\mathrm{Ker} \delta_n$ consists of the morphisms of complexes from C to $D[n]$ whereas $\mathrm{Im} \delta_{n-1}$ is the subgroup of null-homotopic morphisms. Consequently,

$$H^n(\mathrm{Hom}_A^\bullet(C, D)) = \mathrm{Hom}_{\mathrm{Ho}(A\text{-Mod})}(C, D[n]). \tag{1.1}$$

1.3. The derived category of A -Mod. A morphism of complexes $f : C \rightarrow D$ is a *quasi-isomorphism* if the maps $H^n(f) : H^n(C) \rightarrow H^n(D)$ induced on the cohomology groups are isomorphisms for all $n \in \mathbb{Z}$. By Proposition 1.9 this is equivalent to the complex $\mathrm{Cone}(f)$ being acyclic.

The derived category is obtained from the homotopy category by formally inverting the quasi-isomorphisms (equivalently, by taking the quotient by the acyclic complexes). This is analogous to the construction of the fraction field of a domain. More precisely, given two pairs of morphisms in $\mathrm{Ho}(A\text{-Mod})$, say (s, f) and (t, g) with s and t being quasi-isomorphisms, we write $(s, f) \equiv (t, g)$ if there exists a commutative diagram in $\mathrm{Ho}(A\text{-Mod})$

$$\begin{array}{ccccc} & & X & & \\ & s \swarrow & \uparrow a & \searrow f & \\ C & \xleftarrow{r} & Z & \xrightarrow{h} & D \\ & t \swarrow & \downarrow b & \searrow g & \\ & & Y & & \end{array}$$

with r a quasi-isomorphism. This represents the relation “ $fs^{-1} = fa(sa)^{-1} = gb(tb)^{-1} = gt^{-1}$ ”.

Definition 1.11. The *derived category of A -modules*, denoted by $D(A\text{-Mod})$, is the category with

- objects: complexes of A -modules (same as $C(A\text{-Mod})$ and $\mathrm{Ho}(A\text{-Mod})$),
- morphisms:

$$\mathrm{Hom}_{D(A\text{-Mod})}(C, D) := \left\{ \begin{array}{c} \begin{array}{ccc} & X & \\ s \swarrow & & \searrow f \\ C & & D \end{array} \left| \begin{array}{l} X \text{ a complex of } A\text{-modules} \\ f \in \mathrm{Hom}_{\mathrm{Ho}(A\text{-Mod})}(X, D) \\ s \in \mathrm{Hom}_{\mathrm{Ho}(A\text{-Mod})}(X, C) \\ s \text{ a quasi-isomorphism} \end{array} \right. \right\} / \equiv .$$

The natural functor $\mathrm{Ho}(A\text{-Mod}) \rightarrow \mathrm{D}(A\text{-Mod})$ (sending a morphism f to the class of $(1, f)$) sends quasi-isomorphisms to isomorphisms. It is universal for this property. A distinguished triangle in $\mathrm{D}(A\text{-Mod})$ will be by definition the image of a distinguished triangle in $\mathrm{Ho}(A\text{-Mod})$. This endows the derived category with a structure of triangulated category. We can think of distinguished triangles in $\mathrm{D}(A\text{-Mod})$ as analogues of short exact sequences of complexes.

Proposition 1.12.

- (i) *The functor H^\bullet is well-defined on $\mathrm{D}(A\text{-Mod})$.*
- (ii) *Any short exact sequence of complexes $0 \rightarrow C \rightarrow D \rightarrow E \rightarrow 0$ in $\mathrm{C}(A\text{-Mod})$ yields a distinguished triangle $C \rightarrow D \rightarrow E \rightsquigarrow$ in $\mathrm{D}(A\text{-Mod})$.*
- (iii) *The functor $A\text{-Mod} \rightarrow \mathrm{D}(A\text{-Mod})$ sending a module M to the complex $M[0]$ is fully faithful.*

Note however that in general $H^\bullet(C) \simeq H^\bullet(D)$ does not imply $C \simeq D$ in $\mathrm{D}(A\text{-Mod})$. For example $0 \rightarrow \mathbb{C} \xrightarrow{0} \mathbb{C} \rightarrow 0$ and $0 \rightarrow \mathbb{C}[x]/x^2 \xrightarrow{x} \mathbb{C}[x]/x^2 \rightarrow 0$ are not isomorphic in $\mathrm{D}(\mathbb{C}[x]/x^2\text{-mod})$.

1.4. Morphisms in $\mathrm{D}(A\text{-Mod})$. By definition, morphisms in the derived category are equivalence classes of pairs of morphisms in the homotopy category $C \xleftarrow{s} X \xrightarrow{f} D$, representing “ $f \circ s^{-1}$ ”. We explain here how replacing C and D by projective or injective resolutions helps finding nice representatives for these morphisms.

We will denote by $A\text{-Proj}$ the full subcategory of $A\text{-Mod}$ whose objects are the projective A -modules. The corresponding categories $\mathrm{C}(A\text{-Proj})$ and $\mathrm{Ho}(A\text{-Proj})$ correspond to the full subcategories of $\mathrm{C}(A\text{-Mod})$ and $\mathrm{Ho}(A\text{-Mod})$ respectively, whose objects are complexes of projective A -modules. Similarly, $A\text{-Inj}$ will refer to the additive category of injective A -modules, and $\mathrm{C}(A\text{-Inj})$ (resp. $\mathrm{Ho}(A\text{-Inj})$) to the corresponding category of complexes (resp. the homotopy category).

Lemma 1.13. *Let $P \in \mathrm{C}^-(A\text{-Proj})$ be a bounded above complex of projective A -modules, and X, Y be two complexes of A -modules.*

- (i) *If X is an acyclic complex, any morphism $P \rightarrow X$ is null-homotopic.*
- (ii) *Any quasi-isomorphism $Y \rightarrow P$ splits in $\mathrm{Ho}(A\text{-Mod})$.*

Proof. Let X be an acyclic complex and $f : P \rightarrow X$ be a morphism. We can assume without loss of generality that $P_i = 0$ for $i > 0$. We construct a homotopy $h : P \rightarrow X[-1]$ as follows:

$$\begin{array}{ccccccccccc}
 \cdots & \longrightarrow & P_{-2} & \longrightarrow & P_{-1} & \xrightarrow{d_{-1}} & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & \swarrow h_{-1} & \downarrow f_{-1} & \swarrow h_0 & \downarrow f_0 & & \downarrow & & \\
 \cdots & \longrightarrow & X_{-2} & \longrightarrow & X_{-1} & \xrightarrow{\partial_{-1}} & X_0 & \xrightarrow{\partial_0} & X_1 & \longrightarrow & \cdots
 \end{array}$$

Since f is a morphism of complexes, then $\partial_0 f_0 = 0$ hence $\text{Im } \partial_{-1} = \text{Ker } \partial_0 \supset \text{Im } f_0$. Therefore since P_0 is projective there exists $h_0 : P_0 \rightarrow X_{-1}$ such that $f_0 = \partial_{-1} h_0$.

Similarly, using the fact that $\partial_{-1}(f_{-1} - h_0 d_{-1}) = \partial_{-1} f_{-1} - \partial_{-1} h_0 d_{-1} = \partial_{-1} f_{-1} - f_0 d_{-1} = 0$, there exists $h_{-1} : P_{-1} \rightarrow X_{-2}$ such that $f_{-1} - h_0 d_{-1} = \partial_{-2} h_{-1}$. We iterate this construction to get (i).

For (ii) we consider the distinguished triangle $Y \xrightarrow{s} P \rightarrow \text{Cone}(s) \rightsquigarrow$. By (i) the morphism $P \rightarrow \text{Cone}(s)$ is null-homotopic, yielding a map $h : P \rightarrow \text{Cone}(s)[-1] = P[-1] \oplus Y$. If $t : P \rightarrow Y$ denotes the second projection of this map then one can show that t is a morphism of complexes and $st \sim 1_P$. \square

Exercise 1.14. Check that t is a morphism of complexes satisfying $st \sim 1_P$.

Proposition 1.15. *Given $P \in \mathcal{C}^-(A\text{-Proj})$ and $C \in \mathcal{C}(A\text{-Mod})$ there is a natural isomorphism of \mathbb{Z} -modules*

$$\begin{array}{ccc} \text{Hom}_{\text{Ho}(A\text{-Mod})}(P, C) & \xrightarrow{\sim} & \text{Hom}_{\text{D}(A\text{-Mod})}(P, C) \\ f & \mapsto & (1, f) \end{array}$$

Proof. For the injectivity of the map, let $f, g \in \text{Hom}_{\text{Ho}(A\text{-Mod})}(P, C)$ be such that $(1, f) \equiv (1, g)$. This means that there is a commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & & \uparrow & & \\ P & \xleftarrow{s} & Z & \xrightarrow{h} & C \\ & & \downarrow & & \\ & & P & & \end{array}$$

with s a quasi-isomorphism. By Lemma 1.13, there exists $t : P \rightarrow Z$ such that $st \sim 1_P$. From $fs = h = gs$ we get $f = fst = gst = g$ in $\text{Ho}(A\text{-Mod})$.

The surjectivity is another application of Lemma 1.13. Indeed, any pair of morphisms $P \xleftarrow{s} X \xrightarrow{f} C$ can be completed to a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & & \uparrow & & \\ P & \xleftarrow{s} & P & \xrightarrow{ft} & C \\ & & \downarrow t & & \end{array}$$

with t satisfying $st \sim 1_P$. \square

For $? \in \{b, +, -\}$ we will denote by $\text{Ho}^?$ and $\text{D}^?$ the essential images of the categories of bounded/bounded below/bounded above complexes of A -modules. The following theorem follows from Proposition 1.15 and the existence of projective (or injective) resolutions. The proof is left as an exercise.

Theorem 1.16. *The functor $\text{Ho}(A\text{-Mod}) \rightarrow \text{D}(A\text{-Mod})$ induces equivalences*

$$\begin{array}{l} \text{Ho}^-(A\text{-Proj}) \xrightarrow{\sim} \text{D}^-(A\text{-Mod}) \\ \text{and } \text{Ho}^+(A\text{-Inj}) \xrightarrow{\sim} \text{D}^+(A\text{-Mod}). \end{array}$$

Remark 1.17. More generally, if one works with another abelian category \mathcal{A} instead of $A\text{-Mod}$, then the first isomorphism in Theorem 1.16 (resp. the second isomorphism) remains true if \mathcal{A} has enough projective objects (resp. enough injective objects). In these notes we shall often use that $A\text{-mod}$, the category of finitely generated A -modules has enough projectives. This guarantees the existence of projective resolutions of bounded above complexes.

1.5. Derived functors. Let B be another ring with unit. Given an additive functor $F : A\text{-Mod} \rightarrow B\text{-Mod}$ we can form the triangulated functors

$$\begin{aligned} LF : D^-(A\text{-Mod}) &\simeq \text{Ho}^-(A\text{-Proj}) \xrightarrow{F} \text{Ho}^-(B\text{-Mod}) \longrightarrow D^-(B\text{-Mod}) \\ RF : D^+(A\text{-Mod}) &\simeq \text{Ho}^+(A\text{-Inj}) \xrightarrow{F} \text{Ho}^+(B\text{-Mod}) \longrightarrow D^+(B\text{-Mod}) \end{aligned}$$

where the first equivalences are quasi-inverses of the ones given in Theorem 1.16.

When F is right exact and $M \in A\text{-Mod}$, $LF(M)$ is a complex whose terms in positive degrees are zero and which satisfies $H^0(LF(M)) \simeq F(M)$. In that case we refer to LF as the *left derived functor* of F . Similarly, when F is left exact, RF is the *right derived functor* and it satisfies $H^0(RF(M)) \simeq F(M)$ for every A -module M . Historically, only the lower (resp. higher) cohomology groups of $LF(M)$ (resp. $RF(M)$) were considered, not the complex itself. They yield additive functors between the module categories which we will denote by $L^n F := H^{-n} \circ LF$ and $R^n F = H^n \circ RF$. Note that if F is exact then $LF \simeq RF \simeq F$.

Example 1.18. (a) Given an A -module M , the functor $\text{Hom}_A(M, -)$ is an additive, covariant, left exact functor from $A\text{-Mod}$ to $\mathbb{Z}\text{-Mod}$. It yields a right derived functor

$$R\text{Hom}_A(M, -) : D^+(A\text{-Mod}) \longrightarrow D^+(\mathbb{Z}\text{-Mod}).$$

Given another A -module N , the group of degree n extensions between M and N is by definition $\text{Ext}_A^n(M, N) := R^n \text{Hom}_A(M, N)$. Therefore we have

$$\begin{aligned} \text{Ext}_A^n(M, N) &= H^n(R\text{Hom}_A(M, N)) \\ &= H^n(\text{Hom}_A^\bullet(M, I)) && \text{for } I \text{ an injective resolution of } N \\ &= \text{Hom}_{\text{Ho}(A\text{-Mod})}(M, I[n]) && \text{by (1.1)} \\ &= \text{Hom}_{D(A\text{-Mod})}(M, N[n]) && \text{by Proposition 1.15} \end{aligned}$$

(recall that $I \simeq N$ in $D(A\text{-Mod})$). Note that more generally, the definition of $R\text{Hom}_A(M, -)$ makes sense whenever M is a bounded above complex of A -modules.

(b) Given a right A -module M , the functor $M \otimes_A - : A\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is a right exact functor (not exact if M is not flat). The corresponding left derived functor is denoted by

$$M \overset{L}{\otimes}_A - : D^-(A\text{-Mod}) \longrightarrow D^-(\mathbb{Z}\text{-Mod}).$$

It is defined by $M \overset{L}{\otimes}_A N := M \otimes_A P$ where P is any projective resolution of N . This left derived functor yields the torsion groups $\text{Tor}_n^A(M, N) = H^{-n}(M \overset{L}{\otimes}_A N)$ for any non-negative integer n .

1.6. Truncation and applications. Given a complex of A -modules C , one can consider the following truncations of C :

$$\begin{aligned}\tau_{\geq n}(C) &= \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \text{CoKer } d_{n-1} \longrightarrow C_{n+1} \longrightarrow C_{n+2} \longrightarrow \cdots \\ \tilde{\tau}_{\geq n}(C) &= \cdots \longrightarrow 0 \longrightarrow \text{Im } d_{n-1} \longrightarrow C_n \longrightarrow C_{n+1} \longrightarrow C_{n+2} \longrightarrow \cdots \\ \tau_{\leq n}(C) &= \cdots \longrightarrow C_{n-2} \longrightarrow C_{n-1} \longrightarrow \text{Ker } d_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \\ \tilde{\tau}_{\leq n}(C) &= \cdots \longrightarrow C_{n-2} \longrightarrow C_{n-1} \longrightarrow C_n \longrightarrow \text{Im } d_n \longrightarrow 0 \longrightarrow \cdots\end{aligned}$$

We also set $\tau_{< n} = \tau_{\leq n-1}$ (resp. $\tau_{> n} = \tau_{\geq n+1}$) and define analogously $\tilde{\tau}_{< n}$ and $\tilde{\tau}_{> n}$. The truncated complexes are constructed so that they have the same cohomology up to (or starting from) a given degree. For example,

$$H^k(\tau_{\geq n}(C)) = H^k(\tilde{\tau}_{\geq n}(C)) = \begin{cases} H^k(C) & \text{if } k \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition summarizes the relations and properties of truncation operations.

Proposition 1.19. *Let C be a complex of A -modules.*

- (i) *The natural map $\tau_{\leq n}(C) \longrightarrow \tilde{\tau}_{\leq n}(C)$ is a quasi-isomorphism.*
- (ii) *There are short exact sequences of complexes*

$$\begin{aligned}0 \longrightarrow \tilde{\tau}_{< n}(C) \longrightarrow \tau_{\leq n}(C) \longrightarrow H^n(C)[-n] \longrightarrow 0 \\ 0 \longrightarrow \tau_{\leq n}(C) \longrightarrow C \longrightarrow \tilde{\tau}_{> n}(C) \longrightarrow 0\end{aligned}$$

- (iii) *The truncation operations are functorial. They preserve the class of acyclic complexes, null-homotopic complexes, null-homotopic morphisms and quasi-isomorphisms.*

Note that similar statements are obtained by reversing the arrows and swapping $\tau_{\leq n}$ and $\tilde{\tau}_{\leq n}$ with $\tau_{\geq n}$ and $\tilde{\tau}_{\geq n}$ respectively.

We deduce that the truncation functors induce functors at the level of the homotopy and derived categories. The short exact sequences in (ii) together with the quasi-isomorphism in (i) yield distinguished triangles in $\text{D}(A\text{-Mod})$

$$\begin{aligned}\tau_{< n}(C) \longrightarrow \tau_{\leq n}(C) \longrightarrow H^n(C)[-n] \rightsquigarrow \\ \tau_{\leq n}(C) \longrightarrow C \longrightarrow \tau_{> n}(C) \rightsquigarrow 0\end{aligned} \tag{1.2}$$

Consequently, a complex C whose cohomology vanishes outside the degrees $n, n+1, \dots, m$ is quasi-isomorphic to its truncation $\tau_{\geq m}\tau_{\leq n}(C)$, hence to a bounded complex whose terms are zero outside the degrees $n, n+1, \dots, m$. In particular a complex with a unique non-zero cohomology group is quasi-isomorphic to a module shifted in that degree. Another consequence is that the category $\text{D}^b(A\text{-Mod})$ is the full subcategory of $\text{D}(A\text{-Mod})$ with objects satisfying $H^i(C) = 0$ for $|i| \gg 0$.

Another example that will often appear in these notes is the case of a complex C with only two non-zero cohomology groups, say $H^0(C)$ and $H^n(C)$. Then C fits into a distinguished triangle

$$\underbrace{H^0(C)[0]}_{\simeq \tau_{\leq 0}(C)} \longrightarrow C \longrightarrow \underbrace{H^n(C)[-n]}_{\simeq \tau_{> 0}(C)} \rightsquigarrow$$

which means that C is quasi-isomorphic to the cone of the map $H^n(C)[-n] \rightarrow H^0(C)[1]$. This implies that C is determined by $H^0(C)$, $H^n(C)$ and by an element of $\text{Ext}_A^{n+1}(H^n(C), H^0(C)) = \text{Hom}_{\mathbf{D}(A\text{-Mod})}(H^n(C), H^0(C)[n+1])$ (see Example 1.18).

1.7. Examples of derived categories. When A is a semisimple algebra, every injective or surjective map between modules splits. Consequently one can easily show that every complex of A -modules is quasi-isomorphic to the complex formed by its cohomology groups (with zero differential). In other words, the functor $C \mapsto H^\bullet(C)$ induces an equivalence between $\mathbf{D}(A\text{-Mod})$ and the category of \mathbb{Z} -graded modules. This is in particular the case for the derived category of k -vector spaces $D(k\text{-Mod})$ when k is a field, or more generally for the derived category of kG -modules $D(kG\text{-Mod})$ when G is a finite group whose order is invertible in k .

A ring A is said to be *hereditary* if $\text{Ext}_A^n(-, -) = 0$ for all $n \geq 2$. In that case, every bounded complex is again quasi-isomorphic to its cohomology, but not in a canonical way, and the functor $C \mapsto H^\bullet(C)$ is not faithful in general. This is for example the case for the bounded derived category of abelian groups $D^b(\mathbb{Z}\text{-Mod})$. Another example is the bounded derived category of $\mathbb{Z}_\ell G$ -modules $D^b(\mathbb{Z}_\ell G\text{-Mod})$ when G is a finite group whose order is prime to ℓ .

Exercise 1.20. When A is hereditary, show using (1.2) and Example 1.18.a that every bounded complex is quasi-isomorphic to its cohomology.

1.8. The stable category $A\text{-stab}$. Let k be a field. Throughout this section we will assume that A is a finite dimensional k -algebra. In particular every finite dimensional A -module M has a projective cover, which we will denote by P_M (A is said to be *semiperfect*). We shall also assume that A is *symmetric* (i.e. A is isomorphic to its dual $A^* = \text{Hom}_k(A, k)$ as an (A, A) -bimodule). In that case A -modules are projective if and only if they are injective (see for example [2, §1.6]). Consequently any finite dimensional A -module admits an injective resolution. A typical example of such an algebra in these lectures is the group algebra kG of a finite group G .

Definition 1.21. The *stable category of finitely generated A -modules*, denoted by $A\text{-stab}$, is the category with

- objects: finitely generated A -modules (same as $A\text{-mod}$),
- morphisms: $\underline{\text{Hom}}_A(M, N) := \text{Hom}_A(M, N) / \approx$ where $f \approx g$ if and only if $f - g$ factors through a projective module.

In particular, in the stable category any projective module is isomorphic to zero. There is a canonical additive (in fact k -linear) functor $A\text{-mod} \rightarrow A\text{-stab}$, making $A\text{-stab}$ into an additive (k -linear) category. This category has an additional triangulated structure, as we will see below.

Given a finite dimensional A -module M , we define the *Heller operator* Ω by

$$\Omega M = \text{Ker}(P_M \twoheadrightarrow M).$$

We then define inductively $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for $n \geq 1$ with the convention that $\Omega^0(M)$ is the minimal submodule of M such that $M/\Omega^0(M)$ is projective.

Exercise 1.22. Check that $\Omega^n(M)$ is well defined up to isomorphism. Show that Ω is functorial in $A\text{-stab}$ (but not in $A\text{-mod}$).

Similarly, we set $\Omega^{-1}M = \text{Coker}(M \hookrightarrow I_M)$ where I_M is an injective hull of M . One can readily check that $(\Omega^{-1}M)^* \simeq \Omega M^*$ as right A -modules and more generally that $(\Omega^{-n}M)^* \simeq \Omega^n M^*$ for all $n \in \mathbb{Z}$.

Proposition 1.23. *Let M and N be finitely generated A -modules.*

- (i) $M \simeq N$ in $A\text{-stab}$ if and only if there exist finitely generated projective modules P and Q such that $M \oplus P \simeq N \oplus Q$ in $A\text{-mod}$.
- (ii) If M and N are indecomposable non-projective modules with M or N being simple, then $\text{Hom}_A(M, N) \xrightarrow{\sim} \underline{\text{Hom}}_A(M, N)$.
- (iii) If $n > 0$ then

$$\underline{\text{Hom}}_A(\Omega^n M, N) \simeq \underline{\text{Hom}}_A(M, \Omega^{-n} N) \simeq \text{Ext}_A^n(M, N).$$

Proof. (i) If $M \simeq N$ in $A\text{-stab}$ then there exists morphisms $M \xrightarrow{f} N \xrightarrow{g} M$ and projective modules R and T such that $gf - 1_M$ and $fg - 1_N$ factor through R and T respectively. Write $gf - 1_M = \Psi \circ \Phi$ with $\Phi : M \rightarrow R$ and $\Psi : R \rightarrow M$ and consider the following morphisms

$$\begin{array}{ccc} & (f, \Phi) & \\ & \curvearrowright & \\ M & & N \oplus R \\ & \curvearrowleft & \\ & (g - \Psi) & \end{array}$$

which satisfy $(g - \Psi) \circ (f, \Phi) = 1_M$. This shows that M is a direct summand of $N \oplus R$. Similarly, N is a direct summand of $M \oplus T$ and we can invoke the Krull-Schmidt Theorem to conclude.

For (ii) it is enough to see that no injective (resp. surjective) morphism can factor through a projective (hence injective) module if both M and N have no non-trivial projective summands.

For (iii) we start with a minimal projective resolution of M coming from the sequence

$$\cdots \rightarrow P_{\Omega^n M} \rightarrow \cdots \rightarrow P_{\Omega M} \rightarrow P_M \twoheadrightarrow M.$$

By definition (see Example 1.18), the degree n extension group $\text{Ext}_A^n(M, N)$ is the quotient of the subgroup of maps f in $\text{Hom}_A(P_{\Omega^n M}, N)$ such that $\Omega^{n+1}M \subset \text{Ker } f$ by the maps which factor through $P_{\Omega^n M} \rightarrow P_{\Omega^{n-1}M}$. In particular we have a well defined surjective map

$$\begin{aligned} \text{Ext}_A^n(M, N) &\longrightarrow \underline{\text{Hom}}_A(\Omega^n M, N) \\ f &\longmapsto \bar{f} : \underbrace{P_{\Omega^n M} / \Omega^{n+1}M}_{\Omega^n M} \longrightarrow N \end{aligned}$$

Now if $\bar{f} = 0$ there exists a projective (hence injective) module P such that \bar{f} factors through $\Omega^n M \rightarrow P \rightarrow N$. Since P is injective and $\Omega^n M$ is a submodule of $P_{\Omega^{n-1}M}$, the map $\Omega^n M \rightarrow P$ can be extended to a map $P_{\Omega^{n-1}M} \rightarrow P$ in the following commutative diagram

$$\begin{array}{ccc} \Omega^n M & \xrightarrow{\bar{f}} & N \\ \downarrow & \dashrightarrow & \uparrow \\ P_{\Omega^{n-1}M} & \dashrightarrow & P \end{array}$$

Therefore f factors through $P_{\Omega^{n-1}M}$ and hence it is zero in $\text{Ext}_A^n(M, N)$. The case of $\Omega^{-n}N$ is similar. \square

Consequently, any short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ in $A\text{-mod}$ yields a sequence $U \rightarrow V \rightarrow W \rightarrow \Omega^{-1}U$ in $A\text{-stab}$. This, in turn, endows $A\text{-stab}$ with a structure of triangulated category, with suspension functor (or shift) Ω^{-1} , such that the images of short exact sequences of $A\text{-mod}$ are distinguished triangles in $A\text{-stab}$.

The triangulated structure appears in a more natural way from the bounded derived category of $A\text{-mod}$. We say that a complex of A -modules C is *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projective modules. We denote by $A\text{-perf}$ the full subcategory of $\text{D}^b(A\text{-mod})$ of perfect complexes (it is a thick subcategory, *i.e.* stable under direct summands and cones).

Theorem 1.24 (Rickard [61]). *The natural functor $A\text{-mod} \rightarrow \text{D}^b(A\text{-mod})$ induces an equivalence of triangulated categories*

$$\begin{array}{ccc} A\text{-stab} & \xrightarrow{\sim} & \text{D}^b(A\text{-mod})/A\text{-perf} \\ & & \downarrow \text{inverting maps whose cone} \\ & & \text{is a perfect complex} \end{array}$$

Remark 1.25. Using a minimal projective resolution of $M[-n]$ which we truncate in degrees above zero, we obtain a complex

$$C = (\cdots 0 \rightarrow P_{\Omega^{n-1}M} \rightarrow \cdots \rightarrow P_{\Omega M} \rightarrow P_M \rightarrow M \rightarrow 0 \cdots)$$

Then $C \simeq \Omega^n M[0]$ in $D^b(A\text{-mod})$. On the other hand, since all the terms of C are projective modules except the term in degree n we have $C \simeq M[-n]$ in $D^b(A\text{-mod})/A\text{-perf}$. This shows that

$$M[-n] \simeq \Omega^n M \quad \text{in } D^b(A\text{-mod})/A\text{-perf}$$

and proves the compatibility of the suspension functors (Ω^{-1} and $[1]$) under the equivalence given by Rickard’s theorem.

2. Varieties and cohomology

The aim of this chapter is to introduce the geometric tools that we will need to construct the representations of finite reductive groups. They will be obtained from linear invariants (cohomology groups or cohomology complexes) of algebraic varieties acted on by finite groups:

$$\begin{array}{ccc} \text{Algebraic variety } \mathbb{X} & & \text{Family of vector spaces } H^i(\mathbb{X}) \\ + \text{ action of} & \xrightarrow{\text{cohomology}} & \text{or complex of vector spaces } R\Gamma(\mathbb{X}) \\ \text{a finite group } G & & + \text{ linear action of } G. \end{array}$$

For example, if \mathbb{X} is a finite set acted on by G , then we can form the permutation module $\Lambda\mathbb{X}$ over any ring Λ .

Although this construction makes sense for any abstract finite group G , it will be particularly suited for finite reductive groups, since in that case the algebraic variety \mathbb{X} will be constructed from the underlying algebraic group (see §3 for the definition of Deligne–Lusztig varieties).

Since we will be interested in modular representations (with coefficients in fields of positive characteristic) the language of derived categories and derived functors introduced in the previous chapter will be particularly suited for our purpose:

$$\begin{array}{ccccccc} A\text{-modules } M, N & \rightsquigarrow & P \text{ a projective} & \rightsquigarrow & \left\{ \begin{array}{l} P \otimes_A N \\ \text{Hom}_A^\bullet(P, N) \end{array} \right\} & \rightsquigarrow & \left\{ \begin{array}{l} \text{Tor}_i^A(M, N) \\ \text{Ext}_A^i(M, N) \end{array} \right\} \\ & & \text{resolution of } N & & & & \\ \mathcal{F} \text{ a sheaf on } \mathbb{X} & \rightsquigarrow & \mathcal{P} \text{ a flabby} & \rightsquigarrow & \Gamma(\mathcal{P}) & \rightsquigarrow & H^i(\mathbb{X}, \mathcal{F}) \\ & & \text{resolution of } \mathcal{F} & & & & \\ & & & & \downarrow & & \downarrow \\ & & & & \text{derived} & & \text{standard} \\ & & & & \text{setting} & & \text{setting} \end{array}$$

The definition of étale or ℓ -adic cohomology would go far beyond the scope of these notes. For the reader interested in the topic we recommend reading Deligne’s notes [19], or the excellent textbook by Milne [59]. For a more representation-theoretic perspective, most of the properties listed in this chapter are also addressed in [15, Appendix A3] and in [3, Appendix A].

2.1. Definition and first properties. Let \mathbb{X} be a quasi-projective variety over $\overline{\mathbb{F}}_p$ and G be a finite group acting on \mathbb{X} . We fix a prime number $\ell \neq p$ and an ℓ -modular system (K, \mathcal{O}, k) such that K is a finite extension of \mathbb{Q}_ℓ . Finally, we denote by Λ any ring among K , \mathcal{O} and k . We will be interested in representations of G over Λ .

The theory of étale cohomology of sheaves on \mathbb{X} produces two complexes of $\mathcal{O}G$ -modules $R\Gamma(\mathbb{X}, \mathcal{O})$ and $R\Gamma_c(\mathbb{X}, \mathcal{O})$, unique up to quasi-isomorphism, called the *cohomology complex* of \mathbb{X} and the *cohomology complex with compact support* of \mathbb{X} . By extension of scalars, we also have complexes

$$R\Gamma(\mathbb{X}, \Lambda) := R\Gamma(\mathbb{X}, \mathcal{O}) \otimes_{\mathcal{O}}^L \Lambda \quad \text{and} \quad R\Gamma_c(\mathbb{X}, \Lambda) := R\Gamma_c(\mathbb{X}, \mathcal{O}) \otimes_{\mathcal{O}}^L \Lambda.$$

When $\Lambda = K$ (resp. $\Lambda = k$) we will refer to these complexes as the ℓ -adic cohomology complexes (resp. the mod- ℓ cohomology complexes). The groups

$$H^i(\mathbb{X}, \Lambda) := H^i(R\Gamma(\mathbb{X}, \Lambda)) \quad \text{and} \quad H_c^i(\mathbb{X}, \Lambda) := H^i(R\Gamma_c(\mathbb{X}, \Lambda))$$

are the cohomology groups (or cohomology groups with compact support) with coefficients in Λ .

The cohomology is functorial: if $f : \mathbb{Y} \rightarrow \mathbb{X}$ is a G -equivariant morphism of algebraic varieties then it induces a morphism in $\mathbf{D}(\Lambda G\text{-Mod})$

$$f^* : R\Gamma(\mathbb{X}, \Lambda) \rightarrow R\Gamma(\mathbb{Y}, \Lambda)$$

between the cohomology complexes of \mathbb{X} and \mathbb{Y} . If in addition f is proper (e.g. f is a finite morphism), the same holds for the cohomology complexes with compact support.

The cohomology complexes are “small”: the ΛG -modules $H^i(\mathbb{X}, \Lambda)$ and $H_c^i(\mathbb{X}, \Lambda)$ are finitely generated over Λ . Moreover, they vanish for $i < 0$ and $i > 2 \dim \mathbb{X}$. Consequently, $R\Gamma(\mathbb{X}, \Lambda)$ and $R\Gamma_c(\mathbb{X}, \Lambda)$ are quasi-isomorphic to complexes of (Λ -free, see Remark below) finitely generated ΛG -modules with terms in degrees $0, 1, \dots, 2 \dim \mathbb{X}$.

Remark 2.1. The Λ -modules $H^i(\mathbb{X}, \mathcal{O})$ are not free in general, but $H^0(\mathbb{X}, \mathcal{O})$ is. If C is a Λ -free resolution of a given representative of $R\Gamma(\mathbb{X}, \Lambda)$, then C is quasi-isomorphic to $\tau_{>0}(\tau_{\leq 2 \dim \mathbb{X}}(C))$ and the terms of the latter complex are Λ -free since $H^0(\mathbb{X}, \Lambda)$ is (see also Proposition 6.5).

The following theorem, due to Rickard [62] (see also [64]), gives the most satisfactory representative for the cohomology complex of \mathbb{X} from a representation-theoretic perspective.

Theorem 2.2 (Rickard [62]). *$R\Gamma(\mathbb{X}, \Lambda)$ and $R\Gamma_c(\mathbb{X}, \Lambda)$ are quasi-isomorphic to bounded complexes whose terms are direct summands of finite sums of permutation modules $\Lambda G/\text{Stab}_G(x)$ for $x \in \mathbb{X}$.*

Idea of proof. Given a sheaf \mathcal{F} on \mathbb{X} , we can construct the complex $R\Gamma(\mathbb{X}, \mathcal{F})$ from the global sections of the Godement resolution. This resolution involves Λ -modules of the form $\prod_{x \in \mathbb{X}} \mathcal{F}_x$ on which G acts naturally by permuting the stalks.

This suggests that $R\Gamma(\mathbb{X}, \Lambda)$ has a representative in the category generated by the permutation modules $\Lambda G/\text{Stab}_G(x)$ and we can invoke the finiteness property of the cohomology groups to conclude. \square

For $\Lambda = \mathcal{O}$ or k , we denote by $\Lambda G\text{-perm}$ the category of finitely generated ℓ -permutation ΛG -modules. This is the smallest full subcategory of $\Lambda G\text{-mod}$ closed under direct summands and containing the permutation modules. As a consequence of Theorem 2.2, there exist bounded complexes $\widetilde{R}\Gamma(\mathbb{X}, \mathcal{O})$ and $\widetilde{R}\Gamma_c(\mathbb{X}, \mathcal{O})$ in $\text{Ho}^b(\mathcal{O}G\text{-perm})$ of finitely generated ℓ -permutation modules which are quasi-isomorphic to $R\Gamma(\mathbb{X}, \mathcal{O})$ and $R\Gamma_c(\mathbb{X}, \mathcal{O})$ respectively. The following particular case will be intensively used in these notes.

Corollary 2.3. *Assume that for all $x \in \mathbb{X}$ the order of the group $\text{Stab}_G(x)$ is invertible in Λ . Then $R\Gamma(\mathbb{X}, \Lambda)$ and $R\Gamma_c(\mathbb{X}, \Lambda)$ are perfect complexes.*

For affine varieties, the vanishing property of cohomology groups can be refined. If \mathbb{X} is an affine variety of pure dimension (*i.e.* all the irreducible components have the same dimension) then

- $H^i(\mathbb{X}, \Lambda) = 0$ if $i > \dim \mathbb{X}$;
- $H_c^i(\mathbb{X}, \Lambda) = 0$ if $i < \dim \mathbb{X}$.

Consequently, $R\Gamma(\mathbb{X}, \Lambda)$ (resp. $R\Gamma_c(\mathbb{X}, \Lambda)$) has a representative with terms in degrees $0, \dots, \dim \mathbb{X}$ (resp. $\dim \mathbb{X}, \dots, 2 \dim \mathbb{X}$).

We conclude this section by the relation between the compact and non-compact versions of the cohomology complexes. There is a natural map $R\Gamma_c(\mathbb{X}, \Lambda) \rightarrow R\Gamma(\mathbb{X}, \Lambda)$ which is an isomorphism when \mathbb{X} is a projective variety (compare with the case of affine varieties above). In addition, the cohomology complexes of smooth varieties are mutually dual.

Theorem 2.4 (Poincaré–Verdier [19]). *Assume that \mathbb{X} is smooth of pure dimension d . Then*

$$R\Gamma(\mathbb{X}, \Lambda)[2d] \simeq R\text{Hom}_\Lambda(R\Gamma_c(\mathbb{X}, \Lambda), \Lambda)$$

in $D^b(\Lambda G\text{-mod})$.

\downarrow *not necessary to right derive if one works with a representative of $R\Gamma_c(\mathbb{X}, \Lambda)$ with Λ -free terms*

2.2. Tools for computing $R\Gamma_c(\mathbb{X}, \Lambda)$. Unless otherwise stated, all the isomorphisms considered in this section are in the category $D^b(\Lambda G\text{-mod})$ for Λ a ring among K , \mathcal{O} and k .

Theorem 2.5 (Künneth formula). *The cohomology of a product of varieties is given by*

$$R\Gamma_c(\mathbb{X} \times \mathbb{Y}, \Lambda) \simeq R\Gamma_c(\mathbb{X}, \Lambda) \overset{L}{\otimes} R\Gamma_c(\mathbb{Y}, \Lambda).$$

Theorem 2.6 (Open-closed situation). *Let $\mathbb{U} \subset \mathbb{X}$ be an open G -stable subvariety of \mathbb{X} , and $\mathbb{Z} = \mathbb{X} \setminus \mathbb{U}$ be the closed complement. There is a distinguished triangle*

$$R\Gamma_c(\mathbb{U}, \Lambda) \rightarrow R\Gamma_c(\mathbb{X}, \Lambda) \rightarrow R\Gamma_c(\mathbb{Z}, \Lambda) \rightsquigarrow$$

in $D^b(\Lambda G\text{-mod})$. If in addition \mathbb{U} is closed, then this triangle splits and $R\Gamma_c(\mathbb{X}, \Lambda) \simeq R\Gamma_c(\mathbb{U}, \Lambda) \oplus R\Gamma_c(\mathbb{Z}, \Lambda)$.

Taking the cohomology of this distinguished triangle yields a long exact sequence of cohomology groups

$$\cdots \longrightarrow H_c^i(\mathbb{U}) \longrightarrow H_c^i(\mathbb{X}) \longrightarrow H_c^i(\mathbb{Z}) \longrightarrow H_c^{i+1}(\mathbb{U}) \longrightarrow \cdots$$

Theorem 2.7. *The cohomology of the affine space of dimension n is given by*

$$R\Gamma_c(\mathbb{A}_n, \Lambda) \simeq \Lambda[-2n].$$

These three results are enough to compute the cohomology of a large class of varieties. We give below the examples of projective spaces and tori.

Example 2.8. (a) Write $\mathbb{P}_1 = \mathbb{A}_1 \sqcup \{pt\}$. Theorems 2.6 and 2.7 yield a distinguished triangle

$$\begin{array}{ccccccc} R\Gamma_c(\mathbb{A}_1, \Lambda) & \longrightarrow & R\Gamma_c(\mathbb{P}_1, \Lambda) & \longrightarrow & R\Gamma_c(\{pt\}, \Lambda) & \rightsquigarrow & \\ \Lambda[-2] & \longrightarrow & R\Gamma_c(\mathbb{P}_1, \Lambda) & \longrightarrow & \Lambda[0] & \longrightarrow & \Lambda[-1] \\ & & & & \downarrow & & \\ & & & & \text{zero map since} & & \\ & & & & \text{Ext}_{\Lambda G}^{-1}(\Lambda, \Lambda) = 0 & & \end{array}$$

This shows that $R\Gamma_c(\mathbb{P}_1, \Lambda) \simeq \Lambda[0] \oplus \Lambda[-2]$.

More generally, $R\Gamma_c(\mathbb{P}_n, \Lambda) \simeq \Lambda[0] \oplus \Lambda[-2] \oplus \cdots \oplus \Lambda[-2n]$. This method works for any variety paved by affine spaces, e.g. the flag varieties.

(b) Let \mathbb{G}_m be the one-dimensional torus $\mathbb{G}_m = \mathbb{A}_1 \setminus \{0\}$ acted on by multiplication by the group μ_n of n -th roots of unity in $\overline{\mathbb{F}}_p$. Again, we have a distinguished triangle

$$\begin{array}{ccccccc} R\Gamma_c(\mathbb{G}_m, \Lambda) & \longrightarrow & R\Gamma_c(\mathbb{A}_1, \Lambda) & \longrightarrow & R\Gamma_c(\{pt\}, \Lambda) & \rightsquigarrow & \\ R\Gamma_c(\mathbb{G}_m, \Lambda) & \longrightarrow & \Lambda[-2] & \longrightarrow & \Lambda[0] & \rightsquigarrow & \\ & & & & \downarrow & & \\ & & & & \text{element of } \text{Ext}_{\Lambda \mu_n}^2(\Lambda, \Lambda) \simeq H^2(\mu_n, \Lambda) & & \end{array}$$

The long exact sequence in cohomology gives $H_c^\bullet(\mathbb{G}_m, \Lambda) \simeq \Lambda[-1] \oplus \Lambda[-2]$ but we need more information to compute the cohomology complex.

Since μ_n acts freely, the complex $R\Gamma_c(\mathbb{G}_m, \mathcal{O})$ is perfect by Corollary 2.3. Therefore it is quasi-isomorphic to $0 \longrightarrow P \xrightarrow{d} Q \longrightarrow 0$ with P and Q being two finitely-generated projective modules in degrees 1 and 2 respectively (\mathbb{G}_m is an irreducible affine variety of dimension 1). From the previous computation we deduce that $\text{Ker } d \simeq \text{CoKer } d \simeq \mathcal{O}$. Consequently, the trivial module k is in the head of Q and the projective cover P_k of k (as an $\mathcal{O}\mu_n$ -module) is a direct summand of Q . In other words, $R\Gamma_c(\mathbb{G}_m, \mathcal{O})$ has a representative of the form

$$0 \longrightarrow P \xrightarrow{d} Q' \oplus P_k \longrightarrow 0$$

with $Q' \subset \text{Im } d$. This implies that the composition $P \xrightarrow{d} Q' \oplus P_k \longrightarrow Q'$ is surjective. Since Q' is projective, it must split and we can write $P \simeq P' \oplus Q'$ such

that the restriction of d to Q' is the identity. This shows that the previous complex is homotopy equivalent to

$$0 \longrightarrow P' \xrightarrow{d|_{P'}} P_k \longrightarrow 0.$$

To determine P' we can either use the kernel of d or the fact that in the Grothendieck group $[R\Gamma_c(\mathbb{G}_m, \mathcal{O})] = [P_k] - [P'] = \sum (-1)^i [H_c^i(\mathbb{G}_m, \mathcal{O})] = 0$ which forces $P' \simeq P_k$. Finally,

$$R\Gamma_c(\mathbb{G}_m, \mathcal{O}) \simeq (\cdots 0 \longrightarrow P_k \longrightarrow P_k \longrightarrow 0 \cdots).$$

Using Theorem 2.5 we can also compute the cohomology of a higher dimensional torus by $R\Gamma_c((\mathbb{G}_m)^r, \mathcal{O}) \simeq R\Gamma_c(\mathbb{G}_m, \mathcal{O})^{\otimes r}$.

Exercise 2.9. Let ζ be a primitive n -th root of 1. Show that

$$R\Gamma_c(\mathbb{G}_m, \Lambda) \simeq (\cdots 0 \longrightarrow \Lambda\mu_n \xrightarrow{\zeta-1} \Lambda\mu_n \longrightarrow 0 \cdots).$$

2.3. Group action. Recall that G is a finite group acting on the quasi-projective variety \mathbb{X} . In this section we discuss the relation between the cohomology complexes of \mathbb{X} , $G \backslash \mathbb{X}$ and \mathbb{X}^G . Further details can be found in [62], [64] or [15, A3.15].

Theorem 2.10. *Assume that for all $x \in \mathbb{X}$, the order of the group $\text{Stab}_G(x)$ is invertible in Λ (in particular $R\Gamma_c(\mathbb{X}, \Lambda)$ is perfect). Then*

$$R\Gamma_c(G \backslash \mathbb{X}, \Lambda) \simeq \Lambda \otimes_{\Lambda G}^L R\Gamma_c(\mathbb{X}, \Lambda)$$

in $\mathbf{D}^b(\Lambda\text{-mod})$.

Sketch of proof. Let $\pi_*\Lambda$ be the push-forward of the constant sheaf Λ along the quotient map $\pi : \mathbb{X} \rightarrow G \backslash \mathbb{X}$. Since π is finite we have $\pi_* = \pi_!$ and therefore

$$R\Gamma_c(\mathbb{X}, \Lambda) \simeq R\Gamma_c(G \backslash \mathbb{X}, \pi_*\Lambda).$$

Taking the coinvariants we get, using the projection formula (see for example [59, VI.8.14])

$$\Lambda \otimes_{\Lambda G}^L R\Gamma_c(\mathbb{X}, \Lambda) \simeq \Lambda \otimes_{\Lambda G}^L R\Gamma_c(G \backslash \mathbb{X}, \pi_*\Lambda) \simeq R\Gamma_c(G \backslash \mathbb{X}, \Lambda \otimes_{\Lambda G}^L \pi_*\Lambda).$$

It remains to check that the natural map $\Lambda \otimes_{\Lambda G}^L \pi_*\Lambda \rightarrow \Lambda$ is an isomorphism of sheaves. The stalk of $\pi_*\Lambda$ at a point x is the permutation module $\Lambda G / \text{Stab}_G(x)$, which is projective by assumption on the order of $\text{Stab}_G(x)$. Therefore the fact that the previous map is an isomorphism can be checked on the stalks with the usual tensor product. \square

When Λ is a field and $\ell \nmid |G|$ then ΛG is a semisimple algebra and complexes of ΛG -modules are quasi-isomorphic to their cohomology (see §1.7). Furthermore,

invariants and coinvariants are isomorphic as Λ -modules in that case and the previous theorem shows that

$$H_c^i(G \backslash \mathbb{X}, \Lambda) \simeq H_c^i(\mathbb{X}, \Lambda)^G \simeq \Lambda \otimes_{\Lambda G} H_c^i(\mathbb{X}, \Lambda).$$

Assume until the end of this section that Λ is either \mathcal{O} or k . Given $P \subset G$ an ℓ -subgroup of G and V an ℓ -permutation module, we denote by $\mathrm{Br}_P(V)$ the image of the invariants V^P in the coinvariants $k \otimes_{\Lambda P} V$. It induces an additive functor on the homotopy category of ℓ -permutation modules, which we will still denote by Br_P . We refer to [68, §27] for basic results on ℓ -permutation modules and the Brauer functor.

Theorem 2.11 (Rickard). *The inclusion $\mathbb{X}^P \hookrightarrow \mathbb{X}$ induces an isomorphism*

$$\mathrm{Br}_P(\underbrace{\tilde{R}\Gamma_c(\mathbb{X}, \Lambda)}_{\substack{\text{representative in } \mathrm{Ho}^b(\Lambda G\text{-perm)} \text{ (see §2.1)}}}) \xrightarrow{\sim} R\Gamma_c(\mathbb{X}^P, k)$$

in $\mathrm{D}^b(kN_G(P)\text{-mod})$

Sketch of proof. Assume for simplicity that P is a Sylow subgroup of G and that $P \simeq \mathbb{Z}/\ell\mathbb{Z}$. We consider the closed subvariety of \mathbb{X} defined by

$$\mathbb{X}_\ell = \{x \in \mathbb{X} \text{ such that } \ell \mid |\mathrm{Stab}_G(x)|\}.$$

Then $\mathbb{X}_\ell \simeq G \times_{N_G(P)} \mathbb{X}^P$, and hence $R\Gamma_c(\mathbb{X}_\ell, \Lambda) \simeq \mathrm{Ind}_{N_G(P)}^G R\Gamma_c(\mathbb{X}^P, \Lambda)$ by Theorems 2.5 and 2.10. Now by Corollary 2.3, the cohomology complex of $\mathbb{X} \setminus \mathbb{X}_\ell$ is perfect therefore its image by the Brauer functor is zero. Using the distinguished triangle in Theorem 2.6 we deduce that $\mathrm{Br}_P R\Gamma_c(\mathbb{X}, \Lambda) \simeq \mathrm{Br}_P R\Gamma_c(\mathbb{X}_\ell, \Lambda)$ which in turn is isomorphic to $R\Gamma_c(\mathbb{X}^P, k)$.

One can generalize this argument to any ℓ -subgroup P of G by considering a filtration of \mathbb{X} by subvarieties with respect to the size of the ℓ -part of the stabilizer of points. \square

2.4. Trace formula. Assume now that the quasi-projective variety \mathbb{X} is defined over \mathbb{F}_q , and denote by $F : \mathbb{X} \rightarrow \mathbb{X}$ the corresponding Frobenius endomorphism so that in particular $\mathbb{X}(\mathbb{F}_q) = \mathbb{X}^F$. The Frobenius induces a quasi-isomorphism on the complexes $R\Gamma(\mathbb{X}, \Lambda)$ and $R\Gamma_c(\mathbb{X}, \Lambda)$. All the quasi-isomorphisms and triangles listed in §2.2 are compatible with F . The corresponding action on the ℓ -adic cohomology groups can be computed partially from the number of \mathbb{F}_q -points of \mathbb{X} as follows.

Theorem 2.12 (Lefschetz trace formula [19]).

$$\#\mathbb{X}(\mathbb{F}_q) = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(F, H_c^i(\mathbb{X}, K)).$$

From this theorem we can for example derive a formula for the Euler characteristic of \mathbb{X} , given by

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i(\mathbb{X}, K) = - \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \#\mathbb{X}(\mathbb{F}_{q^n}) t^n. \quad (2.1)$$

Exercise 2.13. Show the latter formula (hint: use the eigenvalues of F (and F^n) on $H_c^i(\mathbb{X})$ and Theorem 2.12 for F^n).

Example 2.14. (a) For the affine space of dimension n , we have $H_c^\bullet(\mathbb{A}_n, K) \simeq K[-2n]$ and $\#\mathbb{A}_n(\mathbb{F}_q) = q^n$. Therefore F acts on the cohomology of \mathbb{A}_n by multiplication by q^n (the same actually holds for $H_c^\bullet(\mathbb{A}_n, \Lambda)$).

To take the action of F into account, we will write $R\Gamma_c(\mathbb{A}_n, \Lambda) \simeq \Lambda[-2n](n)$, and (n) will be referred to as a *Tate twist*. With this notation, the F -equivariant form of Poincaré–Verdier duality (see Theorem 2.4) is

$$R\Gamma(\mathbb{X}, \Lambda)[2d](-d) \simeq R\mathrm{Hom}_\Lambda(R\Gamma_c(\mathbb{X}, \Lambda), \Lambda)$$

for \mathbb{X} a smooth variety of pure dimension d .

(b) $H_c^\bullet(\mathbb{P}_1, \Lambda) \simeq \Lambda[0] \oplus \Lambda[-2](1)$ and $\#\mathbb{P}_1(\mathbb{F}_q) = 1 + q$.

(c) $H_c^\bullet(\mathbb{G}_m, \Lambda) \simeq \Lambda[-1] \oplus \Lambda[-2](1)$ and $\#\mathbb{G}_m(\mathbb{F}_q) = -1 + q$.

Formula (2.1) can be extended to the case of a group action. Assume that the action of G on \mathbb{X} is F -equivariant. Then the virtual character of the representation of G afforded by the ℓ -adic cohomology groups is

$$\sum (-1)^i \mathrm{Tr}(g, H_c^i(\mathbb{X}, K)) = - \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \#\mathbb{X}^{g^{F^n}} t^n. \quad (2.2)$$

Since this value is both an algebraic integer (left-hand side) and a rational number independent of ℓ , this shows in particular that it is an integer independent of ℓ . Note however that the individual cohomology groups could depend on ℓ , but it was proved recently that this is not the case for Deligne–Lusztig varieties (see [65]).

3. Deligne–Lusztig varieties and their cohomology

This chapter presents the construction by Deligne–Lusztig of algebraic varieties acted on by finite reductive groups [20]. We discuss several properties of the cohomology complexes of these varieties which we will use to deduce representation-theoretic results for finite reductive groups in the following chapters.

Starting from this chapter, \mathbb{G} will denote a connected reductive algebraic group defined over $\overline{\mathbb{F}}_p$, and $F : \mathbb{G} \rightarrow \mathbb{G}$ a Frobenius endomorphism defining an \mathbb{F}_q -structure on \mathbb{G} . The group of fixed points $\mathbb{G}(\mathbb{F}_q) := \mathbb{G}^F$ is a *finite reductive group*. Given a closed F -stable subgroup \mathbb{H} of \mathbb{G} , the corresponding finite group will be denoted by $H := \mathbb{H}(\mathbb{F}_q) = \mathbb{H}^F$. We refer to Meinolf Geck’s lecture notes [44] for more on these finite groups (see also the textbooks [16, 21]).

The first section of this chapter serves as a motivation for the introduction of Deligne–Lusztig varieties as a generalization of Harish-Chandra induction. It is intentionally very sketchy and will not be used in the rest of these notes.

3.1. Generalizing Harish-Chandra induction. The Harish-Chandra (or parabolic) induction and restriction functors provide an inductive approach to the construction of representations of finite reductive groups. Let \mathbb{P} be an F -stable parabolic subgroup of \mathbb{G} . It has a Levi decomposition $\mathbb{P} = \mathbb{L} \ltimes \mathbb{V}$ where \mathbb{L} is an F -stable Levi complement and \mathbb{V} is the unipotent radical of \mathbb{P} . The finite set $(\mathbb{G}/\mathbb{V})^F = G/V$ is endowed with a left action of G by left multiplication, and a right action of L by right multiplication (since L normalizes V). We can therefore consider the adjoint pair of exact functors

$$R_{LCP}^G = \Lambda G/V \otimes_{\Lambda L} - \quad \text{and} \quad {}^*R_{LCP}^G = \text{Hom}_{\Lambda G}(\Lambda G/V, -)$$

between the categories ΛL -mod and ΛG -mod. Dipper–Du [24] and Howlett–Lehrer [54] showed that these functors depend only on L and not on P , up to isomorphism. Therefore they will be simply denoted by R_L^G and ${}^*R_L^G$.

There are two issues when working with these functors. The first one is that not every representation occurs in a representation induced from a proper Levi subgroup (unlike the usual induction). The second problem is that an F -stable Levi subgroup \mathbb{L} of \mathbb{G} is not necessarily a Levi complement of an F -stable parabolic subgroup. Even though L exists, the finite set G/V might not. However, the variety \mathbb{G}/\mathbb{V} does and one can consider the following subvariety

$$\mathbb{Y}_{\mathbb{V}} := \{g\mathbb{V} \in \mathbb{G}/\mathbb{V} \mid g^{-1}F(g) \in \mathbb{V} \cdot F(\mathbb{V})\}$$

called the *parabolic Deligne–Lusztig variety* associated with \mathbb{V} . As in the case of the set G/V , it has a left action of G by left multiplication, and a right action of L by right multiplication. Consequently, the cohomology complex $R\Gamma_c(\mathbb{Y}_{\mathbb{V}}, \Lambda)$ is a bounded complex of (G, L) -bimodules and we can consider the triangulated functors

$$R_{LCP}^G = R\Gamma_c(\mathbb{Y}_{\mathbb{V}}, \Lambda) \otimes_{\Lambda L}^L - \quad \text{and} \quad {}^*R_{LCP}^G = R\text{Hom}_{\Lambda G}(R\Gamma_c(\mathbb{Y}_{\mathbb{V}}, \Lambda), -)$$

between the derived categories $D^b(\Lambda L\text{-mod})$ and $D^b(\Lambda G\text{-mod})$. They are called Deligne–Lusztig induction and restriction functors. When $F(\mathbb{V}) = \mathbb{V}$ (i.e. when \mathbb{P} is F -stable) then $\mathbb{Y}_{\mathbb{V}}$ is just the finite set $(\mathbb{G}/\mathbb{V})^F \simeq G/V$ and these functors coincide with the Harish-Chandra induction and restriction functors.

In these notes we will focus on the case where \mathbb{L} is a torus. In that case \mathbb{L} is \mathbb{G} -conjugate to a quasi-split torus \mathbb{T} and \mathbb{V} is determined by an element w in $N_{\mathbb{G}}(\mathbb{T})$. The corresponding variety will be denoted by $\mathbb{Y}(w)$ and studied in the next sections. This case corresponds to the original definition of Deligne–Lusztig [20] which was later generalized to the parabolic case in [55].

3.2. Bruhat decomposition. We fix a pair (\mathbb{T}, \mathbb{B}) where \mathbb{T} is a maximal torus of \mathbb{G} contained in a Borel subgroup \mathbb{B} . We assume that both \mathbb{T} and \mathbb{B} are F -stable (such pairs always exist and form a single G -conjugacy class). Such a torus is said to be *quasi-split*. In that case F acts on the Weyl group $W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ of \mathbb{G} .

Given $w \in W$ we define

- $\ell(w) = \dim \mathbb{B}w\mathbb{B} - \dim \mathbb{B} = \dim \mathbb{B}w\mathbb{B}/\mathbb{B}$, the *length* of w with respect to \mathbb{B} ,
- $S = \{w \in W \mid \ell(w) = 1\}$ the set of *simple reflections* of W with respect to \mathbb{B} .

Theorem 3.1 (Bruhat decomposition [21, §1]).

- (i) S generates W and (W, S) is a Coxeter system.
- (ii) \mathbb{G} decomposes as the disjoint union of Bruhat cells

$$\mathbb{G} = \bigsqcup_{w \in W} \mathbb{B}w\mathbb{B}.$$

$$(iii) \quad \mathbb{B}s\mathbb{B}w\mathbb{B} = \begin{cases} \mathbb{B}sw\mathbb{B} & \text{if } \ell(sw) > \ell(w) \\ \mathbb{B}w\mathbb{B} \sqcup \mathbb{B}sw\mathbb{B} & \text{otherwise.} \end{cases}$$

- (iv) The Schubert cell $\mathbb{B}w\mathbb{B}/\mathbb{B}$ is isomorphic to $\mathbb{A}_{\ell(w)}$ (the affine space of dimension $\ell(w)$).

Note that $\ell(w)$ coincides with the length corresponding to the Coxeter system (W, S) . Indeed, $\dim(\mathbb{B}s\mathbb{B}w\mathbb{B}) \leq \dim(\mathbb{B}s\mathbb{B} \times_{\mathbb{B}} \mathbb{B}w\mathbb{B}) = \ell(s) + \ell(w) + \dim \mathbb{B}$, therefore the inequality $\ell(sw) > \ell(w)$ forces $\ell(sw) = \ell(w) + 1$ by (iii). More generally $\ell(w)$ is the smallest integer r such that $w = s_1 s_2 \cdots s_r$ with $s_i \in S$.

The closure $\overline{\mathbb{B}w\mathbb{B}}$ in \mathbb{G} of a Bruhat cell is a closed subvariety of \mathbb{G} stable by left and right multiplication by \mathbb{B} . Therefore by (ii) it must be a finite union of Bruhat cells. We consider a partial order on W , called the *Bruhat order*, defined by $v \leq w$ if $\overline{\mathbb{B}v\mathbb{B}} \subset \overline{\mathbb{B}w\mathbb{B}}$ (or equivalently $\mathbb{B}v\mathbb{B} \subset \overline{\mathbb{B}w\mathbb{B}}$). Then by (ii) we have

$$\overline{\mathbb{B}w\mathbb{B}} = \bigsqcup_{v \leq w} \mathbb{B}v\mathbb{B}.$$

The singularities of these varieties are of considerable interest for the study of representations of semisimple Lie algebras and reductive groups, not only for finite reductive groups.

Example 3.2. (a) For the trivial element of W we have $\mathbb{B}1\mathbb{B} = \mathbb{B}$, which is a closed subvariety of \mathbb{G} . Therefore 1 is the unique minimal element for the Bruhat order.

(b) The variety \mathbb{G} is irreducible, therefore there exists a unique $w_0 \in W$ such that $\overline{\mathbb{B}w_0\mathbb{B}} = \mathbb{G}$. The element w_0 is the unique element of maximal length in W , and its length equals the dimension of the flag variety \mathbb{G}/\mathbb{B} , which is the number of positive roots of W . The element w_0 is also the unique maximal element for the Bruhat order.

Example 3.3. For $\mathbb{G} = \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ we have $\mathbb{G} = \mathbb{B} \sqcup \mathbb{B}s\mathbb{B}$, hence

$$\mathbb{B}s\mathbb{B} = \mathbb{G} \setminus \mathbb{B} = \left\{ \begin{pmatrix} * & * \\ \lambda & * \end{pmatrix} \mid \lambda \neq 0 \right\}.$$

3.3. Deligne–Lusztig varieties. Let $\mathbb{U} = R_u(\mathbb{B})$ be the unipotent radical of the Borel subgroup \mathbb{B} . We fix a set $\{\dot{w}\}_{w \in W}$ of representatives of W in $N_{\mathbb{G}}(\mathbb{T})$. The *Deligne–Lusztig varieties* associated to w are

$$\begin{aligned} \mathbb{Y}_{\mathbb{G}}(\dot{w}) &= \mathbb{Y}(\dot{w}) = \{g\mathbb{U} \in \mathbb{G}/\mathbb{U} \mid g^{-1}F(g) \in \mathbb{U}\dot{w}\mathbb{U}\}, \\ \mathbb{X}_{\mathbb{G}}(w) &= \mathbb{X}(w) = \{g\mathbb{B} \in \mathbb{G}/\mathbb{B} \mid g^{-1}F(g) \in \mathbb{B}w\mathbb{B}\}. \end{aligned}$$

The finite group $G = \mathbb{G}^F$ acts by left multiplication on both $\mathbb{X}(w)$ and $\mathbb{Y}(\dot{w})$. Furthermore, $\mathbb{T}^{\dot{w}F}$ acts by right multiplication on $\mathbb{Y}(\dot{w})$. Indeed, if $g^{-1}F(g) \in \mathbb{U}\dot{w}\mathbb{U}$ then using that \mathbb{T} normalizes \mathbb{U} we have, for every $t \in \mathbb{T}$

$$(gt)^{-1}F(gt) = t^{-1}g^{-1}F(g)F(t) \in \mathbb{U}t^{-1}\dot{w}F(t)\mathbb{U}.$$

Now $t \in \mathbb{T}^{\dot{w}F}$ if and only if ${}^{\dot{w}F}t = t$ which we can rewrite as $t^{-1}\dot{w}F(t) = \dot{w}$.

Using properties of Schubert cells, one can prove that both $\mathbb{X}(w)$ and $\mathbb{Y}(\dot{w})$ are smooth quasi-projective varieties of pure dimension $\ell(w)$. Furthermore, the canonical projection $\pi : \mathbb{G}/\mathbb{U} \rightarrow \mathbb{G}/\mathbb{B}$ induces a G -equivariant isomorphism

$$\mathbb{Y}(\dot{w})/\mathbb{T}^{\dot{w}F} \xrightarrow{\sim} \mathbb{X}(w). \quad (3.1)$$

As in the previous chapter we can consider the cohomology complexes attached to these varieties. The complex $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ is a bounded complex of finitely generated $(G, \mathbb{T}^{\dot{w}F})$ -bimodules, and $R\Gamma_c(\mathbb{X}(w), \Lambda)$ is a bounded complex of finitely generated G -modules. Since $\mathbb{T}^{\dot{w}F}$ acts freely on $\mathbb{Y}(\dot{w})$, we deduce from Theorem 2.10 and (3.1) that

$$R\Gamma_c(\mathbb{X}(w), \Lambda) \simeq R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda) \otimes_{\Lambda\mathbb{T}^{\dot{w}F}}^L \Lambda. \quad (3.2)$$

Example 3.4. If $\dot{w} = w = 1$ then $\mathbb{Y}(1) = \{g\mathbb{U} \mid g^{-1}F(g) \in \mathbb{U}\} = (\mathbb{G}/\mathbb{U})^F$. The latter is just a finite set isomorphic to G/U . Similarly, $\mathbb{X}(1) \simeq G/B$ and therefore the cohomology complexes of $\mathbb{Y}(1)$ and $\mathbb{X}(1)$ are given by a single permutation module in degree 0, namely

$$R\Gamma_c(\mathbb{Y}(1), \Lambda) \simeq \Lambda G/U[0] \quad \text{and} \quad R\Gamma_c(\mathbb{X}(1), \Lambda) \simeq \Lambda G/B[0].$$

Example 3.5. Let $\mathbb{G} = \mathrm{SL}_2(\overline{\mathbb{F}}_p)$ and F be the *standard* Frobenius of \mathbb{G} , raising the entries of a 2×2 matrix to the q th power, so that $G = \mathrm{SL}_2(q)$. The usual subgroups of \mathbb{G} can be chosen as follows:

$$\mathbb{T} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\} \subset \mathbb{B} = \left\{ \begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{pmatrix} \right\}, \quad \mathbb{U} = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}, \quad \dot{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then the varieties \mathbb{G}/\mathbb{U} and \mathbb{G}/\mathbb{B} are given explicitly by

$$\begin{array}{ccc} \mathbb{A}_2 \setminus \{(0, 0)\} & \xrightarrow{\sim} & \mathbb{G}/\mathbb{U} \\ (x, y) & \mapsto & \begin{pmatrix} x & * \\ y & * \end{pmatrix} \mathbb{U} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{P}_1 & \xrightarrow{\sim} & \mathbb{G}/\mathbb{B} \\ [x : y] & \mapsto & \begin{pmatrix} x & * \\ y & * \end{pmatrix} \mathbb{B} \end{array}$$

and the cosets $\mathbb{U}\dot{s}\mathbb{U}$ and $\mathbb{B}\dot{s}\mathbb{B}$ by

$$\mathbb{U}\dot{s}\mathbb{U} = \left\{ \begin{pmatrix} * & * \\ 1 & * \end{pmatrix} \right\} \cap \mathbb{G} \quad \text{and} \quad \mathbb{B}\dot{s}\mathbb{B} = \left\{ \begin{pmatrix} * & * \\ \lambda & * \end{pmatrix} \mid \lambda \neq 0 \right\} \cap \mathbb{G}.$$

Finally, the element $g^{-1}F(g)$ is given by

$$\begin{pmatrix} x & * \\ y & * \end{pmatrix}^{-1} F \begin{pmatrix} x & * \\ y & * \end{pmatrix} = \begin{pmatrix} * & * \\ -y & x \end{pmatrix} \begin{pmatrix} x^q & * \\ y^q & * \end{pmatrix} = \begin{pmatrix} * & * \\ xy^q - yx^q & * \end{pmatrix}.$$

We deduce the following explicit descriptions of the varieties $\mathbb{Y}(\dot{s})$ and $\mathbb{X}(\dot{s})$.

$$\begin{aligned} \mathbb{Y}(\dot{s}) &\simeq \{(x, y) \in \mathbb{A}_2 \mid xy^q - yx^q = 1\}, \\ \mathbb{X}(\dot{s}) &\simeq \{[x : y] \in \mathbb{P}_1 \mid xy^q - yx^q \neq 0\} = \mathbb{P}_1 \setminus \mathbb{P}_1(\mathbb{F}_q). \end{aligned}$$

The variety $\mathbb{Y}(\dot{s})$ is the famous *Drinfeld curve*, discovered and studied by Drinfeld in [27].

Since $\text{char}(K) = 0$, the algebra KG is semisimple and the complex $R\Gamma_c(\mathbb{Y}(\dot{w}), K)$ is quasi-isomorphic to its cohomology $\bigoplus H_c^i(\mathbb{Y}(\dot{w}), K)[-i]$ as a complex of $(G, \mathbb{T}^{\dot{w}F})$ -bimodules. Given an irreducible character θ of $\mathbb{T}^{\dot{w}F}$, we can consider the θ -isotypic part of each individual cohomology group $H_c^i(\mathbb{Y}(\dot{w}), K)_\theta := \text{Hom}_{\mathbb{T}^{\dot{w}F}}(\theta, H_c^i(\mathbb{Y}(\dot{w}), K))$. Since the actions of G and $\mathbb{T}^{\dot{w}F}$ commute, $H_c^i(\mathbb{Y}(\dot{w}), K)_\theta$ is a G -module. Given $g \in G$ we set

$$R_w(\theta)(g) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(g, H_c^i(\mathbb{Y}(\dot{w}), K)_\theta).$$

The function $R_w(\theta)$ is the character of the virtual module $\sum (-1)^i H_c^i(\mathbb{Y}(\dot{w}), K)_\theta$ (or equivalently of the complex $R\Gamma_c(\mathbb{Y}(\dot{w}), K)_\theta$). It is called a *Deligne–Lusztig character*. A particular interesting case is when $\theta = 1$. We have

$$R_w := R_w(1) = \sum_{i \in \mathbb{Z}} (-1)^i [H_c^i(\mathbb{X}(w), K)] = [R\Gamma_c(\mathbb{X}(w), K)]$$

since in that case the isotypic component of $H_c^i(\mathbb{Y}(\dot{w}), K)$ corresponding to the trivial representation of $\mathbb{T}^{\dot{w}F}$ is the invariant part under $\mathbb{T}^{\dot{w}F}$ which by (3.2) is isomorphic to $H_c^i(\mathbb{X}(w), K)$.

3.4. Properties of $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ and $R\Gamma_c(\mathbb{X}(w), \Lambda)$. In the examples 3.4 and 3.5 the varieties $\mathbb{Y}(\dot{w})$ and $\mathbb{X}(w)$ are affine. This was proven in general by Deligne and Lusztig [20] when $q \geq h$ (the Coxeter number of W) and it is conjectured to hold unconditionally (see for example [7] for further examples). In any case, the consequences on the vanishing of the cohomology groups (see §2.1) hold.

Theorem 3.6 (Lusztig [57]). $H_c^i(\mathbb{Y}(\dot{w}), \Lambda) = H_c^i(\mathbb{X}(w), \Lambda) = 0$ for $i < \ell(w)$.

Consequently, the complexes $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ and $R\Gamma_c(\mathbb{X}(w), \Lambda)$ can be represented by complexes with $(\Lambda$ -free) terms in degrees $\ell(w), \ell(w) + 1, \dots, 2\ell(w)$. In addition, one can compute the stabilizer of any point in $\mathbb{Y}(\dot{w})$ under the action of G and invoke Corollary 2.3 to show the following additional properties.

Proposition 3.7. *The complex $\text{Res}_G^{G \times \mathbb{T}^{\dot{w}^F}} R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ is perfect. Furthermore, if the order of $\mathbb{T}^{\dot{w}^F}$ is invertible in Λ then $R\Gamma_c(\mathbb{X}(w), \Lambda)$ is perfect as well.*

More generally, one can show that if Q in an ℓ -subgroup of $G \times \mathbb{T}^{\dot{w}^F}$ such that $\mathbb{Y}(\dot{w})^Q \neq \emptyset$, then Q is necessarily conjugate to a diagonal subgroup of $G \times \mathbb{T}^{\dot{w}^F}$ (with the identification $\mathbb{T}^{\dot{w}^F} \simeq ({}^g\mathbb{T})^F \subset G$ with $g^{-1}F(g)$ a representative of w in $N_{\mathbb{G}}(\mathbb{T})$).

Example 3.8. Let $\mathbb{G} = \text{SL}_2(\overline{\mathbb{F}}_p)$ and $\dot{w} = \dot{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so that

$$\mathbb{T}^{\dot{s}^F} \simeq \left\{ \left(\begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right) \middle| \lambda^{q+1} = 1 \right\} \simeq \mu_{q+1}(\overline{\mathbb{F}}_p).$$

If $(x, y) \in \mathbb{Y}(\dot{s})$ (see Example 3.5) then $g \cdot (x, y) \cdot \text{diag}(\lambda, \lambda^{-1}) = (x, y)$ if and only if (x, y) is an eigenvector of g with eigenvalue λ^{-1} . In particular, either $\pm g$ is unipotent (with eigenvalues ± 1) or g is conjugate to $\text{diag}(\lambda, \lambda^{-1})$.

If $|\mathbb{T}^{\dot{w}^F}|$ is invertible in Λ then there is no non-trivial ℓ -subgroup $G \times \mathbb{T}^{\dot{w}^F}$ such that $\mathbb{Y}(\dot{w})^Q \neq \emptyset$. In particular $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ is perfect as a complex of bimodules in that case. Otherwise we have the following result (see [1, §9] for basic results on vertices and sources).

Proposition 3.9. *Let C be a representative of $R\Gamma_c(\mathbb{Y}(\dot{w}), k)$ as a complex of ℓ -permutation modules with no null-homotopic direct summand. Then the vertices of the terms of C are contained in $\Delta\mathbb{T}^{\dot{w}^F}$.*

Proof. Let P be an ℓ -subgroup of G which is not conjugate to a subgroup of $\Delta\mathbb{T}^{\dot{w}^F}$. Then $\mathbb{Y}(\dot{w})^P = \emptyset$ and therefore by Theorem 2.11 we have $\text{Br}_P(C) \simeq R\Gamma_c(\mathbb{Y}(\dot{w})^P, k) \simeq 0$.

Now assume that there is an indecomposable direct summand M in C_i such that $\text{Br}_P(M) \neq 0$. Without loss of generality we can assume that P is maximal for this property. Then for any other direct summand N of the terms of C , $\text{Br}_P(N)$ is either zero or projective by maximality. Consequently $\text{Br}_P(C)$ is an acyclic complex with projective terms. Take M to be in the largest degree i of C so that $\text{Br}_P(C_j) = 0$ for $j > i$ and $\text{Br}_P(C_{i-1}) \twoheadrightarrow \text{Br}_P(M)$. Then $C_{i-1} \twoheadrightarrow M$ is a split surjection (by [4, Lem. A.1]), which contradicts the minimality of C . \square

3.5. Applications. The following result will be intensively used in the rest of these notes. It was first proved by Lusztig in the case where $\Lambda = K$ [57] and then extended by Bonnafé–Rouquier to the modular setting.

Theorem 3.10 (Bonnafé–Rouquier [5]). *Let M be a simple ΛG -module and $w \in W$ be minimal for the Bruhat order such that $\text{RHom}_{\Lambda G}(R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda), M) \neq 0$. Then there exists a representative $0 \rightarrow P_0 \rightarrow \cdots \rightarrow P_{\ell(w)} \rightarrow 0$ of $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ such that*

- each P_i is a finitely generated projective ΛG -module (in degree $\ell(w) + i$),

- P_M is a direct summand of P_i for $i = 0$ only (middle degree).

In other words, the terms of that complex satisfy $\mathrm{Hom}_{\Lambda G}(P_i, M) \neq 0 \iff i = 0$.

Sketch of proof. The key property shown by Bonnafé–Rouquier is that the cone of the natural map $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda) \rightarrow R\Gamma(\mathbb{Y}(\dot{w}), \Lambda)$ lies in the thick subcategory of $\mathbf{D}^b(\Lambda G\text{-mod})$ generated by the complexes $R\Gamma_c(\mathbb{Y}(\dot{v}), \Lambda)$ for $v < w$. In particular, the minimality of w shows that this map induces an isomorphism

$$\begin{array}{ccc} R\mathrm{Hom}_{\Lambda G}(R\Gamma(\mathbb{Y}(\dot{w}), \Lambda), M) & \xrightarrow{\sim} & R\mathrm{Hom}_{\Lambda G}(R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda), M) \\ \downarrow & & \downarrow \\ \begin{array}{c} \text{terms in degrees} \\ 0, 1, \dots, \ell(w) \end{array} & & \begin{array}{c} \text{terms in degrees} \\ \ell(w), \dots, 2\ell(w) \end{array} \end{array}$$

Consequently, the cohomology of $R\mathrm{Hom}_{\Lambda G}(R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda), M)$ vanishes outside the degree $\ell(w)$. In other words, $\mathrm{Hom}_{\mathrm{Ho}^b(\Lambda G\text{-mod})}(R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda), M[-i]) = 0$ for $i \neq \ell(w)$.

Now let $C = 0 \rightarrow P_0 \rightarrow \dots \rightarrow P_{\ell(w)} \rightarrow 0$ be a representative of $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ with projective terms and with no null-homotopic direct summand. Let i be maximal for the property that P_M is a direct summand of P_i , and assume that $i > 0$. If the composition $P_{i-1} \rightarrow P_i \rightarrow M$ is zero, then the map $P_i \rightarrow M$ induces a morphism between the complexes $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ and $M[-i - \ell(w)]$. Therefore it must be null-homotopic by the above argument, which is impossible since P_M is not a direct summand of P_{i+1} . Therefore the composition $P_{i-1} \rightarrow P_i \rightarrow M$ must be non-zero, hence surjective, which shows that $P_M \xrightarrow{\sim} P_M$ is a (null-homotopic) direct summand of C . By assumption of C , this is again impossible. \square

Corollary 3.11 (Bonnafé–Rouquier [5]). *The triangulated category of perfect complexes $\Lambda G\text{-perf}$ is generated by the cohomology complexes $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ for $w \in W$.*

Proof. We show by induction on the length of w that for every simple ΛG -module M , if $R\mathrm{Hom}_{\Lambda G}(R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda), M) \neq 0$ then P_M lies in the thick subcategory of $\Lambda G\text{-perf}$ generated by the complexes $R\Gamma_c(\mathbb{Y}(\dot{v}), \Lambda)$ for $v \leq w$. This is true for $w = 1$ since $R\Gamma_c(\mathbb{Y}(1), \Lambda) \simeq \Lambda G/U[0]$ (by definition a thick subcategory is stable under direct summands).

Let $w \in W$, and consider a representative C of $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ as a bounded complex of projective modules with no null-homotopic direct summand. It follows from Theorem 3.10 that the indecomposable direct summands of C_i for $i > \ell(w)$ already appear in the cohomology complexes $R\Gamma_c(\mathbb{Y}(\dot{v}), \Lambda)$ for $v < w$. By induction C_i lies in the thick subcategory of $\Lambda G\text{-perf}$ generated by the complexes $R\Gamma_c(\mathbb{Y}(\dot{v}), \Lambda)$ for $v < w$. But the term in middle degree can be written as

$$C_{\ell(w)} = \mathrm{Cone}((C_{\ell(w)+1} \rightarrow \dots \rightarrow C_{2\ell(w)} \rightarrow R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)[\ell(w) + 1])$$

which proves that $C_{\ell(w)}$ lies in the category generated by $R\Gamma_c(\mathbb{Y}(\dot{v}), \Lambda)$ for $v \leq w$.

To conclude, it remains to show that any projective indecomposable module appears as a direct summand of a minimal representative of $R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda)$ for some $w \in W$. It is enough to show it at the level of characters. This follows from the fact that the regular representation of G is *uniform*, which means that it is a linear combination of Deligne–Lusztig characters $R_w(\theta)$. \square

One can invoke Corollary 3.11 to see that a morphism f of bounded complexes of ΛG -modules is a quasi-isomorphism if and only if $\text{Cone}(f) \overset{L}{\otimes}_{\Lambda G} R\Gamma_c(\mathbb{Y}(\dot{w}), \Lambda) = 0$ for all $w \in W$. This was a key step in Bonnafé–Rouquier’s proof of the Jordan decomposition as a Morita equivalence (see [5]).

The analogue of Corollary 3.11 for general bounded complexes (whose terms can have non-trivial vertices) was proved recently by Bonnafé–Dat–Rouquier in [4]. This again was proven essential to show that the Jordan decomposition is a splendid Rickard equivalence. Recall that $\widetilde{R}\Gamma_c(\mathbb{Y}(\dot{w}), k)$ denotes the (unique up to homotopy equivalence) representative of $R\Gamma_c(\mathbb{Y}(\dot{w}), k)$ as a complex of ℓ -permutation modules (see §2.1).

We say that the prime number ℓ is *very good* for \mathbb{G} if ℓ is good for every simple component of \mathbb{G} and $\ell \nmid m + 1$ for every component of \mathbb{G} of type A_m . A sufficient condition for ℓ to be very good is $\ell > h$ where h is the Coxeter number of \mathbb{G} .

Theorem 3.12 (Bonnafé–Dat–Rouquier [4]). *Assume that ℓ is very good. Let \mathcal{X} be the set of complexes $\widetilde{R}\Gamma_c(\mathbb{Y}(\dot{w}), k) \otimes_{kQ} \theta$ where Q runs over the ℓ -subgroups of $\mathbb{T}^{\dot{w}F}$, $\theta \in \text{Irr}kQ$ and $w \in W$. Then*

- (i) *The thick subcategory of $\text{Ho}^b(kG\text{-mod})$ generated by \mathcal{X} coincide with $\text{Ho}^b(\mathcal{B})$, where \mathcal{B} is the additive category generated by the indecomposable modules with one-dimensional sources and abelian vertices.*
- (ii) *The image of \mathcal{X} in $\text{D}^b(kG\text{-mod})$ generates $\text{D}^b(kG\text{-mod})$ as a triangulated category closed under direct summands.*

4. Decomposition numbers from Deligne–Lusztig characters

Recall that (K, \mathcal{O}, k) denotes an ℓ -modular system with K being a finite extension of \mathbb{Q}_ℓ . We will furthermore assume from this chapter on that K and k are big enough for all the groups considered (all the group algebras over K and k will split).

The purpose of this chapter is to explain how one can compute ℓ -decomposition numbers for finite reductive groups using the Deligne–Lusztig characters $R_w(\theta)$ introduced in the previous chapter. We start by recalling elementary results on decomposition matrices (see for example [66, §14]) before explaining the case of finite reductive groups.

4.1. Grothendieck groups and dualities. Let \mathcal{A} be an abelian (resp. additive) category. The *Grothendieck group* of \mathcal{A} , denoted by $K_0(\mathcal{A})$ (or sometimes $[\mathcal{A}]$) is the abelian group generated by the isomorphism classes of objects of \mathcal{A} subject to the relations $[M] = [L] + [N]$ for every short exact sequence (resp. split short exact sequence) $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$. Note that any abelian category is additive. When there is a risk of confusion, the Grothendieck group of \mathcal{A} as an additive category will be referred to as the *split Grothendieck group*. If \mathcal{A} is an abelian category in which every object has finite composition length, then $K_0(\mathcal{A})$ can be identified with the free abelian group with basis $\text{Irr } \mathcal{A}$. Similarly, the split Grothendieck group of a Krull-Schmidt category is the free abelian group generated by the isomorphism classes of the indecomposable objects.

Let \mathcal{T} be a triangulated category. The *Grothendieck group* $K_0(\mathcal{T})$ of \mathcal{T} is the abelian group generated by the isomorphism classes of objects of \mathcal{T} subject to the relations $[M] = [L] + [N]$ for every distinguished triangle $L \rightarrow M \rightarrow N \rightsquigarrow$.

Given an abelian category \mathcal{A} , the fully faithful functor $\mathcal{A} \rightarrow \mathbf{D}^b(\mathcal{A})$ induces an isomorphism $K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathbf{D}^b(\mathcal{A}))$. Under this identification, the class of a bounded complex C is

$$[C] = \sum_{i \in \mathbb{Z}} (-1)^i [C_i] = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(C)].$$

Similarly, for any additive category \mathcal{A} the functor $\mathcal{A} \rightarrow \text{Ho}^b(\mathcal{A})$ induces an isomorphism between the corresponding split Grothendieck groups.

The Grothendieck groups we will be interested in in this chapter are:

- $K_0(KG\text{-mod})$ } \mathbb{Z} -basis given by simple modules
- $K_0(kG\text{-mod})$
- $K_0(kG\text{-proj})$ ← basis given by projective indecomposable modules (PIMs)

Since exact sequences split in $KG\text{-mod}$ and $kG\text{-proj}$ then for M, N in $KG\text{-mod}$ (resp. $kG\text{-proj}$) $[M] = [N]$ if and only if $M \simeq N$. This is not true in general in $kG\text{-mod}$ since kG -modules can have non-trivial extensions when ℓ divides the order of $|G|$ (which is the interesting case from our point of view).

In addition, there are perfect pairings

$$\begin{aligned} \langle -, - \rangle_K &: K_0(KG\text{-mod}) \times K_0(KG\text{-mod}) \longrightarrow \mathbb{Z} \\ \langle -, - \rangle_k &: K_0(kG\text{-proj}) \times K_0(kG\text{-mod}) \longrightarrow \mathbb{Z} \end{aligned}$$

defined by $\langle [P]; [M] \rangle_\Lambda = \dim_\Lambda \text{Hom}_{\Lambda G}(P, M)$ when P and M are actual modules and Λ is the field K or k . Then $\text{Irr}_K G$ is a self-dual basis for the pairing $\langle -, - \rangle_K$, whereas a dual basis of $\text{Irr}_k G$ for $\langle -, - \rangle_k$ is given by the classes of projective indecomposable modules.

4.2. Lifting projective modules. We say that a kG -module M *lifts to characteristic zero* if there exists an $\mathcal{O}G$ -lattice \widetilde{M} (an $\mathcal{O}G$ -module which is free as an \mathcal{O} -module) such that $k\widetilde{M} \simeq M$ as kG -modules. Not every kG -module can be lifted to characteristic zero in general, unless $\ell \nmid |G|$ in which case kG is semisimple. This holds nevertheless for projective modules. Indeed, given a finitely generated projective kG -module P , we can consider the projective cover P_m of P as an $\mathcal{O}/\ell^m \mathcal{O}$ -module. Then $kP_m \simeq P$ and the $\mathcal{O}G$ -module $\widetilde{P} := \varprojlim P_m$ is an $\mathcal{O}G$ -module lifting P . In addition, it is projective and it is – up to isomorphism – the unique projective $\mathcal{O}G$ -module lifting P . Note that if M is a simple kG -module, then \widetilde{P}_M , together with the composition $\widetilde{P}_M \twoheadrightarrow P_M \twoheadrightarrow M$ is a projective cover of M , viewed as a simple $\mathcal{O}G$ -module.

Given $P \in kG\text{-proj}$ and its lift \widetilde{P} to characteristic zero, we can form the KG -module $K\widetilde{P}$. Its character (or rather its image in the Grothendieck group) will be denoted $e([P])$. This defines a group homomorphism

$$e : K_0(kG\text{-proj}) \longrightarrow K_0(KG\text{-mod}).$$

4.3. Decomposing ordinary characters. Let M be a KG -module. One can choose an \mathcal{O} -free \mathcal{O} -submodule \widetilde{M} such that $K\widetilde{M} \simeq M$. By saturating by the action of G one can assume that \widetilde{M} is stable by G , so that \widetilde{M} is an $\mathcal{O}G$ -lattice such that $K\widetilde{M} \simeq M$. Then one can form the kG -module $k\widetilde{M}$ and consider its image in $K_0(kG\text{-mod})$, which we will denote by $d([M])$.

Proposition 4.1.

(i) d is well-defined and extends to a group homomorphism

$$d : K_0(KG\text{-mod}) \longrightarrow K_0(kG\text{-mod})$$

called the decomposition map.

(ii) (Brauer reciprocity) d is the transpose of e for the pairings defined in §4.1. In other words

$$\langle -, d(-) \rangle_k = \langle e(-), - \rangle_K.$$

Proof. It is enough to prove (ii). Let M be a KG -module and P be a projective kG -module. We constructed $\mathcal{O}G$ -lattices \widetilde{M} and \widetilde{P} such that $K\widetilde{M} \simeq M$ and $k\widetilde{P} \simeq P$. Then

$$\begin{aligned} \langle [P]; d([M]) \rangle_k &= \dim_k \operatorname{Hom}_{kG}(P, k\widetilde{M}) = \operatorname{rk}_{\mathcal{O}} \operatorname{Hom}_{\mathcal{O}G}(\widetilde{P}, \widetilde{M}) \\ &= \dim_K \operatorname{Hom}_{KG}(K\widetilde{P}, K\widetilde{M}) = \langle e([P]); [M] \rangle_K. \end{aligned}$$

□

The *decomposition matrix* D (or D_ℓ) is the matrix with entries

$$d_{\chi, S} = \langle [P_S]; d(\chi) \rangle_k = \langle e([P_S]); \chi \rangle_K$$

for $\chi \in \text{Irr}_K G$ and $S \in \text{Irr}_k G$. With this notation we have

$$d(\chi) = \sum_{S \in \text{Irr}_k G} d_{\chi, S} [S] \quad \text{and} \quad e([P_S]) = \sum_{\chi \in \text{Irr}_K G} d_{\chi, S} \chi.$$

Example 4.2. (a) If $\ell \nmid |G|$ then every simple kG -module is projective, and lifts to an irreducible ordinary character. Consequently D is the identity matrix up to reordering.

(b) If G is an ℓ -group, then the only irreducible kG -module is the trivial representation. Since the decomposition map preserves the dimension, we deduce that $d(\chi) = (\dim \chi)[k]$ for every irreducible ordinary character χ of K . The decomposition matrix in that case is a column encoding the dimensions of the irreducible KG -modules.

Dually, the projective cover of the trivial representation is the regular representation $P_k = kG$, which lifts to characteristic zero as $\mathcal{O}G$, and whose character is $[KG] = \sum (\dim \chi)\chi$. This is an example of Brauer reciprocity as stated in Proposition 4.1.ii.

(c) Let us consider the particular case of $G = \mathfrak{S}_3$ and $\ell = 3$. There are three irreducible representations over K : the trivial representation K , the sign ε and the reflection representation, of dimension 2. The latter has an integral version given by $M = \{(x_1, x_2, x_3) \in \mathcal{O}^3 \mid x_1 + x_2 + x_3 = 0\}$. The representations K and ε yield two non-isomorphic representations over k by ℓ -reduction, the trivial representation k and the sign modulo 3, which we still denote by ε . There is a short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & k & \longrightarrow & kM & \longrightarrow & \varepsilon \longrightarrow 0 \\ & & & & x \longmapsto (x, x, x) & & \end{array}$$

which shows that $[kM] = [k] + [\varepsilon]$. We deduce that the decomposition matrix in that case is

$$D = \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \\ 1 & 1 \end{bmatrix}.$$

Consequently, the two PIMs have characters $[K] + [KM]$ and $[\varepsilon] + [KM]$.

4.4. Basic sets of characters. We mentioned in a previous section that not every kG -modules can be lifted to characteristic zero. This is however true at the level of the Grothendieck groups: the class of any kG -module is the ℓ -reduction of a virtual character.

Theorem 4.3. *The decomposition map $d : K_0(KG\text{-mod}) \longrightarrow K_0(kG\text{-mod})$ is surjective.*

By Proposition 4.1, the map e is the transpose of d , hence it is injective. Therefore if P and Q are two projective kG -modules then $P \simeq Q$ if and only if their lifts \tilde{P} and \tilde{Q} have the same character. In other words, a projective module is determined by its character over K .

Since d is surjective, it is natural to search for a set \mathcal{B} of ordinary irreducible characters such that $d(\mathcal{B})$ is a \mathbb{Z} -basis of $K_0(kG\text{-mod})$. Such a set is called a *basic set* (see [45]). If it exists, then the decomposition matrix has the following shape:

$$D = \begin{bmatrix} D_{\mathcal{B}} \\ * \end{bmatrix} \quad \text{with } D_{\mathcal{B}} \in \text{GL}_n(\mathbb{Z}).$$

In that case, a projective kG -module is determined by the projection of its character on \mathcal{B} .

Now assume that $G = \mathbb{G}^F$ is a finite reductive group. By definition, the *unipotent characters* are the irreducible constituents of the virtual characters

$$R_w = \sum_{i \in \mathbb{Z}} (-1)^i [H_c^i(\mathbb{X}(w), K)] = [R\Gamma_c(\mathbb{X}(w), K)]$$

for $w \in W$. The *unipotent blocks* are the ℓ -blocks containing at least one unipotent character, and the irreducible characters in the union of unipotent blocks are the constituents of

$$R_w(\theta) = \sum_{i \in \mathbb{Z}} (-1)^i [H_c^i(\mathbb{Y}(\dot{w}), K)_{\theta}] = [R\text{Hom}_G(\theta, R\Gamma_c(\mathbb{Y}(w), K))]$$

for $w \in W$ and $\theta \in \text{Irr}_{\ell} \mathbb{T}^{\dot{w}F}$ an irreducible ℓ -character of $\mathbb{T}^{\dot{w}F}$.

Exercise 4.4. Given $\theta \in \text{Irr}_{\ell} \mathbb{T}^{\dot{w}F}$, show that $d(R_w(\theta)) = R_w(d(\theta)) = d(R_w)$ in $K_0(kG\text{-mod})$.

Recall that ℓ is said to be very good for \mathbb{G} if ℓ is good for every simple component of \mathbb{G} and $\ell \nmid m+1$ for every component of \mathbb{G} of type A_m . A sufficient condition for ℓ to be very good is $\ell > h$ where h is the Coxeter number of \mathbb{G} .

Theorem 4.5 (Geck–Hiss, Geck [45, 43]). *Assume that ℓ is very good. Then the unipotent characters form a basic set for the union $\mathcal{E}_{\ell}(G, 1)$ of unipotent blocks.*

More generally, under the same assumption on ℓ , given $s \in G^*$ a semisimple ℓ -element, the Lusztig series $\mathcal{E}(G, s)$ (see for example [21, Def. 13.16]) is a basic set for the union of blocks $\mathcal{E}_{\ell}(G, s) := \bigcup \mathcal{E}(G, st)$ where t runs over the set of semisimple ℓ -elements of $C_{G^*}(s)$. This was first proved for finite general linear and unitary groups by Fong–Srinivasan [36] and then generalized to any finite reductive group by Geck–Hiss and Geck [45, 43].

4.5. Decomposition numbers and Deligne–Lusztig characters. This section is the core of this chapter and contains recent results on ℓ -decomposition numbers for unipotent blocks when ℓ is not too small (see for example [29, 32]). We start by listing the different tools and assumptions we are going to use to determine these numbers.

(HC) The Harish-Chandra restriction/induction of a projective ΛG -module remains projective (this follows easily from the biadjointness and the exactness of the functors).

(Uni) When ℓ is very good, the restriction of the decomposition matrix to the set of unipotent characters (a basic set by Theorem 4.5), ordered by increasing a -function, has unitriangular shape. This is only conjectural (see [46, Conj. 3.4]).

(Hecke) The decomposition matrix of the Hecke algebra $\text{End}_{\mathcal{O}G}(\mathcal{O}G/B)$ (corresponding to the unipotent principal series) embeds in the decomposition matrix of the finite group G (see [15, Thm. 5.28]).

(Reg) If $\theta \in \text{Irr}_{\ell} \mathbb{T}^{\dot{w}F}$ is an ordinary irreducible ℓ -character of $\mathbb{T}^{\dot{w}F}$ in general position (i.e. $(-1)^{\ell(w)} R_w(\theta)$ is irreducible) then

$$\langle e([P]); (-1)^{\ell(w)} R_w \rangle_K \geq 0$$

for every projective kG -module P . This gives a non-trivial information since $(-1)^{\ell(w)} R_w$ is only a virtual unipotent character, even though $(-1)^{\ell(w)} R_w(\theta)$ is irreducible.

Proof. Using the fact that $d(R_w(\theta)) = d(R_w)$ when $\theta \in \text{Irr}_{\ell} \mathbb{T}^{\dot{w}F}$ (see Exercise 4.4) and Brauer reciprocity we have

$$\begin{aligned} \langle e([P]); R_w \rangle_K &= \langle [P]; d(R_w) \rangle_k \\ &= \langle [P]; d(R_w(\theta)) \rangle_k \\ &= \langle e([P]); R_w(\theta) \rangle_K. \end{aligned}$$

Since $(-1)^{\ell(w)} R_w(\theta)$ is assumed to be irreducible, the sign of this scalar product coincides with $(-1)^{\ell(w)}$. \square

Recall from Proposition 3.7 that the complex $R\Gamma_c(\mathbb{Y}(\dot{w}), k)$ is perfect as a complex of kG -modules. Let $P_w = [R\Gamma_c(\mathbb{Y}(\dot{w}), k)]$ denote its class in $K_0(kG\text{-proj})$. Then

$$e(P_w) = [R\Gamma_c(\mathbb{Y}(\dot{w}), K)] = \sum_{i \in \mathbb{Z}} (-1)^i [H_c^i(\mathbb{Y}(\dot{w}), K)] = R_w + \text{non-unip. chars.} \quad (4.1)$$

The following property is a character-theoretic consequence of Bonnafé–Rouquier’s theorem 3.10. It is particularly suited for determining decomposition numbers on cuspidal kG -modules, as we will see in the examples of the next section.

(DL) Given a simple kG -module S , let $w \in W$ be minimal (for the Bruhat order) such that $\langle P_w, [S] \rangle \neq 0$ (i.e. $[P_S]$ occurs in P_w). Then $\langle (-1)^{\ell(w)} P_w, [S] \rangle > 0$.

Proof. Let S be a simple kG -module and $w \in W$ be such that $\langle (-1)^{\ell(w)} P_w, [S] \rangle < 0$. Take $C = 0 \rightarrow P_0 \rightarrow \dots \rightarrow P_{\ell(w)} \rightarrow 0$ to be a representative of $R\Gamma_c(\mathbb{Y}(\dot{w}), k)$ with each P_i projective in degree $i + \ell(w)$, and with no null-homotopic direct summand. By assumption, there exists $i > 0$ such that P_S is a direct summand of P_i . Taking i to be maximal, we deduce that

$$\begin{aligned} 0 &\neq \text{Hom}_{\text{Ho}^b(kG\text{-mod})}(C, S[-i - \ell(w)]) \\ &= \text{Hom}_{\text{D}^b(kG\text{-mod})}(R\Gamma_c(\mathbb{Y}(\dot{w}), k), S[-i - \ell(w)]) \\ &= H^{i+\ell(w)}(\text{RHom}_{kG}(R\Gamma_c(\mathbb{Y}(\dot{w}), k), S)). \end{aligned}$$

By Theorem 3.10 if $v \leq w$ is a minimal element (w.r.t. the Bruhat order) for the property that the complex $R\mathrm{Hom}_{kG}(R\Gamma_c(\mathbb{Y}(\dot{v}), k), S)$ is non-zero then $R\Gamma_c(\mathbb{Y}(\dot{v}), k)$ has a representative such that P_S occurs only in middle degree. Therefore $\langle (-1)^{\ell(v)} P_v; [S] \rangle > 0$ and w cannot be minimal for the property $\langle P_w, [S] \rangle \neq 0$. \square

Property (DL) means that if we write $(-1)^{\ell(w)} P_w$ on the basis of characters of PIMs, then the PIMs which do not occur in P_v for $v < w$ have a non-negative coefficient in this decomposition.

4.6. Examples in small rank. We discuss here three examples of small-rank finite reductive groups where the previous tools allow a complete determination of the (unipotent part of the) decomposition matrix. Here the assumption (Uni) was shown to hold by Dipper [23] for finite linear groups, by Geck [40] for finite unitary groups and by White [69] for $\mathrm{Sp}_4(q)$.

(a) We start with $G = \mathrm{SL}_2(q)$. Its order is $|G| = q(q-1)(q+1)$. Assume that ℓ is an odd prime number with $\ell \nmid q$ and $\ell \mid q^2 - 1$. Then the principal block contains the two unipotent characters 1 and St, together with some non-unipotent characters. Using (Uni) and (Hecke) we have

$$D = \begin{bmatrix} 1 & \cdot \\ \alpha & 1 \\ * & * \\ \vdots & \vdots \end{bmatrix} \quad \text{with } \alpha = \begin{cases} 0 & \text{if } \ell \nmid q+1 \\ 1 & \text{otherwise.} \end{cases}$$

Here the value of α can be obtained by (Hecke). Indeed, if $\ell \nmid q+1$ then $\mathrm{End}_{kG}(kG/B)$ is semisimple and $kG/B = k \oplus M$ with M a simple kG -module. Therefore $\alpha = 0$ in this case. Now if $\ell \mid q+1$ then $R_T^G(k) = kG/B$ is a PIM with character $1 + \mathrm{St}$, hence $\alpha = 1$.

(b) Let $G = \mathrm{Sp}_4(q)$, whose order is $q^4(q^2-1)(q^4-1) = q^4(q-1)^2(q+1)^2(q^2+1)$. We will denote by s and t the simple reflections in W corresponding to the short and long simple root respectively. Assume in this example that ℓ is odd and $\ell \mid q+1$. Then the characters in the principal block are

$$\underbrace{\{1, \mathrm{St}, \rho_1, \rho_2, \theta_{10}, \text{non-unipotent}\}}_{\substack{\text{in the principal series} \quad \longleftarrow \\ \longrightarrow \quad \text{cuspidal}}}$$

There is another unipotent character, denoted by χ , which under the assumptions on ℓ is of defect zero and this forms a block by itself. We have $R_1 = R_T^G(K) = 1 + \mathrm{St} + \rho_1 + \rho_2 + 2\chi$, giving all the unipotent characters lying in the principal

series. The first approximation of the decomposition matrix is given by (Uni)

$$D = \begin{array}{c} \left[\begin{array}{ccccc} 1 & \cdot & \cdot & \cdot & \cdot \\ * & 1 & \cdot & \cdot & \cdot \\ * & * & 1 & \cdot & \cdot \\ * & * & * & 1 & \cdot \\ * & * & * & * & 1 \\ \hline * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right] \begin{array}{l} 1 \\ \rho_1 \\ \rho_2 \\ \theta_{10} \\ \text{St} \end{array} \end{array}$$

A (Hecke) argument shows that $R_T^G(k) = kG/B$ is indecomposable. Since it is projective, it gives the first column of the decomposition matrix.

The second and third column can be obtained by (HC). First, let $L = \text{GL}_2(q) \subset \text{Sp}_4(q)$ and P be the PIM of $\text{GL}_2(q)$ such that $e([P]) = \text{St}_{\text{GL}_2(q)} + \text{non-unipotent}$ characters (see Example (a)). Let $b \in \mathcal{O}G$ be the block idempotent corresponding to the principal block. Then $R_L^G(P)$ is projective and its character, cut by the block, is given by

$$\begin{aligned} e(bR_L^G([P])) &= bR_L^G(e([P])) = bR_L^G(\text{St}_{\text{GL}_2(q)} + \text{non-unipotent}) \\ &= \rho_1 + \text{St} + \text{non-unipotent}. \end{aligned}$$

Similarly with the Levi subgroup $L' = \text{SL}_2(q) \times \mathbb{F}_q^\times \subset \text{Sp}_4(q)$ we get

$$e(bR_{L'}^G([P])) = bR_{L'}^G(e([P])) = \rho_2 + \text{St} + \text{non-unipotent}.$$

Consequently the unipotent part of the decomposition matrix is

$$D_{\text{unip}} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \alpha_1 & \alpha_2 & \beta & 1 \end{bmatrix}$$

with $\alpha_1, \alpha_2 \leq 1$.

We use (Reg) to determine the exact value of α_1 and α_2 . Since $|\mathbb{T}^{w_0 F}| = (q+1)^2$ there exists a non-trivial ℓ -character $\theta \in \text{Irr}_K \mathbb{T}^{w_0 F}$. Furthermore, if $(q+1)_\ell > 3$ then one can choose θ to be lying outside the reflection hyperplanes (in the reflection representation of W on the group of characters of \mathbb{T}). In that case it is in general position, and (Reg) yields $\langle e([Q]); R_{w_0} \rangle \geq 0$ for every projective kG -module Q . Now $R_{w_0} = 1 + \text{St} - \rho_1 - \rho_2 - 2\theta_{10}$, so if we apply this to the projective indecomposable modules Q_2, Q_3 and Q_4 corresponding to the second, third and fourth columns of the decomposition matrix we get

$$\begin{aligned} \langle e([Q_2]); R_{w_0} \rangle &= \langle \rho_1 + \alpha_1 \text{St}; 1 + \text{St} - \rho_1 - \rho_2 - 2\theta_{10} \rangle = -1 + \alpha_1 \\ \langle e([Q_3]); R_{w_0} \rangle &= \langle \rho_2 + \alpha_2 \text{St}; 1 + \text{St} - \rho_1 - \rho_2 - 2\theta_{10} \rangle = -1 + \alpha_2 \\ \langle e([Q_4]); R_{w_0} \rangle &= \langle \theta_{10} + \beta \text{St}; 1 + \text{St} - \rho_1 - \rho_2 - 2\theta_{10} \rangle = -2 + \beta \end{aligned}$$

which gives $\alpha_1, \alpha_2 \geq 1$ (and hence $\alpha_1 = \alpha_2 = 1$) and $\beta \geq 2$.

The final ingredient is (DL). To use it we decompose each virtual projective module $P_w = [R\Gamma_c(\mathbb{Y}(\dot{w}), k)]$ on the basis of PIMs. To this end, recall from (4.1) that $e(P_w) = R_w + \text{non-unipotent characters}$. We have

$$\begin{aligned}
e(bP_1) &= 1 + \text{St} + \rho_1 + \rho_2 + \text{non-unip.} = e([Q_1]) \\
e(bP_s) &= 1 - \text{St} + \rho_1 - \rho_2 + \text{non-unip.} \\
&= 1 + \text{St} + \rho_1 + \rho_2 - 2(\rho_2 + \text{St}) + \text{non-unip.} \\
&= e([Q_1] - 2[Q_3]) \\
e(bP_t) &= 1 - \text{St} - \rho_1 + \rho_2 + \text{non-unip.} \\
&= 1 + \text{St} + \rho_1 + \rho_2 - 2(\rho_1 + \text{St}) + \text{non-unip.} \\
&= e([Q_1] - 2[Q_2]) \\
e(bP_{st}) &= 1 + \text{St} + \theta_{10} + \text{non-unip.} \\
&= (1 + \text{St} + \rho_1 + \rho_2) - (\rho_1 + \text{St}) - (\rho_2 + \text{St}) + (\theta_{10} + 2\text{St}) + \text{non-unip.} \\
&= e([Q_1] - [Q_2] - [Q_3]) + \theta_{10} + 2\text{St} + \text{non-unip.}
\end{aligned}$$

Since $\ell(st) = 2$ we deduce from (DL) that $\theta_{10} + 2\text{St} + \text{non-unip.}$ must be a non-negative combination of $e([Q_4])$ and $e([Q_5])$, since Q_4 and Q_5 do not appear in the decomposition of P_w for $w < st$. Writing

$$\theta_{10} + 2\text{St} + \text{non-unip.} = e([Q_4] + (2 - \beta)[Q_5])$$

we deduce that $\beta \leq 2$, which forces $\beta = 2$. We conclude that the unipotent part of the ℓ -decomposition matrix (when ℓ is odd and $(q+1)_\ell > 3$) is given by

$$D_{\text{unip}} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}.$$

Exercise 4.6. We follow the notation of [16, §13] for unipotent characters. Complete the determination of the ℓ -decomposition matrix of $G = G_2(q)$ when $\ell \mid q+1$ and $\ell > 5$, which is given by

$$D_{\text{unip}} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & 1 & 1 & \alpha & \beta & 1 \end{bmatrix} \begin{array}{l} 1 \\ \phi''_{1,3} \\ \phi'_{1,3} \\ G_2[1] \\ G_2[-1] \\ \text{St} \end{array}$$

with $\alpha, \beta \geq 2$ (see [50]). To this end, use (DL) with the following values of the

Deligne–Lusztig characters, cut by the principal block b ,

w	bR_w
1	$1 + \phi'_{1,3} + \phi''_{1,3} + \text{St}$
s, tst	$1 - \phi'_{1,3} + \phi''_{1,3} - \text{St}$
t, sts	$1 + \phi'_{1,3} - \phi''_{1,3} - \text{St}$
st, ts	$1 + G_2[-1] + \text{St}$
$stst$	$1 + G_2[1] + \text{St}$

Here s and t denote the simple reflections in the Weyl group of type G_2 corresponding to the short and long simple root respectively.

(c) Let $G = \text{SU}_5(q) = \text{“SL}_5(-q)”$. Its order is given by

$$\begin{aligned} |G| &= q^{10}((-q)^5 - 1)((-q)^4 - 1)((-q)^3 - 1)((-q)^2 - 1) \\ &= q^{10}(q-1)^3 \underbrace{(q+1)^4}_{\downarrow \text{largest defect}} (q^2+1)(q^2-q+1)(q^4+1). \end{aligned}$$

We will work again in the case where $\ell \mid q+1$. In addition we will assume that $\ell > 5$ to ensure the existence of ℓ -characters in regular position. As in the case of linear groups, the unipotent characters of $\text{SU}_n(q)$ are parametrized by partitions of n . Here, they are $1 = \rho_{(5)}, \rho_{(41)}, \rho_{(32)}, \rho_{(31^2)}, \rho_{(2^21)}, \rho_{(21^3)}, \rho_{(1^5)} = \text{St}$ and are all contained in the principal ℓ -block. By (Uni) the unipotent part of the decomposition matrix has the following shape

$$D_{\text{unip}} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\ * & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 41 \\ * & * & 1 & \cdot & \cdot & \cdot & \cdot & 32 \\ * & * & * & 1 & \cdot & \cdot & \cdot & 31^2 \\ * & * & * & * & 1 & \cdot & \cdot & 2^21 \\ * & * & * & * & * & 1 & \cdot & 21^3 \\ * & * & * & * & * & * & 1 & 1^5 \end{bmatrix}$$

By (Hecke) the projective kG -module $R_T^G(k)$ decomposes as a direct sum of two PIMs. The corresponding decomposition of characters is $R_1 = (\rho_{(5)} + \rho_{(31^2)} + \rho_{(2^21)}) + (\rho_{(32)} + \rho_{(31^2)} + \rho_{(1^5)})$ which gives the first and third columns of the decomposition matrix.

As in the previous examples, other columns can be determined by Harish-Chandra induction of projective modules of various Levi subgroups. The Levi subgroup $L \subset \text{GU}_3(q) \times \mathbb{F}_q^\times$ of type 2A_2 has two interesting PIMs P' and P'' with respective characters

$$\begin{aligned} e([P']) &= \rho_{(21)} + 2\rho_{(1^3)} + \text{non-unip.} \\ e([P'']) &= \rho_{(1^3)} + \text{non-unip.} \end{aligned}$$

yielding by (HC) two projective kG -modules $R_L^G(P')$ and $R_L^G(P'')$ with characters

$$\begin{aligned} e(R_L^G([P'])) &= \rho_{(41)} + \rho_{(21^2)} + 2(\rho_{(31^2)} + \rho_{(2^21)} + \rho_{(1^5)}) + \text{non-unip.} \\ e(R_L^G([P''])) &= \rho_{(31^2)} + \rho_{(2^21)} + \rho_{(1^5)} + \text{non-unip.} \end{aligned}$$

Note that these projective modules might not be indecomposable.

If $\ell > 5$ there exists an ℓ -character θ of $\mathbb{T}^{w_0 F}$ in general position and (Reg) applies. In other words, $\langle e([P]); R_{w_0} \rangle \geq 0$ for every projective kG -module P . With

$$R_{w_0} = \rho_{(5)} + 4\rho_{(41)} + 5\rho_{(32)} - 6\rho_{(31^2)} + 5\rho_{(2^21)} - 4\rho_{(21^2)} + \rho_{(1^5)}$$

we have $\langle e(R_L^G([P'])); R_{w_0} \rangle = \langle e(R_L^G([P''])); R_{w_0} \rangle = 0$. Therefore $\langle e([P]); R_{w_0} \rangle = 0$ for every direct summand P of $R_L^G(P')$ and $R_L^G(P'')$. From this we deduce that

- $R_L^G(P'')$ is indecomposable,
- $R_L^G(P') \simeq Q \oplus R_L^G(P'')^{\oplus m}$ with $m = 0, 1, 2$ and Q indecomposable. But $m \neq 0$ is impossible (the PIM $R_L^G(P'')$ cannot lie in the Harish-Chandra series of both (L, P') and (L, P'')).

This gives the second and fourth column of the decomposition matrix. As in the previous example, application of (Reg) to the fifth and sixth column of the decomposition matrix gives lower bounds for the decomposition numbers and D_{unip} has the following shape

$$D_{\text{unip}} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 2 & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \alpha & 1 & \cdot \\ \cdot & 2 & 1 & 1 & \beta & \gamma & 1 \end{bmatrix} \quad \text{with } \gamma \geq 4 \text{ and } 5 - 4\alpha + \beta \geq 0.$$

The two missing columns correspond to projective covers of cuspidal simple kG -modules. They can be obtained using (DL) from the decomposition of each P_w on the basis of PIMs. The minimal representatives of the F -conjugacy classes of $W \simeq \mathfrak{S}_5$ are ordered as follows under the Bruhat order: $1 \leq s_1, s_2 \leq s_1 s_2, s_2 s_3 s_3 \leq s_1 s_2 s_3 s_2 \leq w_0$. Let $Q_i, i = 1, \dots, 7$, be the PIMs ordered as the columns of the decomposition matrix. Then

$$\begin{aligned} e(P_1) &= e([Q_1] + [Q_3]) \quad (\text{already computed}) \\ e(P_{s_1}) &= \rho_{(5)} - \rho_{(32)} + \rho_{(2^21)} - \rho_{(1^5)} + \text{non-unip.} \\ &= e([Q_1] - [Q_3]) \\ e(P_{s_2}) &= \rho_{(5)} - \rho_{(41)} + \rho_{(32)} - \rho_{(2^21)} - \rho_{(21^3)} - \rho_{(1^5)} + \text{non-unip.} \\ &= e([Q_1] - [Q_2] + [Q_3]). \end{aligned}$$

with each $m_S \geq 0$. Among the simple kG -modules S such that $m_S \neq 0$, choose S_0 with smallest possible $w \in W$ such that $\langle P_w; [S_0] \rangle \neq 0$. In other words, if $v < w$ then $m_S \langle P_v; [S] \rangle = 0$ for every simple kG -module S .

If S is any simple kG -module such that $m_S \neq 0$, then by minimality of w and (DL) we have $(-1)^{\ell(w)} \langle P_w; [S] \rangle > 0$. In particular, $(-1)^{\ell(w)} m_S \langle P_w; [S] \rangle \geq 0$ for every simple kG -module S . Let us write

$$(-1)^{\ell(w w_0)} \langle e(P_w); R_{w_0} \rangle = (-1)^{\ell(w w_0)} \langle P_w; d(R_{w_0}) \rangle = \sum_{S \in \text{Irr}_k G} (-1)^{\ell(w)} m_S \langle P_w; [S] \rangle.$$

By the orthogonality relation of Deligne–Lusztig characters, this sum is zero unless $w = w_0$. Since all the terms are non-negative, and one term is positive (for $S = S_0$) we deduce that $w = w_0$. In other words, if $[P_S]$ occurs in some P_v for $v \neq w_0$ then $m_S = 0$. Now

$$\begin{aligned} \text{rk}_{\mathbb{Z}} \langle P_v, v \neq w_0 \rangle &= \#\{F\text{-conjugacy classes of } W\} - 1 \\ &= \#\{\text{unipotent characters of } G\} - 1 \\ &= \#\{\text{unipotent PIMs of } G\} - 1 \quad (\text{by Theorem 4.5}) \end{aligned}$$

Therefore the \mathbb{Z} -submodule of $K_0(kG\text{-proj})$ spanned by the virtual projective modules P_v for $v \neq w_0$ has codimension 1. We deduce that there is at most one simple kG -module S such that $m_S \neq 0$. It must be $S = S_0 = S_{(1^n)}$ since $e([P_{S_{(1^n)}}]) = \rho_{(1^n)} + \text{non-unipotent characters}$ and

$$\langle [P_{S_{(1^n)}}]; d(R_{w_0}) \rangle = \langle e([P_{S_{(1^n)}}]); R_{w_0} \rangle = \langle \text{St}; R_{w_0} \rangle = (-1)^{\ell(w_0)} \neq 0.$$

This proves that $d(R_{w_0}) = (-1)^{\ell(w_0)} [S_{(1^n)}]$.

Now (Uni) shows that the character of the projective cover of $S_{(21^{n-2})}$ is of the form $e([P_{S_{(21^{n-2})}}]) = \rho_{(21^{n-2})} + \alpha \rho_{(1^n)} + \text{non-unipotent characters}$. Using the fact that it is orthogonal to R_{w_0} we get

$$0 = \langle [P_{S_{(21^{n-2})}}]; d(R_{w_0}) \rangle = \langle e([P_{S_{(21^{n-2})}}]); R_{w_0} \rangle = \langle \rho_{(21^{n-2})}; R_{w_0} \rangle + (-1)^{\ell(w_0)} \alpha$$

which shows that

$$d_{(1^n), (21^{n-2})} := \alpha = (-1)^{\ell(w_0)+1} \langle \rho_{(21^{n-2})}; R_{w_0} \rangle = n - 1.$$

Indeed, the scalar product $\langle \rho_\lambda; R_w \rangle$ is, up to a sign, equal to the value on $w w_0$ of the irreducible character of \mathfrak{S}_n corresponding to the partition λ . So here, it equals the dimension of the representation of \mathfrak{S}_n corresponding to (21^{n-2}) which is $n - 1$ by the hook length formula. \square

4.8. Observations and conjectures. The following conjectures were made by Geck in [39] and Geck–Hiss in [46].

Conjecture 4.8 (Geck–Hiss). *Assume that $\ell \neq p$ and that ℓ is large with respect to $|W|$.*

- (i) *The decomposition matrix has a unitriangular shape.*
- (ii) *If ρ is unipotent and cuspidal then $d(\rho)$ is irreducible i.e. $d(\rho) = [S]$ for some simple kG -module S .*
- (iii) *The unipotent part of the decomposition matrix is independent of q (it depends only on the order of q in \mathbb{F}_ℓ^\times).*

The conjecture was first proven for groups of type A by Dipper [23], for groups of type 2A by Geck [40] and for classical groups when ℓ is *linear* by Gruber–Hiss [49] (building on previous work of Dipper–James [25, 26]). In addition, all the decomposition matrices computed so far for small rank group satisfy the conjecture. In a recent work [34], Malle and the author proved part (ii) of the conjecture under the extra assumption that p is good.

The difficulty in proving this conjecture lies in producing nice projective modules. In the case where the conjecture is known to hold, generalized Gelfand–Graev modules were used. Malle and the author proposed a different strategy in [31], using again the cohomology of Deligne–Lusztig varieties.

Conjecture 4.9 (Dudas–Malle). *For all $w \in W$ there is a sign $\varepsilon_w = \pm 1$ such that*

$$Q_w = \varepsilon_w D_G([IH^\bullet(\overline{\mathbb{X}(w)}, k(\mathbb{T}^{wF})_\ell)])$$

\swarrow *Alvis–Curtis duality* \searrow *intersection cohomology*

is the character of a (non-virtual) projective module.

The Q_w 's are actually virtual projective modules whose character can be explicitly computed using Kazhdan–Lusztig polynomials. They satisfy a unitriangularity property (as we expect for the PIMs) and the multiplicities on unipotent characters do not depend on q .

Example 4.10. For $w = w_0$ the variety $\mathbb{X}(w_0)$ is dense in \mathbb{G}/\mathbb{B} and hence $\overline{\mathbb{X}(w_0)} = \mathbb{G}/\mathbb{B}$. It is a smooth projective variety paved by affine spaces $\mathbb{B}w\mathbb{B}/\mathbb{B}$, therefore

$$[IH^\bullet(\overline{\mathbb{X}(w)}, K)] = [H^\bullet(\mathbb{G}/\mathbb{B}, K)] = |W| \cdot 1_G.$$

Consequently, $e(Q_{w_0}) = \pm |W| \cdot \text{St}_G \pm$ non-unipotent characters. There is indeed a PIM whose character is $\text{St}_G +$ non-unip., hence Q_{w_0} is the character of a projective module as claimed in Conjecture 4.9.

5. Brauer trees of unipotent blocks

This chapter is devoted to the study of unipotent blocks of finite reductive groups with cyclic defect groups. In that case the structure of the block is encoded in a planar embedded tree, the *Brauer tree*. We explain how to use the cohomology complexes of Deligne–Lusztig varieties to get information on the characters of PIMs (which gives the tree as a graph) and extensions between simple modules (which gives the planar embedding). This is based on a recent work of Craven, Rouquier and the author [17].

5.1. Brauer trees. Throughout this chapter, G is any finite group and b is an ℓ -block of $\mathcal{O}G$ with cyclic (and non-trivial) defect groups. We denote by D a defect of b . The results on the structure of b originate in a work of Brauer [9] subsequently completed by Dade [18] and Green [48]. For a self-contained treatment of blocks with cyclic defect groups we recommend [63] and [1, §V].

We denote by $\text{Irr}_K b$ the set of irreducible ordinary characters of G lying in b . There is a set $\text{Exc}_K b \subset \text{Irr}_K b$ called the set of *exceptional characters* of b such that if we define $\chi_{\text{exc}} := \sum_{\chi \in \text{Exc}_K b} \chi$ then the character of any projective indecomposable kG -module P in kb is given by

$$e([P]) = \chi + \chi'$$

with $\chi \neq \chi'$ and $\chi, \chi' \in \{\chi_{\text{exc}}\} \sqcup (\text{Irr}_K b \setminus \text{Exc}_K b)$. In other words, a simple kb -module occurs in the ℓ -reduction of either two distinct non-exceptional characters, or in one non-exceptional character and in every exceptional character.

We define the *Brauer graph* Γ_b of b as the graph with vertices labeled by $\{\chi_{\text{exc}}\} \sqcup (\text{Irr}_K b \setminus \text{Exc}_K b)$ and edges $\chi \text{ --- } \chi'$ for every PIM P such that $e([P]) = \chi + \chi'$. The edges of the Brauer graph are therefore labeled by PIMs or equivalently by simple kb -modules (via their projective cover). The knowledge of the Brauer graph, together with the *multiplicity* $m = \langle \chi_{\text{exc}}; \chi_{\text{exc}} \rangle$ of the exceptional vertex, is equivalent to the knowledge of the decomposition matrix.

Example 5.1. (a) Let $G = \mathbb{Z}/\ell\mathbb{Z}$ be the cyclic group of order ℓ . There is only one PIM $P = kG$ which is the projective cover of the trivial module. One can write its character as

$$e([P]) = 1_G + \sum \text{non-trivial characters} = 1_G + \chi_{\text{exc}}.$$

The convention here is to define the exceptional characters as the non-trivial characters of G . This gives the Brauer graph given in Figure 1.

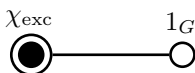


Figure 1. Brauer graph of $\mathbb{Z}/\ell\mathbb{Z}$

(b) Let us now consider the case of $G = \mathbb{Z}/\ell^r\mathbb{Z} \rtimes E$ where E is an ℓ' -subgroup of $\text{Aut}_{\ell'}(\mathbb{Z}/\ell^r\mathbb{Z}) \simeq \mathbb{Z}/(\ell-1)\mathbb{Z}$. In particular the order e of $|E|$ divides $\ell-1$. As in the previous example $\mathcal{O}G$ is indecomposable and hence it forms a single block.

Every simple kG -module S has a trivial action of the ℓ -group $\mathbb{Z}/\ell^r\mathbb{Z}$, and therefore it must be of the form $S = \text{Inf}_E^G(\text{Res}_E^G S)$. In addition, since E is an ℓ' -group, the restriction $\text{Res}_E^G S$ is semisimple hence simple. We deduce that the simple kG -modules are in bijection with the irreducible representations of E (over k or K). More precisely, if we fix a generator x of E and ζ a primitive e -th root of unity in \mathcal{O}^\times , one can consider the simple kE -modules $k_{\bar{\zeta}^i}$ to be the one-dimensional representation of E on which x acts by $\bar{\zeta}^i$, where $\bar{\zeta}$ denotes the class of ζ in k . They

lift to characteristic zero to one-dimensional $\mathcal{O}G$ -modules \mathcal{O}_{ζ^i} and KG -modules K_{ζ^i} on which x acts by ζ^i . Let $T_i := \text{Inf}_E^G K_{\zeta^i}$. Then:

- The kG -modules $T_i = \text{Inf}_E^G k_{\zeta^i}$ for $i = 0, 1, \dots, e - 1$ form a set of representatives of $\text{Irr}_k G$.
- T_i lifts to characteristic zero as $\tilde{T}_i = \text{Inf}_E^G \mathcal{O}_{\zeta^i}$. We denote by θ_i the character of $K\tilde{T}_i \simeq \text{Inf}_E^G K_{\zeta^i}$.
- The projective cover of T_i is $P_i := \text{Ind}_E^G k_{\zeta^i}$, which lifts to characteristic zero as $\tilde{P}_i = \text{Ind}_E^G \mathcal{O}_{\zeta^i}$. It has character

$$e([P_i]) = [\text{Ind}_E^G K_{\zeta^i}] = [\text{Inf}_E^G K_{\zeta^i}] + \theta_{\text{exc}} = \theta_i + \theta_{\text{exc}}$$

where θ_{exc} denotes the sum of the irreducible characters of G which are non-trivial on $\mathbb{Z}/\ell^r\mathbb{Z}$. These are the exceptional characters of G .

We deduce that the Brauer graph is a star-shaped tree as shown in Figure 2.

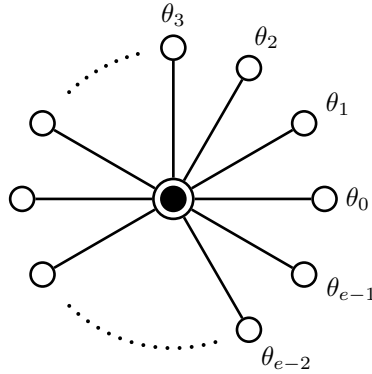


Figure 2. Brauer graph of $\mathbb{Z}/\ell^r\mathbb{Z} \rtimes \mathbb{Z}/e\mathbb{Z}$ with $e \mid \ell - 1$

(c) Let $G = \mathfrak{S}_\ell$ and b be the principal ℓ -block (with defect a Sylow subgroup $\mathbb{Z}/\ell\mathbb{Z}$ of \mathfrak{S}_ℓ). The irreducible ordinary characters of G are labeled by partitions of ℓ . The characters in b correspond to partitions which are ℓ -hooks

$$\text{Irr}_K b = \{1_G = \chi(\ell), \chi(\ell-1,1), \chi(\ell-2,1^2), \dots, \chi(1^\ell) = \varepsilon\}.$$

Here ε denotes the sign character. The subgroup $\mathfrak{S}_{\ell-1}$ of G is an ℓ' -group, therefore every irreducible character is the character of a projective module. Consequently, the same holds for the induction of any representation from $\mathfrak{S}_{\ell-1}$ to \mathfrak{S}_ℓ . Using the branching rules for induction we have

$$\text{Ind}_{\mathfrak{S}_{\ell-1}}^{\mathfrak{S}_\ell} \begin{array}{|c|} \hline \square \dots \square \\ \hline \vdots \\ \hline \square \end{array} = \underbrace{\begin{array}{|c|} \hline \square \dots \square \\ \hline \vdots \\ \hline \square \end{array} + \begin{array}{|c|} \hline \square \dots \square \\ \hline \vdots \\ \hline \square \end{array}}_{\text{projective character}} + \left(\begin{array}{|c|} \hline \square \dots \square \\ \hline \vdots \\ \hline \square \end{array} \right) \downarrow \text{not in } b$$

This shows that the Brauer graph Γ_b is a line as shown in Figure 3. Note that here any vertex can be chosen to be the exceptional vertex.

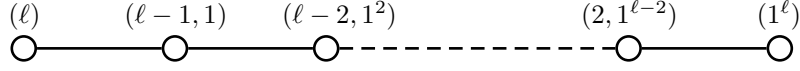


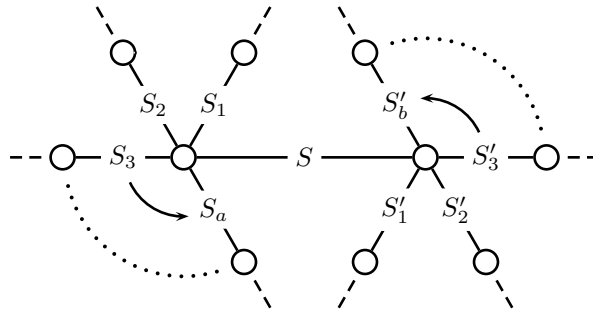
Figure 3. Brauer graph of the principal ℓ -block of \mathfrak{S}_ℓ

Theorem 5.2 ([1, §23]). *The Brauer graph is a tree, called the Brauer tree of b .*

A planar embedding of Γ_b is defined by an ordering of the set of edges incident to any given vertex. Planar embedded trees will be drawn according to the anti-clockwise order around a vertex.

Theorem 5.3 (Structure of PIMs [1, §22]). *Let Γ_b be the Brauer tree of b .*

- (i) *There exists a unique ordering around each vertex of Γ_b such that if S and T are two simple kb -modules labeling edges incident to a given vertex then T follows immediately S if and only if $\text{Ext}_{kG}^1(S, T) \neq 0$.*
- (ii) *Given a simple kb -module S labeling an edge between non-exceptional vertices as follows*



the Loewy structure of P_S is given by

$$P_S = \begin{matrix} & S & \\ & S_1 & S'_1 \\ & S_2 & S'_2 \\ & \vdots & \vdots \\ & S_a & S'_b \\ & S & \end{matrix}$$

Together with this planar embedding, the Brauer tree Γ_b will be referred to as the *planar embedded Brauer tree* of b . This tree encodes the structure of the block b . Indeed, if b' is a block of G' with cyclic defect groups, then kb and kb' are Morita equivalent if and only if the planar embedded Brauer trees of b and b' coincide, and the multiplicity of the exceptional vertices are equal.

Remark 5.4. Note that the structure of P_S can also be described in the case where the edge of S is connected to the exceptional node. In that case one needs to repeat the composition series $S_1/\cdots/S_a/S$ a number of times equal to the multiplicity $m = \langle \chi_{\text{exc}}; \chi_{\text{exc}} \rangle$ of the exceptional vertex. In other words, one needs to turn around the exceptional vertex m times and consider that all the other vertices have multiplicity one.

The proof of the structure theorem consists in constructing a stable equivalence between the block b of $\mathcal{O}G$ and its Brauer correspondent c in $N_G(D)$. This equivalence is built from the Green correspondence between b and its Brauer correspondent in $N_G(\mathbb{Z}/\ell\mathbb{Z})$ (with $\mathbb{Z}/\ell\mathbb{Z}$ the unique subgroup of order ℓ of D) and a Morita equivalence between the latter block and its Brauer correspondent in $N_G(D)$ (which is c). For a block with normal defect group (as in the case of c), the Brauer tree is a star, as shown in Example 5.1.b, and the structure theorem is easily proved. For more details see for example [63] or [1, §V].

The structure theorem for the PIMs has the following consequence, which we will use in the following section. Let χ be an irreducible ordinary character of Kb which labels a leaf of Γ_b (a vertex with only one incident edge). The ℓ -reduction of χ is the simple kG -module S which labels the unique edge incident to that leaf. This is the particular case of Theorem 5.3 where $a = 0$. Then ΩS is a uniserial module with composition factors $S'_1/\cdots/S'_b/S$. Moreover, it lifts to an ordinary character labeling the unique vertex adjacent to the leaf, say χ' , so that $e([P_S]) = \chi + \chi'$. Similarly, $\Omega^2 S$ is a uniserial module which lifts to an ordinary character χ'' such that $e([P_{S'_1}]) = \chi' + \chi''$. The edge labeled by S'_1 is the edge which comes directly after S in the cyclic ordering around χ' . If we iterate this process, we see that $\Omega^i S$ can be obtained by following the edges in a walk around the tree. More precisely, we obtain a sequence of ordinary characters χ_i and simple kb -modules T_i such that:

- $\Omega^i S$ is a uniserial module with head T_i and socle T_{i-1} .
- $\Omega^i S$ lifts to an $\mathcal{O}G$ -lattice with character χ_i (which is either χ_{exc} or irreducible, and hence labels a vertex).
- $e([P_{T_i}]) = \chi_i + \chi_{i+1}$.
- T_{i+1} labels the edge coming directly after T_i in the cyclic ordering around the vertex labeled by χ_{i+1} .

If e denotes the number of isomorphism classes of simple kb -modules, then $\Omega^{2e} S \simeq S$. Each simple kG -module appears exactly twice in the sequence $T_0, T_1, \dots, T_{2e-1}$, called the *Green walk around Γ_b* [48].

Example 5.5. We consider the planar embedded tree as shown in Figure 4. Then for example $\Omega^3 S \simeq S_2$ lifts to a lattice with character ρ_3 whereas $\Omega^8 S$ is uniserial with composition series $S_4/S_1/S_2/S_3$, and it lifts to a lattice with character ρ_2 . The sequence T_0, T_1, \dots, T_9 is $S, S_1, S_2, S_2, S_3, S_3, S_4, S_4, S_1, S$, whereas the sequence χ_0, \dots, χ_9 is $\chi, \rho_1, \rho_2, \rho_3, \rho_2, \chi_{\text{exc}}, \rho_2, \rho_4, \rho_2, \rho_1$.

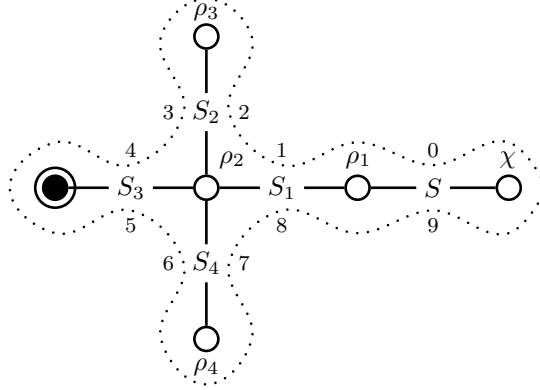


Figure 4. Walking around the Brauer tree

5.2. The case of unipotent blocks. We now focus on the case of finite reductive groups. As before, \mathbb{G} is a connected reductive group over $\overline{\mathbb{F}}_p$ together with a Frobenius endomorphism $F : \mathbb{G} \rightarrow \mathbb{G}$ defining an \mathbb{F}_q -structure. Throughout this section and until the end of this chapter we will assume for simplicity that (\mathbb{G}, F) is *split* (i.e. F acts trivially on the Weyl group).

We denote by d the order of q modulo ℓ , or equivalently the order of the class of q in k . The integer d is minimal for the property that $\ell \mid \Phi_d(q)$, where Φ_d is the d -th cyclotomic polynomial. Recall that when ℓ is very good, the unipotent ℓ -blocks are “generic” and parametrized by d -cuspidal pairs.

$$\begin{aligned} \left\{ \begin{array}{l} \text{Unipotent } \ell\text{-blocks} \\ \text{with defect } D \end{array} \right\} &\longleftrightarrow \left\{ \begin{array}{l} d\text{-cuspidal pairs } (\mathbb{L}, \rho) \\ \text{with } D \simeq (Z(\mathbb{L})^\circ)_\ell^F \end{array} \right\} / G \\ b(\mathbb{L}, \rho) &\longleftarrow (\mathbb{L}, \rho) \end{aligned} \quad (5.1)$$

When D is cyclic, the non-exceptional characters in $b(\mathbb{L}, \rho)$ are the unipotent characters in $b(\mathbb{L}, \rho)$, which are the irreducible constituents of the virtual character

$$R_{\mathbb{L}}^{\mathbb{G}}(\rho) = \sum_{i \in \mathbb{Z}} (-1)^i [H_c^i(\mathbb{Y}_{\mathbb{V}}, K)_\rho]$$

where $\mathbb{V} = R_u(\mathbb{P})$ for some parabolic subgroup \mathbb{P} with Levi complement \mathbb{L} . The variety $\mathbb{Y}_{\mathbb{V}}$ is the parabolic Deligne–Lusztig variety attached to \mathbb{V} (see §3.1). Recall that when \mathbb{L} is a maximal torus of type w and \mathbb{P} is conjugate to the Borel subgroup \mathbb{B} by \dot{w} then $\mathbb{Y}_{\mathbb{V}} \simeq \mathbb{Y}(\dot{w})$. In that case the non-exceptional characters in the block are the constituents of the Deligne–Lusztig character R_w .

We first list the algebraic methods which can be used to determine the Brauer trees of unipotent blocks of G with cyclic defect groups. The first three arguments are not specific to finite reductive groups, whereas the last one relies on results by Geck [42].

(Parity) If $\chi \sim \chi'$ then $\chi(1) \equiv -\chi'(1)$ modulo ℓ (the dimension of a projective module is divisible by ℓ).

(Real stem) If b is stable by complex conjugation, the real characters form a single connected line containing the exceptional node called the *real stem* of Γ_b . The complex conjugation induces a symmetry of Γ_b with respect to that line.

(Degree) The dimension of a non-exceptional character χ equals the sum of the dimensions of the simple modules labeling the edges incident to χ in Γ_b .

(Hecke) The Brauer trees of the blocks of the Hecke algebra associated to a given Harish-Chandra series in b (as defined by Geck in [42]) are subtrees of Γ_b . Each of these subtrees is a line, with dimension (or rather a -function) increasing towards the exceptional vertex.

Example 5.6. Let G be a finite reductive group of type G_2 . We denote by s and t the two simple reflections of its Weyl group W . The degrees of W are 2 and 6, therefore the order of the finite group G is

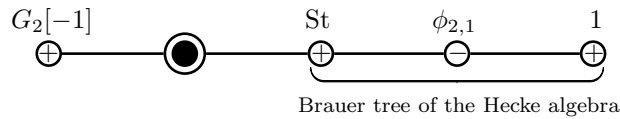
$$\begin{aligned} |G_2(q)| &= q^6(q^6 - 1)(q^2 - 1) \\ &= q^6(q - 1)^2(q + 1)^2(q^2 + q + 1)(q^2 - q + 1). \end{aligned}$$

In this decomposition the exponents of the cyclotomic polynomials $\Phi_3(q) = q^2 + q + 1$ and $\Phi_6(q) = q^2 - q + 1$ are equal to 1. Therefore when $\ell > 3$ (when ℓ is good) and ℓ divides one of these polynomials, the Sylow ℓ -subgroups of G are cyclic and the principal block has cyclic defect groups.

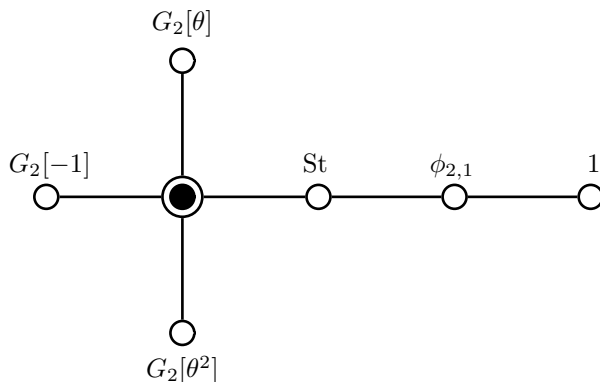
Assume that $\ell > 3$ and $\ell \mid \Phi_6(q) = q^2 - q + 1$, in which case q has order 6 modulo ℓ (the Coxeter number). A torus of type $w = st$ is a Φ_6 -Sylow subgroup since $|\mathbb{T}^{wF}| = q^2 - q + 1$. The trivial character of this torus is 6-cuspidal, and the corresponding block via (5.1) is the principal block, whose characters are

$$\begin{aligned} \text{Irr}_K b(\mathbb{T}_w, 1) &= \left\{ \begin{array}{l} \text{constituents of} \\ R_w = R_w(1) \end{array} \right\} \bigsqcup \left\{ \begin{array}{l} \text{constituents of } R_w(\theta) \\ \text{for } \theta \in \text{Irr}_\ell \mathbb{T}^{wF} \text{ and } \theta \neq 1 \end{array} \right\} \\ &= \underbrace{\{ 1, \text{St}, \phi_{2,1} \}}_{\text{principal series}} \bigsqcup \underbrace{\{ G_2[-1], G_2[\theta], G_2[\theta^2] \}}_{\text{cuspidal characters}} \bigsqcup \text{Exc}_K b \end{aligned}$$

since $R_w = 1 + \text{St} - \phi_{2,1} + G_2[-1] + G_2[\theta] + G_2[\theta^2]$. As in §4.6, we use the notation of [16, §13] for the unipotent characters. In particular θ is a primitive third root of 1 in \mathcal{O}^\times . The only non-real unipotent characters are $G_2[\theta]$ and $G_2[\theta^2]$, therefore by (Degree) or (Parity) the real stem is



By (Parity) the complex conjugate characters $G_2[\theta]$ and $G_2[\theta^2]$ must be connected to either the vertex labeled by $\phi_{2,1}$ or the exceptional vertex. But $\phi_{2,1}(1) - G_2[\theta](1) - G_2[\theta^2](1) < 0$ if $q > 2$. Now $q = 2$ would force $\ell = 3$, which is a prime number that we excluded. Therefore (Degree) forces Γ_b to be as in Figure 5.

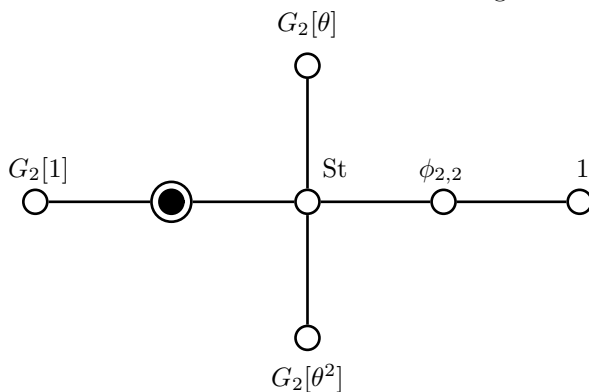
Figure 5. Brauer tree of the principal Φ_6 -block of $G_2(q)$

Exercise 5.7. If $\ell \mid q^2 + q + 1 = \Phi_3(q)$ then $|\mathbb{T}^{wF}| = q^2 + q + 1$, and \mathbb{T}^{wF} is a Φ_3 -Sylow subgroup for $w = stst = (st)^2$. Then the non-exceptional characters of the principal block, with their parity, are

$$\text{Irr}_K b \setminus \text{Exc}_K b = \{1, \text{St}, \phi_{2,2}, G_2[1], G_2[\theta], G_2[\theta^2]\}.$$

+ + - + - -

Here the only non-real characters are again the complex conjugate characters $G_2[\theta]$ and $G_2[\theta^2]$. Show that in that case the Brauer tree is given as in Figure 6.

Figure 6. Brauer tree of the principal Φ_3 -block of $G_2(q)$

Theorem 5.8. *The Brauer trees of unipotent blocks are known for*

- (i) \mathbb{G} of classical type A , B , C and D (Fong–Srinivasan [37, 38]).
- (ii) \mathbb{G} of exceptional type except E_7 and E_8 .

Note that for groups of small rank, the determination of the trees follows from the determination of all the ℓ -decomposition matrices for various ℓ , which were

more specifically solved by Burkhart [14] for 2B_2 , Shamash [67] for G_2 , Geck [41] for 3D_4 , Hiss [51] for 2G_2 and 2F_4 , and Wings [70] for F_4 . The determination of the other trees were obtained by Hiss–Lübeck [52] for F_4 and 2E_6 , and Hiss–Lübeck–Malle [53] for E_6 . In addition to the algebraic arguments used in the example of $G_2(q)$, it is often necessary to use partial information on the character table of the group to determine the tree (in order to decompose tensor products of characters).

The problem for larger exceptional groups such as E_7 and E_8 comes from the increasing number of cuspidal kG -modules. As in the case of decomposition matrices (see Chapter 4), these representations resist to algebraic methods which rely on Harish–Chandra induction and restriction, such as (Hecke). The idea developed in [17] by Craven–Rouquier and the author is to use the cohomology complexes of Deligne–Lusztig varieties to produce perfect complexes satisfying the following proposition.

Proposition 5.9. *Let C be a perfect complex of kG -modules. Assume that there exist integers $a < b$ such that $H^i(C) = 0$ for $i \neq a, b$. Then*

$$H^a(C) \simeq \Omega^{b-a+1}H^b(C) \quad \text{in } kG\text{-stab.}$$

Consequently, the kG -modules $H^a(C)$ and $\Omega^{b-a+1}H^b(C)$ differ only by their projective summands.

Proof. Using the remark following (1.2) we get the following distinguished triangle in $D^b(kG\text{-mod})$

$$H^a(C)[-a] \longrightarrow C \longrightarrow H^b(C)[-b] \rightsquigarrow$$

which we can also write as

$$C \longrightarrow H^b(C)[-b] \longrightarrow H^a(C)[-a+1] \rightsquigarrow$$

Its image in $D^b(kG\text{-mod})/kG\text{-perf}$ is also a distinguished triangle, and $C \simeq 0$ in that category since it is perfect. Using Theorem 1.24 to identify $D^b(kG\text{-mod})/kG\text{-perf}$ with $kG\text{-stab}$ (and the shift [1] with Ω^{-1}) we obtain the following distinguished triangle in $kG\text{-stab}$

$$0 \longrightarrow \Omega^b H^b(C) \longrightarrow \Omega^{a-1} H^a(C) \rightsquigarrow$$

which yields $\Omega^b H^b(C) \xrightarrow{\sim} \Omega^{a-1} H^a(C)$ in $kG\text{-stab}$. \square

Given $\lambda \in \mathcal{O}^\times$, we can consider the complex $C = bR\Gamma_c(\mathbb{X}(w), k)_{\overline{\lambda}}$ obtained from the cohomology complex of $\mathbb{X}(w)$ by cutting by the generalized $\overline{\lambda}$ -eigenspace of F , and by the block b . In other words, the complex C is isomorphic to a direct summand of $R\Gamma_c(\mathbb{X}(w), k)$ such that $H^i(C) = bH_c^i(\mathbb{X}(w), k)_{\overline{\lambda}}$. The requirements to use Proposition 5.9 are

- (1) C must be perfect. This follows from Proposition 3.7 if we assume $\ell \nmid |\mathbb{T}^{wF}|$.
- (2) The condition on the vanishing of $H^i(C)$ should already hold over K by the universal coefficient formula (see §6.3). This supposes a vanishing of many of the groups $H_c^i(\mathbb{X}(w), K)_\mu$ for every eigenvalue μ congruent to λ modulo ℓ .

- (3) The vanishing over k should follow from the vanishing over K and the property that each cohomology group $H_c^i(\mathbb{X}(w), \mathcal{O})$ is \mathcal{O} -free.

In the few examples where $H_c^i(\mathbb{X}(w), \mathcal{O})$ has been explicitly computed, it is torsion-free whenever $\ell \nmid |\mathbb{T}^{wF}|$ (in other words (3) follows from (1)). The reader will find an example of this property in the following chapter, for varieties associated with Coxeter elements.

Example 5.10. Let G be a finite reductive group of type E_7 . We denote by s_1, \dots, s_7 the simple reflections in W . If q has order 14 modulo ℓ , in which case ℓ divides $\Phi_{14}(q)$ then the principal ℓ -block has cyclic defect groups. It corresponds to the cuspidal pair $(\mathbb{T}_w, 1)$ where w is a element of W of order 14 (such an element can be taken to have length 9). The fourteen non-exceptional characters in the block are given by the irreducible constituents of the Deligne–Lusztig character R_w . They consist of eight unipotent characters in the principal series (including 1 and St), four unipotent characters in the D_4 -series and the two cuspidal unipotent characters of E_7 , namely the complex conjugate characters $E_7[i]$ and $E_7[-i]$. The real stem of the Brauer tree is formed by the real characters (all except $E_7[\pm i]$) with increasing dimension towards the exceptional vertex by (Hecke), see Figure 7.

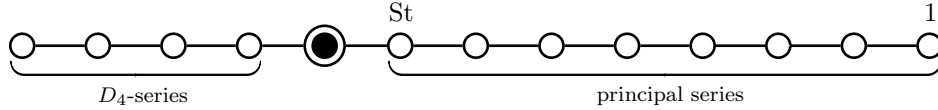


Figure 7. Real stem of Brauer tree of the principal Φ_{14} -block of $E_7(q)$

The missing characters in the tree are the complex conjugate characters $E_7[\pm i]$. Unlike the case of $G_2(q)$ in Example 5.6, a combination of the (Degree) and (Parity) arguments is not enough to determine to which vertex they are attached. To remove the ambiguity, we consider the Deligne–Lusztig variety $\mathbb{X}(c)$ associated to a Coxeter element $c = s_1 s_2 \cdots s_7$ and the corresponding cohomology complex. Here we will consider the generalized eigenspaces of F corresponding to the eigenvalues 1 and -1 . Over K , the cohomology of $\mathbb{X}(c)$, cut by the block, is

$$bH_c^\bullet(\mathbb{X}(c), K) = (\text{St} \oplus E_7[i] \oplus E_7[-i])[-7] \oplus 1[-14]$$

with eigenvalues of F in K	=	1	$iq^{7/2}$	$-iq^{7/2}$	q^7
and eigenvalues of F in k	=	1	-1	1	-1

with the convention that $i \equiv q^{7/2}$ modulo ℓ (here $i^2 = -1$ in K). We obtain

$$bH_c^\bullet(\mathbb{X}(c), K)_{1+\ell\mathcal{O}} \simeq (\text{St} \oplus E_7[-i])[-7],$$

$$bH_c^\bullet(\mathbb{X}(c), K)_{-1+\ell\mathcal{O}} \simeq E_7[i][-7] \oplus 1[-14].$$

If we assume that $H_c^\bullet(\mathbb{X}(c), \mathcal{O})$ is torsion-free, then the universal coefficient theorem shows that $bR\Gamma_c(\mathbb{X}(c), k)_1$ has only one non-zero cohomology group. By using the

truncation functors of Proposition 1.19 it follows that it is quasi-isomorphic to a single projective kG -module in degree 7, which lifts to a lattice with character $\text{St} \oplus E_7[-i]$. This shows that the vertex labeled by $E_7[-i]$ is connected to the vertex labeled by St and gives the Brauer tree of b . Even better, using Proposition 5.9 with the complex $bR\Gamma_c(\mathbb{X}(c), K)_{-1}$ which has only two non-zero cohomology groups yields

$$H_c^7(\mathbb{X}(c), k)_{-1} \simeq \Omega^{14-7+1} H_c^{14}(\mathbb{X}(c), k)_{-1} \simeq \Omega^8 k$$

in the stable category $kG\text{-stab}$. This proves that $\Omega^8 \mathcal{O}$ is an $\mathcal{O}G$ -lattice with character $H_c^7(\mathbb{X}(c), K)_{-1} = E_7[i]$ and the planar embedded Brauer tree shown in Figure 8 is obtained from the Green walk.

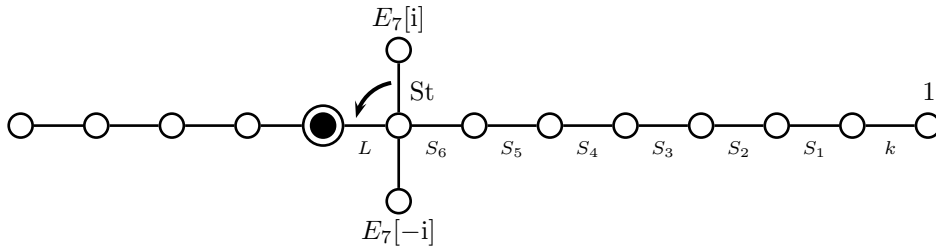


Figure 8. Brauer tree of the principal Φ_{14} -block of $E_7(q)$

Exercise 5.11. Show that the perfect complex $R\Gamma_c(\mathbb{X}(c), k)_{-1}$ is homotopy equivalent to

$$0 \longrightarrow \begin{array}{c} E_7[i] \\ L \\ E_7[-i] \\ S_6 \\ \boxed{E_7[i]} \end{array} \longrightarrow \begin{array}{c} S_6 \\ E_7[i] \\ L \quad S_5 \\ E_7[-i] \\ S_6 \end{array} \longrightarrow \begin{array}{c} S_5 \\ S_6 \quad S_4 \\ S_5 \end{array} \longrightarrow \begin{array}{c} S_4 \\ S_5 \quad S_3 \\ S_4 \end{array} \longrightarrow \cdots \longrightarrow \begin{array}{c} \boxed{k} \\ S_1 \\ k \end{array} \longrightarrow 0.$$

This method has proven very powerful in the case of exceptional groups of type $E_7(q)$ and $E_8(q)$, removing also some ambiguity in the planar embedding for the Ree groups ${}^2F_4(q)$. Luckily, only small-dimensional Deligne–Lusztig varieties were needed to complete the determination of the trees, which made checking the vanishing of the cohomology possible.

Theorem 5.12 (Craven–Dudas–Rouquier [17]). *All the planar embedded Brauer trees of unipotent blocks of finite reductive groups are explicitly known in terms of Lusztig’s parametrization of unipotent characters.*

6. The Coxeter variety

The last chapter of these notes is devoted to the study of Deligne–Lusztig varieties associated to a special class of Weyl group elements, called the Coxeter elements.

These varieties were first studied by Lusztig [56]. Computing their ℓ -adic cohomology turned out to be a key ingredient in Lusztig's subsequent work on the classification of ordinary irreducible characters of finite reductive groups. We explain in this chapter how to extend Lusztig's result to the modular setting, building on work of Bonnafé–Rouquier [6] and the author [28, 30].

Throughout this chapter we will assume for simplicity that (\mathbb{G}, F) is *split* (i.e. F acts trivially on the Weyl group). All the main results of this chapter can be generalized to the case where a power of F acts trivially (which includes the case of the Ree and Suzuki groups).

6.1. Geometry of the Coxeter variety. Let \mathbb{T} be a split maximal torus of \mathbb{G} , contained in an F -stable Borel subgroup \mathbb{B} of \mathbb{G} . Recall that the simple reflections s_1, \dots, s_r of the Weyl group $W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ are exactly the elements $s \in W$ such that $\mathbb{B}s\mathbb{B}/\mathbb{B}$ has dimension 1.

A *Coxeter element* c of W is a product $c = s_1 \cdots s_r$ of all the simple reflections, in any order. Coxeter elements are conjugate under W . The order of any Coxeter element is the *Coxeter number* of W , denoted by h . A *Coxeter variety* is a Deligne–Lusztig variety $\mathbb{X}(c)$ or $\mathbb{Y}(c)$ attached to a Coxeter element. Since Coxeter elements are the elements of minimal length in their conjugacy class, they are actually conjugate by a sequence of cyclic shifts. Consequently, the cohomology of a Coxeter variety does not depend on the choice of a Coxeter element. For that reason we shall denote these varieties simply by \mathbb{X} or \mathbb{Y} . For more details on Coxeter elements see [8, §V.6] and for Coxeter varieties see [56].

Example 6.1. Let $\mathbb{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$. The permutation $c = (1, 2)(2, 3) \cdots (n-1, n) = (1, 2, 3, \dots, n)$ is a Coxeter element of \mathfrak{S}_n . It has length $n-1$ and order $h = n$. For the general linear group the flag variety \mathbb{G}/\mathbb{B} can be identified with the set of flags of vector spaces $V_{\bullet} = (\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \overline{\mathbb{F}}_p^n)$ such that $\dim_{\overline{\mathbb{F}}_p} V_i = i$. With this description, the Deligne–Lusztig variety associated to an element $w \in \mathfrak{S}_n$ is

$$\mathbb{X}(w) \simeq \{V_{\bullet} \in \mathbb{G}/\mathbb{B} \mid V_{\bullet} \text{ and } F(V_{\bullet}) \text{ are in relative position } w\}.$$

Recall that two flags V_{\bullet} and V'_{\bullet} are said to be *in relative position* w if there exist $e_1, \dots, e_n \in \overline{\mathbb{F}}_p^n$ such that e_1, \dots, e_i is a basis of V_i and $e_{w(1)}, \dots, e_{w(i)}$ is a basis of V'_i for each $i = 0, \dots, n$. With $w = c = (1, 2, \dots, n)$ we deduce that $V_{\bullet} \in \mathbb{X}$ if and only if e_1, \dots, e_i is a basis of V_i and e_2, \dots, e_{i+1} is a basis of $F(V_i)$. This can be written $V_{i+1} = V_1 \oplus F(V_i)$, which yields in turn

$$V_{i+1} = V_1 \oplus F(V_1) \oplus \cdots \oplus F^i(V_1).$$

This gives an explicit description of the Coxeter variety in the case of $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ as

$$\begin{aligned} \mathbb{X} &\simeq \{V_1 \in \mathbb{P}(\overline{\mathbb{F}}_p^n) \mid \overline{\mathbb{F}}_p^n = V_1 \oplus F(V_1) \oplus \cdots \oplus F^{n-1}(V_1)\} \\ &\simeq \left\{ [x_1 : x_2 : \cdots : x_n] \in \mathbb{P}_{n-1} \mid \begin{vmatrix} x_1 & x_1^q & \cdots & x_1^{q^{n-1}} \\ x_2 & x_2^q & \cdots & x_2^{q^{n-1}} \\ \vdots & \vdots & & \vdots \\ x_n & x_n^q & \cdots & x_n^{q^{n-1}} \end{vmatrix} \neq 0 \right\}. \end{aligned}$$

If $v \in W$ is such that $v < c$, then it is obtained from c by removing some simple reflections. Therefore it is a product of simple reflections lying in a proper subset I of S , and as such it is a Coxeter element of the parabolic subgroup W_I of W . We shall write $v = c_I$. Consequently,

$$\overline{\mathbb{X}(c)} = \bigsqcup_{v \leq c} \mathbb{X}(v) = \bigsqcup_{I \subset S} \mathbb{X}(c_I). \quad (6.1)$$

Let $\mathbb{P}_I = \mathbb{B}W_I\mathbb{B}$ (resp. \mathbb{L}_I) be the standard parabolic subgroup of \mathbb{G} (resp. standard Levi subgroup of \mathbb{G}) associated to I . Its unipotent radical will be denoted by \mathbb{U}_I . We will write $\mathbb{X}_I = \mathbb{X}_{\mathbb{L}_I}(c_I)$ for the Coxeter variety of \mathbb{L}_I . We have $\mathbb{X}(c_I) \simeq G/U_I \times_{L_I} \mathbb{X}_I$ and therefore

$$R\Gamma_c(\mathbb{X}(c_I), \Lambda) \simeq R_{L_I}^G(R\Gamma_c(\mathbb{X}_I, \Lambda)). \quad (6.2)$$

(Recall that Λ is any ring among K , \mathcal{O} and k).

There are two key properties of the Coxeter variety that are needed to compute its cohomology (and to show that it is torsion-free). The first one is given by (6.1). The second is a result of Lusztig [56] giving the quotient of \mathbb{X} by unipotent subgroups in terms of Coxeter varieties of Levi subgroups.

Proposition 6.2. *Let $I \subset S$. There is a (non-equivariant) isomorphism of varieties*

$$U_I \backslash \mathbb{X} \simeq \mathbb{X}_I \times (\mathbb{G}_m)^{r-|I|}$$

which induces an $L_I \times \langle F \rangle$ -equivariant isomorphism of ℓ -adic cohomology groups

$${}^*R_{L_I}^G(H_c^\bullet(\mathbb{X}, K)) \simeq H_c^\bullet(\mathbb{X}_I, K) \otimes_K H_c^\bullet(\mathbb{G}_m, K)^{\otimes r-|I|}.$$

In particular, with I being the empty set we deduce that $U_\emptyset \backslash \mathbb{X} \simeq (\mathbb{G}_m)^r$. One can actually refine Lusztig's result as follows (see [30, Lem. 4.2]): if $J = S \setminus I$ is the complement of I in S then

$$(U_I \cap U_J) \backslash \mathbb{X} \simeq \mathbb{X}_I \times \mathbb{X}_J. \quad (6.3)$$

Again, this isomorphism is not equivariant for the action of P_I or P_J in general.

6.2. Cohomology over K . Using a combination of (6.1), Proposition 6.2 and computations of $\#\mathbb{X}^{F^n}$ for $n = 1, \dots, h$ (in order to use the Lefschetz trace formula, see Theorem 2.12) Lusztig gave in [56] a complete description of the cohomology of \mathbb{X} over K , with the action of G and F .

Theorem 6.3 (Lusztig).

- (i) A cuspidal character $\rho \in \text{Irr } G$ occurring in the cohomology of \mathbb{X} occurs in the middle degree $H_c^r(\mathbb{X}, K)$ only.
- (ii) The eigenspaces of F on $H_c^\bullet(\mathbb{X}, K)$ give h mutually non-isomorphic irreducible representations of G .
- (iii) The eigenvalues of F on $H_c^\bullet(\mathbb{X}, K)$, restricted to a given Harish-Chandra series, are of the form

$$\frac{i \mid r \quad r+1 \quad \cdots \quad 2(r-m_\zeta)}{H_c^i(\mathbb{X}, K) \mid \zeta q^{m_\zeta} \quad \zeta q^{m_\zeta+1} \quad \cdots \quad \zeta q^{r-m_\zeta}}$$

for some root of unity $\zeta \in \mathcal{O}^\times$ and some $m_\zeta \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. No constituent of $H_c^i(\mathbb{X}, K)$ for $i > 2(r-m_\zeta)$ lies in that Harish-Chandra series.

Sketch of proof. If $v < c$ then $v = c_I$ for some proper subset I of S and from (6.2) we get $H_c^\bullet(\mathbb{X}(v), K) \simeq R_{L_I}^G(H_c^\bullet(\mathbb{X}_I, K))$. Therefore $H_c^\bullet(\mathbb{X}(v), K)$ contains no cuspidal character. By Theorem 3.10, it follows that ρ cannot be a constituent of $H_c^i(\mathbb{X}, K)$ for $i \neq \ell(w) = r$, which proves (i). In addition, one can show using the purity of $\overline{\mathbb{X}}$ that F has eigenvalue $\zeta q^{r/2}$ with $|\zeta| = 1$ on the ρ -isotypic part of $H_c^r(\mathbb{X}, K)$. Furthermore, it is a general property of the cohomology of Deligne–Lusztig varieties that ζ is actually a root of unity.

Let ρ be a cuspidal character of L_I for some $I \subset S$ and set $m = |I|$. The eigenvalue of F on the ρ -isotypic part of $H_c^m(\mathbb{X}_I, K)$ is of the form $\zeta q^{m/2}$ for some root of unity $\zeta \in \mathcal{O}^\times$. If χ is an irreducible character of G lying in the Harish-Chandra series of (L_I, ρ) then $*R_{L_I}^G(\chi)$ is a non-zero multiple of ρ . In particular, the eigenvalues of F on the Harish-Chandra series of (L_I, ρ) correspond to the eigenvalues of F on the ρ -isotypic part of $*R_{L_I}^G(H_c^\bullet(\mathbb{X}, K))$, which by Proposition 6.2 and (i) are $\zeta q^{m/2}$ times the eigenvalues of F on the cohomology of $(\mathbb{G}_m)^{r-m}$. Assertion (iii) follows.

The proof of (ii) requires to compute the endomorphism algebra of $H_c^\bullet(\mathbb{X})$, which would go beyond the scope of these notes. \square

Example 6.4. (a) Let $\mathbb{G} = \text{GL}_n(\overline{\mathbb{F}}_p)$. Recall that the unipotent characters of G are parametrized by partitions of n . We will represent them by their Young diagram. Then the cohomology of the Coxeter variety (given in Example 6.1) with the eigenvalues of F is

$$\frac{i \mid n-1 \quad n \quad n+1 \quad \cdots \quad 2n-3 \quad 2n-2}{H_c^i(\mathbb{X}) \mid \begin{array}{c} \square \\ \vdots \\ \square \end{array} (1) \quad \begin{array}{c} \square \square \\ \vdots \\ \square \end{array} (q) \quad \begin{array}{c} \square \square \square \\ \vdots \\ \square \end{array} (q^2) \quad \cdots \quad \begin{array}{c} \square \cdots \square \\ \vdots \\ \square \end{array} (q^{n-2}) \quad \begin{array}{c} \square \cdots \square \\ \vdots \\ \square \end{array} (q^{n-1})}$$

(b) Let \mathbb{G} a group of type F_4 . Using the notation in [16, §13] for the unipotent characters of G (in particular θ is a primitive third root of 1 and i a primitive fourth root of 1), the cohomology of \mathbb{X} is given by

i	4	5	6	7	8
$H_c^i(\mathbb{X})$	$\text{St}(1)$	$\phi_{4,13}(q)$	$\phi_{6,6}'(q^2)$	$\phi_{4,1}(q^3)$	$1(q^4)$
	$B_{2,\varepsilon}(-q)$	$B_{2,r}(-q^2)$	$B_{2,1}(-q^3)$	} series above B_2	
	$F_4[\pm i](\pm iq^2)$	} cuspidal characters			
	$F_4[\theta](\theta q^2)$				
	$F_4[\theta^2](\theta^2 q^2)$				

6.3. Cohomology over k . Since k is not flat over \mathcal{O} , the mod- ℓ cohomology of a variety is not the ℓ -reduction of the cohomology over \mathcal{O} , but there is still an explicit relation, called the *universal coefficient theorem*, given by the following exact sequence (see for example [2, §2.7])

$$0 \longrightarrow k \otimes_{\mathcal{O}} H_c^i(\mathbb{X}, \mathcal{O}) \longrightarrow H_c^i(\mathbb{X}, k) \longrightarrow \text{Tor}_1^{\mathcal{O}}(H_c^{i+1}(\mathbb{X}, \mathcal{O}), k) \longrightarrow 0.$$

We will use it in the following particular case, for which we can give a direct proof.

Proposition 6.5. *The middle cohomology group $H_c^r(\mathbb{X}, \mathcal{O})$ of the Coxeter variety \mathbb{X} is torsion-free.*

Proof. We consider an \mathcal{O} -free resolution of $R\Gamma_c(\mathbb{X}, \mathcal{O})$ which we truncate using $\tilde{\tau}_{\geq r}$ (see §1.6 for the definition of the truncation). This yields quasi-isomorphisms

$$\begin{aligned} R\Gamma_c(\mathbb{X}, \mathcal{O}) &\simeq (0 \longrightarrow C_{r-1} \xrightarrow{d} C_r \xrightarrow{d'} \dots) \\ R\Gamma_c(\mathbb{X}, k) &\simeq k \otimes_{\mathcal{O}}^L R\Gamma_c(\mathbb{X}, \mathcal{O}) \simeq (0 \longrightarrow kC_{r-1} \xrightarrow{\bar{d}} kC_r \xrightarrow{\bar{d}'} \dots) \end{aligned}$$

where each C_i is \mathcal{O} -free. Furthermore, since $H_c^{r-1}(\mathbb{X}, k) = 0$ the map \bar{d} is injective (note that d is injective by definition). On the other hand, since C_r and therefore $\text{Ker } d'$ is \mathcal{O} -free the exact sequence $0 \longrightarrow C_{r-1} \longrightarrow \text{Ker } d' \longrightarrow H_c^r(\mathbb{X}, \mathcal{O}) \longrightarrow 0$ tensored with k yields an exact sequence

$$0 \longrightarrow \text{Tor}_1^{\mathcal{O}}(H_c^r(\mathbb{X}, \mathcal{O}), k) \longrightarrow kC_{r-1} \xrightarrow{\bar{d}} k\text{Ker } d' \longrightarrow kH_c^r(\mathbb{X}, \mathcal{O}) \longrightarrow 0$$

which forces $\text{Tor}_1^{\mathcal{O}}(H_c^r(\mathbb{X}, \mathcal{O}), k) = 0$. \square

Theorem 6.6. *Assume that $\ell \nmid |G|$. Then $H_c^\bullet(\mathbb{X}, \mathcal{O})$ is torsion-free.*

Proof. We proceed by induction on the semisimple rank of \mathbb{G} . If \mathbb{G} is a torus, then \mathbb{X} is a point and the result holds. Otherwise, if S is non-empty, one can consider a proper subset I of S . Using the isomorphism of varieties $U_I \backslash \mathbb{X} \simeq \mathbb{X}_I \times (\mathbb{G}_m)^{r-|I|}$ given in Proposition 6.2 we get an isomorphism of \mathcal{O} -modules

$${}^*R_{L_I}^G(H_c^\bullet(\mathbb{X}, \mathcal{O})) \simeq H_c^\bullet(\mathbb{X}_I, \mathcal{O}) \otimes_{\mathcal{O}} H_c^\bullet(\mathbb{G}_m, \mathcal{O})^{\otimes r-|I|}.$$

Theorem 6.7 (Dudas–Rouquier [35]). *Let $m \in \mathbb{Z}$ and \bar{q} be the class of q in k . Then*

$$R\Gamma_c(\mathbb{X}, k)_{\bar{q}^m} \simeq \Omega^{2m}k$$

in kG -stab.

Idea of proof. One can compute explicitly the closed subvariety \mathbb{X}_ℓ of \mathbb{X} consisting of the points $x \in \mathbb{X}$ such that ℓ divides the order of $\text{Stab}_G(x)$. Then the cohomology complexes of \mathbb{X}_ℓ and \mathbb{X} are isomorphic in kG -stab. \square

Choose $m \in \mathbb{Z}$ such that $\lambda_\zeta \equiv q^m$ modulo ℓ . Then Theorem 6.7 and (6.4) show that $H_c^r(\mathbb{X}, k)_{\lambda_\zeta} \simeq \Omega^{2m-r}k$ up to projective summands. This proves that $\Omega^{2m-r}k$ lifts to an $\mathcal{O}G$ -lattice with character $H_c^r(\mathbb{X}, K)_{\lambda_\zeta}$, yielding information on the Green walk around the Brauer tree of the principal ℓ -block when $\ell \mid \Phi_h(q)$.

Example 6.8. Let \mathbb{G} be a group of type F_4 , so that $h = 12$. Assume that q has order 12 modulo ℓ . We choose θ (resp. i) to be congruent to q^4 (resp. q^3) modulo ℓ . The various data attached to the representations occurring in the cohomology group of \mathbb{X} in middle degree are listed in the following table.

ζ	1	-1	i	-i	θ	θ^2
λ_ζ	1	-q	iq^2	$-iq^2$	θq^2	$\theta^2 q^2$
q^m	q^0	q^7	q^5	q^{11}	q^6	q^{10}
$2m - r$	-4	10	6	18	8	16
$H_c^r(\mathbb{X})_\lambda$	St	$B_{2,\varepsilon}$	$F_4[i]$	$F_4[-i]$	$F_4[\theta]$	$F_4[\theta^2]$

We get therefore $[\Omega^{-4}\mathcal{O}] = \text{St}$, $[\Omega^{10}\mathcal{O}] = B_{2,\varepsilon}$, $[\Omega^6\mathcal{O}] = F_4[i]$, $[\Omega^8\mathcal{O}] = F_4[\theta]$ and the planar embedded Brauer tree is given in Figure 9.

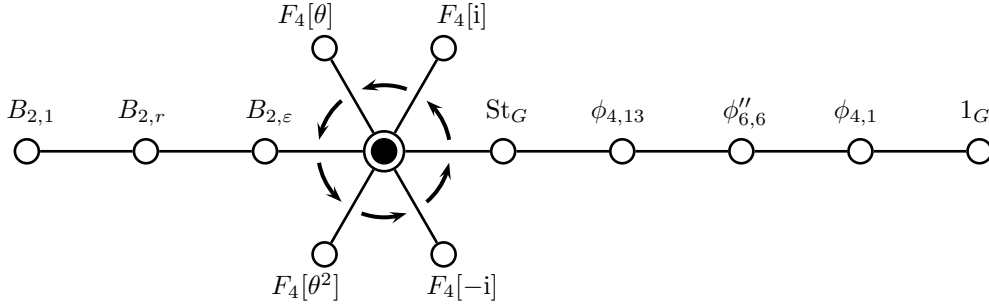


Figure 9. Brauer tree of the principal Φ_{12} -block of $F_4(q)$

Knowing only the characters of the PIMs (in other words, the Brauer tree without the planar embedding) we can finally show that the cohomology of \mathbb{X} over \mathcal{O} is torsion-free when $\ell \mid \Phi_h(q)$.

Theorem 6.9. *Assume $\ell \mid \Phi_h(q)$ and $\ell \nmid h$, so that q has order h modulo ℓ . Then $H_c^\bullet(\mathbb{X}, \mathcal{O})$ is torsion-free.*

Idea of proof. Since the torsion part of the cohomology of \mathbb{X} over \mathcal{O} is cuspidal (see (ii) above), it is enough to show that for every simple cuspidal kG -module M , the complex $R\mathrm{Hom}_{kG}(P_M, R\Gamma_c(\mathbb{X}, k))$ has zero cohomology outside the middle degree $r = \ell(c)$. Indeed, by the universal coefficient formula this shows that the torsion-part of $R\Gamma_c(\mathbb{X}, \mathcal{O})$ is in degree r only, and we can invoke Proposition 6.5 to conclude.

The computation of $R\mathrm{Hom}_{kG}(P_M, R\Gamma_c(\mathbb{X}, k))$ is achieved by using the explicit character of P_M as follows. The shape of the tree, as conjectured in [53] and proved in [35] ensures that M labels an edge incident to the exceptional vertex. In other words, we have $e([P_M]) = \chi_{\mathrm{exc}} + \chi$.

We distinguish two cases: assume first that χ is cuspidal, then P_M does not occur in any of the cohomology complexes $R\Gamma_c(\mathbb{X}(v), k)$ for $v < c$ since otherwise χ would occur in $H_c^\bullet(\mathbb{X}(v), K)$ (recall that $H_c^\bullet(\mathbb{X}(v), \mathcal{O})$ is torsion-free). Consequently, the map $R\Gamma_c(\mathbb{X}, k) \rightarrow R\Gamma(\mathbb{X}, k)$ induces an isomorphism

$$R\mathrm{Hom}_{kG}(P_M, R\Gamma_c(\mathbb{X}, k)) \xrightarrow{\sim} R\mathrm{Hom}_{kG}(P_M, R\Gamma(\mathbb{X}, k))$$

which proves that the cohomology of this complex vanishes in degrees higher than $\dim \mathbb{X} = r$ and lower than $\dim \mathbb{X}$.

Assume now that χ lies in a Harish-Chandra series above a proper Levi subgroup L_I of G . Writing $S = I \sqcup J$, one shows that P_M is a direct summand of $R = \mathrm{Ind}_U^G \mathrm{Inf}_{U/U_I \cap U_J}^U(Q)$ for some (projective) $kU/(U_I \cap U_J)$ -module Q . Now, with \tilde{Q} (resp. \tilde{R}) being a lattice lifting Q (resp. R), we obtain the following isomorphism using adjunction and (6.3)

$$\begin{aligned} R\mathrm{Hom}_{\mathcal{O}G}(\tilde{R}, R\Gamma_c(\mathbb{X}, \mathcal{O})) &\simeq R\mathrm{Hom}_{\mathcal{O}U/U_I \cap U_J}(\tilde{Q}, R\Gamma_c(\mathbb{X}, \mathcal{O})^{U_I \cap U_J}) \\ &\simeq R\mathrm{Hom}_{\mathcal{O}U/U_I \cap U_J}(\tilde{Q}, R\Gamma_c(\mathbb{X}_I, \mathcal{O}) \otimes R\Gamma_c(\mathbb{X}_J, \mathcal{O})). \end{aligned}$$

The cohomology of this complex is torsion-free by Theorem 6.6. Therefore the same holds for $R\mathrm{Hom}_{kG}(\tilde{P}_M, R\Gamma_c(\mathbb{X}, \mathcal{O}))$. Now its cohomology over K vanishes outside of the degree r since χ occurs in the cohomology of \mathbb{X} in middle degree only, and by the universal coefficient theorem the same holds over k . \square

6.4. Applications. Broué's abelian defect group conjecture [10] predicts the existence of a derived equivalence between a block of a finite group with abelian defect and its Brauer correspondent. In the case of finite reductive groups, defect groups of unipotent blocks are generic. When ℓ is large enough and d is the order of q modulo ℓ , they correspond to the ℓ -part of Φ_d -tori in \mathbb{G} , and their centralizers are d -Levi subgroups (see §5.2). Broué suggested in [10] that in this case the derived equivalence should be induced by the cohomology complex of a Deligne–Lusztig variety associated with such a d -Levi subgroup. This was proven by Bonnafé–Rouquier and the author in the case when $d = h$ is the Coxeter number.

Theorem 6.10 (Bonnafé–Rouquier [6], Dudas [28]). *Assume that q has order h modulo ℓ . The action of \mathbb{T}^{cF} on $R\Gamma_c(\mathbb{Y}, \mathcal{O})$ can be extended to an action of $N_{\mathbb{G}^{\mathrm{cF}}}(\mathbb{T})$*

such that the functor

$$R\Gamma_c(\mathbb{Y}, \mathcal{O}) \otimes_{\mathcal{O}N_{\mathbb{G}^{\dot{c}F}}(\mathbb{T})}^L - : \mathbf{D}^b(\mathcal{O}N_{\mathbb{G}^{\dot{c}F}}(\mathbb{T})\text{-mod}) \longrightarrow \mathbf{D}^b(\mathcal{O}G\text{-mod})$$

induces a derived equivalence between the principal blocks of $N_{\mathbb{G}^{\dot{c}F}}(\mathbb{T})$ and G .

The extension of the action of $\mathbb{T}^{\dot{c}F}$ to $N_{\mathbb{G}^{\dot{c}F}}(\mathbb{T})$ is essentially given by twisting the action of the Frobenius endomorphism, since here $N_{\mathbb{G}^{\dot{c}F}}(\mathbb{T})/\mathbb{T}^{\dot{c}F}$ is a cyclic group generated by the image of F . For more general blocks, it is expected that the braid operators constructed in [13, 22] should provide the extension of the action of the finite torus (see [12]).

Once we extended the action, the key point is to prove that $R\Gamma_c(\mathbb{Y}, \mathcal{O})$ is a tilting complex, that is that

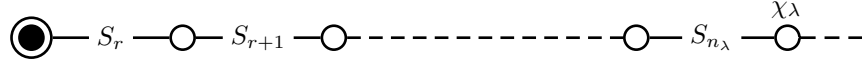
$$R\text{End}_{\mathcal{O}G}(R\Gamma_c(\mathbb{Y}, \mathcal{O})) \simeq \mathcal{O}N_{\mathbb{G}^{\dot{c}F}}(\mathbb{T})[0].$$

Rouquier proved in [64] that it is enough to show the vanishing of the cohomology groups of that complex over k , that is to show that

$$R\text{Hom}_{kG}(R\Gamma_c(\mathbb{Y}, k), R\Gamma_c(\mathbb{Y}, k)[n]) \simeq 0 \quad \text{if } n \neq 0. \tag{6.5}$$

The solution to this problem given in [7] and [28, 30] relies on showing first that the cohomology groups $H_c^\bullet(\mathbb{Y}, \mathcal{O})$ are torsion-free and then to use the cohomology over K (computed by Lusztig) to determine $H_c^\bullet(\mathbb{Y}, k)$ and find an explicit representative for $R\Gamma_c(\mathbb{Y}, k)$.

Proposition 6.11. *Let χ_λ be the unipotent character corresponding to the generalized λ -eigenspace of F on $H_c^{n_\lambda}(\mathbb{X}, K)$ for some $n_\lambda \geq r$. With the following notation for the subtree of the Brauer tree of the principal ℓ -block of kG corresponding to the Harish-Chandra series of χ_λ*



the complex $R\Gamma_c(\mathbb{Y}, k)_{\overline{\chi}}$ is isomorphic to

$$0 \longrightarrow P_{S_r} \longrightarrow P_{S_{r-1}} \longrightarrow \cdots \longrightarrow P_{S_{n_\lambda}} \longrightarrow 0.$$

Sketch of proof. Lusztig’s result on the quotient of \mathbb{X} (Proposition 6.2) can be generalized to \mathbb{Y} as follows. Let us decompose the torus $\mathbb{T}^{\dot{c}F}$ as $\mathbb{T}^{\dot{c}F} = (\mathbb{T}^{\dot{c}F})_\ell \times (\mathbb{T}^{\dot{c}F})_{\ell'}$ as a product of an ℓ -group and an ℓ' -group. We define $\mathbb{Y}_\ell \simeq \mathbb{Y}/(\mathbb{T}^{\dot{c}F})_{\ell'}$. It is an intermediate quotient between \mathbb{Y} and $\mathbb{X} \simeq \mathbb{Y}/\mathbb{T}^{\dot{c}F}$ whose cohomology contains only the principal ℓ -block of $\mathbb{T}^{\dot{c}F}$. Then one shows that for every proper subset I of S there is an isomorphism of \mathcal{O} -modules

$${}^*R_{L_I}^G(H_c^\bullet(\mathbb{Y}_\ell, \mathcal{O})) \simeq H_c^\bullet(\mathbb{X}_I, \mathcal{O}) \otimes_{\mathcal{O}} H_c^\bullet(\mathbb{G}_m, \mathcal{O})^{\otimes r-|I|}.$$

In particular, the torsion part of the cohomology of \mathbb{Y}_ℓ is cuspidal. As in the case of \mathbb{X} , it is enough to show that cuspidal modules occur in the middle degree

only. This was proven for \mathbb{X} along the way to Theorem 6.9. The same property holds for \mathbb{Y} since $R\Gamma_c(\mathbb{Y}_\ell, k) \simeq R\Gamma_c(\mathbb{X}, k(\mathbb{T}^{\dot{c}F})_\ell)$ is built from successive extensions of $R\Gamma_c(\mathbb{X}, k)$ in the same way that $k(\mathbb{T}^{\dot{c}F})_\ell$ is built from extensions of the trivial representation.

By definition, the cohomology groups $H_c^i(\mathbb{Y}_\ell, K)$ are the sum of the cohomology groups $H_c^i(\mathbb{Y}, K)_\theta$ where θ runs over the irreducible ℓ -characters of $\mathbb{T}^{\dot{c}F}$. When $\theta = 1$, $H_c^i(\mathbb{Y}, K)_\theta = H_c^i(\mathbb{X}, K)$ which we know explicitly. When θ is non-trivial, the assumption on ℓ forces θ to be in general position and $H_c^i(\mathbb{Y}, K)_\theta = 0$ except when $i = r = \ell(c)$ in which case it equals the exceptional character $\pm R_w(\theta)$. Consequently, a given eigenspace of F on $H_c^i(\mathbb{Y}_\ell, K)$ is non-zero in at most two degrees, one corresponding to the eigenspace on $H_c^i(\mathbb{X}, K)$, and the other being the middle degree. Since $R\Gamma_c(\mathbb{Y}_\ell, \mathcal{O})$ is a direct summand of the perfect complex $R\Gamma_c(\mathbb{Y}, \mathcal{O})$ (recall that it corresponds to the principal block of $\mathbb{T}^{\dot{c}F}$) then $R\Gamma_c(\mathbb{Y}_\ell, \mathcal{O})$ is also perfect and therefore $[R\Gamma_c(\mathbb{Y}_\ell, K)]$ is the character of a virtual projective module. This forces each generalized λ -eigenspace of F to be of the following form

$$H_c^\bullet(\mathbb{Y}_\ell, K)_\lambda = \chi_{\text{exc}}[-r] \oplus \chi_\lambda[-n_\lambda] \quad (6.6)$$

where n_λ is the unique degree of the cohomology of \mathbb{X} on which F acts by λ and χ_λ is the corresponding unipotent character (we assume here that λ is one of the eigenvalues listed in Theorem 6.3.iii).

From the shape of the Brauer tree we observe that $n_\lambda - r + 1$ is exactly the distance between the node labeling χ_{exc} and the node labeling χ_λ . On the other hand, since the cohomology of \mathbb{Y}_ℓ is torsion-free we deduce from (6.6) and Proposition 5.9 that

$$H_c^r(\mathbb{Y}_\ell, k)_{\overline{\lambda}} \simeq \Omega^{n_\lambda - r + 1} H_c^{n_\lambda}(\mathbb{Y}_\ell, k)_{\overline{\lambda}} \quad (6.7)$$

in the stable category. Both cohomology groups lift to $\mathcal{O}G$ -lattices with characters χ_{exc} and χ_λ , therefore they have no projective summands and the previous isomorphism holds in $kG\text{-mod}$.

Recall from the remark following (1.2) that we have a distinguished triangle in $\mathbf{D}^b(kG\text{-mod})$

$$H_c^r(\mathbb{Y}_\ell, k)_{\overline{\lambda}}[-r] \longrightarrow R\Gamma_c(\mathbb{Y}_\ell, k)_{\overline{\lambda}} \longrightarrow H_c^{n_\lambda}(\mathbb{Y}_\ell, k)_{\overline{\lambda}}[-n_\lambda] \rightsquigarrow$$

which is determined by a map $H_c^{n_\lambda}(\mathbb{Y}_\ell, k)_{\overline{\lambda}}[-n_\lambda] \longrightarrow H_c^r(\mathbb{Y}_\ell, k)_{\overline{\lambda}}[-r + 1]$, which is in turn determined by an element of $\text{Ext}_{kG}^{n_\lambda - r + 1}(H_c^{n_\lambda}(\mathbb{Y}_\ell, k)_{\overline{\lambda}}, H_c^r(\mathbb{Y}_\ell, k)_{\overline{\lambda}})$. By Proposition 1.23 and the isomorphism (6.7), this group of extensions is isomorphic to $\text{End}_{kG}(H_c^{n_\lambda}(\mathbb{Y}_\ell, k)_{\overline{\lambda}})$ and hence it is one-dimensional. Therefore up to isomorphism there is a unique non-zero map $H_c^{n_\lambda}(\mathbb{Y}_\ell, k)_{\overline{\lambda}}[-n_\lambda] \longrightarrow H_c^r(\mathbb{Y}_\ell, k)_{\overline{\lambda}}[-r + 1]$, and the mapping cone of this map can be obtained from a truncated projective resolution of $H_c^{n_\lambda}(\mathbb{Y}_\ell, k)_{\overline{\lambda}}$, which is exactly the complex given in the theorem. \square

Now this representative is exactly the one given by Rickard in [61] to construct a tilting complex for Brauer trees algebras. In particular, it satisfies (6.5) and Theorem 6.10 follows.

References

- [1] J.L. Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics **11**, Cambridge University Press, Cambridge, 1986.
- [2] D.J. Benson, *Representations and cohomology, Vol.1: Basic representation theory of finite groups and associative algebras*, Cambridge Studies in Advanced Mathematics **30**, Cambridge University Press, Cambridge, 1991.
- [3] C. Bonnafé, *Representations of $SL_2(\mathbb{F}_q)$* , Algebra and Applications **13**, Springer-Verlag London, Ltd., London, 2011.
- [4] C. Bonnafé, J.-F. Dat and R. Rouquier, Derived categories and Deligne–Lusztig varieties II, *Ann. Math.* **185** (2017), 609–670.
- [5] C. Bonnafé and R. Rouquier, Catégories dérivées et variétés de Deligne–Lusztig, *Publ. Math. Inst. Hautes Études Sci.* **97** (2003), 1–59.
- [6] C. Bonnafé and R. Rouquier, Coxeter orbits and modular representations, *Nagoya Math. J.* **183** (2006), 1–34.
- [7] C. Bonnafé and R. Rouquier, Affineness of Deligne–Lusztig varieties for minimal length elements, *J. Algebra* **320** (2008), 1200–1206.
- [8] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 4–6. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002.
- [9] R. Brauer, Investigations on group characters, *Ann. of Math.* **42** (1941), 936–958.
- [10] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, *Astérisque* **181–182** (1990), 61–92.
- [11] M. Broué, Isométries de caractères et équivalences de Morita ou dérivées, *Publications mathématiques de l’I.H.É.S.* **71** (1990), 45–63.
- [12] M. Broué and G. Malle, Zyklotomische Heckealgebren, *Astérisque* **212** (1993), 119–189.
- [13] M. Broué and J. Michel, *Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne–Lusztig associées*, in Finite reductive groups (Luminy, 1994), Progr. Math. **141**, 73–139.
- [14] R. Burkhardt, Über die Zerlegungszahlen der Suzukigruppen $Sz(q)$, *J. Algebra* **59** (1979), 421–433.
- [15] M. Cabanes and M. Enguehard, *Representation theory of finite reductive groups*, Cambridge Univ. Press, 2004.
- [16] R.W. Carter, *Finite groups of Lie type. Conjugacy classes and complex characters*, Reprint of the 1985 original. Wiley Classics Library. John Wiley & Sons, Ltd., Chichester, 1993.
- [17] D. Craven, O. Dudas and R. Rouquier, The Brauer trees of unipotent blocks, preprint [arXiv:1701.07097](https://arxiv.org/abs/1701.07097) (2017).
- [18] E. C. Dade, Blocks with cyclic defect groups, *Ann. of Math.* **84** (1966), 20–48.
- [19] P. Deligne, *Cohomologie étale*. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 $\frac{1}{2}$, avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.
- [20] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, *Ann. of Math.* **103** (1976), 103–161.

- [21] F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Math. Soc. Student Texts **21**, Cambridge University Press, 1991.
- [22] F. Digne and J. Michel, Parabolic Deligne-Lusztig varieties, *Adv. Math.* **257** (2014), 136–218.
- [23] R. Dipper, On the decomposition numbers of the finite general linear groups, *Trans. Amer. Math. Soc.* **290** (1985), 315–344.
- [24] R. Dipper and J. Du, Harish-Chandra vertices, *J. Reine Angew. Math.* **437** (1993), 101–130.
- [25] R. Dipper and G. James, The q -Schur algebra, *Proc. London Math. Soc.* **59** (1989), 23–50.
- [26] R. Dipper and G. James, Representations of Hecke Algebras of Type B_n , *J. Algebra* **146** (1992), 454–481.
- [27] V.G. Drinfeld, Elliptic modules, *Mat. Sb. (N.S.)* **94** (1974), 594–627.
- [28] O. Dudas, Coxeter Orbits and Brauer trees, *Adv. Math.* **229** (2012), 3398–3435.
- [29] O. Dudas, A note on decomposition numbers for groups of Lie type of small rank, *J. Algebra* **388** (2013), 364–373.
- [30] O. Dudas, Coxeter Orbits and Brauer trees II. *Int. Math. Res. Not.* **15** (2014), 4100–4123.
- [31] O. Dudas and G. Malle, Projective modules in the intersection cohomology of Deligne–Lusztig varieties, *C. R. Acad. Sci. Paris Sér. I Math.* **352** (2014), 467–471.
- [32] O. Dudas and G. Malle, Decomposition matrices for low rank unitary groups, *Proc. London Math. Soc.* **110** (2015), 1517–1557.
- [33] O. Dudas and G. Malle, Decomposition matrices for exceptional groups at $d = 4$, *J. Pure Appl. Algebra* **220** (2016), 1096–1121.
- [34] O. Dudas and G. Malle, Modular irreducibility of cuspidal unipotent characters, *Invent. Math.* (2017).
- [35] O. Dudas and R. Rouquier, Coxeter orbits and Brauer trees III, *J. Amer. Math. Soc.* **27** (2014), 1117–1145.
- [36] P. Fong and B. Srinivasan, The blocks of finite general linear and unitary groups, *Invent. Math.* **69** (1982), 109–153.
- [37] P. Fong and B. Srinivasan, Brauer trees in $GL(n, q)$, *Math. Z.* **187** (1984), 81–88.
- [38] P. Fong and B. Srinivasan, Brauer trees in classical groups, *J. Algebra* **131** (1990), 179–225.
- [39] M. Geck, *Verallgemeinerte Gelfand–Graev Charaktere und Zerlegungszahlen endlicher Gruppen vom Lie-Typ*, Dissertation, RWTH Aachen, 1990.
- [40] M. Geck, On the decomposition numbers of the finite unitary groups in nondefining characteristic, *Math. Z.* **207** (1991), 83–89.
- [41] M. Geck, Generalized Gelfand–Graev characters for Steinberg’s triality groups and their applications, *Comm. Algebra* **19** (1991), 3249–3269.
- [42] M. Geck, Brauer trees of Hecke algebras, *Comm. Algebra* **20** (1992), 2937–2973.
- [43] M. Geck, Basic sets of Brauer characters of finite groups of Lie type II, *J. London Math. Soc.* **47** (1993), 255–268.

- [44] M. Geck, A first guide to the character theory of finite groups of Lie type, preprint, [arXiv:1705.05083](https://arxiv.org/abs/1705.05083) (2017).
- [45] M. Geck and G. Hiss, Basic sets of Brauer characters of finite groups of Lie type, *J. Reine Angew. Math.* **418** (1991), 173–188.
- [46] M. Geck and G. Hiss, Modular representations of finite groups of Lie type in non-defining characteristic. Finite reductive groups (Luminy, 1994), 195–249, *Progr. Math.* **141**, Birkhäuser Boston, Boston, MA, 1997.
- [47] S.I. Gelfand and Y.I. Manin, *Methods of homological algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.
- [48] J. A. Green. Walking around the Brauer tree, *J. Austral. Math. Soc.*, **17** (1974), 197–213.
- [49] J. Gruber and G. Hiss, Decomposition numbers of finite classical groups for linear primes, *J. Reine Angew. Math.* **485** (1997), 55–91.
- [50] G. Hiss, On the decomposition numbers of $G_2(q)$, *J. Algebra* **120** (1989), 339–360.
- [51] G. Hiss, The Brauer trees of the Ree groups, *Comm. Algebra* **19** (1991), 871–888.
- [52] G. Hiss and F. Lübeck, The Brauer trees of the exceptional Chevalley groups of types F_4 and 2E_6 , *Arch. Math.* **70** (1998), 16–21.
- [53] G. Hiss, F. Lübeck and G. Malle, The Brauer trees of the exceptional Chevalley groups of type E_6 , *Manuscripta Math.* **87** (1995), 131–144.
- [54] R.B. Howlett and G.I. Lehrer, On Harish-Chandra induction and restriction for modules of Levi subgroups. *J. Algebra* **165** (1994), 172–183.
- [55] G. Lusztig, On the finiteness of the number of unipotent classes, *Invent. Math.* **34** (1976), 201–213.
- [56] G. Lusztig, Coxeter orbits and eigenspaces of Frobenius, *Invent. Math.* **38** (1976), 101–159.
- [57] G. Lusztig, *Representations of finite Chevalley groups*, CBMS Regional Conference Series in Mathematics, 39. American Mathematical Society, Providence, R.I., 1978.
- [58] G. Lusztig, *Characters of reductive groups over a finite field*, Ann. Math. Studies **107**, Princeton U. Press, 1984.
- [59] J.S. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton University Press, Princeton, N.J., 1980.
- [60] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton University Press, Princeton, NJ, 2001.
- [61] J. Rickard, Derived categories and stable equivalence, *J. Pure Appl. Algebra* **61** (1989), 303–317.
- [62] J. Rickard, Finite group actions and étale cohomology, *Publ. Math. Inst. Hautes Études Sci.* **80** (1995), 81–94.
- [63] R. Rouquier, The derived category of blocks with cyclic defect groups, in *Derived equivalences for group rings*, Lecture Notes in Math. **1685**, Springer, Berlin, 1998.
- [64] R. Rouquier, Complexes de chaînes étales et courbes de Deligne–Lusztig, *J. Algebra* **257** (2002), 482–508.
- [65] R. Rouquier, Motives of Deligne–Lusztig varieties, preprint (2017).

- [66] J.P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics **42**, Springer-Verlag, New York-Heidelberg, 1977.
- [67] J. Shamash, Brauer trees for blocks of cyclic defect in the groups $G_2(q)$ for primes dividing $q^2 \pm q + 1$, *J. Algebra* **123** (1989), 378–396.
- [68] J. Thévenaz, *G-algebras and modular representation theory*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.
- [69] D.J. White, Decomposition numbers of $\mathrm{Sp}(4, q)$ for primes dividing $q \pm 1$, *J. Algebra* **132** (1990), 488–500.
- [70] E. Wings, *Über die unipotenten Charaktere der Gruppen $F_4(q)$* , Dissertation, Lehrstuhl D für Mathematik, RWTH Aachen, 1995.

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