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Towards a Paley–Wiener theorem for semisimple symmetric spaces

by

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Introduction

In the later years several Paley–Wiener type theorems have been established for Fourier transforms on special classes of non-flat symmetric spaces. In particular this is so for the case of Riemannian symmetric spaces of the non-compact type, see e.g. Ehrenpreis–Mautner [10], Helgason [17] and Gangolli [16], and for the case of the semisimple non-compact Lie groups (which in their own right are non-Riemannian symmetric spaces), see e.g. Zelobenko [31], Ehrenpreis–Mautner [10], Campoli [5], Arthur [1], Delorme [8] and Clozel–Delorme [6] and [7].

For a general non-Riemannian semisimple symmetric space G/H, the question of how the Fourier transform should be defined and in particular how it should be normalized is not definitively clarified. However a fairly explicit Plancherel formula has been announced by Oshima and Sekiguchi. A Paley–Wiener type theorem should either refer to a specific normalization or it should be formulated independent of normalizations. In any case a Paley–Wiener theorem should characterize the image under the Fourier transform of natural classes of compactly supported functions or maybe classes of very rapidly decreasing functions.

The main result of this paper is Theorem 1, which exhibits a large class of functions as Fourier transforms of compactly supported K-finite C^∞-functions on G/H. The proof is in fact rather elementary. However it seems to us, in spite of this, that the content of the theorem is not uninteresting. To illustrate this we specialize in Theorem 2 to the case of a non-compact semisimple Lie group. Theorem 2 was first proved by Campoli [5] for the rank one case and in general by Arthur [1]. For them our Theorem 2 is a simple corollary of their Paley–Wiener theorem, which is rather difficult to prove. E.g. Harish-Chandra's Plancherel formula and the theory of differential equations with regular singularities are used. But in fact our Theorem 2 covers most of those functions in Arthur's Paley–Wiener-space, for which it is manageable directly to verify the conditions.

Clozel and the first author in [6] have a simpler proof of our Theorem 2 than Arthur's, but they still have to rely on the explicit form of Harish-Chandra's Plancherel formula. For them our Theorem 2 is a crucial step in the proof of their invariant Paley–Wiener theorem. This means that our proof leads to a considerable simplification in the proof of their result. Another feature of our proof is that it does not assume linearity of the group.

Following a suggestion by N. Wallach we derive in Theorem 3 an analogue of Theorem 1 for C^∞-functions, which decrease "faster than any exponential". Hereby we generalize Theorem 1.8 in Delorme [9]. This space of functions $\mathscr{S}_0$ was first introduced to the authors in lectures by Casselman.

In the last section we use Theorem 1 to derive a Paley–Wiener type theorem for the isotropic spaces, which are special examples of rank one semisimple symmetric spaces. We use results of Faraut [11] and Kosters [24]. One may remark at this point, that Theorem 1 is not really needed in the proof, since explicit calculations and the Paley–Wiener theorem for the Jacobi transform suffices for these simple cases. We have included these examples of results analogous to the invariant Paley–Wiener theorem, because they reveal some non-trivial features, which in contrast to the Riemannian case and the group case must be taken into account in one way or the other for the general case. In particular only a part of the discrete spectrum can be separated from the continuous spectrum in the Paley–Wiener theorem.

For a general introduction to analysis on non-Riemannian symmetric spaces and for further references see Flensted-Jensen [15].

We should like to thank Schlichtkrull for several helpful comments on the manuscript, Wallach for the suggestion about rapidly decreasing functions and Helgason for pointing out the uniqueness in Theorem 1. The first author has learnt from Duflo, how to see the invariant Paley–Wiener theorem as a particular case of the spherical Paley–Wiener theorem for semisimple symmetric spaces.

§ 1. The main result

In order to state our main result we must recall some notation.

Let G/H be a semisimple symmetric space corresponding to the involution σ of G, i.e. G is a connected, semisimple Lie group and H is a closed subgroup of the fixed point group G^σ for σ containing the identity component H^0 of G^σ. There exists a Cartan involution θ commuting with σ. Let K be the fixed point group for θ, then K is connected and it is also a maximal compact subgroup of G modulo the center $Z=Z(G)$. Let

$$\mathfrak{g} = \mathfrak{h}+\mathfrak{q} = \mathfrak{k}+\mathfrak{p}$$

$$\mathfrak{g} = \mathfrak{h}\cap\mathfrak{k}+\mathfrak{h}\cap\mathfrak{p}+\mathfrak{q}\cap\mathfrak{k}+\mathfrak{q}\cap\mathfrak{p}$$

be the corresponding decompositions of the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{q}$, resp. $\mathfrak{p}$, is the -1 eigenspace of σ, resp. θ, in $\mathfrak{g}$.

Let $\mathfrak{a}\subset\mathfrak{q}$ be a θ-invariant Cartan subspace for G/H, i.e. $\mathfrak{a}$ is a maximal θ-invariant and Abelian subspace of $\mathfrak{q}$. By $\mathfrak{g}_\mathbf{C}$, $\mathfrak{a}_\mathbf{C}$ etc. we denote the complexifications. Let $W=W(\mathfrak{a}_\mathbf{C}, \mathfrak{g}_\mathbf{C})$ be the Weyl group corresponding to the restricted root system $\Sigma=\Sigma(\mathfrak{a}_\mathbf{C}, \mathfrak{g}_\mathbf{C})$. The real form of $\mathfrak{a}_\mathbf{C}$ generated by the co-root vectors is

$$\mathfrak{a}^r = i(\mathfrak{a}\cap\mathfrak{k})+\mathfrak{a}\cap\mathfrak{p}.$$

Clearly $\mathfrak{a}^r$ is W-invariant. Let $\mathbf{D}(G/H)$ be the algebra of G-invariant differential operators on G/H. The complex characters of $\mathbf{D}(G/H)$ are parametrized in the usual way as χ_λ, where $\lambda\in\mathfrak{a}^*_\mathbf{C}$ modulo W.

By $(K^\wedge)_{K\cap H}$ we denote the set of unitary $(K\cap H)$-spherical representations of K, i.e. consisting of equivalence classes of unitary irreducible, and thus finite dimensional, representations (μ, E_μ) of K having a non-trivial $(K\cap H)$-fixed vector $e_0\in E_\mu$. Let $\varkappa$ be the character of Z obtained from μ. We shall call $\varkappa$ the G-central character of μ.

Let (π, V) be a quasisimple representation of G of finite length. (We assume for simplicity that V is a Hilbert space and that π is unitary on K.) Let $V_\infty\subset V$ and $V_{-\infty}\supset V$ be the C^∞-vectors and the distribution vectors respectively for π. Similarly V'_∞ and $V'_{-\infty}$ are defined for the dual representation $\pi^\vee$. So e.g. the representation $\pi_{-\infty}$ on $V_{-\infty}$ is the dual representation to $\pi^\vee_\infty$ on V'_∞, where V'_∞ is given the C^∞-topology. A vector $v_0\in V_{-\infty}$, $v_0\neq 0$, is called H-spherical, if it is $\pi_{-\infty}(H)$-invariant and a joint eigenvector of $\mathbf{D}(G/H)$, i.e. if there exists $\lambda\in\mathfrak{a}^*_\mathbf{C}$ such that

$$\pi_{-\infty}(D)\,v_0 = \chi_\lambda(D)\,v_0 \quad \text{for each} \quad D\in\mathbf{D}(G/H).$$

For $\mu\in(K^\wedge)_{K\cap H}$ we let d_μ be the dimension and $\mathrm{Tr}(\mu)$ the character, then $P_\mu=\pi_{-\infty}(d_\mu\,\mathrm{Tr}(\mu^\vee))$ is well defined as the projection in $V_{-\infty}$ onto the μ-isotypic component $V^\mu\subset V_\infty$, when $V^\mu\neq\{0\}$.

Considering $\mathfrak{a}^r$ as an Euclidean space we denote by $PW(\mathfrak{a}^r)$ the Paley–Wiener space for $\mathfrak{a}^r$, i.e. the image under the classical Fourier transform $\mathcal{F}$ of $C_c^\infty(\mathfrak{a}^r)$, where $\mathcal{F}$ is defined by

$$\mathcal{F}f(\lambda)=\int_{\mathfrak{a}^r} f(x)\exp\langle -i\lambda, x\rangle\, dx, \quad \lambda\in\mathfrak{a}^r_{\mathbf{C}},\ f\in C_c^\infty(\mathfrak{a}^r).$$

Then, as is well known, $PW(\mathfrak{a}^r)$ consists of the entire, rapidly decreasing functions on $(\mathfrak{a}^r)^*_{\mathbf{C}}$ of exponential type. More precisely $\psi\in PW(\mathfrak{a}^r)$ satisfies

$$\exists R\geqslant 0,\ \forall N\in\mathbf{N}:\quad \sup_{\lambda\in(\mathfrak{a}^r)^*_{\mathbf{C}}}(1+\|\lambda\|)^N e^{-R\|\Im\lambda\|}|\psi(\lambda)|<+\infty,$$

where $\Im$ denotes the imaginary part.

Restricting to the W-invariant functions we have in particular that the Fourier transform is a bijection of $C_c^\infty(\mathfrak{a}^r)^W$ onto $PW(\mathfrak{a}^r)^W$. Let finally $C_c^\infty(G/H; K)$ be the space of K-finite C^∞-functions on G/H compactly supported modulo K.

THEOREM 1. *Let* $\psi\in PW(\mathfrak{a}^r)^W$ *and* $\mu\in(K^\wedge)_{K\cap H}$. *There exists a unique function* $f\in C_c^\infty(G/H;K)$ *of type* μ *such that the following holds:*

Let $\varkappa$ *be the G-central character of* μ, let (π, V) *be any quasisimple representation of finite length and with central character* $\varkappa$ *and let* $v_0\in V_{-\infty}$ *be an H-spherical vector corresponding to* $-i\lambda\in\mathfrak{a}^*_{\mathbf{C}}$, *then* $\pi_{-\infty}(f)\,v_0$ *is well defined and we have*

$$\pi_{-\infty}(f)\,v_0=\psi(\lambda)\,P_\mu\,v_0.$$

Thus in particular if v_0' *is any H-invariant vector in* $V'_{-\infty}$ *then*

$$\langle\pi_{-\infty}(f)\,v_0, v_0'\rangle=\psi(\lambda)\,\langle P_\mu\,v_0, v_0'\rangle.$$

Remark. It follows from the proof, that if ψ is of exponential type R, then f has support in a "ball" of radius R, as usual, i.e. $\mathrm{supp}(f)\subset KB_RH$. Notice also, when Z is finite, that the formulas in the theorem are valid for any π without the condition on the central character.

The distribution $f\to\langle\pi_{-\infty}(f)\,v_0, v_0'\rangle$ occuring in the theorem is the prototype of a *spherical distribution* on G/H, i.e. an H-invariant joint eigendistribution on G/H of $\mathbf{D}(G/H)$. The correspondence which associates to $f\in C_c^\infty(G/H)$ the function $F(\pi, v_0, v_0')=\langle\pi_{-\infty}(f)\,v_0, v_0'\rangle$ may be called the *scalar valued spherical Fourier transform* on G/H. In order to describe this transform more explicitly one would need a parametrization of a sufficiently large class of representations (π, V) and for each such a description(¹) of $V^H_{-\infty}$ and $V'^H_{-\infty}$. These two spaces are finite dimensional by van den Ban [3]. Of course one might also define the scalar valued Fourier transform of f as $\pi \to F(\pi)$, $F(\pi) \in \hom_{\mathbf{C}}(V^H_{-\infty} \times V'^H_{-\infty}; \mathbf{C})$, where $F(\pi)(v_0, v_0') = F(\pi, v_0, v_0')$. Similarly we define the *vector valued spherical Fourier transform* of f as $(\pi, v_0) \to \pi_{-\infty}(f)\, v_0$.

Example 1. *The Riemannian case*. Assume that G/H is a Riemannian symmetric space of the non-compact type, then $H=K$. If V is irreducible then $V^H_{-\infty} \subset V_\infty$ and $\dim V^H_{-\infty} = \dim V'^H_{-\infty} \leq 1$. If we normalize such that $\langle v_0, v_0' \rangle = 1$, then the scalar valued Fourier transform is just Harish-Chandra's spherical Fourier transform. The Paley–Wiener theorem corresponding to this transform is proved by Helgason [17] and Gangolli [16]. See also [21].

The vector valued spherical Fourier transform can be identified with Helgason's transform

$$f \to F(\lambda, b) = \int_{G/K} f(x)\, e^{\langle -i\lambda - \varrho, \mathbf{H}(x^{-1}b) \rangle}\, dx,$$

where $F(\lambda, \cdot)$ is an element of $V_\lambda = L^2(K/M)$ for each λ. For this transform Helgason has proved a Paley–Wiener theorem for $C_c^\infty(G/K)$ [19, Theorem 8.3] and one for $C_c^\infty(G/K; K)$ [20, Theorem 7.1], in which the description of the Paley–Wiener space is more explicit.

Theorem 1 in the non-compact Riemannian case is equivalent to the surjectivity statement in the Helgason–Gangolli Paley–Wiener theorem. This statement is the more difficult part to prove. Our proof in the next section consists of a reduction to this case. We should add here, that it is possible along the same lines to reduce the non-compact Riemannian case to the non-compact Riemannian case, for which the group has a complex structure. This case is then reduced to the Euclidean case, cf. [15, Chapter III, Theorem 5].

However before turning to the proof we want to specialize Theorem 1 to the group case. (See Examples 2 and 3 below for more details). So let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$ etc. Let $\mathfrak{a}$ be a θ-invariant Cartan subalgebra and let $\mathfrak{a}^r = i(\mathfrak{a} \cap \mathfrak{k}) + \mathfrak{a} \cap \mathfrak{p}$. Let W be the complex Weyl group of $\mathfrak{a}$ and let $C_c^\infty(G; K)$

(¹) In order for $f \to F(\pi, v_0, v_0')$ to be an eigendistribution of $\mathbf{D}(G/H)$ it is sufficient, that v_0 is an eigenvector. However $V^H_{-\infty}$ being finite dimensional and $\mathbf{D}(G/H)$-invariant implies, since $\mathbf{D}(G/H)$ is Abelian, that any $v_0 \in V^H_{-\infty}$ is a linear combination of generalized eigenvectors. Therefore it may be more convenient to use the H-fixed distribution vectors in the definition of the Fourier transform instead of the $\mathbf{D}(G/H)$-eigenvectors.

denote the space of C^∞-functions on G, which are both left and right finite under K and have compact support modulo Z.

THEOREM 2. *Let $\psi \in PW(\mathfrak{a}^r)^W$ and $\mu \in K^\wedge$. There exists a unique function $f \in C_c^\infty(G;K)$ of type $(\mu,\mu^\vee)$, such that the following holds:*

Let $\varkappa$ be the G-central character of μ, let (π, V) be any quasisimple representation of G of finite length, with central character $\varkappa$ and infinitesimal character $-i\Lambda$, then $\pi(f)$ is well defined and we have

$$\pi(f) = \psi(\Lambda) P_\mu,$$

and hence

$$\mathrm{Trace}(\pi(f)) = \psi(\Lambda)\dim(V^\mu).$$

Example 2. *The matrix valued Fourier transform on G.* Recall that we consider G as the symmetric space $G\times G/\mathbf{d}(G)$, where $\mathbf{d}(G)$ is the diagonal in $G\times G$. The vector valued spherical Fourier transform for $G\times G/\mathbf{d}(G)$ reduces to the usual operator valued transform $f \to \pi(f) \in \hom(V)$, where (V, π) runs through a set of suitable representations of G. If f is K-finite of type $(\mu,\mu^\vee)$ then we may consider $\pi(f)$ as being contained in $\hom(V^\mu)$.

Let $P=MA_pN$ be a minimal parabolic subgroup in G and let $(\pi_{(\delta,\lambda)}, V_\delta)$, where $V_\delta = L^2(K/M;\delta)$, be the principal series representation for $\delta \in M^\wedge, \lambda \in (\mathfrak{a}_p)^*_{\mathfrak{c}}$. Arthur [1] and Campoli [5] defines the Fourier transform of a function f of type $(\mu,\mu^\vee)$ as

$$F(\delta,\lambda) = \pi_{\delta,\lambda}(f) \in \hom(V_\delta^\mu),$$

and they give an intrinsic but rather complicated description of the image space

$$PW(G,K) = \{(\delta,\lambda) \to F(\delta,\lambda) \mid f \in C_c^\infty(G;K)\}.$$

As mentioned earlier, essentially the only functions of (δ,λ), for which Arthur's conditions are simple to verify directly, are the functions of the form $\psi(\Lambda)P_\mu$, where Λ, depending on δ and λ, is the infinitesimal character of $\pi_{\delta,\lambda}$. It follows that our Theorem 2, at least for the linear groups, which essentially are the ones treated by Arthur, is a simple corollary of his Paley–Wiener theorem. One should also mention, that in the case, where G has only one conjugacy class of Cartan subgroups, the first author [8] has proved a Paley–Wiener theorem in which the symmetry conditions in the Paley–Wiener space are explicit and simple. The result in [8] is a non-trivial corollary to Theorem 2 above. The proof given in [8] of Theorem 2 in this particular case uses an idea following Zelobenko, which is mainly a reduction to the K-biinvariant case, by means of multiplication by matrix-coefficients of finite dimensional representations. Apparently this proof does not extend to the general case.

Example 3. *The invariant Paley–Wiener theorem for G.* In the case of $G\times G/\mathbf{d}(G)$ the scalar valued Fourier transform reduces to the invariant Fourier transform

$$f\to \mathrm{Trace}(\pi(f)).$$

Let $P^j=M^jA^jN^j$, $j=1,\dots,r$, be representatives of the different equivalence classes of cuspidal parabolic subgroups in G. For a discrete series representation δ of M^j and $\lambda\in(\mathfrak{a}^j)^*_{\mathbf{C}}$ let $(\pi^j_{\delta,\lambda},V_\delta)$ be the corresponding generalized principal series on $V_\delta=L^2(K/M^j\cap K);\delta|_{(M^j\cap K)})$. Clozel and the first author [6] define the invariant Fourier transform of $f\in C_c^\infty(G;K)$ as the function

$$(j,\delta,\lambda)\to F(j,\delta,\lambda)=\mathrm{Trace}(\pi^j_{\delta,\lambda}(f)).$$

Their invariant Paley–Wiener theorem states that this Fourier transform is onto the space consisting of functions ψ of j,δ and λ of finite support as a function of δ, such that $\lambda\to\psi(j,\delta,i\lambda)$ belongs to $PW(\mathfrak{a}^j)$ and such that $\psi(j,w\delta,w\lambda)=\psi(j,\delta,\lambda)$ for each $w\in W^j$, where W^j is the Weyl group of $\mathfrak{a}^j$ in G. One of the main tools used in their proof is their Theorem 3, which is a corollary of our Theorem 2. See [6], and also [7].

There are many possible variants of a Paley–Wiener theorem corresponding to different choices of function spaces instead of C_c^∞. The following space of very rapidly decreasing functions has been used in some special cases, see f. ex. Oshima–Sekiguchi [25], where it is denoted by $\mathscr{C}_*$, Wallach [30] here denoted by $\mathscr{S}$ and van den Ban–Schlichtkrull [4] here also denoted by $\mathscr{S}$.

For the definition we shall need a little further notation. If we choose $\mathfrak{a}_\mathfrak{p}\subset\mathfrak{p}\cap\mathfrak{q}$ as a maximal Abelian subspace, then we have, cf. Flensted-Jensen [13], with $A_\mathfrak{p}=\exp(\mathfrak{a}_\mathfrak{p})$ that

$$G=KA_\mathfrak{p}H.$$

Let for $x\in G/H$ $|x|$ be defined by

$$|x|=|X|,$$

where $x=k\exp XH$ with $X\in\mathfrak{a}_\mathfrak{p}$ and $k\in K$, and where the norm $|X|$ of X is defined using the Killing form. Finally we let $\mathscr{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}_\mathbf{C}$ and $\mathscr{Z}(\mathfrak{g})$

its center. We can now define the *zero Schwartz space* in the following way, where L_u for $u \in \mathscr{U}(\mathfrak{g})$ denotes u considered as a differential operator acting from the left,

$$\text{(1)} \qquad \mathscr{S}_0(G/H) = \{f \in C^\infty(G/H) \mid \forall u \in \mathscr{U}(\mathfrak{g}),\ \forall m \in \mathbf{R},\ \sup_{x \in G/H} (e^{|x|m} |L_u f(x)|) < +\infty\}.$$

We use the subscript 0 to indicate that $\mathscr{S}_0$ is the limit as $p \to 0$ of the spaces $\mathscr{S}_p$ of rapidly decreasing functions in $L^p(G/H)$. This is a rather simple fact. (For a proof in a slightly less general situation see Delorme [9, Lemma 1.1]). The spaces $\mathscr{S}_0(G/H;K)$ and $\mathscr{S}_0(G;K)$ of K-finite functions are defined in the same way as for the C_c^∞-spaces. When we restrict to K-finite functions as in $\mathscr{S}_0(G/H;K)$ it follows, according to N. Wallach, from general theory, that it is enough in (1) to use $\mathscr{Z}(\mathfrak{g})$ instead of $\mathscr{U}(\mathfrak{g})$. We shall not use this fact.

We shall now define the corresponding Paley–Wiener space $PW_0(\mathfrak{a}^r)$. Identifying $\mathfrak{a}^r$ with $\mathbf{R}^n$ for some n we define $PW_0(\mathbf{R}^n)$ to be the space of Fourier transforms of functions in $\mathscr{S}_0(\mathbf{R}^n)$. It is well known that we then have

$$PW_0(\mathbf{R}^n) = \{\psi : \mathbf{C}^n \to \mathbf{C} \mid \psi \text{ is entire and } \forall N \in \mathbf{N},\ \forall C > 0,$$
$$\sup\{(1 + \|\lambda\|)^N |\psi(\lambda)| \mid \lambda \in \mathbf{C}, \|\Im(\lambda)\| \leq C\} < +\infty\}.$$

THEOREM 3. (a) *Let the notation be as in Theorem* 1. *Let* $\psi \in PW_0(\mathfrak{a}^r)^W$ *and* $\mu \in (K^\wedge)_{K \cap H}$. *There exists a unique function* $f \in \mathscr{S}_0(G/H;K)$ *of type* μ *such that the following holds:*

Let $\varkappa$ *be the G-central character of* μ, *let* (π, V) *be any quasisimple representation of finite length and with central character* $\varkappa$ *and let* $v_0 \in V_{-\infty}$ *be an H-spherical vector corresponding to* $-i\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, *then* $\pi_{-\infty}(f)\,v_0$ *is well defined and we have*

$$\pi_{-\infty}(f)\, v_0 = \psi(\lambda)\, P_\mu v_0.$$

Thus in particular if v_0' *is any H-invariant vector in* $V'_{-\infty}$ *then*

$$\langle \pi_{-\infty}(f)\, v_0, v_0' \rangle = \psi(\lambda)\, \langle P_\mu v_0, v_0' \rangle.$$

(b) *Let the notation be as in Theorem* 2. *Let* $\psi \in PW_0(\mathfrak{a}^r)^W$ *and* $\mu \in K^\wedge$. *There exists a unique function* $f \in \mathscr{S}_0(G;K)$ *of type* $(\mu, \mu^\vee)$, *such that the following holds.*

Let $\varkappa$ *be the G-central character of* μ, *let* (π, V) *be any quasisimple representation of G of finite length, with central character* $\varkappa$ *and infinitesimal character* $-i\Lambda$, *then* $\pi(f)$ *is well defined and we have*

$$\pi(f) = \psi(\Lambda) P_\mu,$$

and hence

$$\mathrm{Trace}(\pi(f)) = \psi(\Lambda) \dim(V^\mu).$$

Remark. Part (b) of the theorem is a slight generalization of Delorme [9, Theorem 1.8], where it is only proved dealing with generalized principal series representations.

§ 2. Proofs of the main theorems

In this chapter we prove the Theorems 1, 2 and 3. The main thing is to prove Theorem 1. Theorem 2 is just a specialization of Theorem 1 and we only give a few remarks about that. The proof of Theorem 3 follows very closely the proofs of Theorems 1 and 2 and we only indicate the necessary adjustments. The proof of Theorem 1 is based on three propositions, which we now describe.

It is clearly sufficient to prove the theorem under the assumption that the symmetric space $X=G/H$ is simply connected, and therefore also that H is connected. Let Z be the center of G, then $Z \subset K$. By factorizing over $Z_H = Z \cap H$ we may assume that $Z \cap H = \{e\}$ and thus also that $K \cap H$ is compact.

Let $G_{\mathbf{C}}$ be the complex adjoint group corresponding to $\mathfrak{g}_{\mathbf{C}}$, then we may consider $G/Z, H, K/Z$ and also the group $H^a = (K \cap H)\exp(\mathfrak{p} \cap \mathfrak{q})$ as subgroups of $G_{\mathbf{C}}$. Let $K_{\mathbf{C}}$ and $H_{\mathbf{C}}$ be the complex analytic subgroups of $G_{\mathbf{C}}$ corresponding to $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{h}_{\mathbf{C}}$. It is then easily seen that $H \subset H_{\mathbf{C}}$ and that K can be embedded into a complex Lie group $K_{\mathbf{C}}^{\sim}$, such that $K_{\mathbf{C}}^{\sim}/Z = K_{\mathbf{C}}$. From the structure theory of symmetric spaces, cf. Flensted-Jensen [13, Theorem 4.1], it follows that

$$G = KH^aH. \tag{1}$$

Let now G^d be the real form of $G_{\mathbf{C}}$ for which $H_{\mathbf{C}} \cap G^d = K^d$ is compact. This means that K^d is the analytic subgroup corresponding to $\mathfrak{k}^d = \mathfrak{h} \cap \mathfrak{k} + i(\mathfrak{h} \cap \mathfrak{p})$ and therefore that K^d is the compact real form of $H_{\mathbf{C}}$ and $G^d = K^d \exp(i(\mathfrak{q} \cap \mathfrak{k}) + (\mathfrak{p} \cap \mathfrak{q}))$. Define $H^d = (G^d \cap K_{\mathbf{C}})^0$, then H^d is a non-compact real form of $K_{\mathbf{C}}$. Similarly to (1) we have

$$G^d = H^d H^a K^d. \tag{2}$$

Notice in particular that H^a, modulo a slight abuse of notation, is equal to the connected component of the identity in $G \cap G^d$. Therefore we have that $H^d \cap K^d = H \cap K$

and that H^d may be considered as a subgroup of $K_{\mathbf{C}}^{\sim}$. This means formally speaking that both G and G^d are "contained" in $K_{\mathbf{C}}^{\sim} H^a H_{\mathbf{C}}$.

If we now consider the K-finite functions $C^\infty(G/H;K)$ on G/H and the H^d-finite functions $C^\infty(G^d/K^d;H^d)$ on G^d/K^d, it follows by holomorphic extension in $K_{\mathbf{C}}^{\sim}$, respectively in $H_{\mathbf{C}}$, and restriction that the following proposition holds.

PROPOSITION 1. *Let the notation be as above. Restriction to $H^a=(G\cap G^d)^0$ and holomorphic extension in $K_{\mathbf{C}}^{\sim}$ and $H_{\mathbf{C}}$ defines an isomorphism $\eta: f\to f^r$ of $C^\infty(G/H;K)$ onto $C^\infty(G^d/K^d;H^d)$ considered as $\mathcal{U}(\mathfrak{g})$-modules from the left and as $\mathcal{U}(\mathfrak{g})^{\mathfrak{h}_{\mathbf{C}}}$-modules from the right.*

For the proof we refer to Schlichtkrull [27] and Flensted-Jensen [14, Theorem 2.3]. The proposition easily extends to finite dimensional vector valued functions.

The symmetric spaces $X=G/H$ and $X^r=G^d/K^d$ may be considered as two different simply connected real forms of the "holomorphic" symmetric space $X_{\mathbf{C}}=G_{\mathbf{C}}/H_{\mathbf{C}}$. Being Riemannian of the non-compact type we shall call X^r the *non-compact Riemannian form* of X.

Let $L^p(K\backslash G/H)$, respectively $L^p(H^d\backslash G^d/K^d)$ denote the space of K-invariant L^p-functions on G/H w.r.t. a G-invariant measure, respectively the K^d-right-invariant L^p-functions on $H^d\backslash G^d$ w.r.t. a G^d-invariant measure. A simple computation of the appropriate Jacobian, cf. Flensted-Jensen [14, Theorem 2.6] leads to the following proposition, in which we assume the measures to be suitably normalized.

PROPOSITION 2. *Let $p\geq 1$. The map $\eta: f\to f^r$ extends from C_c^∞ to an isomorphism of $L^p(K\backslash G/H)$ onto $L^p(H^d\backslash G^d/K^d)$.*

The third result we shall need is the spherical Paley–Wiener theorem for the symmetric space $X^r=G^d/K^d$. Let as in § 1 $\mathfrak{a}\subset\mathfrak{q}$ be a θ-invariant Cartan subspace for G/H, then $\mathfrak{a}^r=i(\mathfrak{a}\cap\mathfrak{k})+\mathfrak{a}\cap\mathfrak{p}$ is a Cartan subspace for $X^r=G^d/K^d$. Let for $\lambda\in\mathfrak{a}_{\mathbf{C}}^{r*}$ ϕ_λ be Harish–Chandra's spherical function, i.e.

$$\phi_\lambda(x)=\int_{K^d} e^{\langle -\lambda-\varrho,\, \mathbf{H}(x^{-1}k)\rangle}\,dk,\quad x\in G^d,$$

where $\mathbf{H}: G^d\to\mathfrak{a}^r$ is the Iwasawa projection, (i.e. $\mathbf{H}$ is defined by $x\in K^d\exp(\mathbf{H}(x))N^d$, where $\mathbf{H}(x)\in\mathfrak{a}^r$ and $G^d=K^dA^rN^d$ is an Iwasawa decomposition of G^d), and ϱ is defined as usual.

PROPOSITION 3. *The spherical Fourier transform on $X^r=G^d/K^d$*

$$f \to f^{\sim}(\lambda) = \int_{G^d} f(x)\, \phi_{-i\lambda}(x)\, dx, \quad \lambda \in \mathfrak{a}^*_{\mathbf{C}},$$

is a bijection of $C_c^\infty(K^d\backslash G^d/K^d)$ *onto* $PW(\mathfrak{a}^r)^W$.

For a good exposition of the original proof by Helgason and Gangolli see Helgason [21]. In [15, Chapter III, Theorem 5] using [13] and a result of Rais [26, Corollary 4.5] the second author gave a rather elementary proof by first reducing it (by the above Proposition 1 for the group case) to the case where G is a complex group and then reducing it (by means of a rather simple known expression for the spherical functions on a complex group) to the classical Paley–Wiener theorem. (See also Clozel–Delorme [7, Appendix B] for a direct proof of Rais' result.)

We can now turn to the proof of Theorem 1.

Proof of Theorem 1. To prove the uniqueness of f assume that for all π and all v_0 we have $\pi(f)\,v_0=0$. Let, for any x in G, f_x be the left translate of f by x^{-1}. Then $\pi(f_x)\,v_0=0$. Therefore we have that

$$\langle \pi(f_x)\, v_0, v_0' \rangle = 0, \quad \text{for all} \quad v_0' \in V'^{\,H}_{-\infty}.$$

Since the Dirac measure at eH on G/H can be expanded in terms of H-invariant distributions of the type $f \to \langle \pi(f)\, v_0, v_0' \rangle$, cf. van Dijk and Poel [32, Proposition 1.4], we conclude, that $f(x)=f_x(e)=0$. Strictly speaking [32, Proposition 1.4] assumes that Z is finite. This assumption stems from Ban [3, Proposition 1.4]. However, Proposition 1.4 in [3] is valid if we instead of the finiteness assumption on Z fix a unitary central character $\varkappa$ of Z. To see this one has to replace $C_c^\infty(X)$ and $L^2(X)$ in [3, Lemma 1.2] by the following spaces of functions:

$$C^\infty_{c,\varkappa}(X) = \{ f \in C^\infty(X) \mid f(zx) = \varkappa(z)^{-1} f(x)\, \forall z \in Z, \forall x \in X, |f| \in C_c(G/ZH) \}$$

and

$$L^2_\varkappa(X) = \{ f\colon X \to \mathbf{C} \mid f(zx) = \varkappa(z)^{-1} f(x)\, \forall z \in Z, \forall x \in X, |f| \in L^2(G/ZH) \}$$

The last space is a Hilbert space with the scalar product

$$(f \mid g)_\varkappa = \int_{G/ZH} f(\dot{x})\, \overline{g(\dot{x})}\, d\dot{x}.$$

The ψ_t of [3, Lemma 2.2] should be defined on $X/Z=G/ZH$ and lifted to $X=G/H$. Then

ψ_t has compact support modulo Z and $\psi_t f \in C^\infty_{c,\varkappa}(X)$ for each $f \in L^2_\varkappa(X)^\infty$. Furthermore we have

$$(Df\,|\,g)_\varkappa = (f\,|\,D^* g)_\varkappa$$

for all $f, g \in C^\infty_{c,\varkappa}(X)$ and $D \in D(G/H)$. With these remarks the proof in [3] is easily adjusted.

Let now $\psi \in PW(\mathfrak{a}^r)^W$ and $\mu \in (K^\wedge)_{K\cap H}$. Let $e_0 \in E_\mu$ and $e_0' \in E_\mu'$ be $(K\cap H)$-fixed vectors such that $\langle e_0, e_0' \rangle = 1$. Let $l = d_\mu - 1$ and extend to dual basis $e_0, e_1, \ldots, e_l$ and $e_0', e_1', \ldots, e_l'$, i.e. assume that $\langle e_i, e_j' \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. To prove existence we proceed in two steps.

Step 1. *Definition of* f. By Proposition 3 we can choose $F \in C_c^\infty(K^d \backslash G^d / K^d)$ such that $F^\sim = \psi$. We let μ and $\mu^\vee$ denote also the holomorphic extension of μ and $\mu^\vee$ to $K_{\mathbf{C}}^\sim$ as well as the restrictions to H^d.

Let $F_\mu\colon G^d/K^d \to E_\mu'$ be defined by

$$F_\mu(x) = d_\mu \int_{H^d} F(hx)\, \mu^\vee(h^{-1})\, e_0'\, dh, \quad x \in G^d/K^d, \tag{3}$$

then clearly

$$F_\mu(yx) = \mu^\vee(y) F_\mu(x), \quad y \in H^d,\ x \in G^d/K^d. \tag{4}$$

Notice that F_μ has compact support modulo H^d. (More precisely assume that ψ is of exponential type R, then F restricted to A^r has support in a ball B_R of radius R. Without loss of generality we may assume that $\mathfrak{a}^r \cap \mathfrak{p} = \mathfrak{a}_\mathfrak{p}$, then F_μ has support in the set $H^d(B_R \cap A_\mathfrak{p})K^d$.) We define $f_\mu\colon G/H \to E_\mu'$ by Proposition 1 and the requirement that

$$\langle e_i, f_\mu(\cdot) \rangle^r = \langle e_i, F_\mu(\cdot) \rangle, \quad i = 0, \ldots, l.$$

Finally we shall define the function f by

$$f(x) = \langle e_0, f_\mu(x) \rangle, \quad x \in G/H.$$

We are now going to show that f fullfils the requirements in Theorem 1. It follows from the definition of f_μ, that it has compact support. (More precisely in a "ball" of radius R, i.e. $\operatorname{supp}(f_\mu) \subset K(B_R \cap A_\mathfrak{p})H$.) From (4) it follows that

$$f_\mu(kx) = \mu^\vee(k) f_\mu(x), \quad k \in K,\ x \in G/H. \tag{5}$$

This shows that f is of type μ. We shall need the following lemma. Notice the analogy with the definition of F_μ in (3).

LEMMA 4. *Let f and f_μ be defined as above, then*

$$f_\mu(x) = d_\mu \int_{K/Z} f(kx)\,\mu^\vee(k^{-1})\,e_0'\,dk.$$

Proof. It suffices to prove for each $i=0,\ldots,l$, that

$$\langle e_i, f_\mu(x)\rangle = d_\mu \int_{K/Z} f(kx)\,\langle e_i, \mu^\vee(k^{-1})\,e_0'\rangle\,dk. \tag{6}$$

By the definition of f and by (5) we get

$$f(kx) = \langle e_0, f_\mu(kx)\rangle = \langle e_0, \mu^\vee(k) f_\mu(x)\rangle.$$

In this formula we substitute

$$f_\mu(x) = \sum_{i=0}^{l} \langle e_j, f_\mu(x)\rangle\, e_j'$$

and insert in the right hand side of (6). Then using that, by Peter–Weyl theory,

$$d_\mu \int_{K/Z} \langle e_0, \mu^\vee(k)\,e_j'\rangle\,\langle e_i, \mu^\vee(k^{-1})\,e_0'\rangle\,dk = \delta_{ij}$$

the result follows. □

Step 2. Computation of $\pi_{-\infty}(f)\,v_0$. Notice first that $\pi_{-\infty}(f)\,v_0$ is well defined, since the function $x \to f(x)\,\pi_{-\infty}(x)\,v_0$ is defined on $Z\backslash G/H$ and has compact support. In the following we write π also for $\pi_{-\infty}$. We shall prove with f as defined above, that for any $v' \in V'$ we have:

$$\langle \pi(f)\,v_0, v'\rangle = \psi(\lambda)\,\langle P_\mu v_0, v'\rangle.$$

Neither side changes if we substitute $v'' = \int_{K\cap H} \pi^\vee(k)\,P_{\mu^\vee}\,v'\,dk$ for v'. This follows since f is $(K\cap H)$-fixed and K-finite of type μ. Therefore we may assume that $v' \in V'^{,\mu^\vee}$, which is the $\mu^\vee$-isotypic component in V', and that v' is $(K\cap H)$-fixed. We then have

$$\langle \pi(f)\,v_0, v'\rangle = \int_{Z\backslash G/H} f(x)\,\langle P_\mu \pi(x)\,v_0, v'\rangle\,dx$$

$$= \int_{K\backslash G/H} \int_{K/Z} f(kx) \langle \pi(k) P_\mu \pi(x) v_0, v' \rangle \, dk \, dx$$
$$= \int_{K\backslash G/H} \left\langle P_\mu \pi(x) v_0, \int_{K/Z} f(kx) \pi^\vee(k^{-1}) v' \, dk \right\rangle dx.$$

Assuming that $v' \neq 0$ we can choose a K-homomorphism α of E'_μ into V' such that $\alpha(e'_0) = v'$. Therefore we have that $\alpha(\mu^\vee(k^{-1}) e'_0) = \pi^\vee(k^{-1}) v'$. From Lemma 4 we then get

$$\langle \pi(f) v_0, v' \rangle = d_\mu^{-1} \int_{K\backslash G/H} \langle P_\mu \pi(x) v_0, \alpha(f_\mu(x)) \rangle \, dx.$$

Let $\phi(x) = P_\mu \pi(x) v_0$, then $\phi: G/H \to V_\mu \subset V_\infty$ is K-finite, such that ϕ^r is defined. In order to apply Proposition 2 we must find the image under the map η of the function $x \to \langle \phi(x), \alpha(f_\mu(x)) \rangle$. But this is clearly

$$\langle \phi(\cdot), \alpha(f_\mu(\cdot)) \rangle^r = \langle \phi^r(\cdot), \alpha(F_\mu(\cdot)) \rangle.$$

We now get using Proposition 2

$$\langle \pi(f) v_0, v' \rangle = d_\mu^{-1} \int_{H^d\backslash G^d/K^d} \langle \phi^r(x), \alpha(F_\mu(x)) \rangle \, dx$$
$$= d_\mu^{-1} \int_{H^d\backslash G^d} \langle \phi^r(x), \alpha(F_\mu(x)) \rangle \, dx$$
$$= d_\mu^{-1} \int_{H^d\backslash G^d} \left\langle \phi^r(x), d_\mu \int_{H^d} F(hx) \pi^\vee(h^{-1}) \alpha(e'_0) \, dh \right\rangle dx$$
$$= \int_{H^d\backslash G^d} \int_{H^d} F(hx) \langle \phi^r(hx), v' \rangle \, dh \, dx$$
$$= \int_{G^d} F(x) \langle \phi^r(x), v' \rangle \, dx$$
$$= \int_{G^d/K^d} F(x) \int_{K^d} \langle \phi^r(kx), v' \rangle \, dk \, dx$$
$$= \int_{G^d/K^d} F(x) \Phi(x) \, dx,$$

where

$$\Phi(x) = \int_{K^d} \langle \phi^r(kx), v' \rangle \, dk.$$

Now recall that v_0 is a spherical vector in $V_{-\infty}$, therefore $\phi(x)=P_\mu\pi(x)v_0$ is an eigenfunction of $\mathbf{D}(G/H)$. It follows that ϕ^r and Φ are eigenfunctions with the same eigenvalue, i.e. corresponding to $-i\lambda$. Since Φ is K^d-bi-invariant, Φ is a constant multiple of Harish-Chandra's spherical function $\phi_{-i\lambda}$ or precisely

$$\Phi(x)=\Phi(e)\,\phi_{-i\lambda}(x).$$

We find that

$$\Phi(e)=\langle\phi^r(e),v'\rangle=\langle\phi(e),v'\rangle=\langle P_\mu v_0,v'\rangle.$$

Finally summing up we find that

$$\begin{aligned}\langle\pi(f)\,v_0,v'\rangle&=\langle P_\mu v_0,v'\rangle\int_{G^d}F(x)\,\phi_{-i\lambda}(x)\,dx\\&=F^{\sim}(\lambda)\,\langle P_\mu v_0,v'\rangle\\&=\psi(\lambda)\,\langle P_\mu v_0,v'\rangle.\end{aligned}$$

This finishes the proof of Theorem 1. □

Proof of Theorem 2. In order to distinguish the notation in Theorem 1 from that of Theorem 2 we let *subscript* "1" *denote the objects in Theorem* 2! So let G_1 be a connected semisimple Lie group, θ_1 a Cartan involution with fixed-points K_1 etc. We define $G=G_1\times G_1$ and $\sigma(x,y)=(y,x)$, then $H=\mathbf{d}(G_1)$ is the diagonal in $G_1\times G_1$ and $K=K_1\times K_1$. We observe that G/H is diffeomorphic to G_1 under the map $(x,y)\,\mathbf{d}(G_1)\to xy^{-1}$ and in this way we can consider G_1 as a symmetric space.

Let (π_1, V_1) be a quasisimple representation of finite length on a Hilbert space V_1. Define $\pi=\pi_1\otimes\pi_1^\vee$ on $V=V_1\hat{\otimes}V_1'$, the Hilbert space tensor product of V_1 with V_1', which is isomorphic to $\hom_{\text{H.S.}}(V_1)$, the Hilbert–Schmidt operators on V_1. Let $\mu_1\in K_1^\wedge$ and define $\mu=\mu_1\otimes\mu_1^\vee\in(K^\wedge)_{K\cap H}$. (Notice that $K\cap H=\mathbf{d}(K_1)$.)

Let $\mathfrak{a}_1$ be a θ_1-invariant Cartan subalgebra and let W_1 be the complex Weyl group. Let $\Lambda_1\in(\mathfrak{a}_1)^*_{\mathbf{C}}$ be the infinitesimal character of π_1. Choosing the opposite ordering $-\Lambda_1$ is the infinitesimal character of $\pi_1^\vee$ and therefore $(\Lambda_1,-\Lambda_1)$ is the infinitesimal character of π. The antidiagonal $\mathfrak{a}=\{X,-X)|X\in\mathfrak{a}_1\}$ is a θ-invariant Cartan subspace for G/H. Therefore $\mathfrak{a}^r$ is isomorphic to $\mathfrak{a}_1^r=i(\mathfrak{a}_1\cap\mathfrak{k}_1)+\mathfrak{a}_1\cap\mathfrak{p}_1$ and W is isomorphic to W_1. The linear functional λ in Theorem 1 is the restriction of $(\Lambda_1,-\Lambda_1)$ to $\mathfrak{a}$.

We may now chose $v_0\in V^H_{-\infty}$ as the identity operator in $\hom(V_1)$ and $v_0'\in V'^{H}_{-\infty}$ as the trace of elements in $\hom(V_1)$. If π_1 is irreducible then the dimensions of $V^H_{-\infty}$ and $V'^{H}_{-\infty}$

are both one and they are spanned by v_0 and v_0' respectively. Since π_1 has infinitesimal character Λ_1, it follows that v_0 is a spherical vector of type $\lambda=(\Lambda_1, -\Lambda_1)|_{\mathfrak{a}}$.

The space

$$C_c^\infty(G/H; K) = C_c^\infty(G_1 \times G_1/\mathbf{d}(G_1); K_1 \times K_1)$$

is isomorphic to $C_c^\infty(G_1; K_1)$. Theorem 2 is now clearly a restatement of Theorem 1 for the above situation. □

Proof of Theorem 3. In order to modify the proof of Theorems 1 and 2 we must be able to replace C_c^∞ with $\mathcal{S}_0$ and $PW(\mathfrak{a}^r)$ with $PW_0(\mathfrak{a}^r)$. The appropriate version of Proposition 3 is the following

PROPOSITION 3*. *The spherical Fourier transform on* $X^r = G^d/K^d$

$$f \to f^{\sim}(\lambda) = \int_{G^d} f(x)\, \phi_{-i\lambda}(x)\, dx, \quad \lambda \in \mathfrak{a}_{\mathbf{C}}^*$$

is a bijection of $\mathcal{S}_0(K^d\backslash G^d/K^d)$ *onto* $PW_0(\mathfrak{a}^r)^W$.

Proof. This follows from Trombi and Varadarajan [29, Theorem 3.10.1], since $\mathcal{S}_0(K^d\backslash G^d/K^d)$ is the intersection for all $p>0$ of $\mathcal{S}_p(K^d\backslash G^d/K^d)$, which Trombi and Varadarajan denotes $\mathcal{I}^p(G)$, and $PW_0(\mathfrak{a}^r)^W$ is the intersection of the spaces $\bar{\mathcal{Z}}(\mathcal{F}^\varepsilon)$ for $\varepsilon>0$. □

Remark. Since the proof in [29] of the Theorem 3.10.1 is rather involved, one should remark, that Proposition 3* similarly to Proposition 3 can be given a rather elementary proof. However Anker [2] has a simple proof of the result of Trombi and Varadarajan via a reduction to the C_c^∞-Paley–Wiener theorem.

We shall also need a couple of lemmas. First define the space of H^d-finite zero-Schwartz-functions on G^d/K^d in the following way

$$\mathcal{S}_0(G^d/K^d; H^d) = \{f \in C^\infty(G^d/K^d; H^d) \mid \forall u \in \mathcal{U}(\mathfrak{g}),\ \forall m \in \mathbf{R},\ \forall \text{ compact set } C \subset H^d,$$
$$\sup\{e^{|H|m} |L_u f(h \exp H)| \mid h \in C,\ H \in \mathfrak{a}_p\} < \infty\}.$$

LEMMA 5. *Let* $F \in \mathcal{C}^\infty(K^d\backslash G^d/K^d)$ *and* $S>0$ *and assume that*

$$|F(g)| \leq C_S\, e^{-2S|g|}, \quad \forall g \in G^d,$$

for some constant C_S. *Let* ϕ *be an* H^d*-finite function on* H^d *and assume that* S *is so* *large that $h \to |\phi(h)|\, e^{-S|h|}$ is integrable. Then we have*

$$\left| \int_{H^d} \phi(h)\, F(ha)\, dh \right| \leqslant C_{S,\phi}\, e^{-S|a|}, \quad \forall a \in A_p,$$

for some constant $C_{S,\phi}$.

Proof. From Flensted-Jensen [15, Chapter IV, Lemma 11] we have that $e^{|ha|} \geqslant e^{|h|}$ and $e^{|ha|} \geqslant e^{|a|}$, therefore we get that

$$\left| \int_{H^d} \phi(h)\, F(ha)\, dh \right| \leqslant C_S \int_{H^d} |\phi(h)|\, e^{-S|h|}\, dh \cdot e^{-S|a|} = C_{S,\phi}\, e^{-S|a|}. \qquad \square$$

Corollary 6. *Let $F \in \mathscr{S}_0(K^d \backslash G^d / K^d)$ and let ϕ be an H^d-finite function on H^d, then f_ϕ defined by*

$$f_\phi(g) = \int_{H^d} \phi(h)\, F(hg)\, dh$$

belongs to $\mathscr{S}_0(G^d/K^d; H^d)$.

Proof. In order to show that $f_\phi \in \mathscr{S}_0(G^d/K^d; H^d)$ we let $m \in \mathbf{R}$ and $u \in \mathscr{U}(\mathfrak{g})$. We have that

$$(L_u f_\phi)(g) = \int_{H^d} \phi(h)\, L_{v_h} F(hg)\, dh,$$

where $v_h = \mathrm{Ad}(h^{-1})\, u$. Let $u_1, u_2, \ldots, u_s$ be a basis of the finite dimensional subspace of $\mathscr{U}(g)$ generated by the action of H on u. We can then write

$$v_h = \sum_{i=1}^{s} \phi_i(h)\, u_i,$$

where $\phi_i, i=1, \ldots, s$, are H^d-finite functions on H^d.

Since $L_{u_i} F$ belongs to $\mathscr{S}_0(K^d \backslash G^d / K^d)$ it follows that $L_u f_\phi$ is a linear combination of functions, which according to Lemma 5 satisfies the relevant growth conditions. Therefore we conclude that $f_\phi \in \mathscr{S}_0(G^d/K^d; H^d)$. $\square$

In particular we have the following

Corollary 7. *Let $F \in \mathscr{S}_0(K^d \backslash G^d / K^d)$ and let $\mu \in (K^\wedge)_{K \cap H}$. Then every component $\langle e, F_\mu(\cdot) \rangle$ for $e \in E_\mu$ of the function F_μ defined by*

$$F_\mu(x) = d_\mu \int_{H^d} F(hx)\mu^\vee(h^{-1})\, e_0'\, dh, \quad x \in G^d$$

belongs to $\mathcal{S}_0(G^d/K^d; H^d)$.

Finally we have the following lemma, which is an easy consequence of the definition of $\mathcal{S}_0$ and the basic properties of the map $\eta: f \to f^r$, cf. Proposition 1.

LEMMA 8. *Let* $f \in C^\infty(G/H; K)$ *then* $f \in \mathcal{S}_0(G/H; K)$ *if and only if* $f^r \in \mathcal{S}_0(G^d/K^d; H^d)$.

Now the proof of Theorem 3 follows exactly the same lines as the proof of Theorems 1 and 2. □

Remark. Let $\mathcal{S}_0^{\mathcal{Z}}(G^d/K^d; H^d)$ denote the space defined with $\mathcal{Z}(g)$ instead of $\mathcal{U}(g)$, and similarily for the spaces of K-finite functions etc.

Clearly $\mathcal{S}_0^{\mathcal{Z}}$ contains $\mathcal{S}_0$. It is easily seen that the spherical Fourier transform on $X^r = G^d/K^d$ maps $\mathcal{S}_0^{\mathcal{Z}}(K^d\backslash G^d/K^d)$ into $PW_0(\mathfrak{a}^r)^W$. It then follows from Proposition 3*, that

$$\mathcal{S}_0^{\mathcal{Z}}(K^d\backslash G^d/K^d) = \mathcal{S}_0(K^d\backslash G^d/K^d).$$

in fact the same argument shows, that it suffices to use powers of the Casimir operator instead of $\mathcal{Z}(g)$.

§ 3. A Paley–Wiener theorem for the isotropic spaces

The spherical distributions and the Fourier transform on the non-Riemannian isotropic spaces has been studied by Faraut [11] for the classical spaces and by M. Kosters [24] for the exceptional space. All these spaces are non-Riemannian semisimple symmetric spaces of rank one. They may each be realized as a projective hyperbolic space with $p \geq 2$ and $q \geq 1$:

$$\begin{aligned} X &= P^{(p-1,q)}(\mathbf{F}) \\ &= \{x \in \mathbf{F}^{p+q} \mid |x_1|^2 + \cdots + |x_p|^2 - \cdots - |x_{p+q}|^2 = 1\}/\{\alpha \in \mathbf{F} \mid |\alpha| = 1\}, \end{aligned}$$

where $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or (only for $p=2$ and $q=1$) $\mathbf{F} = \mathbf{O}$. Here $\mathbf{F}$ is the quaternions and $\mathbf{O}$ is the Cayley numbers. We define $d = \dim_{\mathbf{R}} \mathbf{F}$ and $\varrho = \frac{1}{2}(dp + dq - 2)$.

The K-types having a $(K \cap H)$-fixed vector are in all cases parametrized, as δ_l, by an even non-negative integer l. (Faraut uses two parameters l and m, with $m+l$ even and

$m \leq l$, but $m=0$ when there is a $(K \cap H)$-fixed vector. Kosters uses p, q corresponding to $m=p$ and $l=2q+p$.) The trivial representation of K corresponds to $l=0$.

A spherical distribution is by definition, cf. § 1, an H-invariant eigendistribution of $\mathbf{D}(X)$. Since the rank is one, $\mathbf{D}(X)$ is generated by the Laplace–Beltrami operator on X corresponding to the natural pseudo-Riemannian structure.

From Faraut [11] and Kosters [24] we have the existence of the following spherical distributions:

(i) $\zeta_s, s \in \mathbf{C}$. This is a holomorphic family satisfying $\zeta_s = \zeta_{-s}$.

(ii) $\theta_r, r \in \mathbf{N} = \{0, 1, 2, \dots\}$. These are defind only when dq is even, i.e. when $\mathbf{F} = \mathbf{C}, \mathbf{H}, \mathbf{O}$ or when $\mathbf{F} = \mathbf{R}$ and q is even.

For convenience we denote the parameter space for these distributions by $\Omega = \Omega_s \cup \Omega_r$, where $\Omega_s = \mathbf{C}$ and $\Omega_r = \emptyset$ when dq is odd and $\Omega_r = \mathbf{N}$ otherwise.

Remarks. (a) The spherical distributions ζ_s and θ_r suffice for the decomposition of the Dirac measure at the origin in X. In fact the continuous spectrum corresponds to $s \in i\mathbf{R}$ and the discrete spectrum has two parts: One corresponding to Ω_r and another corresponding to the set of parameters $s \in \Omega_s$ given by $\{s = \varrho + 2r \mid 0 < s < \varrho,\ r \in \mathbf{Z}\}$ if dq is even and $\{s = \varrho + 2r + 1 \mid s > 0,\ r \in \mathbf{Z}\}$ otherwise.

(b) For dq odd any spherical distribution is a multiple of a ζ_s. For dq even the appropriate linear combinations of ζ_s and θ_r does not in general give all spherical distributions. However the remaining ones can be obtained from ζ_s, θ_r and $(d\zeta_s/ds)|_{s=\varrho+2r}$, $r = 0, 1, 2, \dots$. Notice in passing that ζ_s is always non-zero for dq odd. For dq even the parameters $s = \pm(\varrho + 2r)$, $r = 0, 1, 2, \dots$ give exactly the zeros of ζ_s.

(c) The spherical distributions ζ_s are naturally connected with what could be called the principal series representations π_s for the symmetric space X in the following way. One can define spherical distribution vectors u_s for π_s and a pairing between π_s and π_{-s} such that

$$\langle \zeta_s, \phi \rangle = \langle \pi_s(\phi)\, u_s, u_{-s} \rangle, \quad \phi \in \mathbf{C}_c^\infty(X).$$

From this it is easily seen that if χ_δ is the normalized character of a K-type δ, then

$$\langle \zeta_s, \chi_\delta \rangle = \langle \pi_s(\chi_\delta)\, u_s, u_{-s} \rangle$$

is well defined since $\pi_s(\chi_\delta)\, u_s$ is an analytic vector for π_s.

Notice that $\zeta_s(\chi_\delta)$ can only be non-zero if δ has a $(K \cap H)$-fixed vector. Having parametrized these K-types as δ_l, $l = 0, 2, \dots$ we define

$$\gamma_l(s) = \langle \zeta_s, \chi_{\delta_l} \rangle.$$

From [11, p. 407] and [24, p. 72] we get that

$$\gamma_l(s) = c_l \beta_{l0}(s) \beta_{l0}(-s) \tag{1}$$

with

$$\beta_{l0}(s) = c_l' \frac{(s-\varrho)(s-\varrho-2)\dots(s-\varrho-l+2)}{\Gamma((s-\varrho+l+dp)/2)},$$

where c_l and c_l' are constants.

(d) The spherical distributions θ_r, $r \in \Omega_r$, correspond to subrepresentations of π_{-s}, respectively quotient representations of π_s, where $s=\varrho+2r$. The formula corresponding to (1) for θ_r follows from [11, p. 413] and [24, p. 82]

$$\theta_r(\chi_\delta) = c_l \left(\frac{d}{ds} \beta_{l0} \right) (\varrho+2r) \beta_{l0}(-\varrho-2r).$$

From this we get that

$$\langle \theta_r, \phi \rangle = 0 \text{ if } \phi \in C_c^\infty(X) \text{ is of type } \delta_l \text{ with } l < dq+2r$$

and that there exists $\phi \in C_c^\infty(X)$ of type δ_{dq+2r} such that $\langle \theta_r, \phi \rangle \neq 0$.

(e) Any K-finite function in $C_c^\infty(X)$ is a linear combination of functions of the form

$$\phi(ka) = \Phi(k)\phi(a), \quad k \in K \text{ and } a \in A_p,$$

where Φ is K-finite of a specific type δ. We now assume that $\delta=\delta_l$ and that Φ is $(K \cap H)$-invariant and non-zero. For such a function ϕ of the calculation of $\langle \zeta_s, \phi \rangle$ leads to the following explicit formula, cf. [11, p. 402] and [24, p. 72]

$$\langle \zeta_s, \phi \rangle = c(\delta, \Phi) \gamma_l(s) \int_0^\infty \phi(a_t) \phi_{s,\delta}(t) \Delta(t)\, dt, \tag{2}$$

where $c(\delta, \Phi)$ is a constant, a_t is given by $a_t = \exp(tH_0)$ for a normalized choice of $H_0 \in \mathfrak{a}_\mathfrak{p}$ and $\phi_{s,\delta}$ is given by

$$\phi_{s,\delta}(t) = (\cosh t)^l \phi_\lambda^{(\alpha, \beta+l)}(t),$$

where

$$\alpha = \frac{dq}{2} - 1, \quad \beta = \frac{dp}{2} - 1 \quad \text{and} \quad \lambda = is.$$

The function $\phi_\lambda^{(\alpha,\beta)}$ is the Jacobi function. It is given by, cf. Koornwinder [23, p. 5–6],

$$\begin{aligned}\phi_\lambda^{(\alpha,\beta)} &= {}_2F_1(1/2(\alpha+\beta+1-i\lambda), 1/2(\alpha+\beta+1+i\lambda); \alpha+1; -(\sinh t)^2)\\ &= (\cosh t)^{-(\alpha+\beta+1+i\lambda)}{}_2F_1(1/2(\alpha+\beta+1-i\lambda), 1/2(\alpha-\beta+1+i\lambda); \alpha+1; (\tanh t)^2).\end{aligned}$$

Finally $\Delta(t)$ is given by

$$\Delta(t) = (2\cosh t)^{2\beta+1}(2\sinh t)^{2\alpha+1}.$$

Then it follows that $\int_0^\infty \phi(a_t)\,\phi_{s,\delta}(t)\,\Delta(t)\,dt$ is the Jacobi transform of parameter $(\alpha, \beta+l)$ and with variable $\lambda=is$ of the function $t\to(\cosh t)^{-l}\phi(a_t)$. Therefore the conclusion is, cf. (2), that $\langle \zeta_s, \phi\rangle$ is $\gamma_l(i\lambda)$ times the Jacobi transform of $t\to(\cosh t)^{-l}\phi(a_t)$, up to a constant depending on δ and Φ.

From the Paley–Wiener theorem for the Jacobi transform, cf. Koornwinder [22] or [23], we can now derive Theorem 1 for these special cases. In fact we get more precisely, that $s\to\langle \zeta_s, \phi\rangle$ for any K-finite function $\phi\in C_c^\infty(X)$ can be written as $(s=i\lambda)$

$$\sum G_\delta(\lambda)\,\gamma_\delta(s), \tag{3}$$

where the sum is over the finite set of $\delta\in K^\wedge$ related to ϕ and where $G_\delta\in PW_e(\mathbf{R})$, the even functions in $PW(\mathbf{R})$. Conversely any function like (3) can be obtained from a function ϕ in $C_c^\infty(X;K)$ only involving the K-types occuring in (3). One should remark at this point that the expression for $\langle \zeta_s, \phi\rangle$ in (3) is highly non-unique.

Let $\mathcal{G}$ be the vector space spanned by finite linear combinations of the functions $s\to\langle \zeta_s, \chi_\delta\rangle = \langle \pi_s(\chi_\delta)\,u_s, u_{-s}\rangle$, $\delta\in(K^\wedge)_{K\cap H}$ or more precisely by the functions $s\to\gamma_l(s)$, $l=0, 2, 4, \ldots$. Now inspired by Theorem 1 and Remark (e) we define the Paley–Wiener space $PW(X)$ for X to be the vector space of functions $F\colon \Omega\to\mathbf{C}$ such that

(i) F has finite support on Ω_r,

(ii) $\lambda\to F(i\lambda)$, $s=i\lambda\in\Omega_s$ belongs to the natural image of $PW_e(\mathbf{R})\otimes\mathcal{G}$ in the space of functions on Ω.

For $\phi\in C_c^\infty(X)$ the Fourier transform is defined by

$$\phi^\wedge(s) = \langle \zeta_s, \phi\rangle, \quad s\in\Omega_s$$

and

$$\phi^\wedge(r) = \langle \theta_r, \phi\rangle, \quad r\in\Omega_r.$$

One might more precisely call it the scalar valued Fourier transform in contrast to the vector valued Fourier transform, which is defined by

$$\phi \to \pi_s(\phi)\, u_s, \quad s \in \Omega_s, \ \phi \in C_c^\infty(X)$$

and similarly for $r \in \Omega_r$.

THEOREM 4. *Let X be an isotropic non-Riemannian symmetric space and let the notation be as above. A function F on Ω is the Fourier transform of a K-finite function in $C_c^\infty(X)$ if and only if F belongs to the Paley–Wiener space $PW(X)$.*

Proof. First assume that $\phi \in C_c^\infty(X)$ is K-finite then it follows from Remark (e) that $F=\phi^\wedge$ belongs to $PW(X)$.

Next assume that $F \in PW(X)$. In particular we have an expression for $F(s)$, $s \in \Omega$, of the form

$$F(s) = \sum_{j=0}^{n} G_j(is)\, \gamma_{2j}(s)$$

where

$$G_j \in PW_e(\mathbf{R}).$$

From Theorem 1 or from Remark (e) it follows that for $j=1, \ldots, n$ there exist f_j in $C_c^\infty(X)$ of type δ_{2j}, such that if we define $\phi=\Sigma_{j=0}^n f_j$, then

$$\phi^\wedge(s) = F(s), \quad \text{for all} \quad s \in \Omega_s.$$

This means that it will suffice to prove the following lemma.

LEMMA 9. *Let $r_0 \in \mathbf{N}$. There exists $\phi \in C_c^\infty(X)$ which is K-finite and satisfies*

$$\phi^\wedge(r_0) = 1, \ \phi^\wedge(r) = 0 \quad \textit{if} \quad r > r_0 \quad \textit{and} \quad \phi^\wedge(s) = 0 \ \textit{for all} \ s \in \Omega_s.$$

Proof. First we observe, that it follows from Remark (d), that if ϕ is of type δ_0, then $\phi^\wedge(r)=0$ for each $r \in \Omega_r$. It also follows that there exists $\phi_2 \in C_c^\infty(X)$ of type $l_0=dq+2r_0$ such that

$$\theta_{r_0}(\phi_2) = 1 \quad \text{and} \quad \theta_r(\phi_2) = 0 \quad \text{if} \quad r > r_0.$$

From the first part of the proof we know for such ϕ that

$$\phi_2^\wedge(s) = G(is)\gamma_{l_0}(s) \quad \text{for} \quad s \in \Omega_s$$

where $G \in PW_e(\mathbf{R})$.

Using the formula for γ_l and the recursion formula for the Γ-function we can get

$$\phi_2^\wedge(s) = \frac{G(is)\,P(s)}{Q(s)}\gamma_0(s),$$

where $P(s)$ and $Q(s)$ are even polynomials. The precise form of $Q(s)$ is

$$\begin{aligned} Q(s) = 2^{-l_0}&(s-\varrho+dp+l_0-2)\,(s-\varrho+dp+l_0-4)\dots(s-\varrho+dp) \\ &\times(-s-\varrho+dp+l_0-2)\,(-s-\varrho+dp+l_0-4)\dots(-s-\varrho+dp). \end{aligned}$$

Inserting $l_0 = dq+2r_0$ and recalling that $2\varrho = dp+dq-2$ we can write

$$Q(s) = (s+(\varrho+2r_0))\,(-s+(\varrho+2r_0))\,Q_1(s)$$

where $Q_1(s)$ is even and $Q_1(\varrho+2r_0) = Q_1(-\varrho-2r_0) \neq 0$. We may choose an invariant differential operator $D \in \mathbf{D}(X)$ such that

$$(D\phi)^\wedge(s) = Q_1(\varrho+2r_0)^{-1}Q_1(s)\,\phi^\wedge(s)$$

and then we get with $s = \varrho+2r_0$:

$$(D\phi_2)^\wedge(r_0) = Q_1(\varrho+2r_0)^{-1}Q_1(\varrho+2r_0)\,\phi_2^\wedge(r_0) = \phi_2^\wedge(r_0) = 1.$$

Therefore if we put $\phi_1 = D\phi_2$, we get that

$$\begin{aligned} \phi_1^\wedge(s) &= Q_1(\varrho+2r_0)^{-1}Q_1(s)\frac{G(is)\,P(s)}{Q(s)}\gamma_0(s) \\ &= \frac{G(is)\,Q_1(\varrho+2r_0)^{-1}P(s)}{(s+(\varrho+2r_0))\,(-s+(\varrho+2r_0))}\gamma_0(s). \end{aligned} \tag{4}$$

Next we observe that $\phi_1^\wedge(\varrho+2r_0) = \langle \zeta_{\varrho+2r_0}, \phi_1 \rangle = 0$, since $s = \varrho+2r_0$ is a zero of ζ_s. By inspection of the formula for γ_0 it follows that $s = \varrho+2r_0$ is a simple zero of γ_0, but then it follows from (4) that $s = \varrho+2r_0$ must be a zero for $G(is)P(s)$. Similarly for $s = -(\varrho+2r_0)$. This means that $\phi_1^\wedge|_{\Omega_s}$ is of the form $G_0(is)\gamma_0(s)$, where $G_0 \in PW_e(\mathbf{R})$. Now choose ϕ_0 of type δ_0 such that $\phi_0^\wedge(s) = \phi_0^\wedge(s)$, $s \in \Omega_s$. Then $\phi = \phi_1 - \phi_0$ satisfies the lemma. □

This also completes the proof of Theorem 4. □

Remark. It follows from the proof with a few extra considerations, that if F has exponential type R, then the K-finite function can be chosen with support in a K-invariant "ball" of radius R.

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