

Deligne-Lusztig restriction of a Gelfand-Graev module

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Abstract

Using Deodhar's decomposition of a double Schubert cell, we study the regular representations of finite groups of Lie type arising in the cohomology of Deligne-Lusztig varieties associated to tori. We deduce that the Deligne-Lusztig restriction of a Gelfand-Graev module is a shifted Gelfand-Graev module.

INTRODUCTION

Let \mathbf{G} be a connected reductive algebraic group defined over an algebraic closure \mathbb{F} of a finite field of characteristic p . Let F be an isogeny of \mathbf{G} such that some power is a Frobenius endomorphism. The finite group $G = \mathbf{G}^F$ of fixed points under F is called a finite group of Lie type. We fix a maximal torus \mathbf{T} contained in a Borel subgroup \mathbf{B} with unipotent radical \mathbf{U} , all of which assumed to be F -stable. The corresponding Weyl group will be denoted by W .

In an attempt to have a complete understanding of the character theory of G , Deligne and Lusztig have introduced in [DL] a family of biadjoint morphisms R_w and *R_w indexed by W , leading to an outstanding theory of induction and restriction between G and any of its maximal tori. Roughly speaking, they encode, into a virtual character, the different representations occurring in the cohomology of the corresponding Deligne-Lusztig variety. Unfortunately, the same construction does not give enough information in the modular setting, and one has to work at a higher level. More precisely, for a finite extension Λ of the ring \mathbb{Z}_ℓ of ℓ -adic integers, Bonnafé and Rouquier have defined in [BR1] the following functors:

$$\mathcal{R}_w : \mathcal{D}^b(\Lambda \mathbf{T}^{wF}\text{-mod}) \longrightarrow \mathcal{D}^b(\Lambda G\text{-mod})$$

and

$${}^*\mathcal{R}_w : \mathcal{D}^b(\Lambda G\text{-mod}) \longrightarrow \mathcal{D}^b(\Lambda \mathbf{T}^{wF}\text{-mod})$$

between the derived categories of modules, which generalize the definition of Deligne-Lusztig induction and restriction.

In this article we study the action of the restriction functor on a special class of representations: the Gelfand-Graev modules, which are projective modules parametrized by the G -regular characters of U . More precisely, we prove in section 3 the following result:

Theorem. *Let $\psi : U \longrightarrow \Lambda^\times$ be a G -regular linear character, and denote by Γ_ψ the associated Gelfand-Graev module of G . Then, for any w in W , one has*

$${}^*\mathcal{R}_w \Gamma_\psi \simeq \Lambda \mathbf{T}^{wF}[-\ell(w)]$$

in the derived category $\mathcal{D}^b(\Lambda \mathbf{T}^{wF}\text{-mod})$.

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This result was already known for some specific elements of the Weyl group. In the case where w is the trivial element, the Deligne-Lusztig functor ${}^* \mathcal{R}_w$ comes from a functor defined at the level of module categories, and the result can be proved in a complete algebraic setting (see [Car, Proposition 8.1.6]). But more interesting is the case of a Coxeter element, studied by Bonnafé and Rouquier in [BR2]. Their proof relies on the following geometric properties for the corresponding Deligne-Lusztig variety $X(w)$ (see [Lu]):

- $X(w)$ is contained in the maximal Schubert cell $\mathbf{B}w_0\mathbf{B}/\mathbf{B}$;
 - the quotient variety $U \backslash X(w)$ is a product of \mathbf{G}_m 's.
- (★)

Obviously, one cannot expect these properties to hold for any element w of the Weyl group (for instance, the variety $X(1)$ is a finite set of points whose intersection with any F -stable Schubert cell is non-trivial). However, it turns out that for the specific class of representations we are looking at, we can restrict our study to a smaller variety which will be somehow a good substitute for $X(w)$.

Let us give some consequences of this theorem, which are already known but can be deduced in an elementary way from our result. From the quasi-isomorphism one can first obtain a canonical algebra homomorphism from the endomorphism algebra of a Gelfand-Graev module to the algebra $\Lambda \mathbf{T}^{wF}$. Tensoring by the fraction field K of Λ , it can be shown that we obtain the Curtis homomorphism ${}_{\kappa} \text{Cur}_w : \text{End}_{K\mathbf{G}}(K\Gamma_{\psi}) \longrightarrow K\mathbf{T}^{wF}$, thus giving a modular and conceptual version of this morphism (see [BK, Theorem 2.7]).

The character-theoretic version of the theorem is obtained in a drastic way, by tensoring the quasi-isomorphism by K and by looking at the induced equality in the Grothendieck group of the category of $\Lambda \mathbf{T}^{wF}$ -modules. Applying the Alvis-Curtis duality gives then a new method for computing the values of the Green functions at a regular unipotent element (see [DL, Theorem 9.16]). This is the key step for showing that a Gelfand-Graev character has a unique irreducible component in each rational series $\mathcal{E}(G, (s)_{\mathbf{G}^{*F^*}})$.

Beyond these applications, our approach aims at understanding each of the cohomology groups of the Deligne-Lusztig varieties, leading to concentration and disjointness properties, in the spirit of Broué's conjectures. For example, by truncating by unipotent characters, one can deduce that the Steinberg character is concentrated in the half cohomology group. This result was already proved in [DMR, Proposition 3.3.15], by a completely different method, since their proof relies on the computation of eigenvalues of Frobenius. By refining our method, we should be able to deal with some other unipotent characters and enlarge the scope of our result.

This paper is divided into three parts: in the first section, we introduce the basic notations about the modular representation theory of finite groups of Lie type. Then, we focus on an extremely rich decomposition for double Schubert cells, introduced by Deodhar in [Deo]. To this end, we shall use the point of view of [Mo] and use the Bialynicki-Birula decomposition, since it is particularly adapted to our case. This is the crucial ingredient for proving the main theorem. Indeed, we show in the last section that the maximal piece of the induced decomposition on $X(w)$ satisfies the properties (★), and that it is the only one carrying regular characters in its cohomology.

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1 PRELIMINARIES

1 - Cohomology of a quasi-projective variety

Let Λ be a commutative ring and H a finite group. We denote by $\Lambda H\text{-mod}$ the abelian category of finitely generated ΛH -modules, and by $\mathcal{D}^b(\Lambda H\text{-mod})$ the derived category of the corresponding bounded complexes. From now on, we assume that Λ is a finite extension of the ring \mathbb{Z}_ℓ of ℓ -adic integers, for a prime ℓ different from p . To any quasi-projective variety X defined over \mathbb{F} and acted on by H , one can associate a classical object in this category, namely the cohomology with compact support of X , denoted by $R\Gamma_c(X, \Lambda)$. It is quasi-isomorphic to a bounded complex of modules which have finite rank over Λ .

We give here some quasi-isomorphisms we shall use in section 3. The reader will find references or proofs of these properties in [BR1, Section 3] and [DL, Proposition 6.4] for the third assertion. The last one can be deduced from [SGA4, Exposé XVIII, 2.9].

Proposition 1.1. *Let X and Y be two quasi-projective varieties acted on by H . Then one has the following isomorphisms in the derived category $\mathcal{D}^b(\Lambda H\text{-mod})$:*

(i) *The Künneth formula:*

$$R\Gamma_c(X \times Y, \Lambda) \simeq R\Gamma_c(X, \Lambda) \overset{\mathbb{L}}{\otimes} R\Gamma_c(Y, \Lambda)$$

where $\overset{\mathbb{L}}{\otimes}$ denotes the left-derived functor of the tensor product over Λ .

(ii) *The quotient variety $H \backslash X$ exists. Moreover, if the order of the stabilizer of any point of X is prime to ℓ , then*

$$R\Gamma_c(H \backslash X, \Lambda) \simeq \Lambda \overset{\mathbb{L}}{\otimes}_{\Lambda H} R\Gamma_c(X, \Lambda).$$

(iii) *If the action of H on X is the restriction of an action of a connected group and if the order of the stabilizer of any point of X in H is prime to ℓ , then*

$$R\Gamma_c(X, \Lambda) \simeq \Lambda \overset{\mathbb{L}}{\otimes}_{\Lambda H} R\Gamma_c(X, \Lambda).$$

(iv) *Let $\pi : Y \rightarrow X$ be an H -equivariant smooth morphism of finite type. If the fibers of π are isomorphic to affine spaces of constant dimension n , then*

$$R\Gamma_c(Y, \Lambda) \simeq R\Gamma_c(X, \Lambda)[-2n].$$

If N is a finite group acting on X on the right and on Y on the left, we can form the amalgamated product $X \times_N Y$, as the quotient of $X \times Y$ by the diagonal action of N . Assume that the actions of H and N commute and that the order of the stabilizer of any point for the diagonal action of N is prime to ℓ . Then $X \times_N Y$ is an H -variety and we deduce from the above properties that

$$R\Gamma_c(X \times_N Y, \Lambda) \simeq R\Gamma_c(X, \Lambda) \overset{\mathbb{L}}{\otimes}_{\Lambda N} R\Gamma_c(Y, \Lambda) \tag{1.2}$$

in the derived category $\mathcal{D}^b(\Lambda H\text{-mod})$.

2 - Algebraic groups

We keep the basic assumptions of the introduction: \mathbf{G} is a connected reductive algebraic group, together with an isogeny F such that some power is a Frobenius endomorphism. In other words, there exists a positive integer δ such that F^δ defines a split \mathbb{F}_q -structure on \mathbf{G} for a certain power q of the characteristic p . For any algebraic subgroup \mathbf{H} of \mathbf{G} F -stable, we will denote by H the finite group of fixed points \mathbf{H}^F .

We fix a Borel subgroup \mathbf{B} containing a maximal torus \mathbf{T} of \mathbf{G} such that both \mathbf{B} and \mathbf{T} are F -stable. They define a root system Φ with basis Δ , and a set of positive (resp. negative) roots Φ^+ (resp. Φ^-). Note that the corresponding Weyl group W is endowed with an action of F , compatible with the isomorphism $W \simeq N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Therefore, the image by F of a root is a positive multiple of some other root, which will be denoted by $\phi^{-1}(\alpha)$, defining thus a

bijection $\phi : \Phi \rightarrow \Phi$. Since \mathbf{B} is also F -stable, this map preserves Δ and Φ^+ . We will also use the notation $[\Delta/\phi]$ for a set of representatives of the orbits of ϕ on Δ .

Let \mathbf{U} (resp. \mathbf{U}^-) be the unipotent radical of \mathbf{B} (resp. the opposite Borel subgroup \mathbf{B}^-). For any root α , we denote by \mathbf{U}_α the one-parameter subgroup and $u_\alpha : \mathbb{F} \rightarrow \mathbf{U}_\alpha$ an isomorphism. Note that the groups \mathbf{U} and \mathbf{U}^- are F -stable whereas \mathbf{U}_α might not be. However, we may, and we will, choose the family $(u_\alpha)_{\alpha \in \Phi}$ such that the restriction to \mathbf{U}_α of the action of F satisfies $F(u_\alpha(\zeta)) = u_{\phi(\alpha)}(\zeta^{q_\alpha^\circ})$ where q_α° is some power of p defined by the relation $F(\phi(\alpha)) = q_\alpha^\circ \alpha$. We define d_α to be the length of the orbit of α under the action of ϕ and we set $q_\alpha = q_\alpha^\circ q_{\phi(\alpha)}^\circ \cdots q_{\phi^{d_\alpha-1}(\alpha)}^\circ$. Then \mathbf{U}_α is stable by F^{d_α} and $\mathbf{U}_\alpha^{F^{d_\alpha}} \simeq \mathbb{F}_{q_\alpha}$.

Let us consider the derived group $D(\mathbf{U})$ of \mathbf{U} . For any total order on Φ^+ , the product map induces the following isomorphism of varieties:

$$D(\mathbf{U}) \simeq \prod_{\alpha \in \Phi^+ \setminus \Delta} \mathbf{U}_\alpha.$$

Note that it is not a group isomorphism in general, since $D(\mathbf{U})$ might not be abelian. However, the canonical map $\prod_{\alpha \in \Delta} \mathbf{U}_\alpha \rightarrow \mathbf{U}/D(\mathbf{U})$ induces an isomorphism of algebraic groups commuting with F . In order to give a description of the rational points of this group, we introduce the one-parameter group \mathbf{V}_α as the image of the following morphism:

$$v_\alpha : \zeta \in \mathbb{F} \mapsto \prod_{i=0}^{d_\alpha-1} F^i(u_\alpha(\zeta)) = \prod_{i=0}^{d_\alpha-1} u_{\phi^i(\alpha)}(\zeta^{q_\alpha^\circ \cdots q_{\phi^{i-1}(\alpha)}^\circ}) \in \prod_{\alpha \in \Delta} \mathbf{U}_\alpha.$$

Then \mathbf{V}_α is F -stable and we obtain a canonical isomorphism of algebraic groups:

$$U/D(\mathbf{U})^F \simeq (\mathbf{U}/D(\mathbf{U}))^F \simeq \prod_{\alpha \in [\Delta/\phi]} \mathbf{V}_\alpha.$$

We give now a construction of the quotient $D(\mathbf{U})^F \backslash \mathbf{B}$ which will be useful in section 3. Towards this aim, we define the F -group \mathbf{B}_Δ to be

$$\mathbf{B}_\Delta = \left(\prod_{\alpha \in \Delta} \mathbf{U}_\alpha \right) \rtimes \mathbf{T} \simeq D(\mathbf{U}) \backslash \mathbf{B}$$

so that the total order on Φ^+ chosen to describe $D(\mathbf{U})$ gives rise to an isomorphism of varieties $\mathbf{B} \simeq D(\mathbf{U}) \times \mathbf{B}_\Delta$ which commutes with F . If $b \in \mathbf{B}$, we denote by (b_D, b_Δ) its image under this isomorphism. Conversely, we can embed the variety \mathbf{B}_Δ into \mathbf{B} by considering the unique section $\iota : \mathbf{B}_\Delta \rightarrow \mathbf{B}$ such that $b = b_D \iota(b_\Delta)$. We should notice again that if W is not abelian, ι is only a morphism of algebraic varieties. With these notations, we obtain the following realization of $D(\mathbf{U})^F \backslash \mathbf{B}$:

Proposition 1.3. *The map $\varphi : b \in \mathbf{B} \mapsto (b^{-1}F(b), b_\Delta) \in \mathbf{B} \times \mathbf{B}_\Delta$ induces the following isomorphism of varieties:*

$$D(\mathbf{U})^F \backslash \mathbf{B} \simeq \{(\bar{b}, h) \in \mathbf{B} \times \mathbf{B}_\Delta \mid \bar{b}_\Delta = \mathcal{L}_\Delta(h)\}$$

where \mathcal{L}_Δ denote the Lang map corresponding to the F -group structure on \mathbf{B}_Δ .

Proof. Let b be an element of \mathbf{B} ; by definition, we can decompose it into $b = b_D \iota(b_\Delta)$ and hence compute $b^{-1}F(b) = (b_D^{-1}F(b_D)) \iota^{(b_\Delta)} \iota(b_\Delta)^{-1}F(\iota(b_\Delta))$. As a consequence, we get

$$(b^{-1}F(b))_\Delta = (\iota(b_\Delta)^{-1}F(\iota(b_\Delta)))_\Delta = \mathcal{L}_\Delta(b_\Delta).$$

Therefore, the image of φ is the set of pairs $(\bar{b}, h) \in \mathbf{B} \times \mathbf{B}_\Delta$ such that $\bar{b}_\Delta = \mathcal{L}_\Delta(h)$.

As another application of the computation of $b^{-1}F(b)$, one can readily check that the fiber at a point $\varphi(b)$ is exactly the set $D(\mathbf{U})^F b$.

Finally, we prove that $\varphi : \mathbf{B} \rightarrow \text{Im } \varphi$ is étale, which will prove the assertion of the proposition. Since the maps $\mathbf{B} \rightarrow B \backslash \mathbf{B}$ and $\text{Im } \varphi \rightarrow B \backslash \text{Im } \varphi$ are étale, it is sufficient to show that the induced map $\varphi' : B \backslash \mathbf{B} \rightarrow B \backslash \text{Im } \varphi$ is an isomorphism. But by the first projection $B \backslash \text{Im } \varphi \rightarrow \mathbf{B}$, φ' identifies with the canonical isomorphism $Bb \in B \backslash \mathbf{B} \mapsto b^{-1}F(b) \in \mathbf{B}$. \square

3 - Deligne-Lusztig varieties

Following [BR1, Section 11.2], we fix a set of representatives $\{\dot{w}\}$ of W in $N_{\mathbf{G}}(\mathbf{T})$ and we define, for $w \in W$, the Deligne-Lusztig varieties $X(w)$ and $Y(\dot{w})$ by:

$$\begin{array}{c} Y(\dot{w}) = \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}\dot{w}\mathbf{U}\} \\ \pi_w \downarrow / \mathbf{T}^{wF} \\ X(w) = \{g\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid g^{-1}F(g) \in \mathbf{B}w\mathbf{B}\} \end{array}$$

where π_w denotes the restriction to $Y(\dot{w})$ of the canonical projection $\mathbf{G}/\mathbf{U} \rightarrow \mathbf{G}/\mathbf{B}$. They are both quasi-projective varieties endowed with a left action of G by left multiplication. Furthermore, \mathbf{T}^{wF} acts on the right of $Y(\dot{w})$ and π_w is isomorphic to the corresponding quotient map, so that it induces a G -equivariant isomorphism of varieties $Y(\dot{w})/\mathbf{T}^{wF} \simeq X(w)$.

We introduce now the general framework for the modular representation theory of G : we choose a prime number ℓ different from p and we consider an ℓ -modular system (K, Λ, k) consisting on a finite extension K of the field of ℓ -adic numbers \mathbb{Q}_ℓ , the normal closure Λ of the ring of ℓ -adic integers in K and the residue field k of the local ring Λ . We assume moreover that the field K is big enough for G , so that it contains the e -th roots of unity, where e is the exponent of G . In that case, the algebra KG is split semi-simple.

By considering the object $\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda)$ of the category $\mathcal{D}^b(\Lambda G\text{-mod}-\Lambda\mathbf{T}^{wF})$, we define a pair of biadjoint functors between the derived categories $\mathcal{D}^b(\Lambda G\text{-mod})$ and $\mathcal{D}^b(\Lambda\mathbf{T}^{wF}\text{-mod})$, called the **Deligne-Lusztig induction and restriction functors**:

$$\mathcal{R}_{\dot{w}} = \mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda) \overset{\mathbf{L}}{\otimes}_{\mathbf{T}^{wF}} - : \mathcal{D}^b(\Lambda\mathbf{T}^{wF}\text{-mod}) \rightarrow \mathcal{D}^b(\Lambda G\text{-mod})$$

$$\text{and } {}^*\mathcal{R}_{\dot{w}} = \mathrm{RHom}_{\Lambda G}(\mathrm{R}\Gamma_c(Y(\dot{w}), \Lambda), -) : \mathcal{D}^b(\Lambda G\text{-mod}) \rightarrow \mathcal{D}^b(\Lambda\mathbf{T}^{wF}\text{-mod})$$

where $\mathrm{Hom}_{\Lambda G}^\bullet$ denotes the classical bifunctor of the category of complexes of ΛG -modules. Tensoring by K , they induce adjoint morphisms R_w and *R_w between the corresponding Grothendieck groups, known as the original induction and restriction defined in [DL]. Note that when $w = 1$, these functors are the reflect of the Harish-Chandra induction and restriction, which are more simply defined at the level of the modules categories.

4 - Gelfand-Graev modules

In this subsection, we present the basic definitions and results concerning the Gelfand-Graev representations, before stating the main theorem. We discuss then the strategy suggested by Bonnafé and Rouquier in [BR2], coming from some geometric observations in the case of a Coxeter element.

Let $\psi : U \rightarrow \Lambda^\times$ be a linear character of U . We assume that ψ is trivial on $D(\mathbf{U})^F$; it is not a strong condition since the equality $D(\mathbf{U})^F = D(U)$ holds in most of the cases (the only exceptions for quasi-simple groups being groups of type B_2 or F_4 over \mathbb{F}_2 or groups of type G_2 over \mathbb{F}_3 , see [How, Lemma 7]). For any α in $[\Delta/\phi]$, we denote by ψ_α the restriction of ψ through $\mathbb{F}_{q_\alpha} \simeq V_\alpha \hookrightarrow U/D(\mathbf{U})^F$.

Definition 1.4. *A linear character $\psi : U \rightarrow \Lambda^\times$ is said to be **G -regular** if*

- ψ is trivial on $D(\mathbf{U})^F$;
- ψ_α is non-trivial for any α in $[\Delta/\phi]$.

*In that case, we define the **Gelfand-Graev module** of G associated to ψ to be*

$$\Gamma_\psi = \mathrm{Ind}_U^G \Lambda_\psi$$

where Λ_ψ denotes the ΛU -module of Λ -rank 1 on which U acts through ψ .

Remark 1.5. One can perform the same construction without assuming that ψ is regular. However, Steinberg has shown in [St] that in the case of a regular character, the module obtained is always multiplicity free. For example, if \mathbf{G} is a torus, then the unipotent group is trivial, and therefore there is only one Gelfand-Graev module, corresponding to the trivial character: the regular representation ΛG .

The construction of the Gelfand-Graev modules is to some extent orthogonal from the Deligne-Lusztig functors. In this sense, we might try to understand the interactions between the two notions. The following theorem and main result of this article is a partial achievement in this direction; it asserts that the Deligne-Lusztig restriction of a Gelfand-Graev module is a shifted Gelfand-Graev module. By the previous remark, it can be stated as follows:

Theorem 1.6. *Let $\psi : U \longrightarrow \Lambda^\times$ be a G -regular linear character and w be an element of W . Then*

$${}^*\mathcal{R}_{\dot{w}}\Gamma_\psi \simeq \Lambda\mathbf{T}^{wF}[-\ell(w)]$$

in the derived category $\mathcal{D}^b(\Lambda\mathbf{T}^{wF}\text{-mod})$.

Such a result was already known for some specific elements of the Weyl group, namely for the trivial element (see [Car, Proposition 8.1.6]) and for a Coxeter element (see [BR2, Theorem 3.10]). As explained in the introduction, both of these cases suggested to Bonnafé and Rouquier that a stronger result should hold. Before recalling it, we need to introduce some more notations. For $x \in W$, we will denote by $\mathbf{B}x \cdot \mathbf{B}$ the unique \mathbf{B} -orbit of \mathbf{G}/\mathbf{B} containing x , and will refer to it as the **Schubert cell** corresponding to x . Following [BR2], we now define the pieces of the Deligne-Lusztig varieties:

$$Y_x(\dot{w}) = \{g\mathbf{U} \in \mathbf{B}x \cdot \mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}\dot{w}\mathbf{U}\}$$

and

$$X_x(w) = \{g\mathbf{B} \in \mathbf{B}x \cdot \mathbf{B} \mid g^{-1}F(g) \in \mathbf{B}w\mathbf{B}\}.$$

By the Bruhat decomposition and its analog for \mathbf{G}/\mathbf{U} , one can check that this gives a filterable decomposition of the varieties $Y(\dot{w})$ and $X(w)$. Finally, if $\psi : U \longrightarrow \Lambda^\times$ is a linear character of U , we denote by e_ψ the corresponding idempotent, defined by

$$e_\psi = \frac{\psi(1)}{|U|} \sum_{u \in U} \psi(u^{-1})u.$$

Since U is a p -group, its order is invertible in Λ and e_ψ is an central element of ΛU .

With these notations, the result conjectured by Bonnafé and Rouquier in [BR2, Conjecture 2.7] can be stated as follows:

Theorem 1.7. *Let $\psi : U \longrightarrow \Lambda^\times$ be a G -regular linear character and w be an element of W . Then*

$$e_\psi \text{R}\Gamma_c(Y_x(\dot{w}), \Lambda) \simeq \begin{cases} \Lambda\mathbf{T}^{wF}[-\ell(w)] & \text{if } x = w_0 \\ 0 & \text{otherwise} \end{cases}$$

in the derived category $\mathcal{D}^b(\text{mod-}\Lambda\mathbf{T}^{wF})$.

Note that the family of varieties $(Y_x(\dot{w}))_{x \in W}$ is a filterable decomposition of $Y(\dot{w})$ with maximal element $Y_{w_0}(\dot{w})$ and that ${}^*\mathcal{R}_{\dot{w}}\Gamma_\psi \simeq \text{RHom}_{\Lambda U}^\bullet(\text{R}\Gamma_c(Y(\dot{w}), \Lambda), \Lambda_\psi) \simeq (e_\psi \text{R}\Gamma_c(Y(\dot{w}), \Lambda))^\vee$ in the derived category $\mathcal{D}^b(\Lambda\mathbf{T}^{wF}\text{-mod})$, so that theorem 1.6 follows indeed from this result.

To conclude this section, we move back attention to the family of varieties $(Y_x(\dot{w}))_{x \in W}$ and $(X_x(w))_{x \in W}$. By the isomorphism $\mathbf{B} \cap {}^x\mathbf{B}^- \simeq \mathbf{B}x \cdot \mathbf{U}$ which sends b to the coset $bx\mathbf{U}$, and its analog for \mathbf{G}/\mathbf{B} , we obtain a new description of these varieties:

$$Y_x(\dot{w}) \simeq \{b \in \mathbf{B} \cap {}^x\mathbf{B}^- \mid b^{-1}F(b) \in x(\mathbf{U}\dot{w}\mathbf{U})F(x)^{-1}\}$$

and

$$X_x(w) \simeq \{u \in \mathbf{U} \cap {}^x\mathbf{U}^- \mid u^{-1}F(u) \in x(\mathbf{B}w\mathbf{B})F(x)^{-1}\}$$

the right action of \mathbf{T}^{wF} on $Y_x(\dot{w})$ being now twisted by x . With this description, the restriction of π_w sends an element b of $Y_x(\dot{w})$ to its left projection on \mathbf{U} , that is, according to the decomposition $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$.

2 DEODHAR'S DECOMPOSITION

We recall in this section the principal result of [Deo], using a different approach due to Morel (see [Mo, Section 3]) which relies on a general decomposition theorem, namely the Bialynicki-Birula decomposition, applied to Bott-Samelson varieties. We also give an elementary result on the filtration property of this partition. Both are the fundamental tools we will be using in the next section.

1 - Decomposition of a double Schubert cell

Let $w \in W$ be an element of the Weyl group of \mathbf{G} . The Schubert variety S_w associated to w is the closure in \mathbf{G}/\mathbf{B} of the Schubert cell $\mathbf{B}w \cdot \mathbf{B}$. This variety is not smooth in general, but Demazure has constructed in [Dem] a resolution of the singularities, called the Bott-Samelson resolution, which is a projective smooth variety over S_w . The construction is as follows: we fix a reduced expression $w = s_1 \cdots s_r$ of w and we define the Bott-Samelson variety to be

$$\mathbf{BS} = \mathbf{P}_{s_1} \times_{\mathbf{B}} \cdots \times_{\mathbf{B}} \mathbf{P}_{s_r} / \mathbf{B}$$

where $\mathbf{P}_{s_i} = \mathbf{B} \cup \mathbf{B}s_i\mathbf{B}$ is the standard parabolic subgroup corresponding to the simple reflection s_i . The homomorphism $\pi : \mathbf{BS} \rightarrow S_w$ which sends the class $[p_1, \dots, p_r]$ in \mathbf{BS} of an element $(p_1, \dots, p_r) \in \mathbf{P}_{s_1} \times \cdots \times \mathbf{P}_{s_r}$ to the class of the product $p_1 \cdots p_r$ in \mathbf{G}/\mathbf{B} is called the **Bott-Samelson resolution**. It is a proper surjective morphism of varieties and it induces an isomorphism between $\pi^{-1}(\mathbf{B}w \cdot \mathbf{B})$ and $\mathbf{B}w \cdot \mathbf{B}$.

Now the torus \mathbf{T} acts naturally on \mathbf{BS} by left multiplication on the first component, or equivalently by conjugation on each component, so that π becomes a \mathbf{T} -equivariant morphism. There are finitely many fixed points for this action, represented by the classes of the elements of $\Gamma = \{1, s_1\} \times \cdots \times \{1, s_r\}$ in \mathbf{BS} ; such an element will be called a **subexpression** of w .

For a subexpression $\gamma = (\gamma_1, \dots, \gamma_r) \in \Gamma$ of w , we denote by $\gamma^i = \gamma_1 \cdots \gamma_i$ the i -th partial subword and we define the two following sets:

$$I(\gamma) = \{i \in \{1, \dots, r\} \mid \gamma_i = s_i\}$$

and

$$J(\gamma) = \{i \in \{1, \dots, r\} \mid \gamma^i s_i < \gamma^i\}.$$

With these notations, Deodhar's decomposition theorem (see [Deo, Theorem 1.1 and Corollary 1.2]) can be stated as follows:

Theorem 2.1 (Deodhar, 84). *There exists a set $\{D_\gamma\}_{\gamma \in \Gamma}$ of disjoint smooth locally closed subvarieties of $\mathbf{B}w \cdot \mathbf{B}$ such that:*

- (i) D_γ is non-empty if and only if $J(\gamma) \subset I(\gamma)$;
- (ii) if D_γ is non-empty, then it is isomorphic to $(\mathbf{G}_a)^{|I(\gamma)| - |J(\gamma)|} \times (\mathbf{G}_m)^{r - |I(\gamma)|}$ as a variety;
- (iii) for all $v \in W$, the double Schubert cell has the following decomposition:

$$\mathbf{B}w \cdot \mathbf{B} \cap \mathbf{B}^{-1}v \cdot \mathbf{B} = \coprod_{\gamma \in \Gamma_v} D_\gamma$$

where Γ_v is the subset of Γ consisting of all subexpressions γ such that $\gamma^r = v$.

Remark 2.2. In the first assertion, the condition for a cell D_γ to be non-empty, that is $J(\gamma) \subset I(\gamma)$, can be replaced by:

$$\forall i = 2, \dots, r \quad \gamma^{i-1} s_i < \gamma^{i-1} \implies \gamma_i = s_i.$$

A subexpression $\gamma \in \Gamma$ which satisfies this condition is called a **distinguished subexpression**. For example, if $\mathbf{G} = \mathrm{GL}_3(\mathbb{F})$ and $w = w_0 = sts$, then there are seven distinguished subexpressions, the only one being not distinguished is $(s, 1, 1)$.

Sketch of proof: the Bott-Samelson variety is a smooth projective variety endowed with an action of the torus \mathbf{T} . Let us consider the restriction of this action to \mathbf{G}_m through a strictly dominant cocharacter $\chi : \mathbf{G}_m \rightarrow \mathbf{T}$. Since this action has a finite number of fixed points, namely the elements of Γ , there exists a Bialynicki-Birula decomposition of the variety \mathbf{BS} into a disjoint union of affine spaces indexed by Γ (see [BB1, Theorem 4.3]):

$$\mathbf{BS} = \coprod_{\gamma \in \Gamma} C^\gamma$$

In [Hä], Härterich has explicitly computed the cells C^γ . To describe this computation, we need some more notations: the simple roots corresponding to the simple reflections of the reduced expression $w = s_1 \cdots s_r$ will be denoted by β_1, \dots, β_r and we set $\tilde{\beta}_i = \gamma^i(-\beta_i)$. Note that with these notations, one has

$$J(\gamma) = \{i \in \{1, \dots, r\} \mid \tilde{\beta}_i \in \Phi^+\}.$$

Let us consider the open immersion $a_\gamma : \mathbb{A}_r \rightarrow \mathbf{BS}$ defined by

$$a_\gamma(x_1, \dots, x_r) = [u_{\gamma_1(-\beta_1)}(x_1)\gamma_1, \dots, u_{\gamma_r(-\beta_r)}(x_r)\gamma_r].$$

Then one can easily check that $\pi^{-1}(\mathbf{B}w \cdot \mathbf{B}) = \text{Im}(a_{(s_1, \dots, s_r)})$. Moreover, Härterich's computations (see [Hä, Section 1]) show that for any subexpression $\gamma \in \Gamma$, one has:

$$C^\gamma = a_\gamma(\{(x_1, \dots, x_r) \in \mathbb{A}_r \mid x_i = 0 \text{ if } i \in J(\gamma)\}).$$

Taking the trace of this decomposition with $\pi^{-1}(\mathbf{B}w \cdot \mathbf{B})$, one obtains a decomposition of the variety $\pi^{-1}(\mathbf{B}w \cdot \mathbf{B})$. Furthermore, the restriction of π to this variety induces an isomorphism with $\mathbf{B}w \cdot \mathbf{B}$, and thus gives a partition of $\mathbf{B}w \cdot \mathbf{B}$ into disjoint cells:

$$\pi^{-1}(\mathbf{B}w \cdot \mathbf{B}) = \coprod_{\gamma \in \Gamma} \pi^{-1}(\mathbf{B}w \cdot \mathbf{B}) \cap C^\gamma \simeq \coprod_{\gamma \in \Gamma} \mathbf{B}w \cdot \mathbf{B} \cap \pi(C^\gamma) = \mathbf{B}w \cdot \mathbf{B}.$$

If we define D_γ to be the intersection $\mathbf{B}w \cdot \mathbf{B} \cap \pi(C^\gamma)$, then it is explicitly given by:

$$D_\gamma \simeq \pi^{-1}(D_\gamma) = a_\gamma(\{(x_1, \dots, x_r) \in \mathbb{A}_r \mid x_i = 0 \text{ if } i \in J(\gamma) \text{ and } x_i \neq 0 \text{ if } i \notin J(\gamma)\}).$$

This description, together with the inclusion $\pi(C^\gamma) \subset \mathbf{B}^{-\gamma'} \cdot \mathbf{B}$, proves the three assertions of the theorem. \square

Example 2.3. In the case where $\mathbf{G} = \text{GL}_3(\mathbb{F})$, and $w = w_0 = sts$, one can easily describe the double Schubert cell $\mathbf{B}w \cdot \mathbf{B} \cap \mathbf{B}^- \cdot \mathbf{B}$. It is isomorphic to $\mathbf{B}w\mathbf{B} \cap \mathbf{U}^-$ by the map $u \mapsto u\mathbf{B}$. Besides, by Gauss reduction, the set $\mathbf{B}w\mathbf{B}w^{-1} = \mathbf{B}\mathbf{B}^-$ consists of all matrices whose principal minors are non-zero. Hence,

$$\mathbf{B}w\mathbf{B} \cap \mathbf{U}^- = \left\{ \left(\begin{array}{ccc} 1 & 0 & 0 \\ b & 1 & 0 \\ c & a & 1 \end{array} \right) \mid c \neq 0 \text{ and } ab - c \neq 0 \right\}.$$

Considering the alternative $a = 0$ or $a \neq 0$, one has $\mathbf{B}w\mathbf{B} \cap \mathbf{U}^- \simeq (\mathbf{G}_m)^3 \cup \mathbf{G}_a \times \mathbf{G}_m$, which is exactly the decomposition given by the two distinguished expressions $(1, 1, 1)$ and $(s, 1, s)$.

Notations 2.4. For a subexpression $\gamma \in \Gamma$, we define the sequence

$$\begin{aligned} \Phi(\gamma) &= (\gamma^i(-\alpha_i) \mid i = 1, \dots, \ell \text{ and } \gamma^i(\alpha_i) > 0) \\ &= (\tilde{\beta}_i \mid i = 1, \dots, \ell \text{ and } \gamma^i(\alpha_i) > 0). \end{aligned}$$

Using Härterich's computation for the cell C^γ and the definition of π , one can see that each element of $\pi(C^\gamma) \subset \mathbf{B}^{-\gamma'} \cdot \mathbf{B}$ has a representative in \mathbf{U}^- of the form

$$\prod_{\beta \in \Phi(\gamma)} u_\beta(x_\beta) \quad \text{with each } x_\beta \in \mathbf{G}_a,$$

the product being taken with respect to the order on $\Phi(\gamma)$. At the level of D_γ , some of the variables x_β must be non-zero (those corresponding to β_i with $\gamma_i = 1$) but the expression becomes unique. Indeed, the restriction of π to $\pi^{-1}(\mathbf{B}w \cdot \mathbf{B})$ (and then to $\pi^{-1}(D_\gamma)$) is injective. Note that this set of representatives is not contained in the variety $\mathbf{B}w\mathbf{B} \cap (\mathbf{U}^- \cap {}^\nu\mathbf{U}^-)$ in general. However, this is the case for $\nu = 1$, and we obtain in this way a parametrization of the variety $\mathbf{B}w\mathbf{B} \cap \mathbf{U}^-$.

2 - Filtration property

We keep the previous notations: ν and w are two elements of the Weyl group such that $\nu \leq w$ and $w = s_1 \cdots s_r$ is a reduced expression of w . Deodhar has shown (see [Deo, Proposition 5.3.(iv)]) that among all the distinguished subexpressions of w ending by ν , there exists one and only one for which the sets J and I are equal. For example, if $\nu = 1$, it is clear that this subexpression is $(1, \dots, 1)$. By theorem 2.1.(ii) the corresponding Deodhar cell has the following properties:

- it is the unique maximal cell in $\mathbf{B}w \cdot \mathbf{B} \cap \mathbf{B}^{-\nu} \cdot \mathbf{B}$, and it is of dimension $\ell(w) - \ell(\nu)$;
- it is a product of \mathbf{G}_m 's;
- it is dense in $\mathbf{B}w \cdot \mathbf{B} \cap \mathbf{B}^{-\nu} \cdot \mathbf{B}$ (since this variety is irreducible by [Ri]).

In particular, the border of the maximal cell is a union of cells of lower dimensions. Unfortunately, this is not always true for the other cells (see [Du]), and the decomposition is not a stratification in general. However, it is possible to describe some relations between the different closures, showing that it is at least filterable. This is a general property for projective smooth varieties (see [BB2]) but we shall give here a simple method for constructing the filtration. In this aim, we can embed \mathbf{BS} into a product of flag varieties as follows: we define the morphism $\iota : \mathbf{BS} \rightarrow (\mathbf{G}/\mathbf{B})^r$ by

$$\iota([p_1, p_2, \dots, p_r]) = (p_1\mathbf{B}, p_1p_2\mathbf{B}, \dots, p_1p_2 \cdots p_r\mathbf{B}).$$

Note that π is the last component of this morphism. Let $\gamma \in \Gamma$ be a subexpression of w . As a direct consequence of the construction of C^γ , one has

$$\iota(C^\gamma) \subset \prod_{i=1}^r \mathbf{B}^{-\gamma^i} \cdot \mathbf{B}.$$

Since \mathbf{BS} is projective, ι is a closed morphism, and hence it sends the closure of a cell C^γ in \mathbf{BS} to the closure of $\iota(C^\gamma)$. Therefore, it is natural to consider a partial order on the set Γ coming from the Bruhat order on W since it describes the closure relation for Schubert cells. For $\delta \in \Gamma$, we define

$$\delta \preceq \gamma \iff \gamma^i \leq \delta^i \text{ for all } i = 1, \dots, r.$$

Then, by construction: $\overline{C^\gamma} \subset \bigcup_{\delta \preceq \gamma} C^\delta$ and $\overline{D_\gamma} \subset \bigcup_{\delta \preceq \gamma} D_\delta$ (2.5)

where $\overline{D_\gamma}$ denotes the closure of D_γ in the Schubert cell $\mathbf{B}w \cdot \mathbf{B}$. Furthermore, considering subexpressions in Γ_ν only would lead to a similar description for the closure in $\mathbf{B}w \cdot \mathbf{B} \cap \mathbf{B}^{-\nu} \cdot \mathbf{B}$. Therefore,

Lemma 2.6. *Let $w, \nu \in W$ such that $\nu \leq w$ and $w = s_1 \cdots s_r$ be a reduced expression of w . Then there exists a numbering of $\Gamma_\nu = \{\gamma_0, \gamma_1, \dots, \gamma_n\}$ and a sequence $(F_i)_{i=0, \dots, n}$ of closed subvarieties of $\mathbf{B}w \cdot \mathbf{B} \cap \mathbf{B}^{-\nu} \cdot \mathbf{B}$ such that:*

- γ_0 corresponds to the maximal cell;
- $F_n = D_{\gamma_n}$;
- $\forall i = 0, \dots, n-1, F_i \setminus F_{i+1} = D_{\gamma_i}$.

Proof. We can choose any numbering which refines the order \preceq : the (F_i) are then determined by the relation they must verify, and they are closed as a consequence of formula 2.5. \square

3 PROOF OF THE MAIN THEOREM

In this section, we present a proof of theorem 1.7, which, as explained in section 1.4, implies the main result of this article. It is divided into two parts: in the first subsection, we deal with the variety $Y_x(\dot{w})$ for an element $x \in W$ which is assumed to be different from w_0 . We show that there cannot be any regular character in the cohomology of this variety. The key point is to consider a specific quotient of this variety for which one can extend the action of at least one of the finite groups V_α up to the corresponding one-parameter subgroup \mathbf{V}_α . The second subsection is devoted to the remaining piece, the variety $Y_{w_0}(\dot{w})$. The crucial ingredient for studying its cohomology is the Deodhar decomposition, which has been recalled in the previous section. We shall lift this decomposition to obtain a nice partition of $Y_{w_0}(\dot{w})$ and then compute the cohomology of the pieces using the results of [BR2].

1 - The variety $Y_x(\dot{w})$ for $x \neq w_0$

Throughout this section, x will denote any element of the Weyl group W different from w_0 . In order to study the cohomology groups of the Deligne-Lusztig variety $Y_x(\dot{w})$, we define

$$\check{Y}_x(\dot{w}) = \{b \in \mathbf{B} \mid b^{-1}F(b) \in x(\mathbf{U}\dot{w}\mathbf{U})F(x)^{-1}\}.$$

It is a B -variety, endowed with a right action of \mathbf{T}^{wF} obtained by right multiplication by the conjugate $x\mathbf{T}^{wF}x^{-1}$. Besides, it is related to $Y_x(\dot{w})$ via the map $\pi : b \in \check{Y}_x(\dot{w}) \mapsto bx\mathbf{U} \in Y_x(\dot{w})$. One can readily check that this morphism has the following properties:

- it is a surjective smooth $B \times (\mathbf{T}^{wF})^{\text{op}}$ -equivariant morphism of varieties;
- all the fibers of π are isomorphic to an affine space of dimension $\ell(w_0) - \ell(x)$.

It is therefore sufficient to study the cohomology of $\check{Y}_x(\dot{w})$. More precisely, one has by proposition 1.1.(iv)

$$\mathrm{R}\Gamma_c(Y_x(\dot{w}), \Lambda) \simeq \mathrm{R}\Gamma_c(\check{Y}_x(\dot{w}), \Lambda)[2(\ell(w_0) - \ell(w))] \quad (3.1)$$

in the derived category $\mathcal{D}^b(\Lambda B\text{-mod-}\Lambda\mathbf{T}^{wF})$. We now prove the following result:

Proposition 3.2. *Let α be a positive simple root such that $x^{-1}(\alpha) > 0$. Then the action of V_α on the quotient $D(\mathbf{U})^F \backslash \check{Y}_x(\dot{w})$ extends to an action of \mathbf{V}_α .*

Proof. Considering the restriction to $\check{Y}_x(\dot{w})$ of the map $b \in \mathbf{B} \mapsto (b^{-1}F(b), b_\Delta) \in \mathbf{B} \times \mathbf{B}_\Delta$ studied in section 1.2 leads to the following parametrization of the quotient:

$$D(\mathbf{U})^F \backslash \check{Y}_x(\dot{w}) \simeq \{(\bar{b}, h) \in \mathbf{B} \times \mathbf{B}_\Delta \mid \bar{b}_\Delta = \mathcal{L}_\Delta(h) \text{ and } \bar{b} \in x(\mathbf{U}\dot{w}\mathbf{U})F(x)^{-1}\}.$$

We should notice that with this description, U (and then $D(\mathbf{U})^F \backslash U$) acts only on the second coordinate. More precisely, any element u of U acts by $u \cdot (\bar{b}, h) = (\bar{b}, u_\Delta h)$.

Through the quotient map $\mathbf{B}_\Delta \rightarrow \mathbf{T}$, one can extend trivially any character of \mathbf{T} to the group \mathbf{B}_Δ . For the simple root α , we will denote by $\tilde{\alpha} : \mathbf{B}_\Delta \rightarrow \mathbf{G}_m$ the corresponding extension and we define an action of \mathbf{V}_α on the quotient variety $D(\mathbf{U})^F \backslash \check{Y}_x(\dot{w})$ by:

$$\forall \zeta \in \mathbb{F} \quad v_\alpha(\zeta) \cdot (\bar{b}, h) = (u_\alpha(\tilde{\alpha}(h^{-1})(\zeta^{q_\alpha} - \zeta))\bar{b}, v_\alpha(\zeta)h).$$

Let us show that this is a well-defined action on $\check{Y}_x(\dot{w})$: consider $(\bar{b}, h) \in \check{Y}_x(\dot{w})$ and $(\bar{b}', h') = v_\alpha(\zeta) \cdot (\bar{b}, h)$ their image by ζ . Using the fact that $\mathcal{L}_\Delta(v_\alpha(\zeta)) = u_\alpha(\zeta^{q_\alpha} - \zeta)$, we compute in the group \mathbf{B}_Δ :

$$\begin{aligned} \mathcal{L}_\Delta(h') &= h'^{-1}F(h') = h^{-1}\mathcal{L}_\Delta(v_\alpha(\zeta))F(h) \\ &= h^{-1}u_\alpha(\zeta^{q_\alpha} - \zeta)h h^{-1}F(h) \\ \mathcal{L}_\Delta(h') &= u_\alpha(\tilde{\alpha}(h^{-1})(\zeta^{q_\alpha} - \zeta))\mathcal{L}_\Delta(h) \end{aligned}$$

so that $\mathcal{L}_\Delta(h') = \bar{b}'_\Delta$ by definition of \bar{b}' . Moreover, the one-parameter subgroup \mathbf{U}_α is contained in ${}^x\mathbf{U}$ since $x^{-1}(\alpha) > 0$, and hence $\bar{b}' \in {}^x\mathbf{U}\bar{b} \subset x(\mathbf{U}\dot{w}\mathbf{U})F(x)^{-1}$, which proves that $(\bar{b}', h') \in \check{Y}_x(\dot{w})$.

To conclude, we remark that restricting this action to V_α amounts to restrict the parameter ζ to \mathbb{F}_{q_α} . In that case, we clearly recover the natural action of V_α coming from the action of $D(\mathbf{U})^F \backslash U$ we described before. \square

As an application, we show that, as expected in [BR2, Conjecture 2.7], the regular characters do not occur in the cohomology of the variety $Y_x(\dot{w})$. This is the first step for determining the isotypic part of these representations in the cohomology of the variety $Y(\dot{w})$, and hence proving theorem 1.6.

Corollary 3.3. *Let $\psi : U \rightarrow \Lambda^\times$ be a G -regular linear character, and x an element of the Weyl group W different from w_0 . Then*

$$e_\psi \mathrm{R}\Gamma_c(Y_x(\dot{w}), \Lambda) \simeq 0$$

in the derived category $\mathcal{D}^b(\mathrm{mod}\text{-}\Lambda\mathbf{T}^{wF})$.

Proof. By formula 3.1, we can replace $Y_x(\dot{w})$ by $\check{Y}_x(\dot{w})$. Since x is different from w_0 , there exists a positive simple root α such that $x^{-1}(\alpha) > 0$; by proposition 3.2, the action of V_α on $D(\mathbf{U})^F \backslash \check{Y}_x(\dot{w})$ extends to an action of the connected group \mathbf{V}_α . On the other hand, ψ is a regular character, and hence its restriction to V_α is non-trivial. Therefore, we obtain by proposition 1.1.(iii):

$$e_\psi \mathrm{R}\Gamma_c(D(\mathbf{U})^F \backslash \check{Y}_x(\dot{w}), \Lambda) \simeq e_\psi(\Lambda^{\bigotimes_{\Lambda V_\alpha}} \mathrm{R}\Gamma_c(D(\mathbf{U})^F \backslash \check{Y}_x(\dot{w}), \Lambda)) \simeq 0$$

Finally, since ψ is trivial on $D(\mathbf{U})^F$, we get

$$e_\psi \mathrm{R}\Gamma_c(\check{Y}_x(\dot{w}), \Lambda) \simeq \Lambda^{\bigotimes_{\Lambda D(\mathbf{U})^F}} e_\psi \mathrm{R}\Gamma_c(\check{Y}_x(\dot{w}), \Lambda) \simeq e_\psi \mathrm{R}\Gamma_c(D(\mathbf{U})^F \backslash \check{Y}_x(\dot{w}), \Lambda) \simeq 0$$

using the fact that e_ψ is central in ΛU and the proposition 1.1.(ii). \square

2 - The variety $Y_{w_0}(\dot{w})$

In this section we are concerned with the cohomology of the Deligne-Lusztig variety $Y_{w_0}(\dot{w})$ corresponding to the maximal Schubert cell. Our aim is to determine the contribution of the regular characters in these cohomology groups. For this purpose, we define a partition of the variety $X_{w_0}(w)$, coming from Deodhar's decomposition of the double Schubert cell $\mathbf{B}w \cdot \mathbf{B} \cap \mathbf{B}^- \cdot \mathbf{B}$ and we show that these specific characters occur only in the cohomology of the maximal cell. We deduce the result at the level of Y using the same method as in [BR2].

We keep the basic assumptions and the notations of the section 2: $w = s_1 \cdots s_r$ is a reduced expression of w , $\gamma \in \Gamma_1$ is a distinguished subexpression of w ending by 1 and D_γ denotes the corresponding Deodhar cell in $\mathbf{B}w \cdot \mathbf{B} \cap \mathbf{B}^- \cdot \mathbf{B}$. We will be interested in the pull-back Ω_γ of D_γ in \mathbf{U}^- , for which we have a nice parametrization (see notations 2.4). In these terms, Deodhar's decomposition can be written as

$$\mathbf{B}w\mathbf{B} \cap \mathbf{U}^- = \coprod_{\gamma \in \Gamma_1} \Omega_\gamma$$

Now since $F(w_0) = w_0$, the maximal piece of the Deligne-Lusztig variety $X(w)$ has the following expression (see section 1.4):

$$\begin{aligned} X_{w_0}(w) &= \{g\mathbf{B} \in \mathbf{B}w_0 \cdot \mathbf{B} \mid g^{-1}F(g) \in \mathbf{B}w\mathbf{B}\} \\ &\simeq \{u \in \mathbf{U} \mid w_0(u^{-1}F(u)) \in \mathbf{B}w\mathbf{B} \cap \mathbf{U}^-\} \end{aligned}$$

so that if we define the piece X_γ to be:

$$X_\gamma = \{u \in \mathbf{U} \mid w_0(u^{-1}F(u)) \in \Omega_\gamma\}$$

then we get

$$X_{w_0}(w) \simeq \coprod_{\gamma \in \Gamma_1} X_\gamma.$$

Note that each component of this partition is a smooth variety stabilized by the action of U , and that this isomorphism is U -equivariant. The same decomposition holds also for $Y_{w_0}(\dot{w})$ if we define Y_γ to be the pullback $\pi_w^{-1}(X_\gamma)$ where $\pi_w : Y(\dot{w}) \rightarrow X(w)$ is the quotient map by the right action of \mathbf{T}^{wF} defined in section 1.3. One can easily check that this defines a family of locally closed smooth subvarieties of $Y_{w_0}(\dot{w})$ satisfying the filtration property of lemma 2.6.

Example 3.4. Let us go back to our example of $\mathrm{GL}_3(\mathbb{F})$, endowed with the standard \mathbb{F}_q -structure (see example 2.3). Given an element u of \mathbf{U} written as

$$u = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

one has
$$w_0(u^{-1}F(u)) = \begin{pmatrix} 1 & 0 & 0 \\ b^q - b & 1 & 0 \\ c^q - c - a(b^q - b) & a^q - a & 1 \end{pmatrix}.$$

Together with the description of $\mathbf{B}w_0\mathbf{B} \cap \mathbf{U}^-$ given in the example 2.3, we obtain

$$X_{w_0}(w_0) \simeq \left\{ (a, b, c) \in (\mathbf{G}_a)^3 \mid \begin{array}{l} c^q - c - a(b^q - b) \neq 0 \\ c^q - c - a^q(b^q - b) \neq 0 \end{array} \right\}.$$

Since we are concerned with regular characters only, it is more convenient to consider the quotient variety $D(\mathbf{U})^F \backslash X_{w_0}(w_0)$. But taking the quotient by $D(\mathbf{U})^F \simeq \mathbb{F}_q$ amounts to take the expression $C = c^q - c - a(b^q - b)$ as a variable, so that

$$D(\mathbf{U})^F \backslash X_{w_0}(w_0) \simeq \{(a, b, C) \in (\mathbf{G}_a)^2 \times \mathbf{G}_m \mid C - (a^q - a)(b^q - b) \neq 0\}.$$

It remains to apply the partition of $\mathbf{B}w_0\mathbf{B} \cap \mathbf{U}^-$ given in the example 2.3 to deduce a decomposition of this quotient into:

- a closed subvariety X_C defined by the equation $a^q - a = 0$. From the above description, it is clearly isomorphic to $\mathbb{F}_q \times \mathbf{G}_a \times \mathbf{G}_m$ with the product action of $D(\mathbf{U})^F \backslash U \simeq \mathbb{F}_q \times \mathbb{F}_q$ on the first two coordinates;
- an open subvariety X_O defined by the equation $a^q - a \neq 0$. By the change of variables $C' = C/(a^q - a)$, it becomes isomorphic to

$$\begin{aligned} X_O &\simeq \mathcal{L}^{-1}(\mathbf{G}_m) \times \{(b, C') \in \mathbf{G}_a \times \mathbf{G}_m \mid C' \neq b^q - b\} \\ &\simeq \mathcal{L}^{-1}(\mathbf{G}_m) \times ((\mathbf{G}_a \times \mathbf{G}_m) \setminus \mathcal{L}^{-1}(\mathbf{G}_m)) \end{aligned}$$

where we have embedded $\mathcal{L}^{-1}(\mathbf{G}_m)$ into $\mathbf{G}_a \times \mathbf{G}_m$ by the map $b \mapsto (b, b^q - b)$.

It is now a straightforward matter to compute the cohomology groups of the varieties and it turns out that the cohomology of X_C does not carry any regular character whereas there is one occurring in the cohomology of the maximal cell X_O . More precisely, for a regular character ψ , we get $e_\psi H_c^\bullet(D(\mathbf{U})^F \backslash X_{w_0}(w_0)) \simeq e_\psi H_c^\bullet(X_O) \simeq \Lambda_\psi[-3]$ as expected in theorem 1.7.

Keeping in mind the previous example, we give now a description of the quotient $D(\mathbf{U})^F \backslash X_\gamma$ in terms of some combinatorial data, namely the sets $I(\gamma)$ and $J(\gamma)$ introduced in section 2. In order to state this result, we need to introduce some more notations; for $m, n \in \mathbb{N}$ and q a power of p , we define the following variety:

$$X_q(n, m) = \left\{ (\zeta, \mu_1, \dots, \mu_n, \lambda_1, \dots, \lambda_m) \in (\mathbf{G}_a)^{n+1} \times (\mathbf{G}_m)^m \mid \zeta^q - \zeta = \sum_i \mu_i + \sum_j \lambda_j \right\}$$

which is endowed with an action of \mathbb{F}_q by translation on the first variable.

Proposition 3.5. *Let γ be a distinguished subexpression of w . For $\alpha \in \Delta$ a simple positive root and \mathcal{O}_α the corresponding orbit in Δ/ϕ , we consider the following integers:*

- $n_\alpha(\gamma) = |\{i = 1, \dots, r \mid w_0(\tilde{\beta}_i) \in \mathcal{O}_\alpha \text{ and } i \in I(\gamma) \setminus J(\gamma)\}|$;
- $m_\alpha(\gamma) = |\{i = 1, \dots, r \mid w_0(\tilde{\beta}_i) \in \mathcal{O}_\alpha \text{ and } i \notin I(\gamma)\}|$;
- $\bar{n}(\gamma) = |I(\gamma)| - |J(\gamma)| - \sum n_\alpha(\gamma)$;
- $\bar{m}(\gamma) = r - |I(\gamma)| - \sum m_\alpha(\gamma)$.

Then there exists a $D(\mathbf{U})^F \setminus U$ -equivariant morphism of varieties, bijective and finite:

$$(\mathbf{G}_a)^{\bar{n}(\gamma)} \times (\mathbf{G}_m)^{\bar{m}(\gamma)} \times \prod_{\alpha \in [\Delta/\phi]} X_{q_\alpha}(n_\alpha(\gamma), m_\alpha(\gamma)) \longrightarrow D(\mathbf{U})^F \setminus X_\gamma$$

where $D(\mathbf{U})^F \setminus U \simeq \prod_{\alpha \in [\Delta/\phi]} V_\alpha$ acts on the product $\prod_{\alpha \in [\Delta/\phi]} X_{q_\alpha}(n_\alpha(\gamma), m_\alpha(\gamma))$ via the identifications $V_\alpha \simeq \mathbb{F}_{q_\alpha}$. Moreover, if (\mathbf{G}, F) is split, this is an isomorphism of varieties. In general, it induces an equivalence of étale sites.

Proof. Recall that we have chosen in section 1.2 an isomorphism of groups between $D(\mathbf{U}) \setminus \mathbf{U}$ and the abelian group $\prod_{\alpha \in \Delta} \mathbf{U}_\alpha \subset \mathbf{B}_\Delta$. As in the proof of proposition 3.2, we can hence realize the quotient variety as:

$$D(\mathbf{U})^F \setminus X_{w_0}(w) \simeq \{(\bar{u}, h) \in \mathbf{U} \times \prod_{\alpha \in \Delta} \mathbf{U}_\alpha \mid \bar{u}_\Delta = \mathcal{L}_\Delta(h) \text{ and } {}^{w_0}\bar{u} \in \mathbf{B}w\mathbf{B}\}.$$

The restriction of this isomorphism to $D(\mathbf{U})^F \setminus X_\gamma$ gives the following description:

$$\begin{aligned} D(\mathbf{U})^F \setminus X_\gamma &\simeq \{(\bar{u}, h) \in \mathbf{U} \times \prod_{\alpha \in \Delta} \mathbf{U}_\alpha \mid \bar{u}_\Delta = \mathcal{L}_\Delta(h) \text{ and } {}^{w_0}\bar{u} \in \Omega_\gamma\} \\ &\simeq \{(\bar{u}, h) \in {}^{w_0}\Omega_\gamma \times \prod_{\alpha \in \Delta} \mathbf{U}_\alpha \mid \bar{u}_\Delta = \mathcal{L}_\Delta(h)\}. \end{aligned}$$

According to the parametrization of the cell D_γ (see notations 2.4), every element \bar{u} of ${}^{w_0}\Omega_\gamma$ can be uniquely written as

$$\bar{u} = \prod_{\beta \in \Phi(\gamma)} u_{w_0(\beta)}(x_\beta) \in \mathbf{U}$$

with the variables x_β living in \mathbf{G}_a or \mathbf{G}_m whether β is of the form $\tilde{\beta}_i$ with i belonging to $I(\gamma)$ or not. Now calculating \bar{u}_Δ (or equivalently $\bar{u}D(\mathbf{U})$) amounts to keep only the positive simple roots occurring in this expression, that is the negative roots β for which $w_0(\beta) \in \Delta$. On the other hand, for any element $h = (u_\alpha(\zeta_\alpha))_{\alpha \in \Delta}$ of $\prod_{\alpha \in \Delta} \mathbf{U}_\alpha$, one has

$$\mathcal{L}_\Delta(h) = h^{-1}F(h) = (u_{\phi(\alpha)}(\zeta_\alpha^{q_\alpha^\circ} - \zeta_{\phi(\alpha)}))_{\alpha \in \Delta}.$$

Therefore, the quotient variety $D(\mathbf{U})^F \setminus X_\gamma$ has a system of coordinates given by the two sets of variables (x_β) and (ζ_α) satisfying the relations

$$\forall \alpha \in \Delta \quad \zeta_\alpha^{q_\alpha^\circ} - \zeta_{\phi(\alpha)} = \sum_{w_0(\beta)=\phi(\alpha)} x_\beta.$$

Note that we can group the relations and the variables according to the class of the simple root α in Δ/ϕ which is involved in the equations. One deduces that $D(\mathbf{U})^F \setminus X_\gamma$ decomposes into a product of varieties indexed by $[\Delta/\phi]$. Without any loss of generality, we can hence assume that ϕ acts transitively on Δ . In order to simplify the notations, we fix a simple root α and we will denote $\zeta_{\phi^i(\alpha)}$ by ζ_i and $q_{\phi^i(\alpha)}^\circ$ by q_i° . The relations defining X_γ can be then rewritten as

$$\forall i = 1, \dots, d \quad \zeta_i = \zeta_{i-1}^{q_{i-1}^\circ} - \sum_{w_0(\beta)=\phi^i(\alpha)} x_\beta$$

It shows that the value of ζ_i is uniquely determined by ζ_0 and the variables (x_β) . Moreover, by substitution, the last equation becomes

$$\zeta_0^{q_0^\circ} - \zeta_0 = \sum_{w_0(\beta)=\alpha} x_\beta + \sum_{w_0(\beta)=\phi^{-1}(\alpha)} x_\beta^{q_{d-1}^\circ} + \dots + \sum_{w_0(\beta)=\phi(\alpha)} x_\beta^{q_{d-1}^\circ \dots q_2^\circ q_1^\circ}.$$

We should not forget that some of the variables x_β are not involved in this equation, namely the ones for which $w_0(\beta)$ is not a simple root; they correspond to the factor $(\mathbf{G}_a)^{\bar{n}(\gamma)} \times (\mathbf{G}_m)^{\bar{m}(\gamma)}$. Indeed, if we set $q_\beta = q_{d-1}^\circ \cdots q_i^\circ$ provided that $w_0(\beta) = \phi^i(\alpha)$ for $i \in \{1, \dots, d\}$, we obtain the following description of the quotient variety $D(\mathbf{U})^F \backslash X_\gamma$:

$$D(\mathbf{U})^F \backslash X_\gamma \simeq (\mathbf{G}_a)^{\bar{n}(\gamma)} \times (\mathbf{G}_m)^{\bar{m}(\gamma)} \times \{(\zeta, (x_\beta)_{\beta \in -\Delta}) \mid \zeta^{q_\alpha} - \zeta = \sum x_\beta^{q_\beta}\}.$$

Finally, up to a new labelling of the variables $(x_\beta)_{\beta \in -\Delta}$ whether they belong to \mathbf{G}_a or \mathbf{G}_m , the map $(x_\beta) \mapsto (x_\beta^{q_\beta})$ induces the expected morphism of varieties. It is a finite and bijective morphism between smooth varieties, and hence is radicial (see [Bor, AG.18.2]). By [SGA1, Exposé IX, 4.10], it induces an equivalence of étale sites. \square

We want to lift the above description of X_γ up to the variety Y_γ . Unfortunately, one cannot deduce directly from the previous proposition a parametrization of Y_γ since the quotient map by \mathbf{T}^{wF} might not split. Following [BR2], we shall nevertheless construct an abelian covering of $D(\mathbf{U})^F \backslash Y_\gamma$ which will have the expected shape.

Notations 3.6. For the sake of coherence with the notations of the last proposition, we introduce the positive integers $m(\gamma) = r - |I(\gamma)|$ and $n(\gamma) = |I(\gamma)| - |J(\gamma)|$ so that the Deodhar cell is isomorphic to $(\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)}$. The product variety given in the same proposition will be simply denoted by X'_γ .

Let us consider the pullback in Y_γ of a connected component of $U \backslash Y_\gamma$ and denote it by Y_γ° . By construction, Y_γ° is stable by U and the quotient $U \backslash Y_\gamma^\circ$ is connected (whereas Y_γ° might not be). Besides, since $U \backslash Y_\gamma / \mathbf{T}^{wF} \simeq U \backslash X_\gamma$ is connected, the group \mathbf{T}^{wF} permutes transitively the connected components of $U \backslash Y_\gamma$. Therefore, if we define the group H to be the stabilizer of $U \backslash Y_\gamma^\circ$ in \mathbf{T}^{wF} , then the multiplication induces the following $(\mathbf{T}^{wF})^{\text{op}}$ -equivariant isomorphism of varieties:

$$(U \backslash Y_\gamma^\circ) \times_H \mathbf{T}^{wF} \simeq U \backslash Y_\gamma.$$

On the other hand, the actions of U and \mathbf{T}^{wF} commute, so that we can also check that

$$(D(\mathbf{U})^F \backslash Y_\gamma^\circ) \times_H \mathbf{T}^{wF} \simeq D(\mathbf{U})^F \backslash Y_\gamma. \quad (3.7)$$

We now define the analog Y'_γ of X'_γ for the variety $D(\mathbf{U})^F \backslash Y_\gamma^\circ$, which fits into the following commutative diagram, where all the squares are cartesian:

$$\begin{array}{ccccc} Y'_\gamma & \xleftarrow{\text{equ}} & D(\mathbf{U})^F \backslash Y_\gamma^\circ & \longrightarrow & U \backslash Y_\gamma^\circ \\ \pi'_w \downarrow & & \pi_w^\circ \downarrow & & \pi_w^\circ \downarrow \\ X'_\gamma & \xleftarrow{\text{equ}} & D(\mathbf{U})^F \backslash X_\gamma & \longrightarrow & U \backslash X_\gamma \end{array}$$

In this diagram and the following ones, the notation equ stands for an equivalence of étale sites. Note that the morphisms π'_w and π_w° are isomorphic to quotient maps by the right action of $H \subset \mathbf{T}^{wF}$, which is a p' -group. In particular, the map $\pi_w^\circ : U \backslash Y_\gamma^\circ \rightarrow (\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)}$ is a Galois covering which is tamely ramified. By Abhyankar's lemma (see [SGA1, Exposé XIII, 5.3]) there exists a Galois covering $\varpi : (\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)} \rightarrow U \backslash Y_\gamma^\circ$ with group N such that the composition $\pi_w^\circ \circ \varpi : (\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)} \rightarrow (\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)}$ sends $(\mu_1, \dots, \mu_{n(\gamma)}, \lambda_1, \dots, \lambda_{m(\gamma)})$ to $(\mu_1, \dots, \mu_{n(\gamma)}, \lambda_1^d, \dots, \lambda_{m(\gamma)}^d)$ for some positive integer d relatively prime to p . We summarize the different maps involved in the following diagram:

$$\begin{array}{ccc}
(\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)} & & \\
\downarrow \varpi / N & \searrow & \\
U \setminus Y_\gamma^\circ & & / (\mu_d)^{m(\gamma)} \\
\downarrow \pi_w^\circ / H & & \\
U \setminus X_\gamma \simeq (\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)} & &
\end{array}$$

where μ_d denotes the group of the d -th roots of unity in \mathbb{F} . In this setting, N is a subgroup of $(\mu_d)^{m(\gamma)}$ and we have a canonical group isomorphism $(\mu_d)^{m(\gamma)}/N \simeq H$. If we form the fiber product of Y_γ' and $(\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)}$ above $U \setminus Y_\gamma^\circ$, we obtain the following diagram whose squares are cartesian:

$$\begin{array}{ccccc}
Y_\gamma'' & \xrightarrow{\quad} & (\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)} & & \\
\downarrow / N & & \downarrow \varpi / N & \searrow & \\
Y_\gamma' & \xleftarrow{\text{equ}} & D(\mathbf{U})^F \setminus Y_\gamma^\circ & \xrightarrow{\quad} & U \setminus Y_\gamma^\circ \\
\downarrow \pi_w' & & \downarrow \pi_w^\circ & & \downarrow \pi_w^\circ / H \\
X_\gamma' & \xleftarrow{\text{equ}} & D(\mathbf{U})^F \setminus X_\gamma & \xrightarrow{\quad} & U \setminus X_\gamma \simeq (\mathbf{G}_a)^{n(\gamma)} \times (\mathbf{G}_m)^{m(\gamma)}
\end{array} \quad (3.8)$$

so that by definition of X_γ' and the properties of the Galois covering $\pi_w^\circ \circ \varpi$ we get a $D(\mathbf{U})^F \setminus U \times ((\mu_d)^{m(\gamma)})^{\text{op}}$ -equivariant isomorphism of varieties:

$$Y_\gamma'' \simeq (\mathbf{G}_a)^{\bar{n}(\gamma)} \times (\mathbf{G}_m)^{\bar{m}(\gamma)} \prod_{\alpha \in [\Delta/\phi]} Y_{q_{\alpha,d}(n_\alpha(\gamma), m_\alpha(\gamma))} \quad (3.9)$$

where the factors on the right-hand side are defined by

$$Y_{q,s}(n, m) = \left\{ (\zeta, \mu_1, \dots, \mu_n, \lambda_1, \dots, \lambda_m) \in (\mathbf{G}_a)^{n+1} \times (\mathbf{G}_m)^m \mid \zeta^q - \zeta = \sum_i \mu_i + \sum_j \lambda_j^s \right\}$$

and endowed with a natural action \mathbb{F}_q by translation on the first variable together with the action of $(\mu_s)^m$ obtained by multiplication on $(\mathbf{G}_m)^m$. Up to an equivalence of étale sites, we have performed in this way a construction of an abelian covering of the variety $D(\mathbf{U})^F \setminus Y_\gamma^\circ$ which decomposes into a product of varieties. Using the results of [BR2], we shall first compute the cohomology of each of these factors:

Lemma 3.10. *Let ψ be a non trivial character of \mathbb{F}_q . Then*

$$e_\psi \text{R}\Gamma_c(Y_{q,s}(n, m), \Lambda) \simeq \begin{cases} \Lambda_\psi (\mu_s)^m[-m] & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases}$$

in the derived category $D^b(\Lambda \mathbb{F}_q\text{-mod-}\Lambda(\mu_s)^m)$.

Proof. If $n \neq 0$, the equation defining $Y_{q,s}(n, m)$ can be rewritten as

$$\mu_1 = \zeta^q - \zeta - \sum_{i=1}^m \lambda_i - \sum_{j=2}^n \mu_j$$

so that $Y_{q,s}(n, m) \simeq (\mathbf{G}_a)^n \times (\mathbf{G}_m)^m$, with an action of \mathbb{F}_q on the first coordinate. Since the cohomology of the affine line is given by $\text{R}\Gamma_c(\mathbf{G}_a, \Lambda) \simeq \Lambda[-2]$ we get the result by the Künneth formula (see proposition 1.1.(i)).

The variety $Y_{q,s}(0, m)$ can be completely described by the curves $Y_{q,s} = Y_{q,s}(0, 1)$ studied by Bonnafé and Rouquier in [BR2] (see also [Lau]): the map

$$\begin{aligned} Y_{q,s} \times Y_{q,s} \times \cdots \times Y_{q,s} &\longrightarrow Y_{q,s}(0, m) \\ ((\zeta_1, \lambda_1), \dots, (\zeta_m, \lambda_m)) &\longmapsto (\zeta_1 + \cdots + \zeta_m, \lambda_1, \dots, \lambda_m) \end{aligned}$$

induces indeed the following $\mathbb{F}_q \times (\mu_s)^m$ -equivariant isomorphism of varieties:

$$Y_{q,s} \times_{\mathbb{F}_q} Y_{q,s} \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} Y_{q,s} \simeq Y_{q,s}(0, m).$$

To conclude, it remains to translate this isomorphism in the category $\mathcal{D}^b(\Lambda\mathbb{F}_q\text{-mod-}\Lambda(\mu_s)^m)$; we can then deduce the result from the case $m = 1$ which was solved in [BR2, Lemma 3.6]:

$$\begin{aligned} e_\psi \mathrm{R}\Gamma_c(Y_{q,s}(0, m), \Lambda) &\simeq e_\psi \mathrm{R}\Gamma_c(Y_{q,s}, \Lambda) \overset{\mathrm{L}}{\otimes}_{\Lambda\mathbb{F}_q} \cdots \overset{\mathrm{L}}{\otimes}_{\Lambda\mathbb{F}_q} e_\psi \mathrm{R}\Gamma_c(Y_{q,s}, \Lambda) \\ &\simeq \Lambda_{\psi|\mu_s}[-1] \overset{\mathrm{L}}{\otimes}_{\Lambda\mathbb{F}_q} \cdots \overset{\mathrm{L}}{\otimes}_{\Lambda\mathbb{F}_q} \Lambda_{\psi|\mu_s}[-1] \simeq \Lambda_{\psi}(\mu_s)^m[-m]. \end{aligned}$$

the first quasi-isomorphism coming also from the fact that e_ψ is an idempotent. \square

We have now at our disposal all the ingredients we need to compute the isotypic part of a regular character in the cohomology of each variety Y_γ :

Proposition 3.11. *Let γ be a distinguished subexpression of w ending by 1, and $\psi : U \longrightarrow \Lambda^\times$ a G -regular linear character. Then*

$$e_\psi \mathrm{R}\Gamma_c(Y_\gamma, \Lambda) \simeq \begin{cases} \Lambda \mathbf{T}^{wF}[-\ell(w)] & \text{if } \gamma = (1, 1, \dots, 1), \\ 0 & \text{otherwise} \end{cases}$$

in the derived category $\mathcal{D}^b(\mathrm{mod-}\Lambda \mathbf{T}^{wF})$.

Proof. Since ψ is trivial on $D(\mathbf{U})^F$, we can argue as in the proof of the corollary 3.3 to show that it is sufficient to prove the result for the quotient variety $D(\mathbf{U})^F \backslash Y_\gamma$. Furthermore, this variety can be expressed by formula 3.7 as an amalgamated product with \mathbf{T}^{wF} above H so that by formula 1.2 one obtains

$$\mathrm{R}\Gamma_c(D(\mathbf{U})^F \backslash Y_\gamma, \Lambda) \simeq \mathrm{R}\Gamma_c(D(\mathbf{U})^F \backslash Y_\gamma^\circ, \Lambda) \overset{\mathrm{L}}{\otimes}_{\Lambda H} \mathbf{T}^{wF}.$$

We continue the reductions: from the commutative diagram 3.8, we have an equivalence of étale sites between $D(\mathbf{U})^F \backslash Y_\gamma^\circ$ and Y'_γ and an isomorphism between Y'_γ and Y''_γ/N , both being $U \times H^{\mathrm{op}}$ -equivariant. Together, they induce the following quasi-isomorphism:

$$\mathrm{R}\Gamma_c(D(\mathbf{U})^F \backslash Y_\gamma^\circ, \Lambda) \simeq \mathrm{R}\Gamma_c(Y''_\gamma, \Lambda) \overset{\mathrm{L}}{\otimes}_{\Lambda N} \Lambda.$$

Since $H \simeq (\mu_d)^m/N$, we deduce by an adjunction formula that

$$\begin{aligned} \mathrm{R}\Gamma_c(D(\mathbf{U})^F \backslash Y_\gamma, \Lambda) &\simeq (\mathrm{R}\Gamma_c(Y''_\gamma, \Lambda) \overset{\mathrm{L}}{\otimes}_{\Lambda N} \Lambda) \overset{\mathrm{L}}{\otimes}_{\Lambda H} \Lambda \mathbf{T}^{wF} \\ &\simeq (\mathrm{R}\Gamma_c(Y''_\gamma, \Lambda) \overset{\mathrm{L}}{\otimes}_{\Lambda(\mu_d)^m} \Lambda H) \overset{\mathrm{L}}{\otimes}_{\Lambda H} \Lambda \mathbf{T}^{wF} \\ \mathrm{R}\Gamma_c(D(\mathbf{U})^F \backslash Y_\gamma, \Lambda) &\simeq \mathrm{R}\Gamma_c(Y''_\gamma, \Lambda) \overset{\mathrm{L}}{\otimes}_{\Lambda(\mu_d)^m} \Lambda \mathbf{T}^{wF} \end{aligned}$$

Denote by ψ_α the restriction of ψ to the group V_α . The decomposition of the variety Y''_γ (see formula 3.9) translates, by the Künneth formula, into the following quasi-isomorphisms:

$$\begin{aligned} e_\psi \mathrm{R}\Gamma_c(Y''_\gamma, \Lambda) &\simeq \mathrm{R}\Gamma_c((\mathbf{G}_a)^{\bar{n}(\gamma)} \times (\mathbf{G}_m)^{\bar{m}(\gamma)}, \Lambda) \overset{\mathrm{L}}{\otimes} e_\psi \mathrm{R}\Gamma_c\left(\prod_{\alpha \in [\Delta/\phi]} Y_{q_\alpha, d}(n_\alpha(\gamma), m_\alpha(\gamma)), \Lambda\right) \\ &\simeq \mathrm{R}\Gamma_c((\mathbf{G}_a)^{\bar{n}(\gamma)} \times (\mathbf{G}_m)^{\bar{m}(\gamma)}, \Lambda) \overset{\mathrm{L}}{\otimes} \left(\overset{\mathrm{L}}{\otimes}_{\alpha \in [\Delta/\phi]} e_{\psi_\alpha} \mathrm{R}\Gamma_c(Y_{q_\alpha, d}(n_\alpha(\gamma), m_\alpha(\gamma)), \Lambda)\right) \end{aligned}$$

Since ψ is a regular character, every restriction ψ_α is a non trivial character of \mathbb{F}_{q_α} . It follows from the lemma 3.10 that the complex $e_\psi \mathrm{R}\Gamma_c(D(\mathbf{U})^F \backslash Y_\gamma, \Lambda)$ is quasi-isomorphic to zero as

soon as one of the $n_\alpha(\gamma)$ is different from zero. But for $\gamma \neq (1, 1, \dots, 1)$, the set $I(\gamma) \setminus J(\gamma)$ is non-empty, and if i_0 denotes its largest element, then $\gamma^{i_0} = 1$ and γ^{i_0-1} is a simple reflection s_α for some positive simple root $\alpha \in \Delta$. Therefore, $n_{-w_0(\alpha)}(\gamma) \neq 0$ and $e_\psi \mathrm{R}\Gamma_c(Y''_\gamma, \Lambda) \simeq 0$.

If $\gamma = (1, 1, \dots, 1)$, then the invariants $\bar{m}(\gamma)$, $\bar{n}(\gamma)$ and $n_\alpha(\gamma)$ are all equal to zero, and by lemma 3.10, one has

$$\begin{aligned} e_\psi \mathrm{R}\Gamma_c(D(\mathbf{U})^F \setminus Y_\gamma, \Lambda) &\simeq \left(\bigotimes_{\alpha \in [\Delta/\phi]}^{\mathrm{L}} e_{\psi_\alpha} \mathrm{R}\Gamma_c(Y_{q_\alpha, d}(0, m_\alpha(\gamma)), \Lambda) \right) \otimes_{\Lambda(\mathbb{F}_d)^{m(\gamma)}}^{\mathrm{L}} \Lambda \mathbf{T}^{wF} \\ &\simeq \left(\bigotimes_{\alpha \in [\Delta/\phi]}^{\mathrm{L}} \Lambda_{\psi_\alpha}(\mathbb{F}_d)^{m_\alpha(\gamma)}[-m_\alpha(\gamma)] \right) \otimes_{\Lambda(\mathbb{F}_d)^{m(\gamma)}}^{\mathrm{L}} \Lambda \mathbf{T}^{wF} \\ &\simeq \Lambda_\psi(\mathbb{F}_d)^{m(\gamma)}[-m(\gamma)] \otimes_{\Lambda(\mathbb{F}_d)^{m(\gamma)}}^{\mathrm{L}} \Lambda \mathbf{T}^{wF} \\ e_\psi \mathrm{R}\Gamma_c(D(\mathbf{U})^F \setminus Y_\gamma, \Lambda) &\simeq \Lambda_\psi \mathbf{T}^{wF}[-m(\gamma)] \end{aligned}$$

which give the conclusion since $m(\gamma) = \ell(w)$ for this particular subexpression. \square

By combining the last proposition and the filtration property (see lemma 2.6), we can finally state the second part of the result expected in [BR2, Conjecture 2.7], which finishes the proof of theorem 1.7.

Corollary 3.12. *Let $\psi : U \rightarrow \Lambda^\times$ be a G -regular linear character. Then*

$$e_\psi \mathrm{R}\Gamma_c(Y_{w_0}(\dot{w}), \Lambda) \simeq \Lambda_\psi \mathbf{T}^{wF}[-\ell(w)]$$

in the derived category $\mathcal{D}^b(\Lambda U\text{-mod-}\Lambda \mathbf{T}^{wF})$.

Remark 3.13. With the results of [BR2], we can actually compute the cohomology of each variety $D(\mathbf{U})^F \setminus Y_\gamma$, and not only the ψ -isotypic part. However, except in some very special cases, it does not lead to a complete description of the complex for the variety $D(\mathbf{U})^F \setminus Y_{w_0}(\dot{w})$ since it involves highly non-trivial triangles in the category $\mathcal{D}^b(\Lambda U\text{-mod-}\Lambda \mathbf{T}^{wF})$.

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