APPENDIX: NON-UNIQUENESS OF SUPERCUSPIDAL SUPPORT FOR FINITE REDUCTIVE GROUPS

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Abstract. We exhibit a simple representation of the finite symplectic group of $\mathrm{Sp}_8(q)$ in characteristic $\ell \mid q^2+1$ having two non-conjugate supercuspidal supports. This answers a question of M.-F. Vignéras.

Notations. We keep the notation of the article: $\mathbb{G}$ will denote a connected reductive group over $\mathcal{O}_F$, the ring of integers of a p-adic field F. The group $\bar{G} := \mathbb{G}(k_F)$ of points over the residue field k_F is a finite reductive group. We will work with representations of $\bar{G}$ over $\overline{\mathbb{F}}_\ell$ where ℓ is a prime different from p. Unless stated otherwise, all the representations will be assumed to be finitely generated. All the parabolic subgroups of $\mathbb{G}$ that we consider in this appendix will be assumed to be defined over $\mathcal{O}_F$.

1. Supercuspidality for finite reductive groups

Given a parabolic subgroup $\mathbb{P}$ of $\mathbb{G}$ with Levi quotient $\mathbb{M}$, the parabolic induction and restriction functors of the finite reductive groups $\bar{G}$ and $\bar{M}$ are denoted by

$$\mathrm{Rep}_R(\bar{M}) \underset{r_{\bar{P}}^{\bar{G}}}{\overset{i_{\bar{P}}^{\bar{G}}}{\rightleftarrows}} \mathrm{Rep}_R(\bar{G}).$$

They form a pair of biadjoint exact functors when p is invertible in R. Note that in the case of finite reductive groups, these functors preserve the property of a representation to be finitely generated. In particular, induction and restriction of simple modules are finitely generated, and hence have finite length.

Let us focus on the case $R = \overline{\mathbb{F}}_\ell$ with $\ell \neq p$. Given a simple $\overline{\mathbb{F}}_\ell\bar{G}$-module π, one can consider the pairs $(\bar{M}, \sigma)$ such that π occurs as a subquotient in the induction $i_{\bar{P}}^{\bar{G}}(\sigma)$. If $\bar{M}$ is minimal for the inclusion and σ is simple, we say that $(\bar{M}, \sigma)$ is a *supercuspidal support* of π. In that case, the transitivity of induction implies that the simple module σ itself cannot occur as a subquotient of a induced representation from a proper Levi subgroup of $\bar{M}$, and we say that σ is *supercuspidal*.

We shall use a reformulation of these properties using projective modules and the second adjunction between $i_{\bar{P}}^{\bar{G}}$ and $r_{\bar{P}}^{\bar{G}}$. Given a simple $\overline{\mathbb{F}}_\ell\bar{G}$-module π, we will denote by P_π its projective cover. Then π occurs as a subquotient of a (non-necessarily simple) $\overline{\mathbb{F}}_\ell\bar{G}$-module π' if and only if $\mathrm{Hom}_{\bar{G}}(P_\pi, \pi') \neq 0$. A version of the following proposition can be found in [4, Prop. 2.3].

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Proposition 1.1 (Hiss)**.** *Let π be a simple $\overline{\mathbb{F}}_\ell \bar{G}$-module.*

(i) *π is supercuspidal if and only if its projective cover P_π is cuspidal,* i.e. *killed under every proper parabolic restriction.*

(ii) *The pair $(\bar{M}, \sigma)$ is a supercuspidal support of π if and only if σ is a supercuspidal simple $\overline{\mathbb{F}}_\ell \bar{M}$-module and P_σ is a direct summand of $r_{\bar{P}}^{\bar{G}}(P_\pi)$.*

Proof. Let $(\bar{M}, \sigma)$ be a pair where σ is a simple $\overline{\mathbb{F}}_\ell \bar{M}$-module. Then π occurs as a subquotient of $i_{\bar{P}}^{\bar{G}}(\sigma)$ if and only if $\mathrm{Hom}_{\bar{G}}(P_\pi, i_{\bar{P}}^{\bar{G}}(\sigma)) \neq 0$. By the second adjunction, this is equivalent to $\mathrm{Hom}_{\bar{M}}(r_{\bar{P}}^{\bar{G}}(P_\pi), \sigma) \neq 0$. Since $r_{\bar{P}}^{\bar{G}}$ preserves the projectivity of representations, and σ is simple, this is in turn equivalent to P_σ being a direct summand of $r_{\bar{P}}^{\bar{G}}(P_\pi)$. The assertions (i) and (ii) follow. □

In particular, the various cuspidal supports of π can be obtained by looking at the cuspidal indecomposable projective summands of the restrictions of P_π. This will be used in §3 to exhibit two non-conjugate cuspidal supports for a given representation of $\bar{G} = \mathrm{Sp}_8(q)$ when $\ell \mid q^2 + 1$.

2. Decomposition of projective modules

2.1. **Lifting projective modules.** Working with projective modules instead of simple modules has the advantage that they lift to characteristic zero. Therefore one can use the decomposition matrices to decompose them on the basis of projective indecomposable modules, and determine the various cuspidal supports.

Let K be the algebraic closure of the maximal unramified extension of $\mathbb{Q}_l$, and $\mathcal{O} := \mathcal{O}_K$ be the ring of integers of K. It is also the ring of Witt vectors of $\overline{\mathbb{F}}_\ell$ and as such it is a complete discrete valuation ring with residue field $\overline{\mathbb{F}}_\ell$. The tuple $(K, \mathcal{O}, \overline{\mathbb{F}}_\ell)$ forms an ℓ-modular system in the sense of [5, §1.2]. Therefore every finitely generated projective $\overline{\mathbb{F}}_\ell \bar{G}$-module P lifts uniquely, up to isomorphism, to an $\mathcal{O}\bar{G}$-module $\widetilde{P}$. Moreover, two projective $\overline{\mathbb{F}}_\ell \bar{G}$-modules P and Q are isomorphic if and only if the $K\bar{G}$-modules $K\widetilde{P}$ and $K\widetilde{Q}$ are. Since K is an algebraically closed field of characteristic zero, the $K\bar{G}$-modules are determined by their character. In addition, the functors $i_{\bar{P}}^{\bar{G}}$ and $r_{\bar{P}}^{\bar{G}}$ behave well with respect to changing the field of coefficients, so that a projective module P is cuspidal if and only if $K\widetilde{P}$ is.

2.2. **Unipotent blocks.** We now consider the special case of unipotent blocks, which form a special direct summand of the group algebra $\overline{\mathbb{F}}_\ell \bar{G}$. A representation of $\bar{G}$ which factors through this direct summand will be said to be unipotent. If ℓ is good for $\mathbb{G}$ and does not divide the order of the finite group $Z(\mathbb{G})/Z(\mathbb{G})^\circ$ then the unipotent characters of $\mathbb{G}$ form a *basic set of characters* for the unipotent blocks (see [3, 2]). This has the following consequences:

(a) Two unipotent projective modules P and Q are isomorphic if and only if every unipotent character of $\bar{G}$ occurs in $K\widetilde{P}$ and $K\widetilde{Q}$ with the same multiplicity.

(b) A unipotent projective $\overline{\mathbb{F}}_\ell \bar{G}$-module is cuspidal if and only if every unipotent character occuring in $K\widetilde{P}$ is cuspidal.

3. Non-uniqueness of cuspidal support

Throughout this section we will assume that both ℓ and q are odd.

3.1. **Unipotent supercuspidal representations in type C_2.** Let $\mathbb{M}$ be a connected reductive group of semisimple type C_2 defined over $\mathcal{O}_F$. The finite reductive group $\bar{M} := M(k_F)$ has a cuspidal unipotent character, which we will denote by ρ_{C_2} (it was first discovered by Srinivasan in [7]). The other unipotent characters all lie in the principal series, therefore they are parametrized by the irreducible representations of the Weyl group of type C_2, which we can label by bipartitions of 2. The unipotent characters of $\bar{M}$ are therefore

$$\mathrm{Uch}(\bar{M}) = \{\rho_{2.\emptyset}, \rho_{1^2.\emptyset}, \rho_{1.1}, \rho_{\emptyset.1^2}, \rho_{\emptyset.2}, \rho_{C_2}\}.$$

By convention, $\rho_{2.\emptyset}$ is the trivial character whereas $\rho_{\emptyset.1^2}$ is the Steinberg character.

By [8, 9], the reduction modulo ℓ of ρ_{C_2} remains simple. We will denote by σ_{C_2} the ℓ-reduction of ρ_{C_2}, and by P_{C_2} its projective cover. It is a unipotent projective indecomposable $\overline{\mathbb{F}}_\ell\bar{M}$-module. Assuming that ℓ is odd, the character of the lift of P_{C_2} to K was explicitly computed in [8] and [6]. We have

$$K\widetilde{P}_{C_2} = \begin{cases} \rho_{C_2} + 2\rho_{\emptyset.1^2} + \text{non-unipotent characters} & \text{if } \ell \mid q+1 \\ \rho_{C_2} + \text{non-unipotent characters} & \text{otherwise.} \end{cases}$$

Consequently if follows from Proposition 1.1 and §2.2(b) that σ_{C_2} is supercuspidal whenever ℓ is odd and does not divide $q+1$.

3.2. **Non-uniqueness of supercuspidal support in type C_4.** Let $\mathbb{G}$ be a connected reductive group of type C_4 defined over $\mathcal{O}_F$, and let $\mathbb{P}$ be a parabolic subgroup of $\mathbb{G}$ with Levi quotient $\mathbb{M}$ of type C_2. The unipotent characters of $\bar{G}$ fall into two Harish-Chandra series: the principal series (with characters labelled by bipartitions of 4) and the series above ρ_{C_2} (with characters labelled by partitions of 2). When $\ell > 5$ and $\ell \mid q^2+1$, then by [1, Thm. 8.2] there exists a unipotent projective indecomposable $\overline{\mathbb{F}}_\ell\bar{G}$-module P and $a \in \{0,1\}$ such that

$$K\widetilde{P} = \rho_{C_2:\emptyset.2} + a\rho_{1^3.1} + a\rho_{1^4.\emptyset} + (a+2)\rho_{\emptyset.1^4} + \text{non-unipotent characters}.$$

We denote by π the unique simple quotient of P, so that $P \simeq P_\pi$.

Proposition 3.1. *Assume that $\ell > 5$ and $\ell \mid q^2+1$. Let $\mathbb{T}$ be a split torus of $\mathbb{G}$, and $\mathbb{M}$ be a Levi subgroup of type C_2. Then the pairs $(\bar{M}, \sigma_{C_2})$ and $(\bar{T}, \overline{\mathbb{F}}_\ell)$ are supercuspidal supports of π.*

Proof. The parabolic restriction of unipotent characters from $\bar{G}$ to $\bar{M}$ can be easily computed. This gives

$$\begin{aligned} r_{\bar{P}}^{\bar{G}}(K\widetilde{P}_\pi) &= \rho_{C_2} + a\rho_{1.1} + 3a\rho_{1^2.\emptyset} + (a+2)\rho_{\emptyset.1^2} + \text{non-unipotent characters} \\ &= \rho_{C_2} + a(\rho_{\emptyset.1^2} + \rho_{1.1}) + 2\rho_{\emptyset.1^2} + 3a\rho_{1^2.\emptyset} + \text{non-unipotent characters} \end{aligned}$$

By [9], the characters ρ_{C_2}, $\rho_{\emptyset.1^2} + \rho_{1.1}$, $\rho_{\emptyset.1^2}$ and $\rho_{1^2.\emptyset}$ are the unipotent part of characters of projective indecomposable modules. This shows that P_{C_2} is a direct summand of $r_{\bar{P}}^{\bar{G}}(P_\pi)$ (see §2.2(a)), and therefore $(\bar{M}, \sigma_{C_2})$ is a supercuspidal support of π by Proposition 1.1.

The group $\bar{T}$ has a unique unipotent irreducible character, the trivial character $1_{\bar{T}}$ (and it is cuspidal). Also, the projective cover of the trivial $\overline{\mathbb{F}}_\ell\bar{T}$-module has character $K\widetilde{P}_{\overline{\mathbb{F}}_\ell} = 1_{\bar{T}} + \text{non-unipotent characters}$. If $\bar{B}$ is a Borel subgroup of $\bar{G}$ containing $\bar{T}$, then

$$r_{\bar{B}}^{\bar{G}}(K\widetilde{P}_\pi) = (6a+2)1_{\bar{T}} + \text{non-unipotent characters}.$$

Since $6a+2\neq 0$, we deduce again from Proposition 1.1 that $(\bar{T},\overline{\mathbb{F}}_\ell)$ is a supercuspidal support of π. □

3.3. **Non-uniqueness of supercuspidal support in type F_4.** A similar example can be found in a group of type F_4, under the assumption that q is odd and not a power of 3. Again, when $\ell > 5$ and $\ell \mid q^2+1$, there exists a projective $\overline{\mathbb{F}}_\ell \bar{G}$-module P such that

$$K\widetilde{P} = \rho_{C_2:\emptyset.1^2} + 2\mathrm{St} + \text{non-unipotent characters}.$$

See [1, Thm. 8.4]. The restrictions of this character to a Levi subgroup of type C_2 or a split torus can be used to show that the pairs $(\bar{M},\sigma_{C_2})$ and $(\bar{T},\overline{\mathbb{F}}_\ell)$ are supercuspidal supports of the unique simple quotient of P.

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