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The Schwartz algebra of an affine Hecke algebra

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Contents

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1. Introduction

An affine Hecke algebra is associated to a based root datum $\mathscr{R} = (X, Y, R_0, R_0^\vee, F_0)$, where X, Y are lattices with a perfect pairing, $R_0 \subset X$ is a reduced root system, $R_0^\vee \subset Y$ is the coroot system and F_0 is a basis of R_0, together with a length multiplicative function q of the affine Weyl group associated to $\mathscr{R}$. It is denoted by $\mathscr{H}(\mathscr{R}, q)$ or simply $\mathscr{H}$. It admits a natural prehilbertian structure (provided q has values in $\mathbb{R}_+$, which we assume throughout), and it acts on its completion $L_2(\mathscr{H})$ through bounded operators. With this structure $\mathscr{H}$ is a Hilbert algebra in the sense of [9], a remark that gives rise to natural questions from an harmonic analytic and operator algebraic point of view.

The main motivation for considering such matters is the role of affine Hecke algebras in the harmonic analysis of reductive p-adic groups. The most general point of view in this context is provided by the theory of types (see [6]). This theory seeks to describe a given block in the Bernstein decomposition of the category of smooth representations of a p-adic reductive group G via Morita equivalence as the representation category of the Hecke algebra of an associated "type". In many cases this is known, and in many important cases it was shown that the emerging Hecke algebras associated to types are isomorphic to affine Hecke algebras in the above sense (see e.g. [15], [24], [20]). These Morita equivalences respect the harmonic analysis: The spectral measure of the Hilbert algebra of the affine Hecke algebra $\mathscr{H}$ arising as the Hecke algebra of a type of G can be transferred (up to a known positive factor) by the Morita equivalence to the Plancherel measure of G restricted to the corresponding Bernstein block [7]. In this way the affine Hecke algebra may be considered as a tool to disclose parts of the Plancherel measure of a reductive p-adic group, a point of view that was advocated by several authors (e.g. [28], [29], [14]).

Thus we would like to compute the spectral measure of the Hilbert algebra attached to $\mathscr{H}$ (called the "Plancherel measure of $\mathscr{H}$" in the sequel) explicitly. This entails in particular a full description of the set of irreducible representations of $\mathscr{H}$ in the support of the Plancherel measure. From Theorem 3.22 and Theorem 4.3 it follows that the support of the Plancherel measure of $\mathscr{H}$ coincides with the set of *irreducible tempered representations* of $\mathscr{H}$, i.e. the irreducible representations of $\mathscr{H}$ which extend continuously to the Fréchet algebra completion $\mathscr{S} = \mathscr{S}(\mathscr{R}, q)$ (the so-called Schwartz algebra, introduced in [26]) of $\mathscr{H}$. According to Corollary 3.8 this set of irreducible representations of $\mathscr{H}$ can alternatively be described by Casselman's criterion.

The Schwartz algebra $\mathscr{S}$ is instrumental to analyze the set of tempered representations of $\mathscr{H}$. First of all the study of $\mathscr{S}$ gives a detailed understanding of the decomposition in irreducibles of tempered standard modules via the theory of analytic R-groups, similar to the role played by the Harish-Chandra Schwartz algebra $\mathscr{C}(G)$ in the harmonic analysis of a reductive p-adic group G. This type of application follows closely classical arguments due to Harish-Chandra, Knapp-Stein and Silberger. The novelty consists in the fact that the same machinery works equally well for arbitrary q in our space of continuous parameters, a fact that becomes important for the deformation arguments mentioned below. The second application of $\mathscr{S}$ is a comparison theorem of the second author with Solleveld (to appear) stating that for tempered representations V, W there is a natural isomorphism $\mathrm{Ext}^i_{\mathscr{S}}(V,W) \simeq \mathrm{Ext}^i_{\mathscr{H}}(V,W)$. This applies directly to the study of the space of elliptic tempered characters of $\mathscr{H}$ (in the case of reductive p-adic groups see [32], [23]). Finally we mention an important result of Solleveld [34] stating that the q-parameter family of algebra structures on the Fréchet space $\mathscr{S}$ is continuous with respect to the Fréchet topology. This allows, in combination with the above and with the results of this paper, to study the set of irreducible tempered representations by deformation arguments. We claim that for non-simply laced affine Hecke algebras these techniques are essentially sufficient to classify the set of tempered representations and compute the Plancherel measure (to appear elsewhere).

The main theorem of this article is the characterization of the image of $\mathscr{S}$ by the Fourier transform $\mathscr{F}$. This result is reminiscent to Harish-Chandra's results for p-adic groups (see e.g. [36] (but we remark that the proofs of these results are necessarily very different from Harish-Chandra's proofs). We refer the reader to the Appendix 10 for the connection between our Schwartz algebra $\mathscr{S}$ and the Harish-Chandra Schwartz algebra $\mathscr{C}(G)$ in the special case where G is a split semisimple p-adic group and $\mathscr{H} = \mathscr{H}(G,B)$ is the Iwahori-Matsumoto Hecke algebra.

The description of $\mathscr{F}(\mathscr{S})$ has some immediate consequences which are described in Section 5. Let us briefly discuss these applications.

First of all, we obtain the analog of Harish-Chandra's Completeness Theorem for generalized principal series of real reductive groups. The representations involved in the spectral decomposition of $L_2(\mathscr{H})$ are, as representations of $\mathscr{H}$, subrepresentations of certain finite dimensional induced representations from parabolic subalgebras (which are subalgebras of $\mathscr{H}$ which themselves belong to the class of affine Hecke algebras). We call these the standard tempered induced representations. There exist standard interwining operators (see [26]) between the standard induced tempered representations. The Completeness Theorem states that the commutant of the standard tempered induced representations is generated by the self-intertwining operators given by standard intertwining operators.

Next we determine the image of the center of $\mathscr{S}$ and, as a consequence, we obtain the analog of Langlands' Disjointness Theorem for real reductive groups: two standard tempered induced representations are either disjoint, i.e., without simple subquotient in common, or equivalent.

Then we discuss the characterization of the Fourier transform, and of the set of minimal central idempotents of the reduced C^*-algebra $\mathscr{C}_r^*(\mathscr{H})$ of $\mathscr{H}$.

Finally we observe that the dense subalgebra $\mathscr{S} \subset \mathscr{C}_r^*$ is closed for holomorphic calculus. In the case of Hecke algebras $\mathscr{S}(G,K)$ associated with a compact open subgroup K of a reductive p-adic group G this was shown by Vignéras [35].

Let us finally comment on the proof of the Main Theorem. As it is familiar since Harish-Chandra's work on real reductive groups [11], [12], [36], the determination of the image of $\mathscr{S}$ by $\mathscr{F}$ requires a theory of the constant term for coefficients of tempered representations of $\mathscr{H}$. This theory is fairly simple using the decomposition of these linear forms on $\mathscr{H}$ along weights of the action of the abelian subalgebra $\mathscr{A}$ of $\mathscr{H}$. This subalgebra admits as a basis, the family θ_x, $x \in X$, which arises in the Bernstein presentation of $\mathscr{H}$.

There is a natural candidate $\hat{\mathscr{S}}$ for the image of $\mathscr{S}$ by $\mathscr{F}$. The inclusion $\mathscr{F}(\mathscr{S}) \subset \hat{\mathscr{S}}$ is easy to prove, using estimates of the coefficients of standard induced tempered representations.

The only thing that remains to be proved at this point, is that the inverse of the Fourier transform, the wave packet operator $\mathscr{J}$, maps $\hat{\mathscr{S}}$ to $\mathscr{S}$. For this a particular role is played by normalized smooth family of coefficients of standard tempered induced representations: these are smooth families divided by the c-function. Of particular important is the fact that the constant terms of these families are finite sums of normalized smooth families of coefficients for Hecke subalgebras of smaller semisimple rank. This is a nontrivial fact which requires the explicit computation of the constant term of coefficients for generic standard tempered induced representations. If $\mathscr{I}$ is the maximal ideal of the center $\mathscr{Z}$ of $\mathscr{H}$ which annihilates such a representation, its coefficients can be viewed as linear forms on Lusztig's formal completion of $\mathscr{H}$ associated to $\mathscr{I}$. This allows to use Lusztig's First Reduction Theorem [19] which decomposes this algebra. Some results on Weyl groups are then needed to achieve this computation of the constant term.

Once this property of normalized smooth family is obtained, it is easy to form wave packets in the Schwartz space, by analogy with Harish-Chandra's work for real reductive groups [11]. Simple lemmas on spectral projections of matrices and an induction argument, allowed by the theory of the constant term, lead to the desired result.

The paper is roughly structured as follows. First we discuss in Sections 2 to 4 the necessary preliminary material on the affine Hecke algebra and the Fourier transform on $L_2(\mathscr{H})$. We formulate the Main Theorem in Section 5, and we discuss some of its consequences. In Section 6 we compute the constant terms of coefficients of the standard induced representations and of normalized smooth families of such coefficients. Finally, in Section 7 we use this and the material in the Appendix on spectral projections in order to prove the Main Theorem. In two separate appendices we have collected useful fundamental properties of spectral projections and of the Macdonald c-function on which many of our results ultimately rely.

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2. The affine Hecke algebra and the Schwartz algebra

This section serves as a reminder for the definition of the affine Hecke algebra and related analytic structures. We refer the reader to [26], [19] and [25] for further background material.

2.1. The root datum and the affine Weyl group. A reduced root datum is a 5-tuple $\mathcal{R} = (X, Y, R_0, R_0^\vee, F_0)$, where X, Y are lattices with perfect pairing $\langle \cdot , \cdot \rangle$, $R_0 \subset X$ is a reduced root system, $R_0^\vee \subset Y$ is the coroot system (which is in bijection with R_0 via the map $\alpha \to \alpha^\vee$), and $F_0 \subset R_0$ is a basis of simple roots of R_0. The set F_0 determines a subset $R_{0,+} \subset R_0$ of positive roots.

The Weyl group $W_0 = W(R_0) \subset \mathrm{GL}(X)$ of R_0 is the group generated by the reflections s_α in the roots $\alpha \in R_0$. The set $S_0 := \{s_\alpha \mid \alpha \in F_0\}$ is called the set of simple reflections of W_0. Then (W_0, S_0) is a finite Coxeter group.

We define the affine Weyl group $W = W(\mathcal{R})$ associated to $\mathcal{R}$ as the semidirect product $W = W_0 \ltimes X$, which acts as a group of affine transformations on $\mathbb{Q} \otimes_{\mathbb{Z}} X$. The lattice X contains the root lattice Q, and the normal subgroup $W^{\mathrm{aff}} := W_0 \ltimes Q \lhd W$ is a Coxeter group whose Dynkin diagram is given by the affine extension of (each component of) the Dynkin diagram of $R_0^\vee$. The affine root system of W^{aff} is given by $R^{\mathrm{aff}} = R_0^\vee \times \mathbb{Z} \subset Y \times \mathbb{Z}$, whose elements will be denoted $(\check{\alpha}, n)$, or $\check{\alpha} + n$, viewing them as affine functional on $\mathbb{Q} \otimes_{\mathbb{Z}} X$. Note that W acts on R^{aff}.

Let R_+^{aff} be the set of positive affine roots defined by

$$R_+^{\mathrm{aff}} = \{(\alpha^\vee, n) \mid n > 0, \text{ or } n = 0 \text{ and } \alpha \in R_{0,+}\}.$$

Let F^{aff} denote the corresponding set of affine simple roots. Observe that $F_0^\vee \subset F^{\mathrm{aff}}$. If S^{aff} denotes the associated set of affine simple reflections, then $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ is an affine Coxeter group.

In this paper we adhere to the convention $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. We define the length function $l : W \to \mathbb{Z}_+$ on W as usual, by means of the formula $l(w) := |R_+^{\mathrm{aff}} \cap w^{-1}(R_-^{\mathrm{aff}})|$. Let $\Omega \subset W$ denote the set $\{w \in W \mid l(w) = 0\}$. It is a subgroup of W, complementary to W^{aff}. Therefore $\Omega \simeq X/Q$ is a finitely generated abelian subgroup of W.

Let $X^+ \subset X$ denote the cone of dominant elements

$$X^+ = \{x \in X \mid \forall \alpha \in R_{0,+} : \langle x, \alpha^\vee \rangle \geqq 0\}.$$

We put $X^- := -X^+ \subset X$ for the cone of anti-dominant elements in X. Then $Z_X := X^+ \cap X^- \subset X$ is a sublattice which is central in W. In particular it follows that $Z_X \subset \Omega$. The quotient $\Omega_f \simeq \Omega/Z_X$ is a finite abelian group which acts faithfully on S^{aff} by means of diagram automorphisms.

We choose a basis z_i of Z_X, and define a norm $\| \cdot \|$ on the rational vector space $\mathbb{Q} \otimes_{\mathbb{Z}} Z_X$ by $\| \sum l_i z_i \| := \sum |l_i|$. We now define a norm $\mathcal{N}$ on W by

$$\mathcal{N}(w) := l(w) + \|w(0)^0\|, \tag{2.1}$$

where $w(0)$ is the image by the affine transformation w of $0 \in \mathbb{Q} \otimes_{\mathbb{Z}} X$, and where $w(0)^0$ denotes the projection of $w(0)$ onto $\mathbb{Q} \otimes_{\mathbb{Z}} Z_X$ along $\mathbb{Q} \otimes_{\mathbb{Z}} Q$. The norm $\mathcal{N}$ plays an important role in this paper. Observe that it satisfies

$$\mathcal{N}(ww') \leqq \mathcal{N}(w) + \mathcal{N}(w'), \tag{2.2}$$

and that $\mathcal{N}(w) = 0$ if and only if w is an element of Ω of finite order.

We call $\mathcal{R}$ semisimple if $Z_X = 0$.

2.2. Standard parabolic subsystems. A root subsystem $R' \subset R_0$ is called parabolic if $R' = \mathbb{Q}R' \cap R_0$. The Weyl group W_0 acts on the collection of parabolic root subsystems. Let $\mathcal{P}$ be the power set of F_0. With $P \in \mathcal{P}$ we associate a standard parabolic root subsystem $R_P \subset R_0$ by $R_P := \mathbb{Z}P \cap R_0$. Every parabolic root subsystem is W_0-conjugate to a standard parabolic subsystem.

We denote by $W_P = W(R_P) \subset W_0$ the Coxeter subgroup of W_0 generated by the reflections in P. We denote by W^P the set of shortest length representatives of the left cosets W_0/W_P of $W_P \subset W_0$.

Given $P \in \mathcal{P}$ we define a sub root datum $\mathcal{R}^P \subset \mathcal{R}$ simply by $\mathcal{R}^P := (X, Y, R_P, R_P^\vee, P)$. We also define a "quotient root datum" $\mathcal{R}_P$ of $\mathcal{R}^P$ by $\mathcal{R}_P = (X_P, Y_P, R_P, R_P^\vee, P)$ where $X_P := X/\big(X \cap (R_P^\vee)^\perp\big)$ and $Y_P = Y \cap \mathbb{Q}R_P^\vee$. The root datum $\mathcal{R}_P$ is semisimple.

2.3. Label functions and root labels. A positive real label function is a length multiplicative function $q : W \to \mathbb{R}_+$. This means that $q(ww') = q(w)q(w')$ whenever $l(ww') = l(w) + l(w')$, and that $q(\omega) = 1$ for all $\omega \in \Omega$.

Such a function q is uniquely determined by its restriction to the set of affine simple reflections S^{aff}. By the braid relations and the action of Ω_f on S^{aff} it follows easily that $q(s) = q(s')$ whenever $s, s' \in S^{\mathrm{aff}}$ are W-conjugate. Hence there exists a unique W-invariant function $a \mapsto q_a$ on R^{aff} such that $q_{a+1} = q(s_a)$ for all simple affine roots $a \in F^{\mathrm{aff}}$.

We associate a possibly non-reduced root system R_{nr} with $\mathcal{R}$ by

$$R_{\mathrm{nr}} := R_0 \cup \{2\alpha \mid \alpha^\vee \in R_0^\vee \cap 2Y\}. \tag{2.3}$$

If $\alpha \in R_0$ then $2\alpha \in R_{\mathrm{nr}}$ if and only if the affine roots $a = \alpha^\vee$ and $a = \alpha^\vee + 1$ are not W-conjugate. Therefore we can also characterize the label function q on W by means of the following extension of the set of root labels $q_{\alpha^\vee}$ to arbitrary $\alpha \in R_{\mathrm{nr}}$. If $\alpha \in R_0$ with $2\alpha \in R_{\mathrm{nr}}$, then we define

$$q_{\alpha^\vee/2} := \frac{q_{\alpha^\vee+1}}{q_{\alpha^\vee}}. \tag{2.4}$$

With these conventions we have for all $w \in W_0$

$$q(w) = \prod_{\alpha \in R_{\mathrm{nr},+} \cap w^{-1} R_{\mathrm{nr},-}} q_{\alpha^\vee}. \tag{2.5}$$

We denote by $R_1 \subset X$ the reduced root system

$$R_1 := \{\alpha \in R_{\mathrm{nr}} \mid 2\alpha \notin R_{\mathrm{nr}}\}. \tag{2.6}$$

2.3.1. Restriction to parabolic subsystems. Let $P \in \mathscr{P}$. Both the non-reduced root system associated with $\mathscr{R}^P$ and the non-reduced root system associated with $\mathscr{R}_P$ are equal to $R_{P,nr} := \mathbb{Q}R_P \cap R_{\mathrm{nr}}$. We define a collection of root labels $q_{P,\alpha^\vee} = q^P_{\alpha^\vee}$ for $\alpha \in R_{P,\mathrm{nr}}$ by restricting the labels of R_{nr} to $R_{P,\mathrm{nr}} \subset R_{\mathrm{nr}}$. Then q_P denotes the length-multiplicative function on $W(\mathscr{R}_P)$ associated with this label function on $R_{P,\mathrm{nr}}$, and q^P denotes the associated length multiplicative function on $W(\mathscr{R}^P)$.

2.4. The Iwahori-Hecke algebra. Given a root datum $\mathscr{R}$ and a (positive real) label function q on the associated affine Weyl group W, there exists a unique associative complex Hecke algebra $\mathscr{H} = \mathscr{H}(\mathscr{R}, q)$ with $\mathbb{C}$-basis N_w indexed by $w \in W$, satisfying the relations:

(i) $N_{ww'} = N_w N_{w'}$ for all $w, w' \in W$ such that $l(ww') = l(w) + l(w')$.

(ii) $\left(N_s + q(s)^{-1/2}\right)\left(N_s - q(s)^{1/2}\right) = 0$ for all $s \in S^{\mathrm{aff}}$.

Notice that the algebra $\mathscr{H}$ is unital, with unit $1 = N_e$. Notice also that it follows from the defining relations that $N_w \in \mathscr{H}$ is invertible, for all $w \in W$.

By convention we assume that the label function q is of the form

$$q(s) = \mathbf{q}^{f_s}. \tag{2.7}$$

The parameters $f_s \in \mathbb{R}$ are fixed, and the base $\mathbf{q}$ satisfies $\mathbf{q} > 1$.

2.4.1. Isomorphisms between Hecke algebras. Suppose that $\phi : W \to W'$ is a length preserving isomorphism between two affine Weyl groups $W = W_0 \ltimes X$ and $W' = W_0' \ltimes X'$. Then $\phi(\Omega) = \Omega'$ and $\phi(S^{\mathrm{aff}}) = S'^{\mathrm{aff}}$. We define an affine Dynkin diagram isomorphism (also denoted by ϕ) from F^{aff} to F'^{aff} by requiring that $\phi(s_a) = s_{\phi(a)}$ for all $a \in F^{\mathrm{aff}}$. Suppose that we are given label functions q for W and q' for W'. Clearly, if $q'\left(\phi(s)\right) = q(s)$ for all $s \in S$, then ϕ induces an isomorphism of Hecke algebras $\psi : \mathscr{H}(\mathscr{R}, q) \to \mathscr{H}(\mathscr{R}', q')$ by $\psi(N_w) = N_{\phi(w)}$.

Observe that $X \subset W$ is precisely the set of elements in W which have finitely many conjugates. Therefore $\phi(X) = X'$, and $u := \phi|_X : X \to X'$ is an isomorphism of lattices. Let $a = \alpha^\vee + n$ be an affine root, and let $x \in X$ be arbitrary. Using the relation $s_a x s_a^{-1} = s_{\alpha^\vee}(x)$ it is easy to see that there exists an integer n' such that $\phi(s_a) = s_{u^\vee(\alpha^\vee)+n'}$, where $u^\vee$ denotes the inverse transpose of u. Hence there exists a weight λ of R_0 such that the action of ϕ on affine roots is given by $\phi(a)(x) = a\left(u^{-1}(x) + \lambda\right)$. The weight λ is uniquely determined by u. Since ϕ and ψ are thus completely determined by u, we will write ϕ_u and ψ_u.

In the special case where $u(F_0) = F_0'$ we have $\lambda = 0$. In this case u determines an isomorphism between the root data $\mathscr{R} = (X, Y, R_0, R_0^\vee, F_0)$ and $\mathscr{R}' = (X', Y', R_0', R_0'^{\vee}, F_0')$ (with the action on Y and on $R_0^\vee$ being given by $u^\vee$) which is compatible with the label functions. The restriction of ϕ to F^{aff} is now an isomorphism of affine Dynkin diagrams which is obtained by the unique affine extension of the isomorphism $u^\vee$ of finite type Dynkin diagrams. Conversely, every isomorphism u between two root data determines a length preserving isomorphism ϕ_u between the associated affine Weyl groups.

2.4.2. Bernstein presentation. There is another, extremely important presentation of the algebra $\mathscr{H}$, due to Joseph Bernstein (unpublished). Since the length function is additive on the dominant cone X^+, the map $X^+ \ni x \mapsto N_x$ is a homomorphism of the commutative monoid X^+ with values in $\mathscr{H}^\times$, the group of invertible elements of $\mathscr{H}$. Thus there exists a unique extension to a homomorphism $X \ni x \mapsto \theta_x \in \mathscr{H}^\times$ of the lattice X with values in $\mathscr{H}^\times$.

The abelian subalgebra of $\mathscr{H}$ generated by θ_x, $x \in X$, is denoted by $\mathscr{A}$. Let $\mathscr{H}_0 = \mathscr{H}(W_0, q_0)$ be the finite type Hecke algebra associated with W_0 and the restriction q_0 of q to W_0. Then the Bernstein presentation asserts that both the collections $\theta_x N_w$ and $N_w \theta_x$ $(w \in W_0, x \in X)$ are bases of $\mathscr{H}$, subject only to the cross relation (for all $x \in X$ and $s = s_\alpha$ with $\alpha \in F_0$):

$$(2.8) \qquad \theta_x N_s - N_s \theta_{s(x)}$$
$$= \begin{cases} (q_{\alpha^\vee}^{1/2} - q_{\alpha^\vee}^{-1/2}) \dfrac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}} & \text{if } 2\alpha \notin R_{\mathrm{nr}}, \\ \left((q_{\alpha^\vee/2}^{1/2} q_{\alpha^\vee}^{1/2} - q_{\alpha^\vee/2}^{-1/2} q_{\alpha^\vee}^{-1/2}) + (q_{\alpha^\vee}^{1/2} - q_{\alpha^\vee}^{-1/2})\theta_{-\alpha}\right) \dfrac{\theta_x - \theta_{s(x)}}{1 - \theta_{-2\alpha}} & \text{if } 2\alpha \in R_{\mathrm{nr}}. \end{cases}$$

2.4.3. The center $\mathscr{Z}$ of $\mathscr{H}$. An immediate consequence of the Bernstein presentation of $\mathscr{H}$ is the description of the center of $\mathscr{H}$:

Theorem 2.1. *The center $\mathscr{Z}$ of $\mathscr{H}$ is equal to $\mathscr{A}^{W_0}$. In particular, $\mathscr{H}$ is finitely generated over its center.*

As an immediate consequence we see that irreducible representations of $\mathscr{H}$ are finite dimensional by an application of (Dixmier's version of) Schur's Lemma.

We denote by T the complex torus $T = \mathrm{Hom}(X, \mathbb{C}^\times)$ of complex characters of the lattice X. The space $\mathrm{Spec}(\mathscr{Z})$ of complex homomorphisms of $\mathscr{Z}$ is thus canonically isomorphic to the (geometric) quotient $W_0 \backslash T$.

Thus, to an irreducible representation (V, π) of $\mathscr{H}$ we attach an orbit $W_0 t \in W_0 \backslash T$, called the central character of π.

2.4.4. Parabolic subalgebras and their semisimple quotients. We consider another important consequence of the Bernstein presentation of $\mathscr{H}$:

Proposition 2.2. (i) *The Hecke algebra $\mathscr{H}^P := \mathscr{H}(\mathscr{R}^P, q^P)$ is isomorphic to the subalgebra of $\mathscr{H}$ generated by $\mathscr{A}$ and the finite type Hecke subalgebra $\mathscr{H}(W_P) := \mathscr{H}(W_P, q|_{W_P})$.*

(ii) *We can view $\mathscr{H}_P := \mathscr{H}(\mathscr{R}_P, q_P)$ as a quotient of $\mathscr{H}^P$ via the surjective homomorphism $\phi_1 : \mathscr{H}^P \to \mathscr{H}_P$ characterized by* (1) *ϕ_1 is the identity on the finite type subalgebra $\mathscr{H}(W_P)$ and* (2) *$\phi_1(\theta_x) := \theta_{\bar{x}}$, where $\bar{x} \in X_P$ is the canonical image of x in $X_P = X/\big(X \cap (R_P^\vee)^\perp\big)$.*

Let T^P denote the character torus of the lattice $X/(X \cap \mathbb{Q}R_P)$. Then $T^P \subset T$ is a subtorus which is fixed for all the elements $w \in W_P$ and which is inside the simultaneous kernel of the $\alpha \in R_P$. The next result again follows simply from the Bernstein presentation:

Proposition 2.3. *There exists a family of automorphisms ψ_t $(t \in T^P)$ of $\mathscr{H}^P$, defined by $\psi_t(\theta_x N_w) = x(t)\theta_x N_w$.*

We use the above family of automorphisms to twist the projection $\phi_1 : \mathscr{H}^P \to \mathscr{H}_P$. Given $t \in T^P$, we define the epimorphism $\phi_t : \mathscr{H}^P \to \mathscr{H}_P$ by $\phi_t := \phi_1 \circ \psi_t$.

2.5. Intertwining elements. Let $s = s_\alpha \in S_0$ with $\alpha \in F_1$. We define an "intertwining element" $\imath_s \in \mathscr{H}$ as follows:

$$\begin{aligned}\imath_s &= (1-\theta_{-\alpha})N_s + \big((q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1/2} - q_{\alpha^\vee}^{1/2}q_{2\alpha^\vee}^{1/2}) + (q_{2\alpha^\vee}^{-1/2} - q_{2\alpha^\vee}^{1/2})\theta_{-\alpha/2}\big)\\ &= N_s(1-\theta_\alpha) + \big((q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1/2} - q_{\alpha^\vee}^{1/2}q_{2\alpha^\vee}^{1/2})\theta_\alpha + (q_{2\alpha^\vee}^{-1/2} - q_{2\alpha^\vee}^{1/2})\theta_{\alpha/2}\big).\end{aligned}$$

(If $\alpha/2 \notin X$ then we put $q_{2\alpha^\vee} = 1$; see Remark 9.1.) We recall from [25], Theorem 2.8 that these elements satisfy the braid relations, and they satisfy (for all $x \in X$)

$$\imath_s\theta_x = \theta_{s(x)}\imath_s. \tag{2.9}$$

Let $\mathscr{Q}$ denote the quotient field of the centre $\mathscr{Z}$ of $\mathscr{H}$, and let ${}_{\mathscr{Q}}\mathscr{H}$ denote the $\mathscr{Q}$-algebra ${}_{\mathscr{Q}}\mathscr{H} = \mathscr{Q} \otimes_{\mathscr{Z}} \mathscr{H}$. Inside ${}_{\mathscr{Q}}\mathscr{H}$ we normalize the elements $\imath_s$ as follows.

We first introduce

$$n_\alpha := q_{\alpha^\vee}^{1/2}q_{2\alpha^\vee}^{1/2}(1 + q_{\alpha^\vee}^{-1/2}\theta_{-\alpha/2})(1 - q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1}\theta_{-\alpha/2}) \in \mathscr{A}. \tag{2.10}$$

Then the normalized intertwiners $\imath_s^0$ $(s \in S_0)$ are defined by (with $s = s_\alpha$, $\alpha \in R_1$):

$$\imath_s^0 := n_\alpha^{-1}\imath_s \in {}_{\mathscr{Q}}\mathscr{H}. \tag{2.11}$$

It is known that the normalized elements $\imath_s^0$ satisfy $(\imath_s^0)^2 = 1$. In particular, $\imath_s^0 \in {}_{\mathscr{Q}}\mathscr{H}^\times$, the group of invertible elements of ${}_{\mathscr{Q}}\mathscr{H}$. In fact we have:

Lemma 2.4 ([26], Lemma 4.1). *The map $S_0 \ni s \mapsto \imath_s^0 \in {}_{\mathscr{Q}}\mathscr{H}^\times$ extends (uniquely) to a homomorphism $W_0 \ni w \mapsto \imath_w^0 \in {}_{\mathscr{Q}}\mathscr{H}^\times$. Moreover, for all $f \in {}_{\mathscr{Q}}\mathscr{A}$ we have that $\imath_w^0 f \imath_{w^{-1}}^0 = f^w$.*

2.6. Formal completion of $\mathscr{H}$ and Lusztig's Structure Theorem. Let $t \in T$, and let $\mathscr{I}_t$ denote the maximal ideal of $\mathscr{Z}$ associated with the orbit $W_0 t$. We denote by $\bar{\mathscr{Z}}_{W_0 t}$ the $\mathscr{I}_t$-adic completion of $\mathscr{Z}$. In [19] Lusztig considered the structure of the completion

$$\bar{\mathscr{H}}_t := \bar{\mathscr{Z}}_{W_0 t} \otimes_{\mathscr{Z}} \mathscr{H}. \tag{2.12}$$

We will use Lusztig's results on the structure of this formal completion (in a slightly adapted version) for so called R_P-generic points $t \in T$.

2.6.1. R_P-generic points of T. Let $R_P \subset R_0$ be a parabolic subset of roots, i.e., $R_P = \mathbb{R}R_P \cap R_0$. Let us recall the notion of an R_P-generic point $t \in T$ (cf. [26], Definition 4.12). To $t \in T$ we associate $R_{P(t)} \subset R_0$, the smallest parabolic subset containing all roots $\alpha \in R_0$ for which one of the following statements holds (where c_α denotes the Macdonald c-function, cf. equation (9.2)):

(i) $c_\alpha \notin \mathcal{O}_t^\times$ (the invertible holomorphic germs at t).

(ii) $\alpha(t) = 1$.

(iii) $\alpha(t) = -1$ and $\alpha \notin 2X$.

We say that $t_1, t_2 \in T$ are equivalent if there exists a $w \in W_{P(t_1)} := W(R_{P(t_1)})$ such that $t_2 = w(t_1)$. Notice that in this case $R_{P(t_1)} = R_{P(t_2)}$, so that this is indeed an equivalence relation. The equivalence class of $t \in T$ is equal to the orbit $\varpi = W_{P(t)}t \subset W_0 t$.

We define $P(t)$ as the basis of simple roots of $R_{P(t)}$ inside $R_{0,+}$, and we sometimes use the notation $P(\varpi)$ instead of $P(t)$.

Definition 2.5. We call $t \in T$ an R_P-generic point if $wt \in \varpi$ (with $w \in W_0$) implies that $w \in W_P$.

Remark 2.6. Notice that if $t \in T$ is R_P-generic then $R_{P(t)} \subset R_P$, but not conversely.

2.6.2. Lusztig's First Reduction Theorem. Let $P \subset F_0$, and let t be R_P-generic such that $P(t) = P$. This implies in particular that $\varpi = W_P t$. Lusztig ([19], Subsection 8.7) associates idempotents $e_{w\varpi} \in \overline{\mathcal{H}}_t$ with the equivalence classes $w\varpi \in W_0 t$ (in the notation of Lusztig these elements are denoted by $1_{w\varpi}$). By Lusztig's First Reduction Theorem (cf. [19]) we know that if $u, v \in W^P$, then $\imath_u^0 e_\varpi \imath_{v^{-1}}^0$ is a well defined element of $\overline{\mathcal{H}}_t$, and that we have the decomposition (compare with [26], equation (4.46))

$$\overline{\mathcal{H}}_t = \bigoplus_{u,v \in W^P} \imath_u^0 e_\varpi \overline{\mathcal{H}}_t^P \imath_{v^{-1}}^0, \tag{2.13}$$

where $\overline{\mathcal{H}}_t^P$ denotes the completion of $\mathcal{H}^P$ at $\varpi = W_P t$. Moreover, the subspace $\imath_u^0 e_\varpi \overline{\mathcal{H}}_t^P \imath_{v^{-1}}^0$ is equal to $e_{u\varpi}\overline{\mathcal{H}}_t e_{v\varpi}$. When $u = v$ then this is a subalgebra of $\overline{\mathcal{H}}_t$, and when $u = v = e$ then this subalgebra reduces to $e_\varpi \overline{\mathcal{H}}_t^P$, which is isomorphic to $\overline{\mathcal{H}}_t^P$ via $x \mapsto e_\varpi x (= x e_\varpi)$.

Finally, assume that $u = v \in W^P$ is such that $u(P) = Q \subset F_0$. Then u naturally extends to an isomorphism $u : \mathcal{R}^P \to \mathcal{R}^Q$ of root data which is compatible with the label functions q^P and q^Q (since $u \in W$). By 2.4.1 there exists an isomorphism of affine Hecke algebras $\psi_u : \mathcal{H}^P \to \mathcal{H}^Q$. This isomorphism gives rise, by continuity, to an isomorphism (also denoted by ψ_u) $\psi_u : \overline{\mathcal{H}}_t^P \to \overline{\mathcal{H}}_{u(t)}^Q$. Lusztig's Theorem also asserts that for all $x \in \overline{\mathcal{H}}_t^P$, we have the formula

$$\imath_u^0 (e_\varpi x) \imath_{u^{-1}}^0 = e_{u\varpi} \psi_u(x). \tag{2.14}$$

We will use these results of Lusztig in the situation that $t \in T$ is of the form $t = r_P t^P$ with $W_P r_P \subset T_P$ the central character of a discrete series representation (V_δ, δ) (see Definition 2.7), and $t^P \in T^P$ (this is the case if $W_0 t \subset T$ is the central character of a representation which is induced from (V_δ, δ)). In this situation $r_P \in T_P$ is a so-called (R_P, q_P)-residual point (see Definition 9.3, Theorem 2.10). Therefore, $R_{P(t)} \supset R_P$ ([26], Proposition 7.3), and $R_{P(t)} = R_P$ for an open dense subset of T^P (the complement of a subvariety of codimension 1 in T^P). Thus if $t = r_P t^P$ is R_P-generic in this situation, then indeed $P(t) = P$, as required.

2.6.3. Application. We will use the above result (2.13) when analyzing a finite functional $f \in \mathbb{A}$ (cf. section 3.2) or a representation π of $\mathscr{H}$ which contains a power $\mathscr{I}_t^n$ of $\mathscr{I}_t$ in its kernel.

We can then view f (or π) as a linear function on the quotient $\mathscr{H}/\mathscr{I}_t^n\mathscr{H}$. Since this quotient is finite dimensional (by Theorem 2.1), we have

$$\mathscr{H}/\mathscr{I}_t^n\mathscr{H} = \overline{\mathscr{H}}_t/\mathscr{I}_t^n\overline{\mathscr{H}}_t. \tag{2.15}$$

In this way we can view f (resp., π) as a functional (resp., representation) of the completion $\overline{\mathscr{H}}_t$. For example, this applies when W_0t is the central character of an irreducible representation π. We can view π as a representation of the quotient $\mathscr{H}^t := \mathscr{H}/\mathscr{I}_t\mathscr{H}$ (the case $n = 1$ of (2.15)), and the matrix coefficients of π can be viewed as functionals on $\mathscr{H}^t$.

2.7. Hilbert algebra structure on $\mathscr{H}$. The anti-linear map $h \mapsto h^*$ defined by $\left(\sum_w c_w N_w\right)^* = \sum_w \bar{c}_{w^{-1}} N_w$ is an anti-involution of $\mathscr{H}$. Thus it gives $\mathscr{H}$ the structure of an involutive algebra.

In the context of involutive algebras we can also arrange Schur's Lemma for topologically irreducible representations (cf. [9]). Thus the topologically irreducible representations of the involutive algebra $(\mathscr{H}, *)$ are finite dimensional by Theorem 2.1.

The linear functional $\tau : \mathscr{H} \to \mathbb{C}$ given by $\tau\left(\sum_w c_w N_w\right) = c_e$ is a positive trace for the involutive algebra $(\mathscr{H}, *)$. The basis N_w of $\mathscr{H}$ is orthonormal with respect to the pre-Hilbert structure $(x, y) := \tau(x^*y)$ on $\mathscr{H}$. We denote the Hilbert completion of $\mathscr{H}$ with respect to $(\cdot\,,\cdot)$ by $L_2(\mathscr{H})$. This is a separable Hilbert space with Hilbert basis N_w $(w \in W)$.

Let $x \in \mathscr{H}$. The operators $\lambda(x) : \mathscr{H} \to \mathscr{H}$ (given by $\lambda(x)(y) := xy$) and $\rho(x) : \mathscr{H} \to \mathscr{H}$ (given by $\rho(x)(y) := yx$) extend to $B\big(L_2(\mathscr{H})\big)$, the algebra of bounded operators on $L_2(\mathscr{H})$. This gives $\mathscr{H}$ the structure of a Hilbert algebra (cf. [9]).

The operator norm completion of $\lambda(\mathscr{H}) \subset B\big(L_2(\mathscr{H})\big)$ is a C^*-algebra which we call the reduced C^*-algebra $C_r^*(\mathscr{H})$ of $\mathscr{H}$ (cf. [26], Definition 2.4). The natural action of $C_r^*(\mathscr{H})$ on $L_2(\mathscr{H})$ via λ (resp., ρ) is called the left regular (resp., right regular) representation of $C_r^*(\mathscr{H})$. Since it has only finite dimensional irreducible representations by the above remark, $C_r^*(\mathscr{H})$ is of type I.

The norm $\|x\|_o$ of $x \in C_r^*(\mathscr{H})$ is by definition equal to the norm of $\lambda(x) \in B\big(L_2(\mathscr{H})\big)$. Observe that the map $x \mapsto \lambda(x)N_e$ defines an embedding

$$C_r^*(\mathscr{H}) \subset L_2(\mathscr{H}). \tag{2.16}$$

2.8. Discrete series representations.

Definition 2.7. We call an irreducible representation (V_δ, δ) of $(\mathscr{H}, *)$ a discrete series representation if (V_δ, δ) is equivalent to a subrepresentation of $\big(L_2(\mathscr{H}), \lambda\big)$. We denote by $\Delta = \Delta_{\mathscr{R},q}$ a complete set of representatives of the equivalence classes of the irreducible discrete series representations of $(\mathscr{H}, *)$. When $r \in T$ is given, $\Delta_{W_0 r} \subset \Delta$ denotes the subset of Δ consisting of irreducible discrete series representations with central character $W_0 r$ $(r \in T)$.

Corollary 2.8 (of Theorem 2.1). *$\Delta_{W_0 r}$ is a finite set.*

There is an important characterization of the discrete series representations due to Casselman. This characterization has consequences for the growth behaviour of matrix coefficients of discrete series representations. Recall that T denotes the complex algebraic torus of characters of the lattice X. It has polar decomposition $T = T_{rs} T_u$ where T_{rs} is the real split form of T, and T_u the compact form. If $t \in T$ we denote by $|t| \in T_{rs}$ its real split part.

Theorem 2.9 (Casselman's criterion for discrete series representations, cf. [26], Lemma 2.22). *Let (V_δ, δ) be an irreducible representation of $\mathscr{H}$. The following are equivalent:*

(i) *(V_δ, δ) is a discrete series representation.*

(ii) *All matrix coefficients of δ belong to $L_2(\mathscr{H})$.*

(iii) *The character χ of δ belongs to $L_2(\mathscr{H})$.*

(iv) *The weights $t \in T$ of the generalized $\mathscr{A}$-weight spaces of V_δ satisfy: $|x(t)| < 1$, for all $0 \neq x \in X^+$.*

(v) *$Z_X = \{0\}$, and there exists an $\epsilon > 0$ such that for all matrix coefficients m of δ, there exists a $C > 0$ such that the inequality $|m(N_w)| < C\mathbf{q}^{-\epsilon l(w)}$ holds.*

We have the following characterization of the set of central characters of irreducible discrete series representations. For the notion of "residual points" of T we refer the reader to Definition 9.3.

Theorem 2.10 (cf. [26], Lemma 3.31 and Corollary 7.12). *The set $\Delta_{W_0 r}$ is nonempty if and only if $r \in T$ is a residual point. In particular, Δ is finite, and empty unless $Z_X = 0$.*

2.9. The Schwartz algebra; tempered representations. We define norms p_n $(n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\})$ on $\mathscr{H}$ by

$$p_n(h) = \sup_{w \in W} |(N_w, h)| \big(1 + \mathscr{N}(w)\big)^n, \tag{2.17}$$

and we define the Schwartz completion $\mathscr{S}$ of $\mathscr{H}$ by

$$\mathscr{S} := \left\{ x = \sum_w x_w N_w \in \mathscr{H}^* \;\middle|\; p_n(x) < \infty \ \forall n \in \mathbb{Z}_+ \right\}. \tag{2.18}$$

In [26], Theorem 6.5, it was shown that the multiplication operation of $\mathscr{H}$ is continuous with respect to the family p_n of norms. The completion $\mathscr{S}$ is a (nuclear, unital) Fréchet algebra (cf. [26], Definition 6.6).

As an application of [26], Theorem 6.1, it is easy to see that there exist constants $D \in \mathbb{Z}_+$ and $C > 0$ such that $\|h\|_o \leqq Cp_D(h)$ for all $h \in \mathscr{H}$. Thus we have a continuous embedding

$$\mathscr{S} \subset C_r^*(\mathscr{H}). \tag{2.19}$$

The subalgebra $\mathscr{S}$ is clearly dense and symmetric (i.e., $\mathscr{S}^* = \mathscr{S}$). The Main Theorem 5.3 can be viewed as a structure theorem for this Fréchet algebra via the Fourier transform.

Definition 2.11. The topological dual $\mathscr{S}'$ is called the space of tempered functionals. A finite dimensional, continuous representation of $\mathscr{S}$ is called a tempered representation. By abuse of terminology, we call a finite dimensional representation of $\mathscr{H}$ tempered if it extends continuously to $\mathscr{S}$.

In particular, a finite dimensional representation (V, π) of $\mathscr{H}$ is tempered if and only if the matrix coefficient $h \mapsto \phi\big(\pi(h)v\big)$ extends continuously to $\mathscr{S}$ for all $\phi \in V^*$ and $v \in V$.

Remark 2.12. By the proof of Corollary 5.9 the group of units $\mathscr{S}^\times$ is open in the Fréchet algebra $\mathscr{S}$. Therefore automatic continuity applies so that actually any finite dimensional representation of $\mathscr{S}$ is continuous (compare the reasoning in the Appendix by Schneider and Stuhler of [31], p. 205).

Observe that the involution $*$ of $\mathscr{H}$ extends continuously to $\mathscr{S}$. As a consequence we have:

Proposition 2.13. *Let $(V, \pi) \to (V^\circ, \pi^\circ)$ denote the duality functor defined on the category of finite dimensional modules of $\mathscr{H}$ as follows: V° denotes the conjugate linear dual of V, equipped with the $\mathscr{H}$-action defined by $\pi^\circ(h)(\phi)(v) := \phi\big(\pi(h^*)v\big)$. This functor is contravariant exact, and $V^{\circ\circ} \simeq V$. The duality restricts to a duality on the category of tempered modules.*

3. Tempered representations

In this section we collect general facts about tempered representations and their matrix coefficients, the tempered finite functionals. We first discuss Casselman's criterion for temperedness, and then parabolic induction for tempered representations.

3.1. Finite functionals.

3.1.1. Algebraic dual of $\mathscr{H}$. We identify the algebraic dual $\mathscr{H}^*$ of $\mathscr{H}$ with formal linear combinations $f = \sum_{w \in W} d_w N_w$ via the sesquilinear pairing $(\cdot, \cdot)$ defined by $(x, y) = \tau(x^* y)$. Thus $f(x) = (f^*, x)$ and $d_w = f(N_{w^{-1}})$. For $x, y \in \mathscr{H}$ and $f \in \mathscr{H}^*$ we define $R_x(f)(y) := f(yx)$ and $L_x(f)(y) := f(xy)$ (a right representation of $\mathscr{H}$). Note that in terms of multiplication of formal series we have: $R_x(f) = x.f$ and $L_x(f) = f.x$.

3.1.2. Finite functionals. Let $\mathbb{A}(\mathscr{H})$ or simply $\mathbb{A}$ denote the subspace of $\mathscr{H}^*$ consisting of *finite* linear functionals on $\mathscr{H}$:

Definition 3.1. The space $\mathbb{A}$ consists of all the elements $f \in \mathscr{H}^*$ such that the space $V_f := \mathscr{H}.f.\mathscr{H}$ is finite dimensional.

Remark 3.2. $f \in \mathbb{A}$ is a coefficient of (V_f, R). Hence $\mathbb{A}$ is the space of (or equivalently set of) coefficients of finite dimensional representations of $\mathscr{H}$.

Since $\mathscr{H}$ is finitely generated over its center $\mathscr{Z}$, f is finite if and only if $\dim(f.\mathscr{Z}) < \infty$. Recall that $\mathscr{A}$ denotes the abelian subalgebra of $\mathscr{H}$ spanned by the elements θ_x with $x \in X$. Since $\mathscr{Z} \subset \mathscr{A}$ we see that $f \in \mathbb{A}$ if and only if $\dim(\mathscr{A}.f) < \infty$ if and only if $\dim(f.\mathscr{A}) < \infty$.

3.2. Exponents of finite functionals.

Definition 3.3. We say that $t \in T$ is an exponent of $f \in \mathbb{A}$ if the X-module on the finite dimensional space $V = f.\mathscr{H}$ (the space of left translates of f) defined via $x \mapsto L_{\theta_x}|_V$ contains a (generalized) weight space with weight t.

Proposition 3.4. *Let $f \in \mathbb{A}$ and let ϵ denote the set of exponents of f. There exist unique functions E_t^f $(t \in \epsilon)$ on $\mathscr{H} \times X$, polynomial in X, such that*

$$f(\theta_x h) = \sum_{t \in \epsilon} E_t^f(h, x) t(x). \tag{3.1}$$

Proof. Uniqueness: Suppose that we have a finite set ϵ of exponents and for each $t \in \epsilon$ a polynomial function $x \mapsto E_t(x)$ of X such that

$$\sum_{t \in \epsilon} E_t(x) t(x) \equiv 0.$$

Suppose that there exists a $t \in \epsilon$ such that $x \mapsto E_t(x)$ has positive degree. We apply the difference operator $\Delta_{t,y}$ $(t \in \epsilon, y \in X)$ defined by

$$\Delta_{t,y}(f)(x) := t(y)^{-1} f(x+y) - f(x).$$

It is easy to see that for a suitable choice of y this operator lowers the degree of the coefficient of t by 1, and leaves the degrees of the other coefficients invariant. Hence, if we assume that not all of the coefficients E_t are zero, we obtain a nontrivial complex linear relation of characters of X, after applying a suitable sequence of operators $\Delta_{s,z}$. This is a contradiction.

Existence: We fix $h \in \mathscr{H}$ and we decompose f according to generalized L_X-eigenspaces in V. We may replace f by one of its constituents, and thus assume that $\epsilon = \{t\}$. We may replace the action of X by the action $L'_x = t(x)^{-1} L_x$. Therefore it is enough to consider the case $t = 1$. Let N denote the dimension of V. By Engel's theorem applied to the commuting unipotent elements L_{θ_x} acting in V, we see that any product of N or more difference operators of the form $\Delta_y = L_{\theta_y} - 1$ is equal to zero in V. By induction on N this implies that for any h, the function $x \mapsto f(\theta_x h)$ is a polynomial in x of degree at most $N - 1$. $\square$

Corollary 3.5. *We have $E_t^f(\theta_x h, y) = t(x) E_t^f(h, x+y)$. In particular, the degree of the polynomial $E_t^f(h, x)$ is uniformly bounded as a function of h.*

Corollary 3.6. *Put $f_t(h) = E_t^f(h, 0)$. Then f_t is the component of f corresponding to the generalized L_X-eigenspace with eigenvalue t in V. Observe that $f_t(\theta_x h) = t(x)E_t^f(h, x)$, and that $f_t \in f \cdot \mathscr{A} = L_X(f) \subset V \subset \mathbb{A}$.*

3.3. The space $\mathbb{A}^{\mathbf{temp}}$ of tempered finite functionals. If $f \in \mathbb{A}$, we can express the condition $f \in \mathscr{S}'$ (temperedness) or $f \in L_2(\mathscr{H})$ (square integrability) in terms of a system of inequalities on the set of exponents ϵ of f. This is the content of the Casselman conditions for temperedness ([26], Lemma 2.20). We will formulate these results below, adapted to suit the applications we have in mind (Section 3.7).

Given $P \in \mathscr{P}$ we define a partial ordering $\leqq_P$ on exponents as follows:

Definition 3.7. Let $P \in \mathscr{P}$, and let R_P be the standard parabolic subsystem with that subset. For real characters t_1, t_2 on X we say that $t_1 \leqq_P t_2$ if and only if $t_1(x) \leqq t_2(x)$ for all $x \in X^{P,+} := \{x \in X \mid \forall \alpha \in P : \langle x, \alpha^\vee \rangle \geqq 0\}$. In other words, $t_1 \leqq_{F_0} t_2$ means that $t_1(x) \leqq t_2(x)$ for all $x \in X^+$, and in general $t_1 \leqq_P t_2$ if and only if both $t_1 \leqq_{F_0} t_2$ and $t_1|_{X \cap P^\perp} = t_2|_{X \cap P^\perp}$.

Thus $t_1 \leqq_P t_2$ if and only if $t_1 t_2^{-1} = \prod_{\alpha \in P} (d_\alpha \otimes \alpha^\vee)$ with $0 < d_\alpha \leqq 1$, where $d \otimes \alpha^\vee \in T_{rs}$ is the real character defined by $d \otimes \alpha^\vee(x) = d^{\langle x, \alpha^\vee \rangle}$.

Let (V, π) be a finite dimensional representation of $\mathscr{H}$. It follows easily from Definition 3.3 that the union of the sets of exponents of the matrix coefficients $h \mapsto \phi\big(\pi(h)v\big)$ of π coincides with the set of weights t of the generalized $\mathscr{A}$-weight spaces of V. Using [26], Lemma 2.20, we get:

Corollary 3.8 ([26], Lemma 2.20, Casselman's criterion for temperedness). *Let (V, π) be a finite dimensional representation of $\mathscr{H}$. The following statements are equivalent*:

(i) *(V, π) is tempered.*

(ii) *The weights t of the generalized $\mathscr{A}$-weight spaces of V satisfy $|t| \leqq_{F_0} 1$.*

(iii) *The exponents t of the matrix coefficients $h \mapsto \phi\big(\pi(h)v\big)$ of π satisfy $|t| \leqq_{F_0} 1$.*

Let $f \in \mathbb{A}$. The space of matrix coefficients of the finite dimensional representation $\big(V_f := R_{\mathscr{H}}(f), R\big)$ is the space $\mathscr{H} \cdot f \cdot \mathscr{H}$. Hence the union of the sets of exponents of the matrix coefficients of V_f is equal to the set of exponents of f. Hence we obtain:

Corollary 3.9 (Casselman's temperedness condition for functionals). *We have $f \in \mathbb{A}^{\mathrm{temp}} := \mathbb{A} \cap \mathscr{S}'$ if and only if the real part $|t|$ of every exponent t of f satisfies $|t| \leqq_{F_0} 1$.*

Definition 3.10. We put $\mathbb{A}_{2,\mathrm{modc}}$ for the subspace of $\mathbb{A}^{\mathrm{temp}}$ consisting of those f such that all exponents t of f satisfy $|t| = \prod_{\alpha \in F_0} (d_a \otimes \alpha^\vee)$ with $0 < d_\alpha < 1$.

In other words, $f \in \mathbb{A}_{2,\mathrm{modc}}$ if and only if $f \in \mathbb{A}^{\mathrm{temp}}$, and for any of the exponents t of f the following statement holds: If $P \in \mathscr{P}$ is such that $|t| \leqq_P 1$ then $P = F_0$.

Then Theorem 2.9 implies that:

Corollary 3.11. (i) *$\mathbb{A}_2 := \mathbb{A} \cap L_2(\mathscr{H}) \neq 0$ only if $Z_X = 0$, and in this case, $\mathbb{A}_2 = \mathbb{A}_{2,\mathrm{modc}}$.*

(ii) *Let $\omega \in T^{F_0}$. Suppose that $f \in \mathbb{A}(\mathscr{H})$ factors through the morphism $\phi_\omega : \mathscr{H} \to \mathscr{H}_{F_0}$ defined after Proposition 2.3 in an element $f_\omega \in \mathbb{A}(\mathscr{H}_{F_0})$. Then $f \in \mathbb{A}_{2,\mathrm{modc}}(\mathscr{H})$ if and only if $f_\omega \in \mathbb{A}_2(\mathscr{H}_{F_0})$ and $\omega \in T_u^{F_0}$.*

3.4. Induction from standard parabolic subquotient algebras. In this subsection we discuss the technique of parabolic induction of tempered representations.

Let $P \subset F_0$ and let $W_P \subset W_0$ be the standard parabolic subgroup of W_0 generated by the simple reflections s_α with $\alpha \in P$. Let $\mathscr{H}^P \subset \mathscr{H}$ be the subalgebra $\mathscr{H}^P := \mathscr{H}(W_P) \cdot \mathscr{A} \subset \mathscr{H}$, and let $\mathscr{H}_P$ denote the quotient of $\mathscr{H}^P$ by the (two sided) ideal generated by the central elements $\theta_x - 1$ where $x \in X$ is such that $\langle x, \alpha^\vee \rangle = 0$ for all $\alpha \in P$. Then $\mathscr{H}_P$ is again an affine Hecke algebra, with root datum $\mathscr{R}_P = (R_P, X_P, R_P^\vee, Y_P, P)$, where $X_P = X/P^{\vee,\perp}$ and $Y_P = Y \cap \mathbb{R}P^\vee$, and root labels q_P that are obtained by restriction from R_{nr} to $R_{P,\mathrm{nr}}$.

There exists a parameter family of homomorphisms $\phi_{t^P} : \mathscr{H}^P \to \mathscr{H}_P$ with $t^P \in T^P \subset T$, the subtorus with character lattice $X^P = X/(X \cap \mathbb{R}P)$, defined by $\phi_{t^P}(\theta_x T_w) = x(t^P)\theta_{\bar{x}} T_w$, where $\bar{x} \in X_P$ denotes the canonical image of x in X_P. The kernel of ϕ_{t^P} is the two-sided ideal generated by elements of the form $x(t^P)^{-1}\theta_x - 1$, with $x \in X$ such that $\langle x, \alpha^\vee \rangle = 0$ for all $\alpha \in P$.

Let (V_δ, δ) be a discrete series representation of the subquotient Hecke algebra $\mathscr{H}_P$. Let $W_P r_P$ be the central character of δ. It is known that r_P is a residual point of T_P (cf. [26], Lemma 3.31), the subtorus of T with character lattice X_P.

Theorem 3.12 ([26], Proposition 4.19 and Proposition 4.20). *Let $t^P \in T_u^P$, and let δ_{t^P} denote the lift to $\mathscr{H}^P$ of δ via ϕ_{t^P}. Then the induced representation $\pi = \pi(\mathscr{R}_P, W_P r_P, \delta, t^P)$ from the representation δ_{t^P} of $\mathscr{H}^P$ to $\mathscr{H}$ is a unitary, tempered representation.*

Remark 3.13. If (V, δ) is a finite dimensional representation of $\mathscr{H}^P$ let us denote by $(i_P(V), i_P(\delta))$ the induced representation of (V, δ) to $\mathscr{H}$. Then one sees, as in the references for the previous theorem, that, if δ is tempered, then $i_P(\delta)$ is tempered.

3.4.1. Compact realization of $\pi(R_P, W_P r_P, \delta, t^P)$. Put $\mathscr{H}(W^P) \subset \mathscr{H}$ for the finite dimensional linear subspace of $\mathscr{H}$ spanned by the elements N_w with $w \in W^P$. Then

$$\mathscr{H} \simeq \mathscr{H}(W^P) \otimes \mathscr{H}^P, \tag{3.2}$$

where the isomorphism is realized by the product map. Therefore we have the isomorphism

$$\mathscr{H} \otimes_{\mathscr{H}^P} V_\delta \simeq i(V_\delta) := \mathscr{H}(W^P) \otimes V_\delta. \tag{3.3}$$

We will use this isomorphism to identify the representation space of $\pi(P, W_P r_P, \delta, t^P)$ with $i(V_\delta)$. This realization of the induced representation is called the *compact realization*, by analogy with induced representations for reductive groups.

According to [26], Proposition 4.19, the representation $\pi(P, W_P r_P, \delta, t^P)$ is unitary (i.e., a $*$-representation) with respect to the Hermitian inner product

$$\langle h_1 \otimes v_1, h_2 \otimes v_2 \rangle = \tau(h_1^* h_2)(v_1, v_2), \tag{3.4}$$

where (v_1, v_2) denotes the inner product on the representation space V_δ of the discrete series representation (V_δ, δ).

More generally, for $t^P \in T^P$ the Hermitian form $\langle \cdot, \cdot \rangle$ on $i(V_\delta)$ defines a nondegenerate sesquilinear pairing of $\mathscr{H}$-modules as follows:

$$\pi(P, W_P r_P, \delta, \overline{t^P}^{-1}) \times \pi(P, W_P r_P, \delta, t^P) \to \mathbb{C}. \tag{3.5}$$

3.5. Groupoid of tempered standard induction data. Let $\mathscr{P}$ denote the power set of F_0. Let Ξ (respectively Ξ_u) denote the set of all triples $\xi = (P, \delta, t^P)$ with $P \in \mathscr{P}$, δ an irreducible discrete series representation of $\mathscr{H}_P$ (with underlying vector space V_δ), and $t^P \in T^P$ (respectively $t^P \in T_u^P$). We denote the central character of δ by $W_P r_P$.

Let $\mathscr{W}$ denote the finite groupoid whose set of objects is $\mathscr{P}$ and such that the set of arrows from P to Q $(P, Q \in \mathscr{P})$ consists of $K_Q \times W(P, Q)$, where $K_Q = T_Q \cap T^Q$ and $W(P, Q) = \{w \in W_0 \mid w(P) = Q\}$. The composition of arrows is defined by $(k_1, w_1)(k_2, w_2) = \big(k_1 w_1(k_2), w_1 w_2\big)$. This groupoid acts on Ξ as follows. An element $g = k \times n \in K_Q \times W(P, Q)$ of $\mathscr{W}_\Xi$ defines an algebra isomorphism $\psi_g : \mathscr{H}_P \to \mathscr{H}_Q$ as follows. An element $n \in W(P, Q)$ defines an isomorphism from the root datum $(\mathscr{R}_P, q_P)$ to $(\mathscr{R}_Q, q_Q)$ compatible with q_P and q_Q, which determines a Hecke algebra isomorphism ψ_n as in 2.4.1. On the other hand, if $k \in K_Q$ then $\psi_k : \mathscr{H}_Q \to \mathscr{H}_Q$ is the automorphism defined by $\psi_k(\theta_x N_w) = k(x)\theta_x N_w$. Then ψ_g is defined by the composition of these isomorphisms. We obtain a bijection $\Psi_g : \Delta_{W_P r_P} \to \Delta_{k^{-1} W_Q n(r_P)}$ (where $\Delta_{W_P r_P} = \Delta_{P, W_P r_P}$ denotes a complete set of representatives for the equivalence classes of irreducible discrete series representations of $\mathscr{H}_P$ with central character $W_P r_P$) characterized by the requirement $\Psi_g(\delta) \simeq \delta \circ \psi_g^{-1}$. The action of $\mathscr{W}$ on Ξ is defined by: $g(P, \delta, t^P) = \big(Q, \Psi_g(\delta), g(t^P)\big)$, with $g(t^P) := kn(t^P)$.

Definition 3.14. The fibred product $\mathscr{W}_\Xi = \mathscr{W} \times_{\mathscr{P}} \Xi$ is called the groupoid of standard induction data. The full compact subgroupoid $\mathscr{W}_{\Xi_u} = \mathscr{W} \times_{\mathscr{P}} \Xi_u$ is called the groupoid of tempered standard induction data.

Definition 3.15. An element $\xi = (P, \delta, t^P) \in \Xi$ is called generic if $t = r_P t^P$ is R_P-generic (cf. Definition 2.5), where $r_P \in T_P$ is such that $W_P r_P$ is the central character of δ. Notice that the set of non-generic $\xi \in \Xi_{P,\delta}$ is a proper Zariski-closed subset of $\Xi_{P,\delta}$.

The groupoid $\mathscr{W}_{\Xi_u}$ was introduced in [26] (but was denoted by $\mathscr{W}_\Xi$ there) and plays an important role in the theory of the Fourier transform for $\mathscr{H}$. It is easy to see that $\mathscr{W}_\Xi$ is a smooth analytic, étale groupoid, whose set of objects is equal to Ξ. Thus $\mathscr{W}_\Xi$ is a union of complex algebraic tori, and therefore we can speak of polynomial and rational functions on Ξ and on $\mathscr{W}_\Xi$. This also applies to the full compact subgroupoid $\mathscr{W}_{\Xi_u}$.

[26], Theorem 4.38 states that there exists an induction functor

$$\pi : \mathscr{W}_{\Xi_u} \to \mathrm{PRep}_{\mathrm{unit,temp}}(\mathscr{H}),$$

where the target groupoid is the category of finite dimensional, unitary, tempered representations of $\mathscr{H}$ in which the morphisms are given by unitary intertwining isomorphisms modulo the action of scalars. The image of $\xi = (P, \delta, t^P) \in \Xi_u$ is the representation $\pi(\xi) := \pi(P, W_P r_P, \delta, t^P)$ of $\mathscr{H}$, in its compact realization, as was defined in subsection 3.4.1.

The intertwining isomorphism $\pi(g, \xi) : i(V_\delta) \to i(V_{\Psi_g(\delta)})$ associated with

$$g = k \times n \in K_Q \times W(P, Q)$$

is the operator $A(g, \mathscr{R}_P, W_P r_P, \delta, t^P)$ which was defined in [26] (equation (4.82)). In order to explain its construction we need to use Lusztig's Theorem on the structure of the formal completion of $\mathscr{H}$ at the central character of $\pi(\xi)$ (cf. Subsection 2.6). The central character of $\pi(\xi)$ ($\xi = (P, \delta, t^P)$) is equal to $W_0 t$ with $t = r_P t^P$, where $W_P r_P$ denotes the central character of δ. Recall that we can then extend $\pi(\xi)$ to the formal completion $\overline{\mathscr{H}}_t$ of $\mathscr{H}$ with respect to the maximal ideal $\mathscr{I}_t$ of $\mathscr{Z}$ at $W_0 t$ (cf. 2.6.3).

First we consider the case where ξ is generic (Definition 3.15). For $w \in W^P$, $w \neq e$, the idempotent $e_{w\omega}$ (cf. equation (2.13)) vanishes on $1 \otimes V_\delta \subset i(V_\delta)$, where the action is through $\pi(\xi)$ (extended to the completion). Therefore we have the natural isomorphisms of vector spaces:

$$\begin{aligned} (3.6) \qquad i(V_\delta) &\simeq \mathscr{H} \otimes_{\mathscr{H}^P} V_\delta \\ &\simeq \overline{\mathscr{H}}_t \otimes_{e_\varpi \overline{\mathscr{H}}_t^P} V_\delta \\ &\simeq \bigoplus_{u \in W^P} \imath_u^0 e_\varpi \otimes V_\delta, \end{aligned}$$

where $e_\varpi \overline{\mathscr{H}}_t^P \simeq \overline{\mathscr{H}}_t^P$ acts on V_δ via δ_{t^P}, extended to the formal completion at the central character $W_P t$. We will often suppress the subscript $e_\varpi \overline{\mathscr{H}}_t^P$ of $\otimes$.

Let us now define the unitary standard intertwining operators $\pi(g, \xi)$ in this case where ξ is generic. First we *choose* a unitary isomorphism $\tilde{\delta}_g : V_\delta \to V_{\Psi_g(\delta)}$ intertwining the representations $\delta \circ \psi_g^{-1}$ and $\Psi_g(\delta)$. These choices are not canonical, but give rise to a cocycle η_Δ with values in S^1 of the finite groupoid $\mathscr{W}_\Delta = \mathscr{W} \times_{\mathscr{P}} \Delta$ (constructed in a similar way as $\mathscr{W}_\Xi$) such that

$$(3.7) \qquad \widetilde{\Psi_v(\delta)}_u \circ \tilde{\delta}_v = \eta_\Delta(u, v) \tilde{\delta}_{uv}.$$

The cohomology class $[\eta_\Delta] \in H^2(\mathscr{W}_\Delta, S^1)$ is independent of the choices of the intertwiners $\tilde{\delta}_g$. Then we define

$$\begin{aligned} (3.8) \qquad \pi(g, \xi) &: i(V_\delta) \to i(V_{\Psi_g(\delta)}), \\ & h \otimes v \to h \imath_{g^{-1}}^0 e_{g\varpi} \otimes_{e_{g\varpi} \mathscr{H}_{g(t)}^{g(P)}} \tilde{\delta}_g(v), \end{aligned}$$

where we use the isomorphism of equation (3.6) to view the right-hand side as an element of $i(V_{\Psi_g(\delta)})$. It follows easily that $\pi(g, \xi)$ is an intertwining operator between $\pi(\xi)$ and

$\pi(g\xi)$. The composition of these normalized intertwining operators clearly satisfy (for composable arrows u, v of the groupoid $\mathscr{W}_\Xi$, with source of v being ξ):

$$\pi\big(u, v(\xi)\big)\pi(v, \xi) = \eta(u_\Delta, v_\Delta)\pi(uv, \xi) \tag{3.9}$$

(where u_Δ, v_Δ are the images of u, v in $\mathscr{W}_\Delta$ under the natural homomorphism $\mathscr{W}_\Xi \to \mathscr{W}_\Delta$). The appearance of η implies that π is only a projective representation of $\mathscr{W}_\Xi$ (see below).

For general ξ we need the following regularity results from [26]. The matrix elements of $\pi(g, \xi)$ are meromorphic in ξ, with possible poles at the nongeneric ξ. However, it was shown in [26], Theorem 4.33, that for R_P-generic $t = r_P t^P$ with $t^P \in T_u^P$, $\pi(g, \xi)$ is unitary with respect to the Hilbert space structures of $i(V_\delta)$ and $i(V_{\Psi_g(\delta)})$ (which are independent of $t^P \in T_u^P$, cf. equation (3.4)). Together with a description of the locus of the possible singularities of $\pi(g, \xi)$ (as a rational function on $\Xi_{P,\delta}$, the set of induction data of the form (P, δ, t^P) with $t^P \in T^P$), this implies (according to a simple argument, cf. [2], Lemma 8) that $\pi(g, \xi)$ has only removable singularities in a tubular neighbourhood of $\Xi_{P,\delta,u}$ (the subset of triples in $\Xi_{P,\delta}$ with $t^P \in T_u^P$). Thus $\pi(g, \xi)$ has a unique holomorphic extension to a tubular neighbourhood of $\Xi_{P,\delta,u}$. This finally clarifies the definition of $\pi(g, \xi)$ for general $\xi \in \Xi_{P,\delta,u}$ (and in fact in a "tubular neighbourhood" of this subset of $\Xi_{P,\delta}$).

We conclude with the following summary of the above:

Theorem 3.16. *The induction functor $\pi : \mathscr{W}_{\Xi_u} \to \mathrm{PRep}_{\mathrm{unit,temp}}(\mathscr{H})$ is rational and smooth.*

By this we simply mean that on each component $\Xi_{P,\delta,u}$ of Ξ_u, the representations $\pi(\xi)$ can be realized by smooth rational matrices as a function of $\xi \in \Xi_{P,\delta,u}$, and also the matrices of the $\pi(g, \xi)$ are both rational and smooth in $\xi \in \Xi_{P,\delta,u}$. We note that the matrices $\pi(\xi; h) := \pi(\xi)(h)$ (for $h \in \mathscr{H}$ fixed) are in fact even regular functions of ξ, and that the matrices $\pi(k, \xi)$ (for $k \in K_P$) are constant.

3.6. The constant part of a tempered representation. Let (V, π) denote a tempered representation of $\mathscr{H}$, and let $P \in \mathscr{P}$. Given $t \in T$ we denote by t_P the restriction of the character t to the sublattice $P^\perp \cap X$. We define the constant part V^P of V along P by

$$V^P = \bigoplus_{t \in T : |t|_P = 1} V_t \tag{3.10}$$

and its complement

$$V^{P+} = \bigoplus_{t \in T : |t|_P \neq 1} V_t \tag{3.11}$$

where V_t denotes the generalized t-eigenspace for the representation $x \mapsto \pi(x)$ of the lattice X on V. Recall the partial ordering $\leqq_P$ introduced in Definition 3.7. Then, because V is tempered, one has

$$V^P = \bigoplus_{t \in T : |t| \leqq_P 1} V_t. \tag{3.12}$$

Since the sets $t \in T$ defined by $|t|_P = 1$ and $|t|_P \neq 1$ are W_P-invariant it is clear that $V^P \subset V$ and $V^{P+} \subset V$ are subrepresentations of the restriction of (V, π) to the subalgebra $\mathscr{H}^P$ (recall that $\mathbb{C}[X]^{W_P}$ is the center of $\mathscr{H}^P$).

Definition 3.17. Let (V^P, π^P) denote the representation of $\mathscr{H}^P$ on V^P described above. We call this representation the constant part of V along P. We denote by $p_{P,V} : V \to V^P$ the projection of V to V^P along V^{P+}. Observe that this is an $\mathscr{H}^P$-module morphism.

The following proposition is elementary.

Proposition 3.18. (i) *V is the direct sum of the $\mathscr{H}^P$-submodules V^P and V^{P+} which have no irreducible subquotient in common. In particular V^{P+} is the unique complementary H^P-submodule of V^P in V.*

(ii) *The $\mathscr{H}^P$ representation V^P is tempered. It is the unique maximal tempered $\mathscr{H}^P$-subrepresentation of the restriction $V|_{\mathscr{H}^P}$ to $\mathscr{H}^P$ of V.*

(iii) *V^P is a direct summand of $V|_{\mathscr{H}^P}$.*

(iv) *The assignment $V \mapsto V^P$ is functorial, and is an exact functor.*

(v) *Transitivity: If $P \subset Q$, then $V^P = (V^Q)^P$.*

Let ω be a unitary character of the lattice Z_X, and let (V, π) be a finite dimensional representation of $\mathscr{H}$ such that the central subalgebra $\mathscr{A}_Z = \mathbb{C}[\theta_x : x \in Z_X] \simeq \mathbb{C}[Z_X]$ acts on V via the character ω. We call (V, π) an ω-representation. Choose a character $\omega' \in T_u^{F_0}$ such that $\omega'|_{Z_X} = \omega$. By definition (using the notations introduced in Subsection 3.4) the representation π factors through the quotient map $\phi_{\omega'}$. Hence there exists a representation (V, ρ) of the semisimple quotient affine Hecke algebra $\mathscr{H}_{F_0}$ such that $\pi = \rho_{\omega'}$. It is easy to check that the following definition is independent of the choice of the lift ω' of ω:

Definition 3.19. In the above situation we say that (V, π) is an ω-representation which is square integrable modulo Z_X if all matrix coefficients of (V, ρ) belong to $L_2(\mathscr{H}_{F_0})$ or equivalently, if all matrix coefficients of (V, π) belong to $\mathbb{A}_{2,\mathrm{modc}}$.

Proposition 3.20. *Let ω be a unitary character of Z_X and let (V, π) be an ω-representation. Then (V, π) is square integrable modulo Z_X iff (V, π) is tempered and $V^P = 0$ for all proper subsets $P \subset F_0$.*

In this case (V, π) is a direct sum of representations of the form $(U, \delta_{\omega'})$ where the (U, δ) are irreducible discrete series representations of $\mathscr{H}_{F_0}$.

Proof. By the text above Definition 3.19 this reduces to the situation where $Z_X = 0$. So we may and will assume that $\mathscr{H} = \mathscr{H}_{F_0}$ is semisimple from now on.

In view of Definition 3.10, the observation just above Corollary 3.8, and Corollary 3.11 we have $V^P = 0$ for all proper subsets $P \subset F_0$ if and only if all the matrix coefficients of V belong to $\mathbb{A}^{\mathrm{cusp}} = \mathbb{A} \cap L_2(\mathscr{H})$. In particular (V, π) is tempered, and all irreducible subquotients of V are discrete series representations. This implies the first claim of the proposition. The discrete series representations are projective as modules over $\mathscr{S}$ by [26], Proposition 6.10. Using this fact an easy induction argument on the dimension of V shows that (V, π) is actually a direct sum of discrete series representations. $\square$

Taking the constant part of V along P is a right adjoint of the induction functor from tempered representations of $\mathscr{H}^P$ to $\mathscr{H}$ (see [26], Proposition 4.20):

Proposition 3.21 (Frobenius reciprocity). *Let (W, δ) be a tempered representation of $\mathscr{H}^P$, and let (V, π) be a tempered representation of $\mathscr{H}$. Then*

$$\mathrm{Hom}_{\mathscr{H}}\big(\mathrm{Ind}_{\mathscr{H}^P}^{\mathscr{H}}(W), V\big) \simeq \mathrm{Hom}_{\mathscr{H}^P}(W, V^P). \tag{3.13}$$

Proof. By Proposition 3.18(i) we have $\mathrm{Hom}_{\mathscr{H}^P}(W, V^P) = \mathrm{Hom}_{\mathscr{H}^P}(W, V)$. Now use Frobenius reciprocity for induction from $\mathscr{H}^P$ to $\mathscr{H}$. $\square$

3.6.1. Irreducible tempered representations. In this paragraph we prove that the irreducible tempered representations of $\mathscr{H}$ are exhausted by the irreducible summands of the representations $\pi(\xi)$ $(\xi \in \Xi_u)$.

Theorem 3.22. *Let (V, π) be an irreducible tempered $\mathscr{H}$ representation. There exists a tempered standard induction datum ξ such that (V, π) is a summand of $\pi(\xi)$.*

Proof. By Theorem 3.12, $\pi(\xi)$ is unitary (in particular, self-dual). Hence it suffices to show that there exists a tempered standard induction datum $\xi = (P, \delta, t)$ such that $\mathrm{Hom}_{\mathscr{H}}\big(V, \pi(\xi)\big)$ is nonzero. By Proposition 2.13 (duality) it is equivalent to show that $\mathrm{Hom}_{\mathscr{H}}\big(\pi(\xi), W\big)$ is nonzero, where $W = V^{\circ}$. By Proposition 3.21 (Frobenius reciprocity) we need to find a standard parabolic subset $P \subset F_0$, and an irreducible square integrable modulo $Z_X(\mathscr{R}_P) = X \cap P^{\perp}$ representation of the form δ_t for $\mathscr{H}^P$, such that $\mathrm{Hom}_{\mathscr{H}^P}(\delta_t, W^P)$ is nonzero. For this, take P minimal such that W^P is nonzero. By Proposition 3.18 it follows that if Q is a proper subset of P and if U is any submodule of W^P, then $U^Q = 0$. Take any irreducible submodule (U, σ) of W^P. Then there exists a unitary character ω of $X \cap P^{\perp}$ such that (U, σ) is an ω representation. Hence by Proposition 3.20, (U, σ) is of the form (U, δ_t) where (U, δ) is an irreducible discrete series representation for $\mathscr{H}_P$. Hence $\xi = (P, \delta, t)$ is a tempered standard induction datum with the desired properties. $\square$

Corollary 3.23. *An irreducible tempered representation is unitarizable. In particular, it is self-dual in the sense of Proposition* 2.13.

3.7. Definition of the constant terms of $f \in \mathbb{A}^{\mathrm{temp}}$. In this subsection we define the constant term of a tempered finite functional $f \in \mathbb{A}^{\mathrm{temp}}$ along a standard parabolic subalgebra $\mathscr{H}^P$ of $\mathscr{H}$. Recall the notion of exponents (3.2) and the Casselman criteria for tempered finite functionals.

Definition 3.24 (Constant term). Let $P \in \mathscr{P}$ and $f \in \mathbb{A}^{\mathrm{temp}}$. Then we define the constant term of f along P by

$$f^P(h) := \sum_{t \in \epsilon : |t| \leqq_P 1} f_t(h),$$

where (in the notation of Corollary 3.6) $f_t(h) := E_t^f(h,0)$, and the coefficients E_t^f and the set ϵ are defined by the expansion (3.1). We say that an exponent $t \in \epsilon$ of f is P-tempered if it satisfies the condition $|t| \leqq_P 1$.

Hence we have the following characterization of the subspace $\mathbb{A}_{2,\mathrm{modc}}$ (cf. Definition 3.10):

Corollary 3.25. *Let $f \in \mathbb{A}^{\mathrm{temp}}$. Then $f \in \mathbb{A}_{2,\mathrm{modc}}$ if and only if $f^P = 0$ for every proper $P \in \mathscr{P}$.*

Observe the following elementary properties of the constant term:

Corollary 3.26. (i) $f^P \in \mathbb{A}^{\mathrm{temp}}$.

(ii) *L_x commutes with $f \mapsto f^P$ if $x \in \mathscr{H}^P$.*

(iii) *R_y commutes with $f \mapsto f^P$ for all $y \in \mathscr{H}$.*

(iv) $f^P \in L_X(f) = f \cdot \mathscr{A} \subset f \cdot \mathscr{H}$.

(v) *If V is a finite dimensional complex Hilbert space and $T \in \mathrm{End}(V)$ then we denote the adjoint of T by T^*. Now assume that V is a tempered unitary module of $\mathscr{H}$. Given $a, b \in V$ we define the matrix coefficient $f_{a,b} \in \mathbb{A}^{\mathrm{temp}}$ by $f_{a,b}(h) := \langle a, hb \rangle$ $(h \in \mathscr{H})$. For $h \in \mathscr{H}^P$ we have*

$$f_{a,b}^P(h) = f_{(p_{P,V})^*(a), p_{P,V}(b)}(h). \tag{3.14}$$

The projection of f to f^P can be made explicit using an idempotent e^P in a formal completion of $\mathscr{A} \subset \mathscr{H}$. Such completions were introduced and studied by Lusztig [19] (cf. Subsection 2.6). This will be applied to the case were f is a matrix coefficient of a parabolically induced representation in the Section 6.

4. Fourier transform

In this section we briefly review the Fourier transform on $L_2(\mathscr{H})$ as formulated in [26]. The spectral data are organized in terms of the induction functor on the groupoid of tempered standard induction data $\mathscr{W}_{\Xi_u}$.

4.1. Fourier transform on $L_2(\mathscr{H})$. Let V_ξ denote the representation space of $\pi(\xi)$, $\xi \in \Xi$. Thus $V_\xi = i(V_\delta)$ if $\xi = (P, \delta, t^P)$, and this vector space does not depend on the parameter $t^P \in T^P$. We denote by $\mathscr{V}_\Xi$ the trivial fibre bundle over Ξ whose fibre at ξ is V_ξ, thus

$$\mathscr{V}_\Xi := \bigcup_{(P,\delta)} \Xi_{P,\delta} \times i(V_\delta) \tag{4.1}$$

where $\Xi_{P,\delta}$ denotes the component of Ξ associated to $P \in \mathscr{P}$, and $(V_\delta, \delta) \in \Delta_P$. Recall that Δ_P is a complete set of representatives of the irreducible discrete series representations δ of $\mathscr{H}_P$. We denote by $\mathrm{End}(\mathscr{V}_\Xi)$ the endomorphism bundle of $\mathscr{V}_\Xi$, and by $\mathrm{Pol}(\Xi, \mathrm{End}(\mathscr{V}_\Xi))$

the space of polynomial sections in this bundle. Similarly, let us introduce the space $\mathrm{Rat}^{\mathrm{reg}}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$ of rational sections which are regular in a neighbourhood of Ξ_u.

There is an action of $\mathscr{W}$ on $\mathrm{End}(\mathscr{V}_\Xi)$ as follows. If $(P, g) \in \mathscr{W}_P$ (the set of elements of $\mathscr{W}$ with source $P \in \mathscr{P}$) with $g = k \times n \in K_Q \times W(P, Q)$, $\xi \in \Xi_P$, and $A \in \mathrm{End}(V_\xi)$ we define $g(A) := \pi(g, \xi) \circ A \circ \pi(g, \xi)^{-1} \in \mathrm{End}(V_{g(\xi)})$. A section of f of $\mathrm{End}(\mathscr{V}_\Xi)$ is called $\mathscr{W}$-equivariant if we have $f(\xi) = g^{-1}\big(f\big(g(\xi)\big)\big)$ for all $\xi \in \Xi$ and $g \in \mathscr{W}_\xi$ (where $\mathscr{W}_\xi := \mathscr{W}_P$ if $\xi = (P, \delta, t^P)$).

Definition 4.1. We define an averaging projection $p_{\mathscr{W}}$ onto the space of $\mathscr{W}$-equivariant sections by:

$$p_{\mathscr{W}}(f)(\xi) := |\mathscr{W}_\xi|^{-1} \sum_{g \in \mathscr{W}_\xi} g^{-1}\big(f\big(g(\xi)\big)\big). \tag{4.2}$$

Notice that this projection preserves the space $\mathrm{Rat}^{\mathrm{reg}}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$, but not the space $\mathrm{Pol}(\Xi, \mathrm{End}(\mathscr{V}_\Xi))$.

If $h \in \mathscr{W}_\xi$ then $\mathscr{W}_{h(\xi)} = \mathscr{W}_\xi \circ h^{-1}$. Using this one checks simply that $p_{\mathscr{W}}(f)\big(h(\xi)\big) = h\big(p_{\mathscr{W}}(f)(\xi)\big)$ for all $h \in \mathscr{W}_\xi$, or in other words, that $p_{\mathscr{W}}(f)$ is $\mathscr{W}$-equivariant. It is obvious that $p_{\mathscr{W}}$ restricts to the identity on the space of $\mathscr{W}$-equivariant sections. Hence $p_{\mathscr{W}}$ is indeed a projection onto the space of $\mathscr{W}$-equivariant sections.

The Fourier transform $\mathscr{F}_{\mathscr{H}}$ on $\mathscr{H}$ is the following algebra homomorphism

$$\begin{aligned} \mathscr{F}_{\mathscr{H}} : \mathscr{H} &\to \mathrm{Pol}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))^{\mathscr{W}}, \\ h &\mapsto \{\xi \mapsto \pi(\xi; h)\} \end{aligned} \tag{4.3}$$

where $\mathrm{Pol}(\Xi, \mathrm{End}(\mathscr{V}_\Xi))^{\mathscr{W}}$ denotes the space of $\mathscr{W}$-equivariant polynomial sections of $\mathrm{End}(\mathscr{V}_\Xi)$.

We will now describe a $\mathscr{W}$-invariant measure μ_{Pl} on Ξ_u whose push forward to $\mathscr{W}\backslash\Xi_u$ will be the spectral measure of the positive trace τ of $\mathscr{H}$ ([26], Theorem 4.43) (we will call this measure the Plancherel measure of $\mathscr{H}$). Put $\xi = (P, \delta, t^P) \in \Xi_u$ and let $t = r_P t^P$. We write $d\xi := |K_{P,\delta}|\, dt^P$ where dt^P denotes the normalized Haar measure of T_u^P and where $K_{P,\delta}$ denotes the stabilizer of δ under the natural action of K_P on Δ_P. Let $\mathscr{K} \lhd \mathscr{W}$ denote the normal subgroupoid whose set of objects is $\mathscr{P}$, and with $\mathrm{Hom}_{\mathscr{K}}(P, Q) = \emptyset$ unless $P = Q$, in which case we have $\mathrm{Hom}_{\mathscr{K}}(P, P) = K_P$. Thus $\mathscr{W}_P/\mathscr{K}_P = \{w \in W_0 \mid w(P) \subset F_0\}$. Let $\mu_{\mathscr{R}_P, Pl}(\{\delta\})$ denote the Plancherel mass of δ with respect to $\mathscr{H}_P$ (and its trace τ_P). It is known that $\mu_{\mathscr{R}_P, Pl}(\{\delta\}) > 0$ (see [26], Theorem 2.25) and an explicit product formula for $\mu_{\mathscr{R}_P, Pl}(\{\delta\})$ (up to a multiplicative constant independent of $\mathbf{q}$) is known (see [26], Corollary 3.32). We now define the Plancherel measure μ_{Pl}:

Definition 4.2.

$$d\mu_{Pl}(\xi) := q(w^P)^{-1} |\mathscr{W}_P/\mathscr{K}_P|^{-1} \mu_{\mathscr{R}_P, Pl}(\{\delta\}) |c(\xi)|^{-2}\, d\xi \tag{4.4}$$

where $c(\xi)$ is the Macdonald c-function, see Definition 9.7.

This measure is smooth on Ξ_u (Proposition 9.8(v)), and it is invariant for the action of $\mathscr{W}$ on Ξ_u, by Proposition 9.8(ii).

With these notations we have:

Theorem 4.3 ([26], Theorem 4.43). (i) *$\mathscr{F}_{\mathscr{H}}$ extends to an isometric isomorphism*

$$\mathscr{F} : L_2(\mathscr{H}) \to L_2\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl}\big)^{\mathscr{W}}, \tag{4.5}$$

where the Hermitian inner product $(\cdot\,,\cdot)$ on $L_2\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl}\big)^{\mathscr{W}}$ is defined by integrating the Hilbert-Schmidt form $(A, B) := \mathrm{tr}(A^*B)$ in the fibres $\mathrm{End}(V_\xi)$ against the above measure μ_{Pl} on the base space Ξ_u.

(ii) *If $x \in C_r^*(\mathscr{H}) \subset L_2(\mathscr{H})$ then $\mathscr{F}(x)$ is an element of the space $C\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)^{\mathscr{W}}$ of continuous sections of the trivial bundle $\mathrm{End}(\mathscr{V}_\Xi)$.*

(iii) *Let $C_r^*(\mathscr{H})^{\mathrm{o}}$ denote the opposite C^*-algebra of $C_r^*(\mathscr{H})$. Let*

$$(x, y) \in C_r^*(\mathscr{H}) \times C_r^*(\mathscr{H})^{\mathrm{o}}$$

act on $L_2(\mathscr{H})$ via the regular representation $\lambda(x) \times \rho(y)$, and on $L_2\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl}\big)^{\mathscr{W}}$ through fibrewise multiplication from the left with $\mathscr{F}(x)$ and from the right with $\mathscr{F}(y)$. Then $\mathscr{F}$ intertwines these representations of $C_r^*(\mathscr{H}) \times C_r^*(\mathscr{H})^{\mathrm{o}}$.

Proof. As to (ii), first recall that according to equation (4.3),

$$\mathscr{F}_{\mathscr{H}}(\mathscr{H}) \subset \mathrm{Pol}\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)^{\mathscr{W}}.$$

By [26], Theorem 4.43(iii), one easily deduces that $\|h\|_o = \|\mathscr{F}_{\mathscr{H}}(h)\|_{\sup}$ for all $h \in \mathscr{H}$, where $\|\sigma\|_{\sup} := \sup\limits_{\xi \in \Xi_u} \|\sigma(\xi)\|_{\mathrm{o}}$ (where $\|\sigma(\xi)\|_{\mathrm{o}}$ denotes the operatornorm of $\sigma(\xi) \in \mathrm{End}(V_\xi)$). Hence $\mathscr{F}\big(C_r^*(\mathscr{H})\big) \subset C\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)^{\mathscr{W}}$.

Now (iii) follows from (ii) and [26], Theorem 4.43(iii). □

The following result is an immediate consequence of Theorem 4.3 and Theorem 3.22:

Corollary 4.4. *The support of the Plancherel measure is the set of irreducible tempered representations of $\mathscr{H}$.*

The following easy corollary is important in the sequel:

Corollary 4.5 ([26], Corollary 4.45). *The averaging operator $p_{\mathscr{W}}$ defines an orthogonal projection onto the space of $\mathscr{W}$-equivariant sections in $L_2\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl}\big)$. Moreover, if*

$$\mathscr{J} : L_2\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl}\big) \to L_2(\mathscr{H}) \tag{4.6}$$

denotes the adjoint of $\mathscr{F}$ (the wave packet operator), then $\mathscr{J}\mathscr{F} = \mathrm{id}$ and $\mathscr{F}\mathscr{J} = p_{\mathscr{W}}$.

Proof. Theorem 4.3 implies that $\mathscr{J}\mathscr{F} := \mathrm{id}$ and that $\mathscr{F}\mathscr{J}$ is equal to the orthogonal projection onto the space of $\mathscr{W}$-equivariant L_2-sections of $\mathrm{End}(\mathscr{V}_\Xi)$.

On the other hand, since the action of $\mathscr{W}$ on $\mathrm{End}(\mathscr{V}_\Xi)$ is defined in terms of invertible smooth matrices (cf. Theorem 3.16), $p_{\mathscr{W}}$ preserves the space of L_2-sections. By the $\mathscr{W}$-invariance of μ_{Pl}, the projection $p_{\mathscr{W}}$ on $L_2(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl})$ is in fact an orthogonal projection. This finishes the proof. $\square$

5. Main Theorem and its applications

The space of smooth sections of the trivial bundle $\mathrm{End}(\mathscr{V}_\Xi)$ on Ξ_u will be denoted by $C^\infty(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$. We equip this vector space with its usual Fréchet topology. The collection of semi-norms inducing the topology is of the form $p(\sigma) := \sup_{\xi \in \Xi_u} \|D\sigma(\xi)\|_{\mathrm{o}}$, where D is a constant coefficient differential operator on Ξ_u (i.e., one such operator for each connected component of Ξ_u), acting entrywise on the section σ of the trivial bundle $\mathrm{End}(\mathscr{V}_\Xi)$, and where $\|\cdot\|_{\mathrm{o}}$ denotes the operatornorm. It is obvious from the product rule for differentiation that $C^\infty(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$ is a Fréchet algebra.

The projection $p_{\mathscr{W}}$ is continuous on $C^\infty(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$, since it is defined in terms of the action of $\mathscr{W}$ on Ξ_u, and conjugations with invertible smooth matrices. Thus the subalgebra $C^\infty(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))^{\mathscr{W}}$ of $\mathscr{W}$-equivariant sections is a closed subalgebra.

We now define the vector space

Definition 5.1.

$$\mathscr{C}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)) := cC^\infty(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)), \tag{5.1}$$

where c denotes the c-function of Definition 9.7 on Ξ_u. We equip $\mathscr{C}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$ with the Fréchet space topology of $C^\infty(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$ via the linear isomorphism $C^\infty(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)) \to \mathscr{C}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$ defined by $\sigma \mapsto c\sigma$.

Lemma 5.2. *The complex vector space $\mathscr{C}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$ is closed for taking (fibrewise) adjoints, and*

$$\mathscr{C}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)) \subset L_2(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl}). \tag{5.2}$$

Moreover,

$$C^\infty(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)) \subset \mathscr{C}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)) \tag{5.3}$$

is a closed subspace.

Proof. It is closed for taking adjoints by Proposition 9.8(iv) (applied to $d = w^P \in W(P, P')$), and it is a subspace of $L_2(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl})$ by Proposition 9.8(i). The last assertion follows from Proposition 9.8(v). $\square$

Now we are prepared to formulate the main theorem of this paper.

Theorem 5.3. *The Fourier transform restricts to an isomorphism of Fréchet algebras*

$$\mathcal{F}_{\mathcal{S}} : \mathcal{S} \to C^\infty\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}}. \tag{5.4}$$

The wave packet operator $\mathcal{J}$ restricts to a surjective continuous map

$$\mathcal{J}_{\mathcal{C}} : \mathcal{C}\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big) \to \mathcal{S}. \tag{5.5}$$

We have $\mathcal{J}_{\mathcal{C}}\mathcal{F}_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}}$, and $\mathcal{F}_{\mathcal{S}}\mathcal{J}_{\mathcal{C}} = p_{\mathcal{W},\mathcal{C}}$ (the restriction of $p_{\mathcal{W}}$ to $\mathcal{C}\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big)$). In particular, the map $p_{\mathcal{W},\mathcal{C}}$ is a continuous projection of $\mathcal{C}\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big)$ onto $C^\infty\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}}$.

5.1. Applications of the Main Theorem. Before we embark on its proof we discuss some immediate consequences of the Main Theorem. The following corollary of the Main Theorem is the analog for affine Hecke algebras of Harish-Chandra's completeness theorem for real reductive groups.

Corollary 5.4 (Harish-Chandra's completeness Theorem, cf. [12], and [18], Theorem 14.31). *Let $\xi \in \Xi_u$. The complex linear span C_ξ of the set of operators $\{\pi(g,\xi) \mid g \in \mathrm{End}_{\mathcal{W}_\Xi}(\xi)\}$ is a unital, involutive subalgebra of $\mathrm{End}(\mathcal{V}_\xi)$. For all $\xi \in \Xi_u$ we have $C_\xi = \mathrm{End}_{\mathcal{H}}(\mathcal{V}_\xi)$.*

Proof. Let $\xi = (P, \delta, t^P)$ and denote by $C_\xi \subset \mathrm{End}(\mathcal{V}_\xi)$ the complex linear span of the set of operators $\{\pi(g,\xi) \mid g \in \mathrm{End}_{\mathcal{W}_\Xi}(\xi)\}$. By Theorem 3.16, C_ξ is an involutive (i.e., $*$-invariant), unital subalgebra of $\mathrm{End}(\mathcal{V}_\xi)$. Let us show that Theorem 5.3 implies that $\pi(\xi, \mathcal{H})$ is equal to the commuting algebra, C'_ξ, of C_ξ. First observe that the inclusion $\pi(\xi, \mathcal{H}) \subset C'_\xi$ is obvious. Since V_ξ is finite dimensional and since $\pi(\xi, \cdot)$ extends continuously to $\mathcal{S}$ we have $\pi(\xi, \mathcal{H}) = \pi(\xi, \mathcal{S})$. By Theorem 5.3 this last algebra is equal to the algebra of values at ξ of $C^\infty\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}}$. If $A \in C'_\xi$ then we can find a section $\sigma \in C^\infty\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big)$ such that $\sigma(\xi) = A$ and $\sigma(g\xi) = 0$ for all $g \in \mathcal{W}_\xi$ such that $g(\xi) \neq \xi$. Then $p_{\mathcal{W}}(\sigma) \in C^\infty\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}}$ and $p_{\mathcal{W}}(\sigma)(\xi)$ is a non zero scalar multiple of A. We conclude that $A \in \pi(\xi, \mathcal{H})$.

The Bicommutant Theorem therefore implies that C_ξ is equal to the commutant $\pi(\xi, \mathcal{H})'$ of $\pi(\xi, \mathcal{H})$. □

Corollary 5.5. *The center $\mathcal{Z}_{\mathcal{S}}$ of $\mathcal{S}$ is, via the Fourier transform $\mathcal{F}_{\mathcal{S}}$, isomorphic to the algebra $C^\infty(\Xi_u)^{\mathcal{W}}$ of smooth $\mathcal{W}$-invariant functions on Ξ_u.*

Proof. The algebra of scalar sections of $C^\infty\big(\Xi_u, \mathrm{End}(\mathcal{V}_{\Xi_u})\big)^{\mathcal{W}}$ is isomorphic to $C^\infty(\Xi_u)^{\mathcal{W}}$, and is contained in $\mathcal{F}_{\mathcal{S}}(\mathcal{Z}_{\mathcal{S}})$ by Theorem 5.3. To show the equality, observe that Corollary 5.4 implies that an element of $\mathcal{F}_{\mathcal{S}}(\mathcal{Z}_{\mathcal{S}})$ is scalar at all fibers $\mathrm{End}(V_\xi)$ with $\xi \in \Xi_u$ generic (since $\mathrm{End}_{\mathcal{H}}(V_\xi) = \mathbb{C}$ in this case). By the density of the set of generic points in Ξ_u we obtain the desired equality. □

Notice that $\mathcal{Z}_{\mathcal{S}}$ is in general larger than the closure in $\mathcal{S}$ of the center $\mathcal{Z}$ of $\mathcal{H}$.

Corollary 5.6 (Langlands' disjointness Theorem, cf. [18], Theorem 14.90). *Let $\xi, \xi' \in \Xi_u$. If $\pi(\xi)$ and $\pi(\xi')$ are not disjoint, then the objects $\xi, \xi' \in \Xi_u$ of $\mathcal{W}_{\Xi_u}$ are isomorphic (and thus, $\pi(\xi)$ and $\pi(\xi')$ are actually equivalent).*

Proof. Corollary 5.5 implies that $\mathcal{Z}_{\mathcal{S}}$ separates the $\mathcal{W}$-orbits of Ξ_u. Whence the result. □

Corollary 5.7. *The Fourier transform $\mathcal{F}$ restricts to a C^*-algebra isomorphism*

$$\mathcal{F}_C : C_r^*(\mathcal{H}) \to C\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}}, \tag{5.6}$$

where $C_r^*(\mathcal{H})$ denotes the reduced C^*-algebra of $\mathcal{H}$ (cf. 2.7).

Proof. By Theorem 4.3, the restriction of $\mathcal{F}$ to $C_r^*(\mathcal{H})$ is an algebra homomorphism. It is a homomorphism of involutive algebras since $\pi(\xi; x^*) = \pi(\xi; x)^*$ (cf. Subsection 3.4).

The reduced C^*-algebra $C_r^*(\mathcal{H})$ of $\mathcal{H}$ is defined in [26] as the norm closure of $\lambda(\mathcal{H}) \subset B\big(L_2(\mathcal{H})\big)$. By Theorem 4.3, the norm $\|x\|_o$ of $C_r^*(\mathcal{H})$ is equal to the supremum norm $\|\mathcal{F}(x)\|_{\sup}$ of the $\mathcal{W}$-invariant continuous function $\xi \mapsto \|\pi(\xi; x)\|_o$ on Ξ_u (where $\|\pi(\xi; h)\|_o$ denotes the operator norm for operators on the finite dimensional Hilbert space $V_\xi = i(V_\delta)$). Notice that, by the regularity of the standard intertwining operators, the projection operator $p_{\mathcal{W}}$ restricts to a continuous projection on the space of continuous sections of $\mathrm{End}(\mathcal{V}_\Xi)$.

By Theorem 5.3, the closure of $\mathcal{F}(\mathcal{S})$ with respect to $\|\cdot\|_{\sup}$ is equal to $C\big(\Xi_u, \mathrm{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}}$. In view of Theorem 4.3(ii) this finishes the proof. □

Corollary 5.8. *The set of minimal central idempotents of $C_r^*(\mathcal{H})$ is parameterized by the (finite) set of $\mathcal{W}$-orbits of pairs (P, δ) with $P \in \mathcal{P}$ and $\delta \in \Delta_P$. The central idempotents $e_{(P,\delta)}$ are elements of $\mathcal{S}$.*

Proof. This is immediate from Theorem 5.3 and Corollary 5.7. □

Corollary 5.9. *The dense subalgebra $\mathcal{S} \subset C_r^*(\mathcal{H})$ is closed for holomorphic functional calculus.*

Proof. The Fréchet subalgebra $\mathcal{S} \subset C_r^*(\mathcal{H})$ is dense, symmetric, and the embedding is continuous (see (2.19)). In addition, Theorem 5.3 and Corollary 5.7 imply that $\mathcal{S}$ is also spectrally closed, i.e., if $a \in \mathcal{S}$ is invertible in $C_r^*(\mathcal{H})$, then $a^{-1} \in \mathcal{S}$. Hence $\mathcal{S}$ is a Ψ^*-algebra, and thus closed under holomorphic functional calculus [10]. Alternatively, one may verify directly from Theorem 5.3 and the definition of $f(a)$ that $f(a) \in \mathcal{S}$ for all $a \in \mathcal{S}$ and all f holomorphic on the spectrum of a. □

6. Constant terms of matrix coefficients of $\pi(\xi)$

In the remainder of this paper we will prove the Main Theorem, Theorem 5.3. A main tool is the notion of the constant term f^P of a functional $f \in \mathbb{A}^{\mathrm{temp}}$ with respect to a standard parabolic subset $P \in \mathcal{P}$ (see Subsection 3.7).

6.1. Constant terms of coefficients of $\pi(\xi)$ for $\xi \in \Xi_u$ generic. *In this subsection we assume that ξ is generic unless stated otherwise.*

We will compute the constant terms of a matrix coefficient of $\pi = \pi(\xi)$ in the case where $\xi = (P, \delta, t^P) \in \Xi_u$ is generic. Choose $r_P \in T_P$ such that $W_P r_P$ is the central character of δ. We thus assume that $t = r_P t^P \in T$ is R_P-generic in this subsection.

Let $a, b \in i(V_\delta)$, and denote by $f = f_{a,b} = f_{a,b}(\xi)$ the matrix coefficient defined by $f(h) = \langle a, \pi(\xi; h)(b) \rangle$. By [26], Lemma 2.20 and Proposition 4.20, we have: If $t^P \in T_u^P$ then $f_{a,b} \in \mathbb{A}^{\text{temp}}$ for all $a, b \in \mathscr{H}(W^P) \otimes V_\delta$. More precisely:

Proposition 6.1. *The exponents of f (cf. 3.3) are of the form wt' where w runs over the set W^P and where t' runs over the set of weights of δ_{t^P}, thus t^P times the set of X_P-weights of δ.*

Now let $Q \subset F_0$ be another standard parabolic. By the proof of [26], Proposition 4.20 we deduce:

Proposition 6.2. *Let $w \in W^P$ and let $u \in W_P$ such that wut is an exponent of f. If $wu|t| \leqq_Q 1$, then $w(P) \subset R_{Q,+}$.*

Proof. The equivalence class ϖ of $t = r_P t^P$ is equal to $W_P t$ (since we assume genericity). If wut is an exponent then $wut = w't'$ with t' an X-weight of δ_{t^P} and $w' \in W^P$. Thus t' and ut are both in ϖ, the equivalence class of t. Hence by genericity, $w' = w$ and thus $ut = t'$, a weight of δ_{t^P}. But δ is discrete series for $\mathscr{H}_P$, hence $|ut| = \prod_{\alpha \in P} d_\alpha \otimes \alpha^\vee$ with all $0 < d_\alpha < 1$. Thus $wu|t| = \prod_{\alpha \in P} d_\alpha \otimes w(\alpha^\vee) \leqq_Q 1$ implies (since for all $\alpha \in P$: $w(\alpha^\vee) \in R_{0,+}^\vee$) that $w(P) \subset R_{Q,+}$. $\square$

Corollary 6.3. *Recall that the equivalence classes in $W_0 t$ are of the form $w\varpi$ with $\varpi = W_P t$ and $w \in W^P$. If an exponent wt' of f (with $w \in W^P$ and t' a weight of δ_{t^P}) is Q-tempered, then all exponents of f in its class $w\varpi$ are Q-tempered. The class $w\varpi$ ($w \in W^P$) is Q-tempered if and only if $w(P) \subset R_{Q,+}$.*

Proof. Since $w(P) \subset R_{Q,+}$ we have $wW_P w^{-1} \subset W_Q$. Hence $w\varpi \subset W_Q wt$, so that the moduli of all elements of $w\varpi$ have trivial restriction to $X \cap Q^\perp$. $\square$

Now we will express the constant term of a matrix coefficient of $\pi(\xi)$ in terms of the idempotents e_ϖ of the completion $\overline{\mathscr{H}}_t$. Recall the material of Subsection 2.6.

We will use the analog of Lusztig's First Reduction Theorem (2.13) for $\overline{\mathscr{H}}_t$, in combination with the results in [26], Section 4.3 on the Hilbert algebra structure of $\overline{\mathscr{H}}^t$, the quotient of $\mathscr{H}^t$ by the radical of the positive semi-definite Hermitian pairing $(x, y)_t := \chi_t(x^* y)$, in order to express and study the constant term (see Subsection 3.7).

Proposition 6.4. *We have that*

$$f^Q(h) = \sum_{w \in W^P : w(P) \subset R_{Q,+}} f(e_{w\varpi} h).$$

Proof. Let us denote by J_w the ideal in $\mathscr{A}^{W_{w(P)}}$ of elements in this ring vanishing at $w\varpi$. Clearly $\mathscr{I}_t \subset J_w$ for all w. By some elementary algebra (similar to proof of [25], Proposition 2.24(4)) we see that for every $x \in J_w$ and $k \in \mathbb{N}$ there exist a $\bar{x} \in \mathscr{I}_t$ and a unit e in $\bar{\mathscr{A}}_{m_w}$ such that

$$ex \in \bar{x} + m_w^k,$$

where m_w denotes the ideal of all functions in $\mathscr{A}$ which vanish at the points of $w\varpi$. (To be sure, we construct $\bar{x}$ by first adding an element $u \in J_w^k$ such that $x + u$ is nonzero at the other classes $w'\varpi$ with $w' \in W^P$, $w' \neq w$. Take $\bar{x}$ equal to the product of the translates $(x+u)^w$ where w runs over the set of left cosets $W_0/W_{w(P)}$. Let e be equal to the product of these factors $(x+u)^w$ where w runs over the set of left cosets $W_0/W_{w(P)}$ with $w \neq W_{w(P)}$.) Let M be the ideal of functions in $\mathscr{A}$ vanishing at $W_0 t$. Then $M = \prod m_w$ and by genericity the ideals m_w are relatively prime. So $\bar{\mathscr{A}}_M = \bigoplus \bar{\mathscr{A}}_{m_w}$ by the Chinese Remainder Theorem. Then $e_{w\varpi}$ is the unit of the summand $\bar{\mathscr{A}}_{m_w}$ (see [19], 8.7(b)). Let $\bar{e}_{w\varpi}$ be the unit of $R := \bar{\mathscr{A}}_{m_w}/\mathscr{I}_t\bar{\mathscr{A}}_{m_w}$ (the canonical image of $e_{w\varpi} \in \bar{\mathscr{A}}_{m_w}$). Note that R is finite dimensional over $\mathbb{C}$, and thus R is Artinian. By definition of m_w, $m_w\bar{e}_{w\varpi}$ is contained in all the maximal ideals of R. Hence $m_w\bar{e}_{w\varpi}$ is contained in the intersection of the maximal ideals of R, which is nilpotent in R (see the proof of [22], Theorem 3.2). In particular, for sufficiently large k, $m_w^k e_{w\varpi} \subset \mathscr{I}_t\bar{\mathscr{A}}_{m_w}$, whence

$$xe_{w\varpi} \in \mathscr{I}_t\bar{\mathscr{A}}_{m_w}.$$

But then the right-hand side is in the kernel of π, thus we conclude that $J_w e_{w\varpi}$ is in the kernel of π. In particular, the element (for any $z \in X$) $\Theta_z := \prod_{y \in W_{w(P)}z} \big((wt)(z)^{-1}\theta_y - 1\big) \in J_w$ acts by zero on the finite dimensional space of left $\mathscr{A}$-translates of $h \mapsto f(e_{w\varpi}h)$. Thus the exponents of $f \mapsto f(e_{w\varpi}h)$ are contained in $w\varpi$.

We obviously have

$$f(h) = \sum_{w \in W^P} f(e_{w\varpi}h)$$

(splitting of 1 according the decomposition of $\bar{\mathscr{A}}_M$). By the results in this paragraph, an exponent of $h \mapsto f(e_{w\varpi}h)$ is Q-tempered if and only if all exponents of this term are Q-tempered if and only if $w(P) \subset R_{Q,+}$. Hence the result. □

Corollary 6.5. *The constituents $f(e_{w\varpi}h)$ depend on the induction parameter t^P as a rational function.*

Proof. In the proof of Corollary 6.4 we can equally well work over the field K of rational functions on T^P instead of $\mathbb{C}$. Then $\bar{e}_{w\varpi} \in \bar{\mathscr{A}}(K)_M/\mathscr{I}_t\bar{\mathscr{A}}(K)_M = \mathscr{A}(K)/\mathscr{I}_t\mathscr{A}(K)$. Hence the result. □

6.2. Some results for Weyl groups. We want to work with standard parabolics only, and $w(P) \subset R_{Q,+}$ does not need to be standard. We resolve this by combining terms according to left W_Q cosets. We use the following results (see [6], Section 2.7).

Proposition 6.6. *Let $P, Q \in \mathscr{P}$. The set $D^{Q,P} := (W^Q)^{-1} \cap W^P$ intersects every double coset $W_Q w W_P$ in precisely one element $d = d(w)$, which is the unique element of shortest length of the double coset.*

Proposition 6.7 (Kilmoyer). *Let $d \in D^{Q,P}$. Then $W_Q \cap W_{d(P)}$ is the standard parabolic subgroup of W_0 corresponding to the subset $L = Q \cap d(P)$.*

Let $t \in r_P T^P$ be W_P-generic as before, where $W_P r_P \subset T_P$ is the central character of a discrete series representation δ of $\mathscr{H}_P$. Let $\varpi = W_P t$ be the equivalence class of t.

Corollary 6.8. (i) *Let $w \in W^P$ be such that $w\varpi$ is Q-tempered. Then $w(P) \subset R_{Q,+}$. We can write $w = ud$ with $d = d(w) \in D^{Q,P}$ and $u \in W_Q$. Then $d(P) \subset Q$, and $u \in W_Q^{d(P)}$. Conversely, if $d \in D^{Q,P}$ is such that $d(P) \subset Q$, then for all $u \in W_Q^{d(P)}$ we have $|ud(\varpi)| \leqq_Q 1$ (in other words, is Q-tempered).*

(ii) *The classes $\varpi_{u,d} =: ud(\varpi)$ with $d \in D^{Q,P}$ such that $d(P) \subset Q$ and $u \in W_Q^{d(P)}$, are mutually disjoint.*

Proof. (i) According to a result of Howlett (cf. [6], Proposition 2.7.5), we can uniquely decompose w as a product of the form $w = udv$ with $d = d(w) \in D^{Q,P}$, $u \in W_Q \cap W^L$ (with $L = Q \cap d(P)$), and $v \in W_P$. In fact $v = e$, since otherwise there would exist a $\alpha \in R_{P,+}$ with $v(\alpha) = -\alpha_p \in P$. But then $ud(\alpha_p) < 0$, which implies (according to [6], Lemma 2.7.1) that $d(\alpha_p) = \alpha_q \in L$. Hence $u(\alpha_q) < 0$, which contradicts the assumption $u \in W_Q \cap W^L$. Thus we have $d(P) = u^{-1}w(P) \subset R_{Q,+}$, whence $W_{d(P)} \subset W_Q$. By Kilmoyer's result it now follows that $W_{d(P)} = W_{Q \cap d(P)}$. Hence $d(P) \subset Q$ and $L = d(P)$. The converse is clear.

(ii) Suppose that $\varpi_{u,d} \cap \varpi_{u',d'} \neq \emptyset$. The Weyl group W_0 permutes equivalence classes, thus this implies that $(ud)^{-1}u'd'(t) \in \varpi$. Since t is generic, there exists a $v \in W_P$ such that $u'd' = udv$. By Howlett's result [6], Proposition 2.7.5 this implies that $v = 1$, $u = u'$ and $d = d'$. $\square$

Corollary 6.9. *For all $d \in D^{Q,P}$ with $d(P) \subset Q$ we write*

$$e_{W_Q d\varpi} = \sum_{u \in W_Q^{d(P)}} e_{ud\varpi}.$$

This is a collection of orthogonal idempotents of $\overline{\mathscr{H}}_t$. The constant term of $f = f_{a,b}(\xi)$ equals

$$f^Q(h) = \sum_{d \in D^{Q,P} : d(P) \subset Q} f^d(h),$$

where we define $f^d(h) := f(e_{W_Q d\varpi} h)$. This is the contribution to the constant term f^Q of f whose exponents have the same restriction to $X \cap Q^\perp$ as $d(t)$.

6.3. The singularities of f^d. *In this subsection we take the formulae of Corollary 6.9 as a definition of f^Q and f^d, even when $t^P \in T^P$ is not in T_u^P.*

We will now bound the possible singularities of the individual contributions f^d to f^Q, viewed as functions of $t^P \in T^P$. We have seen in Corollary 6.5 that f^d extends to a rational function of $\xi \in \Xi$. To stress this dependence we sometimes write $f^d(\xi, h)$. We write $\xi = (P, \delta, t^P)$ and put $t = t(\xi) = r_P t^P$, where $r_P \in T_P$ is such that $W_P r_P$ is equal to the central character of δ.

Lemma 6.10. *Let $P, Q \in \mathscr{P}$ and let $d \in D^{Q,P}$ be such that $d(P) \subset Q$. Let $h, h' \in \overline{\mathscr{H}}_t$. Then*

$$f^d_{a,b}(\xi; hh') = f^d_{a,\pi(\xi;h')(b)}(\xi; h). \tag{6.1}$$

Proof. This follows immediately from Corollary 3.26. □

Lemma 6.11. *As in Lemma 6.10. Let $g \in \mathscr{W}_P$ and put $P' = g(P)$. According to Corollary 6.8 we can write $dg^{-1} = u'd'$ with $d' \in D^{Q,P'}$ and $u' \in W_Q^{P'}$. We put $t' = g(t)$ and $\varpi' = W_{P'}t' = g(\varpi)$, so that $e_{W_Q d\varpi} = e_{W_Q d'\varpi'}$. With these notations we have the following equality of rational functions of ξ:*

$$f^d_{a,b}(\xi; h) = f^{d'}_{\pi(g,\bar{\xi}^{-1})(a),\pi(g,\xi)(b)}\big(g(\xi); h\big), \tag{6.2}$$

where $\bar{\xi}^{-1} := (P, \delta, \overline{t^{P}}^{-1})$.

Proof. This equation follows from the special case $\xi \in \Xi_{P,\delta,u}$ because the left-hand side and the right-hand side are obviously rational functions of ξ. In this special case the equation simply expresses the unitarity of the intertwiners (cf. Theorem 3.16). □

Lemma 6.12. *Let $P, Q \in \mathscr{P}$. Then $\mathscr{H}$ has the following direct sum decomposition in left $\mathscr{H}^Q$-right $\mathscr{H}(W_P)$-submodules:*

$$\mathscr{H} = \bigoplus_{d \in D^{Q,P}} \mathscr{H}_{Q,P}(d), \tag{6.3}$$

where $\mathscr{H}_{Q,P}(d) := \mathscr{H}^Q N_d \mathscr{H}(W_P)$.

Proof. Using the Bernstein presentation of $\mathscr{H}^Q$ and the definition of the multiplication in $\mathscr{H}(W_0)$ we easily see that

$$\mathscr{H}_{Q,P}(d) = \bigoplus_{w \in W_Q d W_P} \mathscr{A} N_w. \tag{6.4}$$

The result thus follows from the Bernstein presentation of $\mathscr{H}$. □

After these preparations we will now concentrate on an important special case.

Definition 6.13. Let $\pi^d_{Q,P} : \mathscr{H} \to \mathscr{H}_{Q,P}(d)$ denote the projection according to the above direct sum decomposition. Given $Q \in \mathscr{P}$, denote by $w^Q = w_0 w_Q$ the longest element of W^Q, and by $Q' = w^Q(Q) = -w_0(Q) \in \mathscr{P}$. Then $w^{Q'} = (w^Q)^{-1} \in D^{Q,Q'}$, and

$$\mathscr{H}_{Q,Q'}(w^{Q'}) = \mathscr{H}^Q N_{w^{Q'}} = \mathscr{A} N_{w^{Q'}} \mathscr{H}(W_{Q'}). \tag{6.5}$$

Let $p_Q : \mathscr{H} \to \mathscr{H}^Q$ be the left $\mathscr{H}^Q$-module map defined by

$$p_Q(h) := \pi^{w^{Q'}}_{Q,Q'}(h) N^{-1}_{w^{Q'}}. \tag{6.6}$$

(Observe that this map indeed has values in $\mathscr{H}^Q$ by (6.4).)

In (6.5) we have used that $N_{w^{Q'}} N_{w'} = N_w N_{w^{Q'}}$ if $w \in W_Q$, where $w' = w^Q w w^{Q'} \in W_{Q'}$.

Theorem 6.14. *Let $P, Q \in \mathscr{P}$ be such that $P \subset Q$. We put $P' := w^Q(P) \subset Q' \in \mathscr{P}$ and $\xi' = w^Q(\xi) := (P', \delta', t^{P'})$.*

Let $a' \in i(V_{\delta'}) = \mathscr{H}(W^{P'}) \otimes V_{\delta'}$, $b' \in \mathscr{H}(W_{Q'}^{P'}) \otimes V_{\delta'} \subset i(V_{\delta'})$ and let $h \in \mathscr{H}$. We introduce the unitary isomorphism

$$\sigma := \psi_{w^Q} \otimes \tilde{\delta}_{w^Q} : \mathscr{H}(W_Q^P) \otimes V_\delta \to \mathscr{H}(W_{Q'}^{P'}) \otimes V_{\delta'}, \tag{6.7}$$

(see Section 3.5 for the explanations of ψ_{w^Q} and $\tilde{\delta}_{w^Q}$) and the orthogonal projection

$$\rho : i(V_\delta) \to \mathscr{H}(W_Q^P) \otimes V_\delta. \tag{6.8}$$

With these notations, put

$$\begin{aligned} a &:= \rho\big(\pi(w^Q, \bar{\xi}^{-1})^{-1}(a')\big) \in \mathscr{H}(W_Q^P) \otimes V_\delta, \\ b &:= \sigma^{-1}(b') \in \mathscr{H}(W_Q^P) \otimes V_\delta. \end{aligned} \tag{6.9}$$

We then have, with $c^Q(\xi) := \prod_{\alpha \in R_{0,+} \setminus R_{Q,+}} c_\alpha(t)$, the equality of rational functions of ξ

$$f_{a',b'}^{w^{Q'}}(\xi', h) = q(w^Q)^{1/2} c^Q(\xi) f_{Q,a,b}\big(\xi, p_Q(h)\big), \tag{6.10}$$

where $p_Q(h)$ has been defined in (6.6). Here $f_{Q,a,b}(\xi, h) = f_{a,b}(\xi, h)$ (with $h \in \mathscr{H}^Q$, $a, b \in \mathscr{H}(W_Q^P) \otimes V_\delta$) is the matrix coefficient (associated to the pair a, b) of the representation

$$\pi^Q(\xi) := \mathrm{Ind}_{\mathscr{H}^P}^{\mathscr{H}^Q} \delta_{t^P} \tag{6.11}$$

of $\mathscr{H}^Q$ (which is tempered and unitary if $\xi \in \Xi_u$).

Proof. Choose $r_P \in T_P$ such that $W_P r_P$ is the central character of δ, and write $t' = w^Q(t)$ with $t = r_P t^P$. Since we are dealing with rational functions of ξ it is sufficient to assume that ξ is regular, i.e., that t is R_P-regular. We then extend $\pi(\xi')$ to the completion $\overline{\mathscr{H}}_t$ (recall 2.6.3) and study $\pi(\xi')$ in the light of the isomorphisms (2.13) and (3.6).

We combine, in the decomposition (2.13) applied to the parabolic $P' = w^Q(P)$ and parameter t', the idempotents according to left cosets of $W_{Q'}$. In other words, we partition $W_0 t$ into the sets $w(\Omega)$ with $w \in W^{Q'}$ and $\Omega = W_{Q'} t' = W_{Q'}^{P'} \varpi'$ (with $\varpi' = w^Q(\varpi) = W_{P'} t'$). These sets are evidently unions of the original equivalence classes in formula (2.13) (with respect to P' and t'), the left $W_{P'}$-cosets acting on t'. We denote the corresponding idempotents by (for all $w \in W^{Q'}$)

$$e_w^\# := \sum_{x \in W_{Q'}^{w^Q(P)}} e_{wx\varpi'}.$$

Note that t' is P'-generic, and thus certainly Q'-generic. The structure formula (2.13) holds therefore, also in terms of the idempotents $e_w^\#$, where we replace in (2.13) the parabolic P' by Q'.

Remark that $e^{\#}_{w^{Q'}\Omega} N_w e^{\#}_{\Omega} = 0$ for any $w \in W_0$ with length of less than $|R_{0,+} \backslash R_Q|$ (= the length of $w^{Q'}$). Note by the way that $e^{\#}_{w^{Q'}\Omega} = e^{\#}_{W_Q t}$. Thus for all $d \in D^{Q,Q'}$, $d \neq w^{Q'}$ we see that $e^{\#}_{W_Q t} \mathscr{H}^{Q,P}(d) e^{\#}_{w^Q W_Q t} = 0$.

Hence for all $h \in \mathscr{H}$, $a' \in i(V_{\delta'})$ and $b' \in \mathscr{H}(W^{P'}_{Q'})$ we have

$$f^{w^{Q'}}_{a',b'}(\xi', h) = f_{a',b'}\big(\xi', p_Q(h) e^{\#}_{W_Q t} N_{w^{Q'}} e^{\#}_{w^Q W_Q t}\big). \tag{6.12}$$

Since $f^{w^{Q'}}_{a',b'}(\xi', \mathscr{H}\mathscr{I}_t) = 0$ we can use the analog of formula (4.58) of [26] (we use here that the c-function $c^Q(t)$ is W_Q-invariant, together with the argument in the proof of the Proposition 6.4. This makes that we can evaluate the c-factors at t'):

$$f^{w^{Q'}}_{a',b'}(\xi', h) = q(w^Q)^{1/2} c^Q(t) f_{a',b'}\big(\xi', p_Q(h) \imath^0_{w^{Q'}}\big). \tag{6.13}$$

We use Lemma 6.10 and then rewrite the result using (2.13) and Definition 3.8. Assume that $b' = x' \otimes v'$ and $b = \sigma^{-1}(b') = x \otimes v$. Then

$$\begin{aligned}
\imath^0_{w^{Q'}}(b') &= \imath^0_{w^{Q'}}(x' e^{\#}_{W_{Q'} t'} \otimes v') \\
&= e^{\#}_{W_Q t}\big(\psi_{w^{Q'}}(x') \imath^0_{w^{Q'}} \otimes v'\big) \\
&= e^{\#}_{W_Q t} \pi(w^Q, \xi)(x \otimes v) \\
&= \pi(w^Q, \xi)(b).
\end{aligned}$$

(In the first equality we used the identification (3.6), and in the second equality we used equation (2.14).) Thus we obtain

$$\begin{aligned}
f^{w^{Q'}}_{a',b'}(\xi', h) &= q(w^Q)^{1/2} c^Q(\xi) f_{a', \pi(w^Q,\xi)(b)}\big(\xi', p_Q(h)\big) \\
&= q(w^Q)^{1/2} c^Q(\xi) f_{(\pi(w^Q, \bar{\xi}^{-1})^{-1}(a')), b}\big(\xi, p_Q(h)\big) \\
&= q(w^Q)^{1/2} c^Q(\xi) f_{Q,a,b}\big(\xi, p_Q(h)\big).
\end{aligned} \tag{6.14}$$

In the second step we used the unitarity of the intertwining operators $\pi(w^Q, \xi)$ to rewrite the matrix coefficient as a coefficient of the induced representation $\pi(\xi)$ (extended holomorphically as in Lemma 6.11; in fact it is a simple special case of this lemma). Since $b \in \mathscr{H}(W^P_Q) \otimes V_\delta$ and $p_Q(h) \in \mathscr{H}^Q$, we can project the vector $\pi(w^Q, \bar{\xi}^{-1})^{-1}(a')$ onto $\mathscr{H}(W^P_Q) \otimes V_\delta$, and consider the result as a matrix coefficient of $\pi^Q(\xi)$. □

Theorem 6.15. *Fix $P \in \mathscr{P}$ and $\delta \in \Delta_{P, W_P r_P}$. We recall that $\Xi_{P,\delta} \subset \Xi$ is the collection of standard induction data of the form (P, δ, t^P) with $t^P \in T^P$, and we denote by $\Xi_{P,\delta,u} \subset \Xi_{P,\delta}$ the subset of such triples with $t^P \in T^P_u$. Then for all $d \in D^{Q,P}$ such that $d(P) \subset Q$ and for all $a, b \in i(V_\delta)$, the rational function*

$$\xi \mapsto c(\xi)^{-1} f^d_{a,b}(\xi, h) \tag{6.15}$$

is regular in a neighbourhood of $\Xi_{P,\delta,u}$.

Proof. We apply Lemma 6.11 with

$$g = w^Q d \in W(P, P') \quad \text{where } P' = w^Q\big(d(P)\big) \subset Q'.$$

Notice that $d' = w^{Q'}$. Put $\xi' = g(\xi)$ and

$$\begin{aligned} a' &= \pi(g, \bar{\xi}^{-1})(a), \\ \tilde{b}' &= \pi(g, \xi)(b). \end{aligned} \tag{6.16}$$

We obtain

$$c(\xi)^{-1} f^d_{a,b}(\xi, h) = c(\xi)^{-1} f^{w^{Q'}}_{a',\tilde{b}'}(\xi', h). \tag{6.17}$$

Now we can *uniquely* decompose $\tilde{b}'$ in the following way:

$$\tilde{b}' = \pi(\xi', \tilde{h})(b') \tag{6.18}$$

with $\tilde{h} \in \mathscr{H}(W^{Q'})$ and $b' \in \mathscr{H}(W^{P'}_{Q'}) \otimes V_{\delta'} \subset i(V_{\delta'})$. With the help of Lemma 6.10 we get

$$c(\xi)^{-1} f^d_{a,b}(\xi, h) = c(\xi)^{-1} f^{w^{Q'}}_{a',b'}(\xi', h\tilde{h}). \tag{6.19}$$

We can now apply Theorem 6.14 with respect to $d(P) \subset Q$. We put

$$\begin{aligned} a_d &:= \rho\big(\pi(w^Q, d\bar{\xi}^{-1})^{-1}(a')\big) \in \mathscr{H}(W_Q^{d(P)}) \otimes V_{\Psi_d(\delta)}, \\ b_d &:= \sigma^{-1}(b') \in \mathscr{H}(W_Q^{d(P)}) \otimes V_{\Psi_d(\delta)} \end{aligned} \tag{6.20}$$

to obtain:

$$c(\xi)^{-1} f^d_{a,b}(\xi, h) = q(w^Q)^{1/2} c(\xi)^{-1} c(d\xi) c_Q\big((d\xi)_Q\big)^{-1} f_{Q,a_d,b_d}\big(d\xi, p_Q(h\tilde{h})\big), \tag{6.21}$$

where in general for $Q \supset P$ and $\xi \in \Xi_{P,\delta}$ we denote

$$c_Q(\xi_Q) := \prod_{\alpha \in R_{Q,+} \setminus R_{P,+}} c_\alpha(t). \tag{6.22}$$

The regularity of the normalization factor $c_Q\big((d\xi)_Q\big)^{-1}$ as a function of $d\xi$ (and thus as a function of ξ) follows from [26], Theorem 3.25, when we consider the tempered residual coset $r_P T^P_u \subset T$ for the Hecke algebra $\mathscr{H}^Q$ (instead of $\mathscr{H}$ itself). It is a simple special case of Proposition 9.8(v). Similarly, the regularity of $c(\xi)^{-1} c(d\xi)$ is asserted by Proposition 9.8(iv). By the regularity of the various intertwining operators we have used (cf. Theorem 3.16) it is clear that also a_d, b_d are rational and regular on $\Xi_{P,\delta,u}$. We have finished the proof. □

We keep the hypothesis and notations of Theorem 6.14 (with P replaced by $d(P) \subset Q$) and of Theorem 6.15. Define

$$\begin{aligned} \alpha_{\xi,d} : V_{\bar{\xi}^{-1}} &\to \mathscr{H}(W_Q^{d(P)}) \otimes_{\mathscr{H}^{d(P)}} V_{\Psi_d(\delta), d\bar{t}^{-1}}, \\ a &\to \eta(w^{Q'}, g) \rho\big(\pi\big(d, \bar{\xi}^{-1}(a)\big)\big), \end{aligned} \tag{6.23}$$

where η denotes the cocycle (3.7), and

$$(6.24)\qquad \beta_{\xi,d} : V_\xi \to \mathscr{H}(W_Q^{d(P)}) \otimes_{\mathscr{H}^{d(P)}} V_{\Psi_d(\delta),dt},$$
$$b \to \tilde{p}_{\xi,d,Q}\big(\pi(g,\xi)(b)\big),$$

where $\tilde{p}_{\xi,d,Q}$ denotes the $\mathscr{H}^Q$-module map defined by

$$(6.25)\qquad \tilde{p}_{\xi,d,Q} : V_{g\xi} = \mathscr{H} \otimes_{\mathscr{H}^{g(P)}} V_{\Psi_g(\delta),gt} \to \mathscr{H}(W_Q^{d(P)}) \otimes_{\mathscr{H}^{d(P)}} V_{\Psi_d(\delta),dt},$$
$$h \otimes v \to p_Q(h) \otimes \widetilde{\Psi_g(\delta)}_{w^{Q'}}(v).$$

This last map is well defined in view of the following property of the projection map p_Q: For all $h \in \mathscr{H}$ and $h^Q \in \mathscr{H}^Q$ we have

$$(6.26)\qquad p_Q(h)h_Q = p_Q\big(h\psi_{w^Q}(h^Q)\big).$$

The formula for the constant term of $f_{a,b}$ can now be expressed in terms of the maps $\alpha_{\xi,d}$ and $\beta_{\xi,d}$. The resulting formula is the analogue for Hecke algebras of the formula for the weak constant of coefficients of tempered representations of reductive p-adic groups, cf. [36], Proposition V1.1.

Corollary 6.16. *Using the above notations, and assuming the hypotheses of Theorem* 6.15, *we have obtained the following formula for the constant term along Q of the matrix coefficients of $\pi(\xi)$. For all $h \in \mathscr{H}^Q$ and $\xi \in \Xi_{P,\delta,u}$:*

$$(6.27)\qquad c(\xi)^{-1} f_{a,b}^Q(\xi,h) = \sum_{d \in D^{Q,P}: d(P) \subset Q} c_{\xi,d} f_{Q,\alpha_{\xi,d}(a),\beta_{\xi,d}(b)}\big((d\xi)_Q, h\big)$$

where $(d\xi)_Q$ is the induction datum $d\xi$, considered as an induction datum for the Hecke algebra $\mathscr{H}^Q$, and

$$(6.28)\qquad c_{\xi,d} = q(w^Q)^{1/2} c(\xi)^{-1} c(d\xi) c_Q\big((d\xi)_Q\big)^{-1}.$$

In equation (6.27) *we identify the target spaces of $\alpha_{\xi,d}$ and $\beta_{\xi,d}$ with the vector spaces of the "compact realization" of the induced representation $\pi\big((d\xi)_Q\big)$ of $\mathscr{H}^Q$ (notice that by our assumption $\xi \in \Xi_{P,\delta,u}$ we have $\bar{\xi}^{-1} = \xi$). The expressions $f_{Q,\alpha_{\xi,d}(a),\beta_{\xi,d}(b)}\big((d\xi)_Q, h\big)$ and the coefficients $c_{\xi,d}$ of equation* (6.27) *extend to rational functions of $\xi \in \Xi_{P,\delta}$, regular on $\Xi_{P,\delta,u}$. The map $\beta_{\xi,d}$ is an $\mathscr{H}^Q$-module map. This property determines the map $\beta_{\xi,d}$ uniquely up to a scalar function (since the multiplicity of $\pi\big((d\xi)_Q\big)$ as a subquotient of the restriction to $\mathscr{H}^Q$ of $\pi(\xi)$ is one).*

Proof. This is merely a reformulation of the previous theorem in view of the formula

$$(6.29)\qquad f_{a,b}^Q(\xi,h) = \sum_{d \in D^{Q,P}: d(P) \subset Q} f_{a,b}^d(\xi,h)$$

using the definition of the maps $\alpha_{\xi,d}$ and $\beta_{\xi,d}$. Indeed, it is straightforward to see that $\alpha_{\xi,d}(a) = a_d$ (in the notations of the proof of Theorem 6.15) and that $\beta_{\xi,d}(b) = p_Q(\tilde{h})b_d$.

The map $\beta_{\xi,d}$ is a $\mathscr{H}^Q$-module morphism by the properties of p_Q, see Definition 6.13. The last assertions follow from Corollary 11.3 and Proposition 3.18(i). $\square$

Corollary 6.17. *In particular, for all $h \in \mathscr{H}$, $P \in \mathscr{P}$, $\delta \in \Delta_P$, and $a, b \in i(V_\delta)$ fixed, the function*

$$\Xi_{P,\delta,u} \times \mathscr{H}^Q \ni (\xi, h^Q) \mapsto c(\xi)^{-1} f^Q_{a,b}(\xi, h^Q h) \tag{6.30}$$

is a linear combination with coefficient functions which are regular rational functions on $\Xi_{P,\delta,u}$, of normalized matrix coefficients

$$c_Q\big(d(\xi)\big)^{-1} f_{Q,a',b'}\big((d\xi)_Q, h^Q\big) \tag{6.31}$$

of induced representations of $\mathscr{H}^Q$ of the form $\pi^Q\big((d\xi)_Q\big)$ (where d ranges over the Weyl group elements $d \in D^{Q,P}$ such that $d(P) \subset Q$).

7. Proof of the Main Theorem

7.1. Uniform estimates for the coefficients of $\pi(\xi)$. Recall that X^+ is the cone $\{x \in X \mid \langle x, \alpha^\vee \rangle \geqq 0 \text{ for all } \alpha \in R_{1,+}\}$. We put $Z_X = X^+ \cap X^-$. This is a sublattice of elements in X with length 0. Recall that Q denotes the root lattice. The sublattice $Q \oplus Z_X \subset X$ has finite index in X. If $x = x_Q + x_Z \in Q \oplus Z_X$ then

$$\mathscr{N}(x) = x_Q(2\rho^\vee) + \|x_Z\|. \tag{7.1}$$

Let us show that $Q^+ := Q \cap X^+$ is finitely generated over $\mathbb{Z}_+$. For each fundamental weight δ_i, let $q_i = m_i\delta_i$ be the least multiple of δ_i such that $q_i \in Q$ (thus $m_i \in \mathbb{N} = \{1, 2, 3, \ldots\}$ is a divisor of the index of Q in the lattice generated by fundamental weights). These multiples generate over $\mathbb{Z}_+$ a cone $C^+ \subset Q^+$. Let $F = \left\{ \sum_i t_i q_i \,\middle|\, t_i \in [0, 1) \right\}$ and let $F_Q = F \cap Q \subset Q^+$ (a finite set). Clearly F_Q and the $\{q_i\}$ generate Q^+ over $\mathbb{Z}_+$. Let $x_1, \ldots, x_m, x_{m+1}, \ldots, x_N \in X^+$ such that $x_1, \ldots, x_m$ is a set of $\mathbb{Z}_+$-generators of Q^+ and that $x_{m+1}, \ldots, x_N \in Z_X$ is a $\mathbb{Z}$-basis of Z_X. By (7.1) we see that there exists a constant $K > 0$ such that for all $x \in Q^+ + X_Z$ and all decompositions $x = \sum l_i x_i$ with $l_i \geqq 0$ if $i \leqq m$, we have

$$\sum |l_i| \leqq K\mathscr{N}(x) \tag{7.2}$$

(just observe that $x_i(2\rho^\vee) \geqq 1$ if $i = 1, \ldots, m$). We fix such a $K > 0$.

We define a function v on T_{rs} by

$$v(t) = \max\big(\{|x_i(t)| \mid i = m+1, \ldots, N\} \cup \{|x_i(wt)| \mid i = 1, \ldots, m; w \in W_0\}\big). \tag{7.3}$$

The positive real cone spanned by the elements wx_i $(w \in W_0, i = 1, \ldots, m)$ and $\pm x_i$ $(i = m+1, \ldots, N)$ is the full dual of $\mathrm{Lie}(T_{rs})$. Therefore the function $\log(v) \circ \exp$ is a norm on $\mathrm{Lie}(T_{rs})$.

Theorem 7.1. *Let $R > 1$, $P \in \mathscr{P}$, and $\delta \in \Delta_P$ be given. Choose a set $x_i \in X^+$ as above and let K and ν be as above. We use the notation $\nu(|\xi|) := \nu(|t^P|)$ for $\xi = (P, \delta, t^P) \in \Xi_{P,\delta}$. Define a compact neighbourhood $D^P(R) \subset \Xi_{P,\delta}$ of $\Xi_{P,\delta,u} \subset \Xi_{P,\delta}$ by $D^P(R) = \{\xi \in \Xi_{P,\delta} \mid \nu(|\xi|) \leqq R\}$.*

There exists a $d \in \mathbb{N}$, and there exists a constant $c > 0$ (depending on R only) such that for all $w \in W$, for all $a, b \in i(V_\delta)$, and for all $\xi \in D^P(R)$, the matrix coefficient $f_{a,b}(\xi, N_w)$ satisfies

$$|f_{a,b}(\xi, N_w)| \leqq c\|a\|\,\|b\|\big(1 + \mathscr{N}(w)\big)^d \nu(|\xi|)^{K\mathscr{N}(w)}. \tag{7.4}$$

Proof. Using [26], equation (2.27) (also see the proof of Lemma 7.14) we see that it is equivalent to show that $f_{a,b}(\xi, N_u\theta_x N_v)$ can be estimated by the right-hand side of (7.4) with $w = x$, for all $u, v \in W_0$ and $x \in X^+$. By applying right (resp., left) translations of the matrix coefficient $f_{a,b}(\xi)$ by N_v (resp., N_u) and by a set of representatives of the finite quotient $X/(Q + Z_X)$ we see that we may further reduce to proving the estimates (7.4) for $w = x \in Q^+ + Z_X$.

Recall (cf. [26], Proposition 4.20 and its proof) that the eigenvalues of the matrix of $\pi(\xi, \theta_x)$ are of the form $x\big(w_i(r_j t^P)\big)$. Here the $r_j \in T_P$ are the generalized X_P-eigenvalues of the discrete series representation δ. By Casselman's criterion we know therefore that for all $x \in X^+$, $x\big(w_i(r_j)\big) \leqq 1$. This implies that for all $i \in 1, \dots, m$, and for all $\xi \in \Xi_{P,\delta}$, the eigenvalues of $\pi(\xi, \theta_{x_i})$ are bounded by $\nu(|\xi|)$. Then Lemma 8.1 allows one to estimate the norm of $\pi(\xi, \theta_{l_i x_i})$, by dividing $\pi(\xi, \theta_{x_i})$ by $\nu(|\xi|)$.

Taking into account the fact that $D^R(P)$ is compact, one sees that the norm of $\pi(\xi, \theta_{x_i})$ is bounded if ξ is in $D^R(P)$. By a simple product formula, one estimates the norm of $\pi(\xi, \theta_x)$. These estimates together with equation (7.2) imply the desired result. □

Corollary 7.2. *For all constant coefficient differential operators D on $\Xi_{P,\delta}$ there exist constants $d \in \mathbb{N}$ and $c > 0$ such that for all $\xi \in \Xi_{P,\delta,u}$, for all $a, b \in i(V_\delta)$, and for all $w \in W$*

$$|Df_{a,b}(\xi, N_w)| \leqq c\|a\|\,\|b\|\big(1 + \mathscr{N}(w)\big)^d. \tag{7.5}$$

Proof. This is a standard application of the Cauchy integral formula, starting with equation (7.4). Choose a basis $x_1, \dots, x_p$ of the character lattice X^P of T^P, and let $y_1, \dots, y_p$ be the dual basis. Let $C_\epsilon := \{v \in \mathrm{Lie}(T^P) \mid \forall i : |x_i(v)| = \epsilon\}$. We may assume that D is of the form $D = D^\alpha := y_1^{\alpha_1} \dots y_p^{\alpha_p}$. By the holomorphicity of $f_{a,b}$ we have, for a suitable constant $C_\alpha > 0$ and any choice of a sequence of radii $\epsilon(w)$:

$$D^\alpha f_{a,b}(\xi, N_w) = C_\alpha \int_{v \in C_{\epsilon(w)}} \frac{f_{a,b}\big(\exp(v) \cdot \xi, N_w\big)}{\prod_i x_i(v)^{\alpha_i + 1}} dx_1 \wedge \dots \wedge dx_p. \tag{7.6}$$

Now use the estimates of Theorem 7.1 with the sequence $\epsilon(w)$ chosen such that $r(w) := \max\big\{\nu\big(|\exp(v)|\big) \mid v \in C_{\epsilon(w)}\big\}$ is equal to

$$1 + 1/\big(1 + \mathscr{N}(w)\big). \tag{7.7}$$

But $\log(v) \circ \exp$ is a norm on $\mathrm{Lie}(T^P)$, as well as $\operatorname{Sup}_i |x_i(v)|$. They are equivalent. Moreover $\log(1+x) \geqq k'x$ for $x \in [0,1]$, for some $k' > 0$. Together with (7.7) this implies that there exists a constant $k > 0$ such that $\epsilon(w) \geqq k/\big(1 + \mathcal{N}(w)\big)$. So equation (7.6) yields the estimate (for some constant $c' > 0$)

$$(7.8) \qquad |D^\alpha f_{a,b}(\xi, N_w)| \leqq c' \|a\| \, \|b\| \big(1 + \mathcal{N}(w)\big)^{d+|\alpha|} \Big(1 + 1/\big(1 + \mathcal{N}(w)\big)\Big)^{K\mathcal{N}(w)}$$

for all $w \in W$ and for all multi-indices α. This easily leads to the desired result. $\square$

Corollary 7.3. *We have $\mathscr{F}(\mathscr{S}) \subset C^\infty\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)^{\mathscr{W}}$. The restriction $\mathscr{F}_{\mathscr{S}}$ of $\mathscr{F}$ to $\mathscr{S}$ defines a continuous map $\mathscr{F}_{\mathscr{S}} : \mathscr{S} \to C^\infty\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)^{\mathscr{W}}$.*

Proof. The equivariance of the sections in the image is clear. Recall that $\mathscr{F}(N_w) \in \mathrm{Pol}\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)$ is defined by $\mathscr{F}(N_w)(\xi) = \pi(\xi, N_w)$.

Hence by the estimates of Corollary 7.2 we see that for any continuous seminorm p on $C^\infty\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)$ there exist constants $C > 0$ and $d \in \mathbb{Z}_+$ such that $p\big(\mathscr{F}(N_w)\big) \leqq C\big(1 + \mathcal{N}(w)\big)^d$.

Let $b \in \mathbb{Z}_+$ be such that $0 < C_b := \sum_{w \in W} \big(1 + \mathcal{N}(w)\big)^{-b} < \infty$, and let $q = q_p$ denote the continuous seminorm on $\mathscr{S}$ defined by $q(x) := CC_b \sup_w |(x, N_w)| \big(1 + \mathcal{N}(w)\big)^{d+b}$. Then $p\big(\mathscr{F}_{\mathscr{S}}(x)\big) \leqq q(x)$ for all $x \in \mathscr{H}$, implying that $\mathscr{F}_{\mathscr{S}}$ is a continuous map as claimed. $\square$

7.2. Smooth and normalized smooth families of coefficients and their constant terms. We introduce the important notion of a (normalized) smooth family of coefficients:

Definition 7.4. Let $P \in \mathscr{P}$ and let $\delta \in \Delta_P$ be an irreducible discrete series of $\mathscr{H}_P$ with central character $W_P r_P \in W_P \backslash T_P$. We put $\xi = (P, \delta, t^P) \in \Xi_{P,\delta,u}$. A smooth family of coefficients of $\pi(\xi)$, $\xi \in \Xi_{P,\delta,u}$ is a family of linear functionals on $\mathscr{H}$ of the form

$$(7.9) \qquad \mathscr{H} \ni h \mapsto \mathrm{Tr}\big(\sigma(\xi)\pi(\xi)(h)\big),$$

where σ is a section of $C^\infty\big(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi)\big)$.

A smooth section $\sigma \in C^\infty\big(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi)\big)$ is called normalized smooth when it is divisible (in the $C^\infty(\Xi_{P,\delta,u})$-module $C^\infty\big(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi)\big)$) by the smooth function $\{\xi \mapsto c^{-1}(\xi)\} \in C^\infty(\Xi_{P,\delta,u})$ (cf. Proposition 9.8).

A normalized smooth family of coefficients of $\pi(\xi)$, $\xi \in \Xi_{P,\delta,u}$ is a smooth family of coefficients (7.9) for which σ is normalized smooth.

Remark 7.5. We frequently use t^P rather than $\xi = (P, \delta, t^P)$ as the parameter of a family of coefficients.

Corollary 7.6. *It follows directly from the definitions that smooth (resp., normalized smooth) families of coefficients of $\pi(\xi)$, $\xi \in \Xi_{P,\delta,u}$, are stable under left and right translations by elements $h \in \mathscr{H}$.*

Remark 7.7. Using the smoothness of the induction functor (Theorem 3.16) we can exhibit the following equivariance property of smooth families of coefficients with respect to the action of $\mathscr{W}$. Let $g = k \times w \in \mathscr{W}_P$ and let Φ_ξ^σ, $\xi \in \Xi_{P,\delta,u}$ denote the smooth family of coefficients associated with the smooth section $\sigma \in C^\infty(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi))$. Let $g(\sigma) \in C^\infty(\Xi_{w(P),\Psi_g(\delta),u}, \mathrm{End}(\mathscr{V}_\Xi))$ be the smooth section defined by (with $\eta \in \Xi_{w(P),\Psi_g(\delta),u}$)

$$g(\sigma)(\eta) = \pi(g^{-1},\eta)^{-1}\sigma(g^{-1}(\eta))\pi(g^{-1},\eta). \tag{7.10}$$

By the intertwining property of $\pi(g^{-1}, g(\xi))$ we have

$$\Phi_{g(\xi)}^{g(\sigma)} = \Phi_\xi^\sigma. \tag{7.11}$$

7.2.1. Constant term of (normalized) smooth families of coefficients. We start with a general result about the constant terms of smooth and of holomorphic families:

Proposition 7.8. *Let Φ_ξ, $\xi \in \Xi_{P,\delta,u}$, be a smooth family of coefficients for $\mathscr{H}$. Let $Q \in \mathscr{P}$.*

(i) *The constant term $\mathscr{H}^Q \ni h \mapsto \Phi_\xi^Q(h)$ is identically equal to 0, unless there exists a $w \in W^P$ such that $w(P) \subset Q$.*

(ii) *Assume $P \subset Q$. The constant term $\mathscr{H}^Q \ni h \mapsto \Phi_\xi^Q(h)$ is a smooth family of coefficients for $\mathscr{H}^Q$. Here we view $\Xi_{P,\delta,u}$ both as a collection of tempered standard induction data for $\mathscr{H}$ and for $\mathscr{H}^Q$.*

(iii) *As in* (ii). *If Φ_ξ is actually holomorphic in a neighbourhood of $\Xi_{P,\delta,u} \subset \Xi_{P,\delta}$, then $\mathscr{H}^Q \ni h \mapsto \Phi_\xi^Q(h)$ is holomorphic in a neighbourhood of $\Xi_{P,\delta,u}$ as well.*

Proof. (i) Let t denote a generalized weight for $\pi(\xi)$, $\xi \in \Xi_{P,\delta,u}$. By Proposition 6.1, $t = w(rt^P)$ for some $w \in W^P$ and $r \in T_P$ a weight of δ. Suppose that $|t| \leqq_Q 1$. By the argument at the end of the proof of Proposition 6.2 this implies that $w(P) \subset Q$.

(ii) and (iii) Clearly it is enough to prove (iii) for the case where $\Phi_\xi(h) = f_{a,b}(\xi,h)$ for some $a, b \in i(V_\delta)$. Let $k \in \mathbb{N}$ be such that $kP(R_0) \subset X$. Let $(\delta_\alpha)_{\alpha \in F_0}$ be the fundamental weights. Thus for any $t \in T$ one has: $|t| \leqq_Q 1$ if and only if $|t| \leqq_{F_0} 1$ and $|t(k\delta_\alpha)| = 1$ for all $\alpha \in Q$. By definition of the constant term we have

$$f_{a,b}^Q(\xi,h) = \langle P_\xi a, \pi(\xi,h)b\rangle, \tag{7.12}$$

where P_ξ is the product of the spectral projections of the commuting operators $\pi(\xi,\theta_{k\delta_\alpha})^*$, $\alpha \in Q$ corresponding to the eigenvalues of modulus 1. By Proposition 6.1, the eigenvalues of $\pi(\xi,\theta_{k\delta_\alpha}^*)$ are of the form $w(t^P r)(k\delta_{\alpha'})$, with $\alpha' = -w_0\alpha$, r a weight of δ, and $w \in W^P$ (we use the well known formula $\theta_x^* = N_{w_0}\theta_{-w_0x}N_{w_0}^{-1}$). Observe that the moduli of these eigenvalues are constant for $\xi \in \Xi_{P,\delta,u}$. We divide the eigenvalues in two disjoint subsets: those which are of modulus one for $\xi \in \Xi_{P,\delta,u}$ and the others, which are of modulus strictly less than 1 for $\xi \in \Xi_{P,\delta,u}$. Thus if $\varepsilon > 0$ is sufficiently small, there exists a neighbourhood U of $\Xi_{P,\delta,u}$ such that the moduli of the eigenvalues of the first (resp., second) subset are strictly larger (smaller) than $1-\varepsilon$ if $\xi \in U$. The proposition follows using holomorphic functional calculus to express the spectral projections as in Corollary 8.2 (iii). $\square$

Remark 7.9. We may generalize Proposition 7.8(ii) to the case where P is associated with a subset of Q by using Remark 7.7. Choose $g = k \times w \in \mathscr{W}_P$ such that $w(P) \subset Q$. Then (by Remark 7.7 and Proposition 7.8(ii)) $\mathscr{H}^Q \ni h \mapsto \Phi^Q_{g^{-1}(\eta)}(h)$, $\eta \in \Xi_{w(P),\Psi_g(\delta),u}$ is a smooth family of coefficients for $\mathscr{H}^Q$. Here we view $\Xi_{w(P),\Psi_g(\delta),u}$ both as a collection of tempered standard induction data for $\mathscr{H}$ and for $\mathscr{H}^Q$.

For the constant term of a normalized smooth family we have the following consequence of Corollary 6.17:

Proposition 7.10. *The restriction to $\mathscr{H}^Q$ of the constant term of a normalized smooth family of $\pi(\xi) = \pi(P,\delta,t^P)$, $t^P \in T_u^P$, along $Q \in \mathscr{P}$ is a finite sum of terms, each of these being a normalized smooth family of coefficients of $\pi^Q(d(\xi))$, where d is some Weyl group element with $d(P) \subset Q$.*

7.2.2. Uniform estimates of the difference of a smooth family of coefficients and its constant term.

Lemma 7.11. *Assume $Z_X = \{0\}$. Let $\Xi_{P,\delta,t^P} \ni \xi \mapsto \Phi_\xi$ be a smooth family of coefficients.*

Let $\alpha \in F_0$, and put $Q = F_0 \backslash \{\alpha\}$.

Let $\|\cdot\|$ denote a norm which comes from a W_0-invariant euclidean structure on $X \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $a > 0$ and let X_a^+ denote the cone (over $\mathbb{Z}_+$) $X_a^+ = \{x \in X^+ \mid \langle x, \alpha^\vee \rangle > a\|x\|\}$. Then there exists $C, b > 0$ such that

$$|(\Phi_\xi - \Phi_\xi^Q)(N_u \theta_x N_v)| \leqq C e^{-b\|x\|}, \tag{7.13}$$

for all $x \in X_a^+$, $\xi \in \Xi_{P,\delta,u}$, $u \in W_Q$, and $v \in W_0$.

Proof. Recall that the lattice X contains the root lattice $Q(R_0)$, and hence an integral multiple of the weight lattice $P(R_0)$, say $kP(R_0)$. We put $X' = kP(R_0) \subset X$ and we identify X' with $\mathbb{Z}^l$ via a basis of X' consisting of the elements $(k\delta_\beta)$, $\beta \in F_0$ (where the δ_β are the fundamental weights), ordered in such a way that $e_1 = k\delta_\alpha$. The temperedness of $\pi(P,\delta,t^P)$, $t^P \in T_u^P$, and the fact that its central character is given by $t = r_P t^P$ imply that the possible eigenvalues of $\pi(P,\delta,t^P)(\theta_{k\delta_\beta})$ are among the $wt(k\delta_\beta)$ with $w \in W_0$ such that $|wt(k\delta_\beta)| \leqq 1$. Moreover the modulus of $wt(k\delta_\beta)$, hence of $wt(e_1)$, does not depend on $t^P \in T_u^P$.

Hence if $u = v = e$ and $x \in X' = kP(R_0)$, (7.13) follows, in view of the definition of the constant term, from Lemma 8.3.

Let us now derive the general case of (7.13) from this special case. Since $Z_X = \{0\}$, X' is of finite index in X. One can assume that a is small enough, in such a way that X_a^+ is nonempty, otherwise there is nothing to prove. Let $x \in X_a^+$ and let y be the orthogonal projection of x on the line $\mathbb{R}\delta_\alpha$. By definition of X_Q we have $x - y \in X_Q$, and since $\langle x - y, \beta^\vee \rangle = \langle x, \beta^\vee \rangle$ for all $\beta \in Q$ we find that in fact $x - y \in X_Q^+$. Thus $x - y$ is a non-negative linear combination of the fundamental weights δ_β^Q of R_Q. It is a basic fact that the fundamental weights of a root system (with given basis of simple roots) have nonnegative rational coefficients in the basis of its simple roots (indeed, this statement reduces to the case of an irreducible root system, in which case the indecomposability of the Cartan matrix implies that these coefficients are in fact strictly positive). Hence we have $x - y \in X_Q^+ \subset \mathbb{Q}_+ Q$. Since $\langle \beta, \alpha^\vee \rangle \leqq 0$ for all $\beta \in Q$, we see that $\langle y, \alpha^\vee \rangle \geqq \langle x, \alpha^\vee \rangle > a\|x\| \geqq a\|y\|$. Hence $k\delta_\alpha \in X_a^+$.

Let $(x_1, \dots, x_r)$ be a set of representatives in X of X/X'. Let us show that one can choose the x_i in $-X_a^+$. Our claim is a consequence of the following fact. If $y \in X$, one has $y + n\delta_\alpha \in X_a^+$ for n large. In fact by the triangle inequality one has:

$$\langle y + n\delta_\alpha, \alpha^\vee \rangle - a\|y + n\delta_\alpha\| \geqq n(\langle \delta_\alpha, \alpha^\vee \rangle - a\|\delta_\alpha\|) + \langle y, \alpha^\vee \rangle - a\|y\|.$$

Thus, if $x \in X_a^+$ and $x = x' + x_i$, for some $x' \in X'$ and some i, one has $x' \in {X'}_a^+$. To get the estimates, one applies the previous estimates to the translates of the family Φ_ξ by the N_u (from the left), and by $\theta_{x_i} N_v$ (from the right), which are smooth families of coefficients themselves (cf. Corollary 7.6), taking into account Corollary 3.26. $\square$

7.3. Wave packets. Recall that $\mathscr{J}$ was introduced as the adjoint of $\mathscr{F}$. Thus if $\sigma \in L_2(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl})$ then $\mathscr{J}(\sigma) \in L_2(\mathscr{H})$, and is completely characterized by the value of $(\mathscr{J}(\sigma), h)$ where $h \in \mathscr{H}$. We have, using Theorem 4.3, that

$$(7.14) \qquad \mathscr{J}(\sigma)(h) := (\mathscr{J}(\sigma)^*, h) = (h^*, \mathscr{J}(\sigma)) = (\mathscr{F}(h^*), \sigma) = \mu_{\mathscr{R},\delta} \int_{\Xi_{P,\delta,u}} \mathrm{Tr}(\sigma(\xi)\pi(\xi, h)) |c(\xi)|^{-2} \, d\xi,$$

where $\mu_{\mathscr{R},\delta} = q(w^P)^{-1} |\mathscr{W}_P / \mathscr{K}_P|^{-1} \mu_{\mathscr{R}_P, Pl}(\{\delta\}) > 0$.

Recall Definition 5.1. Assume that $\tilde{\sigma} = c(w^P \cdot)\sigma \in \mathscr{C}(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi))$ (in other words, $\sigma \in C^\infty(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi))$).

Denote by Φ^σ the smooth family of coefficients $\Phi_\xi^\sigma(h) = \mathrm{Tr}(\sigma(\xi)\pi(\xi, h))$ associated with σ. Then by (7.14) (with $h \in \mathscr{H}$), we have

$$(7.15) \qquad \mathscr{J}(\tilde{\sigma})(h) = \mu_{\mathscr{R},\delta} W_\sigma(h),$$

where for any $\sigma \in C^\infty(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi))$, we put

$$(7.16) \qquad W_\sigma(h) := \int_{\Xi_{P,\delta,u}} \Phi_\xi^\sigma(h) c^{-1}(\xi) \, d\xi = \int_{\Xi_{P,\delta,u}} \mathrm{Tr}(\sigma(\xi)\pi(\xi, h)) c^{-1}(\xi) \, d\xi.$$

Theorem 7.12. *For every $k \in \mathbb{Z}_+$, there exists a continuous seminorm p_k on $C^\infty(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi))$ such that*

$$|W_\sigma(N_u\theta_x N_v)| \leqq (1 + \|x\|)^{-k} p_k(\sigma), \tag{7.17}$$

for all $x \in X^+$, $u, v \in W_0$ and $\sigma \in C^\infty(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi))$.

Proof. First, by using right and left translations by the N_w, $w \in W_0$, and Corollary 7.6, it is enough to prove (7.17) for $u = v = 1$. Thus, we assume $u = v = 1$ in the following.

The proof is by induction on the rank of X. The statement is clear if the rank of X is zero. One assumes the theorem is true for lattices of rank strictly smaller than the rank of X.

For the induction step we consider two cases, namely the case where $Z_X \neq 0$ (first case), and the case where $Z_X = 0$ (second case).

First case. In this case the semisimple quotient $\mathscr{H}_{F_0}$ of $\mathscr{H} = \mathscr{H}^{F_0}$ has smaller rank than $\mathscr{H}$. Recall the results of Proposition 2.2 and Proposition 2.3. Let us denote the semisimple quotient $\mathscr{H}_{F_0}$ of $\mathscr{H}$ by $\mathscr{H}_0$, its root datum $\mathscr{R}_{F_0}$ by $\mathscr{R}_0$, etc.

We have $T_P \subset T_0$ and $T^P \supset T^0$. Let $T_0^P = (T_0)^P$ be the connected component of e of the intersection $T_0 \cap T^P$. Then the product map $T_P \times T_0^P \to T_0$ is a finite covering, as is the product map $T_0^P \times T^0 \to T^P$. Let $\xi = (P, \delta, t^P)$ and suppose that $t^P = t_0^P t^0$ for $t_0^P \in T_{0,u}^P$ and $t^0 \in T_u^0$. Let $\xi_0 = (P, \delta, t_0^P) \in \Xi_{\mathscr{R}_0, P, \delta, u}$ denote the standard induction datum for $\mathscr{H}_0$. Recall the epimorphism $\phi_{t^0} : \mathscr{H} \to \mathscr{H}_0$ of Proposition 2.3. It is easy to see that $\pi(\xi) = \pi(\xi_0)_{t^0}$, the pull back of the representation $\pi(\xi_0)$ of $\mathscr{H}_0$ to $\mathscr{H}$ via ϕ_{t^0}. This implies that

$$\pi(\xi)(\theta_x) = t^0(x)\pi(\xi_0)(\theta_{x_0}) \tag{7.18}$$

for all $x \in X$, where $x_0 \in X_0$ is the canonical image of x in $X_0 := X/Z_X$.

Hence, since $c(\xi) = c(\xi_0)$ (indeed, use Definition 9.7 and observe that $\alpha(t) = \alpha(r_P t^P) = \alpha(r_P t_0^P t^0) = \alpha(r_P t_0^P)$ for all $\alpha \in R_0$) and since

$$\int_{T_u^P} f(t^P)\, dt^P = \int_{T_{0,u}^P \times T_u^0} f(t_0^P t^0)\, dt_0^P\, dt^0 \tag{7.19}$$

for all integrable functions f on T_u^P, we have

$$W_\sigma(\theta_x) = W_{0,\sigma_x}(\theta_{x_0}), \tag{7.20}$$

where W_{0,σ_x} denotes the wave packet (7.16) for the smooth section

$$\sigma_x \in C^\infty(\Xi_{\mathscr{R}_0, P, \delta, u}, \mathrm{End}(\mathscr{V}_{\Xi_0}))$$

with respect to the root datum $\mathscr{R}_0$, defined by

$$\sigma_x(\xi_0) = \int_{T_u^0} t^0(x)\sigma(t_0^P t^0)\,dt^0 = \int_{T_u^0} t^0(x^0)\sigma(t_0^P t^0)\,dt^0, \tag{7.21}$$

where x^0 is the canonical image of x in $X^0 := X/X \cap F_0$.

From equation (7.21) it is clear, by harmonic analysis on the torus $T_{0,u}^P \times T_u^0$, that for all $k \in \mathbb{Z}_+$ and all continuous seminorms q on $C^\infty\big(\Xi_{\mathscr{R}_0,P,\delta,u}, \mathrm{End}(\mathscr{V}_{\Xi_0})\big)$, there exists a continuous seminorm $p = p_{q,k}$ on $C^\infty\big(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi)\big)$ such that

$$q(\sigma_x) \leqq (1 + \|x^0\|)^{-k} p(\sigma) \tag{7.22}$$

for all $x \in X$ and for all $\sigma \in C^\infty\big(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi)\big)$.

Now apply the induction hypothesis to W_{0,σ_x}. In view of (7.20) and (7.22) this yields the induction step in the first case.

Second case. We now consider the case $Z_X = 0$. If $\alpha \in F_0$ and $a > 0$, one defines $X_{\alpha,a}^+ = \{x \in X^+ \mid \langle x, \alpha^\vee \rangle > a\|x\|\}$. We first prove that:

$$\bigcup_{\alpha \in F_0} X_{\alpha,a}^+ = X^+ \backslash \{0\}, \quad \text{for all } a \text{ small enough.} \tag{7.23}$$

Let $x \in X^+ \backslash \{0\}$. We write

$$x = \sum_{\alpha \in F_0} \langle x, \alpha^\vee \rangle \delta_\alpha,$$

where δ_α are the fundamental weights. For $a > 0$ small enough, one has, by equivalence of norms in finite dimensional vector spaces:

$$2a\|x\| \leqq \sup_{\alpha \in F_0} |\langle x, \alpha^\vee \rangle|, \quad x \in X^+. \tag{7.24}$$

Then, for $x \in X^+ \backslash \{0\}$, choose $\alpha \in F_0$ with $\langle x, \alpha^\vee \rangle$ maximal. From (7.24), one has $\langle x, \alpha^\vee \rangle \geqq 2a\|x\|$, hence, as $\|x\| \neq 0$:

$$\langle x, \alpha^\vee \rangle > a\|x\|, \quad \text{i.e., } x \in X_{\alpha,a}^+$$

which proves (7.23).

Hence it is enough to prove the estimates for $x \in X_{\alpha,a}^+$. Let $Q = F_0 \backslash \{\alpha\}$. Then it follows from Lemma 7.11, that for some $b > 0$, and $C > 0$, one has

$$|\Phi_{t^P}^\sigma(\theta_x) - \Phi_{t^P}^{\sigma,Q}(\theta_x)| \leqq Ce^{-b\|x\|}, \quad \text{for all } x \in X_{\alpha,a}^+,\ t^P \in T_u^P.$$

By integration of this inequality over T_u^P against the continuous function $|c^{-1}(\xi)|$ it suffices to prove the estimates (7.17) after replacing $\Phi_{t^P}^\sigma$ by $\Phi_{t^P}^{\sigma,Q}$. But by Proposition 7.10, the restriction to $\mathscr{H}^Q$ of the constant term $\Phi_{t^P}^{\sigma,Q} c^{-1}(\xi)$ of the normalized smooth family $\Phi_{t^P}^\sigma c^{-1}(\xi)$ of coefficients is a sum of normalized smooth families of coefficients for

$\mathscr{R}_Q = (X, Y, R_Q, R_Q^\vee, Q)$. This brings us back to the first case of the induction step, thus finishing the proof. $\square$

Corollary 7.13. *It follows from Theorem* 7.12, (7.15) *and Lemma* 7.14 *that* $\mathscr{J}(\sigma) \in \mathscr{S}$ *for all* $\sigma \in \mathscr{C}(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi))$, *and that* $\mathscr{J}_\mathscr{C} : \mathscr{C}(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi)) \to \mathscr{S}$ *is continuous.*

In particular, by Lemma 5.2 *we see that* $\mathscr{J}(\sigma) \in \mathscr{S}$ *for all* $\sigma \in C^\infty(\Xi_{P,\delta,u}, \mathrm{End}(\mathscr{V}_\Xi))$.

7.4. End of the proof of the Main Theorem. We start with a basic technical lemma:

Lemma 7.14. *Let* $n \in \mathbb{Z}$. *There exists a constant* C_n *with the following property. For all* $f \in \mathscr{H}^*$ *for which there exists* $C > 0$ *such that*

$$|f(T_u \theta_x T_v)| \leqq C(1 + \|x\|)^{-n}, \quad u, v \in W_0,\ x \in X^+ \tag{7.25}$$

one has

$$|f(N_w)| \leqq C_n C\big(1 + \mathscr{N}(w)\big)^{-n}, \quad w \in W.$$

Proof. As in [26], (2.25), one writes, for $w = uxv$, with $u, v \in W_0$, $x \in X^+$,

$$N_w = \sum_{u, v \in W_0} c_{w,(u',v')} T_{u'} \theta_x T_{v'}, \tag{7.26}$$

where the real coefficients $c_{uxv,(u',v')}$ and $c_{uyv,(u',v')}$ are equal if x and y belong to the same facets of the cone X^+. The number of facets being finite, one sees, by using the assumption (7.25), that there exists C' such that

$$|f(N_{uxv})| \leqq C'C(1 + \|x\|)^{-n}, \quad u, v \in W_0,\ x \in X^+. \tag{7.27}$$

But, from [26], (2.27), one deduces the existence of $r_0 \geqq 0$ such that:

$$\mathscr{N}(x) - r_0 \leqq \mathscr{N}(uxv) \leqq \mathscr{N}(x) + r_0, \quad u, v \in W_0,\ x \in X^+. \tag{7.28}$$

But one has: $\mathscr{N}(x) = \langle x, 2\rho^\vee \rangle + \|x^0\|$, $x \in X^+$ where x^0 is the projection of $x \in X \otimes \mathbb{R}$ on $Z_X \otimes \mathbb{R}$ along $\mathbb{Z}R_0 \otimes \mathbb{R}$. Let us define

$$\|v\|' = \sup_{u \in W_0} |v(2u\rho^\vee)| + \|v^0\|, \quad v \in X \otimes \mathbb{R}.$$

Then $\|\cdot\|'$ is a norm on $X \otimes \mathbb{R}$, which is equivalent to $\|\cdot\|$. Moreover

$$\mathscr{N}(x) = \|x\|', \quad \text{for all } x \in X^+.$$

Hence there exists $C'' > 0$ such that:

$$C''^{-1} \mathscr{N}(x) \leqq \|x\| \leqq C'' \mathscr{N}(x). \tag{7.29}$$

Taking into account (7.27), (7.28) and (7.29), one gets the result. $\square$

End of the proof of the Main Theorem. By Corollary 7.3, the image of $\mathscr{F}_{\mathscr{S}}$ is contained in the space of smooth $\mathscr{W}$-equivariant sections $C^\infty\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)^{\mathscr{W}}$, and $\mathscr{F}_{\mathscr{S}}$ is continuous.

Corollary 7.13 states that the image of $\mathscr{J}_{\mathscr{C}}$ is contained in $\mathscr{S}$, and that $\mathscr{J}_{\mathscr{C}} : \mathscr{C}\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big) \to \mathscr{S}$ is continuous.

Since $C^\infty\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)^{\mathscr{W}} \subset \mathscr{C}\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)$ (see Lemma 5.2) and $\mathscr{S} \subset L_2(\mathscr{H})$ (see (2.16), (2.19)), Corollary 4.5 implies that $\mathscr{J}_{\mathscr{C}}\mathscr{F}_{\mathscr{S}} = \mathrm{id}_{\mathscr{S}}$. It follows that the map $\mathscr{J}_{\mathscr{C}}$ in (5.5) is surjective, and that $\mathscr{F}_{\mathscr{S}}$ is injective.

Since $\mathscr{C} \subset L_2\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi), \mu_{Pl}\big)$ (see Lemma 5.2), Corollary 4.5 also implies that $\mathscr{F}_{\mathscr{S}}\mathscr{J}_{\mathscr{C}} = p_{\mathscr{W},\mathscr{C}}$. It follows, since $p_{\mathscr{W},\mathscr{C}}$ is the identity on

$$C^\infty\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big)^{\mathscr{W}} \subset \mathscr{C}\big(\Xi_u, \mathrm{End}(\mathscr{V}_\Xi)\big),$$

that $\mathscr{F}_{\mathscr{S}}$ is also surjective in (5.4). This finishes the proof of the Main Theorem. □

8. Appendix. Some applications of spectral projections

The following lemma was suggested by [1], Lemma 20.1 and its proof.

Lemma 8.1. *Let V be a complex normed vector space of dimension p. There exists $C > 0$ such that for all $A \in \mathrm{End}(V)$ with eigenvalues of modulus less than or equal to* 1:

$$\|A^n\| \leqq C(1 + \|A\|)^{p-1}(1+n)^p, \quad n \in \mathbb{Z}_+. \tag{8.1}$$

Here $\|A\|$ is the operator norm of A.

Proof. Let D_n be the disk of center 0 and radius $1 + (1+n)^{-1}$. Then

$$A^n = 1/2i\pi \int_{\partial D_n} z^n(z\,\mathrm{Id} - A)^{-1}\,dz. \tag{8.2}$$

From the Cramer rules, there exists a polynomial function from $\mathrm{End}(V)$ into itself, $B \mapsto M(B)$, of degree $p-1$, such that for any invertible B, one has:

$$B^{-1} = \big(\det(B)\big)^{-1}M(B). \tag{8.3}$$

Hence, there exists $C' > 0$ such that:

$$\|M(B)\| \leqq C'(1 + \|B\|)^{p-1}, \quad B \in \mathrm{End}(V).$$

Hence, taking into account:

$$1 + \|z\,\mathrm{Id} - A\| \leqq 2 + (1+n)^{-1} + \|A\| \leqq \big(2 + (1+n)^{-1}\big)(1 + \|A\|), \quad z \in D_n,$$

one has

$$(8.4)\qquad \|M(z\,\mathrm{Id}-A)\| \leqq C'\big(2+(1+n)^{-1}\big)^{p-1}(1+\|A\|)^{p-1}, \quad z \in D_n.$$

Now the eigenvalues of $z\,\mathrm{Id}-A$, $z \in \partial D_n$ are of modulus greater or equal to $(1+n)^{-1}$. Hence

$$(8.5)\qquad |\det(z\,\mathrm{Id}-A)| \geqq (1+n)^{-p}, \quad n \in \mathbb{Z}_+.$$

The length of ∂D_n is $2\pi\big(1+(1+n)^{-1}\big)$. From equations (8.1) to (8.4), one gets:

$$\|A^n\| \leqq \big(1+(1+n)^{-1}\big)^{1+n} C'\big(2+(1+n)^{-1}\big)^{p-1}(1+n)^p(1+\|A\|)^{p-1}.$$

One gets the required estimate with:

$$C = C'e3^{p-1}. \quad \square$$

Corollary 8.2. (i) *Let $r' > r > 0$. There exists $C_{r,r'}$ such that for all $A \in \mathrm{End}(V)$ with eigenvalues of modulus less or equal to r, one has*

$$\|A^n\| \leqq C_{r,r'}(r')^n(1+\|A\|)^{p-1}.$$

(ii) *Let $\epsilon > 0$ and let Ω_ϵ be the set of elements in $\mathrm{End}(V)$ whose eigenvalues are either of modulus one or of modulus less or equal to $1-\epsilon$. Let $P_{<1}$ be the sum of the spectral projections corresponding to the eigenvalues of modulus strictly less than 1. Then $P_{<1}A^n = (A_{<1})^n$, where $A_{<1} = P_{<1}A$. Let $b > 0$ such that $1-\epsilon < e^{-b}$. There exists C depending on b, ϵ and V such that*

$$\|P_{<1}A^n\| \leqq C(1+\|P_{<1}A\|)^{p-1}e^{-bn}, \quad n \in \mathbb{N},\ A \in \Omega_\epsilon.$$

(iii) *If $A(t)$ is a continuous (resp., holomorphic) function with values in Ω_ϵ, then $A_{<1}(t)$ has the same property.*

Proof. (i) One applies the previous result to $r''^{-1}A$, where $r < r'' < r'$ and one uses the fact that $(1+n)^p(r'/r'')^{-n}$ is bounded.

(ii) follows from (i) applied to $A_{<1}$, $r = 1-\epsilon$, $r' = e^{-b}$.

(iii) follows from the formula

$$A_{<1} = 1/2i\pi \int_{\partial D} z(z\,\mathrm{Id}-A)^{-1}\,dz,$$

where D is the disc of center 0 and radius $1-\epsilon/2$. $\square$

Lemma 8.3. *Let $\epsilon, a > 0$, $p, l \in \mathbb{N}$. Let V a normed complex vector space of dimension p and $X = \mathbb{Z}^l$. Let π be a finite dimensional complex representation of X. One denotes by $(e_1, \dots, e_p)$ the canonical basis of X. One sets $A_1 = \pi(e_1), \dots, A_l = \pi(e_l)$. If $n = (n_1, \dots, n_l) \in X$, one sets: $\|n\| = |n_1| + \cdots + |n_l|$, and $A^n = \pi(n)$. Assume that the modulus of the eigenvalues of the A_i are less or equal to one, and the eigenvalues of A_1 are either of modulus one or of modulus less or equal to $1-\epsilon$. Let us denote by $P_{<1}$ (resp., P_1) the sum* *of the spectral projections of A_1 corresponding to the eigenvalues of moduli strictly less than* 1 (*resp., of moduli* 1). *Set* $X_a^+ = \{n \in \mathbb{Z}_+^l \mid n_1 > a\|n\|\}$.

Then there exist a' and C', independent of the representation π of X in V, such that

$$\|P_{<1}A^n\| \leqq C'\biggl(\prod_{i=1,\dots,l}(1+\|A_i\|)^p\biggr)e^{-a'\|n\|}, \quad n \in X_a^+.$$

Proof. From Corollary 8.2(i), (ii), one deduces that, for $b > 0$ such that $1 - \epsilon < e^{-b}$, and $b' > 0$, there exists a constant $C > 0$, depending only on ϵ, b, b' and V such that:

$$\|P_{<1}A^n\| \leqq C'\biggl(\prod_{i=1,\dots,l}(1+\|A_i\|)^p\biggr)e^{-bn_1+b'(n_2+\dots+n_l)}, \quad n \in X^+.$$

If $n \in X_a^+$, one has

$$bn_1 - b'(n_2 + \dots + n_l) \geqq (ab - b')\|n\|;$$

b being chosen, one takes $b' = ab/2$. Then the inequality of the lemma is satisfied for $a' = ab/2$. □

9. Appendix. The c-function

In this appendix we have collected some of the properties of the Macdonald c-function. These properties play a prominent role throughout this paper, and are closely related with the properties of residual cosets as discussed in [26], Appendix: residual cosets (Section 7).

We now define the Macdonald c-function c element of ${}_{\mathcal{Q}}\mathcal{A} = \mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{A}$ the quotient field of $\mathcal{A}$. Set

$$c := \prod_{\alpha \in R_{0,+}} c_\alpha = \prod_{\alpha \in R_{1,+}} c_\alpha, \tag{9.1}$$

where c_α for $\alpha \in R_1$ is equal to

$$c_\alpha := \frac{(1 + q_{\alpha^\vee}^{-1/2}\theta_{-\alpha/2})(1 - q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1}\theta_{-\alpha/2})}{1 - \theta_{-\alpha}} \in {}_{\mathcal{Q}}\mathcal{A}. \tag{9.2}$$

If $\alpha \in R_0 \backslash R_1$ then we define $c_\alpha := c_{2\alpha}$.

Remark 9.1. We have thus associated a c-function c_α to each root $\alpha \in R_{nr}$, but c_α only depends on the direction of α. This convention was used in [26], but differs from the one used in [25]. If $\alpha \in R_1$ and $\alpha/2 \notin R_0$, then the formula for c_α should be interpreted by setting $q_{2\alpha^\vee} = 1$, and then rewriting the numerator as $(1 - q_{\alpha^\vee}^{-1}\theta_{-\alpha})$. Here and below we use this convention.

We view c as a rational function on T via the isomorphism of $\mathcal{A}$ and the ring of regular functions on T sending θ_x to the complex character x of T.

Since the numerator and the denominator of c both are products of irreducible factors whose zero locus is nonsingular (a coset of a codimension 1 subtorus of T), it is straightforward to define the pole order i_t of $\left(c(t)c(w_0 t)\right)^{-1}$ at a point $t \in T$ (see [26], Definition 3.2).

Let $Q = Q(R_0)$ denote the root lattice of R_0. The following theorem is the main property of the c-function:

Theorem 9.2 ([26], Theorem 7.10; [27], Theorem 7.1). *We have*: $i_t \leqq \mathrm{rank}(Q)$ *for all* $t \in T$.

We define the notion of a "residual point" of T (with respect to $(\mathscr{R}, q)$) as follows:

Definition 9.3. A point $t \in T$ is called residual if $i_t = \mathrm{rank}(X)$.

Remark 9.4. There is a complete classification of the residual points [13], [26]. The results on the c-function used in this section can either be proved using this classification (see [13], [26]) or using harmonic analysis (see [27]).

Corollary 9.5. *There exist only finitely many residual points in* T, *for every root datum* $\mathscr{R}$ *and label function* q *for* $\mathscr{R}$, *and the set of residual points is empty unless* $Z_X = 0$.

Proof. Let $n = \mathrm{rank}(R_0)$. From equation (9.1) it is clear that for any $k \in \mathbb{Z}$, the set $S_k := \{t \mid i_t = k\}$ is a finite (possibly empty) union of nonempty Zariski open subsets of cosets L of subtori of T, whose Lie algebra is an intersection of root hyperplanes $\alpha = 0$ of $\mathbb{C} \otimes Y$. If L is such a coset with $\mathrm{codim}(L) = d$, then $R_L := \left\{\alpha \in R_0 \mid \alpha|_L \text{ is constant}\right\}$ is a parabolic subsystem of rank d. Moreover, the projection t_L of L onto T_L is a point with $i_{t_L}^{R_L} = k$. Applying Theorem 9.2 to T_L with respect to $(\mathscr{R}_L, q_L)$ we obtain $k \leqq d$.

Hence if S_n is not finite, then there exists a proper parabolic root subsystem $R_L \subset R_0$ of rank $m < n$ say, such that $n \leqq m$, which is clearly absurd. The remaining part of the corollary is straightforward from Theorem 9.2. □

Another fact of great interest is the following.

Theorem 9.6 ([26], Theorem 7.14; [27], Theorem 7.4). *Let* $r = sc \in T$ *be residual, with* $s \in T_u$ *and* $c \in T_{rs}$. *Then* $r^* := sc^{-1} \in W(R_{s,1})r$, *where* $R_{s,1}$ *is the root subsystem of* R_1 *defined by* $R_{s,1} := \{\alpha \in R_1 \mid \alpha(s) = 1\}$.

We extend the definition of the c-function to arbitrary standard induction data. First recall Theorem 2.10, stating that the central character of a residual discrete series representation is residual.

Definition 9.7. Let $\xi = (P, \delta, t^P)$ be a standard induction datum, and let $r_P \in T_P$ be such that $W_P r_P$ is the central character of δ (thus r_P is a $(\mathscr{R}_P, T_P)$-residual point). Put $t = r_P t^P \in T$. We define:

$$c(\xi) := \prod_{\alpha \in R_{0,+} \setminus R_{P,+}} c_\alpha(t). \tag{9.3}$$

Notice that we recover the original c-function defined on T as the special case where $P = \emptyset$ and δ is the trivial one dimensional representation.

The next proposition goes back to [13], Theorem 3.13 (also see [26], Theorem 3.25).

Proposition 9.8. *Let $P \subset F_0$ and let $\xi = (P, \delta, t^P) \in \Xi_{P,\delta,u}$. Choose $r_P \in T_P$ such that $W_P r_P$ is the central character of δ, and let $t = r_P t^P \in T$.*

(i) $c(\xi^{-1}) = c\big(w^P(\xi)\big) = \overline{c(\xi)}$.

(ii) *The function $\xi \mapsto |c(\xi)|^2$ on Ξ_u is $\mathscr{W}$-invariant.*

(iii) *The function $c(\xi)$ is $\mathscr{K}$-invariant.*

(iv) *Let $P' \in \mathscr{P}$ and $d \in K_{P'} \times W(P, P')$. The rational functions $c(d\xi)c(\xi)^{-1}$ and $c(d\xi)^{-1}c(\xi)$ (of $\xi \in \Xi_{P,\delta}$) are regular in a neighbourhood of $\Xi_{P,\delta,u}$.*

(v) *The rational function $c(\xi)^{-1}$ is regular in a neighbourhood of Ξ_u.*

Proof. (i) A straightforward computation from the definitions, using Theorem 9.6 (cf. [26], (3.58)).

(ii) The $\mathscr{W}$-invariance follows simply from the definitions if we write (using (i)) $|c(\xi)|^{-2} = \big(c(\xi)c(\xi^{-1})^{-1}\big)$ (cf. [26], Proposition 3.27).

(iii) This follows trivially from the definition of the action of $\mathscr{K}$ on ξ: If $k \in K_P$ then $k\xi = k(P, \delta, t^P) = \big(P, \Psi_k(\delta), kt^P\big)$. The central character of $\Psi_k(\delta)$ is equal to $k^{-1}W_P r_P = W_P(k^{-1}r_P)$, thus we need to evaluate the c_α in the product $c(k\xi)$ at the point $k^{-1}r_P k t^P = t$, or any of its images under the action of W_P. Hence $c(k\xi) = c(\xi)$.

(iv) By (i) and (ii) it is clear that these rational functions have modulus 1 on $\Xi_{P,\delta,u}$ (outside their respective singular sets).

The singular sets of these rational functions are of the following form. Choose $r_P \in T_P$ such that $W_P r_P$ is the central character of δ. Then the singular set of $c(d\xi)c(\xi)^{-1}$ is the union of the zero sets of the functions

$$\prod_{\alpha \in R_{1,+} \setminus R_{P,1,+}} (1 - \alpha_{P,\delta}^{-1}) \tag{9.4}$$

and

$$\prod_{\alpha \in R_{1,+} \setminus R_{P,1,+}} (1 + q_{\alpha^\vee}^{-1/2} \alpha_{P,\delta}^{-1/2})(1 - q_{\alpha^\vee}^{-1/2} q_{2\alpha^\vee}^{-1} \alpha_{P,\delta}^{-1/2}) \tag{9.5}$$

on $\Xi_{P,\delta}$, where $\alpha_{P,\delta}$ denotes the function on $\Xi_{P,\delta}$ defined by $\alpha_{P,\delta}(\xi) = \alpha(r_P t^P)$.

In the case of $c(w^P\xi)^{-1}c(\xi)$ the answer is the same, but we need to take the products over the set $\alpha \in d^{-1}R_{1,+} \setminus R_{P,1,-}$.

The intersection of a component of this hypersurface with $\Xi_{P,\delta,u}$ is either empty or it has (real) codimension 1 in $\Xi_{P,\delta,u}$.

By the boundedness of $c(d\xi)c(\xi)^{-1}$ on $\Xi_{P,\delta,u}$, this implies that the pole order of this function at a component of the singular set which meets $\Xi_{P,\delta,u}$ is in fact equal to zero. Hence the poles are removable in a neighbourhood of $\Xi_{P,\delta,u}$. Similarly for $c(d\xi)^{-1}c(\xi)$.

(v) The proof of [13], Theorem 3.13 may be adapted to the present situation. Or we may argue as in (iv) as follows.

Since $|c(\xi)|^{-2} = \big(c(\xi)c\big(w^P(\xi)\big)\big)^{-1}$ is smooth on Ξ_u (cf. [26], Theorem 3.25, equation (3.53), Proposition 3.27 and equation (3.58)), it follows that $c(\xi)^{-1}$ is bounded on $\Xi_{P,\delta,u}$. Hence the argument that was used in the proof of (iv) applies. $\square$

10. Appendix. Relation with the Harish-Chandra Schwartz algebra

In this section we say some words about the explicit interpretation of the Schwartz algebra $\mathscr{S}$ in the situation where $\mathscr{H}$ arises as the algebra of compactly supported spherical functions of a compact open subgroup of a reductive p-adic group G.

We start this discussion with an (admittedly indirect) argument explaining the role of $\mathscr{S}$ for the representation theory of G. Let G be a reductive p-adic group and let $\sigma = (K, \rho)$ be a type for a Bernstein block $\mathscr{B}$ of the category of smooth representations of G (cf. [6]). Let $\mathscr{H}$ be the Hecke algebra of compactly supported σ-spherical functions. Assume further that $\mathscr{H}$ is isomorphic (in the sense of involutive algebras) to an affine Hecke algebra with parameters $\mathscr{H}(\mathscr{R}, q)$ in our sense. We assume moreover that via this isomorphism the trace of $\mathscr{H}$ corresponds to a suitable positive multiple of the tracial state τ of the affine Hecke algebra $\mathscr{H}(\mathscr{R}, q)$ (all this is known e.g. for level zero types of split semisimple groups of adjoint groups, see [24], [14]). In this situation the functor $V \to V^\sigma$ taking σ-spherical vectors defines a Morita equivalence μ from $\mathscr{B}$ to the category of $\mathscr{H}$-modules. It is known that μ preserves the Plancherel measure [7]. On the other hand, it is known that the support of the Plancherel measure of G consists of the irreducible tempered representations of G (cf. [36]). Finally, Corollary 4.4 shows that the support of the Plancherel measure of $\mathscr{H} \simeq \mathscr{H}(\mathscr{R}, q)$ is the set of irreducible representations of the Schwartz algebra $\mathscr{S}(\mathscr{R}, q)$ (restricted to $\mathscr{H}(\mathscr{R}, q)$). Using the well known fact that irreducible smooth representations of G are admissible these arguments yield the following result:

Theorem 10.1. *Let G be a p-adic reductive group and suppose that $\mathscr{H}$ is the Hecke algebra of a type σ of a Bernstein block $\mathscr{B}$ of G. Suppose that $\mathscr{H}$ is isomorphic (as involutive algebra) to an affine Hecke algebra of the form $\mathscr{H}(\mathscr{R}, q)$ such that the trace of $\mathscr{H}$ corresponds to a positive multiple of the trace τ of $\mathscr{H}(\mathscr{R}, q)$. The functor $V \to V^\sigma$ induces an equivalence from the category of admissible tempered representations in the block $\mathscr{B}$ to the category of tempered modules of finite length of $\mathscr{H} \simeq \mathscr{H}(\mathscr{R}, q)$ (i.e. the category of finite dimensional $\mathscr{S}(\mathscr{R}, q)$-modules).*

In the special case of the Iwahori-Matsumoto Hecke algebra $\mathscr{H} = \mathscr{H}(G, B)$ ([15], [4]) of a split semisimple p-adic group G we will show more precisely that the Schwartz completion $\mathscr{S}$ of $\mathscr{H}$ is isomorphic to the algebra $\mathscr{C}(G,B)$ of B-biinvariant Schwartz functions in the sense of Harish-Chandra.

Let $\mathbf{K}$ be a non archimedean local field, q the number of elements of its residue field. Let G the group of $\mathbf{K}$-points of a semisimple algebraic group defined and split over $\mathbf{K}$. Let $(Y,X,R,\check{R})$ its root datum with respect to a split torus A. Let K the isotropy group of a special point of the apartment of the Bruhat-Tits building of G corresponding to A and B an Iwahori subgroup G contained in K. Then K is a maximal compact subgroup of G. The Weyl group of G with respect to A has a set of representative W_0 in K, by definition of special points and one has

$$K = BW_0B. \tag{10.1}$$

The Iwahori subgroup of G determines a set of simple roots of $\check{R}$, F, and also a minimal parabolic subgroup of G, P_0, which contains B, and a Weyl chamber A^+ in A. We denote by $\mathscr{R}$ the reduced based root datum $(X,Y,R,R^\vee,F)$.

We keep the notation of Section 2.4. Then every element, w, of the Weyl group W determines a double coset of BwB of B in G. We denote by 1_{BwB} its characteristic function on G. We denote by $\mathscr{H}(G,B)$ the convolution algebra of compactly supported functions on G which are biinvariant under B. It is endowed with the L^2-scalar product.

We denote also by q the mulitiplicity function on W, $w \mapsto q^{l(w)}$. Then (see [15], Section 3, for Chevalley groups, and [4], Section 3, in general), there is an isomorphism, Φ, between $\mathscr{H}(\mathscr{R},q)$ and $\mathscr{H}(G,B)$ such that

$$\Phi(T_w) = 1_{BwB}, \quad w \in W. \tag{10.2}$$

Moreover [15], Prop. 3.2, shows that:

$$\begin{array}{l}\text{The square norm of } T_w \text{ is equal to the square of the } L^2\text{-norm}\\ \text{of } 1_{BwB},\ q^{l(w)},\ \text{for } w \in W.\end{array} \tag{10.3}$$

The lattice X is identified to a subgroup of A. We denote by X^+ the set of its P_0-dominant elements. One has

$$G = BWB = BW_0X^+W_0B. \tag{10.4}$$

We denote by δ_0 the modulus function of P_0. It is a biinvariant function under B on P_0. One has (cf. [21], Corollary 3.2.5 and Remark 3.2.11):

$$\delta_0(x) = q^{l(x)}, \quad x \in X. \tag{10.5}$$

$$\begin{array}{l}\text{If } X \otimes_{\mathbb{Z}} \mathbb{R} \text{ is endowed with a norm, its restriction to the set } X^+\\ \text{of } P_0\text{-dominant elements, is an equivalent function to}\\ \text{the restriction to } X^+ \text{ of the length function } l.\end{array} \tag{10.6}$$

(See (7.1).)

The transpose, Φ^*, of the isomorphism Φ determines an isomorphism of the dual $\mathcal{H}(G,B)^*$ of $\mathcal{H}(G,B)$ with the dual $\mathcal{H}^*$ of $\mathcal{H} := \mathcal{H}(\mathcal{R},q)$. The scalar product allows to identify $\mathcal{H}(G,B)$ (resp. $\mathcal{H}$) as a subspace of $\mathcal{H}(G,B)^*$ (resp. $\mathcal{H}^*$).

The Harish-Chandra Schwartz algebra of G is the space $\mathcal{C}(G)$ of functions, f, on G which are biinvariant under some compact open subgroup of G and such that for all $n \in \mathbb{N}$:

$$\sup_{g \in G} |f(g)| \Xi(g)^{-1} \big(1 + \log(\|g\|)\big)^n < +\infty.$$

Here the functions $\|\cdot\|$ and Ξ are the biinvariant functions under K, defined for example in [36], I.1, II.1.

From the fact that f is biinvariant under some open compact subgroup of K, hence of B, and from the decomposition (10.4), this is equivalent to

$$\sup_{x \in X^+} |f(bwxw'b')| \Xi(x)^{-1} \big(1 + \log(\|x\|)\big)^n < +\infty \tag{10.7}$$

for all $b, b' \in B$, $w, w' \in W_0$, $n \in \mathbb{N}$. From [36], Lemme II.1.1, there exist $C_1, C_2 > 0$, $d \in \mathbb{N}$ such that

$$C_1 \delta_0^{1/2}(a) \leqq \Xi(a) \leqq C_2 \delta_0^{1/2}(a) \big(1 + \log(\|a\|)\big)^d, \quad a \in A_0^+. \tag{10.8}$$

Using equations (10.8), (10.5), (10.4), (10.6), and taking into account equation [36], I.1 (6), one sees that (10.7) is equivalent to

$$\sup_{x \in X^+} |f(bwxw'b')| q^{l(x)/2} \big(1 + l(x)\big)^n < +\infty \tag{10.9}$$

for all $b, b' \in B$, $w, w' \in W_0$, $n \in \mathbb{N}$. Using the L^2 scalar product, the algebra of B-biinvariant elements of $\mathcal{C}(G)$, $\mathcal{C}(G,B)$, might be viewed as a subspace of $\mathcal{H}(G,B)^*$.

Proposition 10.2. *The image of $\mathcal{C}(G,B)$ by the transpose, Φ^*, of Φ is equal to $\mathcal{S}(\mathcal{H})$.*

Proof. The Schwartz algebra $\mathcal{S}(\mathcal{H})$ is defined by (2.17). Here $\mathcal{N}$ is just the length function, as G is semi-simple: Z_X is reduced to $\{0\}$.

Using our equation (2.2), one sees that, equivalently, $\mathcal{S}(\mathcal{H})$ is the space of $h \in \mathcal{H}^*$ such that

$$\sup_{x \in X^+} |(N_{wxw'}, h)| \big(1 + l(w)\big)^n < \infty \tag{10.10}$$

for all $w, w' \in W_0$, $n \in \mathbb{N}$. Let f be a B-biinvariant function on G, that one views as a linear form on $\mathcal{H}(G,B)$. Let us denote by $h \in \mathcal{H}^*$ its image by Φ^*. Then a simple computation, using (10.2) and (10.3), shows that

$$(h, N_w) = f(w) q^{l(w)}, \quad w \in W.$$

Then conditions (10.9) for f is equivalent to the condition (10.10) for h. The proposition follows. $\square$

11. Appendix. Geometric lemma and the constant term

The referee suggested that our results on the constant term of coefficients should be expressed in terms of representations. The referee mentioned the use of the basic geometric lemma and the reference [36] for similar results for reductive p-adic groups. We will follow the suggestion of the referee and briefly indicate how the results of subsection 6.3 could be viewed in this perspective.

The first result we would like to mention here is the appropriate version in our context of the basic geometric lemma for tempered representations, analogous to [36], Lemme III.3.3:

Lemma 11.1 (Geometric lemma for tempered representations). *Let $P, Q \in \mathscr{P}$ and let (V_δ, δ) be a tempered representation of $\mathscr{H}^P$. Recall the set $D^{Q,P} \subset W_0$ of minimal length representatives of the double cosets $W_Q w W_P$. For $t \in T_u^P$ let $i_P(V_{\delta,t})$ denote the tempered representation of $\mathscr{H}$ obtained by inducing δ_t from $\mathscr{H}^P$ to $\mathscr{H}$ (see* [26], *Proposition* 4.20). *There exists a filtration of the constant part $i_P(V_{\delta,t})^Q$ of $i_P(V_{\delta,t})$ along Q such that*

$$\mathrm{gr}\big(i_P(V_{\delta,t})^Q\big) \simeq \bigoplus_{d \in D^{Q,P}} i^Q_{Q \cap dP}\big(d(V_{\delta,t})^{d^{-1}Q \cap P}\big) \tag{11.1}$$

where $d(V_{\delta,t})^{d^{-1}Q\cap P}$ denotes the pullback of $(V_{\delta,t})^{d^{-1}Q\cap P}$ by the algebra isomorphism $\psi_d^{-1} : \mathscr{H}^{Q\cap dP} \to \mathscr{H}^{d^{-1}Q\cap P}$ (see Subsection 3.5 *for the meaning of ψ_d).*

Proof. We choose a filtration $O_1 \subset O_2 \subset \cdots \subset O_N = W_0$ of W_0 by left W_Q and right W_P invariant subsets such that (with $O_0 = \emptyset$)

$$O_i \backslash O_{i-1} = W_Q d_i W_P \tag{11.2}$$

for $d_i \in D^{Q,P}$ such that the length of d_i is increasing with i. For all $w \in W_{Q\cap d_i P}$ we have

$$N_w N_{d_i} = N_{d_i} N_{d_i^{-1} w d_i} = N_{w d_i} \tag{11.3}$$

since $l(w) = l(d_i^{-1} w d_i)$ and $l(wd_i) = l(w) + l(d_i)$. Using Lustig's relation (2.8) and induction with respect to the length of d_i we see easily that for all $x \in X$:

$$\theta_x N_{d_i} = N_{d_i} \theta_{d_i^{-1} x} + \sum_{w_k \in O_{i-1},\, x_k \in X} a_k N_{w_k} \theta_{x_k}. \tag{11.4}$$

We define a filtration of $i_P(V_{\delta,t})$ by putting

$$i_P(V_{\delta,t})_i := \sum_{j=1,\dots,i} \mathscr{H}^Q N_{d_i} (1 \otimes_{\mathscr{H}^P} V_{\delta,t}). \tag{11.5}$$

By (11.3) and (11.4) we see that this is a filtration by $\mathscr{H}^Q$-submodules, and that

$$i_P(V_{\delta,t})_i / i_P(V_{\delta,t})_{i-1} \simeq i^Q_{Q\cap d_i P}\big(d_i(V_{\delta,t}|_{\mathscr{H}^{d_i^{-1}Q\cap P}})\big). \tag{11.6}$$

Therefore by Proposition 3.18(iv) it is enough to prove that for all d_i:

$$i^Q_{Q\cap d_i P}\big(d_i(V_{\delta,t}^{d_i^{-1}Q\cap P,+})\big)^Q = 0. \tag{11.7}$$

The weights $t \in T$ of $d_i(V_{\delta,t}^{d_i^{-1}Q \cap P,+})$ are of the form

$$t = t' \prod_{\alpha \in d_i P \setminus Q \cap d_i P} d_\alpha \otimes \alpha^\vee \tag{11.8}$$

with $|t'|_{Q \cap d_i P} = 1$ and where for all $\alpha \in d_i P \setminus Q \cap d_i P$, $|d_\alpha| \leqq 1$ and $|d_\alpha| < 1$ for at least one α. From Kilmoyer's result Proposition 6.7 it follows that if $\alpha \in d_i P \setminus Q \cap d_i P$ then $\alpha \in R_{0,+} \setminus R_{Q,+}$. Using the inequalities satisfied by the $|d_\alpha|$ it now follows easily that $|t|_Q \neq 1$ for any weight t of $d_i(V_{\delta,t}^{d_i^{-1}Q \cap P,+})$. The weights $t \in T$ of $i_{Q \cap d_i P}^{Q}\big(d_i(V_{\delta,t}^{d_i^{-1}Q \cap P,+})\big)$ are W_Q translates of the weights of $d_i(V_{\delta,t}^{d_i^{-1}Q \cap P,+})$, hence these weights also all satisfy $|t|_Q \neq 1$. This proves (11.7) and finishes the proof. □

Corollary 11.2. *If δ is a discrete series representation of $\mathscr{H}_P$ and $\xi = (P, \delta, t) \in \Xi_P$ a standard tempered induction datum, then there exists a filtration of $\pi(\xi)^Q$ such that*

$$\mathrm{gr}\big(\pi(\xi)^Q\big) \simeq \bigoplus_{d \in D^{Q,P} : dP \subset Q} i_{dP}^{Q}(d.V_{\delta,t}). \tag{11.9}$$

Proof. This is an easy special case of the previous lemma, using Proposition 3.20. □

Corollary 11.3. *Same conditions as in the previous corollary, but now assume that $t \in T_u^P$ is such that the induction datum ξ is R_P-generic, i.e. such that $tr_\delta \in T$ is R_P-generic where r_δ is such that $W_P r_\delta$ is the central character of δ. Then $\pi(\xi)^Q$ is semisimple and*

$$\pi(\xi)^Q \simeq \bigoplus_{d \in D^{Q,P} : dP \subset Q} i_{dP}^{Q}(d.V_{\delta,t}) \tag{11.10}$$

is the decomposition in irreducibles, all of them being inequivalent to each other.

Proof. Under the assumption of genericity we know that the modules $i_{dP}^{Q}(d.V_{\delta,t})$ with $d \in D^{Q,P}$ such that $dP \subset Q$ are irreducible (see [26], Corollary 4.18) and their central characters are mutually distinct by Corollary 6.8(ii). Then (11.10) follows by the preceding corollary. □

The decomposition of Corollary 11.3 can be easily compared to the decomposition (3.6), by observing that

$$i_{dP}^{Q}(d.V_{\delta,t}) \simeq \mathscr{H}^Q(i_d^0 e_\omega \otimes V_{\delta,t}). \tag{11.11}$$

Now let $Q' = -w^Q(Q)$ and put $P' = -w^Q(P)$. Consider also the decomposition (still assuming that we are in the generic case)

$$\pi(\xi)^{Q'} \simeq \bigoplus_{d' \in D^{Q',P'} : d'(P') \subset Q'} i_{d'(P')}^{Q'}(d'.V_{\delta,t}). \tag{11.12}$$

Notice that this is the geometric lemma for affine Hecke algebras. With respect to the Hermitian inner product on $\pi(\xi)$ defined by (3.4) the subspaces of $\pi(\xi)^{Q'}$ and $\pi(\xi)^Q$ defined by

$$N_{w^Q} i_{d'(P')}^{Q'}(d'.V_{\delta,t}) \tag{11.13}$$

and

$$(11.14) \qquad i^{Q}_{dP}(d.V_{\delta,t})$$

respectively are mutually orthogonal unless $w^Q d' = d$, as one can easily see from the well known formula $\theta_x^* = N_{w_0}\theta_{-w_0(x)}N_{w_0}^{-1}$ and Corollary 6.8(ii). The results of Subsection 6.3, in particular the explicit formula in Corollary 6.16 for the constant term, are based on this orthogonality together with explicit formulas relating the inner products $\langle a, b\rangle$ for $a \in N_{w^Q} i^{Q'}_{d'(P')}(d'.V_{\delta,t}) \subset \pi(\xi)$ and $b \in i^Q_{dP}(d.V_{\delta,t}) \subset \pi(\xi)$ (where $w^Q d' = d$) in terms of standard inner products in standard induced modules of $\mathscr{H}^Q$.

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