Decomposition matrices for groups of Lie type in non-defining characteristic

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Abstract

We determine approximations to the decomposition matrices for unipotent ℓ-blocks of several series of finite reductive groups of classical and exceptional type over $\mathbb{F}_q$ of low rank in non-defining good characteristic ℓ.

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1. Introduction

It is a fundamental problem in representation theory of finite groups to construct all irreducible representations of the finite nearly simple groups, or at least determine their dimensions. This question is open even for the family of symmetric groups. While it was solved by Frobenius more than a hundred years ago for representations over the complex numbers, it remains a challenging problem when the characteristic of the base field is positive. In this manuscript we consider certain families of finite quasi-simple groups of Lie type. For these, the ordinary characters were classified by G. Lusztig in the 1980s (see [**40**]). For positive characteristic representations it is open in general. It comes in two fundamentally different flavours. In the defining characteristic case, the representations of the finite group are closely related to the representations of the ambient algebraic group, to which a highest weight theory applies, but still deep problems remain. Here, we are concerned with the other, the cross characteristic case. The main problem here can be phrased as finding the ℓ-modular decomposition matrices for the so-called unipotent blocks.

We obtain very close approximations to these decomposition matrices for all finite groups of Lie type and Lie rank at most 6, and sometimes even beyond, that is to say, we either determine the matrices completely or up to at most a very small number of undetermined entries.

Our motivation for doing this is at least threefold. First, it is of interest for many applications to know the dimensions of all irreducible modules of the "small" finite simple groups, or at least the few smallest such dimensions. Secondly, we expect that the data obtained in this work may be useful to derive and test conjectures for example on

- the number and labels of cuspidal unipotent Brauer characters,
- the parameters of modular Hecke algebras,
- the subdivision of unipotent characters into ℓ-modular Harish-Chandra series,
- and last not least on the shape of decomposition matrices and on individual decomposition numbers, for example Craven's conjecture [**10**].

For example, the parametrisation of cuspidal Brauer characters and of the distribution into Harish-Chandra series for the class of unitary groups was predicted and proved based on similar data obtained in [**27**]. Thirdly, we expect further applications and connections based on the data obtained here. In fact our previous results on unitary groups [**17**] and on exceptional groups [**18**] have already been used by E. Norton [**44**] to predict and then prove the distribution of characters into modular Harish-Chandra series.

Our results are mostly phrased under the assumption (T_ℓ) that unipotent ℓ-blocks of finite reductive groups G have uni-triangular decomposition matrix. This is known to hold whenever ℓ and the defining characteristic of G are not too small, see [**5**, Thm. A].

The work is structured as follows. The first part contains definitions and some general results. We recall some notions and results from the ordinary and modular representation theory of finite reductive groups in Chapter 1. In Chapter 2 we derive criteria for when the parameters of modular Hecke algebras can be obtained from a corresponding characteristic zero situation. The second part is devoted to the determination of decomposition matrices in the various groups under consideration. It is divided up according to the order $d = d_\ell(q)$ of the size q of the finite field over which our group is defined, modulo ℓ, the characteristic of the representations we consider. We start off with the most difficult case of $d = 2$, for which the ranks of Sylow ℓ-subgroups are the largest. Here, in Section 2.2 we prove a general result on multiplicities in the ℓ-modular reduction of the Steinberg character, before we go on to treat the different families of finite quasi-simple reductive groups.

In the subsequent Chapters 5 and 6 we consider decomposition matrices for the cases $d = 3$ and $d = 6$ respectively. At the end of each section we give an overview of the distribution into Harish-Chandra series observed in the blocks under consideration. Finally, in Chapter 7 we discuss the ℓ-modular Brauer trees when $d \geq 7$. In the last chapter we check that our results are in agreement with a conjecture of Craven, and indicate in Proposition 8.2 which further decomposition numbers would follow if this conjecture holds.

Acknowledgement: The computation of Harish-Chandra and Deligne–Lusztig induction (and thus restriction) of unipotent characters was done in Jean Michel's development version of the Chevie system [**43**] using in particular the command LusztigInductionTable. We thank him for having provided this and various further very convenient functionalities. Chapter 8 was written after an observation by Emily Norton. We thank her for her valuable suggestion as well as for her help in the proof of Proposition 6.15.

Part 1

Setting and general methods

CHAPTER 1

Finite groups of Lie type

1. Basic sets and uni-triangularity

Let ℓ be a prime number and $(K, \mathcal{O}, k)$ be an ℓ-modular system which we assume to be large enough for all the finite groups encountered. More specifically, since we will be working with ℓ-adic cohomology we assume throughout this paper that K is a finite extension of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers.

Let H be a finite group. A representation of H will always be assumed to be finite-dimensional. The set of irreducible characters (resp. irreducible Brauer characters) will be denoted by $\operatorname{Irr} H$ (resp. $\operatorname{IBr} H$). Given a simple kH-module N, we shall denote by φ_N (resp. P_N, resp. Ψ_N) its Brauer character (resp. its projective cover, resp. the ordinary character of its projective cover). The restriction of an ordinary character χ of H to the set of ℓ'-elements will be denoted by χ°, and referred to as the *ℓ-modular reduction* of χ (or sometimes just the *ℓ-reduction* of χ). It decomposes uniquely on the family of irreducible Brauer characters of H as

$$\chi^\circ = \sum_{\varphi \in \operatorname{IBr} H} d_{\chi,\varphi}\, \varphi.$$

The non-negative integral coefficients $(d_{\chi,\varphi})_{\chi\in\operatorname{Irr} H, \varphi\in\operatorname{IBr} H}$ form the *ℓ-modular decomposition matrix* of H. Equivalently, if $\varphi = \varphi_N$ is the Brauer character of a simple kH-module N, then the character of its projective cover satisfies $\Psi_N = \sum_{\chi\in\operatorname{Irr} H} d_{\chi,\varphi}\, \chi$ by Brauer reciprocity.

We denote by $\langle -; - \rangle_H$ the usual inner product on the space of K-valued class functions on H. If N is a simple kH-module with Brauer character φ and P a projective kH-module with ordinary character Ψ then $\langle \Psi; \varphi\rangle = \dim \operatorname{Hom}_{kH}(P, N)$ gives the multiplicity of P_N as a direct summand of P.

Recall that a *basic set of characters* is a set $\mathcal{B}$ of complex irreducible characters of H such that $\mathcal{B}^\circ = \{\chi^\circ \mid \chi \in \mathcal{B}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}\operatorname{IBr} H$. This means that the restriction of the decomposition matrix to $\mathcal{B}$ is invertible over $\mathbb{Z}$.

According to Brauer, the ordinary and ℓ-modular irreducible characters of H are subdivided into ℓ-blocks such that the decomposition matrix is block diagonal with respect to this partition. It then also makes sense to speak of basic sets of individual ℓ-blocks. Now assume that we have a basic set $\mathcal{B}$ for an ℓ-block b of H ordered in such a way that the decomposition matrix of b with respect to $\mathcal{B}$ is known *a priori* to be lower uni-triangular. Then this facilitates very much the explicit determination of this decomposition matrix as explained in the following example.

Example 1.1. Let $\mathcal{B} = \{\rho_1, \ldots, \rho_r\} \subseteq \operatorname{Irr} b$ be a basic set of a block b of H such that the decomposition matrix is known to be uni-triangular with respect to this ordering, and let $\{\Psi_1, \dots, \Psi_r\}$ be the corresponding projective indecomposable modules (PIMs for short) in b. Then:

(a) If Ψ is a projective character in b that involves ρ_i but no ρ_j with $j < i$ then Ψ_i is a direct summand of Ψ.
(b) More generally, let Ψ, Ψ' be two projective characters such that $\Psi - \Psi'$ involves ρ_i with positive multiplicity, but no ρ_j with $j < i$. Then Ψ_i is a direct summand of Ψ.

Thus, under certain conditions known PIMs can be subtracted from given projectives, and in this way a set of projective characters can be partially echelonised.

2. Unipotent characters and unipotent blocks

Let $\mathbf{G}$ be a connected reductive linear algebraic group over $\overline{\mathbb{F}}_p$, and $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism with respect to an $\mathbb{F}_q$-structure. (In particular, we do not consider Ree and Suzuki groups here.) We let $(\mathbf{G}^*, F)$ be a Langlands dual group to $(\mathbf{G}, F)$. We let $G := \mathbf{G}^F$ and $G^* := \mathbf{G}^{*F}$ denote the associated finite reductive groups. Throughout, whenever $\mathbf{H} \leq \mathbf{G}$ is an F-stable subgroup we write $H = \mathbf{H}^F$.

Recall from [**12**, §13] that the irreducible characters of G over K fall into rational Lusztig series attached to conjugacy classes of semisimple elements in G^*. Given s a semisimple element of G^* we denote by $\mathcal{E}(G, s)$ the corresponding Lusztig series. The *unipotent characters* of G are by definition the elements of $\mathcal{E}(G, 1)$, and the *unipotent ℓ-blocks* are the blocks that contain at least one unipotent character. By Broué–Michel (see [**7**, Thm. 9.12]), the irreducible characters lying in the unipotent blocks are exactly those lying in $\cup_t \mathcal{E}(G, t)$, for t running over the ℓ-elements of G^*.

Unipotent characters were classified by Lusztig [**40**]. When working with explicit characters later, we will use the notation in [**8**] for exceptional groups. For classical groups, we will denote by $[\lambda]$ a unipotent character parametrised by a partition or a bipartition λ.

ASSUMPTION 1.2. *Throughout this work we shall make the following assumptions on ℓ:*

- *$\ell \neq p$ (non-defining characteristic),*
- *ℓ is* very good *for* $\mathbf{G}$. *(*i.e., *ℓ is good for $\mathbf{G}$ and ℓ divides neither $|Z(\mathbf{G})/Z^\circ(\mathbf{G})|$ nor $|Z(\mathbf{G}^*)/Z^\circ(\mathbf{G}^*)|$).*

Note that both conditions are inherited by all Levi subgroups of $\mathbf{G}$, so that inductive arguments can be applied.

In this situation, the unipotent characters lying in a given unipotent ℓ-block b of G form a basic set of this block (see [**24, 22**]). Consequently, the restriction of the decomposition matrix of b to the unipotent characters is invertible and the whole decomposition matrix of the unipotent blocks can be recovered from these unipotent parts together with the ordinary character table. This square matrix will be referred to as the *unipotent ℓ-decomposition matrix* of G. In particular every (virtual) unipotent character is a virtual projective character, up to adding and removing some non-unipotent characters. In addition, under our Assumption 1.2 on ℓ, not only the parametrisation of unipotent characters but also the decomposition matrix of the unipotent ℓ-blocks is independent of the isogeny type of $\mathbf{G}$ (see [**7**, Thm. 17.7]). Therefore, we will not specify it in our statements and proofs.

Furthermore, under these assumptions on ℓ, the basic set is widely expected to be uni-triangular. This means that one can order the set of unipotent characters and unipotent Brauer characters in such a way that the unipotent decomposition matrix has uni-triangular shape (see [**25**, Conj. 3.4]). More precisely, in many of our proofs we will assume the following property of a finite group of Lie type G at a prime ℓ:

(T_ℓ) *The decomposition matrix of the unipotent ℓ-blocks of G has a uni-triangular shape with respect to any total ordering of the unipotent characters compatible with the order on families.*

This has recently been shown when p is a good prime for $\mathbf{G}$ (see [**5**, Thm. A]).

The unipotent ℓ-blocks of the finite reductive groups are known by the work of Fong–Srinivasan [**19**], Broué–Malle–Michel [**4**] and Cabanes–Enguehard [**6**]. We will use these results throughout without further reference.

3. Deligne–Lusztig theory

We fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Such a torus is said to be *maximally split* (or sometimes *quasi-split* as in [**12**]). Given an F-stable Levi complement $\mathbf{L}$ of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, we denote by R_L^G and ${}^*R_L^G$ the Deligne–Lusztig induction and restriction maps

$$R_L^G : \mathbb{Z}\mathrm{Irr}_{\,\Lambda} L \longrightarrow \mathbb{Z}\mathrm{Irr}_{\,\Lambda} G \quad \text{and} \quad {}^*R_L^G : \mathbb{Z}\mathrm{Irr}_{\,\Lambda} G \longrightarrow \mathbb{Z}\mathrm{Irr}_{\,\Lambda} L$$

where Λ is the field K or k. We will only use them when in a situation where they do not depend on $\mathbf{P}$, which justifies our notation. We refer to [**1**] for conditions ensuring the independence from the parabolic subgroup. When $\mathbf{P}$ can be chosen to be F-stable — in which case we will say that $\mathbf{L}$ is a 1-*split* Levi subgroup of $\mathbf{G}$ — these coincide with the Harish-Chandra induction and restriction map induced by the Harish-Chandra induction and restriction functors. These will be also denoted by R_L^G and ${}^*R_L^G$.

Let t be a semisimple element of G^*. Assume that $\mathbf{L}^* := C_{\mathbf{G}^*}(t)$ is a Levi subgroup of $\mathbf{G}^*$, in duality with an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$. Since $t \in Z(\mathbf{L}^*)^F$, the element t corresponds to a linear character of $L = \mathbf{L}^F$, which we will denote by $\hat{t}$. By [**12**, Thm. 13.25] there is a sign $\varepsilon \in \{\pm 1\}$ which depends only on $\mathbf{L}$, $\mathbf{G}$ and F such that the maps

$$\mathcal{E}(L,1) \xrightarrow{-\otimes \hat{t}} \mathcal{E}(L,t) \xrightarrow{\varepsilon R_L^G} \mathcal{E}(G,t)$$

are bijections. When t is an ℓ-element and ρ a unipotent character of L, the irreducible character $\varepsilon R_L^G(\rho \otimes \hat{t})$ lies in a unipotent block of G. Consequently, its ℓ-reduction can be written uniquely as a linear combination of the ℓ-reductions of unipotent characters, which form a basic set. On the other hand, the ℓ-reduction of $\hat{t}$ is trivial, therefore by [**12**, Prop. 12.2] the character $\varepsilon R_L^G(\rho \otimes \hat{t})$ has the same ℓ-reduction as the virtual unipotent character $\varepsilon R_L^G(\rho)$. This will prove to be a powerful tool to derive relations on the decomposition numbers of G, as shown in Example 1.4.

Lemma 1.3. *Let t be a semisimple ℓ-element of G^* such that $C_{\mathbf{G}^*}(t) = \mathbf{L}^*$ is a Levi subgroup of $\mathbf{G}^*$ with dual $\mathbf{L} \leq \mathbf{G}$. Then for every unipotent character ρ of L, $\varepsilon R_L^G(\rho)^\circ$ is a non-negative linear combination of irreducible Brauer characters.*

EXAMPLE 1.4. Let $G = \mathrm{PGL}_2(q)$ and hence $G^* = \mathrm{SL}_2(q)$. The unipotent characters of G are St_G and 1_G. There are semisimple element $t \in G^*$ with eigenvalues $\{\lambda, \lambda^{-1}\}$ for every $\lambda \in \mathbb{F}_{q^2}$ satisfying $\lambda^q = \lambda^{-1}$. When $\lambda \neq \pm 1$, such an element is regular. If $\ell \mid (q+1)$ is an odd prime number, then one can take λ to be a non-trivial ℓ-element. The corresponding semisimple element t is a regular ℓ-element of G^*, therefore $\mathbf{L}^* = C_{\mathbf{G}^*}(t)$ is a maximal torus of $\mathbf{G}^*$. By Lemma 1.3, the ℓ-modular reduction of the virtual character $-R_L^G(1_L) = \mathrm{St}_G - 1_G$ is a non-negative combination of irreducible Brauer characters. This shows that when ℓ is an odd prime number dividing $q+1$, then the ℓ-reduction of St_G contains at least one copy of the trivial Brauer character and the unipotent ℓ-decomposition matrix has the following shape

$$\begin{array}{r|cc} 1_G & 1 & \cdot \\ \mathrm{St}_G & \alpha & 1 \end{array}$$

with $\alpha \geq 1$ (the coefficient 1 in the bottom right corner comes from the fact that this square matrix is invertible since the unipotent characters form a basic set for the unipotent blocks). We have $-R_L^G(1_L)^\circ = (\alpha - 1)\varphi_k + \varphi_{\mathrm{St}}$.

Let $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ denote the Weyl group of $\mathbf{G}$. It is a finite Coxeter group with distinguished set of generators S and associated length function l. Recall from [**8**, Prop. 3.3.3] that the G-conjugacy classes of F-stable maximal tori of $\mathbf{G}$ are parametrised by the F-conjugacy classes of W. Given $w \in W$, we will denote by $\mathbf{T}_w$ a torus of type w with respect to $\mathbf{T}$ and we will write $R_w := R_{T_w}^G(1_{T_w})$. This is a virtual character of G all of whose constituents are unipotent characters.

As pointed out above, under our Assumption 1.2 the unipotent characters form a basic set for the unipotent blocks. Therefore R_w can be seen as the unipotent part of a virtual projective module $\widetilde{R}_w$. The decomposition of $\widetilde{R}_w$ on the basis of characters of PIMs can be read off from the decomposition of R_w on the family of unipotent parts of characters of PIMs. Given a simple kG-module N, the *coefficient of* Ψ_N *on* R_w will refer to the coefficient of Ψ_N on $\widetilde{R}_w$, which is given by $\langle \widetilde{R}_w; \varphi_N \rangle$. The following proposition gives some control on the sign of this coefficient (see [**15**, Prop. 1.5]).

PROPOSITION 1.5. *Let $w \in W$. If Ψ_N does not occur in R_v for any $v < w$ in the Bruhat order on W then the coefficient of Ψ_N in $(-1)^{l(w)} R_w$ is non-negative.*

EXAMPLE 1.6. As in Example 1.4, we work with $G = \mathrm{PGL}_2(q)$ and an odd prime number ℓ dividing $q+1$. There are two elements in the Weyl group of G, namely the trivial element e and the simple reflection s. The Deligne–Lusztig characters are $R_e = 1_G + \mathrm{St}_G$ and $R_s = 1_G - \mathrm{St}_G$. With the decomposition matrix given in Example 1.4 they decompose on the basis of PIMs as

$$\widetilde{R}_e = \Psi_k + (1-\alpha)\Psi_{\mathrm{St}} \qquad \text{and} \qquad \widetilde{R}_s = \Psi_k - (1+\alpha)\Psi_{\mathrm{St}}.$$

By Proposition 1.5 we have $1 - \alpha \geq 0$, which forces $\alpha = 1$. Therefore Ψ_{St} does not occur in R_e; it has a cuspidal head and Ψ_{St} occurs with multiplicity $1 + \alpha = 2$ in $-R_s$, a non-negative number, as predicted by Proposition 1.5. The unipotent ℓ-decomposition matrix is hence

$$\begin{array}{r|cc} 1_G & 1 & \cdot \\ \mathrm{St}_G & 1 & 1 \\ \hline & ps & c \end{array}$$

The projective character corresponding to the first column lies in the principal series (indicated by the label “ps”), while the second one is cuspidal (and therefore labelled “c”).

If $w \in W$ is such that wF is a d-regular element (in the sense of Springer [**47**]) for some $d \geq 1$ then $\mathbf{T}_w$ contains a (Sylow) Φ_d-torus $\mathbf{S}$ of $\mathbf{G}$ such that $C_{\mathbf{G}}(\mathbf{S}) = \mathbf{T}_w$. For d the order of q modulo ℓ, under suitable conditions on the ℓ-part $\Phi_d(q)_\ell$ of the dth cyclotomic polynomial Φ_d evaluated at q, the dual torus $(\mathbf{S}^*)^F$ contains an ℓ-element t such that $C_{\mathbf{G}^*}(t) = \mathbf{T}_w^*$ (see Proposition 4.1 for the case $d = 2$ and the examples below). In that case by Lemma 1.3, $(-1)^{l(w)}(R_w)^\circ$ is a non-negative linear combination of irreducible Brauer characters.

Example 1.7. (a) Let G be a group of type $C_n(q)$. Then a regular semisimple element of G is an element of $\mathbf{G}$ with $2n$ distinct eigenvalues of the form $\{\lambda_1^{\pm 1}, \dots, \lambda_n^{\pm 1}\}$ in the natural $2n$-dimensional matrix representation that are permuted under the Frobenius map $\lambda \longmapsto \lambda^q$. If the eigenvalues are ℓ-elements and d is the order of q modulo ℓ, then each eigenvalue has an orbit under F of size d, therefore d must divide $2n$. In that case regular ℓ-elements of G exist whenever $\Phi_d(q)_\ell > 2n$.

(b) Let G be a group of type $D_n(q)$. There are two types of regular semisimple elements of odd order: elements of $\mathbf{G}$ with $2n$ distinct eigenvalues of the form $\{\lambda_1^{\pm 1}, \dots, \lambda_n^{\pm 1}\}$, or elements with $2n-2$ distinct eigenvalues $\{\lambda_1^{\pm 1}, \dots, \lambda_{n-1}^{\pm 1}\}$ and two eigenvalues both equal to 1. If such an element is F-stable then d, the order of q modulo ℓ, divides $2n$ or $2n-2$ respectively. It can be chosen to be an ℓ-element provided that $\Phi_d(q)_\ell > 2n$ or $\Phi_d(q)_\ell > 2n-2$ respectively.

(c) Let G be a group of type ${}^2D_n(q)$ with n odd. Then w_0F acts as $-q$ on the set of characters and cocharacters of $\mathbf{T}$. Let e be the order of $-q$ modulo ℓ. Using Ennola duality we deduce from (b) that when e divides $2n$ and $\Phi_e(q)_\ell > 2n$ (resp. when e divides $2n-2$ and $\Phi_e(q)_\ell > 2n-2$) then there exists a regular ℓ-element of G.

(d) Let $\mathbf{S}$ be an F-stable torus of $\mathbf{G}^*$ with $S = \mathbf{S}^F$ of order $\Phi_d(q)$ (a Φ_d-torus of rank 1). If ℓ is good and does not divide the order of $(Z(\mathbf{G}^*)/Z^\circ(\mathbf{G}^*))^F$ then by [**7**, Lemma 13.17], the d-split Levi subgroup $\mathbf{L}^* = C_{\mathbf{G}^*}(\mathbf{S})$ of $\mathbf{G}^*$ is also equal to $C_{\mathbf{G}^*}(S_\ell)$, the centraliser of a Sylow ℓ-subgroup S_ℓ of S. Consequently, if $t \in S_\ell$ is any generator of the cyclic group S_ℓ then $\mathbf{L}^* = C_{\mathbf{G}^*}(t)$.

CHAPTER 2

Hecke algebras

Let Λ be one of K, $\mathcal{O}$ or k. Throughout this chapter, let $\mathbf{L}$ be a 1-split Levi subgroup of $\mathbf{G}$, that is, an F-stable Levi complement of an F-stable parabolic subgroup of $\mathbf{G}$. In this case the maps R_L^G and ${}^*R_L^G$ introduced in §1.3 are induced by the Harish-Chandra induction and restriction functors, which we still denote by R_L^G and ${}^*R_L^G$.

Throughout this section we shall assume that $Z(\mathbf{G})$ is connected so that the results in [**40**, §8] apply.

1. Reduction stability

A ΛL-module X is *cuspidal* if it is killed under every proper Harish-Chandra restriction. In that case the algebra $\mathcal{H}_G(L,X) := \mathrm{End}_{\Lambda G}(R_L^G(X))$ has a very specific structure. When $\Lambda = K$ and X is simple, Lusztig showed in [**40**, §8.6] that the group $W_G(L,X) := N_G(\mathbf{L},X)/L$ admits a structure of a Coxeter system and $\mathcal{H}_G(L,X)$ is an Iwahori–Hecke algebra associated to $W_G(L,X)$. More precisely, if $S_G(L,X)$ denotes the set of simple reflections associated to the Coxeter structure then $\mathcal{H}_G(L,X)$ has a K-basis $\{T_w\}_{w\in W_G(L,X)}$ satisfying, for all $s \in S_G(L,X)$ and $w \in W_G(L,X)$

$$T_sT_w = \begin{cases} T_{sw} & \text{if } sw > w, \\ (q_s-1)T_{sw} + q_sT_w & \text{otherwise.} \end{cases}$$

In addition the parameters $\{q_s\}_{s\in S_G(L,X)}$ can be computed explicitly. They are actually already determined by the Hecke algebras of the cuspidal module X inside the minimal Levi overgroups of L in G. When $\Lambda = k$, the endomorphism algebra was studied for example in [**27**]; is can still be shown to be closely related to an Iwahori–Hecke algebra, but much less is known about the parameters. The best situation occurs when the normaliser of the cuspidal lattice is invariant under change of scalars, as studied in [**21**, §2.6]. More precisely, given an $\mathcal{O}L$-lattice X we will say that $R_L^G(X)$ is *reduction stable* if

(1) KX is irreducible, and
(2) $N_G(\mathbf{L},X) = N_G(\mathbf{L},KX) = N_G(\mathbf{L},kX)$.

In that case $\mathcal{H}_G(L,kX)$ is an Iwahori–Hecke algebra whose parameters are the ℓ-reduction of the parameters of the Iwahori–Hecke algebra $\mathcal{H}_G(L,KX)$.

Proposition 2.1 (Geck). *Let X be a cuspidal $\mathcal{O}L$-module. If $R_L^G(X)$ is reduction stable then there is an $\mathcal{O}$-basis $\{T_w\}_{w\in W_G(L,X)}$ of $\mathcal{H}_G(L,X)$ endowing it with a structure of an Iwahori–Hecke algebra.*

Furthermore, if Λ is one of K or k then $\{1_\Lambda \otimes_{\mathcal{O}} T_w\}_{w\in W_G(L,X)}$ is a Λ-basis of $\mathcal{H}_G(L,\Lambda X)$.

The standard setup for reduction stability is when both KX and kX are simple modules. In that case it is enough to check that $N_G(\mathbf{L}, KX) = N_G(\mathbf{L}, kX)$. We will often need to work with a slightly more general setup.

PROPOSITION 2.2. *Let X be a non-zero cuspidal $\mathcal{O}L$-lattice. We assume that*

(1) *the head Y of kX is a simple module;*
(2) $N_G(\mathbf{L}, KX) = N_G(\mathbf{L}, Y)$*; and*
(3) $\mathrm{Hom}_{kL}({}^w(kX), \mathrm{rad}(kX)) = 0$ *for all* $w \in N_G(\mathbf{L})$.

Then $R_L^G(X)$ is reduction stable. Furthermore, $\mathcal{H}_G(L, kX) \simeq \mathcal{H}_G(L, Y)$.

PROOF. First note that $N_G(\mathbf{L}, X) \subset N_G(\mathbf{L}, kX) \subset N_G(\mathbf{L}, Y) = N_G(\mathbf{L}, KX)$ where the second inclusion comes from the fact that the head of kX is the simple module Y. Therefore we only need to show that $N_G(\mathbf{L}, KX) \subset N_G(\mathbf{L}, X)$ to prove the reduction stability.

Let $w \in N_G(\mathbf{L})$. Consider the short exact sequence of kL-modules

$$0 \longrightarrow \mathrm{rad}(kX) \longrightarrow kX \longrightarrow Y \longrightarrow 0.$$

Since $\mathrm{Hom}_{kL}({}^w(kX), \mathrm{rad}(kX)) = 0$ by (4), the covariant functor $\mathrm{Hom}_{kL}({}^w(kX), -)$ gives an injective map

$$\psi_w : \mathrm{Hom}_{kL}({}^w(kX), kX) \hookrightarrow \mathrm{Hom}_{kL}({}^w(kX), Y).$$

On the other hand, with the head of ${}^w(kX)$ being ${}^wY$ we have a natural isomorphism

$$\phi_w : \mathrm{Hom}_{kL}({}^wY, Y) \simeq \mathrm{Hom}_{kL}({}^w(kX), Y)$$

induced by the map ${}^w(kX) \twoheadrightarrow {}^wY$. This shows that $\mathrm{Hom}_{kL}({}^w(kX), kX)$ has dimension at most 1. Now let $w \in N_G(\mathbf{L}, KX)$. Since $K\mathrm{Hom}_{\mathcal{O}L}({}^wX, X) \simeq \mathrm{Hom}_{KL}({}^w(KX), KX)$ we must have that $\mathrm{Hom}_{\mathcal{O}L}({}^wX, X)$ is non-zero. On the other hand, the natural map

$$k\mathrm{Hom}_{\mathcal{O}L}({}^wX, X) \hookrightarrow \mathrm{Hom}_{kL}({}^w(kX), kX)$$

is an embedding. By the previous observation on the dimension of $\mathrm{Hom}_{kL}({}^w(kX), kX)$, we deduce that it must be an isomorphism and that $\mathrm{Hom}_{kL}({}^w(kX), kX) \simeq k$. Consequently ψ_w is also an isomorphism. In particular, any non zero morphism from ${}^w(kX)$ to kX will send the head of ${}^wkX$ to the head of kX and therefore must be an isomorphism. This shows that ${}^w(kX) \simeq kX$. In addition, since $k\mathrm{Hom}_{\mathcal{O}L}({}^wX, X) \simeq \mathrm{Hom}_{kL}({}^w(kX), kX)$ then we also have ${}^wX \simeq X$ by Nakayama's lemma. This proves that $R_L^G(X)$ is reduction stable. Note that the fact that $\mathrm{End}_{kL}(kX)$ has dimension 1 forces KX to be a simple KL-module.

Note that if $w \notin N_G(\mathbf{L}, Y)$ then ψ_w is the zero map, therefore it is also an isomorphism in that case. Let $\pi : kX \twoheadrightarrow Y$. By adjunction and the Mackey formula, the natural map $\mathrm{End}_{kG}(R_L^G(kX)) \longrightarrow \mathrm{Hom}_{kG}(R_L^G(kX), R_L^G(Y))$ induced by $R_L^G(\pi)$ is an isomorphism since for all $w \in N_G(\mathbf{L})$ each map $\psi_w : \mathrm{Hom}_{kL}({}^w(kX), kX) \longrightarrow \mathrm{Hom}_{kL}({}^w(kX), Y)$ is an isomorphism. The same holds for the natural map $\mathrm{End}_{kG}(R_L^G(Y)) \longrightarrow \mathrm{Hom}_{kG}(R_L^G(kX), R_L^G(Y))$ since ϕ_w is an isomorphism for all $w \in N_G(\mathbf{L})$. The combination of the two gives the asserted isomorphism $\mathrm{End}_{kG}(R_L^G(kX)) \simeq \mathrm{End}_{kG}(R_L^G(Y))$. $\square$

2. Embedding of decomposition matrices

Let X be a cuspidal simple kL-module. Then the simple representations of the algebra $\mathcal{H}_G(L,X)$ encode the simple kG-modules occurring in the head of $R_L^G(X)$, that is the simple modules lying in the Harish-Chandra series above the cuspidal pair (L,X). In addition, when X comes from a reduction stable $\mathcal{O}L$-lattice $\widetilde{X}$, one can compute the decomposition of $R_L^G(P_X)$ into PIMs using the decomposition matrix of $\mathcal{H}_G(L,\widetilde{X})$, as explained in [**14**, §3]. This gives a powerful method to compute the decomposition numbers of G for PIMs whose heads lie in the series above (L,X). As in the previous section, we explain how to generalise this method when X is no longer simple.

Proposition 2.3. *Let X be a cuspidal $\mathcal{O}L$-lattice. We assume that*

(1) *the head Y of kX is a simple module;*
(2) $N_G(\mathbf{L},KX)=N_G(\mathbf{L},Y)$*; and*
(3) $\mathrm{Hom}_{kL}({}^wP_Y,\mathrm{rad}(kX))=0$ *for all* $w\in N_G(\mathbf{L})$.

Then $R_L^G(X)$ is reduction stable, $\mathcal{H}_G(L,kX)\simeq\mathcal{H}_G(L,Y)$ and the decomposition matrix of $\mathcal{H}_G(L,X)$ embeds into that of G.

Proof. Let $w\in N_G(\mathbf{L})$. Condition (3) implies that ${}^wY$ is not a composition factor of $\mathrm{rad}(kX)$ which shows in particular that $\mathrm{Hom}_{kL}({}^w(kX),\mathrm{rad}(kX))=0$. Therefore the conditions of Proposition 2.2 are satisfied and we deduce that $R_L^G(X)$ is reduction stable and $\mathcal{H}_G(L,kX)\simeq\mathcal{H}_G(L,Y)$.

To show the statement on the decomposition matrices we only need to check the condition (D) given in [**29**, 4.1.13, 4.1.14]. Since the head of kX is simple, equal to Y, we have a surjective map $P_Y\twoheadrightarrow kX$, which induces a surjective map $R_L^G(P_Y)\twoheadrightarrow R_L^G(kX)$. Let $\widetilde{P}_Y$ be the projective $\mathcal{O}L$-module which is the (unique) lift of P_Y, and let $P:=R_L^G(\widetilde{P}_Y)$. Since P is projective we have a map $P\longrightarrow R_L^G(X)$ which lifts the surjective map $R_L^G(P_Y)\twoheadrightarrow R_L^G(kX)$. By Nakayama's lemma it should also be surjective. Now condition (D) is equivalent to

$$\langle KP;R_L^G(KX)\rangle_G=\langle R_L^G(KX);R_L^G(KX)\rangle_G.$$

To verify it we use the condition (3), the Mackey formula and cuspidality. We have

$$\langle KP;R_L^G(KX)\rangle_G=\langle R_L^G(KP_Y);R_L^G(KX)\rangle_G=\sum_{w\in N_G(\mathbf{L})/L}\langle KP_Y;{}^wKX\rangle_G.$$

Now $\langle KP_Y;{}^wKX\rangle_G$ equals the multiplicity of Y^w as a composition factor of kX, which by (3) is also the multiplicity of Y^w in $Y=kX/\mathrm{rad}(kX)$. It is therefore 1 if $w\in N_G(\mathbf{L},Y)$ or 0 otherwise. By (2) we have $N_G(\mathbf{L},Y)=N_G(\mathbf{L},KX)$ hence $\langle KP_Y;{}^wKX\rangle_G=\langle KX;{}^wKX\rangle_G$ and we are done. □

For convenience we state a version of Proposition 2.3 when $N_G(\mathbf{L},kX)$ is as big as possible, in which case condition (3) becomes simpler.

Corollary 2.4. *Let X be a cuspidal $\mathcal{O}L$-lattice in a block b of $\mathcal{O}L$ such that*

(1) *the head Y of kX is a simple module;*
(2) $N_G(\mathbf{L},Y)=N_G(\mathbf{L},KX)=N_G(\mathbf{L},b)$*; and*
(3) *Y occurs only once as a composition factor of kX.*

Then $R_L^G(X)$ is reduction stable, $\mathcal{H}_G(L,kX) \simeq \mathcal{H}_G(L,Y)$ and the decomposition matrix of $\mathcal{H}_G(L,X)$ embeds into that of G.

EXAMPLE 2.5. Let χ be a cuspidal ordinary irreducible character of a 1-split Levi subgroup L of G.

(a) Assume that χ lies in a block b with cyclic defect groups. Let Y be any kL-composition factor of the ℓ-reduction of χ. Then by Green [**32**] there exists an $\mathcal{O}L$-lattice X with character χ such that kX is uniserial, has mutually distinct composition factors, and has X as its head (the various composition factors are labelling the edges incident to the vertex labelled by χ in the Brauer tree of b). Then the assumptions of Corollary 2.4 are for example satisfied whenever $N_G(\mathbf{L},b)$ acts trivially on the Brauer tree and on the character χ.

(b) Assume that the ℓ-modular reduction of χ has only two composition factors, say Y and Z, and that Y is self-dual. Then there exists an $\mathcal{O}L$-lattice X with character χ or χ^* such that kX is uniserial with head Y. In that case it is enough to check assumption (2) of Corollary 2.4. Note that it can be checked equivalently on Y or Z.

(c) More generally, assume that Y is a simple kL-module occurring in the ℓ-modular reduction of χ with multiplicity one. In other words we assume that $\langle \Psi_Y;\chi\rangle_L = 1$. Denote by $\widetilde{P}_Y$ a projective $\mathcal{O}L$-module lifting P_Y. Let
$$e := \sum_{\rho\in\mathrm{Irr}\,_K b\smallsetminus\{\chi\}} e_\rho$$
where e_ρ is the central idempotent associated to the irreducible character ρ. Then $N := e\widetilde{P}_Y \cap \widetilde{P}_Y$ is a pure $\mathcal{O}L$-submodule of $\widetilde{P}_Y$ and $X := \widetilde{P}_Y/N$ is an $\mathcal{O}L$-lattice with character χ such that kX has simple head Y. In particular, X satisfies conditions (1) and (3) of Corollary 2.4.

3. Verifying reduction stability

Let us comment on how to guarantee the assumptions of Corollary 2.4 in certain situations. In this work, we will solely be concerned with unipotent blocks of finite reductive groups $G = \mathbf{G}^F$. By [**7**, Thm. 17.7] unipotent characters and unipotent Brauer characters are insensitive to the centre of the group whenever ℓ is very good, therefore we may assume that $\mathbf{G}$ has connected centre. Now let $\mathbf{L}$ be an F-stable Levi subgroup with a cuspidal $\mathcal{O}L$-lattice X such that the head of kX is simple. If X lies in a unipotent block then it is trivial on $Z(L)$. Now $N_G(\mathbf{L}) = N_{\mathbf{G}}(\mathbf{L})^F$ induces algebraic automorphisms of $\mathbf{L}$, hence inner, diagonal and graph automorphisms. If $\mathbf{G}$ has connected centre, then so has $\mathbf{L}$, and all diagonal automorphisms of L are inner; hence $N_{\mathbf{G}}(\mathbf{L})^F$ induces graph automorphisms on L. Then there exists an $N_G(\mathbf{L})$-stable $\mathcal{O}L$-lattice X' with $kX' = kX$ if one of the following holds:

- $[\mathbf{L},\mathbf{L}]$ does not have non-trivial graph automorphisms;
- $[\mathbf{L},\mathbf{L}]$ is a product of simple factors regularly permuted by the graph automorphisms induced by $N_G(\mathbf{L})$ (since then we may choose a lift in one of the factors and then take the product over the $N_G(\mathbf{L})$-orbit).

This will deal with most of the cases we encounter. It thus remains to discuss situations in which $N_{\mathbf{G}}(\mathbf{L})^F$ induces non-trivial graph automorphisms on a simple factor of $\mathbf{L}^F$.

Part 2

Decomposition matrices

CHAPTER 3

Description of the strategy

We keep the notation and setup from the previous chapter. In particular, $\mathbf{G}$ is a connected reductive linear algebraic group, $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism with respect to an $\mathbb{F}_q$-rational structure and $G = \mathbf{G}^F$ is the corresponding finite group of Lie type.

In our proofs we shall use the following tools, most of which already served well in our papers [**17, 18**]:

(HCi) Harish-Chandra inducing PIMs from proper Levi subgroups and cutting by a block of G gives projective characters, which are hence non-negative integral linear combinations of PIMs of G. Similarly, projective characters can also be obtained by Harish-Chandra restricting projectives from a larger group containing G as a Levi subgroup, or by a succession of such steps.

(HCr) If Ψ is a projective character of G such that no non-zero proper subcharacter has the property that its Harish-Chandra restriction to any Levi subgroup L of G decomposes non-negatively on the PIMs of L, then Ψ is the character of a PIM.

($\mathcal{H}$) The number of Brauer characters in a Harish-Chandra series equals the number of simple modules of the Hecke algebra $\mathcal{H}$ of the corresponding cuspidal Brauer character. More precisely, the decomposition of induced PIMs in that series can be read off from the corresponding decomposition for $\mathcal{H}$ (see Proposition 2.3).

(Csp) There exist cuspidal unipotent Brauer characters for G if and only if the centraliser of a Sylow ℓ-subgroup of G is not contained in any proper 1-split Levi subgroup (see [**27**, Cor. 2.7]).

(St) The *ℓ-modular Steinberg character of* G, i.e., the unique unipotent Brauer constituent in the ℓ-modular reduction of an ordinary Gelfand–Graev character of G, is cuspidal if and only if a Sylow ℓ-subgroup of G is not contained in any proper 1-split Levi subgroup of G (see [**27**, Thm. 4.2]).

(DL) Let $w \in W$ and Ψ be the character of a PIM of G. If Ψ does not occur in the Deligne–Lusztig character R_v for any $v < w$ then the coefficient of Ψ in $(-1)^{l(w)}R_w$ is non-negative (see Proposition 1.5).

(Red) Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ such that $\mathbf{L}^*$ is the centraliser of a semisimple ℓ-element of G^*. Then there is a sign $\varepsilon \in \{\pm 1\}$ such that for any unipotent character ρ of L, $\varepsilon R_L^G(\rho)^\circ$ is a non-negative linear combination of irreducible Brauer characters (see Lemma 1.3).

(Tri) Assume that the (unipotent) decomposition matrix of G is uni-triangular with respect to some total ordering of unipotent characters compatible with increasing a-values. Then, we can partially echelonise any set of projective characters of G (as explained in Example 1.1).

Remark 3.1. When ℓ is very good for G there is an F-equivariant bijection between the conjugacy classes of ℓ-elements of G and of G^* preserving the centralisers (see [**24**, Prop. 4.2]). Therefore in that case the assumptions of (Red) hold for G whenever they hold for the dual group G^*.

Throughout, for a prime ℓ and a finite reductive group G defined over $\mathbb{F}_q$ we will denote by $d = d_\ell(q)$ the order of q modulo ℓ. Our results turn out to be uniform in d, not depending on the prime ℓ (once ℓ is large enough with respect to the root system of G).

We will not consider the case $d = 1$, since there by a result of Puig for all large enough primes the unipotent decomposition matrix is always the identity matrix. Indeed, in this case the ℓ-blocks are unions of Harish-Chandra series and all Hecke algebras are semisimple after reduction modulo ℓ. Furthermore, we will not deal with the case when $d = 4$ since this was already considered in our predecessor paper [**18**]; we shall only indicate how some of the remaining unknowns can be computed using (T_ℓ), see Remark 8.4. Finally, we will not consider the general linear groups, as their unipotent decomposition matrices were determined up to rank 10 by James [**37**], nor the general unitary groups whose unipotent decomposition matrices up to rank 10 were computed in [**27**] for linear primes and in [**17**] for unitary primes.

CHAPTER 4

Decomposition matrices at $d_\ell(q) = 2$

In this section we determine decomposition matrices for unipotent ℓ-blocks of various classical and exceptional groups over $\mathbb{F}_q$ for primes ℓ dividing $q+1$. This is by some measure the most complicated case since the ranks of Sylow ℓ-subgroups for such primes are generally larger than for any prime ℓ with $d_\ell(q) \geq 3$. Nevertheless, we obtain almost complete results in the cases considered.

1. Centralizers of ℓ-elements

Recall that $\mathbf{G}$ is connected reductive with Frobenius map F, $\mathbf{T}$ is a maximally split torus of $\mathbf{G}$ and $\mathbf{B}$ is an F-stable Borel subgroup containing it. We denote by Φ the set of roots of $\mathbf{G}$ with respect to $\mathbf{T}$, by Φ^+ the set of positive roots corresponding to $\mathbf{B}$, and by Δ the set of simple roots. Given $\alpha \in \Phi$ we write $\mathrm{ht}(\alpha)$ for the height of α, that is the sum of the coefficients of α expressed in the basis Δ. The Weyl group of Φ can be identified with the Weyl group W of $\mathbf{G}$.

To any subset I of S there is a corresponding parabolic subgroup W_I of W and a standard Levi subgroup $\mathbf{L}_I = \langle \mathbf{T}, W_I \rangle$ of $\mathbf{G}$. When I is F-stable then $\mathbf{L}_I$ is F-stable and is a 1-split Levi subgroup of $\mathbf{G}$. In this section we shall rather be interested in the 2-split Levi subgroups as defined, e.g., in [**4**, p. 17]. They are obtained as centralisers of Φ_2-tori, which are F-stable tori of $\mathbf{G}$ of order $(q+1)^r$ for some $r \geq 0$. Let w_0 be the longest element of W and by S the set of simple reflections in W corresponding to Δ. In the case where w_0F acts trivially on $S \smallsetminus I$ then the pair $(\mathbf{L}_I, w_0F)$ is conjugate to a pair $(\mathbf{L}, F)$ where $\mathbf{L}$ is a 2-split Levi subgroup. We give here some further conditions on ℓ to ensure the existence of an ℓ-element whose centralizer is $\mathbf{L}$. This will be needed in order to use the method (Red) from Chapter 3.

PROPOSITION 4.1. *Let $I \subseteq S$ be a subset of the set of simple reflections of W and $\mathbf{L}_I$ be the corresponding standard Levi subgroup of $\mathbf{G}$. We assume that*

(1) *w_0F acts trivially on $S \smallsetminus I$;*
(2) *ℓ is very good for $\mathbf{G}$; and*
(3) *$(q+1)_\ell > \mathrm{ht}(\pi_I(\alpha))$ for all $\alpha \in \Phi^+$, where π_I is the projection of $\mathbb{Z}\Phi^+$ to $\mathbb{Z}\Phi^+_{S\smallsetminus I}$.*

Then there exists $t \in Z^\circ(\mathbf{L}_I)^{w_0F}$ such that $C_\mathbf{G}(t) = \mathbf{L}_I$.

PROOF. Let $\pi_{\mathrm{ad}} : \mathbf{G} \twoheadrightarrow \mathbf{G}_{\mathrm{ad}} := \mathbf{G}/Z(\mathbf{G})$. From [**12**, Prop. 2.3] it follows that $\pi_{\mathrm{ad}}(C^\circ_\mathbf{G}(t)) = C^\circ_{\mathbf{G}_{\mathrm{ad}}}(\pi_{\mathrm{ad}}(t))$ for any semisimple element $t \in \mathbf{G}$. Since ℓ is very good, both $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ and $Z(\mathbf{G}^*)/Z^\circ(\mathbf{G}^*)$ are ℓ'-groups. The first one ensures that any ℓ-element of G_{ad} lifts to an ℓ-element of G, whereas the second implies that the centralisers of ℓ-elements are connected by [**12**, 13.14(iii) and 13.15(i)]. Finally, by [**7**, Prop. 13.12] we also have that $Z(\mathbf{L})/Z^\circ(\mathbf{L})$ is an ℓ'-group for any Levi subgroup of $\mathbf{G}$. This shows that we can assume that $\mathbf{G}$ is semisimple of adjoint type without loss of generality.

Write $\Delta = \{\alpha_1,\ldots,\alpha_n\} \subset X(\mathbf{T})$ and denote by $\{\varpi_1,\ldots,\varpi_n\} \subset Y(\mathbf{T})$ the dual basis for the pairing between characters and cocharacters. Here $X(\mathbf{T})$ is the lattice of characters of $\mathbf{T}$ whereas $Y(\mathbf{T})$ is the lattice of cocharacters. We reorder the simple roots so that $\{\alpha_1,\ldots,\alpha_m\}$ are the simple roots corresponding to the simple reflections in $S \smallsetminus I$.

Given $\lambda \in \overline{\mathbb{F}}_p^\times$ we consider the semisimple element

$$t(\lambda) := \varpi_1(\lambda)\varpi_2(\lambda)\cdots\varpi_m(\lambda)$$

of $\mathbf{G}$. If $\alpha = \sum_{i=1}^n x_i\alpha_i \in X(\mathbf{T})$ then

$$\alpha\big(t(\lambda)\big) = \lambda^{\sum_{i=1}^m x_i} = \lambda^{\mathrm{ht}(\pi_I(\alpha))}, \tag{1}$$

where π_I is the projection of $\mathbb{Z}\Phi^+$ to $\mathbb{Z}\Phi^+_{S\smallsetminus I}$. In particular $t(\lambda)$ lies in the kernel of every root $\alpha \in \Phi_I$, therefore it lies in $Z(\mathbf{L}_I)$. In addition, $w_0F(\varpi_i) = -q\varpi_i$ for every $i \leq m$. Indeed, by assumption (1) w_0F permutes the elements in I but fixes the elements in $S\smallsetminus I$. Therefore we have

$$\begin{aligned}\langle w_0F(\varpi_i) + q\varpi_i, \alpha_j\rangle &= \langle w_0F(\varpi_i),\alpha_j\rangle + q\langle \varpi_i,\alpha_j\rangle\\ &= \langle \varpi_i, w_0F(\alpha_j)\rangle + q\langle\varpi_i,\alpha_j\rangle\\ &= \langle \varpi_i, w_0F(\alpha_j)\rangle + q\delta_{i,j}.\end{aligned}$$

If $j > m$ then $w_0F(\alpha_j) \in \mathbb{Z}\Phi_I$ therefore $\langle \varpi_i, w_0F(\alpha_j)\rangle = 0$ whereas if $j \leq m$ we have $w_0F(\alpha_j) = -q\alpha_j$ and hence $\langle\varpi_i, w_0F(\alpha_j)\rangle = -q\delta_{i,j}$. In each case $\langle w_0F(\varpi_i)+q\varpi_i,\alpha_j\rangle = 0$ which proves that $w_0F(\varpi_i) = -q\varpi_i$ for all $i \leq m$. We deduce that $t(\lambda)$ is w_0F-stable whenever λ is an ℓ-element satisfying $\lambda^q = \lambda^{-1}$.

By (1) and assumption (3), there exists an ℓ-element λ of $\overline{\mathbb{F}}_p^\times$ such that $\lambda^{q+1} = 1$ and $\alpha\big(t(\lambda)\big) \neq 1$. By [**12**, Prop. 2.3] this implies that $C^\circ_{\mathbf{G}}(t(\lambda)) = \mathbf{L}_I$. But since ℓ is very good the centraliser of every ℓ-element is connected by [**12**, 13.14(iii)]. $\square$

2. Multiplicities in the Steinberg character

Before considering individual series of groups, we first prove a general result. It demonstrates how Deligne–Lusztig characters can be used to obtain relations between the entries of decomposition matrices of unipotent Φ_2-block, yielding new decomposition numbers in the "bottom right corner" of the matrix. This is the part of the decomposition matrix about which Harish-Chandra methods usually yield the least information.

We call

$$h := 1 + \max\{\mathrm{ht}(\alpha) \mid \alpha \in \Phi^+\}$$

the *Coxeter number* of $\mathbf{G}$. When Φ is irreducible, it equals the order of any Coxeter element of W (see [**3**, Prop. VI.1.31]).

THEOREM 4.2. *Let $\mathbf{G}$ be connected reductive. We assume that*

(1) *w_0F acts trivially on W;*
(2) *ℓ is very good for $\mathbf{G}$; and*
(3) *$(q+1)_\ell \geq h$, where h is the Coxeter number of $\mathbf{G}$.*

Then there exists a linear character θ of T_{w_0} in general position such that $\theta^\circ = 1$ and

$$(-1)^{l(w_0)}R^G_{T_{w_0}}(\theta)^\circ = \varphi_{\mathrm{St}}.$$

PROOF. Let $(\mathbf{G}^*, \mathbf{T}^*, F)$ be in duality with $(\mathbf{G}, \mathbf{T}, F)$. By Proposition 4.1 applied to $\mathbf{G}^*$ and $I = \emptyset$, there exists an ℓ-element $t \in T^*_{w_0}$ such that $C_{\mathbf{G}^*}(t) = \mathbf{T}^*$. Therefore t is a regular ℓ-element. Under the duality, this shows that there exists an irreducible character θ of T_{w_0} in general position such that $\theta^\circ = 1$.

By [**39**, Cor. 2.10] the property that θ is regular implies that $\chi_\theta = (-1)^{l(w_0)} R^G_{T_{w_0}}(\theta)$ is an irreducible character. Furthermore, since w_0F is central w_0 lies in a cuspidal F-conjugacy class of W which implies that χ_θ is a cuspidal character by [**39**, Cor. 2.19]. In particular, any irreducible Brauer character φ occurring in χ°_θ is also cuspidal.

Since θ is an ℓ-character then $\chi^\circ_\theta = (-1)^{l(w_0)} R^G_{T_{w_0}}(1)^\circ$, which gives the expression of χ°_θ on the basic set of unipotent characters. We denote by P_{St} the unique projective indecomposable summand of a Gelfand–Graev character which contains the Steinberg character St of G, and by Ψ_{St} its character. The Steinberg character occurs with multiplicity 1 in Ψ_{St} and any other constituent is non-unipotent. Since $\langle R^G_{T_{w_0}}(1); \mathrm{St}\rangle = \langle R^G_{T_{w_0}}(1); \Psi_{\mathrm{St}}\rangle = (-1)^{l(w_0)}$ (see for example [**12**, Cor. 12.18(ii)]) we deduce that φ_{St} occurs with multiplicity one in χ°_θ.

We need to show that no other Brauer character can occur. Recall from §3 that for $w \in W$, we denote by $\widetilde{R}_w$ a virtual projective character obtained by adding and removing suitable non-unipotent characters to $R_w = R^G_{T_w}(1)$. The orthogonality relations for Deligne–Lusztig characters, together with the fact that R_{w_0} contains only unipotent constituents, yield the following relation for $w \neq w_0$:

$$0 = \langle R^G_{T_w}(1); (-1)^{l(w_0)} R^G_{T_{w_0}}(1)\rangle = \langle \widetilde{R}_w; \chi^\circ_\theta\rangle = \sum_{\varphi \in \mathrm{IBr}\, G} \langle \widetilde{R}_w; \varphi\rangle \langle \Psi_\varphi; \chi^\circ_\theta\rangle. \tag{2}$$

Note that since χ_θ is cuspidal, the Brauer characters contributing to this sum are all cuspidal. Let $w \neq w_0$ be of minimal length in its conjugacy class. We prove by induction on its length $l(w)$ that for every cuspidal Brauer character φ, if $\langle \widetilde{R}_w; \varphi\rangle \neq 0$ then $\langle \Psi_\varphi; \chi^\circ_\theta\rangle = 0$. This already holds for any element w lying in a proper F-stable parabolic subgroup since in that case there are no cuspidal Brauer characters φ such that $\langle \widetilde{R}_w; \varphi\rangle \neq 0$. Assume now that the property holds for any $v \in W$ such that $l(v) < l(w)$. If φ is an irreducible Brauer character such that $\langle \widetilde{R}_w; \varphi\rangle\langle \Psi_\varphi; \chi^\circ_\theta\rangle \neq 0$, then by induction, Ψ_φ cannot occur in any $\widetilde{R}_v$ for $l(v) < l(w)$. It follows from Proposition 1.5 that $(-1)^{l(w)}\langle \widetilde{R}_w; \varphi\rangle > 0$, so that $(-1)^{l(w)}\langle \widetilde{R}_w; \varphi\rangle\langle \Psi_\varphi; \chi^\circ_\theta\rangle > 0$ which contradicts (2).

In other words, if Ψ_φ occurs in some $\widetilde{R}_w$ for $w \neq w_0$ then φ is not a constituent of χ°_θ. It remains to see that all the PIMs but one will actually occur. Note that since $\langle \chi^\circ_\theta; \varphi_{\mathrm{St}}\rangle = 1$ we already know that Ψ_{St} occurs only in $\widetilde{R}_{w_0}$. Let $\varphi \neq \varphi_{\mathrm{St}}$ be an irreducible Brauer character. Let us consider the virtual projective character

$$\widetilde{Q} = \sum_{w \in W} (-1)^{l(w)} \widetilde{R}_w.$$

Its unipotent part equals St, therefore we must have $\widetilde{Q} = \Psi_{\mathrm{St}}$. In particular, Ψ_φ does not occur in $\widetilde{Q}$. We deduce that if Ψ_φ does not occur in $\widetilde{R}_w$ for all $w \neq w_0$, then it does not occur in any $\widetilde{R}_w$, which contradicts [**2**, Thm. A]. $\square$

We can use the previous theorem to compute non-trivial decomposition numbers of the Steinberg character.

COROLLARY 4.3. *Let* $(\mathbf{G}, F)$ *be simple of classical type* ${}^2A_{n-1}(q)$, $D_{2n}(q)$, ${}^2D_{2n+1}(q)$ *with* $n \geq 2$, *or of exceptional type* ${}^2E_6(q)$, $E_7(q)$, *or* $E_8(q)$. *Assume that* p *is good and* ℓ *is very good for* $\mathbf{G}$. *Then:*

(a) *There is a unique unipotent character* ρ *of* $\mathbf{G}^F$ *with* a*-value* 1.
(b) *Let* ρ^* *be the Alvis–Curtis dual of* ρ. *If* $(q+1)_\ell \geq h$, *there exists a PIM of* $\mathbf{G}^F$ *whose unipotent part is given by* $\rho^* + (\operatorname{rank} \mathbf{G})\,\mathrm{St}$.

PROOF. When ℓ is very good unipotent characters and Brauer characters are insensitive to the centre by [**7**, Thm. 17.7]. Therefore we may and we will assume that $\mathbf{G}$ has trivial centre. Let $\mathcal{O}$ be the subregular unipotent class of $\mathbf{G}$, that is the maximal unipotent class outside the regular unipotent class. It is the unique class of codimension 2 in the variety of unipotent elements in $\mathbf{G}$ and thus F-stable. From the classification of unipotent classes in good characteristic (see e.g. [**46**]) one can check that it is special. We list below for each type considered the class (with Jordan form in the natural matrix representation for classical types), the special unipotent character ρ of $\mathbf{G}^F$ with unipotent support $\mathcal{O}$ and its Alvis–Curtis dual ρ^*. Recall from §2 that a unipotent character corresponding to a bipartition λ is denoted by $[\lambda]$.

Type	$\mathcal{O}$	ρ	ρ^*
${}^2A_{n-1}$	$(n-1,1)$	$[(n-1,1)]$	$[21^{n-2}]$
D_n	$(2n-3,3)$	$[n-1.1]$	$[1.1^{n-1}]$
${}^2D_n$	$(2n-3,3)$	$[(n-2,1).]$	$[.21^{n-3}]$
${}^2E_6$	$E_6(a_1)$	$\phi'_{2,4}$	$\phi''_{2,16}$
E_7	$E_7(a_1)$	$\phi_{7,1}$	$\phi_{7,46}$
E_8	$E_8(a_1)$	$\phi_{8,1}$	$\phi_{8,91}$

In each of these cases, the a-value of ρ equals 1, and ρ (as well as ρ^*) is alone in its family. By maximality of $\mathcal{O}$, every other non-trivial unipotent character has a-value at least 2, which proves (a).

Since the Springer correspondence sends the trivial local system on $\mathcal{O}$ to the reflection representation ϕ_{ref} of W, then ρ is equal to the almost character associated with ϕ_{ref} (see for example [**8**, §13.3]). More precisely, there exists an extension $\widetilde{\phi}_{\mathrm{ref}}$ of ϕ_{ref} to $W \rtimes \langle F \rangle$ such that

$$\rho = R_{\widetilde{\phi}_{\mathrm{ref}}} := \frac{1}{|W|} \sum_{w \in W} \widetilde{\phi}_{\mathrm{ref}}(wF) R_w. \tag{3}$$

When $(\mathbf{G}, F)$ is split, ρ is just the principal series character corresponding to ϕ_{ref}.

By [**41**, Thm. 11.2] (see [**48**] for the generalisation to the case where p is good), there exists a generalised Gelfand–Graev module Γ of $\mathcal{O}G$ whose character has unipotent part $\rho^* + x\mathrm{St}$ for some non-negative integer x (depending on q). Let P be the unique direct summand of Γ whose character Ψ has ρ^* as a constituent. Then the unipotent part of Ψ equals $\rho^* + y\mathrm{St}$ for some non-negative integer y. Let φ be the irreducible Brauer character such that $\Psi = \Psi_\varphi$. By Theorem 4.2, the multiplicity of φ in $(R_{w_0})^\circ$ is zero, yielding the equation

$$0 = \langle \Psi; R_{w_0}\rangle = \langle \rho^*; R_{w_0}\rangle + (-1)^{l(w_0)}y. \tag{4}$$

Now, the Alvis–Curtis dual of R_{w_0} is $(-1)^{l(w_0)}R_{w_0}$ (see for example [**12**, Thm. 12.8]). Using equation (3) we get

$$\langle \rho^*; R_{w_0}\rangle = (-1)^{l(w_0)}\langle \rho; R_{w_0}\rangle = (-1)^{l(w_0)}\widetilde{\phi}_{\mathrm{ref}}(w_0F) = -(-1)^{l(w_0)}\mathrm{rank}\,\mathbf{G}.$$

Then (b) follows from (4). □

For the groups not listed in Corollary 4.3, but for which w_0F acts trivially on W, there are several unipotent characters with minimal non-zero a-value and they form a non-trivial family. We can still give an approximation of the previous decomposition number as we illustrate for groups of type B and C.

Corollary 4.4. *Let $(\mathbf{G}, F)$ be simple of type $B_n(q)$ or $C_n(q)$, $n \geq 2$. Assume that p and ℓ are odd. If $(q+1)_\ell > 2n$, then there exist two PIMs whose unipotent parts are given by*

$$[1^n.] + (n-\delta)\,\mathrm{St} \quad \textit{and} \quad [B_2{:}\,.1^{n-2}] + (n-1+\delta)\,\mathrm{St},$$

where $\delta = 1$ if n is even, and 0 otherwise.

Proof. As above, we may and we will assume that the centre of $\mathbf{G}$ is trivial. The unique family of unipotent characters of G with a-value 1, which corresponds to the subregular unipotent class, is $\mathcal{F} = \{[n-1.1], [.n], [(n-1,1).], [B_2{:}\,n-2.]\}$. In terms of symbols, these unipotent characters are given in the same order by

$$\mathcal{F} = \left\{ \begin{pmatrix} 0\ n \\ 1 \end{pmatrix}, \begin{pmatrix} 0\ 1 \\ n \end{pmatrix}, \begin{pmatrix} 1\ n \\ 0 \end{pmatrix}, \begin{pmatrix} 0\ 1\ n \\ - \end{pmatrix} \right\}.$$

Let us focus on the characters $[.n]$ and $[B_2{:}\,n-2.]$. Their uniform parts can be expressed in terms of almost characters (see [**40**, Chap. 4]), from which we can compute the scalar product with any Deligne–Lusztig character R_w, $w \in W$. This yields

$$\langle [.n]; R_w\rangle = \frac{1}{2}\big(\phi^{n-1.1}(w) - \phi^{(n-1,1).}(w) + \phi^{.n}(w)\big),$$
$$\langle [B_2{:}\,n-2.]; R_w\rangle = \frac{1}{2}\big(\phi^{n-1.1}(w) - \phi^{(n-1,1).}(w) - \phi^{.n}(w)\big),$$

where ϕ^μ denotes the irreducible character of W corresponding to the bipartition μ of n. One can compute easily the values of these characters at the central element w_0: the character $\phi^{n-1.1}$ is the reflection character, therefore $\phi^{n-1.1}(w_0) = -n$. The character $\phi^{.n}$ is linear, with value -1 on the first simple reflection and 1 on the others, and thus $\phi^{.n}(w_0) = (-1)^n$. Finally, for the value of $\phi^{(n-1,1).}$ we use that the induction of $\phi^{n-1.}$ from B_{n-1} to B_n decomposes as $\phi^{(n-1,1).} + \phi^{n.} + \phi^{n-1.1}$ which gives $\phi^{(n-1,1).}(w_0) = n-1$. This yields

$$\begin{aligned} \langle [.n]; R_{w_0}\rangle &= -n + \delta_{n\ \mathrm{even}}, \\ \langle [B_2{:}\,n-2.]; R_{w_0}\rangle &= -n + \delta_{n\ \mathrm{odd}}. \end{aligned} \tag{5}$$

The Alvis–Curtis duals of $[.n]$ and $[B_2{:}\,n-2.]$ are $[1^n.]$ and $[B_2{:}\,.1^{n-2}]$ respectively. By [**5**, Thm. B], there exist Kawanaka modules K' and K'' which are projective modules whose characters have unipotent parts $[1^n.] + x\mathrm{St}$ and $[B_2{:}\,.1^{n-2}] + y\mathrm{St}$, for some non-negative integers x and y. Therefore there is an indecomposable summand P' (resp. P'') of K' (resp. K''), with $[1^n.]$ (resp. $[B_2\colon .1^{n-2}]$) occurring in the character of P' (resp. P''). Now using Theorem 4.2 and (5) we can compute the multiplicity of St in both of these projective characters as given in the statement. $\square$

3. Unipotent decomposition matrix of $D_4(q)$

We are now ready to compute decomposition matrices for specific series of finite reductive groups. We start with the orthogonal groups $D_4(q)$. As customary, we will label the unipotent characters in the principal series by characters of the Weyl group of type D_4, that is, by unordered pairs of partitions of 4, while the (unique) cuspidal unipotent character will be denoted "D_4". This is also the labelling used in the Chevie system [**43**].

In our tables of decomposition matrices the second column lists the degrees of the unipotent characters as a product of cyclotomic polynomials Φ_e evaluated at q. In the last line, we give the ℓ-modular Harish-Chandra series of the Brauer characters (and, by abuse of notation, also of the corresponding PIMs), by either writing "ps" for characters in the principal series, or a type of Levi subgroup if that Levi subgroup has a unique cuspidal unipotent Brauer character, or by the label of a cuspidal unipotent Brauer character of that Levi subgroup. The root system of type D_4 has three conjugacy classes of subsystems of type A_1^2, cyclically permuted by the graph automorphism of order 3; we denote them by D_2, A_1^2 and $A_1^{2\prime}$. The symbol "c" denotes cuspidal PIMs. Also, in all of our tables for better readability we print "." in place of "0".

The groups of type D_4 have 14 unipotent characters. For primes $\ell > 2$ with $\ell | (q+1)$, thirteen of them lie in the principal ℓ-block, while the one with label 1.21 is of ℓ-defect zero.

THEOREM 4.5. *The decomposition matrix for the unipotent ℓ-blocks of $D_4(q)$, q odd, $q \equiv -1 \pmod{\ell}$, $\ell \geq 11$, is as given in Table 1.*

PROOF. By [**24**, Thm. 5.1] the unipotent characters form a basic set for the unipotent ℓ-blocks of $G = D_4(q)$ when $\ell \neq 2$. It was shown in [**31**], by constructing projective characters by Harish-Chandra induction and from generalised Gelfand–Graev characters, that when q is odd the decomposition matrix of the unipotent blocks of G is uni-triangular. This provides a unique labelling for the irreducible unipotent Brauer characters; we denote them by φ_x if x is the label of the corresponding ordinary unipotent character.

Let us denote by Ψ_i, $1 \leq i \leq 14$, the linear combinations of unipotent characters given by the columns in Table 1. Harish-Chandra induction from proper Levi subgroups now yields these projectives except for Ψ_9, Ψ_{13} and Ψ_{14}. The Steinberg-PIM is cuspidal by (St) since no proper Levi subgroup contains a Sylow ℓ-subgroup of G.

The first two columns correspond to PIMs as can be seen from the decomposition matrix of the Hecke algebra for the principal series. By Harish-Chandra theory, if any of the listed induced projective characters is decomposable, it can only contain constituents from lower Harish-Chandra series. It is then easily seen that all other projective characters given in Table 1 must also be indecomposable.

It remains to show that Ψ_9 and Ψ_{13} are the unipotent parts of PIMs. The graph automorphism of G of order 3 fixes the unique cuspidal unipotent character but permutes

TABLE 1. $D_4(q)$, $11 \leq \ell|(q+1)$

$.4$	1	1													
1.3	$q\Phi_4^2$	2	1												
$2+$	$q^2\Phi_3\Phi_6$	1	1	1											
$2-$	$q^2\Phi_3\Phi_6$	1	1	.	1										
$.31$	$q^2\Phi_3\Phi_6$	1	1	.	.	1									
$.2^2$	$\frac{1}{2}q^3\Phi_4^2\Phi_6$	.	1	1	1	1	1								
$1^2.2$	$\frac{1}{2}q^3\Phi_3\Phi_4^2$	2	2	1	1	1	.	1							
1.21	$\frac{1}{2}q^3\Phi_2^4\Phi_6$	.	.	.	.	.	.	.	1						
D_4	$\frac{1}{2}q^3\Phi_1^4\Phi_3$	.	.	.	.	.	.	.	.	1					
1^2+	$q^6\Phi_3\Phi_6$	1	1	1	1	1	1	1	.	2	1				
1^2-	$q^6\Phi_3\Phi_6$	1	1	1	1	1	1	1	.	2	.	1			
$.21^2$	$q^6\Phi_3\Phi_6$	1	1	1	1	1	1	1	.	2	.	.	1		
1.1^3	$q^7\Phi_4^2$	2	1	1	1	1	1	2	.	4	1	1	1	1	
$.1^4$	q^{12}	1	.	.	.	.	1	1	.	6	1	1	1	4	1
		ps	ps	A_1^2	$A_1^{2\prime}$	D_2	D_2A_1	A_1	ps	c	A_1^2	$A_1^{2\prime}$	D_2	c	c

the characters with labels $1^2+, 1^2-, .21^2$ cyclically. It follows that the first three entries below the diagonal in the 9th column of the decomposition matrix must be equal. We denote them by a_1. The other two unknown entries in this column will be denoted by a_2 and a_3. The last entry in the 13th column will be denoted by a_4 so that

$$\begin{aligned}\Psi_9 =&[D_4] + a_1([1^2+] + [1^2-] + [.21^2]) + a_2[1.1^3] + a_3[.1^4],\\ \Psi_{13} =&[1.1^3] + a_4[.1^4].\end{aligned}$$

By Theorem 4.2 we know that $R_{w_0}^\circ$ has $\varphi_{.1^4}$ as only Brauer constituent. The relation $\langle \Psi_{1.1^3}; R_{w_0}^\circ \rangle = 0$ shows that $a_4 = 4$, and the relation $\langle \Psi_{D_4}; R_{w_0}^\circ \rangle = 0$ yields $a_3 = 8 - 9a_1 + 4a_2$.

Let $w \in W$ be a Coxeter element. The coefficient of Ψ_{13} on R_w equals $-1 + 3a_1 - a_2$. By (DL) this forces $-1 + 3a_1 - a_2 \geq 0$. On the other hand by Proposition 4.1 for $\ell > 4$ a 2-split Levi subgroup ${}^2A_2(q).(q+1)^2$ of G has a linear ℓ-character in general position. Then (Red) with ρ the unique cuspidal unipotent character of L gives $3a_1 - a_2 \leq 3$. Therefore there exists $c \in \{-1, 0, 1\}$ such that $a_2 = 3a_1 - 2 + c$. With this notation we have $a_3 = 4c + 3a_1$. We will see in the proof of Theorem 4.8 that $a_1 = 2$ and in the proof of Theorem 4.9 that $c = 0$. □

REMARK 4.6. Assuming (T_ℓ), the result in Table 1 remains true for even q. In fact, it was shown by Paolini [**45**] that (T_ℓ) holds for $D_4(q)$ in characteristic 2 and thus the unipotent decomposition matrix of $D_4(2^f)$ agrees with the one given in Table 1, up to the knowledge of a_1 and c which will require (T_ℓ) for the types D_5 and D_6.

4. Unipotent decomposition matrix of $D_5(q)$

We now turn to the orthogonal groups $D_5(q)$. For the determination of the decomposition matrices for unipotent blocks of non-cyclic defect we will need to know the structure and parameters of the Hecke algebras attached to cuspidal ℓ-modular Brauer characters of certain Levi subgroups. Here and later, by convention, $\mathcal{H}(X; q_1)$ for $X \in \{A_n, D_n\}$ denotes the Iwahori–Hecke algebra over k of type X with parameters $(q_1, -1)$, and $\mathcal{H}(B_n; q_1; q_2)$ denotes the Iwahori–Hecke algebra over k of type B_n with parameters $(q_1, -1)$ at the type B_1-node, and parameters $(q_2, -1)$ at the remaining nodes on the A_{n-1}-branch. In Table 2 we have collected the number $|\mathrm{Irr}\,\mathcal{H}|$ for some small rank modular Iwahori–Hecke $\mathcal{H}$ algebras occurring later. They can easily be computed using for example the programme of Jacon [**36**].

TABLE 2. $|\mathrm{Irr}\,\mathcal{H}|$ for some modular Hecke algebras

$n =$	1	2	3	4	5	6
$\mathcal{H}(B_n; 1; 1)$	2	5	10	20	36	65
$\mathcal{H}(B_n; -1; 1)$	1	2	3	5	7	?
$\mathcal{H}(B_n; 1; -1)$	2	2	4	6	8	12
$\mathcal{H}(B_n; -1; -1)$	1	2	3	4	6	9
$\mathcal{H}(D_n; 1)$	1	2	3	13	18	37
$\mathcal{H}(D_n; -1)$	1	1	2	3	4	6

In this as well as in all later tables, $W(D_1)$ has to be interpreted as the trivial group.

LEMMA 4.7. *Let q be a prime power and $2 < \ell|(q+1)$. The Hecke algebras of various ℓ-modular cuspidal pairs (L, λ) of Levi subgroups L in $D_n(q)$ and their respective numbers of simple modules are as given in Table 3.*

TABLE 3. Hecke algebras and $|\mathrm{Irr}\,\mathcal{H}|$ in $D_n(q)$ for $d_\ell(q) = 2$

(L, λ)	$\mathcal{H}$	$n = 4$	5	6
(A_1, φ_{1^2})	$\mathcal{H}(A_1; q) \otimes \mathcal{H}(D_{n-2}; q)$	1	2	$2+1$
$(A_1^2, \varphi_{1^2}^{\boxtimes 2})$	$\mathcal{H}(B_2; q^2; q) \otimes \mathcal{H}(D_{n-4}; q)$	2	2	2
$(A_1^3, \varphi_{1^2}^{\boxtimes 3})$	$\mathcal{H}(B_3; q; q^2) \otimes \mathcal{H}(D_{n-6}; q)$	–	–	3
$(D_2, \varphi_{.1^2})$	$\mathcal{H}(B_{n-2}; q^2; q)$	2	4	$4+2$
$(D_2A_1, \varphi_{.1^2} \boxtimes \varphi_{1^2})$	$\mathcal{H}(A_1; q) \otimes \mathcal{H}(B_{n-4}; q^2; q)$	1	2	2
$(D_2A_1^2, \varphi_{.1^2} \boxtimes \varphi_{1^2}^{\boxtimes 2})$	$\mathcal{H}(B_2; q; q) \otimes \mathcal{H}(B_{n-6}; q^2; q)$	–	–	2
(D_4, D_4)	$\mathcal{H}(B_{n-4}; q^4; q)$	1	2	2
$(D_4, \varphi_{1.1^3})$	$\mathcal{H}(B_{n-4}; q^2; q)$	1	2	2
$(D_4, \varphi_{.1^4})$	$\mathcal{H}(B_{n-4}; q^2; q)$	1	2	2
$(D_4A_1, D_4 \boxtimes \varphi_{1^2})$	$\mathcal{H}(A_1; q^9) \otimes \mathcal{H}(B_{n-6}; q^4; q)$	–	–	1

PROOF. The relative Weyl group of a Levi subgroup L of $D_n(q)$ of type A_1 has type $D_{n-2}A_1$ (see either [**35**, p. 72] or use Chevie [**26**]). As the modular Steinberg character φ_{1^2} of $A_1(q)$ is liftable to a cuspidal character in characteristic 0, the parameters of the

Hecke algebra are the same as in characteristic 0 by Proposition 2.2. They can hence be determined locally inside the minimal Levi overgroups of L: in type A_1^2 the parameter is q, and similarly in a Levi subgroup of type D_2A_1.

The relative Weyl group of a Levi subgroup of type A_1^2 has type B_2D_{n-4}. The modular Steinberg character of $A_1(q)^2$ is liftable, so we can determine the parameters locally inside $A_3(q)$ (here the parameter is q^2), in $D_2(q)A_1(q)$ and in $A_1(q)^3$ (where the parameters clearly are q). The relative Weyl group of a Levi subgroup of type A_1^3 has type B_3D_{n-6}, and it is contained in minimal Levi overgroups of types A_3A_1 $D_2A_1^2$ and A_1^4.

The relative Weyl group of a Levi subgroup of type D_2 (the two end nodes interchanged by the graph automorphism) has type B_{n-2} by [**35**, p. 71]. Again the modular Steinberg character $\varphi_{.1^2}$ of $D_2(q)$ is liftable to a cuspidal character in characteristic 0, and the parameters of the Hecke algebra can hence be determined locally: inside a Levi subgroup of type D_3 the parameter is q^2, while inside a Levi subgroup of type D_2A_1, it clearly is q.

The relative Weyl group of a Levi subgroup D_2A_1 is of type A_1B_{n-4}; the parameters of the Hecke algebra for its cuspidal ℓ-modular (liftable) Steinberg character $\varphi_{.1^2} \boxtimes \varphi_{1^2}$ are again determined locally inside proper Levi subgroups D_4, D_3A_1 and $D_2A_1^2$ of G. In all cases discussed so far, the assumptions of reduction stability in Corollary 2.4 are satisfied as all graph automorphisms of L just permute simple components. The same considerations apply to the cuspidal Brauer character of a Levi subgroup of type $D_2A_1^2$.

The ordinary cuspidal unipotent character of $G = D_4(q)$ remains irreducible upon reduction modulo ℓ, by Table 1, hence the corresponding cuspidal Brauer character is reduction stable. The parameters of the Hecke algebra in characteristic 0 are known, see [**8**, p. 464]. The parameters in the remaining cases are determined analogously. The cuspidal modular Steinberg character is the ℓ-modular reduction of a Deligne–Lusztig character $R_T^G(\theta)$ for $\theta \in \operatorname{Irr} T$ in general position with T a Sylow Φ_2-torus. The normaliser of a Levi subgroup of type D_4 in larger groups of type D_n induces the graph automorphism of order 2 on G. (This can be seen from the inclusion $(\mathrm{GO}_8\mathrm{GO}_{2n-8}) \cap \mathrm{SO}_{2n} \leq \mathrm{SO}_{2n}$.) It is clear that there exist ℓ-characters $\theta \in \operatorname{Irr} T$ in general position invariant under this graph automorphism. Finally, the cuspidal Brauer character $\varphi_{1.1^3}$ occurs as one of two composition factors in the ℓ-modular reduction of a character $R_L^G(\theta)$, with $\mathbf{L}^F = {}^2A_2(q).(q+1)^2$ and $\theta = \theta_1 \boxtimes \theta_2$ with θ_1 the cuspidal unipotent character of ${}^2A_2(q)$, and $\theta_2 \in \operatorname{Irr} Z$ an ℓ-character in general position of the Φ_2-torus $Z = Z(\mathbf{L})^F$. Here, θ_1 is certainly stable under all automorphisms, and for the two dimension torus Z it is immediate that there is a stable ℓ-character in general position. Thus, Corollary 2.4 applies. □

The groups of type D_5 have 20 unipotent characters. All of them lie in the principal ℓ-block for primes ℓ dividing $q+1$.

THEOREM 4.8. *Assume* (T_ℓ). *The decomposition matrix for the principal* ℓ*-block of* $D_5(q)$, $11 \leq \ell | (q+1)$, *is as given in Table 4, where* $d \in \{0, 1\}$.

For reasons of space, in the 2nd column of Table 4 we just give the leading coefficient and q-power of the degree polynomials of the corresponding unipotent characters. The Harish-Chandra series of the cuspidal PIMs of $D_4(q)$ are indicated by the label of the unipotent character corresponding to it by triangularity of the decomposition matrix given in Table 1.

TABLE 4. $D_5(q)$, $11 \le \ell | (q+1)$, principal block

$.5$	1	1																			
1.4	q	1	1																		
$.41$	q^2	.	1	1																	
2.3	q^2	1	1	.	1																
$1^2.3$	$\frac{1}{2}q^3$	1	1	1	1	1															
$.32$	$\frac{1}{2}q^3$	.	.	.	1	.	1														
1.31	$\frac{1}{2}q^3$	1	1	1	.	.	.	1													
$D_4{:}\,2$	$\frac{1}{2}q^3$	.	.	.	.	.	.	.	1												
2.21	q^4	1	1	1	1	1	.	1	.	1											
1.2^2	q^5	.	.	1	.	.	.	1	.	1	1										
$.31^2$	q^6	1	.	.	1	1	1	.	2	.	.	1									
$1^2.21$	q^6	1	1	1	1	1	1	1	$2-d$	1	1	.	1								
$1^3.2$	$\frac{1}{2}q^7$	1	1	.	1	1	1	.	$2-d$	.	.	1	1	1							
$.2^21$	$\frac{1}{2}q^7$	.	.	.	1	1	1	.	2	1	.	1	.	.	1						
1.21^2	$\frac{1}{2}q^7$	1	1	2	.	1	.	1	$2-d$	1	1	.	1	.	.	1					
$D_4{:}\,1^2$	$\frac{1}{2}q^7$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1				
$1^2.1^3$	q^{10}	1	1	1	1	2	1	.	$4-d$	1	1	1	1	1	1	.	2	1			
$.21^3$	q^{12}	.	1	1	.	.	.	.	$2-d$	.	1	.	1	3	.	1	2	.	1		
1.1^4	q^{13}	1	1	1	.	1	.	.	$4-d$	.	1	.	1	3	.	1	2	1	1	1	
$.1^5$	q^{20}	1	.	.	.	1	.	.	4	.	.	1	.	1	1	.	2	1	.	3	1
		ps	ps	D_2	ps	A_1	D_2	ps	D_4	A_1^2	D_2A_1	D_2	A_1	1.1^3	D_2A_1	D_2	D_4	A_1^2	$.1^4$	1.1^3	$.1^4$

PROOF. Let $G = D_5(q)$. Again, the unipotent characters form a basic set for the principal ℓ-block. Note that the ℓ-modular decomposition matrices for unipotent blocks of all proper Levi subgroups are known, except for the one PIM in the decomposition matrix of the principal ℓ-block of $D_4(q)$ whose unipotent part is given by

$$[D_4] + a_1([1^2+] + [1^2-] + [.21^2]) + a_2[1.1^3] + a_3[.1^4]$$

with $a_2 = c + 3a_1 - 2$, $a_3 = 4c + 3a_1$ and $c \in \{-1, 0, 1\}$ (see the proof of Theorem 4.5). Let us again denote by Ψ_i, $1 \le i \le 20$, the linear combinations of unipotent characters given by the columns in Table 4. We shall show that these are the unipotent parts of projective indecomposable characters of G.

The columns Ψ_1, Ψ_2, Ψ_4 and Ψ_7 are obtained from the decomposition matrix of the principal series Hecke algebra $\mathcal{H}(D_5; q)$, which in turn is easily deduced from the one for $\mathcal{H}(B_5; 1; q)$; see the programme of Jacon [**36**]. Harish-Chandra induction of PIMs from proper Levi subgroups gives Ψ_9, Ψ_{12}, Ψ_{17} and the projectives with unipotent parts

$$\tilde{\Psi}_3 = \Psi_3 + 2\Psi_6, \qquad \tilde{\Psi}_5 = \Psi_5 + 2\Psi_{12},$$

$$\tilde{\Psi}_8 = \Psi_8' + a_1\Psi_{12} + (a_2 - a_1)\Psi_{15} + \Psi_{16}',$$

$$\tilde\Psi_{10} = \Psi_{10} + \Psi_{14}, \qquad \tilde\Psi_{11} = \Psi_{11} + 2\Psi_{15},$$
$$\tilde\Psi_{13} = \Psi_{13} + \Psi_{15} + \Psi_{19}, \qquad \tilde\Psi_{18} = \Psi_{18} + \Psi_{20}.$$

The Hecke algebra for the cuspidal ℓ-modular Steinberg character of a Levi subgroup of G of type D_2 (the two end nodes interchanged by the graph automorphism) was determined in Lemma 4.7 to be $\mathcal{H} = \mathcal{H}(B_3; q^2; q)$. It has four PIMs, thus there will be four PIMs of G in the Harish-Chandra series D_2. Now the projectives $\tilde\Psi_3$ and $\tilde\Psi_{11}$ contain summands in that series. The only proper subsums of $\tilde\Psi_3$ satisfying (HCr) are Ψ_3, Ψ_6, $\Psi_3 + \Psi_6$ and $2\Psi_6$. On the other hand, from the decomposition matrix of $\mathcal{H}$ we see that this projective has to have two distinct summands in the D_2-Harish-Chandra series. It follows that Ψ_3 and Ψ_6 are PIMs for G. Exactly the same reasoning applies to the projective $\tilde\Psi_{11}$: it must decompose as $\Psi_{11} + 2\Psi_{15}$, and both Ψ_{11} and Ψ_{15} are PIMs by (HCr).

The Hecke algebra $\mathcal{H}(A_1; q) \otimes \mathcal{H}(A_1; q^2)$ for the cuspidal modular Steinberg character of a Levi subgroup D_2A_1 has two irreducible characters (see Table 3). It follows that the corresponding Harish-Chandra series contains two PIMs. The projective $\tilde\Psi_{10}$ contains summands in that series, and the only splitting satisfying (HCr) is $\Psi_{10} + \Psi_{14}$; this yields that Ψ_{10} and Ψ_{14} are PIMs.

For the Harish-Chandra series above the Steinberg PIM of a Levi subgroup of type A_1, the Hecke algebra $\mathcal{H} = \mathcal{H}(A_1; q) \otimes \mathcal{H}(D_3; q)$ determined in Lemma 4.7 has two PIMs; by (HCr) and the decomposition matrix of $\mathcal{H}$ the only admissible splitting of the projective $\tilde\Psi_5$ in this series is as $\Psi_5 + 2\Psi_{12}$, thus we find the PIM Ψ_5. The Harish-Chandra induction $\tilde\Psi_{18}$ of the Steinberg PIM from D_4 contains the Steinberg PIM of G by (St); this gives Ψ_{18} and Ψ_{20}, both of which are indecomposable by (HCr).

The Hecke algebra for the ordinary unipotent cuspidal character of D_4 is $\mathcal{H}(A_1; q^4)$, so $\tilde\Psi_8$ has at least two projective summands $\tilde\Psi'_8, \tilde\Psi_{16}$, the first containing $[D_4\colon 2]$, the second $[D_4\colon 1^2]$. On the other hand by (Tri) there is a PIM involving $[D_4\colon 1^2]$, $[.2^21]$, and unipotent characters of larger a-value. As $\tilde\Psi_{16}$ must be a summand of this, it only contains unipotent characters with a-value at least 7. The only possible summand of $\tilde\Psi_8$ satisfying (HCr) and containing none of $[1^3.2]$ and $[1.21^2]$ is Ψ'_{16} as listed below, with a parameter $b \geq 0$ satisfying $a_2 \leq b \leq a_3$, and so $\tilde\Psi'_8 = \Psi'_8 + a_1\Psi_{12} + (a_2 - a_1)\Psi_{15}$ is projective:

D_4:2	1	
$.31^2$	a_1	
$1^2.21$	a_1	
$1^3.2$	$a_2{-}a_1$	.
$.2^21$	a_1	.
1.21^2	a_1	.
D_4:1^2	.	1
$1^2.1^3$	a_2	a_1
$.21^3$	$b{-}a_2$	$a_1{+}a_3{-}b$
1.1^4	$b{+}a_1{-}a_2$	$a_2{+}a_3{-}a_1{-}b$
$.1^5$	$a_2{+}a_3{-}b$	$b{-}a_2$
	Ψ'_8	Ψ'_{16}

As the centraliser of a Sylow ℓ-subgroup of G is contained inside a proper parabolic subgroup of type D_4, G cannot have cuspidal Brauer characters by (Csp). Since we already accounted for all Harish-Chandra series except for the one above $\varphi_{1.1^3}$ of D_4, Ψ_{19} must lie in that series, and must hence be a summand of the Harish-Chandra induction $\tilde\Psi_{13}$ of the cuspidal PIM $\Psi_{1.1^3}$ of $D_4(q)$. The only such subsum of $\tilde\Psi_{13}$ satisfying (HCr) is Ψ_{19}, so this gives the PIM Ψ_{19}. Harish-Chandra inducing Ψ_{19} to $D_6(q)$, restricting back again and decomposing shows that $\tilde\Psi_{13} - \Psi_{19} = \Psi_{13} + \Psi_{15}$ must be decomposable, and (HCr) then yields Ψ_{13}.

Modulo the knowledge of a_1, b and c we have now obtained all columns in Table 4 except for Ψ_8. When Harish-Chandra inducing Ψ'_{16} to $D_6(q)$ and restricting back again, the decomposition in terms of the projectives obtained so far has coefficient 1 on $\tilde\Psi'_8$ and negative coefficients $-a_1$ on Ψ_{12} and $a_1 - a_2$ on Ψ_{15}. So $\tilde\Psi'_8$ is not indecomposable, but contains Ψ_{12} at least a_1 times and Ψ_{15} at least $a_2 - a_1$ times. This shows that Ψ'_8 is projective. In addition, (HCr) shows the inequalities $2(a_2 - a_1) \leq b \leq a_3$. Note however that at this stage we cannot show that Ψ'_8 is indecomposable since it could still contain some copies of Ψ_{11}, Ψ_{12} and Ψ_{15}.

Let us consider the Deligne–Lusztig character R_w associated to the Coxeter element $w \in W$. The PIMs Ψ_{18} and Ψ_{20} do not occur in any R_v for $v < w$. The multiplicity of R_w on Ψ_{20} is equal to $4 - 8a_1 + 2b - 8c$ whereas the multiplicity on Ψ_{18} is the opposite. We deduce from (DL) that both quantities are zero and therefore $b = -2 + 4a_1 + 4c$. Now since $b \leq a_3$ and $a_3 = 4c + 3a_1$ we obtain $a_1 \leq 2$. On the other hand, if $\ell > 6$ then by Proposition 4.1 there exists a linear ℓ-character of a Levi subgroup L of G of type $D_2(q).(q-1)(q+1)^2$. Using (Red) with the trivial character of L we get $a_1 \geq 2$, therefore $a_1 = 2$ and $b = a_3 = 4c + 6$ and thus $a_2 = c + 4$. In addition, the relation $a_2 \leq b$ forces $c \geq 0$ hence $c \in \{0, 1\}$.

Finally we use (Red) with the following pairs (L, ρ) to show that Ψ'_8 is almost indecomposable:

- $(A_3(q).(q+1)^2, [4])$ shows that Ψ_{11} cannot be a direct summand of Ψ'_8;
- $({}^2D_4(q).(q+1), [1.2])$ shows that Ψ_{15} cannot be a direct summand of Ψ'_8;
- $(A_1(q)A_3(q).(q+1), [2] \boxtimes [31])$ shows that Ψ_{12} can be a direct summand of Ψ'_8 with multiplicity at most 1.

Therefore $\Psi_8 := \Psi'_8 - d\Psi_{12}$ is indecomposable for some $d \in \{0, 1\}$. □

5. Unipotent decomposition matrix of $D_6(q)$

The groups of type D_6 have 42 unipotent characters. For primes $\ell > 2$ with $\ell | (q+1)$, 37 of them lie in the principal ℓ-block, the other five lie in a block of defect $(q+1)^2_\ell$.

Theorem 4.9. *Assume (T_ℓ). The decomposition matrices for the unipotent ℓ-blocks of $D_6(q)$, for $11 \leq \ell | (q+1)$, are as given in Tables 5–7, where $d \in \{0, 1\}$ is as in Theorem 4.8 and the unknown entries satisfy moreover*

$$c_{20} = 24 - 15\, c_{17} - 5\, c_{18} + 6\, c_{19},$$

$5 \leq 4\, c_{17} + c_{18} - c_{19} \leq 7$, *$c_4, c_{12}, c_{17} \geq 2$, and $c_{18} \geq 4$.*

Proof. For the non-principal unipotent block all columns as given in Table 7 are obtained directly from (HCi) and are indecomposable by (HCr). So we are left to consider the principal block.

TABLE 5. $D_6(q)$, $11 \leq \ell | (q+1)$, principal block

$.6$	1	1																					
1.5	q	2	1																				
$.51$	q^2	1	1	1																			
2.4	q^2	2	2	.	1																		
$3+$	q^3	1	1	.	1	1																	
$3-$	q^3	1	1	.	1	.	1																
$.42$	$\frac{1}{2}q^3$	.	1	1	1	.	.	1															
$1^2.4$	$\frac{1}{2}q^3$	2	2	1	1	.	.	.	1														
$D_4{:}\,2.$	$\frac{1}{2}q^3$	.	.	.	.	.	.	.	.	1													
$.3^2$	$\frac{1}{2}q^4$	.	.	.	1	1	1	1	.	.	1												
2.31	$\frac{1}{2}q^4$	2	3	1	2	1	1	.	1	.	.	1											
$D_4{:}\,1^2.$	$\frac{1}{2}q^4$	.	.	.	.	.	.	.	.	1	.	.	1										
1.32	q^5	.	1	1	1	1	1	1	.	.	1	1	.	1									
$.41^2$	q^6	1	1	1	1	.	.	1	1	2	.	.	.	.	1								
2.2^2	q^6	.	1	1	1	1	1	.	1	.	.	1	.	1	.	1							
$1^2.31$	q^6	2	3	2	2	1	1	1	2	$2{-}d$	.	1	.	1	.	.	1						
$1^3.3$	$\frac{1}{2}q^7$	2	2	1	2	1	1	1	2	$2{-}d$	.	.	.	.	1	.	1	1					
1.31^2	$\frac{1}{2}q^7$	2	2	2	1	1	1	1	2	$4{-}d$	1	1	2	1	1	.	1	.	1				
$D_4{:}\,1.1$	$\frac{1}{2}q^7$	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	1			
$21+$	q^7	1	2	1	2	2	2	1	2	$2{-}d$	1	1	2	1	.	1	1	.	.	.	1		
$21-$	q^7	1	2	1	2	2	2	1	2	$2{-}d$	1	1	2	1	.	1	1	.	.	.	.	1	
$1^2.2^2$	q^8	.	1	1	1	2	2	1	1	$2{-}d$	2	1	4	2	.	1	1	.	.	.	1	1	
2.21^2	q^8	2	3	2	2	2	2	1	3	$4{-}2d$	1	1	4	1	1	1	2	1	1	.	1	1	
1.2^21	q^9	.	1	2	1	2	2	1	2	$4{-}d$	2	1	6	2	1	2	1	.	1	.	1	1	
$.2^3$	$\frac{1}{2}q^{10}$	.	.	.	1	1	1	1	1	2	1	.	2	.	1	1	.	.	.	.	.	.	
$1^2.21^2$	$\frac{1}{2}q^{10}$	2	3	3	2	3	3	2	4	$8{-}3d$	3	1	10	3	1	2	3	1	1	2	2	2	
$D_4{:}\,.2$	$\frac{1}{2}q^{10}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	
$.31^3$	q^{12}	1	1	1	1	1	1	1	1	$4{-}d$	1	.	2	1	1	.	1	3	1	2	.	.	
$.2^21^2$	$\frac{1}{2}q^{13}$	.	1	1	1	1	1	1	1	$4{-}d$	2	.	8	1	1	1	1	3	1	2	1	1	
$1^4.2$	$\frac{1}{2}q^{13}$	2	2	1	1	1	1	1	2	$6{-}2d$	1	.	4	1	1	.	2	4	1	2	1	1	
$D_4{:}\,.1^2$	$\frac{1}{2}q^{13}$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	
1^3+	q^{15}	1	1	1	1	1	1	1	2	$4{-}d$	1	.	6	1	1	1	1	1	.	2	1	1	
1^3	q^{15}	1	1	1	1	1	1	1	2	$4{-}d$	1	.	6	1	1	1	1	1	.	2	1	1	
$1^2.1^4$	q^{16}	2	2	2	1	1	1	1	3	$8{-}2d$	2	.	14	2	1	1	2	4	1	4	2	2	
$.21^4$	q^{20}	1	1	1	.	.	.	.	1	$6{-}d$	1	.	8	1	1	.	1	4	1	4	1	1	
1.1^5	q^{21}	2	1	1	.	.	.	.	2	$8{-}d$	1	.	10	1	1	.	1	4	1	4	1	1	
$.1^6$	q^{30}	1	.	.	.	.	.	.	1	4	.	.	6	.	1	.	.	1	.	2	.	.	
		ps	ps	D_2	ps	A_1^3	$A_1^{3'}$	D_2	A_1	D_4	A_1^4	ps	D_4A_1	D_2A_1	D_2	A_1^2	A_1	1.1^3	D_2	D_4	A_1^3	$A_1^{3'}$	
dec										$*$			$*$										

Here, we write A_1^4 for the HC-series of type $D_2A_1^2$,

Denote by Ψ_i, $1 \leq i \leq 37$, the linear combinations corresponding to the columns of Tables 5 and 6. All Ψ_i apart from the ones with

$$i \in \{3, 9, 10, 25, 27, 31, 34, 36, 37\}$$

TABLE 6. $D_6(q)$, $11 \leq \ell|(q+1)$, cntd.

$1^2.2^2$	q^8	1															
2.21^2	q^8	.	1														
1.2^21	q^9	1	1	1													
$.2^3$	$\frac{1}{2}q^{10}$	.	.	1	1												
$1^2.21^2$	$\frac{1}{2}q^{10}$	1	1	1	.	1											
D_4: $.2$	$\frac{1}{2}q^{10}$	.	.	.	.	.	1										
$.31^3$	q^{12}	.	.	.	.	.	.	1									
$.2^21^2$	$\frac{1}{2}q^{13}$	1	4	1	1	.	2	1	1								
$1^4.2$	$\frac{1}{2}q^{13}$	.	1	.	.	1	.	1	.	1							
D_4: $.1^2$	$\frac{1}{2}q^{13}$	.	.	.	2	.	1	.	.	.	1						
1^3+	q^{15}	1	1	1	c_4	1	c_{12}	.	.	.	.	1					
1^3-	q^{15}	1	1	1	c_4	1	c_{12}	.	.	.	.	.	1				
$1^2.1^4$	q^{16}	2	5	1	$2c_4$	2	$2c_{12}$	1	1	1	c_{17}	1	1	1			
$.21^4$	q^{20}	1	4	1	3	1	2	1	1	3	c_{18}	.	.	.	1		
1.1^5	q^{21}	1	5	1	$2c_4$	2	$2c_{12}$	1	1	4	c_{19}	1	1	4	1	1	
$.1^6$	q^{30}	1	4	1	$3{+}2c_4$	1	$4{+}2c_{12}$	.	1	3	c_{20}	1	1	9	1	6	1
		A_1^4	$A_1 1.1^3$	D_2A_1	c	A_1^2	c	$.1^4$	$A_1.1^4$	1.1^3	c	A_1^3	$A_1^{3\prime}$	c	$.1^4$	c	c

TABLE 7. $D_6(q)$, $11 \leq \ell|(q+1)$, block of defect Φ_2^2

1.41	$\frac{1}{2}q^3\Phi_2^4\Phi_3\Phi_6^2\Phi_{10}$	1				
21.3	$\frac{1}{2}q^4\Phi_2^4\Phi_5\Phi_6^2\Phi_{10}$	1	1			
$.321$	$\frac{1}{2}q^7\Phi_2^4\Phi_6^2\Phi_8\Phi_{10}$	.	1	1		
$1^3.21$	$\frac{1}{2}q^{10}\Phi_2^4\Phi_5\Phi_6^2\Phi_{10}$	1	1	1	1	
1.21^3	$\frac{1}{2}q^{13}\Phi_2^4\Phi_3\Phi_6^2\Phi_{10}$	1	.	.	1	1
		ps	ps	D_2	A_1	D_2

are obtained by (HCi) from Levi subgroups of types D_5, A_5, D_4A_1 and D_2A_3, up to the knowledge of $c \in \{0, 1\}$ which we will determine later in the proof. Moreover we obtain $\Psi_{10} + \Psi_{11}$ and $\Psi_{10} + \Psi_{22}$ from which (Tri) yields Ψ_{10}. We can also determine the Harish-Chandra series of all the Ψ_i obtained so far, and comparing with Table 3 it ensures that all non-cuspidal series have been accounted for except for one PIM in series D_2, which must be a summand of $\Psi_3 + \Psi_{11}$, and one PIM in series D_4 which must be a summand of Ψ_9.

Thus the principal block must contain six cuspidal Brauer characters corresponding to the columns with indices $25, 27, 31, 34, 36, 37$. By our assumption (T_ℓ) there will be projectives for these columns with non-zero entries only on and below the diagonal, for characters lying in smaller families. We denote these unknown entries by $c_1, \ldots, c_{24}$ according to the following table, where we have used that the non-trivial graph automorphism of $D_6(q)$ of order 2 interchanges the unipotent characters $[1^3+]$ and $[1^3-]$ but fixes the six cuspidal

PIMs, which means that their multiplicities in these PIMs must agree.

$.2^3$	1				
$1^2.21^2$	.				
D_4: $.2$	.	1			
$.31^3$	.	.			
$.2^21^2$	c_1	c_9			
$1^4.2$	c_2	c_{10}			
D_4: $.1^2$	c_3	c_{11}	1		
$1^3+, 1^3-$	c_4	c_{12}	.		
$1^2.1^4$	c_5	c_{13}	c_{17}	1	
$.21^4$	c_6	c_{14}	c_{18}	c_{21}	
1.1^5	c_7	c_{15}	c_{19}	c_{22}	1
$.1^6$	c_8	c_{16}	c_{20}	c_{23}	c_{24}
	Ψ_{25}	Ψ_{27}	Ψ_{31}	Ψ_{34}	Ψ_{36}

Note that the missing cuspidal columns can not occur as summands of any of the projectives obtained so far. The value $c_{24} = 6$ follows from Corollary 4.3(b). To derive conditions on the other unknowns we start by looking at the coefficients of the various PIMs on the Deligne–Lusztig character R_w attached to a Coxeter element $w \in W$. The PIMs corresponding to the cuspidal simple modules, as well as the PIMs Ψ_{35} and Ψ_{29} do not occur in R_v for $v < w$, therefore their coefficients must be non-negative by (DL).

The coefficients on Ψ_{29} and Ψ_{31} are $3-c_1-c_9$ and $3-c_3-c_{11}$ respectively. On the other hand, if $\ell > 4$ we can invoke (Red) with Proposition 4.1 for ρ the unipotent character labelled by $.2^2$ (resp. the cuspidal unipotent character) of $D_4(q).(q+1)^2$ to get $c_9 \geq 2$ and $c_1 \geq 1$ (resp. $c_3 \geq 2$ and $c_{11} \geq 1$). This shows that $c_1 = c_{11} = 1$ and $c_3 = c_9 = 2$.

At this point, Harish-Chandra inducing the cuspidal PIM $\Psi_{.2^3}$ to $D_7(q)$ and restricting the result to $D_5(q)A_1(q)$ only decomposes non-negatively when the parameter c left open in the decomposition matrix for $D_4(q)$ satisfies $c = 0$. This furnishes the final step in the proofs of Theorems 4.5 and 4.8.

The coefficient on Ψ_{34} is $c_{10} + 2c_{12} - c_{13} + c_2 + 2c_4 - c_5$. If $\ell > 4$, (Red) applied to the cuspidal unipotent character of a 2-split Levi subgroup ${}^2A_2(q)^2.(q+1)^2$ gives

$$c_{10} + 2c_{12} - c_{13} \leq 0 \quad \text{and} \quad c_2 + 2c_4 - c_5 \leq 0.$$

Therefore they must both be zero.

The coefficient on Ψ_{35} is $5 + 3c_{10} - c_{14} + 3c_2 - c_6$. If $\ell > 8$ then by (Red) for the trivial character of $A_1(q)^2.(q+1)^4$ we get

$$2 + 3c_{10} - c_{14} \leq 0 \quad \text{and} \quad 3 + 3c_2 - c_6 \leq 0.$$

Hence $c_{14} = 2 + 3c_{10}$ and $c_6 = 3 + 3c_2$.

Finally, the coefficient on Ψ_{36} is $4c_{10} + 2c_{12} - c_{15} + 4c_2 + 2c_4 - c_7$, and hence is non-negative. On the other hand, the trivial character of $A_1(q).(q+1)^5$ when $\ell > 10$ gives, by (Red), the relations

$$4c_{10} + 2c_{12} - c_{15} \leq 0 \quad \text{and} \quad 4c_2 + 2c_4 - c_7 \leq 0,$$

and therefore these expressions must both vanish. Using this first set of relations we obtain by Theorem 4.2 the relations

$$c_8 = 3 + 3c_2 + 2c_4 \quad \text{and} \quad c_{16} = 4 + 3c_{10} + 2c_{12}$$

for the multiplicities in the Steinberg character.

We continue by analysing the coefficients of the Deligne–Lusztig character R_w for $w = s_1s_3s_4s_3s_1s_2s_3s_4s_5s_6$. The coefficients on Ψ_{35} and Ψ_{36} are $-4c_{21}$ and $16 + 4c_{21} - 4c_{22}$ respectively. Hence $c_{21} = 0$ by (DL). On the other hand, when $\ell > 10$, (Red) with the trivial character of $A_1(q).(q + 1)^5$ gives $4 - c_{22} \leq 0$, and we deduce that $c_{22} = 4$. Theorem 4.2 gives $c_{23} = 9$.

For the last relation we consider the coefficient of Ψ_{36} in the Deligne–Lusztig character R_w for $w = s_1s_2s_3s_1s_2s_3s_4s_3s_1s_2s_3s_4s_5s_6$. It gives

$$X := -120 + 96c_{18} + 24c_{19} - 24c_{20} \geq 0.$$

On the other hand (Red) applied to the cuspidal unipotent character of ${}^2A_2(q).(q + 1)^4$ gives $X/24 \leq 2$. Theorem 4.2 finally yields $c_{20} = 24 - 15c_{17} - 5c_{18} + 6c_{19}$.

The Harish-Chandra series above $[.2^3]$ in $D_7(q)$ has two summands; decomposing the Harish-Chandra induction shows that we must have $c_2 = 0$. Similarly, decomposing the Harish-Chandra series above $[D_4\colon .2]$ in $D_7(q)$ shows that we must have $c_{10} = 0$.

Now (HCr) proves that all Ψ_i are indecomposable, apart possibly from Ψ_9 and Ψ_{12}: The projective Ψ_9 can only contain $\Psi_{14}, \Psi_{16}, \Psi_{18}$. When $\ell > 4$ one can invoke (Red) for the trivial character of a Levi subgroup $D_4(q).(q + 1)^2$ to exclude the possibility that Ψ_{14} occurs. Moreover, using (Red) with the unipotent character $[2]\boxtimes[2-]$ of $A_1(q)D_4(q).(q+1)$ and the character $[2.2]$ of ${}^2D_5(q).(q+1)$ we can check that Ψ_{16} and Ψ_{18} each can occur at most once.

The projective module of character Ψ_{12} can only contain $\Psi_{20}, \Psi_{21}, \Psi_{22}, \Psi_{26}, \Psi_{27}, \Psi_{32}, \Psi_{33}$. When $\ell > 6$ one can use (Red) with the trivial character of a Levi subgroup $A_1(q)^3.(q + 1)^3$ to check that in fact only Ψ_{22} and Ψ_{26} might occur. In addition, they can occur only once, which follows from (Red) with the unipotent characters $[1^2.]$ of ${}^2A_4(q).(q + 1)^2$ and $[.1^2] \boxtimes [2] \boxtimes [2]$ of ${}^2A_3(q)A_1(q)^2.(q + 1)$.

The lower bounds on the c_i's are obtained using (Red) in the following cases:

- $L^* = {}^2A_3(q).A_1(q).(q + 1)^2$ with $\rho = [.2] \boxtimes [2]$ gives $c_4 \geq 2$;
- $L^* = A_1(q)^3.(q + 1)^3$ with the trivial representation gives $c_{12} \geq 2$; and
- $L^* = A_1(q)^2.(q + 1)^4$ (two non-conjugate) with the trivial representation gives $c_{17} \geq 2$ and $c_{18} \geq 4$.

Note that the previous relations imply $c_{19} \geq 5$. □

REMARK 4.10. In [**16**] we introduced, for any $w \in W$ a virtual character Q_w representing the Alvis–Curtis dual of the intersection cohomology of the Deligne–Lusztig variety corresponding to w. Let $w = s_1s_3s_1s_2s_3s_4s_5s_6$ and $w' = (s_1s_2s_3)^2s_4s_3s_1s_2s_3s_4s_5s_6$. Then

$$\begin{aligned} \langle Q_w; \varphi_{1^3+} \rangle &= 13 - 5c_{12} - c_4, \\ \langle Q_{w'}; \varphi_{1^2.1^4} \rangle &= 48 - 24c_{17}, \\ \langle Q_{w'}; \varphi_{.21^4} \rangle &= 96 - 24c_{18}. \end{aligned}$$

If the conjecture in [**16**, Conj. 1.2] holds then these multiplicities must be non-negative. With the lower bounds on the c_i's given in Theorem 4.9 this would imply $c_{12} = c_{17} = 2$, $c_4 \in \{2, 3\}$, $c_{18} = 4$ and $c_{19} \in \{5, \dots, 7\}$.

6. Unipotent decomposition matrix of $E_6(q)$

For the groups $E_6(q)$ we again first determine some Hecke algebras:

LEMMA 4.11. *Let q be a prime power and ℓ odd such that $(q+1)_\ell > 5$. The Hecke algebras of various ℓ-modular cuspidal pairs (L, λ) of Levi subgroups L in $E_6(q)$ and their respective numbers of simple modules are as given in Table 8.*

TABLE 8. Hecke algebras in $E_6(q)$ for $d_\ell(q) = 2$

(L, λ)	$\mathcal{H}$	$\lvert\mathrm{Irr}\,\mathcal{H}\rvert$
(A_1, φ_{1^2})	$\mathcal{H}(A_5; q)$	$3+1$
$(A_1^2, \varphi_{1^2}^{\boxtimes 2})$	$\mathcal{H}(B_3; q^2; q)$	4
$(A_1^3, \varphi_{1^2}^{\boxtimes 3})$	$\mathcal{H}(A_1; q) \otimes \mathcal{H}(A_2; q^2)$	3
(D_4, D_4)	$\mathcal{H}(A_2; q^4)$	3
$(D_4, \varphi_{1.1^3})$	$\mathcal{H}(A_2; q^2)$	3
$(D_4, \varphi_{.1^4})$	$\mathcal{H}(A_2; q^2)$	3

PROOF. The arguments are very similar to the ones used in the proof of Lemma 4.7. In fact, most of the parameters of the relevant Hecke algebras were already determined there. As an example let us consider the ℓ-modular Steinberg character of a Levi subgroup of type A_1. By [**35**, p. 75] its relative Weyl group in $E_6(q)$ is of type A_5, and as the character is liftable, its parameters are determined locally, inside a Levi subgroup of type A_1^2, to be equal to q. For the (liftable) modular Steinberg character $.1^4$ of a Levi subgroup of type D_4 the relative Weyl group has type A_2, and the parameter was again already in Lemma 4.7 shown to be equal to q^2.

Reduction stability for the first four cases has already been argued in the proof of that lemma. The normaliser of a Levi subgroup of type D_4 inside E_6 induces the full group $\mathfrak{S}_3$ of graph automorphisms. Thus, by our description of $\varphi_{.1^4}$ in Lemma 4.7 we need to see that there exist ℓ-characters in general position for a Sylow Φ_2-torus which are stable under the graph automorphisms. For this we use that the extension of D_4 by $\mathfrak{S}_3$ is realised inside the groups of type F_4 (see e.g. [**42**, Exmp. 13.9]): the long root subgroups generate a subgroup of type D_4, and it is normalised by the Weyl group of type A_2 generated by two reflections at short roots. By [**38**, Tab. T.A.133] there exist semisimple ℓ-elements with centralisers a short root A_2 in $F_4(q)$ whenever $q > 5$ and thus there is an ℓ-character θ of a Sylow Φ_2-torus T of $G = D_4(q)$ such that $R_T^G(\theta)$ lifts the modular Steinberg character $\varphi_{.1^4}$. The argument for $\varphi_{1.1^3}$ is similar. $\square$

The groups of type E_6 have 30 unipotent characters. For primes $\ell > 3$ with $\ell | (q+1)$, 25 of them lie in the principal ℓ-block and three are of defect zero. The remaining two lie in a unipotent block of cyclic defect, with Brauer tree

$$\phi_{64,4} \underset{ps}{\text{---}} \phi_{64,13} \underset{A_1}{\text{---}} \bigcirc$$

THEOREM 4.12. *Assume* (T_ℓ). *The decomposition matrix for the principal* ℓ*-block of* $E_6(q)$, $11 \leq \ell|(q+1)$, *is as given in Table 9 where* $d \in \{0, 1\}$ *is as in Theorem 4.8. Here, the projectives in columns 7 and 11 might be decomposable.*

TABLE 9. $E_6(q)$, $11 \leq \ell|(q+1)$, principal block

$\phi_{1,0}$	1	1																								
$\phi_{6,1}$	q	.	1																							
$\phi_{20,2}$	q^2	.	1	1																						
$\phi_{30,3}$	$\frac{1}{2}q^3$	.	1	1	1																					
$\phi_{15,5}$	$\frac{1}{2}q^3$	1	.	1	.	1																				
$\phi_{15,4}$	$\frac{1}{2}q^3$	1	.	.	.	.	1																			
D_4: 3	$\frac{1}{2}q^3$	.	.	.	.	.	.	1																		
$\phi_{60,5}$	q^5	.	.	1	.	.	.	.	1																	
$\phi_{24,6}$	q^6	.	.	1	1	1	.	.	.	1																
$\phi_{81,6}$	q^6	1	1	1	.	1	1	$4{-}d$	1	.	1															
$\phi_{20,10}$	$\frac{1}{6}q^7$	.	1	.	.	.	1	$2{-}d$	.	.	1	1														
$\phi_{90,8}$	$\frac{1}{3}q^7$	.	1	2	1	1	.	$6{-}3d$	1	1	1	.	1													
$\phi_{10,9}$	$\frac{1}{3}q^7$	.	.	.	1	.	.	.	.	1	.	.	.	1												
$\phi_{60,8}$	$\frac{1}{2}q^7$	.	.	.	.	.	1	$4{-}d$	1	.	1	.	.	.	1											
D_4: 21	$\frac{1}{2}q^7$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1										
$\phi_{81,10}$	q^{10}	1	1	1	.	1	1	$10{-}4d$	1	.	2	1	1	.	1	2	1									
$\phi_{60,11}$	q^{11}	.	.	1	.	1	.	$8{-}3d$	1	1	1	.	1	.	1	2	1	1								
$\phi_{24,12}$	q^{12}	.	.	1	1	.	.	$2{-}2d$	.	1	.	2	1	1	.	2	.	.	1							
$\phi_{30,15}$	$\frac{1}{2}q^{15}$	.	1	1	1	1	.	$6{-}3d$	.	2	1	3	1	1	.	2	.	.	1	1						
$\phi_{15,17}$	$\frac{1}{2}q^{15}$	1	.	1	.	1	.	$4{-}2d$	.	.	.	2	1	.	.	2	1	.	1	.	1					
$\phi_{15,16}$	$\frac{1}{2}q^{15}$	1	.	.	.	1	1	$4{-}d$	.	.	1	1	.	.	1	2	1	.	.	.	.	1				
D_4: 1^3	$\frac{1}{2}q^{15}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1			
$\phi_{20,20}$	q^{20}	.	1	1	.	1	.	$8{-}3d$	.	1	1	4	1	.	.	4	1	1	1	1	3	.	2	1		
$\phi_{6,25}$	q^{25}	.	1	.	.	.	.	$4{-}d$	.	.	1	2	.	.	.	2	.	.	.	1	2	.	.	1	1	
$\phi_{1,36}$	q^{36}	1	.	.	.	1	.	2	.	.	.	.	.	.	.	2	1	.	.	.	1	1	.	.	2	1
		ps	ps	ps	ps	A_1	ps	D_4	ps	A_1^2	A_1	1.1^3	A_1	A_1^3	A_1^2	D_4	A_1^2	A_1^3	$.1^4$	A_1^2	1.1^3	A_1^3	D_4	$.1^4$	1.1^3	$.1^4$
dec?								$*$				$*$														

PROOF. Let $G = E_6(q)$. We denote by Ψ_i, $1 \leq i \leq 25$, the linear combinations of unipotent characters given by the columns in Table 9. We shall show that these are the unipotent parts of projective indecomposable characters of G.

The columns Ψ_i with $i \in \{1, 2, 3, 4, 6, 8\}$ are obtained from the decomposition matrix of the Hecke algebra of type E_6 which has been determined by Geck [**23**, Table D].

By Table 8 the A_1-Harish-Chandra series contains three Brauer characters in the principal block. Harish-Chandra induction of the two PIMs of $D_5(q)$ in that series yields $\tilde{\Psi}_5 = \Psi_5 + \Psi_8 + 2\Psi_{10} + \Psi_{12}$ and $\tilde{\Psi}_{10} = \Psi_{10} + 2\Psi_{12}$. From the decomposition matrix of the Hecke algebra $\mathcal{H}(A_5; q)$ it follows that $\tilde{\Psi}_{10}$ must contain one PIM in that series plus two copies of another one. But the only splitting of $\tilde{\Psi}_{10}$ compatible with (HCr) is into

Ψ_{10} plus two times Ψ_{12}. So these two lie in the A_1-series. Then, $\tilde{\Psi}_5$ has to contain two copies of Ψ_{10} and one copy of Ψ_{12}, so that $\tilde{\Psi}_5' = \Psi_5 + \Psi_8$ is projective, and it contains one indecomposable summand from the A_1-series. Inducing the first A_1-PIM from $A_5(q)$ we see that the same holds for $\tilde{\Psi}_5'' = \Psi_4 + \Psi_5$. Hence neither of $\tilde{\Psi}_5', \tilde{\Psi}_5''$ is indecomposable, and all of their summands apart from the one in the A_1-series lie in the principal series. This yields Ψ_5.

The A_1^2-series contains four Brauer characters by Table 8. Harish-Chandra induction of the two PIMs of $A_5(q)$ in that series yields $\tilde{\Psi}_9 = \Psi_9 + 2\Psi_{14}$ and $\tilde{\Psi}_{16} = \Psi_{16} + 2\Psi_{19}$. Thus, both $\tilde{\Psi}_9$ and $\tilde{\Psi}_{16}$ contain two summands from that series, one with multiplicity 2. The only splitting consistent with (HCr) is just given by Ψ_9 plus $2\Psi_{14}$, respectively Ψ_{16} plus $2\Psi_{19}$.

The A_1^3-series contains three Brauer characters. From (HCi) and using (Tri) we obtain $\Psi_{13}+\Psi_{14}$, $\Psi_{13}+\Psi_{14}+\Psi_{17}$ (which together yield Ψ_{17}), $\Psi_{14}+\Psi_{17}+\Psi_{19}$ and Ψ_{21}. By (Tri) we have that $\Psi_{13}+\Psi_{14}$ cannot be indecomposable, and the only admissible splitting with (HCr) is into Ψ_{13} plus Ψ_{14}.

The Harish-Chandra series above the three cuspidal unipotent Brauer characters of $D_4(q)$ all contain three Brauer characters by Table 8. For 1.1^3, (HCi) gives $\Psi_{11}+\Psi_{12}+\Psi_{20}$ and $\Psi_{20}+\Psi_{24}$, which both must contain two PIMs from this series. Now, by (Tri), $\Psi_{11}+\Psi_{12}$ cannot be indecomposable, and the only possible splitting with (HCr) leads to Ψ_{11}, Ψ_{20} and Ψ_{24}. The situation for $.1^4$ is entirely similar, giving Ψ_{18}, Ψ_{23} and Ψ_{25}, all three of which are indecomposable by (HCr). We also find $\Psi_7 + \Psi_{15} + 2\Psi_{19}$ and $\Psi_{15} + 2\Psi_{19} + \Psi_{22}$ which split up at least into $\Psi_7, \Psi_{15} + 2\Psi_{19}$ and Ψ_{22}.

Harish-Chandra inducing Ψ_{22} to $E_7(q)$, cutting by the principal block, and restricting back shows that Ψ_{19} is twice contained in $\Psi_{15} + 2\Psi_{19}$.

We have thus accounted for all columns in the table. An application of (HCr) shows that all of them are indecomposable, except possibly for Ψ_7 and Ψ_{11}: Ψ_{11} might contain Ψ_{19} once, and Ψ_7 might contains copies of Ψ_{10} and of Ψ_{12}. □

7. Unipotent decomposition matrix of ${}^2D_4(q)$

We now turn to the groups of twisted type, where again we start with the orthogonal groups. Note that ${}^2D_3(q) \cong {}^2A_3(q)$ is a unitary group, whose unipotent decomposition matrix we already determined in [**17**].

The groups of type ${}^2D_4$ have 10 unipotent characters, all of which lie in the principal ℓ-block for primes ℓ dividing $q+1$. For the following result, we need no restriction on q:

THEOREM 4.13. *The decomposition matrix for the principal ℓ-block of ${}^2D_4(q)$, with $2 < \ell|(q+1)$, is as given in Table 10.*

Here $\alpha + 1$ is the multiplicity of the cuspidal Brauer character $\varphi_{.2}$ in the ℓ-modular reduction of the Steinberg character $[.1^2]$ of ${}^2D_3(q)$; in particular $\alpha = 2$ if $(q+1)_\ell > 3$ by [**17**, Tab. 1], and $\alpha \leq 2$ always.

PROOF. We employ similar arguments as in the untwisted case. All unipotent characters of $G = {}^2D_4(q)$ lie in the principal ℓ-block, and they form a basic set. Let $\Psi_1, \ldots, \Psi_{10}$ denote the linear combinations of unipotent characters corresponding to the ten columns of Table 10. We construct projective characters as follows: the decomposition matrix of

TABLE 10. ${}^2D_4(q)$, $2 < \ell | (q + 1)$

3.	1	1									
21.	$q\Phi_8$	.	1								
2.1	$q^2\Phi_3\Phi_6$	.	1	1							
1^3.	$\frac{1}{2}q^3\Phi_6\Phi_8$	1	.	.	1						
.3	$\frac{1}{2}q^3\Phi_6\Phi_8$	.	.	1	.	1					
1^2.1	$\frac{1}{2}q^3\Phi_3\Phi_8$	.	1	1	.	.	1				
1.2	$\frac{1}{2}q^3\Phi_3\Phi_8$	1	.	.	.	.	.	1			
1.1^2	$q^6\Phi_3\Phi_6$	1	.	.	1	α	.	1	1		
.21	$q^7\Phi_8$	.	.	.	.	α	.	1	1	1	
.1^3	q^{12}	.	.	1	.	1	1	.	.	α	1
		ps	ps	ps	A_1	.2	A_1	ps	.1^2	.2	.1^2

the Hecke algebra $\mathcal{H}(B_3; q^2; q)$ for the principal series gives the four PIMs $\Psi_1, \Psi_2, \Psi_3, \Psi_7$ labelled by "ps" in Table 10. Next Harish-Chandra induction of the A_1-series PIM of a Levi subgroup of type A_2 gives a projective character with unipotent part $\tilde{\Psi}_4 = \Psi_4 + \Psi_6$, but the induction of neither of the A_1-series PIMs from a Levi subgroup of type ${}^2D_2A_1$ contains this character, so $\tilde{\Psi}_4$ must be decomposable. The only subsums compatible with (HCr) are Ψ_4 and Ψ_6, so these are the two PIMs of G in the A_1-series.

Note that the centraliser of a Sylow Φ_2-torus of G lies inside a Levi subgroup of type ${}^2D_3$, so by (Csp), all Brauer characters of G have a Harish-Chandra vertex contained in that Levi subgroup. Furthermore, by [**27**, Thm. 4.2], the Harish-Chandra vertex of the modular Steinberg character of G is the (cuspidal) modular Steinberg character $\varphi_{.1^2}$ of ${}^2D_3(q)$. But the Harish-Chandra induction of this PIM has unipotent part $\tilde{\Psi}_8 = \Psi_8 + \Psi_{10}$, so $\tilde{\Psi}_8$ is decomposable and yields the two PIMs Ψ_8 and Ψ_{10}. We have now accounted for all Harish-Chandra series except for that of the cuspidal character $\varphi_{.2}$ of ${}^2D_3(q)$. Hence the two remaining PIMs must lie in that series.

The column Ψ_9 (with a yet undetermined entry $a \geq 0$ in the last row) comes from the tensor product of the unipotent character [21.] with an irreducible Deligne–Lusztig character for a Coxeter torus (which is projective). This can be computed using the table of unipotent characters for ${}^2D_4(q)$ available in Chevie [**26**]. Harish-Chandra induction of $\Psi_{.2}$ from ${}^2D_3(q)$ gives a projective character with unipotent part $\tilde{\Psi}_5 = \Psi_5 + \Psi_7 + \Psi_9$. As the remaining two PIMs must lie in the Harish-Chandra series above $\Psi_{.2}$, $\tilde{\Psi}_5$ has at least two summands, and it has three if Ψ_7 is a summand of $\tilde{\Psi}_5$. We thus obtain the following lower right-hand corner of the decomposition matrix, for some $b \in \{0, 1\}$ (with $b = 0$ if

$\tilde{\Psi}_5$ has two summands, and $b = 1$ if it has three).

$.3$	1					
$1^2.1$	.	1				
1.2	$1-b$	.	1			
1.1^2	$u-b$	.	1	1		
$.21$	$u-b$	.	1	1	1	
$.1^3$	$u-v$	1	.	.	v	1
	$.2$	A_1	ps	$.1^2$	$.2$	$.1^2$

Note that for $(q+1)_\ell = 3$, the result in [**17**, Tab. 1] only gives an upper bound 3 for the multiplicity of the Brauer character $\varphi_{.2}$ in the 3-modular reduction of the Steinberg character. We write u for this multiplicity (which equals 3 unless $(q+1)_\ell = 3$), and we have $0 \le v \le u$.

The Harish-Chandra restriction to $^2D_3(q)$ of the two projectives in the $\varphi_{.2}$-series decomposes as $(1-2b+u)\Psi_{1.1} + \Psi_{1^2.} + (u-v-1)\Psi_{.1^2}$, respectively $\Psi_{1^2.} + (1+v-u)\Psi_{.1^2}$ (see [**17**, Table 1]). This shows that $v = u-1$. Finally, Harish-Chandra induction of Ψ_5 to $^2D_5(q)$ and restriction back to $^2D_4(q)$ should decompose non-negatively into PIMs. This forces $b = 1$. Our claim follows by setting $\alpha := v$. $\square$

Remark 4.14. From the known 3-modular decomposition matrices it can be seen that $\alpha = 1$ for $\mathrm{SO}_6^-(2) \cong \mathrm{U}_4(2)$ and $\mathrm{SO}_8^-(2)$, so the case $(q+1)_\ell = 3$ does indeed behave differently from the generic one where $\alpha = 2$.

8. Unipotent decomposition matrix of $^2D_5(q)$

We now turn to the groups $^2D_5(q)$, where we first need to determine the structure of certain Hecke algebras.

Lemma 4.15. *Let q be a prime power and $\ell \ne 2$ with $(q+1)_\ell \ge 7$. The Hecke algebras of various ℓ-modular cuspidal pairs (L, λ) of Levi subgroups L in $^2D_n(q)$ and their respective numbers of irreducible characters are as given in Table 11.*

Table 11. Hecke algebras and $|\mathrm{Irr}\,\mathcal{H}|$ in $^2D_n(q)$ for $d_\ell(q) = 2$

(L, λ)	$\mathcal{H}$	$n=4$	5	6	7
(A_1, φ_{1^2})	$\mathcal{H}(A_1; q) \otimes \mathcal{H}(B_{n-3}; q^2; q)$	2	2	4	6
$(A_1^2, \varphi_{1^2}^{\boxtimes 2})$	$\mathcal{H}(B_2; q^2; q) \otimes \mathcal{H}(B_{n-5}; q^2; q)$	–	2	4	4
$(^2D_3, \varphi_{.2})$	$\mathcal{H}(B_{n-3}; q^2; q)$	2	2	4	6
$(^2D_3, \varphi_{.1^2})$	$\mathcal{H}(B_{n-3}; q^2; q)$	2	2	4	6
$(^2D_3A_1, \varphi_{.2} \boxtimes \varphi_{1^2})$	$\mathcal{H}(A_1; q^3) \otimes \mathcal{H}(B_{n-5}; q^2; q)$	–	1	2	2
$(^2D_3A_1, \varphi_{.1^2} \boxtimes \varphi_{1^2})$	$\mathcal{H}(A_1; q) \otimes \mathcal{H}(B_{n-5}; q^2; q)$	–	1	2	2

Proof. Recall that $^2D_n(q)$ has Weyl group of type B_{n-1}. The relative Weyl group of a Levi subgroup of type A_1 inside $^2D_n(q)$ is of type A_1B_{n-3}, see [**35**, p. 70]. Since the modular Steinberg character φ_{1^2} of $A_1(q)$ is liftable, we may determine the parameters locally, inside minimal Levi overgroups of types $^2D_3$, $^2D_2A_1$ and A_1^2. The relative Weyl group for a Levi of type A_1^2 is of type ${}^2D_3B_{n-5}$, and the minimal Levi overgroups have types A_3, ${}^2D_3A_1$ ${}^2D_2A_1^2$ and A_1^3. For the modular Steinberg character of a Levi subgroup ${}^2D_3(q)$ the relative Weyl group has type B_{n-3} and the minimal Levi overgroups are of types ${}^2D_4$ and ${}^2D_3A_1$, with corresponding parameters q^2 and q. Finally, the minimal Levi overgroups for a Levi subgroup of type ${}^2D_3A_1$ have types ${}^2D_5$, ${}^2D_4A_1$ and ${}^2D_3A_1^2$, with parameters q^2 and q in the latter two. The parameter for the containment in ${}^2D_5(q)$ will be determined in the proof of Theorem 4.16.

The cuspidal Brauer character $\varphi_{.2}$ of $G = {}^2D_3(q)$ lift to an ordinary character $R_L^G(\theta)$ for $\theta \in \operatorname{Irr} L$ by [**17**, Prop. 5.4], where $L = {}^2A_2(q).(q+1)$ and θ is the product of the cuspidal unipotent character θ_1 of ${}^2A_2(q)$ with an ℓ-character θ_2 of $Z(L)$ in general position. Now $R_L^G(\theta_1 \boxtimes \theta_2) = R_L^G(\theta_1 \boxtimes \theta_2^{-1})$, so $R_L^G(\theta)$ is invariant under the graph automorphism of ${}^2D_3(q)$ and hence reduction stable. Similarly, the cuspidal modular Steinberg character $\varphi_{.1^2}$ lifts to $R_T^G(\theta)$ for θ an ℓ-character in general position of a Sylow Φ_2-torus T of G. Here the normaliser of G in a larger type D-group induces the Weyl group W of type B_3 on T, and direct calculation then shows that there is an ℓ-character of T in general position stabilised by a short root reflection of W when $(q+1)_\ell \geq 7$. □

The groups of type ${}^2D_5$ have 20 unipotent characters. For primes $\ell > 2$ with $\ell | (q+1)$, 18 of them lie in the principal ℓ-block, and there is a further unipotent ℓ-block of cyclic defect, with Brauer tree

$$21.1 \underset{ps}{\text{---}} \bigcirc \underset{ps}{\text{---}} 1.21$$

THEOREM 4.16. *Assume (T_ℓ). The decomposition matrix for the principal ℓ-block of ${}^2D_5(q)$, $11 \leq \ell | (q+1)$, is as given in Table 12, where $b \geq 2$ and $d \in \{0,1\}$.*

PROOF. The group $G = {}^2D_5(q)$ has 18 unipotent characters in its principal ℓ-block. Since $\ell > 2$ these unipotent characters form a basic set. As in the previous proofs let $\Psi_1, \ldots, \Psi_{18}$ denote projective characters with unipotent part as in the columns of the matrix in Table 12. All projectives Ψ_i except for $i \in \{12, 15, 17, 18\}$ are found by (HCi).

Since the Sylow ℓ-subgroups of G are not contained in any proper Levi subgroup of G, the PIM Ψ_{18} is cuspidal by (St). Application of (HCr) now shows that all projectives obtained so far are in fact indecomposable, except possibly for Ψ_5. (For Ψ_4 and Ψ_8, there are two possible splittings consistent with (HCr), but in both cases, neither summand occurs among the other PIMs, so that a splitting would lead to a non-independent set of PIMs.) This implies in particular that the series above the two cuspidal unipotent Brauer characters of ${}^2D_3(q)A_1(q)$ both just contain one Brauer character and hence that the parameter of the Hecke algebra has to be $-1 \in k$, as claimed in Lemma 4.15.

As $\ell > 10$, by Theorem 4.2 there exists a non-unipotent cuspidal character ρ with $\rho^\circ = R_{w_0}^\circ = \varphi_{1^4.}$. Assume $\varphi \in \{\varphi_{1^4.}, \varphi_{.2^2}, \varphi_{.21^2}\}$ is not cuspidal. Then the corresponding projective cover can be obtained from the other columns of the decomposition matrix. The condition $\langle P_\varphi; \rho\rangle = 0$ would then force $\varphi = \varphi_{.2^2}$ and the unipotent part of Ψ_{15} to be either $\rho_{.2^2} + \rho_{1.1^3} + 3\rho_{.21^2} + \rho_{1^4.}$ or $\rho_{.2^2} + 2\rho_{1.1^3} + 4\rho_{.21^2} + 2\rho_{1^4.}$. This contradicts the fact that the entry at $[1.1^3]$ in Ψ_{15} vanishes by (Tri).

Now, denote by $b_1, \ldots, b_7$ the yet unknown decomposition numbers below the diagonal in the columns corresponding to $\varphi_{1^4.}, \varphi_{.2^2}$ and $\varphi_{.21^2}$ (recall that the first entry below the

TABLE 12. ${}^2D_5(q)$, $11 \leq \ell|(q+1)$, principal block

$4.$	1	1																	
$31.$	$q\Phi_3\Phi_{10}$	1	1																
3.1	$q^2\Phi_4\Phi_8$	1	1	1															
$2^2.$	$q^2\Phi_8\Phi_{10}$	.	1	.	1														
$.4$	$\frac{1}{2}q^3\Phi_6\Phi_8\Phi_{10}$	.	.	1	.	1													
$21^2.$	$\frac{1}{2}q^3\Phi_3\Phi_8\Phi_{10}$	1	1	.	1	.	1												
2.2	$\frac{1}{2}q^3\Phi_3\Phi_4^2\Phi_{10}$	1	1	1	1	.	.	1											
1.3	$q^4\Phi_4\Phi_8\Phi_{10}$	1	.	1	.	1	.	1	1										
$1^2.2$	$q^5\Phi_3\Phi_8\Phi_{10}$	1	1	1	1	.	.	1	1	1									
2.1^2	$q^6\Phi_3\Phi_6\Phi_8$	1	1	1	1	2	1	1	.	.	1								
$1^3.1$	$q^6\Phi_4\Phi_8\Phi_{10}$	1	1	1	1	.	1	.	.	1	.	1							
$1^4.$	$\frac{1}{2}q^7\Phi_6\Phi_8\Phi_{10}$	1	.	.	.	.	1	.	.	.	.	1	1						
$.31$	$\frac{1}{2}q^7\Phi_3\Phi_8\Phi_{10}$	.	.	1	.	3	.	1	1	.	1	.	.	1					
$1^2.1^2$	$\frac{1}{2}q^7\Phi_3\Phi_4^2\Phi_{10}$	1	1	1	2	2	1	1	3	1	1	1	.	.	1				
$.2^2$	$q^{10}\Phi_8\Phi_{10}$	.	.	.	1	2	.	1	3	.	1	.	b	1	1	1			
1.1^3	$q^{12}\Phi_4\Phi_8$	1	.	1	1	3	1	1	3	1	1	1	3	2	1	.	1		
$.21^2$	$q^{13}\Phi_3\Phi_{10}$	.	.	1	1	3	.	1	4	1	1	.	$3b-d$	3	1	3	1	1	
$.1^4$	q^{20}	.	.	1	.	1	.	.	3	1	.	1	$3+5b-5d$	2	1	5	1	5	1
		ps	ps	ps	A_1^2	$.2$	A_1	ps	$.2{\cdot}A_1$	A_1	$.1^2$	A_1^2	c	$.2$	$.1^2{\cdot}A_1$	c	$.1^2$	c	c

diagonal in Ψ_{12} and in Ψ_{15} vanishes). Thus, we have

$$\begin{aligned} \Psi_{12} &= [1^4.] + b_1[.2^2] + b_2[1.1^3] + b_3[.21^2] + b_4[.1^4], \\ \Psi_{15} &= [.2^2] + b_5[.21^2] + b_6[.1^4], \\ \text{and} \qquad \Psi_{17} &= [.21^2] + b_7[.1^4]. \end{aligned}$$

From Theorem 4.2 we obtain the relations

$$-15 + 10b_1 + 4b_2 - 5b_3 + b_4 = 0, \quad 10 - 5b_5 + b_6 = 0, \quad b_7 = 5.$$

To obtain further relations we decompose suitable Deligne–Lusztig characters R_w in terms of projective characters and then apply (DL). We start with a Coxeter element $w = s_2s_3s_4s_5$, whose coefficient on Ψ_{17} is $3 - b_5$, forcing $b_5 \geq 3$. On the other hand, if $\ell > 8$ then by Proposition 4.1 there exists a linear ℓ-character in general position of a 2-split Levi subgroup L of type $A_1(q).(q+1)^4$. Then (Red) with ρ being the Steinberg character of L shows that $b_5 \geq 3$, which yields $b_5 = 3$ and $b_6 = 5$ using the previous equations. Note that as a consequence neither Ψ_{12}, Ψ_{17} nor Ψ_{18} occurs in R_w.

With $w = s_1s_2s_3s_1s_2s_3s_4s_5$, the coefficient of Ψ_{17} on R_w equals $-3 + 3b_1 + b_2 - b_3$ and must be non-negative by (DL). On the other hand, if $\ell > 6$ one can use (Red) with a Levi subgroup of type ${}^2A_2(q).(q+1)^3$ and its cuspidal unipotent character to find that $5 - 3b_1 - b_2 + b_3 \geq 0$. This shows that there exists $d \in \{0, 1, 2\}$ such that $b_3 = -3 + 3b_1 + b_2 - d$.

At this stage we can only find lower bounds on the remaining unknowns b_1, b_2, d. We have already seen that $d \in \{0, 1, 2\}$. Using (Red) for the two non-conjugate pairs of the form $(L, \rho) = (A_1(q)^2.(q+1)^3, [2]^{\boxtimes 2})$ gives $b_1 \geq 2$ and $b_2 \geq 3$ whenever $\ell > 6$. The other relations will be obtained in the proof of Theorem 4.18. In the table we have set $b := b_1$. □

Remark 4.17. As in Remark 4.10 consider the virtual character Q_w introduced in [**16**], for $w = (s_1 s_2 s_3)^2 s_4 s_5 s_4 s_3 s_1 s_2 s_3$. Then in the principal block B_0 of ${}^2D_5(q)$ we have

$$B_0 Q_w = 48([1^4.] + [.31] + 3[.2^2] + 5[1.1^3] + 9[.21^2] + 15[.1^4]).$$

If the conjecture in [**16**, Conj. 1.2] holds, this should have $\Psi_{1^4.}$ as a direct summand, which would prove that $\Psi_{1^4.}$ is a direct summand of $[1^4.] + 2[.2^2] + 3[1.1^3] + 6[.21^2] + 13[.1^4]$. This would force in particular $b \leq 2$ hence $b = 2$. We deduce that assuming the conjecture, there are the following two possibilities left for the character of $\Psi_{1^4.}$:

$1^4.$	1	1
$.2^2$	2	2
1.1^3	3	3
$.21^2$	6	5
$.1^4$	13	8

9. Unipotent decomposition matrix of ${}^2D_6(q)$

The groups of type ${}^2D_6$ have 36 unipotent characters, all of which lie in the principal ℓ-block for primes ℓ dividing $q+1$.

Theorem 4.18. *Assume (T_ℓ). The decomposition matrix for the principal ℓ-block of ${}^2D_6(q)$, $11 \leq \ell | (q+1)$, is as given in Table 13, where $b \geq 2$ and $d \in \{0, 1\}$ are as in Theorem 4.16. All columns but possibly Ψ_5 are indecomposable.*

Proof. Since $\ell > 2$ the unipotent characters form a basic set for the unipotent blocks. As before, we denote by Ψ_i, $1 \leq i \leq 36$, (virtual) projective characters of G whose unipotent parts decompose as given in the respective columns of Table 13, and we propose to show that these are the unipotent PIMs of G.

Note that the decomposition matrices of the unipotent blocks of all proper Levi subgroups are known, up to the undetermined entries in the PIM $\Psi_{1^4.}$ of ${}^2D_5(q)$. The Ψ_i, $i \in \{1, 2, 3, 4, 6, 7, 10, 18\}$ are the PIMs in the principal series, so are obtained from the decomposition matrix of the Hecke algebra $\mathcal{H}(B_5; q^2; q)$ (which can be computed with [**36**]). Now consider the Harish-Chandra series above the cuspidal ℓ-modular Steinberg character φ_{1^2} of a Levi subgroup of type A_1. Harish-Chandra inducing the two PIMs in that series from a Levi subgroup of type ${}^2D_5$ gives $\Psi_8 + 2\Psi_{16}$ and $\Psi_{13} + 2\Psi_{22}$. The decomposition matrix of the corresponding Hecke algebra $\mathcal{H}(A_1; q) \otimes \mathcal{H}(B_3; q^2; q)$ (see Lemma 4.15) shows that each of these must have two summands in the A_1-series, one with multiplicity 2. The only possible splitting compatible with (HCr) gives the four PIMs Ψ_8, Ψ_{13}, Ψ_{16} and Ψ_{22}.

Next consider the Harish-Chandra series of the cuspidal ℓ-modular Steinberg character $\varphi_{1^2}^{\boxtimes 2}$ of a Levi subgroup of type A_1^2. HC-inducing the two PIMs in that series from a Levi

TABLE 13. ${}^2D_6(q)$, $11 \le \ell | (q+1)$

5.	1	1																			
41.	q	.	1																		
4.1	q^2	.	1	1																	
32.	q^2	.	.	.	1																
.5	$\frac{1}{2}q^3$	.	.	1	.	1															
3.2	$\frac{1}{2}q^3$	1	.	.	1	.	1														
31.1	$\frac{1}{2}q^3$	.	1	1	.	.	.	1													
$31^2.$	$\frac{1}{2}q^3$	1	.	.	1	.	.	.	1												
1.4	$\frac{1}{2}q^4$	1	.	.	.	.	1	.	.	1											
2.3	$\frac{1}{2}q^4$	.	1	1	.	1	.	.	.	.	1										
$2^2.1$	$\frac{1}{2}q^4$	.	.	.	.	.	.	1	.	.	.	1									
$2^21.$	$\frac{1}{2}q^4$	.	.	.	1	.	.	.	1	.	.	.	1								
21.2	q^5	.	1	1	.	.	.	1	.	.	1	1	.	1							
3.1^2	q^6	1	.	.	1	2	1	.	1	.	.	.	.	.	1						
$1^2.3$	q^6	.	1	1	.	1	.	.	.	.	1	.	.	1	.	1					
$21^2.1$	q^6	.	1	1	.	.	.	1	.	.	.	1	.	1	.	.	1				
.41	$\frac{1}{2}q^7$	.	.	.	.	2	1	.	.	1	.	.	.	.	1	.	.	1			
2.21	$\frac{1}{2}q^7$	1	.	.	1	2	1	.	1	.	.	.	1	.	1	.	.	.	1		
21.1^2	$\frac{1}{2}q^7$	.	1	1	.	.	.	1	.	2	1	1	.	1	.	.	1	.	.	1	
$21^3.$	$\frac{1}{2}q^7$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1
1.31	q^8	1	.	.	.	2	1	.	.	1	.	.	.	.	1	.	.	1	1	.	.
$1^3.2$	q^8	1	.	.	1	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$1^2.21$	q^9	1	.	.	1	2	1	.	1	.	.	.	1	.	1	2	.	.	1	.	.
.32	$\frac{1}{2}q^{10}$	.	.	.	.	1	.	.	.	2	1	.	.	.	.	.	.	.	.	1	.
1.2^2	$\frac{1}{2}q^{10}$	.	.	.	.	2	.	.	.	.	.	.	1	.	1	2	.	1	1	.	b
$1^3.1^2$	$\frac{1}{2}q^{10}$	1	.	.	1	2	1	.	2	.	.	.	1	.	1	2	.	.	.	.	.
$1^4.1$	$\frac{1}{2}q^{10}$	.	1	1	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	1
2.1^3	q^{12}	.	1	1	.	1	.	.	.	2	1	.	.	1	.	.	1	2	.	1	3
$.31^2$	$\frac{1}{2}q^{13}$	.	.	1	.	2	.	.	.	2	1	.	.	1	.	1	.	2	.	1	$2b$–d
1.21^2	$\frac{1}{2}q^{13}$	1	.	.	.	4	1	.	1	1	.	.	1	.	2	2	.	1	1	.	b
$1^2.1^3$	$\frac{1}{2}q^{13}$	.	1	1	.	1	.	.	.	2	1	1	.	2	.	1	1	2	.	1	3
$1^5.$	$\frac{1}{2}q^{13}$	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
$.2^21$	q^{16}	.	.	.	.	1	.	.	.	2	1	1	.	1	.	1	.	2	.	1	$2b$–d
1.1^4	q^{20}	1	.	.	.	2	1	.	1	1	.	.	.	.	1	2	.	.	.	.	$3b$–$4d$
$.21^3$	q^{21}	.	.	.	.	2	1	.	.	1	.	.	.	.	1	2	.	1	.	.	$4b$–$4d$
$.1^5$	q^{30}	.	.	1	.	1	.	.	.	.	.	.	.	1	.	1	.	2	.	.	$3+2b$–d
		ps	ps	ps	ps	$.2$	ps	ps	A_1	$.2$	ps	A_1^2	A_1^2	A_1	$.1^2$	$.2{\cdot}A_1$	A_1	$.2$	ps	$.1^2$	$1^4.$
dec?						$*$															

subgroup of type A_4 gives $\Psi_{11} + \Psi_{12}$ and $\Psi_{26} + \Psi_{27}$. Comparison with the corresponding Hecke algebra $\mathcal{H}(B_2; q^2; q) \otimes \mathcal{H}(B_1; q^2)$ shows that both have to have two PIMs as summands in this series. Again splitting via (HCr) yields $\Psi_{11}, \Psi_{12}, \Psi_{26}$ and Ψ_{27}.

TABLE 14. ${}^2D_6(q)$, $11 \leq \ell|(q+1)$, cntd.

1.31	q^8	1															
$1^3.2$	q^8	.	1														
$1^2.21$	q^9	1	1	1													
$.32$	$\frac{1}{2}q^{10}$	.	.	.	1												
1.2^2	$\frac{1}{2}q^{10}$	1	.	1	.	1											
$1^3.1^2$	$\frac{1}{2}q^{10}$	.	1	1	.	.	1										
$1^4.1$	$\frac{1}{2}q^{10}$	.	.	.	.	.	.	1									
2.1^3	q^{12}	.	.	.	.	.	.	.	1								
$.31^2$	$\frac{1}{2}q^{13}$	.	.	.	1	2	.	.	1	1							
1.21^2	$\frac{1}{2}q^{13}$	1	1	1	2	1	.	.	.	.	1						
$1^2.1^3$	$\frac{1}{2}q^{13}$	2	.	.	.	.	.	1	1	.	.	1					
$1^5.$	$\frac{1}{2}q^{13}$	.	.	.	.	.	1	.	.	.	.	.	1				
$.2^21$	q^{16}	2	.	.	1	2	.	.	1	1	.	1	b	1			
1.1^4	q^{20}	.	1	1	2	3	1	.	.	4	1	.	3	.	1		
$.21^3$	q^{21}	1	1	1	2	4	.	.	.	4	1	.	$2b$–d	2	1	1	
$.1^5$	q^{30}	2	.	.	.	2	.	1	1	1	.	1	$3b$–$4d$	3	.	4	1
		$.2{\cdot}A_1$	A_1	$.1^2{\cdot}A_1$	$.2$	$.2^2$	A_1^2	A_1^2	$.1^2$	$.21^2$	$.1^2$	$.1^2{\cdot}A_1$	$1^4.$	$.2^2$	$.1^4$	$.21^2$	$.1^4$

Now consider the Harish-Chandra series of the cuspidal ℓ-modular Steinberg character $\varphi_{.1^2}$ of a Levi subgroup of type ${}^2D_3$. Here, Harish-Chandra inducing the two PIMs in that series from a Levi subgroup of type ${}^2D_5$ gives $\Psi_{14} + 2\Psi_{19}$ and $\Psi_{28} + 2\Psi_{30}$. Comparison with the decomposition matrix of the Hecke algebra $\mathcal{H}(B_3; q^2; q)$ and application of (HCr) then shows that $\Psi_{14}, \Psi_{19}, \Psi_{28}$ and Ψ_{30} are projective.

The situation for the Harish-Chandra series of the cuspidal unipotent Brauer character $\varphi_{.2}$ of a Levi subgroup of type ${}^2D_3$ is precisely as in the previous case, yielding the columns $\Psi_5, \Psi_9, \Psi_{17}$ and Ψ_{24}. Now consider the Harish-Chandra series above the cuspidal unipotent character $\varphi_{.2} \boxtimes \varphi_{1^2}$ of a Levi subgroup of type ${}^2D_3A_1$. By Table 11 this series will contain two PIMs, whence the Harish-Chandra induction of the unique PIM of a Levi subgroup of type ${}^2D_3A_2$ in that series, viz. $\Psi_{15} + \Psi_{21} + \Psi_{22}$, must have at least two summands in that series. The only minimal summands compatible with (HCr) are Ψ_{15}, Ψ_{21}, Ψ_{22}, of which the later lies in a lower Harish-Chandra series. But Ψ_{22} cannot be a summand in the PIMs induced from ${}^2D_4A_1$, so this yields Ψ_{15} and Ψ_{21}. Next consider the Harish-Chandra series of the modular Steinberg character $\varphi_{.1^2} \boxtimes \varphi_{1^2}$ of a Levi subgroup of type ${}^2D_3.A_1$. The Harish-Chandra induction of the corresponding PIM from ${}^2D_3.A_2$ gives $\tilde{\Psi}_{23} = \Psi_{23} + \Psi_{31}$. As a Levi subgroup of type ${}^2D_4.A_1$ has two PIMs in this series, $\tilde{\Psi}_{23}$ must have at least two summands in this series, and the only possibility according to (HCr) is Ψ_{23} plus Ψ_{31}.

Since a Sylow ℓ-subgroup of G is contained in a Levi subgroup of G of type ${}^2D_5$, G does not possess cuspidal unipotent Brauer characters by (Csp) and moreover, by (St) the Harish-Chandra induction of the Steinberg PIM from ${}^2D_5(q)$ must contain the Steinberg PIM of G, which gives Ψ_{34} and Ψ_{36}.

We have now accounted for all Harish-Chandra series except for those above the three cuspidal Brauer characters $\varphi_{1^4.}$, $\varphi_{.2^2}$ and $\varphi_{.21^2}$ of $^2D_5(q)$. As we are still missing six projectives, each of these three series will contribute two PIMs, and they must occur as summands of the Harish-Chandra induction $\tilde\Psi_{20}, \tilde\Psi_{25}$ and $\tilde\Psi_{29}$ of the respective cuspidal PIMs from $^2D_5(q)$. The only way that $\tilde\Psi_{25}$ can split according to (HCr) is as $\tilde\Psi_{25} = \Psi_{25} + \Psi_{33}$. By (Tri) there is a PIM of G containing $[.21^3]$ once plus some copies of the Steinberg character. This cannot be a summand of $\tilde\Psi_{20}$, so it must occur in $\tilde\Psi_{29}$. Then (HCr) shows that Ψ_{35} is a summand of $\tilde\Psi_{29}$. Harish-Chandra restriction of a PIM from $^2D_7(q)$ which contains the unipotent character $[.31^3]$ once and other unipotent characters of at least the same a-value decomposes as $\tilde\Psi_{29} - \Psi_{30}$ plus Ψ_i's with higher a-value, so Ψ_{30} is a summand of $\tilde\Psi_{29}$ as well. This yields Ψ_{29}.

Finally, the PIM involving $[1^2.1^3]$ and constituents with higher a-value must be a summand of $\tilde\Psi_{20}$. Again, (HCr) then yields two projective modules Ψ'_{20} and Ψ_{32} as given below:

$21^3.$	1	
$.32$	b_1	
1.2^2	b_1	
$1^3.1^2$	.	
$1^4.1$	1	
2.1^3	b_2	
$.31^2$	$3b_1{+}b_2{-}d{-}3$	
1.21^2	$3b_1{+}2b_2{-}d{-}3$	
$1^2.1^3$	b_2	
$1^5.$	.	1
$.2^21$	$3b_1{+}b_2{-}d{-}3$	b_1
1.1^4	$5b_1{+}b_2{-}5d{-}c$	$b_2{+}c$
$.21^3$	$6b_1{+}b_2{-}5d{-}c$	$2b_1{+}b_2{-}d{+}c{-}3$
$.1^5$	$2b_1{+}2b_2{-}d{+}c{-}3$	$3{+}3b_1{-}b_2{-}4d{-}c$
	Ψ'_{20}	Ψ_{32}

where c is some non-negative integer.

Let $w = s_2s_1s_3s_1s_2s_3s_4s_5s_6$. The coefficients of Ψ_{34} and Ψ_{36} on R_w since these PIMs cannot occur in Ψ'_{20} or Ψ_{32}. We find that the coeeficients are opposed therefore they must be zero by (DL). This gives $-24 + 8b_2 + 8c = 0$, hence $c = 3 - b_2$. But we showed that $b_2 \geq 3$ at the end of the proof of Theorem 4.16, therefore we must have $b_2 = 3$ and $c = 0$. In addition, if $\ell > 4$ then (Red) with $(^2A_2(q)A_1(q^2).(q + 1)^2, {}^2A_2 \boxtimes [2])$ gives $d \leq 1$.

All columns are indecomposable, except possibly for Ψ_5 which might contain once Ψ_{18}, and for Ψ'_{20} which could contain the PIMs Ψ_{24}, Ψ_{28} and Ψ_{30}. Now Harish-Chandra inducing Ψ_{32} to $^2D_7(q)$, restricting it back and decomposing it in the Ψ_i shows that Ψ'_{20} must indeed contain Ψ_{24} at least b_1 times, and Ψ_{30} at least $3-d$ times. The new character could only contain Ψ_{28}. However, when $\ell > 6$, one can use (Red) with the trivial character of a Levi subgroup of type $A_3(q).(q+1)^3$ to see that Ψ_{28} cannot be a summand of Ψ_{20}. $\square$

10. Unipotent decomposition matrix of ${}^2E_6(q)$

LEMMA 4.19. *Let q be a prime power and $2 < \ell|(q+1)$. The Hecke algebras of various ℓ-modular cuspidal pairs (L, λ) of Levi subgroups L in ${}^2E_6(q)$ and their respective numbers of irreducible characters are as given in Table 15.*

TABLE 15. Hecke algebras in ${}^2E_6(q)$ for $d_\ell(q) = 2$

(L, λ)	$\mathcal{H}$	$\lvert\mathrm{Irr}\,\mathcal{H}\rvert$
(A_1, φ_{1^2})	$\mathcal{H}(B_3; q; q^2)$	3
$({}^2D_3, \varphi_{.2})$	$\mathcal{H}(B_2; q^2; q)$	2
$({}^2D_3, \varphi_{.1^2})$	$\mathcal{H}(B_2; q^2; q)$	2
$({}^2A_5, \varphi_{321})$	$\mathcal{H}(A_1; q^9)$	1
$({}^2A_5, \varphi_{2^21^2})$	$\mathcal{H}(A_1; q)$	1
$({}^2A_5, \varphi_{21^4})$	$\mathcal{H}(A_1; q)$	1
$({}^2A_5, \varphi_{1^6})$	$\mathcal{H}(A_1; -1)$	1

PROOF. For type A_1 the minimal Levi overgroups in ${}^2E_6(q)$ are of types ${}^2D_3$ and $A_1{}^2D_2$; for ${}^2D_3$ the minimal overgroups are of types ${}^2D_4$ and ${}^2A_5$. For the ordinary cuspidal unipotent character [321] of ${}^2A_5(q)$ the Hecke algebra inside ${}^2E_6(q)$ is known to have parameter q^9 (see [**8**, p. 464]). The induction of the Steinberg PIM of ${}^2A_5(q)$ to ${}^2E_6(q)$ is indecomposable by (HCr), so the parameter here is $-1 \in k$. By [**17**, Prop. 5.4], the cuspidal Brauer character φ_{21^4} of ${}^2A_5(q)$ lifts to an ordinary cuspidal character in the Lusztig series of an ℓ-element with centraliser ${}^2A_2(q)(q+1)^4$, whose centraliser in ${}^2E_6(q)$ is of type ${}^2A_2(q)A_1(q)(q+1)^3$. Similarly, $\varphi_{2^21^2}$ lifts to an ordinary cuspidal character in the Lusztig series of an ℓ-element with centraliser ${}^2A_2(q)^2(q+1)^2$ (resp. ${}^2A_2(q)^2A_1(q)(q+1)$) in ${}^2A_5(q)$ (resp. ${}^2E_6(q)$). Thus the parameter of the Hecke algebra is q in either case.

The reduction stability of the cuspidal unipotent Brauer characters of ${}^2D_3(q)$ was already discussed in Lemma 4.15. From the extended Dynkin diagram it can be seen that the relative Weyl group of an A_5-Levi subgroup inside $E_6(q)$ acts trivially on the A_5-factor, so all cuspidal Brauer characters of ${}^2A_5(q)$ are reduction stable. $\square$

The groups ${}^2E_6(q)$ have 30 unipotent characters. For primes $\ell > 3$ dividing $q+1$, 25 of these lie in the principal ℓ-block and three more, namely those labelled $\phi'_{8,3}, \phi''_{8,9}, \phi_{16,5}$, lie in a block of defect $(q+1)^2_\ell$. The unipotent part of the decomposition matrix of this latter ℓ-block is easily shown to be the identity matrix. The last two unipotent characters, both of which are cuspidal, are of defect 0. We obtain the following approximation to the decomposition matrix of the principal block:

THEOREM 4.20. *The decomposition matrix for the principal ℓ-block of ${}^2E_6(q)$, $\ell|(q+1)$, $\ell > 3$ and $(q+1)_\ell > 11$, is approximated as given in Table 16. The unknown entries satisfy*

$f \in \{0,1,2\}$ and

$$\begin{aligned} x_5 &= 6x_1 - x_2 + x_3 + 2x_4 - 7\\ x_7 &= 3x_1 - 3x_2 + x_3 + 2x_4 - 5\\ x_8 &= 6x_1 - 3x_2 + x_3 + 2x_4 + 2x_6 - 12\\ x_9 &= 6x_1 - 5x_2 + 4x_4 + 5x_6 - 15,\\ x_{10} &= 15x_1 - 15x_2 + x_3 + 10x_4 + 5x_6 - 25,\\ v_2 &= 5v_1 + f - 9,\\ v_3 &= 10v_1 + 6f - 24. \end{aligned}$$

All projectives listed are indecomposable except possibly for Ψ_6 and Ψ_7.

TABLE 16. $^2E_6(q)$, $\ell|(q+1)$, $\ell > 3$ and $(q+1)_\ell > 11$

$\phi_{1,0}$	1	1																								
$\phi'_{2,4}$	q	.	1																							
$\phi_{4,1}$	q^2	.	1	1																						
$\phi_{9,2}$	$\frac{1}{2}q^3$	1	1	1	1																					
$\phi'_{1,12}$	$\frac{1}{2}q^3$	1	.	.	.	1																				
$\phi''_{2,4}$	$\frac{1}{2}q^3$	.	.	1	.	.	1																			
$^2A_5{:}\,2$	q^4	.	.	.	.	.	.	1																		
$\phi'_{4,7}$	q^5	.	1	1	.	.	.	.	1																	
$\phi''_{8,3}$	q^6	1	.	1	1	.	1	2	.	1																
$\phi'_{9,6}$	q^6	1	1	1	1	1	2	2	1	.	1															
$^2E_6[1]$	$\frac{1}{6}q^7$	.	.	.	.	.	.	.	.	.	.	1														
$\phi_{12,4}$	$\frac{1}{6}q^7$	.	1	2	1	.	2	4	1	.	1	.	1													
$\phi'_{6,6}$	$\frac{1}{3}q^7$	1	.	.	1	1	2	2	.	1	1	.	.	1												
$\phi''_{6,6}$	$\frac{1}{3}q^7$	1	.	.	1	.	.	2	.	1	.	.	.	.	1											
$\phi_{4,8}$	$\frac{1}{2}q^7$	.	.	.	1	.	2	2	.	.	1	.	.	.	.	1										
$\phi''_{9,6}$	q^{10}	.	.	1	1	.	3	6	.	1	1	x_1	1	3	.	1	1									
$\phi''_{4,7}$	q^{11}	.	.	1	.	.	1	4	.	.	.	x_2	1	2	.	.	1	1								
$\phi'_{8,9}$	q^{12}	1	.	1	1	1	3	4	1	1	1	x_3	.	1	1	2	.	.	1							
$^2A_5{:}\,1^2$	q^{13}	.	.	.	.	.	.	1	.	.	.	x_4	.	.	.	.	.	2	.	1						
$\phi_{9,10}$	$\frac{1}{2}q^{15}$	.	.	1	1	.	5	12	1	1	2	x_5	1	6	1	3	5	3	1	2	1					
$\phi''_{1,12}$	$\frac{1}{2}q^{15}$	.	.	.	.	.	1	2	.	1	.	x_6	.	3	.	.	1	.	.	z	.	1				
$\phi'_{2,16}$	$\frac{1}{2}q^{15}$	.	.	1	.	.	1	2	1	.	.	x_7	.	.	.	2	.	1	1	2	.	.	1			
$\phi_{4,13}$	q^{20}	.	.	1	.	.	3	8	1	.	1	x_8	1	6	.	2	5	1	1	$2z{+}2$	1	2	v_1	1		
$\phi''_{2,16}$	q^{25}	.	.	.	.	.	2	8	.	.	1	x_9	1	8	.	1	6	3	.	$5z{+}4$	1	5	v_2	5	1	
$\phi_{1,24}$	q^{36}	.	.	.	.	.	1	6	.	1	.	x_{10}	.	6	1	2	5	5	1	$5z{+}10$	1	5	v_3	10	6	1
		ps	ps	ps	ps	A_1	$.2$	321	A_1	ps	$.1^2$	c	ps	2^21^2	A_1	$.2$	21^4	c	$.1^2$	c	1^6	c	c	c	c	c
dec?							$*$	$*$																		

Here the symbols ".2", ".1^2", "321", "2^21^2", "21^4" and "1^6" stand for Harish-Chandra series of cuspidal Brauer characters of $^2D_3(q)$, $^2A_5(q)$ respectively, see [**17**, Table 1].

Proof. As usual, we write Ψ_i for linear combinations of unipotent characters corresponding to the column i of Table 16. For $i \in \{6, 7, 10, 12, 15, 16, 18, 20\}$ they are obtained by (HCi). We also find $\Psi_2 + \Psi_4$ and $\Psi_2 + \Psi_5$ which by (Tri) and (HCr) yield Ψ_2, Ψ_4 and Ψ_5. Then $\Psi_1 + 2\Psi_2$ and $\Psi_1 + \Psi_3$ give Ψ_1 and Ψ_3. From this, $\Psi_3 + \Psi_9$ allows to isolate Ψ_9. Also $\Psi_4 + \Psi_8$ again by (Tri) yields Ψ_8. Then $\Psi_8 + \Psi_{14}$ leads to Ψ_{14} and $\Psi_{12} + \Psi_{13}$ gives Ψ_{13}.

Counting the columns obtained so far and comparing with the number of characters in non-cuspidal Harish-Chandra series with Lemma 4.19 shows that the eight columns with indices $i \in \{11, 17, 19, 21, 22, 23, 24, 25\}$ must be cuspidal.

By Corollary 4.3 the second to last entry in the last row equals $6 = \operatorname{rank} \mathbf{G}$. We then use (DL) to find some of the entries in columns corresponding to the other cuspidal Brauer characters. Note that by Proposition 4.1 we can use (Reg) for any d-split Levi subgroup whenever $(q+1)_\ell > 12$. We start with Ψ_{17}: the decomposition numbers in that column will be denoted by $y_1, \ldots, y_8$ so that the unipotent part of Ψ_{17} equals

$$\Psi_{17} = \phi''_{4,7} + y_1\phi'_{8,9} + \cdots + y_7\phi''_{2,16} + y_8\phi_{1,24}.$$

Let us consider the Deligne–Lusztig character R_w associated to a Coxeter element w. One first checks that Ψ_{18} does not occur in any R_v for $v < w$. Its coefficient at Ψ_{18} is $-y_1$, therefore by (DL) the entry y_1 must be zero. The coefficient of Ψ_{19} equals $2 - y_2$ which forces $y_2 \leq 2$. One can use (Red) with the character ${}^2A_2 \boxtimes [2]$ of a Levi subgroup ${}^2A_2(q)^2.(q+1)^2$ to see that we also have $y_2 \geq 2$, which proves that $y_2 = 2$. The coefficients of Ψ_{20}, Ψ_{21} and Ψ_{22} are $3 - y_3$, $-y_4$ and $1 - y_5$ respectively. Using (Red) for a Levi subgroup $D_4(q).(q+1)^2$ and the character $[.2^2]$ (resp. the cuspidal unipotent character) we have $y_3 \geq 3$ (resp. $y_5 \geq 1$). We deduce that

$$y_3 = 3, \quad y_4 = 0 \quad \text{and} \quad y_5 = 1.$$

The coefficient of Ψ_{23} is $1 - y_6$, and (Red) for the trivial character of a Levi subgroup $A_1(q)^2.(q+1)^4$ gives $y_6 \geq 1$, whence $y_6 = 1$. Finally, the coefficient of Ψ_{24} is $3 - y_7$; one can invoke (Red) with the trivial character of a Levi subgroup $A_1(q).(q+1)^5$ to ensure that $y_7 \geq 3$, hence $y_7 = 3$. We conclude that $y_8 = 5$ using Theorem 4.2.

We now turn to Ψ_{21} using the Deligne–Lusztig character R_w with $w = s_1s_2s_3s_1s_4s_3$. We denote by u_i the unknown decomposition numbers in the 21st column: under the assumption (Tri) there are $u_i \geq 0$ such that

$$\Psi_{21} = \phi''_{1,12} + u_1\phi_{4,13} + u_2\phi''_{2,16} + u_3\phi_{1,24}.$$

The coefficient of Ψ_{23} on R_w is $2 - u_1$; on the other hand, (Red) with the trivial character of a Levi subgroup $A_1(q)^2.(q+1)^4$ gives $u_2 \geq 2$. The coefficient of Ψ_{24} is $5 - u_2$ and (Red) with the trivial character of $A_1(q).(q+1)^5$ gives $u_5 \geq 5$. This shows that $u_1 = 2$ and $u_2 = 5$. Theorem 4.2 gives $u_3 = 5$.

We continue with the PIM Ψ_{19} and the Deligne–Lusztig character R_w with $w = s_4s_5s_4s_2s_3s_1s_4s_5$. The unknown entries will be denoted by $z_1, \ldots, z_6$. The coefficients of the PIMs Ψ_{20} and Ψ_{22} on R_w are $4 - 2z_1$ and $4 - 2z_3$ respectively. The unipotent characters $[.2^2]$ and $[D_4]$ of $D_4(q).(q+1)^2$ give, by (Red), the relations $z_1, z_3 \geq 2$. We deduce that $z_1 = z_3 = 2$. The PIM Ψ_{23} has coefficient $4 + 4z_2 - 2z_4$; using (Red) with the cuspidal unipotent character of ${}^2A_2(q)^2.(q+1)^2$ we get $-2 - 2z_2 + z_4 \geq 0$, which proves that $z_4 = 2 + 2z_2$. Next, Ψ_{24} has coefficient $8 + 10z_2 - 2z_5$ and by (Red) with the trivial character of $A_1(q).(q+1)^5$ we must also have $-4-5z_2+z_5 \geq 0$, whence $z_5 = 4+5z_2$. Finally, Theorem 4.2 shows that $z_6 = 10+5z_2$. We set $z := z_2$.

Let us now consider the PIM Ψ_{11} for which many entries are unknown. Using (Tri), only 10 entries below the diagonal can be non-zero, starting with the row corresponding to $\phi''_{9,6}$. We denote these by x_i. Let $w = s_1s_2s_4s_3s_1s_5s_4s_3s_6s_5s_4s_3$. The coefficients of Ψ_{20} and Ψ_{22} in R_w are

$$X := 6x_1 - x_2 + x_3 + 2x_4 - x_5 - 7 \quad \text{and} \quad Y := 3x_1 - 3x_2 + x_3 + 2x_4 - x_7 - 5$$

respectively. By (Red) with the trivial character of ${}^2A_2(q).(q+1)^4$, the sum $X+Y$ must be non-positive. Therefore (DL) forces $X = Y = 0$ which gives the values of x_5 and x_7 in terms of $x_1, \ldots, x_4$. The coefficient of Ψ_{23} is $-12+6x_1-3x_2+x_3+2x_4+2x_6-x_8$. By (Red) applied to the cuspidal unipotent character of ${}^2A_2(q)^2.(q+1)^2$ it is also non-positive, therefore it must be zero, so that

$$x_8 = 6x_1 - 3x_2 + x_3 + 2x_4 + 2x_6 - 12.$$

Finally, the coefficient of Ψ_{24} is $-15+6x_1-5x_2+4x_4+5x_6-x_9$; it is non-negative by (DL) and non-positive by (Red) applied to the trivial character of $A_1(q).(q+1)^5$, forcing

$$x_9 = 6x_1 - 5x_2 + 4x_4 + 5x_6 - 15.$$

Theorem 4.2 then gives $x_{10} = 15x_1 - 15x_2 + x_3 + 10x_4 + 5x_6 - 25$.

Let us denote by w_1 and w_2 the two unknown entries in the 23rd column. Let $w = s_2s_3s_4s_2s_3s_4s_6s_5s_4s_2s_3s_4s_5s_6$. The coefficient of Ψ_{23} in R_w is $60-12w_1$ therefore $w_1 \leq 5$ by (DL). On the other hand, (Red) applied to the trivial character of $A_1(q).(q+1)^5$ forces $w_1 \geq 5$, hence $w_1 = 5$. We get $w_2 = 10$ using Theorem 4.2.

The last Deligne–Lusztig character R_w we look at is the one associated to the element $w = s_1s_2s_3s_1s_4s_3s_1s_5s_4s_3s_1s_6s_5s_4s_3s_1$. Let v_1, v_2, v_3 be the three unknown entries in the 22nd column. The coefficient of Ψ_{24} on R_w equals $-252+180v_1-36v_2 = 36(-7+5v_1-v_2)$. By (DL) this implies that $v_2 \leq 5v_1 - 7$. On the other hand, (Red) for the cuspidal unipotent character of ${}^2A_2(q).(q+1)^4$ yields the relation $9-5v_1+v_2 \geq 0$. We deduce that $v_2 = 5v_1 - 9 + f$ with some $f \in \{0,1,2\}$. Theorem 4.2 then gives $v_3 = 30-20v_1+6v_2 = 10v_1+6f-24$.

Now all projectives in the table are indecomposable except possibly for Ψ_6 which might contain Ψ_{12} once, and for Ψ_7 which might contain Ψ_9 (twice), Ψ_{12} (four times), Ψ_{14} and Ψ_{15} (once each). □

11. Unipotent decomposition matrices of $B_4(q)$ and $C_4(q)$

Here, we find the decomposition matrices for the unipotent blocks of odd-dimensional orthogonal groups $B_4(q)$ and symplectic groups $C_4(q)$, assuming (T_ℓ).

The decomposition matrix for the principal ℓ-block of $\mathrm{SO}_7(q)$, $2 < \ell | (q+1)$, was determined by Himstedt–Noeske [**34**]. Again we first record the parameters of certain Hecke algebras.

LEMMA 4.21. *Let q be a prime power and $2 < \ell | (q+1)$. The Hecke algebras of various ℓ-modular cuspidal pairs (L, λ) of Levi subgroups L in $B_4(q)$ and their respective numbers of irreducible characters are as given in Table 17.*

TABLE 17. Hecke algebras in $B_4(q)$ for $d_\ell(q) = 2$

(L, λ)	$\mathcal{H}$	$\lvert\operatorname{Irr}\mathcal{H}\rvert$
(A_1, φ_{1^2})	$\mathcal{H}(A_1; q) \otimes \mathcal{H}(B_2; q; q)$	2
$(B_1, \varphi_{.1})$	$\mathcal{H}(B_3; q; q)$	3
$(A_1^2, \varphi_{1^2}^{\otimes 2})$	$\mathcal{H}(B_2; q^2; q)$	2

PROOF. The relative Weyl groups of the relevant Levi subgroups can be found in [**35**, p. 70]. All three cuspidal characters are the ℓ-modular Steinberg characters of the respective groups and thus liftable. The minimal Levi overgroups for type A_1 are of types B_2, B_1A_1 and A_1^2 and thus lead to parameters q in all three cases. For a Levi subgroup of type B_1 the minimal overgroups have types B_2 and B_1A_1, with parameters q as just seen. Finally for a Levi subgroup of type A_1^2 the minimal overgroups are of types A_3 and B_2A_1, with parameters q^2, q respectively.

As for reduction stability, by the considerations in Section 2.3 we only need to worry about the last case. Here, the embedding $\mathrm{SO}_5\mathrm{SO}_4 \leq \mathrm{SO}_9$ shows that the relative Weyl group B_2 centralises the A_1^2 Levi subgroup and so any lift of the cuspidal Brauer character will be reduction stable. □

The groups of type B_4 have 25 unipotent characters. For primes $\ell > 2$ dividing $q+1$, 20 of these lie in the principal ℓ-block, the other five lie in a block of defect $(\Phi_2^2)_\ell$.

THEOREM 4.22. *Assume (T_ℓ). The ℓ-modular decomposition matrices of the unipotent ℓ-blocks of $G = B_4(q)$, $7 < \ell|(q+1)$, are as given in Table 18 and 19.*

Here B_3^a denotes the Harish-Chandra series of the cuspidal unipotent Brauer character $B_2{:}\,1^2$ of $B_3(q)$, B_2^b, B_2^c the ones of the cuspidal unipotent Brauer characters $B_2 \boxtimes \varphi_{1^2}$ and $\varphi_{.1^2} \boxtimes \varphi_{1^2}$ of $B_2(q)A_1(q)$ and A_1^{2*} the one of the ℓ-modular Steinberg character $\varphi_{1^2} \boxtimes \varphi_{1^2}$ of $B_1(q)A_1(q)$.

PROOF. The five projectives in the non-principal unipotent ℓ-block are obtained by (HCi) and are easily seen to be indecomposable. This proves Table 19. So now consider the principal block. The three columns labelled "ps" come from the decomposition matrix of the Hecke algebra $\mathcal{H}(B_4; q; q)$. Harish-Chandra inducing the unipotent PIMs from proper Levi subgroups L of G and cutting by the principal block we obtain projectives which are non-negative integral linear combinations of the sixteen columns $\Psi_1, \ldots, \Psi_{14}, \Psi_{16}, \Psi_{19}$ in our table. (For Levi subgroups of type B_3 the decomposition matrix, depending on $(q+1)_\ell$, was obtained in [**34**, Table 5].) Furthermore, among these induced projectives we actually find all of the Ψ_i not labelled "ps", except that instead of Ψ_6 we obtain $\Psi_6 + \Psi_7$ and $\Psi_6 + \Psi_{19}$. Since the space spanned by these projectives together with Ψ_7 and Ψ_{19} is only 3-dimensional, we conclude that Ψ_6 is also a projective character.

We next claim that all of the Ψ_i are indecomposable. For $i \notin \{3, 6, 8\}$ this follows by application of (HCr). For the remaining three columns there is only one possible non-trivial decomposition each, into

$$\Psi_3 = \Psi_3' + \Psi_7', \quad \Psi_6 = \Psi_6' + \Psi_8', \quad \Psi_8 = \Psi_8' + \Psi_{10}'$$

(with $\Psi_7' + \Psi_8' = \Psi_7$). The three projective characters are induced from Steinberg PIMs in the series B_1, A_1^2, A_1. The corresponding Hecke algebras were determined in Lemma 4.21,

TABLE 18. $B_4(q)$, $7 < \ell|(q+1)$, principal block

4.	1	1																			
B_2:2.	$\frac{1}{2}q$	.	1																		
.4	$\frac{1}{2}q$	1	.	1																	
31.	$\frac{1}{2}q$	1	.	.	1																
B_2:1^2.	$\frac{1}{2}q^2$	.	1	.	.	1															
2^2.	$\frac{1}{2}q^2$	.	.	.	1	.	1														
2.2	$\frac{1}{2}q^2$	1	.	1	.	.	.	1													
21.1	q^3	1	.	.	1	.	1	1	1												
.31	$\frac{1}{2}q^4$	1	.	1	1	.	.	.	.	1											
21^2.	$\frac{1}{2}q^4$	1	.	.	1	.	1	.	1	.	1										
2.1^2	$\frac{1}{2}q^4$	1	2	1	.	.	.	1	1	.	.	1									
$1^2.2$	q^4	1	.	1	.	.	.	1	1	.	.	.	1								
1.21	q^5	1	2	2	1	2	1	1	1	1	.	1	1	1							
B_2:.2	$\frac{1}{2}q^6$	.	1	.	.	.	.	.	.	.	.	.	.	.	1						
$.2^2$	$\frac{1}{2}q^6$	.	.	.	1	2	1	.	.	1	.	.	.	1	.	1					
$1^2.1^2$	$\frac{1}{2}q^6$	1	2	1	.	2	1	1	2	.	.	1	1	1	.	.	1				
B_2:.1^2	$\frac{1}{2}q^9$	.	1	.	.	1	.	.	.	.	.	.	.	.	1	2	.	1			
1^4.	$\frac{1}{2}q^9$	1	.	.	.	.	.	.	1	.	1	.	.	.	.	.	1	.	1		
$.21^2$	$\frac{1}{2}q^9$	1	2	1	1	2	1	.	1	1	3	1	1	1	2	1	.	.	.	1	
$.1^4$	q^{16}	1	2	1	.	2	.	.	1	.	3	1	1	1	2	3	1	4	3	1	1
		ps	B_2	B_1	ps	B_2^b	A_1^2	ps	A_1	B_1	1^3.	.1^2	A_1^{2*}	B_2^c	B_3^a	c	A_1^2	c	c	.1^3	c

TABLE 19. $B_4(q)$, $5 \le \ell|(q+1)$, block of defect Φ_2^2

3.1	$\frac{1}{2}q\Phi_2^2\Phi_4\Phi_6$	1				
1.3	$\frac{1}{2}q^2\Phi_2^2\Phi_6\Phi_8$	1	1			
B_2: 1.1	$\frac{1}{2}q^4\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	.	.	1		
$1^3.1$	$\frac{1}{2}q^6\Phi_2^2\Phi_6\Phi_8$	1	.	.	1	
1.1^3	$\frac{1}{2}q^9\Phi_2^2\Phi_4\Phi_6$	1	1	2	1	1
		ps	B_1	B_2	A_1	.1^2

by Table 11 they have 3,2,2 modular irreducibles, respectively. We already saw that the other unipotent block of G has one PIM in each of the series B_1 and A_1. If Ψ_3 decomposes, its summands must lie in the B_1-series, which would produce four PIMs in that series, but we just argued that there should be exactly three. The same argument applies to the other two series. Thus all columns not labelled "c" in Table 18 correspond to PIMs. Since $(q+1)_\ell > 8$, Corollary 4.4 gives the columns Ψ_{17} and Ψ_{18}.

Now by uni-triangularity there are $x_i \geq 0$ such that

$$\Psi_{15} = [.2^2] + x_1[B_2\colon .1^2] + x_2[1^4.] + x_3[.21^2] + x_4[.1^4].$$

For determining the x_i's we proceed as usual. We first use Theorem 4.2 to get $x_4 = 4x_1 + 3x_2 + x_3 - 6$. Then we decompose the Deligne–Lusztig character R_w for w a Coxeter element on the basis of PIMs. We have, up to adding and removing non-unipotent characters

$$\begin{aligned} R_w = {} & \Psi_1 + \Psi_2 - \Psi_3 - \Psi_4 - \Psi_5 + \Psi_6 - \Psi_8 + \Psi_9 + \Psi_{12} - \Psi_{14} + \Psi_{15} - \Psi_{16} \\ & + (2 - x_1)\Psi_{17} - x_2\Psi_{18} + (1 - x_3)\Psi_{19}. \end{aligned}$$

By decomposing R_v for $v < w$, one checks that none of Ψ_{17}, Ψ_{18} Ψ_{19} occurs (for Ψ_{17} and Ψ_{18} one could also invoke the fact that they correspond to cuspidal modules). Then (DL) yields $x_1 \leq 2$, $x_2 = 0$ and $x_3 \leq 1$. On the other hand, for a 2-split Levi subgroup $B_2(q).(q+1)^2$ and $\ell > 4$ one can use (Red) with the unipotent characters $[B_2]$ and $[.2]$ to get respectively $x_1 \geq 2$ and $x_3 \geq 1$, showing $x_1 = 2$ and $x_3 = 1$. □

As Lusztig has shown, the unipotent characters of groups of types B_n and C_n are parametrised by the same combinatorial objects. With this we may state:

THEOREM 4.23. *Assume (T_ℓ). Then the ℓ-modular decomposition matrices for the unipotent blocks of $G = C_4(q)$, for $8 < \ell | (q+1)$, are as given in Tables 18 and 19 for $B_4(q)$ above.*

PROOF. All of the arguments in the proof of Theorem 4.22 go through for $C_4(q)$ as well. □

12. Unipotent decomposition matrix of $F_4(q)$

The groups of type F_4 have 37 unipotent characters. For $d_\ell(q) = 2$ and $\ell > 3$, 25 of these lie in the principal ℓ-block, five more lie in a block of defect $(q+1)_\ell^2$, and the last seven are of defect 0. For p good, the decomposition matrices of the unipotent ℓ-blocks of $F_4(q)$ were partially computed by Köhler in [**38**]. He completely determined the matrix for the non-principal block of positive defect. We obtain here most of the entries that were left undetermined for the principal block.

THEOREM 4.24. *The decomposition matrix for the principal ℓ-block of $F_4(q)$, $(q, 6) = 1$, with $3 < \ell | (q+1)$ and $(q+1)_\ell > 11$ is approximated as given in Table 20.*

Here, the unknown parameters satisfy

$$x_6 = 4 - 2x_1 - 2x_2 + 2x_3 \quad \textit{and} \quad x_7 = 4 - x_3 + 2x_4 + 2x_5.$$

Furthermore, $x_1, x_2, x_3, z \geq 2$.

Here B_3^a denotes the Harish-Chandra series of the cuspidal unipotent Brauer character $B_2{:}\varphi_{1^2}$ of $B_3(q)$, and A_1^{2*} the one of the ℓ-modular Steinberg character $\varphi_{1^2} \boxtimes \varphi_{1^2}$ of $\tilde{A}_1(q)A_1(q)$.

PROOF. The values a, b, c, d left undetermined in [**38**, T.A.157] are obtained by Harish-Chandra induction from the decomposition matrices of the Levi factors $B_3(q)$ and $C_3(q)$: using the values given in [**34**, Table 5 and Thm. 4.3], we get $b, d \leq \gamma = 2$ and $a, c \leq \beta - 1 = 2$ (recall that $(q+1)_\ell > 5$). This forces $a = b = c = d = 2$. Furthermore

TABLE 20. $F_4(q)$, $\ell > 3$ and $(q+1)_\ell > 11$

$\phi_{1,0}$	1	1																								
$\phi''_{2,4}$	$\frac{1}{2}q$	.	1																							
$\phi'_{2,4}$	$\frac{1}{2}q$	.	.	1																						
$B_2{:}2.$	$\frac{1}{2}q$	.	.	.	1																					
$\phi_{9,2}$	q^2	.	1	1	.	1																				
$\phi'_{8,3}$	q^3	1	.	1	.	1	1																			
$\phi''_{8,3}$	q^3	1	1	.	.	1	.	1																		
$F_4^{II}[1]$	$\frac{1}{24}q^4$	.	.	.	.	.	.	.	1																	
$\phi''_{6,6}$	$\frac{1}{12}q^4$	1	.	.	2	1	1	1	.	1																
$\phi'_{9,6}$	$\frac{1}{8}q^4$	.	1	1	.	1	1	.	.	.	1															
$\phi'_{1,12}$	$\frac{1}{8}q^4$	1	.	.	.	.	1	.	.	.	.	1														
$\phi''_{9,6}$	$\frac{1}{8}q^4$	.	1	1	.	1	.	1	.	.	.	.	1													
$F_4^{I}[1]$	$\frac{1}{8}q^4$	.	.	.	.	.	.	.	.	.	.	.	.	1												
$\phi''_{1,12}$	$\frac{1}{8}q^4$	1	.	.	.	.	.	1	.	.	.	.	.	.	1											
$F_4[-1]$	$\frac{1}{4}q^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1										
$B_2{:}1^2.$	$\frac{1}{4}q^4$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1									
$B_2{:}.2$	$\frac{1}{4}q^4$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1								
$\phi'_{6,6}$	$\frac{1}{3}q^4$	1	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	1							
$\phi'_{8,9}$	q^9	1	1	.	2	1	2	1	x_1	1	1	3	.	.	.	2	2	.	1	1						
$\phi''_{8,9}$	q^9	1	.	1	2	1	1	2	x_2	1	.	.	1	.	3	2	.	2	1	.	1					
$\phi_{9,10}$	q^{10}	.	1	1	2	1	1	1	x_3	1	1	2	1	y	2	z	2	2	1	1	1	1				
$\phi''_{2,16}$	$\frac{1}{2}q^{13}$	.	.	1	.	.	.	.	x_4	.	.	.	1	$y{+}2$	3	z	.	2	.	.	1	1	1			
$\phi'_{2,16}$	$\frac{1}{2}q^{13}$	.	1	.	.	.	.	.	x_5	.	1	3	.	$y{+}2$	.	z	2	.	.	1	.	1	.	1		
$B_2{:}.1^2$	$\frac{1}{2}q^{13}$	.	.	.	1	.	.	.	x_6	.	.	.	.	$2y$	.	$2z{-}4$	1	1	.	.	.	2	.	.	1	
$\phi_{1,24}$	q^{24}	1	.	.	2	.	1	1	x_7	1	.	3	.	$3y{+}4$	3	$3z$	2	2	1	1	1	3	2	2	4	1
		ps	ps	ps	C_2	ps	A_1	C_1	c	$.1^2$	A_1	$1^3.$	C_1	c	$1^3.$	c	B_3^a	B_3^a	B_2^b	$.1^3$	$.1^3$	c	c	c	c	c

[**38**, T.A.157] gives the following approximation to the lower right hand corner of the decomposition matrix:

$\phi_{9,10}$	1				
$\phi''_{2,16}$	1	1			
$\phi'_{2,16}$	1	.	1		
$B_2{:}.1^2$	b_1	.	.	1	
$\phi_{1,24}$	b_2	b_3	b_4	b_5	1

Now, Theorem 4.2 yields the values of b_3, b_4 and b_5, together with the relation $b_2 = 4b_1 - 5$.

Let w be a Coxeter element of the Weyl group of $G = F_4(q)$. The corresponding Deligne–Lusztig character R_w has the following decomposition in terms of projective characters in the principal block B_0:

$$B_0 R_w = \Psi_1 + \Psi_4 - \Psi_6 - \Psi_7 + \Psi_8 - \Psi_{13} + \Psi_{14} + \Psi_{17} - \Psi_{18} + \Psi_{21} + (2 - b_1)\Psi_{24}.$$

By [**15**, Prop. 1.5], we deduce that $b_1 \leq 2$, and therefore $b_1 = 2$ and $b_2 = 3$.

We now focus on the columns corresponding to the ordinary cuspidal unipotent characters, whose entries will be denoted by $x_1, \ldots, x_{21}$ as follows:

$$\begin{array}{l|lll}
F_4^{II}[1] & 1 & & \\
F_4^{I}[1] & . & 1 & \\
F_4[-1] & . & . & 1 \\
\phi'_{8,9} & x_1 & x_8 & x_{15} \\
\phi''_{8,9} & x_2 & x_9 & x_{16} \\
\phi_{9,10} & x_3 & x_{10} & x_{17} \\
\phi''_{2,16} & x_4 & x_{11} & x_{18} \\
\phi'_{2,16} & x_5 & x_{12} & x_{19} \\
B_2{:}.1^2 & x_6 & x_{13} & x_{20} \\
\phi_{1,24} & x_7 & x_{14} & x_{21}
\end{array}$$

Let $w = s_1s_2s_3s_4s_2s_3$. The coefficients of the PIMs Ψ_{19}, Ψ_{22} and Ψ_{24} are respectively $2 - x_{15}$, $2 - x_{15} + x_{17} - x_{18}$ and $4 - 2x_{15} - 2x_{16} + 2x_{17} - x_{20}$. They are all non-negative by (DL) and they add up to $8 - 4x_{15} - 2x_{16} + 3x_{17} - x_{18} - x_{20}$. But when $(q+1)_\ell > 7$, this sum should also be non-positive by (Red) applied to the trivial character of $\widetilde{A}_1(q).(q+1)^3$ (see Proposition 4.1). Therefore we must have

$$x_{15} = 2, \quad x_{18} = x_{17} \quad \text{and} \quad x_{20} = 2x_{17} - 2x_{16}.$$

Similarly, the coefficients of Ψ_{20}, Ψ_{23} and Ψ_{24} are respectively $2 - x_{16}$, $2 - x_{16} + x_{17} - x_{19}$ and $4 - 2x_{15} - 2x_{16} + 2x_{17} - x_{20}$. They add up to a number which is both non-negative by (DL) and non-positive by (Red) applied to the trivial character of $A_1(q).(q+1)^3$. Therefore

$$x_{16} = 2, \quad x_{19} = x_{17} \quad \text{and} \quad x_{20} = 2x_{17} - 4.$$

Then by Theorem 4.2 we get $x_{21} = 3x_{17}$. We set $z := x_{17}$.

Let $w = (s_1s_2s_3s_4)^2$ and R_w be the corresponding Deligne–Lusztig character. We proceed as in the previous paragraph. The sum of the coefficients of Ψ_{19}, Ψ_{22} and Ψ_{24} is

$$(-x_8) + (2 - x_8 + x_{10} - x_{11}) + (-2x_8 - 2x_9 + 2x_{10} - x_{13})$$

and the sum of the coefficients of Ψ_{20}, Ψ_{24} and Ψ_{24} is

$$(-x_9) + (2 - x_9 + x_{10} - x_{12}) + (-2x_8 - 2x_9 + 2x_{10} - x_{13}).$$

Both are sums of non-negative integers by (DL) and are non-positive by (Red) (for the same unipotent characters as before). We deduce that

$$x_8 = x_9 = 0, \quad x_{11} = x_{12} = x_{10} + 2, \quad \text{and} \quad x_{13} = 2x_{10}.$$

With Theorem 4.2 we get $x_{14} = 3x_{10} + 4$. We set $y := x_{10}$.

We now consider the Deligne–Lusztig character associated with $w = (s_1s_2s_3s_4)^3$. Only the PIMs Ψ_{24} and Ψ_{25} do not occur in R_v for $v < w$. The coefficient of Ψ_{24} on R_w equals $4 - 2x_1 - 2x_2 + 2x_3 - x_6$ and therefore it must be non-negative by (DL). On the other hand, if $(q+1)_\ell > 4$ on can invoke (Red) for the cuspidal unipotent character of the Levi subgroup $B_2(q).(q+1)^2$ to ensure that it is also non-positive. We deduce that $x_6 = 4 - 2x_1 - 2x_2 + 2x_3$. The relation $x_7 = 4 - x_3 + 2x_4 + 2x_5$ now follows from Theorem 4.2.

Finally, we use (Red) to obtain lower bounds on some of the missing entries. With the trivial character of the 2-split Levi subgroups of type ${}^2A_2(q).(q+1)^2$ we find $x_1 \geq 2$ and $x_2 \geq 2$. Using the character $[21]\boxtimes[2]$ of a 2-split Levi subgroup of type ${}^2A_2(q)A_1(q).(q+1)$ we find $x_3 \geq x_1 + x_2 - 2$ hence $x_3 \geq 2$. ☐

Remark 4.25. As before we can obtain upper bounds on the missing decomposition numbers if [**16**, Conj. 1.2] holds. Let $w_1 = s_2s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_4$. We have

$$\langle Q_{w_1}; \varphi_{19}\rangle = 4 - 2x_1 \quad \text{and} \quad \langle Q_{w_1}; \varphi_{20}\rangle = 4 - 2x_2,$$

which would show that $x_1 = x_2 = 2$. Now with $w_2 = s_1s_2s_1s_3s_2s_1s_4s_3s_2s_1s_3s_2s_4s_3s_2s_1$ and $w_3 = s_3s_4s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_2s_3s_4$ we have

$$\begin{aligned}
\langle Q_{w_2}; \varphi_{22}\rangle &= 72 + 12x_3 - 12x_4,\\
\langle Q_{w_3}; \varphi_{21}\rangle &= 156 - 12z - 12x_3,\\
\langle Q_{w_3}; \varphi_{23}\rangle &= 72 + 12x_3 - 12x_5.
\end{aligned}$$

This gives $x_4, x_5 \leq x_3 + 6$ and $x_3 + z \leq 13$. Finally with $w_4 = s_3s_2s_3s_4s_3s_2s_1s_3s_4$ and $w_5 = s_2s_1s_4s_3s_2s_1s_3s_2s_3$ we find

$$\langle Q_{w_4}; \varphi_{21}\rangle = 18 - 6z \quad \text{and} \quad \langle Q_{w_4}; \varphi_{21}\rangle = 14 - 2y.$$

We conclude that $y \leq 7$ and $z \in \{2, 3\}$. Using the previous inequalities we obtain $x_3 \leq 11$ and $x_4, x_5 \leq 17$.

CHAPTER 5

Decomposition matrices at $d_\ell(q) = 3$

Here, we consider decomposition matrices of unipotent blocks of groups of Lie type $G = G(q)$ for primes ℓ with $d_\ell(q) = 3$, so $\ell|(q^2 + q + 1)$ and in particular $\ell \geq 7$. If G is of classical type, then such primes ℓ with are *linear* for G, and so the decomposition numbers are known by work of Gruber and Hiss [**33**] to be given by suitable q-Schur algebras. (This implies, for example, that the block distribution of unipotent characters refines the subdivision into ordinary Harish-Chandra series, and that (T_ℓ) is always satisfied.) Nevertheless, to our knowledge they have never been written out explicitly, so we derive them here, also as an induction base for blocks of groups of exceptional type for which the theory of linear primes from [**33**] does not apply.

1. Even-dimensional split orthogonal groups

We begin with groups of type D_n for $n \leq 7$. The Brauer trees of unipotent blocks with cyclic defect were first determined by Fong and Srinivasan [**20**] and in our situation can also easily be obtained by Harish-Chandra induction:

Proposition 5.1. *Let q be a prime power and ℓ a prime with $d_\ell(q) = 3$. The Brauer trees of the unipotent ℓ-blocks of $D_n(q)$, $3 \leq n \leq 7$, with cyclic defect are as given in Table 1.*

Here and later on, we label the ordinary unipotent characters by their Harish-Chandra series; for the principal series this means by the irreducible characters of a Weyl group of type D_n, hence by unordered pairs of partitions of n, and for characters in the Harish-Chandra series of the cuspidal unipotent character of a Levi subgroup of type D_4 by the symbol "D_4" and a character of the relative Weyl group, which is of type B_{n-4}, hence by a bipartition of $n - 4$.

Under the edges of the Brauer trees, which represent the irreducible Brauer characters (or equivalently the PIMs) of the block, we have indicated their corresponding ℓ-modular Harish-Chandra series; here "ps" stands for the principal series, while "A_2" stands for the series of the cuspidal ℓ-modular Steinberg character of a Levi subgroup of type A_2.

Again, we first need to determine the parameters of the Hecke algebras attached to cuspidal ℓ-modular Brauer characters of certain Levi subgroups:

Lemma 5.2. *Let q be a prime power and $\ell|(q^2 + q + 1)$.*

(a) *The Hecke algebra for the cuspidal ℓ-modular Steinberg character φ_{1^3} of a Levi subgroup of type A_2 inside $D_n(q)$, $n \geq 4$, is $\mathcal{H}(B_{n-3}; 1; q)$.*
(b) *The Hecke algebra for the cuspidal ℓ-modular Steinberg character $\varphi_{1^3}^{\boxtimes 2}$ of a Levi subgroup of type A_2^2 inside $D_n(q)$ is $\mathcal{H}(A_1; q^3) \otimes \mathcal{H}(A_1; q^3)$ if $n = 6$ and $\mathcal{H}(B_2; q^3; 1) \otimes \mathcal{H}(D_{n-6}; q)$ when $n \geq 7$ (where D_1 has to be interpreted as the trivial group).*

TABLE 1. Brauer trees for $D_n(q)$ $(3 \le n \le 7)$, $7 \le \ell | (q^2+q+1)$

$$\begin{array}{lccccccc}
D_3(q): & .3 & \text{---} & .21 & \text{---} & .1^3 & \text{---} & \bigcirc \\
D_5(q): & 1.4 & \text{---} & 1.2^2 & \text{---} & 1.1^4 & \text{---} & \bigcirc \\
D_7(q): & 2.5 & \text{---} & 2.2^21 & \text{---} & 2.21^3 & \text{---} & \bigcirc \\
 & 1^2.41 & \text{---} & 1^2.32 & \text{---} & 1^2.1^5 & \text{---} & \bigcirc \\
 & & ps & & ps & & A_2 &
\end{array}$$

$$\begin{array}{lccccccccccccc}
D_4(q): & 1.3 & \text{---} & 1.21 & \text{---} & 1.1^3 & \text{---} & \bigcirc & \text{---} & .1^4 & \text{---} & .2^2 & \text{---} & .4 \\
D_5(q): & .5 & \text{---} & .2^21 & \text{---} & .21^3 & \text{---} & \bigcirc & \text{---} & 2.1^3 & \text{---} & 2.21 & \text{---} & 2.3 \\
 & .41 & \text{---} & .32 & \text{---} & .1^5 & \text{---} & \bigcirc & \text{---} & 1^2.1^3 & \text{---} & 1^2.21 & \text{---} & 1^2.3 \\
D_6(q): & 1.5 & \text{---} & 1.2^21 & \text{---} & 1.21^3 & \text{---} & \bigcirc & \text{---} & 2.1^4 & \text{---} & 2.2^2 & \text{---} & 2.4 \\
 & 1^2.4 & \text{---} & 1^2.2^2 & \text{---} & 1^2.1^4 & \text{---} & \bigcirc & \text{---} & 1.1^5 & \text{---} & 1.32 & \text{---} & 1.41 \\
D_7(q): & .61 & \text{---} & .32^2 & \text{---} & .31^4 & \text{---} & \bigcirc & \text{---} & 1^3.31 & \text{---} & 21.31 & \text{---} & 3.31 \\
 & .51^2 & \text{---} & .3^21 & \text{---} & .21^5 & \text{---} & \bigcirc & \text{---} & 1^3.21^2 & \text{---} & 21.21^2 & \text{---} & 3.21^2 \\
 & 1^2.5 & \text{---} & 1^2.2^21 & \text{---} & 1^2.21^3 & \text{---} & \bigcirc & \text{---} & 2.1^5 & \text{---} & 2.32 & \text{---} & 2.41 \\
 & & ps & & ps & & A_2 & & A_2 & & ps & & ps &
\end{array}$$

$$\begin{array}{lccccccccccccc}
D_7(q): & D_4{:}\,3. & \text{---} & D_4{:}\,21. & \text{---} & D_4{:}\,1^3. & \text{---} & \bigcirc & \text{---} & D_4{:}\,.1^3 & \text{---} & D_4{:}\,.21 & \text{---} & D_4{:}\,.3 \\
 & & D_4 & & D_4 & & D_4A_2 & & D_4A_2 & & D_4 & & D_4 &
\end{array}$$

PROOF. First, by [**35**, p. 72] the relative Weyl group of A_2 inside D_n is of type B_{n-3}. The parameters of the corresponding Hecke algebra are determined by the parameters inside the minimal Levi subgroups above A_2, viz. those of types D_4 and A_2A_1. Clearly the parameter inside the product $A_2(q)A_1(q)$ is equal to q, by [**28**, Lemma 3.19]. Now the ℓ-modular cuspidal Steinberg character of $L = A_2(q)$ is liftable to an ordinary cuspidal character λ by [**27**, Thm. 7.8], lying in the Lusztig series of a regular semisimple ℓ-element of L^*. Then the parameters of the relative Hecke algebra inside $M = D_4(q)$ can be determined as the ℓ-modular reduction of the quotient of the degrees of the two ordinary constituents of $R_L^M(\lambda)$, see [**28**, Lemma 3.17]. This equals 1 as claimed in (a). Reduction stability holds by Example 2.5(a).

In (b), the cuspidal modular Steinberg character of a Levi subgroup of type A_2^2 is the exterior tensor product of those of the two factors, so is again liftable to characteristic zero by [**27**, Thm. 7.8]. Again by [**35**, p. 72] the relative Weyl group has type as stated. The minimal Levi overgroups here are of types A_5 (twice) inside D_6, and of type D_4A_2 and $A_2^2A_1$ in D_7, D_8 respectively, and the parameters in either case can again be determined using [**28**, Lemma 3.17]. Here, reduction stability holds if we choose the same lift to characteristic 0 in both factors. □

The only unipotent ℓ-blocks of non-cyclic defect of $D_6(q)$ and $D_7(q)$ are their principal blocks.

PROPOSITION 5.3. *The decomposition matrix for the principal ℓ-block of $D_6(q)$ for primes $7 \le \ell | (q^2+q+1)$ is as given in Table 2.*

Here, in the second column we print the degrees of the corresponding unipotent characters as products of cyclotomic polynomials evaluated at q.

TABLE 2. $D_6(q)$, $7 \le \ell | (q^2+q+1)$

$.6$	1	1																	
$.51$	$q^2\Phi_5\Phi_{10}$	1	1																
$3+$	$q^3\Phi_5\Phi_8\Phi_{10}$	.	.	1															
$3-$	$q^3\Phi_5\Phi_8\Phi_{10}$	.	.	.	1														
$.3^2$	$\frac{1}{2}q^4\Phi_5\Phi_6^2\Phi_8\Phi_{10}$	.	1	.	.	1													
21.3	$\frac{1}{2}q^4\Phi_2^4\Phi_5\Phi_6^2\Phi_{10}$	.	.	1	1	.	1												
$.41^2$	$q^6\Phi_5\Phi_8\Phi_{10}$	.	1	.	.	.	.	1											
$1^3.3$	$\frac{1}{2}q^7\Phi_4^2\Phi_5\Phi_6^2\Phi_8$	.	.	.	.	.	1	.	1										
$.321$	$\frac{1}{2}q^7\Phi_2^4\Phi_6^2\Phi_8\Phi_{10}$	1	1	.	.	1	.	1	.	1									
$21+$	$q^7\Phi_4^2\Phi_5\Phi_8\Phi_{10}$	.	.	1	.	.	1	.	.	.	1								
$21-$	$q^7\Phi_4^2\Phi_5\Phi_8\Phi_{10}$	.	.	.	1	.	1	.	.	.	.	1							
$.2^3$	$\frac{1}{2}q^{10}\Phi_5\Phi_6^2\Phi_8\Phi_{10}$	1	.	.	.	.	.	.	.	1	.	.	1						
$1^3.21$	$\frac{1}{2}q^{10}\Phi_2^4\Phi_5\Phi_6^2\Phi_{10}$	.	.	.	.	.	1	.	1	.	1	1	.	1					
$.31^3$	$q^{12}\Phi_5\Phi_8\Phi_{10}$	.	.	.	.	.	.	1	.	1	.	.	.	.	1				
1^3+	$q^{15}\Phi_5\Phi_8\Phi_{10}$	.	.	.	.	.	.	.	.	.	1	.	.	1	.	1			
1^3-	$q^{15}\Phi_5\Phi_8\Phi_{10}$	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1		
$.21^4$	$q^{20}\Phi_5\Phi_{10}$	.	.	.	.	1	.	.	.	1	.	.	1	.	1	.	.	1	
$.1^6$	q^{30}	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1
		ps	ps	ps	ps	ps	ps	ps	A_2	ps	ps	ps	A_2^2	A_2	A_2	A_2^2	A_2^2	A_2	A_2^2

PROOF. By Harish-Chandra induction we obtain the projectives in the principal series as well as those in the A_2-series, which are easily seen to be indecomposable by (HCr). Since the centraliser of a Sylow ℓ-subgroup of $D_6(q)$ is contained in a Levi subgroup of type A_5, there are no cuspidal Brauer characters by (Csp) so the remaining four PIMs must belong to Brauer characters lying in the Harish-Chandra series above the cuspidal Steinberg PIM of a Levi subgroup of type A_2^2. The corresponding Hecke algebra $\mathcal{H}(A_1;q^3)^{\boxtimes 2}$, determined in Lemma 5.2(b), remains semisimple modulo ℓ. Now Harish-Chandra induction also yields the projectives $\Psi_{10}+\Psi_{12}+\Psi_{15}$, $\Psi_{16}+\Psi_{18}$, $\Psi_{11}+\Psi_{12}+\Psi_{16}$, $\Psi_{15}+\Psi_{18}$, and $\Psi_{12}+\Psi_{15}+\Psi_{16}+\Psi_{18}$, where Ψ_i denotes the linear combination of unipotent characters with coefficients as given in the ith column of our table. The only way that these can split into four characters all satisfying (HCr) is the one given. □

PROPOSITION 5.4. *The decomposition matrix for the principal ℓ-block of $D_7(q)$ for primes $7 \leq \ell | (q^2 + q + 1)$, is as given in Table 3.*

TABLE 3. $D_7(q)$, $7 \leq \ell | (q^2 + q + 1)$

.7	1	1																										
1.6	q	.	1																									
1.51	$\frac{1}{2}q^3$	.	1	1																								
.52	$\frac{1}{2}q^3$	1	.	.	1																							
3.4	q^3	.	.	.	.	1																						
.43	$\frac{1}{2}q^4$	.	.	.	1	.	1																					
21.4	$\frac{1}{2}q^4$	.	.	.	.	1	.	1																				
$2^2.3$	$\frac{1}{2}q^6$	.	.	.	.	1	.	.	1																			
1.3^2	$\frac{1}{2}q^6$	.	.	1	.	.	.	.	.	1																		
.421	$\frac{1}{2}q^7$	1	.	.	1	.	1	.	.	.	1																	
$1^3.4$	$\frac{1}{2}q^7$	.	.	.	.	.	.	1	.	.	.	1																
1.41^2	$\frac{1}{2}q^7$	.	.	1	.	.	.	.	.	.	.	.	1															
21.2^2	q^9	.	.	.	.	1	.	1	1	.	.	.	.	1														
1.321	q^9	.	1	1	.	.	.	.	.	1	.	.	1	.	1													
1.2^3	$\frac{1}{2}q^{12}$	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1												
$1^3.2^2$	$\frac{1}{2}q^{12}$	.	.	.	.	.	.	1	.	.	.	1	.	1	.	.	1											
$.41^3$	q^{12}	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1										
1.31^3	$\frac{1}{2}q^{13}$	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	1									
$.321^2$	$\frac{1}{2}q^{13}$	1	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	1	.	1								
$1^4.3$	$\frac{1}{2}q^{13}$	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	1							
$.2^31$	$\frac{1}{2}q^{16}$	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1						
$1^4.21$	$\frac{1}{2}q^{16}$	.	.	.	.	.	.	.	1	.	.	1	.	1	.	.	.	.	.	.	1	.	1					
1.21^4	$\frac{1}{2}q^{21}$	.	.	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	1	.	.	.	.	1				
$.2^21^3$	$\frac{1}{2}q^{21}$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1	.	.	1			
$1^3.1^4$	q^{21}	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	1	.	.	1		
1.1^6	q^{31}	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	
$.1^7$	q^{42}	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1
		ps	ps	ps	ps	ps	ps	ps	ps	ps	ps	A_2	ps	ps	ps	A_2^2	A_2	A_2	A_2	ps	A_2	A_2^2	A_2	A_2	A_2	A_2^2	A_2^2	A_2^2

PROOF. Again, since the centraliser of a Sylow ℓ-subgroup of $D_7(q)$ is contained in a Levi subgroup of type A_5, there are no cuspidal Brauer characters by (Csp) so all Brauer characters must lie in the Harish-Chandra series above cuspidal characters of proper Levi subgroups. We investigate these in turn.

The Hecke algebra $\mathcal{H}(D_7; q)$ for the principal series can be considered as a subalgebra of index 2 of an Iwahori–Hecke algebra $\mathcal{H}(B_7; 1; q)$. The latter is Morita equivalent to a sum of tensor products of Hecke algebras of type A_n, $n \leq 7$, by [**13**, 4.7], and the decomposition matrices of the latter are known by the work of James [**37**, p. 259]. This gives the 14 columns corresponding to the principal series, for all $\ell \geq 7$. Next, the columns Ψ_i, for $i \in \{18, 20, 22, 23, 25\}$ are obtained by Harish-Chandra induction. Further, the projectives $\Psi_3+\Psi_7+\Psi_{17}$, $\Psi_{11}+\Psi_{20}$ and $\Psi_{17}+\Psi_{18}$ yield Ψ_{11} and Ψ_{17}; while $\Psi_9+\Psi_{16}+\Psi_{19}+\Psi_{24}$, $\Psi_{14}+\Psi_{16}+\Psi_{20}+2\Psi_{22}+\Psi_{23}$ and $\Psi_{23}+\Psi_{24}$ yield Ψ_{16} and Ψ_{24}. Harish-Chandra induction also gives the projectives $\Psi_{15}+\Psi_{21}$, $\Psi_{15}+\Psi_{26}$, $\Psi_{21}+\Psi_{27}$ and $\Psi_{26}+\Psi_{27}$ (modulo addition of known projectives). The Hecke algebra $H(B_2;q^3;1)$ for the cuspidal ℓ-modular Steinberg character $\varphi_{1^3}^{\boxtimes 2}$ of a Levi subgroup of type A_2^2 remains semisimple modulo ℓ by Lemma 5.2(b). This yields the remaining four columns of the asserted decomposition matrix. Then (HCr) shows that all columns are indecomposable. □

2. Unipotent decomposition matrix of $E_6(q)$

The triangular shape of the decomposition matrix for the unipotent blocks of $E_6(q)$, when q is a power of a good prime for E_6, has been shown by Geck and Hiss [**25**, 7.5] in the case when $d_\ell(q)=3,6$, by using generalised Gelfand–Graev characters. This shows that (T_ℓ) is satisfied under these assumptions on q. In fact, for $d_\ell(q)=3$, Geck and Hiss [**25**, Table 3] give an approximation to the decomposition matrix involving four unknown entries, plus the unknown decomposition of the two ordinary cuspidal characters.

Lemma 5.5. *Let q be a prime power and $\ell|(q^2+q+1)$. The Hecke algebra for the cuspidal ℓ-modular Steinberg character $\varphi_{1^3}^{\boxtimes 2}$ of a Levi subgroup of type A_2^2 inside $E_6(q)$ is $\mathcal{H}(G_2;q^3;q)$, with four irreducible characters.*

This is easily seen as in the previous cases; reduction stability holds since the normaliser just interchanges the two A_2-factors and we can choose the same Steinberg module in both factors.

Theorem 5.6. *Let $(q,6)=1$. Then the decomposition matrix for the principal ℓ-block of $E_6(q)$ for primes $\ell>3$ with $(q^2+q+1)_\ell>7$ is as given in Table 4.*

Here, the unknown parameters satisfy $a_8\le -1-a_3-a_4+a_6+a_7$ and the relations

$$a_5=1-a_1-a_2+a_3,\quad a_9=3+2a_3+3a_4-3a_6-2a_7+3a_8\quad\textit{and}\quad b_3=3b_2-2b_1-6.$$

There is a further unipotent ℓ-block of cyclic defect with Brauer tree

$$D_4:3 \underset{D_4}{\text{---}} D_4:21 \underset{D_4}{\text{---}} D_4:1^3 \underset{c}{\text{---}} \bigcirc$$

Proof. We explain how to find projective characters Ψ_i, $1\le i\le 24$, with unipotent parts as given in the columns of Table 4. First, (HCi) gives all columns except for those with index $i=9,12,14,15,19,20,21,23,24$. (Alternatively, the ten PIMs in the principal series can also be read off from the decomposition matrix of the Hecke algebra $\mathcal{H}(E_6;q)$, given in [**29**, Tab. 7.13].) Furthermore (HCi) yields $\Psi_9+\Psi_{11}+\Psi_{12}$ and $\Psi_9+\Psi_{11}+\Psi_{21}$. From these and (Tri) we get $\Psi_9+\Psi_{11}$, Ψ_{12} and Ψ_{21}. Harish-Chandra inducing the 16th PIM from a Levi subgroup of type D_6 to $E_7(q)$ and restricting it back to $E_6(q)$ yields $\Psi_9+\Psi_{12}+\Psi_{16}$, which shows that Ψ_9 is a projective character. Finally, (HCi) from a Levi subgroup of type A_5 also yields a projective character $\Psi_{19}'=\Psi_{19}+\Psi_{21}$. Now the decomposition matrix of the Hecke algebra for the A_2^2-series, determined in Lemma 5.5 shows that Ψ_{19}' must have two summands in that series, and then (HCr) leads to the only admissible splitting $\Psi_{19}\oplus\Psi_{21}$.

At this point we have accounted for projectives in all non-cuspidal Harish-Chandra series, so the remaining five PIMs must belong to cuspidal Brauer characters. (A priori,

TABLE 4. $E_6(q)$, $7 < (q^2+q+1)_\ell$, $(q,6)=1$

$\phi_{1,0}$	1	1																							
$\phi_{6,1}$	q	1	1																						
$\phi_{20,2}$	q^2	1	1	1																					
$\phi_{30,3}$	$\frac{1}{2}q^3$	.	1	.	1																				
$\phi_{15,5}$	$\frac{1}{2}q^3$	.	1	.	.	1																			
$\phi_{15,4}$	$\frac{1}{2}q^3$	.	.	1	.	.	1																		
$\phi_{64,4}$	q^4	.	1	1	1	1	.	1																	
$\phi_{60,5}$	q^5	.	1	1	1	.	1	1	1																
$\phi_{24,6}$	q^6	.	.	1	.	.	.	1	.	1															
$\phi_{20,10}$	$\frac{1}{6}q^7$	.	.	.	.	1	.	1	.	.	1														
$\phi_{80,7}$	$\frac{1}{6}q^7$	.	1	.	1	1	.	1	1	.	.	1													
$\phi_{10,9}$	$\frac{1}{3}q^7$	.	1	.	.	.	.	.	1	.	.	.	1												
$\phi_{60,8}$	$\frac{1}{2}q^7$	1	1	1	.	1	1	1	1	1	.	.	.	1											
$E_6[\zeta_3]$	$\frac{1}{3}q^7$	.	.	.	.	.	.	.	.	.	.	.	.	.	1										
$E_6[\zeta_3^2]$	$\frac{1}{3}q^7$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1									
$\phi_{60,11}$	q^{11}	1	1	.	.	1	.	.	1	.	.	1	1	1	a_1	a_1	1								
$\phi_{24,12}$	q^{12}	.	.	.	.	1	.	.	.	.	.	.	.	1	a_2	a_2	.	1							
$\phi_{64,13}$	q^{13}	.	.	.	.	1	.	1	1	1	1	1	.	1	a_3	a_3	1	1	1						
$\phi_{15,16}$	$\frac{1}{2}q^{15}$	1	.	.	.	.	.	.	.	1	.	.	.	1	a_4	a_4	1	.	.	1					
$\phi_{15,17}$	$\frac{1}{2}q^{15}$	.	.	.	.	.	.	1	1	1	1	.	.	.	a_5	a_5	.	.	1	.	1				
$\phi_{30,15}$	$\frac{1}{2}q^{15}$	.	.	.	.	.	.	.	1	.	.	1	1	.	a_6	a_6	1	.	1	.	.	1			
$\phi_{20,20}$	q^{20}	.	.	.	.	.	1	.	1	1	.	.	1	1	a_7	a_7	1	1	1	1	b_1	.	1		
$\phi_{6,25}$	q^{25}	.	.	.	.	.	1	.	1	.	.	.	1	.	a_8	a_8	.	.	1	.	b_2	1	1	1	
$\phi_{1,36}$	q^{36}	.	.	.	.	.	1	.	.	.	.	.	.	.	a_9	a_9	.	.	.	1	b_3	.	1	3	1
		ps	ps	ps	ps	ps	ps	ps	ps	A_2^2	A_2	ps	A_2^2	ps	c	c	A_2	A_2	A_2	A_2^2	c	A_2^2	A_2	c	c

by (St), the modular Steinberg character is known to be cuspidal.) Let us denote the unknown entries below the diagonal of those PIMs as follows:

$E_6[\zeta_3]$	1			
$E_6[\zeta_3^2]$	.	1		
$\phi_{60,11}$	a_1	a_1		
$\phi_{24,12}$	a_2	a_2		
$\phi_{64,13}$	a_3	a_3		
$\phi_{15,16}$	a_4	a_4		
$\phi_{15,17}$	a_5	a_5	1	
$\phi_{30,15}$	a_6	a_6	.	
$\phi_{20,20}$	a_7	a_7	b_1	
$\phi_{6,25}$	a_8	a_8	b_2	1
$\phi_{1,36}$	a_9	a_9	b_3	b_4

Note here that since the ordinary cuspidal unipotent characters $E_6[\zeta_3]$ and $E_6[\zeta_3^2]$ are Galois conjugate, while all other unipotent characters are rational valued, the unknown decomposition numbers in the two corresponding columns must agree.

We now use the combination of (DL) and (Red) to determine some of the unknown entries. We start with the Deligne–Lusztig character R_w associated to a Coxeter element w. The coefficient on Ψ_{21} is $2-2a_1-2a_2+2a_3-2a_5$ and hence is non-negative by (DL). On the other hand, if $\ell > 3$ then (Red) for the cuspidal unipotent character of the 3-split Levi subgroup ${}^3D_4(q).(q^2+q+1)$ gives the relation $-1+a_1+a_2-a_3+a_5 \geq 0$ (see Example 1.7(d)). We deduce that $a_5 = 1-a_1-a_2+a_3$. The coefficients of Ψ_{23} and Ψ_{24} are

$$\begin{aligned} & X = -1-2a_3-2a_4+2a_6+2a_7-2a_8 \geq 0 \\ \text{and} \qquad & Y = 3-2a_3+2a_7-2a_9-b_4X \geq 0. \end{aligned}$$

Note that X cannot be equal to zero by parity. Using explicit computations in Chevie [**26**], one can see that there exists a regular ℓ-element in G^* whenever $(q^2+q+1)_\ell > 7$. Then (Red) applied to the case of a maximal torus gives $b_4 \geq 3$ and $-3-2a_3-3a_4+3a_6+2a_7-3a_8+a_9 \geq 0$. Then

$$\begin{aligned} 0 \leq Y &= 3-2a_3+2a_7-2a_9-b_4X \\ &\leq 3-2a_3+2a_7-2a_9-3X \\ &= 2(3+2a_3+3a_4-3a_6-2a_7+3a_8-a_9) \leq 0 \end{aligned}$$

forces $b_4 = 3$ and $3+2a_3+3a_4-3a_6-2a_7+3a_8-a_9 = 0$.

Finally, the coefficient of Ψ_{24} on the Deligne–Lusztig character R_w associated with $w = s_1s_2s_3s_1s_5s_4s_6s_5s_4s_2s_3s_4$ is equal to $-18-6b_1+9b_2-3b_3$. With (Red) applied to a maximal torus again we have also $6+2b_1-3b_2+b_3 \leq 0$ hence (DL) forces $b_3 = -6-2b_1+3b_2$.

It now follows with (HCr) that all projectives in the table are indecomposable. □

Remark 5.7. As before, we can use the virtual characters Q_w for $w \in W$ to get conjectural upper bounds for the unknown entries. With w being the Coxeter element, we find the following inequalities:

$$\begin{array}{lll} a_1 \leq 13, & a_2 \leq 1, & a_3 \leq 4+a_1+a_2, \\ a_4 \leq 9+a_1, & a_6 \leq 2-a_2+a_3, & a_7 \leq 21-a_1+a_3+a_4, \\ a_8 \leq -1-a_3-a_4+a_6+a_7. & & \end{array}$$

On the other hand, the trivial character of a 2-split Levi subgroup of type $A_2(q)A_2(q).(q^2+q+1)$ (resp. $A_2(q).(q^2+q+1)^2$) gives $a_2 \geq 1$ (resp. $a_6 \geq 2-a_2+a_3$) by (Red). Therefore $a_2 = 1$ and $a_6 = 1+a_3$. From this we deduce that we should have $a_1 \leq 13, a_3 \leq 18, a_4 \leq 22, a_5 \leq 5, a_6 \leq 14, a_7 \leq 48$ and $a_8 \leq 26$.

For the bounds on b_1 and b_2 we use $Q_{w'}$ with $w' = s_1s_2s_3s_1s_4s_3s_1s_5s_4s_3s_1s_6s_5s_4s_3s_1$ from which we get $b_1 \leq 5$ and $b_2 - b_1 \leq 8$ if [**16**, Conj. 1.2] holds. This shows that $b_2 \leq 13$ and $b_3 \leq 23$. Most of these upper bounds are probably not sharp.

3. Even-dimensional non-split orthogonal groups

Next, we consider the unipotent blocks of twisted orthogonal groups ${}^2D_n$, $4 \leq n \leq 7$. Again, the Brauer trees were described in [**20**] (and can readily be determined):

PROPOSITION 5.8. *Let q be a prime power and ℓ a prime with $d_\ell(q) = 3$. The Brauer trees of the unipotent ℓ-blocks of ${}^2D_n(q)$, $4 \le n \le 7$, with cyclic defect are as given in Table 5.*

TABLE 5. Brauer trees for ${}^2D_n(q)$ $(4 \le n \le 7)$, $7 \le \ell|(q^2 + q + 1)$

$$
\begin{array}{rccccccccccccc}
{}^2D_4(q): & 3. & — & 21. & — & 1^3. & — & \bigcirc & — & .1^3 & — & .21 & — & .3 \\
{}^2D_5(q): & 4. & — & 2^2. & — & 1^4. & — & \bigcirc & — & 1.1^3 & — & 1.21 & — & 1.3 \\
 & .4 & — & .2^2 & — & .1^4 & — & \bigcirc & — & 1^3.1 & — & 21.1 & — & 3.1 \\
{}^2D_6(q): & 5. & — & 2^21. & — & 21^3. & — & \bigcirc & — & 2.1^3 & — & 2.21 & — & 2.3 \\
 & .5 & — & .2^21 & — & .21^3 & — & \bigcirc & — & 1^3.2 & — & 21.2 & — & 3.2 \\
 & 41. & — & 32. & — & 1^5. & — & \bigcirc & — & 1^2.1^3 & — & 1^2.21 & — & 1^2.3 \\
 & .41 & — & .32 & — & .1^5 & — & \bigcirc & — & 1^3.1^2 & — & 21.1^2 & — & 3.1^2 \\
 & 4.1 & — & 2^2.1 & — & 1^4.1 & — & \bigcirc & — & 1.1^4 & — & 1.2^2 & — & 1.4 \\
{}^2D_7(q): & 5.1 & — & 2^21.1 & — & 21^3.1 & — & \bigcirc & — & 2.1^4 & — & 2.2^2 & — & 2.4 \\
 & 1.5 & — & 1.2^21 & — & 1.21^3 & — & \bigcirc & — & 1^4.2 & — & 2^2.2 & — & 4.2 \\
 & 41.1 & — & 32.1 & — & 1^5.1 & — & \bigcirc & — & 1^2.1^4 & — & 1^2.2^2 & — & 1^2.4 \\
 & 1.41 & — & 1.32 & — & 1.1^5 & — & \bigcirc & — & 1^4.1^2 & — & 2^2.1^2 & — & 4.1^2 \\
 & & ps & & ps & & A_2 & & A_2 & & ps & & ps &
\end{array}
$$

Here and later, the unipotent characters of ${}^2D_n(q)$ in the principal series are denoted by the corresponding character of the Weyl group, which in this case is of type B_{n-1}, hence by bipartitions of $n - 1$. Note that the order of ${}^2D_n(q)$ with $n \le 3$ is not divisible by primes ℓ with $d_\ell(q) = 3$.

Again, we first collect some information on Hecke algebras of cuspidal characters.

LEMMA 5.9. *Let q be a prime power and $\ell|(q^2 + q + 1)$.*

(a) *The Hecke algebra for the cuspidal ℓ-modular Steinberg character φ_{1^3} of a Levi subgroup of type A_2 inside ${}^2D_n(q)$, $n \ge 4$, is $\mathcal{H}(A_1; 1) \otimes \mathcal{H}(B_{n-4}; q^2; q)$.*

(b) *The Hecke algebra for the cuspidal ℓ-modular Steinberg character $\varphi_{1^3}^{\boxtimes 2}$ of a Levi subgroup of type A_2^2 inside ${}^2D_n(q)$, $n \ge 7$, is $\mathcal{H}(B_2; q^3; 1) \otimes \mathcal{H}(B_{n-7}; q^2; q)$.*

PROOF. Note that ${}^2D_n(q)$ has Weyl group of type B_{n-1}. By [**35**, p. 70] the relative Weyl group of A_2 inside B_{n-1} is of type A_1B_{n-4}, and that of A_2^2 is of type B_2B_{n-7}. The cuspidal characters in question are the same as those in Lemma 5.2 and thus lift to characteristic zero. The relevant minimal Levi overgroups in (a) are of types ${}^2D_4$ when $n = 4$, ${}^2D_2A_2$ when $n = 5$ and A_2A_1 when $n = 6$, leading to the parameters $1, q^2, q$

respectively. In (b) they are of types A_5, ${}^2D_4A_2$ when $n = 7$, ${}^2D_2A_2^2$ when $n = 8$ and $A_2^2A_1$ when $n = 9$, leading to the parameters $q^3, 1, q^2$ and q. The argument is now completely analogous to the one employed in the proof of Lemma 5.2.

The Steinberg character of a Levi subgroup of type A_2 is reduction stable by Example 2.5(a), and for a Levi subgroup A_2^2 we take the same Steinberg module in both factors. $\square$

The Sylow ℓ-subgroups of ${}^2D_6(q)$ are cyclic when $d_\ell(q) = 3$, so the smallest rank not covered by Proposition 5.8 is $n = 7$:

PROPOSITION 5.10. *The decomposition matrix for the principal ℓ-block of ${}^2D_7(q)$ for primes $7 \leq \ell | (q^2 + q + 1)$ is as given in Table 6.*

TABLE 6. ${}^2D_7(q)$, $7 \leq \ell | (q^2 + q + 1)$

6.	1	1																										
51.	q	1	1																									
.6	$\frac{1}{2}q^3$	.	.	1																								
41^2.	$\frac{1}{2}q^3$	.	1	.	1																							
3^2.	q^3	.	1	.	.	1																						
3.3	$\frac{1}{2}q^4$	.	.	.	.	.	1																					
321.	$\frac{1}{2}q^4$	1	1	.	1	1	.	1																				
2^3.	$\frac{1}{2}q^6$	1	.	.	.	.	.	1	1																			
21.3	$\frac{1}{2}q^6$	.	.	.	.	.	1	.	.	1																		
.51	$\frac{1}{2}q^7$	.	.	1	.	.	.	.	.	.	1																	
31^3.	$\frac{1}{2}q^7$	.	.	.	1	.	.	1	.	.	.	1																
3.21	$\frac{1}{2}q^7$	.	.	.	.	.	1	.	.	.	.	.	1															
1^3.3	q^9	.	.	.	.	.	.	.	.	1	.	.	.	1														
21.21	q^9	.	.	.	.	.	1	.	.	1	.	.	1	.	1													
$.3^2$	$\frac{1}{2}q^{12}$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1												
1^3.21	$\frac{1}{2}q^{12}$	.	.	.	.	.	.	.	.	1	.	.	.	1	1	.	1											
3.1^3	q^{12}	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1										
$.41^2$	$\frac{1}{2}q^{13}$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1									
21^4.	$\frac{1}{2}q^{13}$	.	.	.	.	1	.	1	1	.	.	1	.	.	.	.	.	.	.	1								
21.1^3	$\frac{1}{2}q^{13}$	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	.	.	1							
$1^3.1^3$	$\frac{1}{2}q^{16}$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	1	1						
.321	$\frac{1}{2}q^{16}$	.	.	1	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	.	.	1					
1^6.	$\frac{1}{2}q^{21}$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1				
$.31^3$	$\frac{1}{2}q^{21}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1			
$.2^3$	q^{21}	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1		
$.21^4$	q^{31}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	1	1	1	
$.1^6$	q^{42}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	1
		ps	ps	ps	ps	ps	ps	ps	A_2^2	ps	ps	A_2	ps	A_2	ps	ps	A_2	A_2	ps	A_2	A_2	A_2^2	ps	A_2^2	A_2	A_2^2	A_2	A_2^2

PROOF. First, observe that the centraliser of a Sylow ℓ-subgroup of ${}^2D_7(q)$ is contained in a Levi subgroup of type A_5, so there do not exist cuspidal unipotent ℓ-modular Brauer characters by (Csp). The Hecke algebra for the principal series is $\mathcal{H}(B_6; q^2; q)$ (see e.g. [**8**, p. 464]). By [**13**, 4.7] it is Morita equivalent to a sum of products of Hecke algebras of type A_n, $n \leq 6$, whose decomposition matrices are known by [**37**, p. 259] for all $\ell \geq 7$. This yields the columns of the principal series PIMs. All columns labelled A_2 are obtained by (HCi). Then (HCr) shows that these are indeed indecomposable. Finally, by Lemma 5.9(b) the Hecke algebra for the cuspidal ℓ-modular Steinberg character of a Levi subgroup of type A_2^2 is $\mathcal{H}(B_2; q^3; 1)$ and hence semisimple modulo ℓ. Then splitting up the Harish-Chandra induction of the corresponding PIMs using (HCr) yields the last five missing columns. □

4. Symplectic and odd-dimensional orthogonal groups

Next, we consider the symplectic groups $C_n(q)$ and the odd-dimensional orthogonal groups $B_n(q)$ with $n \leq 6$. Note that according to Lusztig's classification, the unipotent characters of both series of groups are parametrised in the same way, see e.g. [**8**, §13.8].

PROPOSITION 5.11. *Let q be a prime power and ℓ a prime with $d_\ell(q) = 3$. The Brauer trees of the unipotent ℓ-blocks of $B_n(q)$ and $C_n(q)$, $3 \leq n \leq 6$, with cyclic defect are the same as for the unipotent ℓ-blocks of ${}^2D_{n+1}(q)$ in Table 5, plus the three additional trees given in Table 7.*

TABLE 7. Brauer trees for $B_n(q)$ and $C_n(q)$ $(5 \leq n \leq 6)$, $7 \leq \ell | (q^2 + q + 1)$

$B_5(q)$:	$B_2{:}\,3.$	—	$B_2{:}\,21.$	—	$B_2{:}\,1^3.$	—	$\bigcirc$	—	$B_2{:}\,.1^3$	—	$B_2{:}\,.21$	—	$B_2{:}\,.3$
$B_6(q)$:	$B_2{:}\,4.$	—	$B_2{:}\,2^2.$	—	$B_2{:}\,1^4.$	—	$\bigcirc$	—	$B_2{:}\,1.1^3$	—	$B_2{:}\,1.21$	—	$B_2{:}\,1.3$
	$B_2{:}\,3.1$	—	$B_2{:}\,21.1$	—	$B_2{:}\,1^3.1$	—	$\bigcirc$	—	$B_2{:}\,.1^4$	—	$B_2{:}\,.2^2$	—	$B_2{:}\,.4$
	B_2		B_2		B_2A_2		B_2A_2		B_2		B_2		

Here, B_2A_2 denotes the Harish-Chandra series of the cuspidal unipotent ℓ-modular Brauer character $B_2 \boxtimes \varphi_{1^3}$ of a Levi subgroup of type B_2A_2.

PROPOSITION 5.12. *The decomposition matrix for the principal ℓ-block of $B_6(q)$ and of $C_6(q)$ for primes $7 \leq \ell | (q^2 + q + 1)$ is as given in Table 8.*

PROOF. The arguments are exactly the same for $G = B_6(q)$ and $G = C_6(q)$. All columns and their respective Harish-Chandra series except those labelled A_2^2 are directly obtained by (HCi). Since the centraliser of a Sylow ℓ-subgroup of G is contained inside a Levi subgroup of type A_5, there are no cuspidal PIMs by (Csp). Thus the five missing PIMs belong to Brauer characters in the series of the cuspidal Steinberg character of a Levi subgroup of type A_2^2. With (HCi) we find projectives with unipotent parts

$$\Psi_{12} + \Psi_{21} + \Psi_{25}, \qquad \Psi_{21} + \Psi_{22} + \Psi_{27},$$
$$\Psi_{11} + \Psi_{12} + \Psi_{21} + \Psi_{22} \quad \text{and} \quad \Psi_{21} + \Psi_{25} + \Psi_{27}.$$

TABLE 8. $B_6(q)$ and $C_6(q)$, $7 \leq \ell|(q^2+q+1)$

6.	1																										
.6	.	1																									
51.	1	.	1																								
$3^2.$	.	.	1	1																							
3.3	.	.	.	.	1																						
.51	.	1	.	.	.	1																					
$41^2.$	.	.	1	.	.	.	1																				
321.	1	.	1	1	.	.	1	1																			
3.21	.	.	.	.	1	.	.	.	1																		
21.3	.	.	.	.	1	.	.	.	.	1																	
21.21	.	.	.	.	1	.	.	.	1	1	1																
$2^3.$	1	.	.	.	.	.	.	1	.	.	.	1															
$1^3.3$	.	.	.	.	.	.	.	.	.	1	.	.	1														
$.3^2$	.	.	.	.	.	1	.	.	.	.	.	.	.	1													
$31^3.$	.	.	.	.	.	.	1	1	.	.	.	.	.	.	1												
3.1^3	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1											
$.41^2$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1										
.321	.	1	.	.	.	1	.	.	.	.	.	.	.	1	.	.	1	1									
21.1^3	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.	1								
$1^3.21$	.	.	.	.	.	.	.	.	.	1	1	.	1	.	.	.	.	.	.	1							
$1^3.1^3$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	1	1						
$.2^3$	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1					
$21^4.$	.	.	.	1	.	.	.	1	.	.	.	1	.	.	1	.	.	.	.	.	.	.	1				
$.31^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	1			
$1^6.$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1		
$.21^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	1	.	1	.	1	
$.1^6$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	1
	ps	ps	ps	ps	ps	ps	ps	ps	ps	ps	ps	A_2^2	A_2	ps	A_2	A_2	ps	ps	A_2	A_2	A_2^2	A_2^2	A_2	A_2	A_2^2	A_2	A_2^2

The only way these can decompose into sums of unipotent characters consistent with (HCr) is the one given in the table. The indecomposability of all listed projectives also readily follows with (HCr). □

5. Unipotent decomposition matrix of $F_4(q)$

THEOREM 5.13. *Let $(q,6)=1$. Then the decomposition matrix for the principal ℓ-block of $F_4(q)$ for primes $\ell \geq 7$ with $(q^2+q+1)_\ell > 7$ is approximated as given in Table 9.*

Here the unknown parameters satisfy $x_1 \geq 2$, $y_1, y_2 \geq 1$, $y_6 = 3-2y_1-2y_2+y_3+y_4+2y_5$ and $y_5 \leq y_1+y_2-1$.

PROOF. The triangular shape of the decomposition matrix when $(q,6)=1$ was shown by Köhler [**38**, Tab. A.160]. He also proves that the ordinary cuspidal character $F_4^{II}[1]$ can only occur in the modular reduction of $\phi_{4,13}$ or $\phi_{1,24}$. We will denote their multiplicities as x_1 and x_2 respectively. When $(q^2+q+1)_\ell > 7$ they satisfy $x_1 \geq 1$ and $3+x_2-2x_1 \geq 0$.

TABLE 9. $F_4(q)$, $(q^2+q+1)_\ell > 7$, $(q,6)=1$

$\phi_{1,0}$	1	1																				
$\phi'_{2,4}$	$\frac{1}{2}q$	1	1																			
$\phi''_{2,4}$	$\frac{1}{2}q$	1	.	1																		
$\phi_{4,1}$	$\frac{1}{2}q$	.	.	.	1																	
$\phi'_{8,3}$	q^3	.	.	.	1	1																
$\phi''_{8,3}$	q^3	.	.	.	1	.	1															
$F_4^{II}[1]$	$\frac{1}{24}q^4$	.	.	.	.	.	.	1														
$\phi'_{1,12}$	$\frac{1}{8}q^4$	.	1	.	.	.	.	.	1													
$\phi''_{1,12}$	$\frac{1}{8}q^4$	.	.	1	.	.	.	.	.	1												
$\phi_{4,8}$	$\frac{1}{8}q^4$	1	1	1	.	.	.	.	.	.	1											
$\phi'_{4,7}$	$\frac{1}{4}q^4$	.	.	.	.	1	.	.	.	.	.	1										
$\phi''_{4,7}$	$\frac{1}{4}q^4$	.	.	.	.	.	1	.	.	.	.	.	1									
$\phi_{16,5}$	$\frac{1}{4}q^4$	.	.	.	1	1	1	.	.	.	.	.	.	1								
$F_4[\zeta_3]$	$\frac{1}{3}q^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	1							
$F_4[\zeta_3^2]$	$\frac{1}{3}q^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1						
$\phi'_{8,9}$	q^9	.	.	.	.	1	.	.	.	.	.	1	.	1	y_1	y_1	1					
$\phi''_{8,9}$	q^9	.	.	.	.	.	1	.	.	.	.	.	1	1	y_2	y_2	.	1				
$\phi'_{2,16}$	$\frac{1}{2}q^{13}$	.	1	.	.	.	.	.	1	.	1	.	.	.	y_3	y_3	.	.	1			
$\phi''_{2,16}$	$\frac{1}{2}q^{13}$	.	.	1	.	.	.	.	.	1	1	.	.	.	y_4	y_4	.	.	.	1		
$\phi_{4,13}$	$\frac{1}{2}q^{13}$	.	.	.	.	.	.	x_1	.	.	.	.	.	1	y_5	y_5	1	1	.	.	1	
$\phi_{1,24}$	q^{24}	.	.	.	.	.	.	$2x_1{-}3$	.	.	1	.	.	.	y_6	y_6	.	.	1	1	2	1
		ps	ps	ps	ps	ps	ps	c	A_2	$\widetilde{A}_2$	ps	A_2	$\widetilde{A}_2$	ps	c	c	A_2	$\widetilde{A}_2$	A_2	$\widetilde{A}_2$	c	c

Here $\widetilde{A}_2$ denotes the Harish-Chandra series above the Steinberg PIM of $A_2 < C_3$ in F_4.

Similarly, he proved that the Brauer character for $\phi_{4,13}$ can only possibly occur in the modular reduction of the Steinberg character $\phi_{1,24}$, with a multiplicity z that satisfies $z \geq 2$ whenever $(q^2+q+1)_\ell > 7$. The multiplicities of unipotent characters in the PIMs corresponding to the two Galois conjugate cuspidal characters $F_4[\zeta_3]$ and $F_4[\zeta_3^2]$ will be denoted by $y_1, \ldots, y_6$, so that we have

$$\begin{aligned} &\Psi_{14} = F_4[\zeta_3] + y_1\phi'_{8,9} + y_2\phi''_{8,9} + y_3\phi'_{2,16} + y_4\phi''_{2,16} + y_5\phi_{4,13} + y_6\phi_{1,24} \\ \text{and}\quad &\Psi_{15} = F_4[\zeta_3^2] + y_1\phi'_{8,9} + y_2\phi''_{8,9} + y_3\phi'_{2,16} + y_4\phi''_{2,16} + y_5\phi_{4,13} + y_6\phi_{1,24}. \end{aligned}$$

Under the assumption on ℓ, we can use (Red) in the case of a maximal torus to get the relation $-3 + 2y_1 + 2y_2 - y_3 - y_4 - 2y_5 + y_6 \geq 0$ which will be useful in the sequel.

Let w be a Coxeter element and R_w be the corresponding Deligne–Lusztig character. The multiplicities of Ψ_{20} and Ψ_{21} in R_w are respectively

$$\begin{aligned} 2X &= -2 + 2y_1 + 2y_2 - 2y_5 \geq 0 \\ \text{and}\qquad Y &= 2 + 2y_3 + 2y_4 - 2y_6 - zX \geq 0. \end{aligned}$$

Since $z \geq 2$ we have

$$Y \geq 2 + 2y_3 + 2y_4 - 2y_6 - 2X = 6 - 4y_1 - 4y_2 + 2y_3 + 2y_4 + 4y_5 - 2y_6$$

which is non-positive by (Red). Therefore it must be zero, and we must also have $X = 0$ or $z = 2$. In any case, the coefficient of Ψ_{21} is zero.

The coefficient of Ψ_{21} in R_w with $w = s_1s_2s_3s_4s_2s_3$ is $2 - z$, hence $z = 2$ by (DL). The coefficient on R_w with $w = s_1s_2s_3s_4s_1s_2s_3s_4s_1s_2s_3s_4$ is $-3 + 2x_1 - x_2$ and therefore $x_2 = 2x_1 - 3$ by (DL) and the previous inequality on the x_i's.

Finally, using (Red) with a the trivial character of the 2-split Levi of type $A_2.(q^2+q+1)$ we get $y_1 \geq 1$ and $y_2 \geq 1$. ☐

REMARK 5.14. As before the virtual characters Q_w for $w \in W$ provide conjectural upper bounds for the remaining unknown entries. We start with $w = s_1s_2s_1s_4s_3s_2s_1$, and we compute

$$\begin{aligned}
\langle Q_w; \varphi_{16}\rangle &= 12 - 12y_1,\\
\langle Q_w; \varphi_{17}\rangle &= 12 - 12y_2, ,\\
\langle Q_w; \varphi_{18}\rangle &= 12 - 12y_3,\\
\langle Q_w; \varphi_{19}\rangle &= 18 - 12y_4.
\end{aligned}$$

If [**16**, Conj. 1.2] holds then this forces $y_1, y_2, y_3, y_4 \leq 1$. Using the relations in Theorem 5.13 we deduce that $y_1 = y_2 = 1$, $y_5 \leq 1$ and $y_6 \leq 3$. Furthermore, with $w' = s_2s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_2s_4s_3s_2$ we have $\langle Q_{w'}; \varphi_{20}\rangle = 176 - 8x_1$ from which we get $x_1 \leq 22$.

We close this section by collecting in Table 10 some data on the Φ_3-modular Harish-Chandra series in the principal ℓ-blocks B_0 considered above. Here $W_G(B_0)$ denotes the relative Weyl group of a Sylow Φ_3-torus of the ambient group G (which contains a Sylow ℓ-subgroup of G and hence a defect group of B_0). This is known to be a complex reflection group; more concretely, here it is one of two imprimitive complex reflection groups of rank 2, denoted $G(6,1,2)$ and $G(6,2,2)$, respectively the primitive reflection groups G_5, G_{25} and G_{26}.

TABLE 10. Modular Harish-Chandra-series in Φ_3-blocks

G	$W_G(B_0)$	$\lvert\mathrm{IBr}\, B_0\rvert$	ps	A_2	A_2^2	E_6	c
D_6	$G(6,2,2)$	18	10	4	4		
$B_6, C_6, D_7, {}^2D_7$	$G(6,1,2)$	27	14	8	5		
E_6	G_{25}	24	10	5	4	5	
E_7	G_{26}	48	20	10	8	10	
$F_4, {}^2E_6$	G_5	21	8	4	4		5

It can be observed from the decomposition matrices that the principal ℓ-blocks of $D_7(q)$ and ${}^2D_7(q)$ are not Morita equivalent, even though their modular Harish-Chandra series distribution agrees. On the other hand, the decomposition matrices for the principal ℓ-blocks of ${}^2D_7(q)$, $B_6(q)$ and $C_6(q)$ agree after a suitable simultaneous permutation of rows and columns of the one of ${}^2D_7(q)$, so these blocks might be Morita equivalent.

CHAPTER 6

Decomposition matrices at $d_\ell(q) = 6$

Now, we consider unipotent decomposition matrices for groups $G = G(q)$ at primes ℓ with $d_\ell(q) = 6$, so $\ell|(q^2 - q + 1)$ and $\ell \geq 7$. For groups of classical type, such primes are what is called *unitary* for G, so the theory of q-Schur algebras does not apply and the decomposition matrices are not even understood theoretically.

While our assumption already implies that $\ell \geq 7$, we will often need to make a further assumption on $(q^2 - q + 1)_\ell$ in order to use (Red).

1. Even-dimensional split orthogonal groups

As in the previous sections we begin with groups of type D_n, $4 \leq n \leq 7$. The Brauer trees for the unipotent ℓ-blocks of cyclic defect can easily be obtained by Harish-Chandra induction and are well-known [**20**]. For completeness and better reference we collect them here:

PROPOSITION 6.1. *Let q be a prime power and ℓ a prime with $d_\ell(q) = 6$. The Brauer trees of the unipotent ℓ-blocks of $D_n(q)$, $4 \leq n \leq 7$, with cyclic defect are as given in Table 1.*

TABLE 1. Brauer trees for $D_n(q)$ $(4 \leq n \leq 7)$, $7 \leq \ell|(q^2 - q + 1)$

$D_4(q)$:	$.4$	—	1.3	—	$1^2.2$	—	$1^3.1$	—	$1^4.$	—	$\bigcirc$	—	D_4
$D_5(q)$:	$5.$	—	3.2	—	2.21	—	1.21^2	—	$.21^3$	—	$\bigcirc$	—	$D_4{:}\,2$
	$.41$	—	1.31	—	$1^2.21$	—	$1^3.1^2$	—	$1^5.$	—	$\bigcirc$	—	$D_4{:}\,1^2$
$D_6(q)$:	$.42$	—	1.32	—	$1^2.2^2$	—	$1^2.1^4$	—	1.1^5	—	$\bigcirc$	—	$D_4{:}\,2.$
	1.5	—	2.4	—	2.2^2	—	1.2^21	—	$.2^21^2$	—	$\bigcirc$	—	$D_4{:}\,.1^2$
$D_7(q)$:	$.61$	—	3.31	—	31.21	—	$31^2.1^2$	—	$.31^4$	—	$\bigcirc$	—	$D_4{:}\,1.2$
	$.51^2$	—	2.31^2	—	21.21^2	—	$21^2.1^3$	—	$21^5.$	—	$\bigcirc$	—	$D_4{:}\,1^2.1$
	5.1^2	—	4.21	—	3.2^2	—	1.2^3	—	$.2^31$	—	$\bigcirc$	—	$D_4{:}\,.1^3$
	$.43$	—	1.3^2	—	$1^3.2^2$	—	$1^4.21$	—	$1^5.2$	—	$\bigcirc$	—	$D_4{:}\,3.$
		ps		ps		ps		ps		$.1^4$		D_4	

Here and later on, "$.1^4$" stands for the cuspidal unipotent ℓ-modular Brauer character of a Levi subgroup of type D_4 labelled by the unordered bipartition $(-; 1^4)$, and "D_4" for the ℓ-modular reduction of the cuspidal unipotent character of a Levi subgroup of type D_4.

To treat the blocks with non-cyclic defect, again we now first determine the parameters of certain relative Hecke algebras:

LEMMA 6.2. *Let q be a prime power and $\ell|(q^2 - q + 1)$. The Hecke algebras of various ℓ-modular cuspidal pairs (L, λ) of Levi subgroups L in $D_n(q)$ and their respective numbers of irreducible characters are as given in Table 2.*

TABLE 2. Hecke algebras and $|\mathrm{Irr}\,\mathcal{H}|$ in $D_n(q)$ for $d_\ell(q) = 6$

(L, λ)	$\mathcal{H}$	$n = 6$	7	8
(D_4, D_4)	$\mathcal{H}(B_{n-4}; q^4; q)$	2	4	10
$(D_4, \varphi_{.1^4})$	$\mathcal{H}(B_{n-4}; q^2; q)$	2	4	10
(A_5, φ_{1^6})	$\mathcal{H}(A_1; q^3) \otimes \mathcal{H}(D_{n-6}; q)$	1	1	4
$(D_6, \varphi_{.1^6})$	$\mathcal{H}(B_{n-6}; q; q)$	1	2	5

PROOF. In the first three cases, the cuspidal Brauer character lies in a block with cyclic defect (see the Brauer tree in Table 1), and hence reduction stability follows from Example 2.5(a).

A Levi subgroup L of type D_4 has relative Weyl group of type B_{n-4} inside D_n. The cuspidal Brauer character $\varphi_{.1^4}$ is a constituent of the ℓ-modular reduction of an ordinary cuspidal character λ of L lying in the Lusztig series of an ℓ-element s with centraliser $(q^3 + 1)(q + 1)$. By Corollary 2.4, the Hecke algebra for $\varphi_{.1^4}$ is the same as for λ. The minimal Levi overgroups are of types D_5 when $n \geq 5$, where s has centraliser ${}^2D_2(q)(q^3+1)$, and D_4A_1 when $n \geq 6$, which gives the parameters q^2 and q.

The relative Weyl group of a Levi subgroup of type A_5 in D_n is of type A_1D_{n-6}, by [**35**, p. 72]. As can be seen from the Brauer tree, the ℓ-modular Steinberg character φ_{1^6} of A_5 is liftable, so again the parameter q^3 for the Hecke algebra is determined inside the minimal Levi overgroups of types D_6 and A_5A_1.

Finally, the relative Weyl group of D_6 inside D_7 has type A_1. The modular Steinberg character $\varphi_{.1^6}$ lifts to a cuspidal Deligne–Lusztig character, so it is reduction stable by Example 2.5(c), and its Hecke algebra is the ℓ-modular reduction of an Iwahori–Hecke algebra in characteristic 0. The parameters are determined already inside Levi subgroups of types D_7, D_6A_1. □

PROPOSITION 6.3. *Assume (T_ℓ). The decomposition matrix for the principal ℓ-block of $D_6(q)$ for primes ℓ with $(q^2 - q + 1)_\ell > 7$ is as given in Table 3.*

PROOF. Let us write Ψ_i, $1 \leq i \leq 18$, for the linear combinations of unipotent characters given by the columns of Table 3. We need to show that these are restrictions to the principal block of ℓ-modular PIMs of $G = D_6(q)$. Note that the unipotent decomposition matrices of all proper Levi subgroups are known, either by Proposition 6.1 or by [**37**].

TABLE 3. $D_6(q)$, $(q^2-q+1)_\ell > 7$

$.6$	1	1																	
$.51$	$q^2\Phi_5\Phi_{10}$	1	1																
$3+$	$q^3\Phi_5\Phi_8\Phi_{10}$	1	.	1															
$3-$	$q^3\Phi_5\Phi_8\Phi_{10}$	1	.	.	1														
2.31	$\frac{1}{2}q^4\Phi_3^2\Phi_5\Phi_8\Phi_{10}$	1	1	1	1	1													
$D_4\colon 1^2.$	$\frac{1}{2}q^4\Phi_1^4\Phi_3^2\Phi_5\Phi_{10}$	.	.	.	.	.	1												
$.41^2$	$q^6\Phi_5\Phi_8\Phi_{10}$	.	1	.	.	.	.	1											
1.31^2	$\frac{1}{2}q^7\Phi_3^2\Phi_4^2\Phi_8\Phi_{10}$	.	1	.	.	1	.	1	1										
$D_4\colon 1.1$	$\frac{1}{2}q^7\Phi_1^4\Phi_3^2\Phi_5\Phi_8$	.	.	.	.	.	1	.	.	1									
$21+$	$q^7\Phi_4^2\Phi_5\Phi_8\Phi_{10}$	.	.	1	.	1	.	.	.	.	1								
$21-$	$q^7\Phi_4^2\Phi_5\Phi_8\Phi_{10}$	.	.	.	1	1	.	.	.	.	.	1							
$1^2.21^2$	$\frac{1}{2}q^{10}\Phi_3^2\Phi_5\Phi_8\Phi_{10}$	.	.	.	.	1	.	.	1	.	1	1	1						
$D_4\colon .2$	$\frac{1}{2}q^{10}\Phi_1^4\Phi_3^2\Phi_5\Phi_{10}$	.	.	.	.	.	.	.	.	1	.	.	.	1					
$.31^3$	$q^{12}\Phi_5\Phi_8\Phi_{10}$	.	.	.	.	.	.	1	1	.	.	.	.	.	1				
1^3+	$q^{15}\Phi_5\Phi_8\Phi_{10}$	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1			
1^3-	$q^{15}\Phi_5\Phi_8\Phi_{10}$	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	1		
$.21^4$	$q^{20}\Phi_5\Phi_{10}$	.	.	.	.	.	.	.	1	.	.	.	1	.	1	.	.	1	
$.1^6$	q^{30}	.	.	.	.	.	.	.	.	.	.	.	1	2	.	1	1	1	1
		ps	ps	ps	ps	ps	D_4	ps	ps	D_4	ps	ps	ps	c	$.1^4$	A_5	A_5'	$.1^4$	c

Projectives Ψ_i with $i \neq 13, 18$ are obtained by Harish-Chandra induction of PIMs from proper Levi subgroups. This accounts, in particular, for all PIMs in the principal series, which can be seen to be indecomposable from the decomposition matrix of the Hecke algebra, computed with the programme of N. Jacon [**36**]. As a Sylow ℓ-subgroup of G is not contained in any proper Levi subgroup, by (St) the ℓ-modular reduction of the Steinberg character contains a cuspidal Brauer character. The relative Hecke algebra $\mathcal{H}(B_2; q^4; q)$ for the cuspidal Brauer character D_4 of a Levi subgroup of type D_4 (see Lemma 6.2) has two simple modules in characteristic ℓ, so the corresponding modular Harish-Chandra series only contains two PIMs. Furthermore, the two projectives obtained by Harish-Chandra induction of $\Psi_{.1^4}$ from $D_5(q)$ are indecomposable by (HCr), as are those in the A_5-series, so there are no further PIMs in those series. Hence the remaining PIM must be cuspidal. From uni-triangularity of the decomposition matrix, we deduce that it will have the form

$$[D_4\colon .2] + a_1[1^3+] + a_1[1^3-] + a_2[.21^4] + a_3[.1^6].$$

Here we use that the graph automorphism of G interchanges the two unipotent characters labelled 1^3+, 1^3- but fixes $[D_4\colon .2]$ and hence the corresponding two entries in that column must agree. Note also that the family consisting of the unipotent character $[.31^3]$ is not comparable to the family containing $[D_4 : .2]$ which explains why it does not appear in the previous projective character. The coefficient of Ψ_{18} in the Deligne–Lusztig character R_w for w a Coxeter element is $2+2a_1+a_2-a_3$, and therefore by (DL) it must be non-negative.

On the other hand we have $2 + 2a_1 + a_2 - a_3 \leq 0$ and therefore $a_3 = a_2 + 2a_1 + 2$ provided that we can use (Red), that is when there exists a semisimple regular ℓ-element of G^*. By Example 1.7(b) such an element exists whenever $(q^2-q+1)_\ell > 12$ (in particular whenever $(q^2 - q + 1)_\ell > 7$ since we also assumed $\ell \geq 7$).

It will be a consequence of the determination of the decomposition matrices for $D_8(q)$ in Proposition 6.5 that in fact $a_1 = a_2 = 0$ and thus $a_3 = 2$, completing the proof. □

PROPOSITION 6.4. *Assume* (T_ℓ). *The decomposition matrix for the principal ℓ-block of $D_7(q)$ for primes ℓ with $(q^2 - q + 1)_\ell > 7$ is as given in Table 4.*

Here D_6^s denotes the Harish-Chandra series of the cuspidal ℓ-modular constituent of the ℓ-modular reduction of the unipotent character $[D_4\colon .2]$ of $D_6(q)$.

PROOF. As before, let's denote by Ψ_i, $1 \leq i \leq 27$, the linear combinations of unipotent characters corresponding to the columns of Table 4. Those Ψ_i with i not equal to

$$1,\ 15,\ 16,\ 19,\ 22,\ 24,\ 26 \text{ and } 27$$

are obtained by (HCi). Furthermore, by uni-triangularity the projectives $\Psi_{17} + \Psi_{19}$, $\Psi_{18} + \Psi_{19}$ yield Ψ_{19}, and $\Psi_{21} + \Psi_{24}$, $\Psi_{23} + \Psi_{24}$ yield Ψ_{24}. Since a Levi subgroup L of type D_6 contains the centraliser of a Sylow ℓ-subgroup of $D_7(q)$, there are no cuspidal Brauer characters by (Csp) and in particular the Harish-Chandra induction of the Steinberg PIM of L splits off the Steinberg PIM of G; this yields Ψ_{26} and Ψ_{27}.

The projective cover of the trivial character lies in the principal series and is determined by the decomposition matrix of the Hecke algebra $\mathcal{H}(D_7; q)$; this yields Ψ_1 (HC-induction only gives $\Psi_1 + \Psi_2$), and similarly we obtain Ψ_{15} (HC-induction only yields $\Psi_{15} + \Psi_{21}$). Now the Harish-Chandra induction to G of the PIM of the ℓ-modular cuspidal unipotent character D_6^s of a Levi subgroup of type D_6 equals

$$\Psi = [D_4\colon .3] + [D_4\colon .21] + a_2[1.21^4] + a_2[.2^21^3] + 2a_1[1^3.1^4] + a_3[1.1^6] + a_3[.1^7].$$

As we have accounted for all other Harish-Chandra series, and there are no cuspidal unipotent Brauer characters, this projective character has to have two summands in that series, which by uni-triangularity must start at $[D_4\colon .3]$ and $[D_4\colon .21]$ respectively. The only possibility for splitting Ψ into two summands compatible with (HCr) is

$$\tilde\Psi_{16} = [D_4\colon .3] + (a_2 - z_2)[1.21^4] + z_2[.2^21^3] + a_1[1^3.1^4] + (a_3 - z_3)[1.1^6] + z_3[.1^7],$$
$$\tilde\Psi_{22} = [D_4\colon .21] + z_2[1.21^4] + (a_2 - z_2)[.2^21^3] + a_1[1^3.1^4] + z_3[1.1^6] + (a_3 - z_3)[.1^7],$$

with suitable non-negative integers $z_2 \leq a_2$ and $z_3 \leq a_3$. It will be a consequence of the determination of the decomposition matrices for the unipotent blocks of $D_8(q)$ in Proposition 6.5 that in fact $a_1 = a_2 = z_2 = 0$ and thus $a_3 = 2$. This accounts for the last two missing columns Ψ_{16} and Ψ_{22}.

Finally, the coefficient of Ψ_{26} on R_w when w is a Coxeter element is $2z_4$. Since Ψ_{26} does not occur in any R_v for $v < w$, we deduce from (DL) that $z_4 \leq 0$ since $l(w) = 7$ is odd. Therefore $z_4 = 0$. Then (HCr) shows that all of the Ψ_i are in fact indecomposable. □

Table 4. $D_7(q)$, $(q^2-q+1)_\ell > 7$

$.7$	1	1																										
1.6	q	.	1																									
1.51	$\frac{1}{2}q^3$	.	1	1																								
$.52$	$\frac{1}{2}q^3$	1	.	.	1																							
3.4	q^3	1	1	.	.	1																						
2.41	$\frac{1}{2}q^4$	.	1	1	.	1	1																					
$D_4\colon 21.$	$\frac{1}{2}q^4$	.	.	.	.	.	.	1																				
2.32	$\frac{1}{2}q^6$	1	.	.	1	1	.	.	1																			
$D_4\colon 1^3.$	$\frac{1}{2}q^6$	.	.	.	.	.	.	.	.	1																		
$.421$	$\frac{1}{2}q^7$	.	.	.	1	.	.	.	.	.	1																	
1.41^2	$\frac{1}{2}q^7$	.	.	1	.	.	1	.	.	.	.	1																
$D_4\colon 2.1$	$\frac{1}{2}q^7$	.	.	.	.	.	.	1	.	.	.	.	1															
21.2^2	q^9	.	.	.	.	1	1	.	1	.	.	.	.	1														
1.321	q^9	.	.	.	1	.	.	.	1	.	1	.	.	.	1													
$1^2.2^21$	$\frac{1}{2}q^{12}$	.	.	.	.	.	.	.	1	.	.	.	.	1	1	1												
$D_4\colon .3$	$\frac{1}{2}q^{12}$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1											
$.41^3$	q^{12}	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1										
1.31^3	$\frac{1}{2}q^{13}$	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	1	1									
$.321^2$	$\frac{1}{2}q^{13}$	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	1								
$D_4\colon 1.1^2$	$\frac{1}{2}q^{13}$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1							
$1^2.21^3$	$\frac{1}{2}q^{16}$	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1						
$D_4\colon .21$	$\frac{1}{2}q^{16}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1					
1.21^4	$\frac{1}{2}q^{21}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	1	.	1				
$.2^21^3$	$\frac{1}{2}q^{21}$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	1	.	.	.	.	1			
$1^3.1^4$	q^{21}	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	1	.	.	.	1		
1.1^6	q^{31}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	1	.	1	.	1	1	
$.1^7$	q^{42}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	2	.	1	1	.	1
		ps	ps	ps	ps	ps	ps	D_4	ps	D_4	ps	ps	D_4	ps	ps	ps	D_6^s	$.1^4$	ps	$.1^4$	D_4	ps	D_6^s	$.1^4$	$.1^4$	A_5	$.1^6$	$.1^6$

Proposition 6.5. *Assume* (T_ℓ). *The decomposition matrices for the three unipotent* ℓ*-blocks of* $D_8(q)$ *of non-cyclic defect for primes* ℓ *with* $(q^2-q+1)_\ell > 7$ *are as given in Tables 5 and 6.*

Here the blocks are labelled by the 6-cuspidal unipotent characters of the centraliser $D_2(q).\Phi_6^2$ of a Sylow Φ_6-torus of $D_8(q)$ (according to the parametrisation of ℓ-blocks in [**4**]).

TABLE 5. $D_8(q)$, $(q^2 - q + 1)_\ell > 7$, blocks $\binom{2}{0}$ and $\binom{1\ 2}{0\ 1}$

$.8$	1																										
2.6	.	1																									
3.5	1	1	1																								
2.51	.	1	1	1																							
$.53$	1	.	.	.	1																						
$D_4\colon 31.$	.	.	.	.	.	1																					
1.51^2	.	.	.	1	.	.	1																				
$D_4\colon 3.1$	.	.	.	.	.	1	.	1																			
2.41^2	.	.	1	1	.	.	1	.	1																		
$.431$	.	.	.	.	1	.	.	.	.	1																	
2.33	1	.	1	.	1	.	.	.	.	.	1																
1.331	.	.	.	.	1	.	.	.	.	1	1	1															
$D_4\colon 1^3.1$	.	.	.	.	.	.	.	.	.	.	.	.	1														
$.51^3$	.	.	.	.	.	.	1	.	.	.	.	.	.	1													
$21^2.2^2$	.	.	1	.	.	.	.	.	1	.	1	.	.	.	1												
$D_4\colon .4$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1											
2.31^3	.	.	.	.	.	.	1	.	1	.	.	.	.	1	.	.	1										
$.331^2$	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	1									
$D_4\colon 1^2.1^2$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1								
$1^3.2^21$	.	.	.	.	.	.	.	.	.	.	1	1	.	.	1	.	.	.	.	1							
21.21^3	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	1	.	.	.	1						
$1^4.21^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	1	1					
$D_4\colon .2^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1				
2.21^4	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	1	.	.	1			
$.2^21^4$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	1	.	.	.	.	1		
$1^6.2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	1	1	.	1	.	1	
$.21^6$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	2	.	1	.	1
	ps	ps	ps	ps	ps	D_4	ps	D_4	ps	ps	ps	ps	D_4	$.1^4$	ps	D_6^s	ps	$.1^4$	D_4	ps	ps	A_5	D_6^s	$.1^4$	$.1^4$	$.1^6$	$.1^6$

$.71$	1																										
$1^2.6$	.	1																									
$.62$	1	.	1																								
31.4	1	1	.	1																							
$D_4\colon 2^2.$	.	.	.	.	1																						
$1^2.51$	.	1	.	.	.	1																					
21.41	.	1	.	1	.	1	1																				
3.32	1	.	1	1	.	.	.	1																			
$2^2.31$	.	.	.	1	.	.	1	1	1																		
$D_4\colon 1^4.$	.	.	.	.	.	.	.	.	.	1																	
$1^2.41^2$	.	.	.	.	.	1	1	.	.	.	1																
$.42^2$	.	.	1	.	.	.	.	.	.	.	.	1															
$D_4\colon 2.2$	.	.	.	.	1	.	.	.	.	.	.	.	1														
1.32^2	.	.	1	.	.	.	.	1	.	.	.	1	.	1													
$D_4\colon 1.3$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1												
$1^2.31^3$	.	.	.	.	.	.	1	.	1	.	1	.	.	.	.	1											
$.32^21$	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1										
$1^2.2^3$	.	.	.	.	.	.	.	1	1	.	.	.	.	1	.	.	.	1									
$.41^4$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1								
1.31^4	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	1							
$D_4\colon 1.1^3$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1						
$1^2.21^4$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	1	.	1					
$.2^31^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	1	.	.	.	.	1				
$D_4\colon .21^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1			
$1^3.1^5$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	1		
$1^2.1^6$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	1	.	1	.	.	1	1	
$.1^8$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	2	1	.	1
	ps	ps	ps	ps	D_4	ps	ps	ps	ps	D_4	ps	ps	D_4	ps	D_6^s	ps	$.1^4$	ps	$.1^4$	$.1^4$	D_4	ps	$.1^4$	D_6^s	A_5	$.1^6$	$.1^6$

TABLE 6. $D_8(q)$, $(q^2-q+1)_\ell > 7$, block $\binom{1}{1}$

1.7	1																	
$4+$	1	1																
$4-$	1	.	1															
1.52	1	.	.	1														
2.42	1	1	1	1	1													
D_4: $21^2.$	.	.	.	.	.	1												
1.421	.	.	.	1	1	.	1											
2^2+	.	1	.	.	1	.	.	1										
2^2-	.	.	1	.	1	.	.	.	1									
$.421^2$	.	.	.	.	.	.	1	.	.	1								
D_4: 2.1^2	.	.	.	.	.	1	.	.	.	.	1							
1.321^2	.	.	.	.	1	.	1	.	.	1	.	1						
$1^2.2^21^2$	.	.	.	.	1	.	.	1	1	.	.	1	1					
D_4: $.31$	.	.	.	.	.	.	.	.	.	.	1	.	.	1				
1.2^21^3	.	.	.	.	.	.	.	.	.	1	.	1	1	.	1			
1^4+	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1		
1^4-	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	1	
1.1^7	.	.	.	.	.	.	.	.	.	.	.	.	1	2	1	1	1	1
	ps	ps	ps	ps	ps	D_4	ps	ps	ps	$.1^4$	D_4	ps	ps	D_6^s	$.1^4$	A_5	A_5	$.1^6$

PROOF. In the principal block, labelled by the trivial character $\binom{2}{0}$ of $D_2(q)$, the columns Ψ_i, $i \neq 16, 23$, are obtained by Harish-Chandra inducing PIMs from Levi subgroups of type D_7 and A_7. (HCi) also yields projectives

$$\tilde{\Psi}_{16} = [D_4\colon .1^4] + a_1[1^4.21^2] + (a_2 - z_2)[2.21^4] + z_2[.2^21^4] + (a_3 - z_3)[1^6.2] + z_3[.21^6],$$
$$\tilde{\Psi}_{23} = a_1[1^4.21^2] + [D_4\colon .2^2] + z_2[2.21^4] + (a_2 - z_2)[.2^21^4] + z_3[1^6.2] + (a_3 - z_3)[.21^6],$$

with a_i, z_i as in the proof of Proposition 6.4. By (T_ℓ), we must have that

$$\tilde{\Psi}_{23} - a_1\Psi_{22} = [D_4\colon .2^2] + z_2[2.21^4] + (a_2 - z_2)[.2^21^4] + (z_3 - a_1)[1^6.2] + (a_3 - z_3 - a_1)[.21^6]$$

is a projective character. Now Harish-Chandra restriction of this to a Levi subgroup of type D_6A_1 yields negative multiplicities in PIMs unless $a_1 = 0$.

For the block labelled $\binom{1}{1}$ again all columns except the 15th and 24th are obtained by Harish-Chandra inducing suitable PIMs from proper Levi subgroups. We also obtain

$$\tilde{\Psi}_{15} = [D_4\colon 1.3] + (a_2 - z_2)[1.31^4] + (a_2 - z_2)[1^2.21^4] + z_2[.2^31^2] + (a_3 - z_3)[1^2.1^6] + z_3 3[.1^8],$$
$$\tilde{\Psi}_{24} = z_2[1.31^4] + z_2[1^2.21^4] + (a_2 - z_2)[.2^31^2] + [D_4\colon .21^2] + z_3[1^2.1^6] + (a_3 - z_3)[.1^8].$$

Triangularity shows that

$$\tilde{\Psi}_{24} - z_2\Psi_{20} - (a_2 - z_2)\Psi_{23} = [D_4\colon .21^2] + (z_3 - z_2)[1^2.1^6] + (a_3 - z_3 - a_2 + z_2)[.1^8]$$

must be a projective character. Harish-Chandra restriction of this to a Levi subgroup of type D_6A_1 yields negative multiplicities in PIMs unless $z_2 = a_2 = 0$. Using that

$a_3 = a_2 + 2a_1 + 2 = 2$ this completes the determination of the decomposition matrices for blocks 1 and 3, as well as for the principal blocks of $D_6(q)$ and $D_7(q)$.

For the second block of $D_8(q)$, labelled $\binom{1\ 2}{0\ 1}$, all columns are obtained directly by (HCi). Finally, (HCr) shows that all projectives constructed in the three blocks are in fact indecomposable. $\square$

2. Unipotent decomposition matrix of $E_6(q)$

For $E_6(q)$ and $d_\ell(q) = 6$, the triangular shape of the decomposition matrix and thus property (T_ℓ) for the unipotent blocks of G and primes $\ell > 3$ has been shown by Geck–Hiss [**25**, Thm. 7.4] under the additional assumption that q is a power of a good prime for E_6.

LEMMA 6.6. *Let q be a prime power and $\ell|(q^2 - q + 1)$. The Hecke algebras of various ℓ-modular cuspidal pairs (L, λ) of Levi subgroups L in $E_6(q)$ and their respective numbers of irreducible characters are as given in Table 7.*

TABLE 7. Hecke algebras in $E_6(q)$ for $d_\ell(q) = 6$

(L, λ)	$\mathcal{H}$	$\lvert\mathrm{Irr}\,\mathcal{H}\rvert$
(D_4, D_4)	$\mathcal{H}(A_2; q^4)$	2
$(D_4, \varphi_{.1^4})$	$\mathcal{H}(A_2; q^2)$	2
(A_5, φ_{1^6})	$\mathcal{H}(A_1; q^3)$	1

PROOF. Reduction stability holds in all cases as Sylow ℓ-subgroups of L are cyclic. The proof is now as in the previous cases. For example, the ℓ-modular Steinberg character φ_{1^6} of $A_5(q)$ lifts to an ordinary cuspidal character in the Lusztig series of a regular ℓ-element with centraliser a maximal torus of order $\Phi_2\Phi_3\Phi_6$. In $E_6(q)$ such an element has centraliser ${}^2A_2(q).\Phi_3\Phi_6$, whence we find the parameter q^3. $\square$

The only unipotent block of positive ℓ-defect of $E_6(q)$ is the principal block.

THEOREM 6.7. *Assume (T_ℓ). The decomposition matrix for the principal ℓ-block of $E_6(q)$ for primes $\ell > 3$ with $(q^2 - q + 1)_\ell > 13$ is approximated by Table 8. Here the unknown entries satisfy $a_5 \leq 1$, $b_4 \leq 2$,*

$$a_7 = -1 - a_1 - a_2 + a_6, \quad \textit{and} \quad a_8 = -2 - a_1 - a_2 - a_3 + a_4 + 2a_5 + a_6.$$

PROOF. Let Ψ_i denote the linear combinations corresponding to the columns of Table 8. (HCi) yields all Ψ_i except for those with number

$$i \in \{1, 4, 8, 12, 13, 14, 17, 18, 20, 21\}.$$

Further, $\Psi_4 + \Psi_5$ and $\Psi_4 + \Psi_7$ yield Ψ_4, $\Psi_4 + \Psi_8$ and $\Psi_7 + \Psi_8$ yield Ψ_8. The two principal series PIMs Ψ_1 and Ψ_{14} are obtained via the theorem of Dipper from the decomposition matrix of the Hecke algebra $\mathcal{H} = \mathcal{H}(E_6; q)$ which has been determined by Geck [**23**, Table D]. By a result of Geck–Müller [**30**, Thm. 3.10] the decomposition matrix of $\mathcal{H}$ does not depend on ℓ for all $\ell \geq 7$.

TABLE 8. $E_6(q)$, $\ell > 3$, $(q^2-q+1)_\ell > 13$

$\phi_{1,0}$	1	1																				
$\phi_{6,1}$	q	.	1																			
$\phi_{20,2}$	q^2	1	1	1																		
$\phi_{30,3}$	$\frac{1}{2}q^3$	.	1	1	1																	
$\phi_{15,4}$	$\frac{1}{2}q^3$	1	.	.	.	1																
$D_4{:}\,3$	$\frac{1}{2}q^3$	.	.	.	.	.	1															
$\phi_{60,5}$	q^5	1	.	1	.	1	.	1														
$\phi_{24,6}$	q^6	.	.	1	.	.	.	.	1													
$\phi_{80,7}$	$\frac{1}{6}q^7$	.	.	1	1	.	.	1	1	1												
$\phi_{60,8}$	$\frac{1}{2}q^7$	.	.	.	.	1	.	1	.	.	1											
$D_4{:}\,21$	$\frac{1}{2}q^7$	.	.	.	.	.	1	.	.	.	.	1										
$E_6[\zeta_3]$	$\frac{1}{3}q^7$	.	.	.	.	.	.	.	.	.	.	.	1									
$E_6[\zeta_3^2]$	$\frac{1}{3}q^7$	.	.	.	.	.	.	.	.	.	.	.	.	1								
$\phi_{60,11}$	q^{11}	.	.	.	.	.	.	1	.	1	1	.	a_1	a_1	1							
$\phi_{24,12}$	q^{12}	.	.	.	1	.	.	.	.	1	.	.	a_2	a_2	.	1						
$\phi_{30,15}$	$\frac{1}{2}q^{15}$	.	.	.	.	.	.	.	1	1	.	.	a_3	a_3	.	.	1					
$\phi_{15,16}$	$\frac{1}{2}q^{15}$	.	.	.	.	.	.	.	.	.	1	.	a_4	a_4	1	.	.	1				
$D_4{:}\,1^3$	$\frac{1}{2}q^{15}$	.	.	.	.	.	.	.	.	.	.	1	a_5	a_5	.	.	.	.	1			
$\phi_{20,20}$	q^{20}	.	.	.	.	.	.	.	.	1	.	.	a_6	a_6	1	1	1	.	b_1	1		
$\phi_{6,25}$	q^{25}	.	.	.	.	.	.	.	.	.	.	.	a_7	a_7	.	.	1	.	b_1	1	1	
$\phi_{1,36}$	q^{36}	.	.	.	.	.	.	.	.	.	.	.	a_8	a_8	1	.	.	1	$b_1{+}2$	1	b_4	1
		ps	ps	ps	ps	ps	D_4	ps	ps	ps	ps	D_4	c	c	ps	$.1^4$	ps	A_5	c	$.1^4$	c	c

To determine some of the remaining entries we use the combination of (DL) and (Red). We denote the three unknown entries below the diagonal in the 18th column by b_1, b_2, b_3, and the eight unknown entries below the diagonal in the 12th and 13th column by $a_1, \ldots, a_8$ (they agree in the two columns since the cuspidal characters $E_6[\zeta_3]$ and $E_6[\zeta_3^2]$ are Galois conjugate). Let us first note that explicit computations in Chevie [**26**] show that there exists a regular ℓ-element in G^* whenever $(q^2-q+1)_\ell > 13$. In addition, the condition $\ell > 3$ forces ℓ to be good and therefore by Example 1.7(d) one can also use (Red) for centralisers of Φ_d-tori of rank 1. Let w be a Coxeter element and R_w be the corresponding Deligne–Lusztig character. We consider the generalised 1-eigenspace of F on R_w. The coefficient of the PIM Ψ_{20} is $b_1 - b_2$, hence $b_1 \geq b_2$ by (DL). On the other hand, the cuspidal unipotent character of the 6-split Levi subgroup ${}^2A_2(q).\Phi_3\Phi_6$ gives $b_2 - b_1 \geq 0$ by (Red), hence $b_2 = b_1$. The coefficient of the PIM Ψ_{21} is $2 + b_1 - b_3$. This number is non-positive by (Red) applied to a maximal torus. Therefore $b_3 = 2 + b_1$.

We now turn to the generalised q-eigenspace of F on R_w. The coefficient of the PIM Ψ_{18} is $1 - a_5$, therefore $a_5 \leq 1$ by (DL). The coefficient of Ψ_{20} is $-1 - a_1 - a_2 + a_6 - a_7$. It is non-negative by (DL) but also non-positive by (Red) applied to the cuspidal unipotent character of ${}^2A_2(q).\Phi_3\Phi_6$. Therefore $a_7 = -1 - a_1 - a_2 + a_6$. Finally, the coefficient of Ψ_{21}

is $-2 - a_1 - a_2 - a_3 + a_4 + 2a_5 + a_6 - a_8$. Again, it must be zero by the combination of (DL) and (Red) for a maximal torus. Consequently $a_8 = -2 - a_1 - a_2 - a_3 + a_4 + 2a_5 + a_6$.

We finish with the Deligne–Lusztig character R_w for $w = s_1s_2s_3s_1s_5s_4s_6s_5s_4s_2s_3s_4$. The coefficient of the PIM Ψ_{21} is $48 - 24b_4$, which forces $b_4 \le 2$. □

3. A non-principal block of $E_8(q)$

THEOREM 6.8. *Assume* (T_ℓ). *The decomposition matrix for the unipotent* ℓ*-block of* $E_8(q)$ *for primes* $\ell > 3$ *with* $(q^2-q+1)_\ell > 13$, *of defect* $(\Phi_6)^2_\ell$, *is approximated by Table 9. Here the unknown entries* $a_1, \ldots, a_8, b_1, b_4$ *are as in Theorem 6.7, and starred entries in the column* E_6^b *are understood to be the same as in the column for* E_6^a.

TABLE 9. $E_8(q)$, $\ell > 3$, $(q^2 - q + 1)_\ell > 13$, block of defect Φ_6^2

	p	p	p	p	D_4	p	p	p	p	E_6^a	E_6^b	p	D_4	$.1^4$	p	p	E_6^c	$.1^4$	A_5	E_6^d	E_6^e
$\phi_{112,3}$	1																				
$\phi_{160,7}$	.	1																			
$\phi_{400,7}$	1	.	1																		
$\phi_{1344,8}$	1	1	.	1																	
D_4: $\phi'_{8,3}$	.	.	.	.	1																
$\phi_{2240,10}$	1	.	1	1	.	1															
$\phi_{3360,13}$	.	1	.	1	.	.	1														
$\phi_{3200,16}$	.	.	.	1	.	.	.	1													
$\phi_{7168,17}$	.	.	.	1	.	1	1	1	1												
$E_6[\zeta_3]$: $\phi_{2,2}$	.	.	.	.	.	.	.	.	.	1											
$E_6[\zeta_3^2]$: $\phi_{2,2}$	.	.	.	.	.	.	.	.	.	.	1										
$\phi_{1344,19}$	.	.	1	.	.	1	.	.	.	.	.	1									
$D_4 : \phi_{16,5}$	.	.	.	.	1	.	.	.	.	.	.	.	1								
$\phi_{3200,22}$	.	.	.	.	.	.	1	.	1	$2a_2-a_1-a_3+a_6$	$*$	.	.	1							
$\phi_{3360,25}$	.	.	.	.	.	.	.	1	1	$2a_3-a_2+a_4+a_6$	$*$	.	.	.	1						
$\phi_{2240,28}$	.	.	.	.	.	1	.	.	1	$a_1-a_2+a_3+a_4+a_6$	$*$	1	.	.	.	1					
D_4: $\phi''_{8,9}$	.	.	.	.	.	.	.	.	.	a_5	$*$	.	1	.	.	.	1				
$\phi_{1344,38}$	.	.	.	.	.	.	.	.	1	$a_4-a_1+4a_6+a_7+a_8$	$*$	.	.	1	1	1	$3b_1+2$	1			
$\phi_{400,43}$	.	.	.	.	.	.	.	.	.	$a_3-a_2+2a_4+a_6$	$*$	1	.	.	.	1	.	.	1		
$\phi_{160,55}$	.	.	.	.	.	.	.	.	.	$a_6+2a_7+a_8$	$*$	.	.	.	1	.	$3b_1+2$	1	.	1	
$\phi_{112,63}$	.	.	.	.	.	.	.	.	.	$a_6+a_7+2a_8$	$*$	.	.	.	.	1	$3b_1+4$	1	1	b_4	1

PROOF. The principal series PIMs can be found in [**29**, Table 7.15]. Then (HCi) yields the other listed projectives. □

4. Even-dimensional non-split orthogonal groups

We continue with the unipotent blocks of twisted orthogonal groups ${}^2D_n(q)$, $n \leq 7$, for primes ℓ with $d_\ell(q) = 6$. Again the Brauer trees in the case of cyclic defect were first described by Fong and Srinivasan [**20**] and are easily obtained:

PROPOSITION 6.9. *Let q be a prime power and ℓ a prime with $d_\ell(q) = 6$. The Brauer trees of the unipotent ℓ-blocks of ${}^2D_n(q)$, $3 \leq n \leq 7$, with cyclic defect are as given in Table 10.*

TABLE 10. Brauer trees for ${}^2D_n(q)$ $(3 \leq n \leq 7)$, $7 \leq \ell | (q^2 - q + 1)$

$$
\begin{array}{lcccccccccccccc}
{}^2D_3(q): & 2. & — & 1.1 & — & .1^2 & — & \bigcirc \\
{}^2D_5(q): & 31. & — & 1^2.2 & — & .21^2 & — & \bigcirc \\
{}^2D_7(q): & 42. & — & 1^2.31 & — & 1.31^2 & — & \bigcirc \\
 & 31^2.1 & — & 21^2.2 & — & .2^21^2 & — & \bigcirc \\
 & & ps & & ps & & .1^2 & \\
{}^2D_4(q): & 3. & — & 1.2 & — & .21 & — & \bigcirc & — & .1^3 & — & 1^2.1 & — & 21. \\
{}^2D_5(q): & 4. & — & 1.3 & — & .31 & — & \bigcirc & — & 1.1^3 & — & 1^2.1^2 & — & 2^2. \\
 & 3.1 & — & 2.2 & — & .2^2 & — & \bigcirc & — & .1^4 & — & 1^3.1 & — & 21^2. \\
{}^2D_6(q): & 5. & — & 1.4 & — & .41 & — & \bigcirc & — & 2.1^3 & — & 21.1^2 & — & 2^2.1 \\
 & 41. & — & 1^2.3 & — & .31^2 & — & \bigcirc & — & 1.21^2 & — & 1^2.21 & — & 32. \\
 & 4.1 & — & 2.3 & — & .32 & — & \bigcirc & — & 1.1^4 & — & 1^3.1^2 & — & 2^21. \\
 & 31^2. & — & 1^3.2 & — & .21^3 & — & \bigcirc & — & .2^21 & — & 21.2 & — & 31.1 \\
 & 3.1^2 & — & 2.21 & — & 1.2^2 & — & \bigcirc & — & .1^5 & — & 1^4.1 & — & 21^3. \\
{}^2D_7(q): & 5.1 & — & 2.4 & — & .42 & — & \bigcirc & — & 2.1^4 & — & 21^2.1^2 & — & 2^21.1 \\
 & 41.1 & — & 21.3 & — & .321 & — & \bigcirc & — & 1.21^3 & — & 1^3.21 & — & 321. \\
 & 4.2 & — & 3.3 & — & .3^2 & — & \bigcirc & — & 1^2.1^4 & — & 1^3.1^3 & — & 2^3. \\
 & 4.1^2 & — & 2.31 & — & 1.32 & — & \bigcirc & — & 1.1^5 & — & 1^4.1^2 & — & 2^21^2. \\
 & & ps & & ps & & .1^2 & & .1^2 & & ps & & ps &
\end{array}
$$

Here, $.1^2$ denotes the Harish-Chandra series of the cuspidal ℓ-modular Steinberg character of ${}^2D_3(q)$.

In Table 11 we have collected the number $|\mathrm{Irr}\,\mathcal{H}|$ for some small rank modular Iwahori–Hecke $\mathcal{H}$ algebras occurring later. Again, they can be computed using the programme of Jacon [**36**].

TABLE 11. $|\mathrm{Irr}\,\mathcal{H}|$ for some modular Hecke algebras

$n =$	1	2	3	4	5
$\mathcal{H}(B_n; 1; q)$	2	5	10	18	32
$\mathcal{H}(B_n; q; q)$	2	5	9	18	30
$\mathcal{H}(B_n; q^2; q)$	2	4	8	15	26
$\mathcal{H}(B_n; q^3; q)$	1	3	5	10	16
$\mathcal{H}(B_n; q^4; q)$	2	4	8	15	26

LEMMA 6.10. *Let q be a prime power and $\ell|(q^2-q+1)$. The Hecke algebras of various ℓ-modular cuspidal pairs (L, λ) of Levi subgroups L in ${}^2D_n(q)$, $n \geq 4$, and their respective numbers of irreducible characters are as given in Table 12.*

TABLE 12. Hecke algebras and $|\mathrm{Irr}\,\mathcal{H}|$ in ${}^2D_n(q)$ for $d_\ell(q) = 6$

(L, λ)	$\mathcal{H}$	6	7	8	9
$({}^2D_3, \varphi_{.1^2})$	$\mathcal{H}(B_{n-3}; 1; q)$	10	18	32	54
(A_5, φ_{1^6})	$\mathcal{H}(A_1; q^3) \otimes \mathcal{H}(B_{n-7}; q^2; q)$	–	1	2	4
$({}^2D_7, \varphi_{1^6.})$	$\mathcal{H}(B_{n-7}; q^4; q)$	–	1	2	4
$({}^2D_7, \varphi_{.1^6})$	$\mathcal{H}(B_{n-7}; q^4; q)$	–	1	2	4

PROOF. Note that ${}^2D_n(q)$ has Weyl group of type B_{n-1}. Now the relative Weyl group of a Levi subgroup $B_2(q)$ inside $B_{n-1}(q)$ has type B_{n-3}, and the one of a Levi subgroup $A_5(q)$ has type B_{n-6} by [**35**, p. 70]. The modular Steinberg character $\varphi_{.1^2}$ of ${}^2D_3(q)$ is liftable by the ℓ-modular Brauer tree in Table 10, and the one of $A_5(q)$ by the Brauer tree for $\mathrm{GL}_6(q)$, so we can argue as usual, with reduction stability following by Example 2.5(a).

Next, the ℓ-modular Steinberg character $\varphi_{.1^6}$ of ${}^2D_7(q)$ occurs with multiplicity 1 in the reduction of an ordinary Deligne–Lusztig character $R_T^G(\theta)$ from a maximal torus T centralising a Sylow Φ_6-torus as can be seen from the decomposition matrix in Table 13. Thus we obtain reduction stability with Example 2.5(c).

Finally, again using the decomposition matrix one checks that $\varphi_{1^6.}$ lifts to a non-unipotent character whose Jordan correspondent is the cuspidal unipotent character of $D_4(q)\Phi_2\Phi_6$. The latter is invariant under all automorphisms, and thus $\varphi_{1^6.}$ is also reduction stable. $\square$

Observe that Sylow ℓ-subgroups of ${}^2D_n(q)$, $n \leq 6$, are cyclic for primes ℓ with $d_\ell(q) = 6$, so we only need to deal with the case $n \geq 7$.

PROPOSITION 6.11. *Assume (T_ℓ). The decomposition matrix for the principal ℓ-block of ${}^2D_7(q)$ for primes ℓ with $(q^2 - q + 1)_\ell > 7$ is as given in Table 13.*

Here, the unknown entries satisfy $y_2 \leq 5$ and $y_3 \leq 2$.

TABLE 13. ${}^2D_7(q)$, $(q^2-q+1)_\ell > 7$

$6.$	1	1																										
$51.$	q	1	1																									
$.6$	$\frac12 q^3$	.	.	1																								
$41^2.$	$\frac12 q^3$	.	1	.	1																							
$3^2.$	q^3	.	.	.	.	1																						
1.5	$\frac12 q^4$	1	.	1	.	.	1																					
32.1	$\frac12 q^4$	.	.	.	.	1	.	1																				
$1^2.4$	$\frac12 q^6$	1	1	.	.	.	1	.	1																			
$2^2.2$	$\frac12 q^6$	.	.	.	.	.	.	1	.	1																		
$.51$	$\frac12 q^7$	.	.	1	.	.	1	.	.	.	1																	
31.1^2	$\frac12 q^7$	.	.	.	.	.	.	1	.	.	.	1																
$31^3.$	$\frac12 q^7$	.	.	.	1	.	.	.	.	.	.	.	1															
$1^3.3$	q^9	.	1	.	1	.	.	.	1	.	.	.	.	1														
21.21	q^9	.	.	.	.	1	.	1	.	1	.	1	.	.	1													
$1^2.2^2$	$\frac12 q^{12}$	.	.	.	.	1	.	.	.	.	.	.	.	.	1	1												
$1^4.2$	$\frac12 q^{12}$	.	.	.	1	.	.	.	.	.	.	.	1	1	.	.	1											
3.1^3	q^{12}	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1										
$.41^2$	$\frac12 q^{13}$	.	.	.	.	.	1	.	1	.	1	.	.	.	.	.	.	.	1									
2.21^2	$\frac12 q^{13}$	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	1	.	1								
$21^4.$	$\frac12 q^{13}$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1							
1.2^21	$\frac12 q^{16}$	.	.	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	.	1	.	1						
$1^5.1$	$\frac12 q^{16}$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	1	.	1					
$1^6.$	$\frac12 q^{21}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	1				
$.31^3$	$\frac12 q^{21}$	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	1			
$.2^3$	q^{21}	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1		
$.21^4$	q^{31}	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	.	1	y_3	1	
$.1^6$	q^{42}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	y_2	.	y_3+2	1	1
		ps	ps	ps	ps	ps	ps	ps	ps	ps	$.1^2$	ps	ps	ps	ps	$.1^2$	ps	$.1^2$	$.1^2$	$.1^2$	ps	$.1^2$	A_5	c	$.1^2$	c	$.1^2$	c

PROOF. All columns in Table 13 except for the 23rd, the 25th and the 27th are obtained by Harish-Chandra induction of PIMs from Levi subgroups of types ${}^2D_6$ and A_5. With Lemma 6.10 this accounts for all Harish-Chandra series from proper Levi subgroups, so the remaining three PIMs must be cuspidal. The ordinary Gelfand-Graev character gives the Steinberg PIM in column 27. The uni-triangular shape of the decomposition matrix shows that there must exist PIMs with unipotent part of the form

$$\Psi_{23} = [1^6.] + y_1[.21^4] + y_2[.1^6], \quad \Psi_{25} = [.2^3] + y_3[.21^4] + y_4[.1^6],$$

with suitable coefficients $y_i \geq 0$. Not that $[1^6.]$ and $[.2^3]$ lie in families which are not comparable, which explains why $[.2^3]$ does not occur in Ψ_{23}. It will turn out in the determination of the decomposition matrices for ${}^2D_8(q)$ in Proposition 6.12 that $y_1 = 0$.

It remains to compute y_2 and y_4. To this end we consider the decomposition of Deligne–Lusztig characters R_w. For w being a Coxeter element the coefficient of Ψ_{27} in R_w is equal to $2 + y_3 - y_4$ which by (DL) forces $y_4 \leq 2 + y_3$. On the other hand, by Example 1.7(c) there exists a regular ℓ-element of G^* whenever $(q^2 - q + 1)_\ell > 12$ (in particular whenever $(q^2 - q + 1)_\ell > 7$ since we also assumed $\ell \geq 7$). Therefore one can use (Red) to show that $y_4 \geq 2 + y_3$, which proves that $y_4 = 2 + y_3$. Now let $w = s_2s_1s_3s_4s_5s_4s_3s_1s_2s_3s_4s_5s_6s_7$. By explicit computations and using that $y_4 = 2 + y_3$, the PIM Ψ_{27} does not occur in R_v for $v < w$. In addition, the coefficient of Ψ_{27} in R_w is $60 - 12y_2$, which by (DL) gives $y_2 \leq 5$. (HCr) and the decomposition matrix of the Hecke algebra for the principal series show that all columns correspond to PIMs. Moreover this proves that $y_3 \leq 2$. $\square$

It seems that in order to complete the proof of Proposition 6.11 we need to also study some decomposition matrices for ${}^2D_8(q)$ and ${}^2D_9(q)$:

PROPOSITION 6.12. *Assume (T_ℓ). The decomposition matrices for the unipotent ℓ-blocks of ${}^2D_8(q)$ and ${}^2D_9(q)$ of defect $(q^2 - q + 1)_\ell^2$ for primes ℓ with $(q^2 - q + 1)_\ell > 7$ are as given in Table 14.*

Here, the unknown entries $y_2 \leq 5$ and $y_3 \leq 2$ are as in Proposition 6.11.

Here, the blocks are labelled by the 6-cuspidal unipotent characters of the centraliser ${}^2D_2(q).(q^3 + 1)^2$, respectively ${}^2D_3(q).(q^3 + 1)^2$, of a Φ_6-torus of rank 2, in accordance with [**4**].

PROOF. First consider ${}^2D_8(q)$. All columns Ψ_i in the principal block apart from Ψ_{23} and Ψ_{25} are obtained by Harish-Chandra induction, and we find two further projectives with unipotent parts

$$\tilde{\Psi}_{23} = [.32^2] + y_3[1.21^4] + y_4[1.1^6],$$
$$\tilde{\Psi}_{25} = [1^7.] + y_1[1.21^4] + y_2[1.1^6].$$

Since $[1.21^4]$ and $[1^7.]$ lie in families which are not comparable, the character $\tilde{\Psi}_{25}$ must involve y_1 copies of Ψ_{26} by (T_ℓ). Then (HCr) shows that $y_1 = 0$.

The columns in the block labelled $\binom{0\ 1\ 2}{1}$ except for the 20th and 25th are again obtained by Harish-Chandra induction, as well as

$$\Psi_{20} = [21^5.] + [1^6.1] + y_2[.1^7],$$
$$\Psi_{25} = [.2^31] + y_3[.2^21^3] + y_4[.1^7].$$

We now turn to ${}^2D_9(q)$. For the principal block with (HCi) we find all columns Ψ_i but Ψ_1, Ψ_{11}, Ψ_{13}, Ψ_{22} and Ψ_{26}, as well as the projectives

$$\tilde{\Psi}_{22} = [.3^22] + y_3[1^2.21^4] + y_4[1^2.1^6],$$
$$\tilde{\Psi}_{26} = [1^8.] + y_2[1^2.1^6].$$

Furthermore, we get $\Psi_1 + \Psi_{11}$, $\Psi_1 + \Psi_5$ and $\Psi_{11} + \Psi_{13}$, which yield Ψ_1, Ψ_{11} and Ψ_{13} by triangularity.

The block for ${}^2D_9(q)$ labelled $\binom{0\ 1\ 2}{2}$ is now obtained as usually. An application of (HCr) shows that all columns correspond to PIMs. This completes the proof of Propositions 6.12 and 6.11. $\square$

Table 14. ${}^2D_8(q)$, $(q^2-q+1)_\ell > 7$, blocks $\binom{0\ 2}{}$ and $\binom{0\ 1\ 2}{1}$

7.	1																										
52.	1	1																									
43.	.	.	1																								
421.	.	1	.	1																							
42.1	.	.	1	.	1																						
1.6	1	.	.	.	.	1																					
41.1^2	.	.	.	.	1	.	1																				
.61	.	.	.	.	.	1	.	1																			
$321^2.$	.	.	.	1	.	.	.	.	1																		
1.51	1	.	.	.	.	1	.	1	.	1																	
$2^2.3$	.	.	.	.	1	.	.	.	.	.	1																
21.31	.	.	1	.	1	.	1	.	.	.	1	1															
$1^2.41$	1	1	.	.	.	.	.	.	.	1	.	.	1														
4.1^3	.	.	.	.	.	.	1	.	.	.	.	.	.	1													
$1^3.31$	.	1	.	1	.	.	.	.	.	.	.	.	1	.	1												
$1^2.32$	.	.	1	.	.	.	.	.	.	.	.	1	.	.	.	1											
2.31^2	.	.	.	.	.	.	1	.	.	.	.	1	.	1	.	.	1										
1.41^2	.	.	.	.	.	.	.	1	.	1	.	.	1	.	.	.	.	1									
$2^21^3.$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1								
1.321	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	1	1	.	.	1							
$1^4.21$	.	.	.	1	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	1						
1.31^3	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	.	.	.	1					
$.32^2$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	1				
$1^5.1^2$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1			
$1^7.$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	1		
1.21^4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	1	y_3	.	.	1	
1.1^6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	$y_3{+}2$	1	y_2	1	1
	ps	ps	ps	ps	ps	ps	ps	$.1^2$	ps	ps	ps	ps	ps	$.1^2$	ps	$.1^2$	$.1^2$	$.1^2$	ps	$.1^2$	ps	$.1^2$	$.2^3$	A_5	$1^6.$	$.1^2$	$.1^6$

6.1	1																										
.7	.	1																									
51.1	1	.	1																								
$3^21.$	.	.	.	1																							
2.5	1	1	.	.	1																						
$41^2.1$	.	.	1	.	.	1																					
21.4	1	.	1	.	1	.	1																				
321.1	.	.	.	1	.	.	.	1																			
$2^21.2$	.	.	.	.	.	.	.	1	1																		
$21^2.3$	.	.	1	.	.	1	1	.	.	1																	
$31^2.1^2$	.	.	.	.	.	.	.	1	.	.	1																
.52	.	1	.	.	1	.	.	.	.	.	.	1															
$31^3.1$	.	.	.	.	.	1	.	.	.	.	.	.	1														
$21^3.2$	.	.	.	.	.	1	.	.	.	1	.	.	1	1													
$21^2.21$	.	.	.	1	.	.	.	1	1	.	1	.	.	.	1												
$21^4.1$	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	1											
.421	.	.	.	.	1	.	1	.	.	.	.	1	.	.	.	.	1										
$1^3.2^2$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.	1									
3.1^4	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1								
$21^5.$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1							
2.21^3	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	1	.	1						
$1^6.1$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	1	.	1					
1.2^21^2	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1	.	.	1	.	1				
$.321^2$	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	1			
$.2^31$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1		
$.2^21^3$	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	1	y_3	1	
$.1^7$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	y_2	.	1	.	.	$y_3{+}2$	1	1
	ps	ps	ps	ps	ps	ps	ps	ps	ps	ps	ps	$.1^2$	ps	ps	ps	ps	$.1^2$	$.1^2$	$.1^2$	$1^6.$	$.1^2$	A_5	$.1^2$	$.1^2$	$.2^3$	$.1^2$	$.1^6$

TABLE 15. ${}^2D_9(q)$, $(q^2 - q + 1)_\ell > 7$, blocks $\binom{1\ 2}{}$ and $\binom{0\ 1\ 2}{2}$

	ps	ps	ps	ps	ps	ps	ps	ps	ps	ps	ps	$.1^2$	$.1^2$	$.1^2$	$.1^2$	ps	ps	$.1^2$	ps	$.1^2$	ps	$.2^3$	$.1^2$	A_5	$.1^2$	$1^6.$	$.1^6$
$71.$	1																										
$62.$	1	1																									
$4^2.$	.	.	1																								
$42^2.$	.	1	.	1																							
42.2	.	.	1	.	1																						
$1^2.6$	1	.	.	.	.	1																					
32.3	.	.	.	.	1	.	1																				
41.21	.	.	.	.	1	.	.	1																			
$32^21.$	.	.	.	1	.	.	.	.	1																		
31.31	.	.	1	.	1	.	1	1	.	1																	
$1^2.51$	1	1	.	.	.	1	.	.	.	.	1																
4.21^2	.	.	.	.	.	.	.	1	.	.	.	1															
$.61^2$	.	.	.	.	.	1	.	.	.	.	.	.	1														
3.31^2	.	.	.	.	.	.	.	1	.	1	.	1	.	1													
1.51^2	.	.	.	.	.	1	.	.	.	.	1	.	1	.	1												
$2^31^2.$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1											
$1^2.41^2$	.	1	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1										
$1^2.3^2$	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1									
$1^3.31^2$	.	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1								
1.3^21	.	.	.	.	.	.	1	.	.	1	.	.	.	1	.	.	.	1	.	1							
$1^4.21^2$	.	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	1						
$.3^22$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1					
$1^2.31^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1	.	.	.	1				
$1^5.1^3$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1			
$1^2.21^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	y_3	1	.	1		
$1^8.$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	1	
$1^2.1^6$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	$y_3{+}2$	.	1	1	y_2	1

	ps	ps	ps	ps	ps	ps	ps	ps	ps	$.1^2$	ps	ps	ps	ps	$.1^2$	ps	ps	$1^6.$	$.1^2$	$.1^2$	$.1^2$	$.1^2$	$.1^2$	A_5	$.2^3$	$.1^2$	$.1^6$
$.8$	1																										
6.2	.	1																									
51.2	.	1	1																								
3.5	1	1	.	1																							
$3^22.$	.	.	.	.	1																						
31.4	.	1	1	1	.	1																					
$41^2.2$	.	.	1	.	.	.	1																				
$32^2.1$	.	.	.	.	1	.	.	1																			
$31^2.3$	.	.	1	.	.	1	1	.	1																		
$.53$	1	.	.	1	.	.	.	.	.	1																	
$31^3.2$	.	.	.	.	.	.	1	.	1	.	1																
$2^3.2$	.	.	.	.	.	.	.	1	.	.	.	1															
$31^4.1$	.	.	.	.	.	.	.	.	.	.	1	.	1														
$31^2.1^3$	.	.	.	.	.	.	.	1	.	.	.	.	.	1													
$.431$	.	.	.	1	.	1	.	.	.	1	.	.	.	.	1												
$21^2.21^2$	.	.	.	.	1	.	.	1	.	.	.	1	.	1	.	1											
$21^4.2$	.	.	.	.	.	.	.	.	1	.	1	.	1	.	.	.	1										
$31^5.$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1									
31.1^4	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1								
$1^3.2^21$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1							
21.21^3	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	.	1						
$.3^21^2$	.	.	.	.	.	1	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	1					
$1^2.2^21^2$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	1	1	.	1				
$1^6.2$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	1	.	.	.	.	.	1			
$.2^4$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1	.	1		
$.2^21^4$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	y_3	1	
$.21^6$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	y_2	.	.	.	.	.	1	$y_3{+}2$	1	1

5. Odd-dimensional orthogonal groups

PROPOSITION 6.13. *Let q be a prime power and ℓ a prime with $d_\ell(q) = 6$. The Brauer trees of the unipotent ℓ-blocks of $B_n(q)$ and of $C_n(q)$, $3 \le n \le 6$, with cyclic defect are as given in Table 16.*

Here B_2 denotes the Harish-Chandra series of the ordinary cuspidal unipotent character of $B_2(q)$, and B_3^s the cuspidal ℓ-modular constituent of the unipotent character $[B_2{:}\,.1]$ of $B_3(q)$.

TABLE 16. Brauer trees for $B_n(q)$ and $C_n(q)$ $(3 \le n \le 6)$, $7 \le \ell | (q^2 - q + 1)$

$B_3(q)$: $3.$ — 2.1 — 1.1^2 — $.1^3$ — ○ — $B_2{:}\,.1$ — $B_2{:}\,1.$

$B_4(q)$: $4.$ — 2.2 — 1.21 — $.21^2$ — ○ — $B_2{:}\,.1^2$ — $B_2{:}\,1^2.$

$31.$ — 21.1 — $1^2.1^2$ — $.1^4$ — ○ — $B_2{:}\,.2$ — $B_2{:}\,2.$

$B_5(q)$: $5.$ — 2.3 — 1.31 — $.31^2$ — ○ — $B_2{:}\,1.1^2$ — $B_2{:}\,1^2.1$

$41.$ — 21.2 — $1^2.21$ — $.21^3$ — ○ — $B_2{:}\,.21$ — $B_2{:}\,21.$

$31^2.$ — $21^2.1$ — $1^3.1^2$ — $.1^5$ — ○ — $B_2{:}\,1.2$ — $B_2{:}\,2.1$

$32.$ — 22.1 — $1^2.1^3$ — 1.1^4 — ○ — $B_2{:}\,.3$ — $B_2{:}\,3.$

4.1 — 3.2 — 1.22 — $.221$ — ○ — $B_2{:}\,.1^3$ — $B_2{:}\,1^3.$

$B_6(q)$: 5.1 — 3.3 — 1.32 — $.321$ — ○ — $B_2{:}\,1.1^3$ — $B_2{:}\,1^3.1$

$42.$ — $2^2.2$ — $1^2.21^2$ — 1.21^3 — ○ — $B_2{:}\,.31$ — $B_2{:}\,31.$

$33.$ — $2^2.1^2$ — 21.1^3 — 2.1^4 — ○ — $B_2{:}\,.4$ — $B_2{:}\,4.$

41.1 — 31.2 — $1^2.2^2$ — $.2^21^2$ — ○ — $B_2{:}\,.21^2$ — $B_2{:}\,21^2.$

4.1^2 — 3.21 — 2.2^2 — $.2^3$ — ○ — $B_2{:}\,.1^4$ — $B_2{:}\,1^4.$

$321. \underset{ps}{\text{—}} 2^21.1 \underset{ps}{\text{—}} 1^3.1^3 \underset{ps}{\text{—}} 1.1^5 \underset{.1^3}{\text{—}} \bigcirc \underset{B_3^s}{\text{—}} B_2{:}\,1.3 \underset{B_2}{\text{—}} B_2{:}\,3.1$

$B_5(q) : \; .5 \underset{ps}{\text{—}} 1.4 \underset{ps}{\text{—}} 1^2.3 \underset{ps}{\text{—}} 1^3.2 \underset{ps}{\text{—}} 1^4.1 \underset{ps}{\text{—}} 1^5. \underset{1^5.}{\text{—}} \bigcirc$

We will encounter the following Hecke algebras:

LEMMA 6.14. *Let q be a prime power and $\ell | (q^2 - q + 1)$. The Hecke algebras of various ℓ-modular cuspidal pairs (L, λ) of Levi subgroups L in $B_n(q)$ and their respective numbers of irreducible characters are as given in Table 17.*

TABLE 17. Hecke algebras and $|\mathrm{Irr}\,\mathcal{H}|$ in $B_n(q)$ for $d_\ell(q) = 6$

(L,λ)	$\mathcal{H}$	$n=5$	6	7	8
(B_2, B_2)	$\mathcal{H}(B_{n-2}; q^3; q)$	$5+0$	$6+2+2$	$12+4$	$13+15$
$(B_3, \varphi_{.1^3})$	$\mathcal{H}(B_{n-3}; q; q)$	5	$6+3$	$12+6$	$10+20$
(B_3, B_3^s)	$\mathcal{H}(B_{n-3}; q; q)$	5	$6+3$	$12+6$	$10+20$
(A_5, φ_{1^6})	$\mathcal{H}(B_{n-5}; q^3; q)$	–	1	$1+2$	5
$(B_5, \varphi_{1^5.})$	$\mathcal{H}(B_{n-5}; q^3; q)$	1	1	$1+2$	5

PROOF. In all cases, reduction stability for the cuspidal Brauer characters follows with Example 2.5(a) as Sylow ℓ-subgroups of L are cyclic. The cuspidal ℓ-modular character B_2 is the ℓ-modular reduction of the cuspidal unipotent character of $B_2(q)$, thus its Hecke algebra is the reduction of the one in characteristic 0. The Brauer tree in Table 16 shows that the cuspidal ℓ-modular Brauer character B_3^s of $B_3(q)$ is a constituent of the ℓ-modular reduction of an ordinary cuspidal character λ in the Lusztig series of an ℓ-element with centraliser a torus of order q^3+1. The minimal Levi overgroups are of types B_4 and B_3A_1, hence the parameters of the Hecke algebra for λ are as given.

Again, by the shape of the Brauer tree the cuspidal ℓ-modular Brauer character $\varphi_{1^5.}$ of $B_5(q)$ is liftable to an ordinary cuspidal character lying in the Lusztig series of an ℓ-element with centraliser $B_2(q)(q^3+1)$, and with centraliser $B_3(q)(q^3+1)$ in $B_6(q)$. Thus its Hecke algebra in $B_6(q)$ is $\mathcal{H}(B_1; q^3)$. Similarly, the cuspidal ℓ-modular Steinberg character φ_{1^6} of $A_5(q)$ is liftable to an ordinary cuspidal character in the Lusztig series of an ℓ-element with centraliser $(q^6-1)/(q-1)$, and with centraliser $\mathrm{GU}_2(q^3)$ in $B_6(q)$. Thus again its Hecke algebra in $B_6(q)$ is $\mathcal{H}(B_1; q^3)$. □

PROPOSITION 6.15. *Assume (T_ℓ). The decomposition matrix for the principal ℓ-block of $B_6(q)$ for primes ℓ with $(q^2-q+1)_\ell > 7$ is as given in Table 18.*

PROOF. The Hecke algebras for the cuspidal ℓ-modular Brauer character $1^5.$ of $B_5(q)$ and the cuspidal ℓ-modular Steinberg character of $A_5(q)$ were determined in Lemma 6.14. As neither is semisimple modulo ℓ, there is just one PIM of $B_6(q)$ in either series.

All columns except for those numbered 14, 22, 26 and 27 are obtained by Harish-Chandra induction from a Levi subgroup of type B_5. According to the description of the Hecke algebras in Lemma 6.14, with this we have accounted for all PIMs from proper Harish-Chandra series, so the remaining four PIMs must be cuspidal. By triangularity, their non-zero entries lie below the diagonal, and we denote them by $a_1, \ldots, a_{13}$ for Ψ_{14}, by $a_{14}, \ldots, a_{18}$ for Ψ_{22} and by a_{19} for Ψ_{26}. Using the order on families we actually have $a_i = 0$ for $i \in \{1, 2, 3, 14, 15\}$.

To get relations we use the combination of (DL) and (Red). Let w be a Coxeter element. The coefficients of R_w on Ψ_{26} and Ψ_{27} are equal to $-a_{17}$ and $2+a_{16}-a_{18}+a_{17}a_{19}$ respectively. Therefore by (DL) we must have $a_{17} = 0$ and $a_{18} \leq 2 + a_{16}$. On the other hand, if $(q^2-q+1)_\ell > 7$ and $\ell > 7$ then $(q^2-q+1)_\ell > 12$. By Example 1.7(a) there exists a regular ℓ-element of G^*. Therefore one can invoke (Red) to get $a_{18} \geq 2 + a_{16}$ hence $a_{18} = 2 + a_{16}$. We go on to the next Deligne–Lusztig character R_w for which Ψ_{26} and Ψ_{27} can potentially occur. It corresponds to $w = s_1s_2s_3s_2s_1s_2s_3s_4s_5s_6$. By (DL), the

TABLE 18. $B_6(q)$, $(q^2-q+1)_\ell > 7$

$6.$	1	1																										
$.6$	$\frac{1}{2}q$	.	1																									
$51.$	$\frac{1}{2}q$	1	.	1																								
2.4	$\frac{1}{2}q^3$	1	1	.	1																							
$B_2\colon 2^2.$	$\frac{1}{2}q^3$	.	.	.	.	1																						
$.51$	$\frac{1}{2}q^4$	.	1	.	.	.	1																					
$41^2.$	$\frac{1}{2}q^4$	.	.	1	.	.	.	1																				
1.41	$\frac{1}{2}q^5$	.	1	.	1	.	1	.	1																			
$B_2\colon 21.1$	$\frac{1}{2}q^5$	.	.	.	.	1	.	.	.	1																		
21.3	q^5	1	.	1	1	.	.	.	.	.	1																	
$1^2.31$	$\frac{1}{4}q^7$	.	.	.	1	.	.	.	1	.	1	1																
$21^2.2$	$\frac{1}{4}q^7$	.	.	1	.	.	.	1	.	.	1	.	1															
$B_2\colon 1^2.2$	$\frac{1}{4}q^7$	.	.	.	.	.	.	.	.	1	.	.	.	1														
B_6	$\frac{1}{4}q^7$	.	.	.	.	.	.	.	.	.	.	.	.	.	1													
$31^3.$	$\frac{1}{2}q^9$	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1												
$.41^2$	$\frac{1}{2}q^9$	.	.	.	.	.	1	.	1	.	.	.	.	.	.	.	1											
$B_2\colon 2.1^2$	$\frac{1}{2}q^9$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1										
$21^3.1$	$\frac{1}{2}q^{11}$	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	.	1									
$B_2\colon 1.21$	$\frac{1}{2}q^{11}$	.	.	.	.	1	.	.	.	1	.	.	.	1	.	.	.	1	.	1								
$1^3.21$	q^{11}	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	1							
$1^4.1^2$	$\frac{1}{2}q^{15}$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	1	1						
$B_2\colon .2^2$	$\frac{1}{2}q^{15}$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1					
$21^4.$	$\frac{1}{2}q^{16}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	1				
$.31^3$	$\frac{1}{2}q^{16}$	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.	.	.	.	.	.	1			
$.21^4$	$\frac{1}{2}q^{25}$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	.	1	1		
$1^6.$	$\frac{1}{2}q^{25}$	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	1	.	.	1	.	1	.	.	1	
$.1^6$	q^{36}	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	2	.	.	1	.	1
		ps	ps	ps	ps	B_2	ps	ps	ps	B_2	ps	ps	ps	B_3^s	c	ps	$.1^3$	B_3^s	ps	B_3^s	ps	A_5	c	$1^5.$	$.1^3$	$.1^3$	c	c

coefficients on these two PIMs give $2 - a_4 - a_6 + a_7 + a_9 - a_{12} \geq 0$ and

$$
\begin{aligned}
-(1+a_{19})\,&\underbrace{(2 - a_4 - a_6 + a_7 + a_9 - a_{12})}_{\geq 0 \text{ by (DL)}} \\
&- \underbrace{(-2 + 2a_4 + 2a_5 + 2a_6 - 2a_7 - 2a_8 - a_9 + a_{10} - a_{11} + a_{12} + a_{13})}_{\geq 0 \text{ by (Red)}} \geq 0.
\end{aligned}
$$

Therefore we must have $a_{12} = 2 - a_4 - a_6 + a_7 + a_9$ and $a_{13} = 2 - 2a_4 - 2a_5 - 2a_6 + 2a_7 + 2a_8 + a_9 - a_{10} + a_{11} - a_{12}$, hence $a_{13} = -a_4 - 2a_5 - a_6 + a_7 + 2a_8 - a_{10} + a_{11}$. Finally, the coefficient of Ψ_{27} in R_w for $w = s_1s_2s_3s_4s_3s_2s_1s_2s_3s_4s_5s_6$ is $48 - 24a_{19}$. By (DL) this forces $a_{19} \leq 2$.

It will be shown in the proofs for $B_7(q)$ and $B_8(q)$ in Proposition 6.16 that $a_i = 0$ for $i \in \{4, \dots, 11, 16, 19\}$. All columns are indecomposable by (HCr), respectively by the decomposition numbers for the principal series Hecke algebras. □

As already for ${}^2D_7(q)$ in order to obtain further information on the decomposition matrix for $B_6(q)$ we will consider blocks of larger groups.

Proposition 6.16. *Assume (T_ℓ). The decomposition matrices for the unipotent ℓ-blocks of $B_7(q)$ and $B_8(q)$ of non-cyclic defect for primes ℓ with $(q^2-q+1)_\ell > 7$ are as given in Tables 19, 20, 21 and 22, except that for the block labelled $\binom{0\ 1\ 2}{1\ 2}$ the 9th column, labelled B_6, might not be indecomposable.*

Here, a is as in Proposition 6.15 and we have $b_3 = 2 - b_1 + b_2$.

Here B_6^t denotes the Harish-Chandra series of the cuspidal ℓ-modular constituent of the unipotent character $B_2{:}\,.2^2$ of $B_6(q)$, B_2^a the series of the ℓ-modular cuspidal unipotent character $B_2 \boxtimes \varphi_{1^6}$ of B_2A_5.

Proof. Let us first consider the blocks of $B_7(q)$. In the principal block, (HCi) yields all columns except for the 19th, and a projective character Ψ with unipotent part $a_4[2^21^2.1] + a_5[B_2{:}\,1.31] + [B_6{:}\,1^2]$ plus further constituents with larger a-value. By our assumption of triangularity, $\Psi - a_4\Psi_{17} - a_5\Psi_{18}$ must then also be a projective character. By (HCr) this is only the case when $a_4 = a_5 = 0$.

Next, consider the blocks of $B_8(q)$. Here, the principal block shows that we must have $a_6 = a_8 = 0$, the block with label $\binom{0\ 1\ 2}{1\ 2}$ gives $a_{16} = 0$, and the block with label $\binom{1\ 2}{0}$ forces that $a_9 = a_{10} = a_{19} = 0$. Thus we have obtained all information that had been missing in the proof of Proposition 6.15. The correctness of the six printed decomposition matrices is now verified as in our previous proofs. The relation between the b_i in the block labelled $\binom{0\ 1\ 2}{\ }$ is obtained from (DL) using the Coxeter element.

Finally, to conclude that $a_{11} = 0$ (resp. $a_7 = 0$) we have to go up to the unipotent block of $B_9(q)$ (resp. $B_{10}(q)$) labelled by $\binom{1\ 2\ 3}{0\ 1}$ (resp. by $\binom{2\ 3}{0}$) and invoke (T_l) (we do not print the corresponding decomposition matrices). □

6. Symplectic groups

Finally, we consider the symplectic groups. The Brauer trees here are the same as for the odd-dimensional orthogonal groups and had already been given in Proposition 6.13. The arguments used to determine the decomposition matrices for blocks of defect 2 involve only Harish-Chandra induction/restriction, unipotent Deligne–Lusztig characters (which depend only on (W, F)) and the existence of ℓ-regular elements (which are similar for G and G^* when ℓ is odd, see Remark 3.1). Consequently, under our assumptions on ℓ, the unipotent part of the decomposition matrices of the unipotent ℓ-blocks of $C_n(q)$ and $B_n(q)$ are identical.

Corollary 6.17. *Assume (T_ℓ). The decomposition matrices for the unipotent ℓ-blocks of $C_n(q)$, $n = 6, 7, 8$, of non-cyclic defect for primes ℓ with $(q^2-q+1)_\ell > 7$ are the same as for $B_n(q)$ and hence as given in Tables 18–22, except possibly for the values of the yet unknown entries b_1 and b_2 (which need not be the same as for type B_n).*

TABLE 19. $B_7(q)$, $(q^2-q+1)_\ell > 7$, blocks $\binom{1}{}$ and $\binom{0\ 1}{\ 1\ }$

	ps	ps	ps	ps	B_2	ps	ps	B_2	ps	$.1^3$	B_3^s	ps	B_3^s	ps	ps	ps	ps	B_3^s	B_6	ps	B_6^t	$.1^3$	$1^5.$	A_5	$.1^3$	$1^6.$	$.1^6$
$7.$	1																										
$52.$	1	1																									
1.6	.	.	1																								
2.5	1	.	1	1																							
$B_2\colon 32.$	.	.	.	.	1																						
1.51	.	.	1	1	.	1																					
$421.$	.	1	.	.	.	.	1																				
$B_2\colon 31.1$	.	.	.	.	1	.	.	1																			
$2^2.3$	1	1	.	1	.	.	.	.	1																		
$.51^2$	.	.	.	.	.	1	.	.	.	1																	
$B_2\colon 1^2.3$	.	.	.	.	.	.	.	1	.	.	1																
$2^21.2$	.	1	.	.	.	.	1	.	1	.	.	1															
$B_2\colon 3.1^2$	.	.	.	.	.	.	.	1	.	.	.	.	1														
$321^2.$	.	.	.	.	.	.	1	.	.	.	.	.	.	1													
1.41^2	.	.	.	1	.	1	.	.	.	1	.	.	.	.	1												
$1^2.31^2$	.	.	.	1	.	.	.	.	1	.	.	.	.	.	1	1											
$2^21^2.1$	.	.	.	.	.	.	1	.	.	.	.	1	.	1	.	.	1										
$B_2\colon 1.31$	.	.	.	.	1	.	.	1	.	.	1	.	1	.	.	.	.	1									
$B_6\colon 1^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1								
$1^3.21^2$	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	1	.	.	.	1							
$B_2\colon .32$	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1						
1.31^3	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	1					
2^21^3	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	.	.	.	.	1				
$1^4.1^3$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	.	.	1			
1.21^4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1	.	.	1		
$1^7.$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	2	.	.	.	1	1	.	1	
1.1^6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	2	.	.	1	1	.	1

	ps	ps	ps	ps	ps	B_2	ps	ps	ps	ps	B_2	B_6	ps	B_3^s	$.1^3$	ps	ps	$1^5.$	B_3^s	$.1^3$	ps	B_3^s	A_5	B_6^t	$.1^3$	$1^6.$	$.1^6$
$.7$	1																										
6.1	.	1																									
51.1	.	1	1																								
3.4	1	1	.	1																							
31.3	.	1	1	1	1																						
$B_2\colon 2^21.$	.	.	.	.	.	1																					
$.52$	1	.	.	.	.	.	1																				
$41^2.1$	.	.	1	.	.	.	.	1																			
1.42	1	.	.	1	.	.	1	.	1																		
$31^2.2$	.	.	1	.	1	.	.	1	.	1																	
$B_2\colon 21^2.1$	.	.	.	.	.	1	.	.	.	.	1																
$B_6\colon 2$	.	.	.	.	.	.	.	.	.	.	.	1															
$1^2.32$	.	.	.	1	1	.	.	.	1	.	.	.	1														
$B_2\colon 1^3.2$	.	.	.	.	.	.	.	.	.	.	1	.	.	1													
$.421$	.	.	.	.	.	.	1	.	1	.	.	.	.	.	1												
$31^3.1$	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	1											
$1^3.2^2$	.	.	.	.	1	.	.	.	.	1	.	.	1	.	.	.	1										
$31^4.$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1									
$B_2\colon 2.1^3$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1								
$.321^2$	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1	.	.	.	.	1							
$21^4.1$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	1	.	.	1						
$B_2\colon 1.21^2$	.	.	.	.	.	1	.	.	.	.	1	.	.	1	.	.	.	.	1	.	.	1					
$1^5.1^2$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	1	.	1				
$B_2\colon .2^21$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1			
$.2^21^3$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	1	.	.	.	.	1		
$1^6.1$	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	.	.	1	.	.	1	.	1	.	.	1	
$.1^7$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	2	1	.	1

TABLE 20. $B_8(q)$, $(q^2-q+1)_\ell > 7$, blocks $\binom{2}{}$ and $\binom{0\ 1\ 2}{1\ 2}$

$8.$	1																										
$53.$	1	1																									
2.6	1	.	1																								
$B_2{:}\,42.$	.	.	.	1																							
1.61	.	.	1	.	1																						
$B_2{:}\,41.1$	.	.	.	1	.	1																					
$431.$	.	1	.	.	.	.	1																				
2.51	1	.	1	.	1	.	.	1																			
$.61^2$	.	.	.	.	1	.	.	.	1																		
$B_2{:}\,4.1^2$	.	.	.	.	.	1	.	.	.	1																	
$2^2.31$	1	1	.	.	.	.	.	1	.	.	1																
$B_2{:}\,1^2.4$	.	.	.	.	.	1	.	.	.	.	.	1															
$3^21^2.$	.	.	.	.	.	.	1	.	.	.	.	.	1														
2.41^2	.	.	.	.	1	.	.	1	1	.	.	.	.	1													
21.31^2	.	.	.	.	.	.	.	1	.	.	1	.	.	1	1												
$B_2{:}\,1.41$	.	.	.	1	.	1	.	.	.	1	.	1	.	.	.	1											
$2^21.21$	.	1	.	.	.	.	1	.	.	.	1	.	.	.	.	.	1										
$2^21^2.1^2$	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	1	1									
$21^2.21^2$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1	.	1								
$B_2{:}\,.42$	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1							
$B_6{:}\,.2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1						
2.31^3	.	.	.	.	.	.	.	.	1	.	.	.	.	1	1	.	.	.	.	.	.	1					
$21^3.1^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	1				
$2^21^4.$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	.	.	.	.	1			
2.21^4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	1	.	.	1		
$21^6.$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	2	.	1	1	.	1	
2.1^6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	2	.	.	1	.	1	.	1
	ps	ps	ps	B_2	ps	B_2	ps	ps	$.1^3$	B_3^s	ps	B_3^s	ps	ps	ps	B_3^s	ps	ps	ps	B_6^t	B_6	$.1^3$	A_5	$1^5.$	$.1^3$	$1^6.$	$.1^6$

$.71$	1																										
6.1^2	.	1																									
51.1^2	.	1	1																								
$.62$	1	.	.	1																							
3.41	1	1	.	.	1																						
31.31	.	1	1	.	1	1																					
2.42	1	.	.	1	1	.	1																				
$B_2{:}\,2^21^2.$	.	.	.	.	.	.	.	1																			
$B_6{:}\,1^2.$	.	.	.	.	.	.	.	.	1																		
$41^2.1^2$	.	.	1	.	.	.	.	.	.	1																	
$31^2.21$	.	.	1	.	.	1	.	.	.	1	1																
$B_2{:}\,21^3.1$	.	.	.	.	.	.	.	1	.	.	.	1															
21.32	.	.	.	.	1	1	1	.	.	.	.	.	1														
$21^2.2^2$	.	.	.	.	.	1	.	.	.	.	1	.	1	1													
$B_2{:}\,1^4.2$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1												
$31^3.1^2$	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	1											
$.42^2$	.	.	.	1	.	.	1	.	.	.	.	.	.	.	.	.	1										
$21^4.1^2$	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	1	.	1									
$.32^21$	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	1	.	1								
$31^5.$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1							
$B_2{:}\,2.1^4$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1						
$21^5.1$	.	.	.	.	.	.	.	.	2	.	.	.	.	.	.	1	.	1	.	1	.	1					
$B_2{:}\,1.21^3$	.	.	.	.	.	.	.	1	.	.	.	1	.	.	1	.	.	.	.	.	1	.	1				
$1^6.1^2$	.	.	.	.	.	.	.	.	2	.	.	.	.	1	.	.	.	1	.	.	.	1	.	1			
$.2^31^2$	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	1	.	.	.	.	.	1		
$B_2{:}\,.2^21^2$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	
$.1^8$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	1	1	2	1
	ps	ps	ps	ps	ps	ps	ps	B_2	B_6	ps	ps	B_2	ps	ps	B_3^s	ps	$.1^3$	ps	$.1^3$	$1^5.$	B_3^s	$1^6.$	B_3^s	A_5	$.1^3$	B_6^t	$.1^6$

TABLE 21. $B_8(q)$, $(q^2-q+1)_\ell > 7$, blocks $\binom{0\ 2}{\ 1\ }$ and $\binom{1\ 2}{\ 0\ }$

7.1	1																										
1.7	.	1																									
3.5	1	1	1																								
52.1	1	.	.	1																							
B_2: 321.	.	.	.	.	1																						
421.1	.	.	.	1	.	1																					
1.52	.	1	1	.	.	.	1																				
B_2: $31^2.1$	.	.	.	.	1	.	.	1																			
32.3	1	.	1	1	.	.	.	.	1																		
321.2	.	.	.	1	.	1	.	.	1	1																	
.521	.	.	.	.	.	.	1	.	.	.	1																
$321^2.1$	.	.	.	.	.	1	.	.	.	1	.	1															
1.421	.	.	1	.	.	.	1	.	.	.	1	.	1														
B_6: 1.1	.	.	.	.	.	.	.	.	.	.	.	.	.	1													
B_2: $1^3.3$	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1												
$1^2.321$	.	.	1	.	.	.	.	.	1	.	.	.	1	.	.	1											
B_2: 3.1^3	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1										
$321^3.$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1									
$2^21^3.1$	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	1	1								
1.321^2	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	.	.	.	1							
B_2: 1.31^2	.	.	.	.	1	.	.	1	.	.	.	.	.	.	1	.	1	.	.	.	1						
$1^3.2^21$	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	1	.	.	.	.	.	1					
B_2: .321	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1				
$1^5.1^3$	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	1	.	.	1	.	1			
1.2^21^3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1	.	.	1		
$1^7.1$	.	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	1	1	.	.	.	.	1	.	1	
1.1^7	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	2	1	1	.	1
	ps	ps	ps	ps	B_2	ps	ps	B_2	ps	ps	$.1^3$	ps	ps	B_6	B_3^s	ps	B_3^s	$1^5.$	ps	$.1^3$	B_3^s	ps	B_6^t	A_5	$.1^3$	$1^6.$	$.1^6$

71.	1																										
62.	1	1																									
$1^2.6$	.	.	1																								
B_2: $3^2.$	.	.	.	1																							
21.5	1	.	1	.	1																						
$1^2.51$	.	.	1	.	1	1																					
$42^2.$	.	1	.	.	.	.	1																				
B_2: 31.2	.	.	.	1	.	.	.	1																			
$2^2.4$	1	1	.	.	1	.	.	.	1																		
B_2: 21.3	.	.	.	.	.	.	.	1	.	1																	
B_2: 3.21	.	.	.	.	.	.	.	1	.	.	1																
$32^21.$	.	.	.	.	.	.	1	.	.	.	.	1															
$1^2.41^2$	.	.	.	.	1	1	.	.	1	.	.	.	1														
B_2: 2.31	.	.	.	1	.	.	.	1	.	1	1	.	.	1													
$2^3.2$	.	1	.	.	.	.	1	.	1	.	.	.	.	.	1												
$2^31.1$	.	.	.	.	.	.	1	.	.	.	.	1	.	.	1	1											
$.51^3$	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	1										
1.41^3	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	1	1									
$2^31^2.$	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	1								
$1^2.31^3$	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	.	.	1	.	1							
B_6: $.1^2$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1						
B_2: $.3^2$	.	.	.	1	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1					
$1^3.21^3$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	1	.	.	1				
$1^2.21^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	1			
$1^4.1^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	1	.	1		
$1^2.1^6$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	1	1	1	1	
$1^8.$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	1	.	2	.	.	.	1	.	1
	ps	ps	ps	B_2	ps	ps	ps	B_2	ps	B_3^s	B_3^s	ps	ps	B_3^s	ps	ps	$.1^3$	$.1^3$	$1^5.$	ps	B_6	B_6^t	ps	$.1^3$	A_5	$.1^6$	$1^6.$

TABLE 22. $B_8(q)$, $(q^2 - q + 1)_\ell > 7$, blocks $\binom{0\ 1}{2}$ and $\binom{0\ 1\ 2}{}$

$.8$	1																										
6.2	.	1																									
51.2	.	1	1																								
4.4	1	1	.	1																							
41.3	.	1	1	1	1																						
$.53$	1	.	.	.	.	1																					
$41^2.2$	.	.	1	.	1	.	1																				
$B_6\colon 2.$	.	.	.	.	.	.	.	1																			
$B_2\colon 2^3.$	.	.	.	.	.	.	.	.	1																		
1.43	1	.	.	1	.	1	.	.	.	1																	
$41^3.1$	.	.	.	.	.	.	1	.	.	.	1																
$.431$	.	.	.	.	.	1	.	.	.	1	.	1															
$31^3.2$	.	.	.	.	1	.	1	.	.	.	1	.	1														
$B_2\colon 21^2.1^2$	.	.	.	.	.	.	.	.	1	.	.	.	.	1													
$1^2.3^2$	.	.	.	1	1	.	.	.	.	1	.	.	.	.	1												
$B_2\colon 1^3.21$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1											
$41^4.$	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1										
$B_2\colon 21.1^3$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1									
$21^4.2$	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	1	.	1								
$.3^21^2$	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	.	.	.	.	1							
$B_2\colon 1^2.21^2$	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	1	.	1	.	.	1						
$1^4.2^2$	.	.	.	.	1	.	.	.	.	.	.	.	1	.	1	.	.	.	.	.	.	1					
$1^5.21$	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	1	1				
$1^6.2$	.	.	.	.	.	.	.	2	.	.	.	.	.	.	.	.	1	.	1	.	.	.	1	1			
$B_2\colon .2^3$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1		
$.2^21^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	1	.	.	.	1	
$.21^6$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	2	1	1
	ps	ps	ps	ps	ps	ps	ps	B_6	B_2	ps	ps	$.1^3$	ps	B_2	ps	B_3^s	$1^5.$	B_3^s	ps	$.1^3$	B_3^s	ps	A_5	$1^6.$	B_6^t	$.1^3$	$.1^6$

$B_2\colon 6.$	1																										
$B_2\colon 51.$	1	1																									
$4^2.$	.	.	1																								
$B_2\colon 41^2.$	.	1	.	1																							
43.1	.	.	1	.	1																						
42.1^2	.	.	.	.	1	1																					
$B_2\colon .6$	1	.	.	.	.	.	1																				
$B_2\colon 31^3.$	.	.	.	1	.	.	.	1																			
$3^2.2$	.	.	.	.	1	.	.	.	1																		
32.21	.	.	1	.	1	1	.	.	1	1																	
41.1^3	.	.	.	.	.	1	.	.	.	.	1																
31.21^2	.	.	.	.	.	1	.	.	.	1	1	1															
$B_2\colon .51$	1	1	.	.	.	.	1	.	.	.	.	.	1														
$B_2\colon 21^4.$	.	.	.	.	.	.	.	1	.	.	.	.	.	1													
$2^2.2^2$	.	.	1	.	.	.	.	.	.	1	.	.	.	.	1												
21.2^21	.	.	.	.	.	.	.	.	1	1	.	1	.	.	1	1											
4.1^4	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1										
3.21^3	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	1	1									
2.2^21^2	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1	1								
$B_2\colon 1^6.$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1							
$B_2\colon .41^2$	.	1	.	1	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1						
$1^2.2^3$	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	1	.	.	.	.	.	1					
1.2^31	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	1	.	.	1	1				
$.2^4$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	1	1			
$B_2\colon .31^3$	.	.	.	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	b_1	1		
$B_2\colon .21^4$	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	b_2	1	1	
$B_2\colon .1^6$	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	1	.	.	.	b_3	.	1	1
	B_2	B_2	ps	B_2	ps	ps	B_3^s	B_2	ps	ps	ps	ps	B_3^s	B_2	ps	ps	$.1^3$	$.1^3$	$.1^3$	B_2^a	B_3^s	$.1^3$	$.1^3$	c	B_3^s	B_3^s	c

7. Unipotent decomposition matrix of $F_4(q)$

THEOREM 6.18. *Let $(q,6)=1$. The decomposition matrix for the principal ℓ-block of $F_4(q)$ for primes $\ell \geq 7$ with $(q^2-q+1)_\ell > 7$ is approximated by Table 23.*

Here, the unknown entries satisfy $y_6 = 1+y_1+y_2$, $z_3 \leq 1$ and $x_1 + 3(z_1+z_2) \leq 5$ (in particular all the z_i are equal to either 0 or 1).

TABLE 23. $F_4(q)$, $(q^2-q+1)_\ell > 7$, $(q,6)=1$

$\phi_{1,0}$	1	1																				
$\phi'_{2,4}$	$\frac{1}{2}q\Phi_4\Phi_8\Phi_{12}$	.	1																			
$\phi''_{2,4}$	$\frac{1}{2}q\Phi_4\Phi_8\Phi_{12}$	.	.	1																		
$B_2{:}\,2.$	$\frac{1}{2}q\Phi_1^2\Phi_3^2\Phi_8$	.	.	.	1																	
$\phi'_{8,3}$	$q^3\Phi_4^2\Phi_8\Phi_{12}$	1	1	.	.	1																
$\phi''_{8,3}$	$q^3\Phi_4^2\Phi_8\Phi_{12}$	1	.	1	.	.	1															
$\phi_{12,4}$	$\frac{1}{24}q^4\Phi_2^4\Phi_3^2\Phi_8\Phi_{12}$	1	.	.	.	1	1	1														
$\phi'_{9,6}$	$\frac{1}{8}q^4\Phi_3^2\Phi_4^2\Phi_8\Phi_{12}$	.	1	.	.	1	.	.	1													
$\phi''_{9,6}$	$\frac{1}{8}q^4\Phi_3^2\Phi_4^2\Phi_8\Phi_{12}$	.	.	1	.	.	1	.	.	1												
$F_4[1]$	$\frac{1}{8}q^4\Phi_1^4\Phi_3^2\Phi_8\Phi_{12}$	.	.	.	.	.	.	.	.	.	1											
$B_2{:}\,.2$	$\frac{1}{4}q^4\Phi_1^2\Phi_3^2\Phi_4\Phi_8\Phi_{12}$	.	.	.	1	.	.	.	.	.	.	1										
$B_2{:}\,1^2.$	$\frac{1}{4}q^4\Phi_1^2\Phi_3^2\Phi_4\Phi_8\Phi_{12}$	.	.	.	1	.	.	.	.	.	.	.	1									
$F_4[-1]$	$\frac{1}{4}q^4\Phi_1^4\Phi_3^2\Phi_4^2\Phi_{12}$	.	.	.	.	.	.	.	.	.	.	.	.	1								
$F_4[\zeta_3]$	$\frac{1}{3}q^4\Phi_1^4\Phi_2^4\Phi_4^2\Phi_8$	.	.	.	.	.	.	.	.	.	.	.	.	.	1							
$F_4[\zeta_3^2]$	$\frac{1}{3}q^4\Phi_1^4\Phi_2^4\Phi_4^2\Phi_8$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1						
$\phi'_{8,9}$	$q^9\Phi_4^2\Phi_8\Phi_{12}$	.	.	.	.	1	.	1	1	.	.	.	.	.	y_1	y_1	1					
$\phi''_{8,9}$	$q^9\Phi_4^2\Phi_8\Phi_{12}$	.	.	.	.	.	1	1	.	1	.	.	.	.	y_2	y_2	.	1				
$\phi'_{2,16}$	$\frac{1}{2}q^{13}\Phi_4\Phi_8\Phi_{12}$	.	.	.	.	.	.	.	1	.	.	.	.	.	y_1	y_1	1	.	1			
$\phi''_{2,16}$	$\frac{1}{2}q^{13}\Phi_4\Phi_8\Phi_{12}$	.	.	.	.	.	.	.	.	1	.	.	.	.	y_2	y_2	.	1	.	1		
$B_2{:}\,.1^2$	$\frac{1}{2}q^{13}\Phi_1^2\Phi_3^2\Phi_8$	.	.	.	1	.	.	.	.	.	.	1	1	.	1	1	.	.	.	.	1	
$\phi_{1,24}$	q^{24}	.	.	.	.	.	.	1	.	.	x_1	.	.	2	y_6	y_6	1	1	z_1	z_2	z_3	1
		ps	ps	ps	B_2	ps	ps	ps	ps	ps	c	B_3	C_3	c	c	c	B_3	C_3	c	c	c	c

PROOF. Köhler [**38**, Table A.162] has shown uni-triangularity and determined the ℓ-modular Harish-Chandra series of the unipotent Brauer characters under the stated assumptions on ℓ. He has also proved that the ordinary cuspidal characters $F_4[1]$ and $F_4[-1]$ can only occur in the reduction of the Steinberg character $\phi_{1,24}$. We denote these multiplicities by x_1 and x_2. Similarly, he showed that the cuspidal irreducible Brauer characters belonging to $\phi'_{2,16}$, $\phi''_{2,16}$ and $B_2{:}\,.1^2$ can occur only in the reduction of the Steinberg character, and we denote by z_1, z_2, z_3 their respective multiplicities. The projective characters corresponding to the Galois conjugate cuspidal characters $F_4[\zeta_3]$ and $F_4[\zeta_3^2]$ can have potentially several unipotent constituents. We will write the unipotent part of these projective characters as

$$\begin{aligned} &\Psi_{14} = F_4[\zeta_3] + y_1\phi'_{8,9} + y_2\phi''_{8,9} + y_3\phi'_{2,16} + y_4\phi''_{2,16} + y_5 B_2{:}\,.1^2 + y_6\phi_{1,24} \\ \text{and}\quad &\Psi_{15} = F_4[\zeta_3^2] + y_1\phi'_{8,9} + y_2\phi''_{8,9} + y_3\phi'_{2,16} + y_4\phi''_{2,16} + y_5 B_2{:}\,.1^2 + y_6\phi_{1,24}. \end{aligned}$$

We now use (DL) to compute some of these coefficients. We start with the Deligne–Lusztig character R_w where w is a Coxeter element. The coefficients of Ψ_{18}, Ψ_{19} and Ψ_{20} in R_w are $2y_1 - 2y_3$, $2y_2 - 2y_4$ and $2 - 2y_5$ respectively and are all non-negative by (DL). The coefficient on Ψ_{21} is

$$X = 2 + 2y_1 + 2y_2 - 2y_6 - z_1(2y_1 - 2y_3) - z_2(2y_2 - 2y_4) - z_3(2 - 2y_5) \geq 0.$$

Now if $(q^2 - q + 1)_\ell > 7$, we can use (Red) for the maximal torus of order $(q^2 - q + 1)^2$ to obtain $-3 - 2y_1 - 2y_2 + y_3 + y_4 + 2y_5 + y_6 \geq 0$. Adding twice this non-negative number to X we get

$$-(1 + z_1)(2y_1 - 2y_3) - (1 + z_2)(2y_2 - 2y_4) - (2 + z_3)(2 - 2y_5) \geq 0$$

which forces $2y_1 - 2y_3 = 2y_2 - 2y_4 = 2 - 2y_5 = 0$ and $X = 0$. This gives

$$y_1 = y_3, \quad y_2 = y_4, \quad y_5 = 1 \quad \text{and} \quad y_6 = 1 + y_1 + y_2.$$

We continue with the Deligne–Lusztig character associated with $w = s_1s_2s_3s_4s_2s_3$. The coefficient of Ψ_{21} in R_w is $2 - x_2$, hence $x_2 \leq 2$ by (DL). On the other hand, (Red) applied to the case of a torus gives $x_2 \geq 2$ hence $x_2 = 2$.

We finish with the Deligne–Lusztig character R_w with $w = s_1s_2s_3s_4s_1s_2s_3s_4$. The coefficient of the PIM Ψ_{21} is $25 - x_1 - 9z_1 - 9z_2 - 14z_3$. Therefore by (DL) we must have $z_3 \leq 1$ and $z_1 + z_2 \leq 2$. If all of the z_i are zero then we can only deduce that $x_1 \leq 25$, which is not satisfactory. However, if one considers the generalised q^4-eigenspace of F on R_w, then the coefficient of Ψ_{21} is $5 - x_1 - 3z_1 - 3z_2$, which gives $x_1 \leq 5$ if $z_1 = z_2 = 0$ or $x_1 \leq 2$ if z_1 or z_2 is non-zero. □

Again, we collect data on the Φ_6-modular Harish-Chandra series in the blocks b considered in this section in Tables 24, 25 and 26. None of the decomposition matrices determined here agree after any permutations of rows and columns, so none of the blocks can be Morita equivalent.

TABLE 24. Modular Harish-Chandra-series in Φ_6-blocks

G	b	$W_G(b)$	$\lvert\mathrm{IBr}\, b\rvert$	ps	D_4	$.1^4$	A_5	$.1^6$	D_6^s	E_6
D_6		$G(6,2,2)$	18	10	2	2	2	1	1	
D_8	2	$G(6,2,2)$	18	10	2	2	2	1	1	
D_7		$G(6,1,2)$	27	14	4	4	1	2	2	
D_8	1,3	$G(6,1,2)$	27	14	4	4	1	2	2	
E_6		G_5	21	11	2	2	1			5
E_8	2	G_5	21	11	2	2	1			5
				ps	B_2	B_3	C_3	c		
F_4		G_5	21	8	1	2	2	8		

Table 25. Modular Harish-Chandra-series in Φ_6-blocks, twisted groups

G	b	$W_G(b)$	$\lvert\mathrm{IBr}\, b\rvert$	ps	$.1^2$	A_5	$1^6.$	$.2^3$	$.1^6$
${}^2D_7$		$G(6,1,2)$	27	15	8	1	1	1	1
${}^2D_8$	$1,2$	$G(6,1,2)$	27	15	8	1	1	1	1
${}^2D_9$	$1,2$	$G(6,1,2)$	27	15	8	1	1	1	1

Table 26. Modular Harish-Chandra-series in Φ_6-blocks, type B_n

G	b	$W_G(b)$	$\lvert\mathrm{IBr}\, b\rvert$	ps	B_2	$.1^3$	B_3^s	A_5	B_6^t	$1^5.$	B_6	$1^6.$	$.1^6$	B_2^a	c
B_6, B_7		$G(6,1,2)$	27	13	2	3	3	1	1	1	1	1	1		
B_8	1-5	$G(6,1,2)$	27	13	2	3	3	1	1	1	1	1	1		
B_8	6	$G(6,1,2)$	27	9	5	5	5	0	0	0	0	0	0	1	2

CHAPTER 7

Decomposition matrices at $d_\ell(q) = 5, 7, 8, 10, 12, 14$

In this final section we collect the decomposition matrices for primes ℓ with $d_\ell(q) \notin \{1, 2, 3, 4, 6\}$ for the classical groups considered in this work. (The Brauer trees for exceptional groups can be found in [**10**].) In all cases, the corresponding cyclotomic polynomial divides the order polynomial of the groups in question at most once, so all such blocks are of cyclic defect. We thus give their Brauer trees; they were all first determined by Fong and Srinivasan [**20**].

Proposition 7.1. *The Brauer trees of the unipotent ℓ-blocks, for $11 \le \ell | \Phi_5(q)$, of $B_5(q)$, $D_n(q)$ with $5 \le n \le 7$ and of ${}^2D_n(q)$ with $6 \le n \le 7$, are as given in Table 1.*

Table 1. Brauer trees for $11 \le \ell | \Phi_5(q)$

$$D_5(q):\quad .5 \text{ --- } .41 \text{ --- } .31^2 \text{ --- } .21^3 \text{ --- } .1^5 \text{ --- } \bigcirc$$

$$D_7(q):\quad 1.6 \underset{ps}{\text{---}} 1.42 \underset{ps}{\text{---}} 1.321 \underset{ps}{\text{---}} 1.2^21^2 \underset{ps}{\text{---}} 1.1^6 \underset{1^4}{\text{---}} \bigcirc$$

$$D_6(q):\quad .6 \text{ --- } .42 \text{ --- } .321 \text{ --- } .2^21^2 \text{ --- } .1^6 \text{ --- } \bigcirc \text{ --- } 1.1^5 \text{ --- } 1.21^3 \text{ --- } 1.31^2 \text{ --- } 1.41 \text{ --- } 1.5$$

$$D_7(q):\quad .7 \text{ --- } .43 \text{ --- } .3^21 \text{ --- } .2^21^3 \text{ --- } .21^5 \text{ --- } \bigcirc \text{ --- } 1^5.2 \text{ --- } 2.21^3 \text{ --- } 2.31^2 \text{ --- } 2.41 \text{ --- } 2.5$$

$$.61 \underset{ps}{\text{---}} .52 \underset{ps}{\text{---}} .32^2 \underset{ps}{\text{---}} .2^31 \underset{ps}{\text{---}} .1^7 \underset{1^4}{\text{---}} \bigcirc \underset{1^4}{\text{---}} 1^2.1^5 \underset{ps}{\text{---}} 1^2.21^3 \underset{ps}{\text{---}} 1^2.31^2 \underset{ps}{\text{---}} 1^2.41 \underset{ps}{\text{---}} 1^2.5$$

$$B_5(q):\quad 5. \text{ --- } 41. \text{ --- } 31^2. \text{ --- } 21^3. \text{ --- } 1^5. \text{ --- } \bigcirc \text{ --- } .1^5 \text{ --- } .21^3 \text{ --- } .31^2 \text{ --- } .41 \text{ --- } .5$$

$${}^2D_6(q):\quad 5. \text{ --- } 41. \text{ --- } 31^2. \text{ --- } 21^3. \text{ --- } 1^5. \text{ --- } \bigcirc \text{ --- } .1^5 \text{ --- } .21^3 \text{ --- } .31^2 \text{ --- } .41 \text{ --- } .5$$

$${}^2D_7(q):\quad 6. \text{ --- } 42. \text{ --- } 321. \text{ --- } 2^21^2. \text{ --- } 1^6 \text{ --- } \bigcirc \text{ --- } 1.1^5 \text{ --- } 1.21^3 \text{ --- } 1.31^2 \text{ --- } 1.41 \text{ --- } 1.5$$

$$5.1 \underset{ps}{\text{---}} 41.1 \underset{ps}{\text{---}} 31^2.1 \underset{ps}{\text{---}} 21^3.1 \underset{ps}{\text{---}} 1^5.1 \underset{1^4}{\text{---}} \bigcirc \underset{1^4}{\text{---}} .1^6 \underset{ps}{\text{---}} .2^21^2 \underset{ps}{\text{---}} .321 \underset{ps}{\text{---}} .42 \underset{ps}{\text{---}} .6$$

Here, "1^4" denotes the cuspidal Steinberg PIM of $A_4(q)$, and similarly in the subsequent Brauer trees, the Harish-Chandra series are labelled by names of unipotent characters in which the corresponding cuspidal Brauer character first appears.

TABLE 2. Brauer trees for $17 \leq \ell | \Phi_8(q)$

$B_4(q)$: $4. \;\text{—}\; 3.1 \;\text{—}\; 2.1^2 \;\text{—}\; 1.1^3 \;\text{—}\; .1^4 \;\text{—}\; \bigcirc \;\text{—}\; B_2{:}\,.1^2 \;\text{—}\; B_2{:}\,1.1 \;\text{—}\; B_2{:}\,2.$

$B_5(q)$: $5. \;\text{—}\; 3.2 \;\text{—}\; 2.21 \;\text{—}\; 1.21^2 \;\text{—}\; .21^3 \;\text{—}\; \bigcirc \;\text{—}\; B_2{:}\,.1^3 \;\text{—}\; B_2{:}\,1^2.1 \;\text{—}\; B_2{:}\,21.$

$41. \;\underset{ps}{\text{—}}\; 31.1 \;\underset{ps}{\text{—}}\; 21.1^2 \;\underset{ps}{\text{—}}\; 1^2.1^3 \;\underset{ps}{\text{—}}\; .1^5 \;\underset{.1^4}{\text{—}}\; \bigcirc \;\underset{B_2:\,.1^2}{\text{—}}\; B_2{:}\,.21 \;\underset{B_2}{\text{—}}\; B_2{:}\,1.2 \;\underset{B_2}{\text{—}}\; B_2{:}\,3.$

$D_5(q)$: $.5 \;\text{—}\; 1.4 \;\text{—}\; 1^2.3 \;\text{—}\; 1^3.2 \;\text{—}\; 1.1^4 \;\text{—}\; .1^5 \;\text{—}\; \bigcirc \;\text{—}\; D_4{:}\,1^2 \;\text{—}\; D_4{:}\,2$

$D_6(q)$: $.6 \;\text{—}\; 2.4 \;\text{—}\; 21.3 \;\text{—}\; 21^2.2 \;\text{—}\; 21^3.1 \;\text{—}\; 21^4. \;\text{—}\; \bigcirc \;\text{—}\; D_4{:}\,.1^2 \;\text{—}\; D_4{:}\,1^2.$

$.51 \;\text{—}\; 1.41 \;\text{—}\; 1^2.31 \;\text{—}\; 1^3.21 \;\text{—}\; 1^4.1^2 \;\text{—}\; 1^6. \;\text{—}\; \bigcirc \;\text{—}\; D_4{:}\,.2 \;\text{—}\; D_4{:}\,2.$

$D_7(q)$: $.7 \;\text{—}\; 3.4 \;\text{—}\; 31.3 \;\text{—}\; 31^2.2 \;\text{—}\; 31^3.1 \;\text{—}\; 31^4. \;\text{—}\; \bigcirc \;\text{—}\; D_4{:}\,1.1^2 \;\text{—}\; D_4{:}\,1^2.1$

$1.6 \;\text{—}\; 2.5 \;\text{—}\; 2^2.3 \;\text{—}\; 2^21.2 \;\text{—}\; 2^21^2.1 \;\text{—}\; 2^21^3. \;\text{—}\; \bigcirc \;\text{—}\; D_4{:}\,.1^3 \;\text{—}\; D_4{:}\,1^3.$

$.61 \;\text{—}\; 2.41 \;\text{—}\; 21.31 \;\text{—}\; 21^2.21 \;\text{—}\; 21^3.1^2 \;\text{—}\; 21^5. \;\text{—}\; \bigcirc \;\text{—}\; D_4{:}\,.21 \;\text{—}\; D_4{:}\,21.$

$.52 \;\text{—}\; 1.42 \;\text{—}\; 1^2.32 \;\text{—}\; 1^3.2^2 \;\text{—}\; 1^5.1^2 \;\text{—}\; 1^6.1 \;\text{—}\; \bigcirc \;\text{—}\; D_4{:}\,.3 \;\text{—}\; D_4{:}\,3.$

$.51^2 \;\underset{ps}{\text{—}}\; 1.41^2 \;\underset{ps}{\text{—}}\; 1^2.31^2 \;\underset{ps}{\text{—}}\; 1^3.21^2 \;\underset{ps}{\text{—}}\; 1^4.1^3 \;\underset{ps}{\text{—}}\; 1^7. \;\underset{.1^5}{\text{—}}\; \bigcirc \;\underset{D_4:\,.1^2}{\text{—}}\; D_4{:}\,1.2 \;\underset{D_4}{\text{—}}\; D_4{:}\,2.1$

${}^2D_5(q)$: $4. \;\text{—}\; 2.2 \;\text{—}\; 1.21 \;\text{—}\; .21^2 \;\text{—}\; \bigcirc \;\text{—}\; .1^4 \;\text{—}\; 1^2.1^2 \;\text{—}\; 21.1 \;\text{—}\; 31.$

${}^2D_6(q)$: $5. \;\text{—}\; 2.3 \;\text{—}\; 1.31 \;\text{—}\; .31^2 \;\text{—}\; \bigcirc \;\text{—}\; 1.1^4 \;\text{—}\; 1^2.1^3 \;\text{—}\; 2^2.1 \;\text{—}\; 32.$

$4.1 \;\text{—}\; 3.2 \;\text{—}\; 1.2^2 \;\text{—}\; .2^21 \;\text{—}\; \bigcirc \;\text{—}\; .1^5 \;\text{—}\; 1^3.1^2 \;\text{—}\; 21^2.1 \;\text{—}\; 31^2.$

${}^2D_7(q)$: $6. \;\text{—}\; 2.4 \;\text{—}\; 1.41 \;\text{—}\; .41^2 \;\text{—}\; \bigcirc \;\text{—}\; 2.1^4 \;\text{—}\; 21.1^3 \;\text{—}\; 2^2.1^2 \;\text{—}\; 3^2.$

$51. \;\text{—}\; 21.3 \;\text{—}\; 1^3.31 \;\text{—}\; .31^3 \;\text{—}\; \bigcirc \;\text{—}\; 1.21^3 \;\text{—}\; 1^2.21^2 \;\text{—}\; 2^2.2 \;\text{—}\; 42.$

$5.1 \;\text{—}\; 3.3 \;\text{—}\; 1.32 \;\text{—}\; .321 \;\text{—}\; \bigcirc \;\text{—}\; 1.1^5 \;\text{—}\; 1^3.1^3 \;\text{—}\; 2^21.1 \;\text{—}\; 321.$

$41^2. \;\text{—}\; 21^2.2 \;\text{—}\; 1^3.21 \;\text{—}\; .21^4 \;\text{—}\; \bigcirc \;\text{—}\; .2^21^2 \;\text{—}\; 1^2.2^2 \;\text{—}\; 31.2 \;\text{—}\; 41.1$

$4.1^2 \;\underset{ps}{\text{—}}\; 3.21 \;\underset{ps}{\text{—}}\; 2.2^2 \;\underset{ps}{\text{—}}\; .2^3 \;\underset{.1^3}{\text{—}}\; \bigcirc \;\underset{.1^3}{\text{—}}\; .1^6 \;\underset{ps}{\text{—}}\; 1^4.1^2 \;\underset{ps}{\text{—}}\; 21^3.1 \;\underset{ps}{\text{—}}\; 31^3.$

${}^2D_4(q)$: $3. \;\text{—}\; 2.1 \;\text{—}\; 1.1^2 \;\text{—}\; .1^3 \;\text{—}\; \bigcirc$

${}^2D_6(q)$: $41. \;\underset{ps}{\text{—}}\; 21.2 \;\underset{ps}{\text{—}}\; 1^2.21 \;\underset{ps}{\text{—}}\; .21^3 \;\underset{.1^3}{\text{—}}\; \bigcirc$

PROPOSITION 7.2. *The Brauer trees of the unipotent ℓ-blocks, for $17 \leq \ell | \Phi_8(q)$, of $B_4(q)$, $B_5(q)$, $D_n(q)$ with $5 \leq n \leq 7$ and of ${}^2D_n(q)$ with $4 \leq n \leq 7$, are as given in Table 2.*

PROPOSITION 7.3. *The Brauer tree of the principal ℓ-block of $D_7(q)$, for $29 \leq \ell | \Phi_7(q)$, is as given by*

$$.7 \underset{ps}{\text{---}} .61 \underset{ps}{\text{---}} .51^2 \underset{ps}{\text{---}} .41^3 \underset{ps}{\text{---}} .31^4 \underset{ps}{\text{---}} .21^5 \underset{ps}{\text{---}} .1^7 \underset{c}{\text{---}} \bigcirc$$

PROPOSITION 7.4. *The Brauer tree of the principal ℓ-block of ${}^2D_7(q)$, for $29 \leq \ell | \Phi_{14}(q)$, is as given by*

$$6. \underset{ps}{\text{---}} 5.1 \underset{ps}{\text{---}} 4.1^2 \underset{ps}{\text{---}} 3.1^3 \underset{ps}{\text{---}} 2.1^4 \underset{ps}{\text{---}} 1.1^5 \underset{ps}{\text{---}} .1^6 \underset{c}{\text{---}} \bigcirc$$

PROPOSITION 7.5. *The Brauer trees of the unipotent ℓ-blocks, for $11 \leq \ell | \Phi_{10}(q)$, of $B_5(q)$, $D_n(q)$ with $6 \leq n \leq 7$ and of ${}^2D_n(q)$ with $5 \leq n \leq 7$, are as given in Table 3.*

TABLE 3. Brauer trees for $11 \leq \ell | \Phi_{10}(q)$

$$B_5(q) : \; 5. \underset{ps}{\text{---}} 4.1 \underset{ps}{\text{---}} 3.1^2 \underset{ps}{\text{---}} 2.1^3 \underset{ps}{\text{---}} 1.1^4 \underset{ps}{\text{---}} .1^5 \underset{c}{\text{---}} \bigcirc \underset{c}{\text{---}} B_2{:}\,.1^3 \underset{B_2}{\text{---}} B_2{:}\,1.1^2 \underset{B_2}{\text{---}} B_2{:}\,2.1 \underset{B_2}{\text{---}} B_2{:}\,3.$$

$$D_6(q) : \; .6 \text{---} 1.5 \text{---} 1^2.4 \text{---} 1^3.3 \text{---} 1^4.2 \text{---} 1^5.1 \text{---} 1^6. \text{---} \bigcirc \text{---} D_4{:}\,.1^2 \text{---} D_4{:}\,1.1 \text{---} D_4{:}\,2.$$

$$D_7(q) : \; .7 \text{---} 2.5 \text{---} 21.4 \text{---} 21^2.3 \text{---} 21^3.2 \text{---} 21^4.1 \text{---} 21^5. \text{---} \bigcirc \text{---} D_4{:}\,.1^3 \text{---} D_4{:}\,1^2.1 \text{---} D_4{:}\,21.$$

$$.61 \underset{ps}{\text{---}} 1.51 \underset{ps}{\text{---}} 1^2.41 \underset{ps}{\text{---}} 1^3.31 \underset{ps}{\text{---}} 1^4.21 \underset{ps}{\text{---}} 1^5.1^2 \underset{ps}{\text{---}} 1^7. \underset{1^6.}{\text{---}} \bigcirc \underset{D_4:\,.1^2}{\text{---}} D_4{:}\,.21 \underset{D_4}{\text{---}} D_4{:}\,1.2 \underset{D_4}{\text{---}} D_4{:}\,3.$$

$${}^2D_6(q) : \; 5. \text{---} 3.2 \text{---} 2.21 \text{---} 1.21^2 \text{---} .21^3 \text{---} \bigcirc \text{---} .1^5 \text{---} 1^2.1^3 \text{---} 21.1^2 \text{---} 31.1 \text{---} 41.$$

$${}^2D_7(q) : \; 6. \text{---} 3.3 \text{---} 2.31 \text{---} 1.31^2 \text{---} .31^3 \text{---} \bigcirc \text{---} 1.1^5 \text{---} 1^2.1^4 \text{---} 2^2.1^2 \text{---} 32.1 \text{---} 4.2$$

$$5.1 \underset{ps}{\text{---}} 4.2 \underset{ps}{\text{---}} 2.2^2 \underset{ps}{\text{---}} 1.2^21 \underset{ps}{\text{---}} .2^21^2 \underset{.1^4}{\text{---}} \bigcirc \underset{.1^4}{\text{---}} .1^6 \underset{ps}{\text{---}} 1^3.1^3 \underset{ps}{\text{---}} 21^2.1^2 \underset{ps}{\text{---}} 31^2.1 \underset{ps}{\text{---}} 41^2.$$

$${}^2D_5(q) : \; 4. \text{---} 3.1 \text{---} 2.1^2 \text{---} 1.1^3 \text{---} .1^4 \text{---} \bigcirc$$

$${}^2D_7(q) : \; 51. \underset{ps}{\text{---}} 31.2 \underset{ps}{\text{---}} 21.21 \underset{ps}{\text{---}} 1^2.21^2 \underset{ps}{\text{---}} .21^4 \underset{.1^4}{\text{---}} \bigcirc$$

PROPOSITION 7.6. *The Brauer trees of the unipotent ℓ-blocks, for $13 \leq \ell | \Phi_{12}(q)$, of $B_6(q)$, $D_7(q)$, ${}^2D_6(q)$ and ${}^2D_7(q)$ are as given in Table 4.*

Table 4. Brauer trees for $13 \leq \ell | \Phi_{12}(q)$

$$\begin{array}{lccccccccccccccc}
D_7(q): & .7 & — & 1.6 & — & 1^2.5 & — & 1^3.4 & — & 1^4.3 & — & 1^5.2 & — & 1^6.1 & — & 1^7. \\
 & & & & & & & & & & & & & & & | \\
 & & & & & & & D_4{:}\,3. & — & D_4{:}\,2.1 & — & D_4{:}\,1.1^2 & — & D_4{:}\,.1^3 & — & \bigcirc
\end{array}$$

$$\begin{array}{lccccccccccccc}
{}^2D_6(q): & 5. & \underset{ps}{—} & 4.1 & \underset{ps}{—} & 3.1^2 & \underset{ps}{—} & 2.1^3 & \underset{ps}{—} & 1.1^4 & \underset{ps}{—} & .1^5 & \underset{c}{—} & \bigcirc
\end{array}$$

$$\begin{array}{lccccccccccccc}
{}^2D_7(q): & 6. & — & 4.2 & — & 3.21 & — & 2.21^2 & — & 1.21^3 & — & .21^4 & — & \bigcirc \\
 & & & & & & & & & & & & & | \\
 & & & 51. & — & 41.1 & — & 31.1^2 & — & 21.1^3 & — & 1^2.1^4 & — & .1^6
\end{array}$$

CHAPTER 8

On a conjecture of Craven

Let $G = \mathbf{G}^F$ for $\mathbf{G}$ connected reductive defined over $\mathbb{F}_q$. By [**40**] for any unipotent character ρ of G there exists a *degree polynomial* $R_\rho \in \mathbb{Q}[x]$ such that, in particular, $\rho(1) = R_\rho(q)$. The degree (resp. the valuation at x) of R_ρ is denoted A_ρ (resp. a_ρ). By the main theorem of [**5**], under suitable conditions the decomposition matrix of the unipotent blocks of G is unitriangular with respect to the ordering on families. This induces a bijection $\rho \mapsto \varphi_\rho$ between the unipotent characters and the irreducible Brauer characters lying in unipotent blocks. Since both the a-function and A-function are decreasing with respect to the ordering on families, we have, for unipotent characters $\rho \neq \rho'$

$$\langle \rho, \Psi_{\varphi_{\rho'}} \rangle \neq 0 \Longrightarrow a_\rho > a_{\rho'} \text{ and } A_\rho > A_{\rho'}.$$

In [**10**], Craven predicts the existence of perverse equivalences between unipotent ℓ-blocks of G and their Brauer correspondents, for primes ℓ such that Sylow ℓ-subgroups of G are abelian. A consequence of the existence of such equivalences is that the decomposition matrix would be unitriangular with respect to the perversity function (see [**9**, Prop. 8.1]).

Let us recall how the perversity function of Craven's conjectural perverse equivalence is defined. For a root of unity $\zeta \in \mu(\mathbb{C})$ we denote by $\mathrm{Arg}(\zeta) \in]0, 2\pi]$ the argument of ζ. We write $m(\zeta, R)$ for the multiplicity of ζ as a root of a polynomial R. For any positive integer d and unipotent character ρ of G we then define

$$\pi_d(\rho) := \frac{A_\rho + a_\rho}{d} + \frac{m(1, R_\rho)}{2} + \sum_{\substack{\zeta \in \mu(\mathbb{C}) \smallsetminus \{1\} \\ \mathrm{Arg}(\zeta) < 2\pi/d}} m(\zeta, R_\rho).$$

It is shown in [**10**] that $\pi_d(\rho) - \pi_d(\rho')$ is an integer whenever ρ and ρ' lie in the same d-Harish-Chandra series of G. In particular $\pi_d(\rho)$ is an integer for every unipotent character ρ in the principal d-Harish-Chandra series, as $\pi_d(1_G) = 0$.

Conjecture 8.1 (Craven). *Let ℓ be a prime such that Sylow ℓ-subgroups of G are abelian, and let $d = d_\ell(q)$ be the order of q in $\mathbb{F}_\ell^\times$. Then for any two unipotent characters $\rho \neq \rho'$ of G we have*

$$\langle \rho, \Psi_{\varphi_{\rho'}} \rangle \neq 0 \Longrightarrow \pi_d(\rho) > \pi_d(\rho').$$

When $d = 2$, every non-real root ζ of R_ρ satisfies $\mathrm{Arg}(\zeta, R_\rho) < 2\pi/d$ or $\mathrm{Arg}(\overline{\zeta}, R_\rho) < 2\pi/d$. Since R_ρ has real coefficients, this shows that

$$\pi_2(\rho) = A_\rho - \frac{m(-1, R_\rho)}{2}.$$

In addition, for ℓ a good prime for $\mathbf{G}$, $m(-1, R_\rho)$ is constant on the unipotent characters in a fixed unipotent ℓ-block since they form a single 2-Harish-Chandra series, therefore in this case Conjecture 8.1 follows from the result of [**5**].

On the other hand, when $d > 2$ the order coming from π_d might be quite different from the one given by the families. As an example, let us consider the principal Φ_6-block of $B_6(q)$ whose decomposition matrix is given in Table 18. The unipotent characters

$$21^3.1, B_2{:}\, 1.21, 1^3.21, 1^4.1^2, B_2{:}\, .2^2, 21^4., .31^3, .21^4, 1^6., \text{ and } .1^6$$

all lie in families which are smaller than the family containing the cuspidal unipotent character B_6. However the π_6-function on these characters is given by

ρ	B_6	$21^3.1$	$B_2{:}\, 1.21$	$1^3.21$	$1^4.1^2$	$B_2{:}\, .2^2$	$21^4.$	$.31^3.$	$.21^4$	$1^6.$	$.1^6$
$\pi_6(\rho)$	11	10	10	10	11	11	11	10	11	12	12

which predicts that the character of the PIM P_{B_6} can only have $1^6.$ and $.1^6$ as unipotent constituents.

We have checked that the non-zero entries in the decomposition matrices computed in the previous chapters are compatible with Conjecture 8.1. Furthermore, the conjecture predicts the following about yet unknown decomposition numbers in our tables:

Proposition 8.2. *Assume that Craven's Conjecture 8.1 holds. Then the following decomposition numbers in our tables vanish:*

- *for $E_6(q)$ with $d = 3$: $b_1 = 0$,*
- *for $E_6(q)$ and $E_8(q)$ with $d = 6$: $a_1 = a_2 = a_4 = b_1 = b_4 = 0$,*
- *for ${}^2D_7(q)$ with $d = 6$, $\ell > 7$: $y_2 = y_3 = 0$,*
- *for $F_4(q)$ with $d = 6$: $x_1 = y_1 = y_2 = z_1 = z_2 = z_3 = 0$.*

Moreover, several of our proofs would become easier given the conjecture. Observe that Sylow ℓ-subgroups of G are abelian in all cases of the proposition.

In [**18**] decomposition matrices for $d_\ell(q) = 4$ were computed up to a few unknown entries. Craven's conjecture also predicts that some of them vanish.

Proposition 8.3. *Assume that Craven's Conjecture 8.1 holds. Then the following decomposition numbers computed in* [**18**] *vanish:*

- *for $D_7(q)$: $b_1 = b_3 = b_4 = b_5 = b_7 = c = d = 0$,*
- *for ${}^2D_5(q)$, ${}^2D_7(q)$: $a = 0$,*
- *for ${}^2E_6(q)$: $c_1 = c_4 = c_6 = d_2 = 0$,*
- *for $C_4(q)$: $a = 0$.*

Remark 8.4. Note that we did not use the assumption (T_ℓ) when computing the matrices in [**18**], but rather proved a weaker property for the matrices we looked at. Using (T_ℓ) for larger groups and Harish-Chandra induction/restriction, we can actually deduce that all the entries listed in 8.3 vanish except possibly for c_4 and c_6 in ${}^2E_6(q)$.

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