BOUNDING HARISH-CHANDRA SERIES

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Abstract. We use the progenerator constructed in [8] to give a necessary condition for a simple module of a finite reductive group to be cuspidal, or more generally to obtain information on which Harish-Chandra series it can lie in. As a first application we show the irreducibility of the smallest unipotent character in any Harish-Chandra series. Secondly, we determine a unitriangular approximation to part of the unipotent decomposition matrix of finite orthogonal groups and prove a gap result on certain Brauer character degrees.

1. Introduction

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over a finite field $\mathbb{F}_q$, with corresponding Frobenius endomorphism F. The unipotent characters of the finite reductive group $\mathbf{G}^F$ were classified by Lusztig in [22]. One feature of this classification is that cuspidal unipotent characters belong to a unique family, whose numerical invariant—Lusztig's a-function—gets bigger as the rank of $\mathbf{G}$ increases.

For representations in positive characteristic $\ell \nmid q$, many more cuspidal representations can occur and they need not all lie in the same family. The purpose of this paper is to show that nevertheless the statement about the a-function still holds. In other words, we show that unipotent representations with small a-value must lie in a Harish-Chandra series corresponding to a small Levi subgroup of G (see Theorem 2.2). The proof uses the progenerator constructed in [8] which ensures that cuspidal unipotent modules appear in the head of generalised Gelfand–Graev representations attached to cuspidal classes. Our result then follows from the computation of lower bounds for the numerical invariants attached to theses classes (see Proposition 3.1). As a consequence we deduce in Theorem 4.3 that the unipotent characters with smallest a-value in any ordinary Harish-Chandra series remain irreducible under ℓ-modular reduction. This generalises our result for unitary groups from [6].

In Sections 5 and 6 we apply these considerations to Harish-Chandra series of unipotent representations of the finite spin groups $\mathrm{Spin}_n^{(\pm)}(q)$ for a-value at most 3 respectively 4 and determine approximations of the corresponding partial decomposition matrices, see Theorems 5.4, 5.5 and 6.3. From the partial triangularity of these decomposition matrices we then derive a gap result for the corresponding unipotent Brauer character degrees, see Corollaries 5.8 and 6.5.

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2. Bounding Harish-Chandra series

Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$, with corresponding Frobenius endomorphism F. Throughout this section we assume that p, the characteristic of $\mathbb{F}_q$, is good for $\mathbf{G}$. We let ℓ be a prime different from p and $(K, \mathcal{O}, k)$ denote a splitting ℓ-modular system for $G = \mathbf{G}^F$. The irreducible representations of kG are partitioned into Harish-Chandra series, but this partition is not known in general. In this section we give a necessary condition for a unipotent module to lie in a given series, see Theorem 2.2. It involves a numerical invariant coming from Lusztig's a-function.

2.1. **Unipotent support.** Given $\rho \in \mathrm{Irr}(G)$ and C an F-stable unipotent class of $\mathbf{G}$, we denote by $\mathrm{AV}(C, \rho) = |C^F|^{-1} \sum_{g \in C^F} \rho(g)$ the average value of ρ on C^F. We say that C is a *unipotent support* of ρ if C has maximal dimension for the property that $\mathrm{AV}(C, \rho) \neq 0$. Geck [11, Thm. 1.4] has shown that whenever p is good for G every irreducible character ρ of G has a unique unipotent support, which we will denote by C_ρ. By Lusztig [23, §11] (see Taylor [29, §14] for the extension to any good characteristic), unipotent supports of unipotent characters are *special classes* (see below). They can be computed as follows: any family $\mathcal{F}$ in the Weyl group of $\mathbf{G}$ contains a unique special representation, which is the image under the Springer correspondence of the trivial local system on a special unipotent class $C_{\mathcal{F}}$. Then $C_{\mathcal{F}}$ is the common unipotent support of all the unipotent characters in $\mathcal{F}$.

2.2. **Duality.** In [27, III.1] Spaltenstein studied an order-reversing map d on the set of unipotent classes of $\mathbf{G}$ partially ordered by inclusion of closures. When p is good, the image of d consists of the so-called special unipotent classes, and d restricts to an involution on this subset of classes.

The effect of d on unipotent supports of unipotent characters (which are special classes) can be computed as follows. Let ρ be a unipotent character of G and let ρ^* be its Alvis–Curtis dual (see [5, §8]). If $\mathcal{F}$ is the family of $\mathrm{Irr}(W)$ attached to ρ, then $\mathcal{F} \otimes \mathrm{sgn}$ is the family attached to ρ^*. Let ϕ be the unique special character in $\mathcal{F}$. Via the Springer correspondence, it corresponds to the trivial local system on C_ρ. Then by [2, §3], the character $\phi \otimes \mathrm{sgn}$ corresponds to some local system on $d(C_\rho)$. Moreover, $\phi \otimes \mathrm{sgn}$ is special and therefore that local system is trivial except when $\mathcal{F}$ is one of the exceptional families in type E_7 and E_8 (see for example [4, §11.3 and §12.7]). Consequently, for any unipotent character ρ of G we have

$$d(C_\rho) = C_{\rho^*} \tag{1}$$

This property does not hold in general for other series of characters.

2.3. **Wave front set.** Given a unipotent element $u \in G$, we denote by Γ_u^G, or simply Γ_u, the *generalised Gelfand–Graev representation* associated with u. It is an $\mathcal{O}G$-lattice. The construction is given for example in [20, §3.1.2] (with some extra assumption on p) or in [29, §5]. The first elementary properties that can be deduced are

- if $\ell \neq p$, then Γ_u is a projective $\mathcal{O}G$-module;
- if u and u' are conjugate under G then $\Gamma_u \cong \Gamma_{u'}$.

The character of $K\Gamma_u$ is the *generalised Gelfand–Graev character* associated with u. We denote it by γ_u^G, or simply γ_u. It depends only on the G-conjugacy class of u. When u is a regular unipotent element then γ_u is a usual Gelfand–Graev character as in [5, §14].

The following theorem by Lusztig [23, Thm. 11.2] and Achar–Aubert [1, Thm. 9.1] (see Taylor [29] for the extension to good characteristic) gives a condition on the unipotent support of a character to occur in a generalised Gelfand-Graev character.

Theorem 2.1 (Lusztig, Achar–Aubert, Taylor). *Let $\rho \in \mathrm{Irr}(G)$ and $\rho^* \in \mathrm{Irr}(G)$ its Alvis-Curtis dual. Then*

(a) *there exists $u \in C_{\rho^*}^F$ such that $\langle \gamma_u; \rho\rangle \neq 0$;*
(b) *if C is an F-stable unipotent conjugacy class of $\mathbf{G}$ such that $\langle \gamma_u; \rho\rangle \neq 0$ for some $u \in C^F$ then $C \subset \overline{C_{\rho^*}}$.*

Here $\overline{C}$ denotes the Zariski closure of a conjugacy class C.

2.4. **Lusztig's a-function.** Let $\rho \in \mathrm{Irr}(G)$. By [22, 4.26.3], there exist nonnegative integers n_ρ, N_ρ, a_ρ with $n_\rho \geq 1$ and $N_\rho \equiv \pm 1 \pmod q$ such that

$$\dim \rho = \frac{1}{n_\rho} q^{a_\rho} N_\rho.$$

Moreover, by [22, 13.1.1], the integer a_ρ is equal to the dimension of the Springer fibre at any element of the unipotent support C_ρ of ρ. More precisely, for any $u \in C_\rho$ we have

$$a_\rho = \frac{1}{2}(\dim C_{\mathbf{G}}(u) - \mathrm{rk}(\mathbf{G})). \tag{2}$$

Let $\chi \in \mathbb{Z}\mathrm{Irr}(G)$. We define a_χ to be the minimum over the a-values of the irreducible *unipotent* constituents of χ (and ∞ if there is none). If $\varphi \in \mathrm{IBr}(G)$ is unipotent we define a_φ to be the a-value of its projective cover. By extension we set $a_S = a_\varphi$ if S is a simple kG-module with Brauer character φ.

2.5. **Harish-Chandra series and a-value.** It results from Lusztig's classification of unipotent characters that there is at most one family of $\mathrm{Irr}(G)$ containing a cuspidal unipotent character. In addition, the unipotent support attached to such a family is self-dual with respect to d and has a large a-value compared to the rank of the group. We give a generalisation of this second statement to positive characteristic using the progenerator constructed in [8].

For this, recall from [15] that an F-stable unipotent class C of $\mathbf{G}$ is said to be *cuspidal* if there exists no proper 1-split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and $u \in C \cap \mathbf{L}^F$ such that the natural map $C_{\mathbf{L}}(u)/C_{\mathbf{L}}(u)^\circ \to C_{\mathbf{G}}(u)/C_{\mathbf{G}}(u)^\circ$ is an isomorphism. For C an F-stable cuspidal unipotent class of $\mathbf{G}$ and $d(C)$ the dual class we set (see (2))

$$a_{d(C)} := \frac{1}{2}(\dim C_{\mathbf{G}}(u) - \mathrm{rk}(\mathbf{G})) \qquad \text{where } u \in d(C).$$

Theorem 2.2. *Let $\mathbf{G}$ be connected reductive. Let S be a simple $k\mathbf{G}^F$-module lying in the Harish-Chandra series above a cuspidal pair $(\mathbf{L}^F, X)$ of $\mathbf{G}^F$. Then there exists an F-stable unipotent class C of $\mathbf{L}$ which is cuspidal for $\mathbf{L}_{\mathrm{ad}}$ and such that $a_{d(C)} \leq a_S$.*

Proof. By [8, Thm. 2.3], there exists an F-stable unipotent class C of $\mathbf{L}$ which is cuspidal for $\mathbf{L}_{\mathrm{ad}}$ and $u \in C^F$ such that the generalised Gelfand–Graev module Γ_u^L maps onto X. Let P_X be the projective cover of X. Since Γ_u^L is projective, it must contain P_X as a direct summand. In particular, any irreducible unipotent constituent χ of a lift $\tilde{P}_X$ to $\mathcal{O}$ of P_X

is also a constituent of γ_u^L. For such a character χ we have $C \subseteq \overline{C_{\chi^*}}$ by Theorem 2.1. By applying Spaltenstein's duality d, we deduce from (1) that $\overline{d(C)} \supseteq d(C_{\chi^*}) = C_\chi$. Consequently $a_{d(C)} \leq a_\chi$ for every unipotent constituent χ of $\tilde{P}_X$, therefore $a_{d(C)} \leq a_X$.

It remains to see that $a_X \leq a_S$. This follows from the fact that P_S is a direct summand of $R_L^G(P_X)$ and that Harish-Chandra induction can not decrease the a-value (see for example [22, Cor. 8.7]). □

In Sections 5 and 6 we will apply Theorem 2.2 to determine the Harish-Chandra series of unipotent characters with small a-value in the finite orthogonal groups. For this, it will be useful to derive lower bounds for $a_{d(C)}$.

3. Cuspidal classes: minimality and induction

3.1. **Cuspidal classes and a-value.** Cuspidal unipotent classes for simple groups of adjoint type in good characteristic were classified by Geck and the second author [15, Prop. 3.6]. From the classification we can see that they are all special classes of rather large dimension (compared to the rank of the group). We give here, for every classical type, the minimal dimension of the Springer fibre over the dual of any cuspidal unipotent class in good characteristic.

Proposition 3.1. *Assume that p is good for $\mathbf{G}$ and $Z(\mathbf{G})$ is connected. Then there is a unique F-stable cuspidal class $C_{\min}$ which is contained in the closure of every F-stable cuspidal class of $\mathbf{G}$. Furthermore, the values of $a_{\min} := a_{d(C_{\min})}$ for simple classical types are given as follows:*

(a) *for type A_m: $a_{\min} = \frac{1}{2}m(m+1)$;*

(b) *for type ${}^2A_m$, where $m+1 = \binom{s+1}{2} + d$ with $0 \leq d \leq s$:*

$$a_{\min} = \frac{1}{6}s(s^2-1) + \frac{1}{2}d(2s+1-d);$$

(c) *for type B_m, where $m = s(s+1) + d$ with $0 \leq d \leq 2s+1$:*

$$a_{\min} = \frac{1}{6}s(s+1)(4s-1) + \begin{cases} d(2s+1-d) & \text{if } d \leq s, \\ d(4s+2-d) - s(2s+1) & \text{if } s \leq d; \end{cases}$$

(d) *for type C_m, where $m = s(s+1) + d$ with $0 \leq d \leq 2s+1$:*

$$a_{\min} = \frac{1}{6}s(s+1)(4s-1) + \begin{cases} d(2s+2-d) & \text{if } d \leq s+1, \\ d(4s+3-d) - (s+1)(2s+1) & \text{if } s+1 \leq d; \end{cases}$$

(e) *for types D_m and ${}^2D_m$, where $m = s^2 + d$ with $0 \leq d \leq 2s$:*

$$a_{\min} = \frac{1}{6}s(s-1)(4s+1) + \begin{cases} d(2s+1-d) & \text{if } d \leq s, \\ d(4s-d) - s(2s-1) & \text{if } s \leq d. \end{cases}$$

Proof. Arguing as in [15, §3.4] we may and will assume that $\mathbf{G}$ is simple of adjoint type. For exceptional types there always is a cuspidal family, and then our claim is in [15, Thm. 3.3]. For type A_m the regular class C is the only cuspidal class by [15, Prop. 3.6]. Its dual class is the trivial class, for which $a_{d(C)} = \frac{1}{2}(\dim \mathrm{PGL}_{m+1} - m) = m(m+1)/2$.

For the other classical types, to any conjugacy class C corresponds a partition λ coming from the elementary divisors of any element in C in the natural matrix representation of a classical group isogenous to $\mathbf{G}$. If C is cuspidal, then depending on the type of $\mathbf{G}^F$, the partition λ satisfies the following properties (see [15, Prop. 3.6]):

- type ${}^2A_m$: λ is a partition of $m+1$ into distinct parts;
- type B_m: λ is a partition of $2m+1$ into odd parts, each occurring at most twice;
- type C_m: λ is a partition of $2m$ into even parts, each occurring at most twice.
- type D_m and ${}^2D_m$: λ is a partition of $2m$ into odd parts, each occurring at most twice, and $\lambda \neq (m,m)$ in type D_m.

(It was claimed erroneously in loc. cit. that for type D_m, m odd, classes with label (m,m) are cuspidal, but in fact they do lie in a Levi subgroup of type A_{m-1}.)

First consider type ${}^2A_m$. We claim that the class $C_{\min}$ labelled by the partition $\lambda = (s+1,\ldots,s-d+2,s-d,\ldots,1)$ is contained in the closure of all other cuspidal classes. So let λ be a partition with all parts distinct and first assume that $\lambda_i \geq \lambda_{i+1}+2$ and $\lambda_j \geq \lambda_{j+1}+2$ for some $i<j$. Then the partition λ' with

$$\lambda'_i = \lambda_i - 1,\ \lambda'_{j+1} = \lambda_{j+1}+1,$$

and $\lambda'_l = \lambda_l$ for all other indices l, is smaller in the dominance order, hence labels a cuspidal class which is smaller in the partial order of unipotent classes. Similarly, if $\lambda_i \geq \lambda_{i+1}+3$ then λ' with

$$\lambda'_i = \lambda_i - 1,\ \lambda'_{i+1} = \lambda_{i+1}+1,$$

again labels a smaller cuspidal class. Application of these two operations eventually leads to the label of $C_{\min}$, so our claim follows.

In type C_m, we claim that the cuspidal class $C_{\min}$ such that $r_i \in \{0,2\}$ for all i, and in that range there is at most one i with $r_i = r_{i+1} = 2$, is containd in the closure of all other cuspidal classes. Here, $\lambda^* = (1^{r_1}, 2^{r_2}, \ldots, h^{r_h})$ denotes the conjugate partition of λ written in exponential notation. Indeed, if C has a label with $r_i \geq 4$ for some i, then the partition μ with μ^* such that

$$r'_{i-1} = r_{i-1}+2,\ r'_i = r_i - 4,\ r'_{i+1} = r_{i+1}+2,$$

labels a smaller cuspidal class. Similarly, if $r_i = r_{i+1} = 2$ and $r_j = r_{j+1} = 2$ for some $i<j$, then the partition μ with

$$r'_{i-1} = r_{i-1}+2,\ r'_i = r_i - 2,\ r'_{j+1} = r_{j+1} - 2,\ r'_{j+2} = r_{j+2}+2,$$

labels a smaller class.

In type B_m, respectively D_m and ${}^2D_m$, let $C_{\min}$ denote the cuspidal class such that removing the biggest part $h = 2s+1$ (resp. $h = 2s$) of λ^* we obtain the label of the minimal cuspidal class in type C_{m-h}. Then as before it is easy to see that $C_{\min}$ lies in the closure of all cuspidal classes. This completes the proof of the first assertion.

To determine $a_{d(C_{\min})}$ note that in all types but D_m and ${}^2D_m$, the dual of the special class corresponding to λ is the special class corresponding to the conjugate λ^* of λ (see [4, §12.7 and §13.4]). Then the claimed expression for $a_{d(C_{\min})}$ follows from the centraliser orders given in [4, §13.1]: in type ${}^2A_m$

$$\dim C_{\mathbf{G}}(u) = \sum_{i\geq 1}(r_i + r_{i+1} + \cdots)^2 - 1$$

for $u \in d(C)$ labelled by $\lambda^* = (1^{r_1}, 2^{r_2}, \dots, h^{r_h})$, in type C_m by

$$\dim C_{\mathbf{G}}(u) = \frac{1}{2}\Big(\sum_{i\geq 1}(r_i + r_{i+1} + \cdots)^2 + \sum_{i\equiv 1(2)} r_i\Big),$$

and in type B_m by

$$\dim C_{\mathbf{G}}(u) = \frac{1}{2}\Big(\sum_{i\geq 1}(r_i + r_{i+1} + \cdots)^2 - \sum_{i\equiv 1(2)} r_i\Big).$$

In types D_m, ${}^2D_m$ there is no elementary operation to deduce the effect of Lusztig–Spaltenstein duality on unipotent classes. However one can use (1) to compute $a_{d(C)}$ from the a-value of the Alvis–Curtis dual ρ^* of the unique special unipotent character with unipotent support C. Let

$$\mathcal{S} = \begin{pmatrix} a_1 & \dots & a_{2s-1} \\ a_2 & \dots & a_{2s} \end{pmatrix}$$

be the symbol of an irreducible character of $W(D_m)$. It is special if and only if $a_1 \leq a_2 \leq \ldots \leq a_{2s}$. By [25, (5.15)], for example (noting that the last "$-$" sign in that formula should be a "$+$"), the A-value of the corresponding unipotent principal series character $\rho_{\mathcal{S}}$ is given by

$$A(\mathcal{S}) = \sum_{i<j} \max\{a_i, a_j\} - \sum_{i=1}^{s-1}\binom{2i}{2} - 2\sum_{i=1}^{2s}\binom{a_i+1}{2} + m^2,$$

and the a-value of the Alvis–Curtis dual $\rho_{\mathcal{S}^*} = D_G(\rho_{\mathcal{S}})$ of $\rho_{\mathcal{S}}$ is

$$a(\mathcal{S}^*) = m(m-1) - A(\mathcal{S}).$$

Now let λ be a partition of $2m$ labelling a cuspidal unipotent class C for D_m. By the description of cuspidal unipotent classes, λ has an even number of non-zero parts, say $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2l} > 0)$. The Springer correspondent of C is obtained as follows, see [16, §2D]: it is labelled by a symbol $\mathcal{S}$ as above with entries $a_i = (\lambda_{2l+1-i} - 1)/2 + \lfloor \frac{i}{2} \rfloor$, $1 \leq i \leq 2l$. The symbol $\mathcal{S}_{\min}$ of the character $\rho_{\min}$ corresponding to the class $C_{\min}$ is then given by

$$\begin{pmatrix} & 0 \quad \dots \quad 2(s-1) & \\ 1 \ \dots \ 2(s-d)-1 & 2(s-d+1) \ \dots \ 2s \end{pmatrix} \qquad \text{for } 0 \leq d \leq s, \text{ resp.}$$

$$\begin{pmatrix} 0 \ \dots \ 2(2s-d-1) & 2(2s-d)+1 \ \dots 2s-1 \\ & 2 \quad \dots \quad 2s \end{pmatrix} \qquad \text{for } s \leq d \leq 2s.$$

(Here, dots signify that the entries in between are meant to increase in steps of 2. In particular, all $a_i - a_{i-2} = 2$ except at one single position.) Application of the above formula for $a(\mathcal{S}^*_{\min})$ yields the assertion. $\square$

Remark 3.2. Theorem 2.2 and Proposition 3.1 show that unipotent Brauer characters with "small" a-value must lie in "small" Harish-Chandra series, that is, in Harish-Chandra series corresponding to Levi subgroups of small semisimple rank. In particular, cuspidal modules have a large a-value compared to the rank of the group.

More precisely, by Proposition 3.1 the a-values of cuspidal unipotent classes in classical groups grow roughly like $cm^{\frac{3}{2}}$ with the rank m, where $c = \frac{\sqrt{2}}{3}$ for type ${}^2A_m$ and $c = \frac{2}{3}$

for types B_m, C_m, D_m and ${}^2D_m$. This is the same order of magnitude as the a-value of a cuspidal unipotent character. Since the latter is a fixed point of Alvis–Curtis duality, this shows that cuspidal unipotent Brauer characters only occur "in the lower half" of the decomposition matrix of a group of classical type.

In Table 1 we give the minimal a-value for classical groups of rank at most 10.

m	1	2	3	4	5	6	7	8	9	10
A_m	1	3	6	10	15	21	28	36	45	55
${}^2A_m$	–	1	3	4	4	7	9	10	10	14
B_m	–	1	3	6	7	7	11	13	18	21
C_m	–	1	4	5	7	7	12	15	16	20
$D_m, {}^2D_m$	–	–	–	3	7	9	12	13	13	19

TABLE 1. $a_{d(C_{\min})}$ in classical types

3.2. **Induction of cuspidal classes.** Let us recall the construction of induced unipotent classes by Lusztig–Spaltenstein [24]. Let $\mathbf{P} = \mathbf{LU}$ be a parabolic subgroup of $\mathbf{G}$ with Levi complement $\mathbf{L}$. Given any unipotent class C of $\mathbf{L}$, the variety $C\mathbf{U}$ is irreducible and consists of unipotent elements only. Therefore there exists a unique unipotent class $\widetilde{C}$ of $\mathbf{G}$ such that $\widetilde{C} \cap C\mathbf{U}$ is dense in $C\mathbf{U}$. The class $\widetilde{C}$ is called the induction of C from $\mathbf{L}$ to $\mathbf{G}$. It is shown in [24] that it depends only on $(C, \mathbf{L})$ and not on $\mathbf{P}$. As in [24] we will denote it by $\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(C)$.

Proposition 3.3. *Assume that p is good for $\mathbf{G}$ and $Z(\mathbf{G})$ is connected. Let $\mathbf{L}$ be a 1-split Levi subgroup of $\mathbf{G}$. If C is an F-stable cuspidal unipotent class of $\mathbf{L}$ then $\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(C)$ is an F-stable cuspidal unipotent class of $\mathbf{G}$.*

Proof. By standard reductions, we can assume that $\mathbf{G}$ is simple of adjoint type (see for example [15, §3.4]). Also, by transitivity of the induction of unipotent classes, we can assume without loss of generality that $\mathbf{L}$ is a maximal proper 1-split Levi subgroup of $\mathbf{G}$. Assume that $\mathbf{L}$ has an untwisted type A_r-factor. Its only cuspidal unipotent class is the regular class, which is obtained by induction of the trivial class of a maximal torus. By transitivity of induction we may thus replace this factor by a maximal torus, and then replace that Levi subgroup by any other maximal 1-split Levi subgroup of $\mathbf{G}$ containing it. In particular, our claim holds for groups of untwisted type A.

Thus, if $\mathbf{G}$ is simple of classical type X_m (where m is the rank), then we may assume that $\mathbf{L}$ is a group of type $X_{m-\delta}$, where $\delta = 2$ if $X_m = {}^2A_m$ and $\delta = 1$ otherwise. Let λ be a partition labelling a cuspidal unipotent class C of $\mathbf{L}$. It satisfies the properties listed in the beginning of the proof of Proposition 3.1, depending on the type of $\mathbf{L}$. Under these conditions, if follows from [27, §7.3] that the partition labelling $\mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(C)$ is $(\lambda_1 + 2, \lambda_2, \ldots)$, and so the corresponding class is also cuspidal.

If $\mathbf{G}$ is simple of exceptional type, then by our previous reduction we are in one of the cases listed in Table 2. The induction of classes is then easily determined using for example the Chevie [26] command `UnipotentClasses` and the fact that induction of classes is just j-induction of the corresponding Springer representations, see [28, 4.1]. $\square$

B_3	7	51^2	3^21
C_3	6	42	
F_4	F_4	$F_4(a_1)$	$F_4(a_2)$

D_5	91	73	531^2
E_6	E_6	$E_6(a_1)$	$E_6(a_3)$

${}^2A_5$	6	51	42		321
${}^2D_4$	71	53		3^21^2	
${}^2E_6$	E_6	$E_6(a_1)$	D_5	$E_6(a_3)$	$D_5(a_1)$

D_6	11.1	93	75	731^2	5^21^2	53^21	
E_6	E_6	$E_6(a_1)$		$E_6(a_3)$			$D_4(a_1)$
E_7	E_7	$E_7(a_1)$	$E_7(a_2)$	$E_7(a_3)$	$E_6(a_1)$	$E_7(a_4)$	$E_7(a_5)$

D_7	13.1	11.3	95	931^2	751^2	73^21	5^231		
E_7	E_7	$E_7(a_1)$	$E_7(a_2)$	$E_7(a_3)$	$E_6(a_1)$	$E_7(a_4)$		$E_7(a_5)$	A_4+A_1
E_8	E_8	$E_8(a_1)$	$E_8(a_2)$	$E_8(a_3)$	$E_8(a_4)$	$E_8(b_4)$	$E_8(a_5)$	$E_8(b_5)$	$E_6(a_1)+A_1$

Table 2. Some induced cuspidal classes

4. Irreducibility of the smallest character in a Harish-Chandra series

We extend our irreducibility result in [8, Thm. A].

Lemma 4.1. *Assume that* $\mathbf{G}$ *is simple of adjoint type. Let* $\mathbf{L} \leq \mathbf{G}$ *be a* 1*-split Levi subgroup with a cuspidal unipotent character* λ*. Then* $a_\lambda \leq a_{d(C)}$ *for all* F*-stable cuspidal unipotent classes* C *of* $\mathbf{G}$*, with equality occurring if and only if* $\mathbf{L} = \mathbf{G}$*.*

Proof. For classical types, see Proposition 3.1. For exceptional types, the minimal $a_{d(C)}$ and the possible a_λ are easily listed using [15, Prop. 3.6] and the tables in [4, §13]; they are given in Table 3. $\square$

Remark 4.2. For an alternative proof of the lemma based on Proposition 3.3 consider the (cuspidal) unipotent support C_λ of λ and the induced class $\widetilde{C}_\lambda = \mathrm{Ind}_{\mathbf{L}}^{\mathbf{G}}(C_\lambda)$. Then $a_\lambda = a_{d(C_\lambda)} = a_{C_\lambda} = a_{\widetilde{C}_\lambda}$ by [24, Thm. 1.3]. Moreover, by Proposition 3.3, the class $\widetilde{C}_\lambda$ is cuspidal. Therefore the minimal cuspidal class $C_{\min}$ from Proposition 3.1 lies in the closure of $\widetilde{C}_\lambda$, and hence $a_\lambda \leq a_{C_{\min}} = a_{d(C_{\min})}$ with equality only when $C_{\min} = \widetilde{C}_\lambda$. In that case the unipotent support of λ is self-dual, therefore the character λ is its own Alvis-Curtis dual and $\mathbf{L} = \mathbf{G}$.

The following extends our result [6, Prop. 4.3] for unitary groups (which was shown using the known unitriangularity of the decomposition matrix) to all reductive groups; here instead we use the irreducibility of cuspidal unipotent characters from [8, Thm. A]:

Theorem 4.3. *Assume that* p *is good for* $\mathbf{G}$*. Let* $\ell \neq p$ *be a prime which is good for* $\mathbf{G}$ *and not dividing* $|Z(\mathbf{G})^F/Z^\circ(\mathbf{G})^F|$*. Let* ρ *be a unipotent character of* G *which has minimal* a*-value in its (ordinary) Harish-Chandra series. Then the* ℓ*-modular reduction of* ρ *is irreducible.*

	$C_{\min}$	$a_{d(C_{\min})}$	possible a_λ
G_2	$G_2(a_1)$	1	$0,1$
F_4	$F_4(a_3)$	4	$0,1,4$
$E_6, {}^2E_6$	$D_4(a_1)$	7	$0,3,4,7$
E_7	A_4+A_1	11	$0,3,7,11$
E_8	$E_8(a_7)$	16	$0,3,7,11,16$

TABLE 3. Minimal $a_{d(C)}$ in exceptional types

Proof. As in the proof of [8, Thm. A] we can assume that $\mathbf{G}$ is simple of adjoint type.

Let $\mathbf{L} \leq \mathbf{G}$ be a 1-split Levi subgroup and $\lambda \in \mathrm{Irr}(L)$ a cuspidal unipotent character such that ρ lies in the Harish-Chandra series of (L, λ). Since $\mathbf{G}$ is adjoint, the centre of $\mathbf{L}$ is connected and by [8, Thm. A] there exists a unique PIM of L containing λ. In addition, it satisfies $\langle \lambda, P_\lambda \rangle = 1$ and $a_{P_\lambda} = a_\lambda$.

Recall that the characters in the Harish-Chandra series of (L, λ) are in bijection with the irreducible representations of the Hecke algebra corresponding to the cuspidal pair. Under this bijection, the character ρ of minimal a-value corresponds to the representation that specialises to the trivial character at $q = 1$. Furthermore, the degree of ρ equals the product of $\lambda(1)$ with the Poincaré polynomial of the Hecke algebra. In particular, $a_\lambda = a_\rho$. In addition, $\langle \rho, R_L^G(P_\lambda) \rangle = 1$ by the Howlett–Lehrer comparison theorem.

Let P_ρ denote the (projective) indecomposable summand of $R_L^G(P_\lambda)$ containing ρ (once). Then $a_\rho \geq a_{P_\rho} \geq a_{P_\lambda}$. Since $a_{P_\lambda} = a_\lambda = a_\rho$ we deduce that $a_{P_\rho} = a_\rho$.

We claim that ρ does not occur in any other PIM of G. For this, assume that ρ is a constituent of $R_M^G(P)$ for some 1-split Levi subgroup $\mathbf{M} \leq \mathbf{G}$ and some PIM P of M. Then P has to have some constituent $\psi \in \mathrm{Irr}(M)$ from the (L, λ)-series, whence $a_\lambda \leq a_\psi \leq a_\rho = a_\lambda$. So we have that $a_\psi = a_\lambda$ and ψ is the character of M with minimal a-value in the (L, λ)-series. If M is proper, then by induction there is exactly one PIM P_ψ of M containing ψ which is a direct summand of $R_L^M(P_\lambda)$; as P_ρ occurs exactly once in $R_L^G(P_\lambda)$, we see that the summand of $R_M^G(P) = R_M^G(P_\psi)$ containing ρ is just P_ρ.

To conclude, finally assume that $M = G$, so P is a cuspidal PIM of G. But then by [8, Prop. 2.2] it occurs as a summand of a GGGR Γ_C for some F-stable cuspidal unipotent class C of $\mathbf{G}$. According to Lemma 4.1 we have that $a_{\Gamma_C} \geq a_\lambda$, with equality only when $\mathbf{L} = \mathbf{G}$. Thus, γ_C can contain ρ only when $\mathbf{L} = \mathbf{G}$, and so $\rho = \lambda$ is cuspidal. But then our claim is just [8, Thm. A]. □

5. Small Harish-Chandra series in even-dimensional orthogonal groups

We start by considering the two families of even-dimensional orthogonal groups. As Theorem 2.2 only holds in good characteristic, we shall restrict ourselves to the case where q is odd. As before, ℓ will denote a prime not dividing q.

5.1. **Small cuspidal Brauer characters.** We first determine the cuspidal unipotent Brauer characters with small a-value. For a connected reductive group $\mathbf{G}$ we set $\mathbf{G}_{\mathrm{ad}} := \mathbf{G}/Z(\mathbf{G})$.

Lemma 5.1. *Let* $\mathbf{G}$ *be connected reductive and assume that*

(1) *G_{ad} is simple of type $A_m(q)$ ($m \geq 1$), $D_m(q)$ (q odd, $m \geq 4$) or ${}^2D_m(q)$ (q odd, $m \geq 2$);*
(2) *ℓ does not divide $2q|Z(\mathbf{G})/Z^\circ(\mathbf{G})|$; and*
(3) *if $\ell = 3|(q+1)$ then $G_{\mathrm{ad}} \not\cong {}^2D_3(q)$.*

Then any cuspidal unipotent Brauer character $\varphi \in \mathrm{IBr}(G)$ with $a_\varphi \leq 3$ occurs in Table 4.

TABLE 4. Some cuspidal unipotent Brauer characters in types A_m, D_m and ${}^2D_m$

G_{ad}	φ	a_φ	condition on ℓ
$A_1(q)$	1^2	1	$\ell\|(q+1)$
$A_2(q)$	1^3	3	$\ell\|(q^2+q+1)$
${}^2D_2(q)$	$-.1$	2	$\ell\|(q^2+1)$
${}^2D_3(q)$	$-.2$	3	$\ell\|(q+1)$
$D_4(q)$	D_4	3	always

The Brauer characters are labelled, via the triangularity of the decomposition matrix, by the Harish-Chandra labels of the ordinary unipotent characters.

Proof. Let $\varphi \in \mathrm{IBr}(G)$ be as in the statement. By Theorem 2.2, there exists a cuspidal unipotent class C of $\mathbf{G}_{\mathrm{ad}}$ with $a_{d(C)} \leq 3$. By Proposition 3.1, G_{ad} must then be of one of the types

$$A_1(q),\ A_2(q),\ {}^2D_3(q) \cong A_1(q^2),\ {}^2D_3(q) \cong {}^2A_3(q),\ {}^2D_4(q) \text{ or } D_4(q).$$

Under our assumptions on ℓ for all listed groups the unipotent characters form a basic set for the union of unipotent ℓ-blocks of both G and G_{ad} [12, 10], and the decomposition matrix with respect to these is uni-triangular (see [19, 6, 17]). Moreover, from the explicit knowledge of these decomposition matrices, it follows that φ must be as in Table 4. □

5.2. **Small Harish-Chandra series.** We can now determine those unipotent Harish-Chandra series that may contain Brauer characters of small a-value.

Lemma 5.2. *Assume G_{ad} is simple of type $A_m(q)$ and $\ell > 2$.*
(a) *If $m \geq 2$ and $\ell|(q+1)$ then $a_\varphi \geq 3$ for all unipotent Brauer characters $\varphi \in \mathrm{IBr}(G)$ in the $(A_1, 1^2)$-Harish-Chandra series.*
(b) *If $m \geq 3$, and $\ell|(q^2+q+1)$, then $a_\varphi \geq 6$ for all unipotent Brauer characters $\varphi \in \mathrm{IBr}(G)$ in the $(A_2, 1^3)$-Harish-Chandra series.*
(c) *If $m \geq 4$ and $\ell|(q+1)$ then $a_\varphi \geq 4$ for all unipotent Brauer characters $\varphi \in \mathrm{IBr}(G)$ in the $(A_1^2, 1^2 \otimes 1^2)$-Harish-Chandra series.*

Proof. It follows from the known unipotent decomposition matrices (see [19, App. 1]), or from the known distribution into modular Harish-Chandra series, that $a_\varphi = 3$ for $m = 2$ in (a), $a_\varphi = 6$ for $m = 3$ in (b), and $a_\varphi \geq 4$ for $m = 4$ in (c) respectively. Since Harish-Chandra induction does not diminish the a-value (see for example [22, Cor. 8.7]), the claim also holds for all larger m. □

Proposition 5.3. *Assume that $n \geq 4$ and $\ell \nmid 6q$. Then all unipotent Brauer characters of $\mathrm{Spin}_{2n}^{\pm}(q)$, q odd, with $a_\varphi \leq 3$ lie in one of the Harish-Chandra series given in Table 5.*

	(L,λ)	conditions
$D_n(q)$	$(\emptyset,1),\ (D_4,D_4)$	always
	$(A_1,1^2),\ (D_2,-.2)$	$\ell\|(q+1)$
	$(A_1^2,1^2\otimes 1^2),\ (D_2A_1,-.2\otimes 1^2)$	$\ell\|(q+1),\ n=4$
${}^2D_n(q)$	$(\emptyset,1)$	always
	$(A_1,1^2),\ ({}^2D_3,-.2)$	$\ell\|(q+1)$
	$({}^2D_2,-.1)$	$\ell\|(q^2+1)$
	$(A_1^2,1^2\otimes 1^2)$	$\ell\|(q+1),\ n=5$
	$(A_2,1^3)$	$\ell\|(q^2+q+1),\ n=4$

TABLE 5. Unipotent Harish-Chandra series in ${}^{(2)}D_n(q)$ with small a-value

Proof. First assume that $G=\mathrm{Spin}_{2n}^+(q)$. Let $\varphi\in\mathrm{IBr}(G)$ be an ℓ-modular unipotent Brauer character of a-value at most 3. It lies in the Harish-Chandra series associated to a cuspidal pair (L,λ), with $\lambda\in\mathrm{IBr}(L)$. Note that for Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ we have that $Z(\mathbf{L})/Z^\circ(\mathbf{L})$ is a 2-group, so that under our assumptions on ℓ the unipotent Harish-Chandra series and unipotent cuspidal characters of L are as in its adjoint quotient. But the latter is a direct product of simple groups of adjoint type. Thus, since the a-value is additive over outer tensor products of characters we may apply Lemma 5.1. By the shape of parabolic subgroups of a Weyl group of type D_n, L is of one of the types

$$A_1^m(q)\ (m\le 3),\ D_2(q),\ D_2(q)A_1(q),\ A_2(q),\ \text{or } D_4(q),$$

λ is a cuspidal unipotent Brauer character of L as in Table 4, and ℓ is as given there. We consider these Harish-Chandra series (L,λ) in turn.

Assume that $\ell|(q^2+q+1)$, so L is of type $A_2(q)$. For $n\ge 4$ the group G contains a Levi subgroup of type A_3, and so by Lemma 5.2(b) the $(A_2,1^3)$-series of G only contains Brauer characters of a-value at least 6.

It remains to consider the case that $\ell|(q+1)$. By Lemma 5.2 neither the A_1^2-series nor the D_2A_1-series can contribute if $n\ge 5$, since these Levi subgroups are contained in a Levi subgroup of G of type A_4, resp. of type D_2A_2. Similarly, A_1^3 (which occurs for $n\ge 6$) is contained in a Levi subgroup of type A_3A_1 and so the a-values in that series are at least 4.

Now let $G=\mathrm{Spin}_{2n}^-(q)$ and φ as in the statement. So by Theorem 2.2, φ lies in a Harish-Chandra series (L,λ) with $a_\lambda\le 3$. Thus, by Lemma 5.1, L is of one of the types

$$A_1^m(q)\ (m\le 3),\ {}^2D_2(q),\ A_2(q),\ \text{or } {}^2D_3(q),$$

and $\lambda\in\mathrm{IBr}(L)$ is a cuspidal Brauer character of L as in Table 4. We consider the contributions by these Harish-Chandra series in turn. For $\ell|(q^2+q+1)$ the known Brauer trees show that for $n=4$ there is one Brauer character in the $(A_2,1^3)$-series with a-value 3 , but for $n=5$ (and hence for all larger n) all a-values are at least 6 .

Now consider the case when $\ell|(q+1)$. Since ${}^2D_6$ contains a Levi subgroup of type A_4 if $n\ge 6$ we see by Lemma 5.2 that the Harish-Chandra series of type A_1^2 cannot contain φ in that case. Similarly A_1^3 lies in a Levi subgroup of type A_3A_1 and so the A_1^3-series does not contain φ. □

5.3. **A triangular subpart of the decomposition matrix.** We give an approximation to the ℓ-modular decomposition matrix of $\mathrm{Spin}_{2n}^{\pm}(q)$ for certain unipotent Brauer characters of a-value at most 3. For simplicity, in view of Table 5 we only consider primes ℓ not dividing $q+1$. Recall that unipotent characters of $\mathrm{Spin}_{2n}^{+}(q)$ are labelled by symbols of rank n and defect congruent to 0 modulo 4 (see [4, §13.8]).

Theorem 5.4. *Let $G = \mathrm{Spin}_{2n}^{+}(q)$ with q odd and $n \geq 5$, and $\ell > 5$ a prime not dividing $q(q+1)$. Then the first eight rows of the decomposition matrix of the unipotent ℓ-blocks of G are approximated from above by Table 6, where $k = n-5$ and*

$$(a,b,c,d,e) = \begin{cases} (1,0,0,0,0) & \textit{when } \ell|(q^2+q+1), \\ (0,1,0,0,0) & \textit{when } \ell|(q^2+1), \\ (0,0,1,0,0) & \textit{when } \ell|(q^4+q^3+q^2+q+1), \\ (0,0,0,1,0) & \textit{when } \ell|(q^2-q+1), \\ (0,0,0,0,1) & \textit{when } \ell|(q^4+1), \\ (0,0,0,0,0) & \textit{otherwise.} \end{cases}$$

ρ	a_ρ								
$\binom{n}{0}$	0	1							
$\binom{n-1}{1}$	1	$e+k$	1						
$\binom{1,n}{0,1}$	2	$c+k$	.	1					
$\binom{3}{2}$	2	$b+d+ke+\binom{k}{2}$	$b+k$	.	1				
$\binom{0,1,2,n-1}{-}$	3	.	.	.	.	1			
$\binom{0,n-1}{1,2}$	3	$ke+\binom{k}{2}$	$e+k$	.	.	.	1		
$\binom{1,n-1}{0,2}$	3	$k(c+e+k-1)$	$b+k$	$d+k$	b	.	.	1	
$\binom{2,n-1}{0,1}$	3	$b+kc+\binom{k}{2}$	.	$a+k$	.	.	.	.	1
HC-series		ps	ps	ps	ps	D_4	ps	ps	ps

TABLE 6. Approximate decomposition matrices for $\mathrm{Spin}_{2n}^{+}(q)$, $n \geq 5$

Proof. Let $\varphi \in \mathrm{IBr}(G)$ be an ℓ-modular constituent of one of the eight unipotent characters $\rho_1, \ldots, \rho_8$ listed in Table 6 (in that order). Then by Brauer reciprocity, ρ_i is a constituent of the projective cover of φ. The degree formula [4, §13.8] shows that $a_{\rho_i} \leq 3$ for all i. Thus, by Proposition 5.3, φ lies in one of the Harish-Chandra series (L, λ) given in Table 5. We consider these in turn.

The Hecke algebra for the principal series is the Iwahori–Hecke algebra $\mathcal{H}(D_n; q)$ of type D_n, whose ℓ-decomposition matrix is known to be uni-triangular with respect to a canonical basic set given by FLOTW bipartitions, see [14, Thm. 5.8.19]. All of the ρ_i apart from ρ_5 are contained in this basic set, for all relevant primes ℓ. Thus the corresponding part of the decomposition matrix is indeed lower triangular. The given upper bounds on the entries in this part of the decomposition matrix are obtained by Harish-Chandra inducing the PIMs from $\mathrm{Spin}_8^{+}(q)$. For this group we can consider each cyclotomic factor in turn. The Sylow ℓ-subgroups for primes $\ell > 3$ dividing q^2+q+1, $q^4+q^3+q^2+q+1$, q^2-q+1

or q^4+1 are cyclic, and the Brauer trees of unipotent blocks are easily determined. The unipotent decomposition matrix for primes $3 < \ell | (q-1)$ is the identity matrix by a result of Puig, see [3, Thm. 23.12]. Finally, for $(q^2+1)_\ell > 5$ the unipotent decomposition matrix of $\mathrm{Spin}_8^+(q)$ was determined in [7, Thm. 3.3].

Next consider the Harish-Chandra series of the ordinary cuspidal unipotent character λ of a Levi subgroup L of type D_4. Then $R_L^G(\lambda)$ only contains unipotent characters in the ordinary Harish-Chandra series of type D_4, hence among $\rho_1, \ldots, \rho_8$ only ρ_5, just once. So there is exactly one PIM in this series involving one of the ρ_i, namely the projective cover of ρ_5. □

We now consider the similar question for the non-split orthogonal groups $\mathrm{Spin}_{2n}^-(q)$. Recall that its unipotent characters are labelled by symbols of rank n and defect congruent to 2 modulo 4. Since all considered characters lie in the principal series, they can also be indexed by irreducible characters of the Weyl group of type B_{n-1}, hence by suitable bipartitions of $n-1$, which we will do here.

Theorem 5.5. *Let $G = \mathrm{Spin}_{2n}^-(q)$ with q odd and $n \geq 5$, and $\ell > 5$ a prime not dividing $q(q+1)$. Then the first eight rows of the decomposition matrix of the unipotent ℓ-blocks of G are approximated from above by Table 7, where $k = n-5$ and*

$$(a,b,c,d,e) = \begin{cases} (1,0,0,0,0) & \textit{when } \ell | (q^2+q+1), \\ (0,1,0,0,0) & \textit{when } \ell | (q^2+1), \\ (0,0,1,0,0) & \textit{when } \ell | (q^2-q+1), \\ (0,0,0,1,0) & \textit{when } \ell | (q^4+1), \\ (0,0,0,0,1) & \textit{when } \ell | (q^4-q^3+q^2-q+1), \\ (0,0,0,0,0) & \textit{otherwise.} \end{cases}$$

The eight corresponding PIMs all lie in the principal series, except that for $\ell | (q^2+1)$ the eighth (and the seventh when $n=4$) is in the ${}^2D_2$-series.

ρ	a_ρ								
$(n-1;-)$	0	1							
$(n-2,1;-)$	1	$b+k$	1						
$(n-2;1)$	2	$e+k$	.	1					
$(n-3,2;-)$	2	$a+kb+\binom{k}{2}$	k	.	1				
$(n-3,1^2;-)$	3	$kb+\binom{k}{2}$	$b+k$	.	.	1			
$(n-3,1;1)$	3	$k(b+e+k-1)$	$d+k$	$a+k$	b	.	1		
$(n-3;2)$	3	$d+ke+\binom{k}{2}$	.	$c+k$	.	.	.	1	
$(-;n-1)$	3	b	.	.	.	.	.	.	1

TABLE 7. Approximate decomposition matrices for $\mathrm{Spin}_{2n}^-(q)$, $n \geq 5$

Proof. This is completely analogous to the proof of Theorem 5.4. The eight unipotent characters $\rho_1, \ldots, \rho_8$ displayed in Table 7 all have a-value at most 3. Thus, arguing as in the proof of Theorem 5.4 and using Table 5 we see that any ℓ-modular constituent φ of the ρ_i either lies in the principal series, or $\ell|(q^2+1)$ and φ lies in the ${}^2D_2$-series.

For $n=4$ the Sylow ℓ-subgroups of G are cyclic for all relevant primes ℓ, and the claim follows from the known Brauer trees. So now assume that $n \geq 5$. By [14, Thm. 5.8.13] the Iwahori-Hecke algebra $\mathcal{H}(B_{n-1}; q^2; q)$ for the principal series has a canonical basic set indexed by suitable Uglov bipartitions. For $n \geq 5$ the bipartitions indexing $\rho_1, \ldots, \rho_7$, as well as ρ_8 when $\ell \not| (q^2+1)$, are Uglov, so these characters lie in the basic set. Hence, there are at most eight PIMs in the principal series (resp. seven when $\ell \mid (q^2+1)$) involving one of the ρ_i; since the unipotent characters form a basic set for the unipotent blocks by [12], it must be exactly eight (resp. seven).

Finally, assume that $\ell|(q^2+1)$ and consider the series of the cuspidal Brauer character $(-;1)$ of a Levi subgroup of type ${}^2D_2$. It follows from the known Brauer trees that there are two PIMs of ${}^2D_4(q)$ in the ${}^2D_2$ Harish-Chandra series, with unipotent parts $(-;3)$ and $(1;2)$ respectively. Harish-Chandra induction to ${}^2D_5(q)$ shows that there is exactly one PIM in this series containing one of the ρ_i, namely ρ_8 with label $(-;4)$ (see [7, Thm. 7.1]). The Harish-Chandra induction of $(-;4)$ from ${}^2D_5(q)$ to ${}^2D_n(q)$ only contains ρ_8 among our ρ_i, so there is at most one PIM in this series that contributes to the first eight rows of the decomposition matrix. □

Remark 5.6. To complete the determination of the part of the decomposition matrix displayed in Table 7 it would be enough to compute the corresponding part of the decomposition matrix of the Hecke algebra $\mathcal{H}(B_{n-1}; q^2; q)$. When ℓ is large, this can be done using canonical basis elements of Fock spaces (see [14, §6.4]).

5.4. **Unipotent Brauer characters of low degree.** As a consequence we can determine those irreducible Brauer characters of low degree which occur as constituents of $\rho_1, \ldots, \rho_8$. For this we need the ℓ-modular decomposition of the smallest three unipotent characters $\rho_1 = 1_G$, ρ_2 and ρ_3; here, for an integer m, we set

$$\kappa_{\ell,m} := \begin{cases} 1 & \text{if } \ell|m, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.7 (Liebeck). *Let $G = \mathrm{Spin}_{2n}^{\pm}(q)$, $n \geq 3$ and let ℓ be a prime not dividing $q(q+1)$. The ℓ-modular decomposition of ρ_2, ρ_3 is given by*

$$\begin{array}{c|ccc} 1 & 1 & & \\ \rho_2 & a & 1 & \\ \rho_3 & b & . & 1 \end{array}$$

where $a = \kappa_{\ell,q^{n-1}\pm 1}$, $b = \kappa_{\ell,q^n \mp 1}$.

Proof. First note that if $\ell \nmid q^{n-1} \pm 1$ (resp. $\ell \nmid q^n \mp 1$) then ρ_3 (resp. ρ_2) does not lie in the principal block (see [9, §13]).

The unipotent characters ρ_2, ρ_3 are the two non-trivial constituents of the Harish-Chandra induction of the trivial character from a Levi subgroup of type $\mathrm{Spin}_{2n-2}^{\pm}(q)$. This induced character is the permutation character of the rank 3 permutation module of G on singular 1-spaces. The claim thus follows from Liebeck [21, Thm. 2.1 and pp.14/15]. Indeed, in the case (1) of [21, p. 10] there are obviously just three ℓ-modular composition factors of that permutation module, so all ρ_i remain irreducible. In the case (2) there are four composition factors, and those in head and socle clearly are trivial, so one of the ρ_i is reducible. Which one that is follows from the block distribution. □

Note that the zeroes in the decomposition matrix in Proposition 5.7 also follow from Tables 6 and 7, and the multiplicity of the trivial character in the ℓ-modular reductions of ρ_2 and ρ_3 could also easily be computed from the corresponding Hecke algebras.

Corollary 5.8. *Under the hypotheses of Theorem 5.4 resp. 5.5 assume that $\varphi \in \mathrm{IBr}(G)$ is a constituent of one of the unipotent characters ρ_i, $1 \leq i \leq 8$, in Table 6 resp. 7. If*

$$\varphi(1) < \begin{cases} q^{4n-10} & \textit{for } G = \mathrm{Spin}_{2n}^{+}(q), \ \textit{resp.} \\ q^{4n-10} - q^9 & \textit{for } G = \mathrm{Spin}_{2n}^{-}(q), \end{cases}$$

then $i \leq 3$. In particular, φ is as described in Proposition 5.7.

Proof. Let φ_j, $1 \leq j \leq 8$, denote the Brauer character of the head (and socle) of the PIM given by column j of the approximate decomposition matrices in Table 6, respectively 7. By the proven partial triangularity, any constituent φ of one of the ρ_i is among the φ_j, $1 \leq j \leq 8$.

Now for each i, $4 \leq i \leq 8$, a lower bound for $\varphi_i(1)$ can be computed as follows: we certainly have $\varphi_j(1) \leq \rho_j(1)$ for $1 \leq j \leq i-1$. Then from the ith row of the given approximation of the decomposition matrix with that vector of upper bounds we obtain a lower bound for $\varphi_i(1)$. This is a polynomial in q (depending on n) of degree at least $4n-10$, and subtracting the bound B in the statement we obtain a polynomial of positive degree and positive highest coefficient. It is now a routine calculation to see that this difference is positive for all relevant i, q, n, so $\varphi_i(1) \geq B$ for $i \geq 4$. □

6. Small Harish-Chandra series in odd-dimensional orthogonal groups

We now consider the odd-dimensional spin groups $\mathrm{Spin}_{2n+1}(q)$. Again we assume that q is odd in order to be able to apply the results from Section 2.

6.1. Small Harish-Chandra series.

Again we first determine cuspidal modules with small a-value.

Lemma 6.1. *Assume that G_{ad} is simple of type $B_m(q)$ (q odd, $m \geq 2$), and let ℓ be a prime not dividing $2q$. If $\varphi \in \mathrm{IBr}(G)$ is a cuspidal unipotent Brauer character with $a_\varphi \leq 4$ then φ occurs in Table 8.*

G_{ad}	φ	a_φ	condition
$B_2(q)$	B_2	1	always
	$-.1^2$	4	$\ell \mid (q+1)(q^2+1)$
$B_3(q)$	$1^3.-$	4	$\ell \mid (q+1)$
	$B_2 : 1^2$	4	$\ell \mid (q+1)(q^2-q+1)$

Table 8. Cuspidal unipotent Brauer characters in type B_m

Proof. Let $\varphi \in \mathrm{IBr}(G)$ be as in the statement. By Theorem 2.2, the projective cover of φ is a direct summand of Γ_C^G for a cuspidal unipotent class C of G_{ad} with $a_{d(C)} \leq 4$. By Proposition 3.1 this implies $m \in \{2,3\}$. For these groups the unipotent characters form a basic set for the union of unipotent ℓ-blocks of G [10], and the decomposition matrix with respect to these is uni-triangular (see [18]). From the explicit knowledge of these decomposition matrices, it follows that φ must be as in Table 8. □

Proposition 6.2. *Assume that $n \geq 2$ and $\ell \not| q(q+1)$. Then the unipotent Brauer characters of* $\mathrm{Spin}_{2n+1}(q)$, *q odd, with $a_\varphi \leq 4$ lie in a Harish-Chandra series as given in Table 9.*

(L,λ)	conditions
$(\emptyset,1),\ (B_2,B_2)$	always
$(B_2,-.1^2)$	$\ell\|(q^2+1)$
$(A_2,1^3)$	$\ell\|(q^2+q+1),\ n=3$
$(B_2A_2,B_2\otimes 1^3)$	$\ell\|(q^2+q+1),\ n=5$
$(B_3,B_2:1^2)$	$\ell\|(q^2-q+1),\ n=3$

TABLE 9. Unipotent Harish-Chandra series in $B_n(q)$ with small a-value

Proof. Let $\varphi \in \mathrm{IBr}(G)$ be an ℓ-modular unipotent Brauer character with $a_\varphi \leq 4$. So φ lies in the Harish-Chandra series associated to a cuspidal pair (L,λ) with $a_\lambda \leq 4$. By Theorem 2.2 and Lemmas 5.1 and 6.1, L is either a torus or of one of the types

$$A_2(q),\ B_2(q),\ B_2(q)A_2(q),\ \text{or } B_3(q),$$

with λ the corresponding outer tensor product of the cuspidal Brauer characters given in Tables 4 and 8. (Note that Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ have $|Z(\mathbf{L})/Z^\circ(\mathbf{L})| \leq 2$ and so one can invoke both Lemma 5.1 and Lemma 6.1) We deal with the various series in turn. First note that a Levi subgroup of G of type A_2 is contained in one of type A_3 when $n \geq 4$, and so by Lemma 5.2(b) the A_2-series (for $\ell|(q^2+q+1)$) can only contribute when $n=3$. Similarly, the B_2A_2-series of $\lambda = B_2 \otimes 1^3$ can only contribute when $n=5$. From the known Brauer trees it follows that for $n=4$ all ℓ-modular Brauer characters in the $(B_3, B_2:1^2)$-series for $\ell|(q^2-q+1)$ have a-value at least 6. □

For primes $\ell|(q+1)$ there are many further modular HC-series with a-value at most 4; we will discuss those elsewhere.

6.2. **A triangular subpart of the decomposition matrix.** We give an approximation to the ℓ-modular decomposition matrix of $\mathrm{Spin}_{2n+1}(q)$ for certain unipotent Brauer characters of a-value at most 4. Again we only consider primes ℓ not dividing $q+1$. The induction basis is given by the cases $n=3$ computed in [18] and the case $n=4$ treated below. Recall that unipotent characters of $\mathrm{Spin}_{2n+1}(q)$ are labelled by symbols of rank n and odd defect (see [4, §13.8]).

Theorem 6.3. *Let $G = \mathrm{Spin}_{2n+1}(q)$ with q odd and $n \geq 4$, and $\ell > 5$ a prime not dividing $q(q+1)$. Then the first fourteen rows of the decomposition matrix of the unipotent ℓ-blocks of G are approximated from above by Table 10, where $k = n-4$ and*

$$(a,b,c,d) = \begin{cases} (1,0,0,0) & \text{when } \ell|(q^2+q+1), \\ (0,1,0,0) & \text{when } \ell|(q^2+1), \\ (0,0,1,0) & \text{when } \ell|(q^2-q+1), \\ (0,0,0,1) & \text{when } \ell|(q^4+1), \\ (0,0,0,0) & \text{otherwise.} \end{cases}$$

The Harish-Chandra series are indicated in the last line; here the twelfth PIM is in the principal series unless $\ell|(q^2+1)$.

ρ	a_ρ														
$\binom{n}{-}$	0	1													
$\binom{0,1,n}{-}$	1	.	1												
$\binom{0,1}{n}$	1	.	.	1											
$\binom{1,n}{0}$	1	$b+k$	.	.	1										
$\binom{0,n}{1}$	1	$d+k$	.	.	.	1									
$\binom{0,2,n-1}{-}$	2	.	k	.	.	.	1								
$\binom{0,2}{n-1}$	2	b	.	$b+k$	.	.	.	1							
$\binom{2,n-1}{0}$	2	$a+kb+\binom{k}{2}$	.	.	k	.	.	.	1						
$\binom{0,n-1}{2}$	2	$c+kd+\binom{k}{2}$	.	.	.	$b+k$	.	.	.	1					
$\binom{1,n-1}{1}$	3	$k(b+d+k-1)$	.	.	$c+k$	$a+k$	.	.	.	.	1				
$\binom{0,1,2,n}{1}$	4	.	$d+k$	.	.	.	b	.	.	.	.	1			
$\binom{0,1,2}{1,n}$	4	.	.	$b+k$	.	.	.	b	.	.	.	.	1		
$\binom{1,2,n}{0,1}$	4	$kb+\binom{k}{2}$	.	.	$b+k$	.	.	.	.	.	.	.	.	1	
$\binom{0,1,n}{1,2}$	4	$kd+\binom{k}{2}$	.	.	.	$d+k$	.	.	.	.	.	.	.	.	1
HC-series		ps	B_2	ps	ps	ps	B_2	ps	ps	ps	ps	B_2	$ps/.1^2$	ps	ps

TABLE 10. Approximate decomposition matrices for $\mathrm{Spin}_{2n+1}(q)$, $n \geq 4$

Proof. Let first consider the case $n = 4$. For primes $\ell > 3$ dividing one of q^2+q+1, q^2-q+1 or q^4+1, the Sylow ℓ-subgroups of $G = \mathrm{Spin}_9(q)$ are cyclic, and the Brauer trees of unipotent blocks are easily determined. In particular the decomposition matrix is triangular in those cases. The unipotent decomposition matrix for primes $3 < \ell|(q-1)$ is the identity matrix by [3, Thm. 23.12]. Finally, for $\ell|(q^2+1)$ projective characters of G with the indicated decomposition into ordinary characters can be obtained by Harish-Chandra inducing PIMs from a Levi subgroup of type $\mathrm{Spin}_7(q)$, for which Sylow ℓ-subgroups are cyclic. Harish-Chandra restriction then shows that all of the non-zero entries "b" under the diagonal have to be positive. This construction also gives the claim about the modular Harish-Chandra series. (See also [6, Thm. 8.2] for an analogous argument for $\mathrm{Sp}_8(q)$.)

Now assume $n \geq 5$. All character $\rho_1, \ldots, \rho_{14}$ occurring in the table have a-value at most 4. So as before, their ℓ-modular constituents must lie in one of the Harish-Chandra series listed in Table 9. From the known Brauer trees it follows that for $n = 5$ none of our characters lies in the $(B_2A_2, B_2 \otimes 1^3)$-series when $\ell|(q^2+q+1)$, so this will not contribute. Harish-Chandra inducing the corresponding columns of the approximate unipotent decomposition matrix from a Levi subgroup of semisimple type $\mathrm{Spin}_{2n-1}(q)$ to G gives the columns stated in Table 10, with the induction basis given by the case $n = 4$ treated before. We just obtain one column with a character in the $(B_2, .1^2)$-series, which contains exactly one of the ρ_i, so there is at most one PIM in that series in the range of our table. As in the case $n = 4$ we find three projectives containing one of the ρ_i above the cuspidal unipotent character of B_2.

All characters marked "ps" lie in the canonical basic set given by FLOTW bipartitions of the Hecke algebra $\mathcal{H}(B_n; q; q)$, whose ℓ-decomposition matrix is uni-triangular, see [14, Thm. 5.8.5]. Similarly, according to [14, Thm. 5.8.2] the relative Hecke algebra $\mathcal{H}(B_{n-2}; q^3; q)$ of the cuspidal unipotent character of $B_2(q)$ in G has a canonical basic set given by FLOTW bipartitions and all characters marked "B_2" lie in that basic set. As the ordinary cuspidal unipotent character of B_2 is irreducible modulo all primes and reduction stable, the decomposition matrix of that Hecke algebra embeds into the decomposition matrix of G by Dipper's theorem, so the corresponding part of our decomposition matrix is indeed lower triangular. □

6.3. **Unipotent Brauer characters of low degree.** We now describe the decomposition of the first five unipotent characters from Theorem 6.3.

Proposition 6.4. *Let $G = \mathrm{Spin}_{2n+1}(q)$, $n \geq 2$. Assume that ℓ is prime to $2q(q + 1)$. Then the ℓ-modular decomposition matrix of $\rho_1 = 1_G, \ldots, \rho_5$ from Table 10 is given as follows:*

$$\begin{array}{c|ccccc} 1 & 1 \\ \rho_2 & . & 1 \\ \rho_3 & . & . & 1 \\ \rho_4 & a & . & . & 1 \\ \rho_5 & b & . & . & . & 1 \end{array}$$

where $a = \kappa_{\ell,q^n-1}$ and $b = 1 - a$ if $\ell|(q^{2n} - 1)/(q - 1)$, and $a = b = 0$ otherwise.

Proof. The claim holds when $n = 3$ by [18]. For $n \geq 4$, all the zeroes in the table are correct by Table 10, so we only need to discuss the last two entries in the first column.

The entries a, b are determined as follows: The unipotent characters ρ_4, ρ_5 are constituents of the rank 3 permutation module of G on singular 1-spaces, and thus the claim for $\ell \nmid (q^{2n} - 1)/(q - 1)$ follows from [21, Thm. 2.1]. Moreover, else we have that $a + b = 1$. If $\ell|(q^n - 1)$ then ρ_5 does not lie in the principal block (see [9, §12]), so that certainly $b = 0$, and hence $a = 1$. On the other hand, if $\ell \nmid (q^n - 1)$, but $\ell|(q^{2n} - 1)/(q - 1)$, then ρ_4 does not lie in the principal block, whence $a = 0, b = 1$. □

We obtain the following analogue of Corollary 5.8:

Corollary 6.5. *Under the hypotheses of Theorem 6.3 assume that $\varphi \in \mathrm{IBr}(G)$ is an ℓ-modular constituent of one of the unipotent characters ρ_i, $1 \leq i \leq 14$, in Table 10. If $\varphi(1) < \frac{1}{2}q^{4n-6} - q^{3n-3}$ then $i \leq 5$. In particular, φ is as described in Proposition 6.4.*

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