Wave packets in the Schwartz space of a reductive p-adic symmetric space

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Abstract

We form wave packets in the Schwartz space of a reductive p-adic symmetric space for certain famillies of tempered functions. We show how to construct such families from Eisenstein integrals.

*Mathematics Subject Classification 2000:*MSC classification 22E50

Keywords and phrases: p-adic reductive groups, symmetric spaces, Schwartz packets, Eisenstein integrals.

1 Introduction

Let G be the group of $\mathbf{F}$-points of an algebraic group, $\underline{G}$, defined over $\mathbf{F}$, where $\mathbf{F}$ is a nonarchimedean local field of characteristic different from 2. Let H be the group of $\mathbf{F}$-points of an open $\mathbf{F}$-subgroup of the fixed point group of a rational involution of $\underline{G}$ defined over $\mathbf{F}$.

We introduce the space $\mathcal{A}_{temp}(H\backslash G)$ of smooth tempered functions on $H\backslash G$. They are the tempered functions which are generalized coefficients of an H-fixed linear form ξ on an admissible G-module V, when V and ξ varies.

Using the theory of the constant term (cf. [L], [KT1]), we introduce the weak constant term of elements of $\mathcal{A}_{temp}(H\backslash G)$ as it was made in [W] for tempered functions on the group.

Then, we introduce families of elements of $\mathcal{A}_{temp}(H\backslash G)$ of type I, by conditions on their exponents. Then the conditions are strengthened to introduce families of type I', and we add conditions on the weak constant term to define families of type II'. This is the analogue of the families used in [BaCD] for the real case.

Important examples of such families are given (cf. Theorem 5.1) in terms of Eisenstein integrals, due to the main results of [CD].

Then, following [BaCD] for the real case, which was largely inspired by the work of Harish-Chandra [H-C], and [W], we show that one can form wave packets in the Schwartz space for such families (Theorem 4.6). Notice also that the intermediate Proposition 3.12 is the analogue of the important Lemma 7.1 of [A].

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The recent work of Sakerallidis and Venkatesh [SaV] on spherical varieties includes in particular the L^2-Plancherel formula for $H\backslash G$, when G is split and the characteristic of $\mathbf{F}$ is equal to zero. It should be possible using our result to determine the Fourier transform of the Schwartz space for these symmetric spaces. This should be entirely analogous to the work [DO] for affine Hecke algebras.

Acknowledgments. We thank warmly the referee for his very pertinent mathematical comments and his careful remarks on our presentation. We thank also Omer Offen for his collaboration when this work was intended to be used for truncation on some particular symmetric spaces.

2 The map H_G and the real functions Θ_G, $\|.\|$ and N_d on $H\backslash G$

2.1. Notation.

If E is a vector space, E' will denote its dual. If E is real, $E_{\mathbb{C}}$ will denote its complexification.

If G is a group, $g \in G$ and X is a subset of G, $g.X$ will denote gXg^{-1}. If J is a subgroup of G, $g \in G$ and (π, V) is a representation of J, V^J will denote the space of invariant elements of V under J and $(g\pi, gV)$ will denote the representation of $g.J$ on $gV := V$ defined by:

$$(g\pi)(gxg^{-1}) := \pi(x), x \in J.$$

We will denote by (π', V') the dual representation of G in the algebraic dual vector space V' of V.

If V is a vector space of vector valued functions on G which are invariant by right (resp., left) translations, we will denote by ρ (resp., λ) the right (resp., left) regular representation of G in V.

If G is locally compact, $d_l g$ will denote a left invariant Haar measure on G and δ_G will denote the modulus function.

Let $\mathbf{F}$ be a non archimedean local field with finite residue field $\mathbf{F}_q$. Unless specified we assume:

The characteristic of $\mathbf{F}$ is different from 2. (2.1)

Let $|.|_{\mathbf{F}}$ be the normalized absolute value of $\mathbf{F}$.

We will use conventions like in [W]. One considers various algebraic groups defined over $\mathbf{F}$, and a sentence like:

" let A be a split torus " will mean " let A be the group of $\mathbf{F}$-points of a torus, $\underline{A}$, defined and split over $\mathbf{F}$ ". (2.2)

With these conventions, let G be a connected reductive linear algebraic group. Let $\tilde{A}_G$ be the maximal split torus of the center of G. The change to standard notation will become clear later.

Let A be a split torus of G. Let $X_*(A)$ be the group of one-parameter subgroups of A. This is a free abelian group of finite type. Such a group will be called a lattice.

One fixes a uniformizer ϖ of $\mathbf{F}$. One denotes by $\Lambda(A)$ the image of $X_*(A)$ in A by the morphism of groups $\underline{\lambda} \mapsto \underline{\lambda}(\varpi)$. By this morphism $\Lambda(A)$ is isomorphic to $X_*(A)$.

If J is an algebraic group, one denotes by Rat(J) the group of its rational characters defined over $\mathbf{F}$. Let us define:

$$\tilde{\mathfrak{a}}_G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat(G)}, \mathbb{R}).$$

The restriction of rational characters from G to $\tilde{A}_G$ induces an isomorphism:

$$\mathrm{Rat(G)} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Rat}(\tilde{\mathrm{A}}_{\mathrm{G}}) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.3}$$

Notice that $\mathrm{Rat}(\tilde{\mathrm{A}}_{\mathrm{G}})$ appears as a generating lattice in the dual space $\tilde{\mathfrak{a}}'_G$ of $\tilde{\mathfrak{a}}_G$ and:

$$\tilde{\mathfrak{a}}'_G \simeq \mathrm{Rat(G)} \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.4}$$

One has the canonical map $\tilde{H}_G : G \to \tilde{\mathfrak{a}}_G$ which is defined by:

$$e^{\langle \tilde{H}_G(x), \chi \rangle} = |\chi(x)|_{\mathbf{F}},\ x \in G, \chi \in \mathrm{Rat(G)}. \tag{2.5}$$

The kernel of $\tilde{H}_G$, which is denoted by $\tilde{G}^1$, is the intersection of the kernels of $|\chi|_{\mathbf{F}}$ for all character $\chi \in \mathrm{Rat(G)}$ of G. One defines $X(G) = \mathrm{Hom}(\mathrm{G}/\tilde{\mathrm{G}}^1, \mathbb{C}^*)$, which is the group of unramified characters of G. One will use similar notation for Levi subgroups of G.

One denotes by $\tilde{\mathfrak{a}}_{G,\mathbf{F}}$ (resp., $\tilde{\tilde{\mathfrak{a}}}_{G,\mathbf{F}}$) the image of G (resp., $\tilde{A}_G$) by $\tilde{H}_G$. Then $G/\tilde{G}^1$ is isomorphic to the lattice $\tilde{\mathfrak{a}}_{G,\mathbf{F}}$.

There is a surjective map:

$$(\tilde{\mathfrak{a}}'_G)_{\mathbb{C}} \to X(G) \to 1 \tag{2.6}$$

denoted by $\nu \mapsto \chi_\nu$ which associates to $\chi \otimes s$, with $\chi \in \mathrm{Rat(G)}$, $s \in \mathbb{C}$, the character $g \mapsto |\chi(g)|^s_{\mathbf{F}}$ (cf. [W], I.1.(1)). In other words:

$$\chi_\nu(g) = e^{\langle \nu, \tilde{H}_G(g) \rangle}, g \in G, \nu \in (\tilde{\mathfrak{a}}'_G)_{\mathbb{C}}. \tag{2.7}$$

The kernel is a lattice and it defines a structure of a complex algebraic variety on $X(G)$ of dimension $dim_{\mathbb{R}}\tilde{\mathfrak{a}}_G$. Moreover $X(G)$ is an abelian complex Lie group whose Lie algebra is equal to $(\tilde{\mathfrak{a}}'_G)_{\mathbb{C}}$.

If χ is an element of $X(G)$, let ν be an element of $\tilde{\mathfrak{a}}'_{G,\mathbb{C}}$ such that $\chi_\nu = \chi$. The real part $\mathrm{Re}\, \nu \in \tilde{\mathfrak{a}}'_{\mathrm{G}}$ is independent from the choice of ν. We will denote it by $\mathrm{Re}\, \chi$. If $\chi \in \mathrm{Hom(G}, \mathbb{C}^*)$ is continuous, the character $|\chi|$ of G is an element of $X(G)$. One sets $\mathrm{Re}\, \chi = \mathrm{Re}\, |\chi|$. Similarly, if $\chi \in \mathrm{Hom}(\tilde{\mathrm{A}}_{\mathrm{G}}, \mathbb{C}^*)$ is continuous, the character $|\chi|$ of $\tilde{A}_G$ extends uniquely to an element of $X(G)$ with values in $\mathbb{R}^{*+}$, that we will denote again by $|\chi|$ and one sets $\mathrm{Re}\, \chi = \mathrm{Re}\, |\chi|$.

If P is a parabolic subgroup of G with Levi subgroup M, we keep the same notation with M instead of G.

The inclusions $\tilde{A}_G \subset \tilde{A}_M \subset M \subset G$ determine a surjective morphism $\tilde{\mathfrak{a}}_{M,\mathbf{F}} \to \tilde{\mathfrak{a}}_{G,\mathbf{F}}$ (resp., an injective morphism, $\tilde{\tilde{\mathfrak{a}}}_{G,\mathbf{F}} \to \tilde{\tilde{\mathfrak{a}}}_{M,\mathbf{F}}$) which extends uniquely to a surjective linear map between $\tilde{\mathfrak{a}}_M$ and $\tilde{\mathfrak{a}}_G$, (resp., injective map between $\tilde{\mathfrak{a}}_G$ and $\tilde{\mathfrak{a}}_M$). The second map allows to identify $\tilde{\mathfrak{a}}_G$ with a subspace of $\tilde{\mathfrak{a}}_M$ and the kernel of the first one, $\tilde{\mathfrak{a}}_M^G$, satisfies:

$$\tilde{\mathfrak{a}}_M = \tilde{\mathfrak{a}}_M^G \oplus \tilde{\mathfrak{a}}_G. \tag{2.8}$$

If an unramified character of G is trivial on M, it is trivial on any maximal compact subgroup of G and on the unipotent radical of P, hence on G. It allows to identify $X(G)$ with a subgroup of $X(M)$. Then $X(G)$ is the analytic subgroup of $X(M)$ with Lie algebra $(\tilde{\mathfrak{a}}'_G)_{\mathbb{C}} \subset (\tilde{\mathfrak{a}}'_M)_{\mathbb{C}}$. This follows easily from (2.7) and (2.8).

One has (cf. [D2], (4.5)),

The map $\Lambda(\tilde{A}_G) \to G/\tilde{G}^1$ is injective and allows to identify $\Lambda(\tilde{A}_G)$ with the subgroup $\tilde{H}_G(\tilde{A}_G)$ of $\tilde{\mathfrak{a}}_G$. (2.9)

Let $\underline{G}$ be the algebraic group defined over $\mathbf{F}$ whose group of $\mathbf{F}$-points is G. Let σ be a rational involution of $\underline{G}$ defined over $\mathbf{F}$. Let H be the group of $\mathbf{F}$-points of an open $\mathbf{F}$-subgroup of the fixed point set of σ. We will also denote by σ the restriction of σ to G.

A split torus A of G is said to be σ-split if A is contained in the set of elements of G which are antiinvariant by σ. Now we explain the change to standard notation: A_G will denote the maximal σ-split torus of the center of G.

Let $\tilde{A}$ be a σ-stable split torus of G. The involution σ induces an involution, denoted in the same way, on $\tilde{\mathfrak{a}} := \tilde{\mathfrak{a}}_{\tilde{A}}$. Let $\tilde{A}^\sigma$ (resp., $\tilde{A}_\sigma$) be the maximal split torus in the group of elements of $\tilde{A}$ which are invariant (resp., antiinvariant) by σ. Then $\tilde{\mathfrak{a}}^\sigma$ (resp., $\tilde{\mathfrak{a}}_\sigma$) is identified with the set of invariant (resp., antiinvariant) of $\tilde{\mathfrak{a}}$ by σ and $\tilde{A}_\sigma$ is the maximal σ-split torus of $\tilde{A}$.

In particular, one has $A_G = (\tilde{A}_G)_\sigma$ and $\tilde{\mathfrak{a}}_G = \tilde{\mathfrak{a}}_G^\sigma \oplus \mathfrak{a}_G$ where $\tilde{\mathfrak{a}}_G^\sigma$ (resp., $\mathfrak{a}_G$) is the space of invariant (resp., antiinvariant) elements of $\tilde{\mathfrak{a}}_G$ by σ.

We define a morphism of groups $H_G : G \to \mathfrak{a}_G$ which is the composition of $\tilde{H}_G$ with the projection on $\mathfrak{a}_G$ parallel to $\tilde{\mathfrak{a}}_G^\sigma$. We remark that, as is seen easily, $\tilde{H}_G$ commutes with σ. Hence H_G is trivial on H.

We denote by G^1 the kernel of H_G, which contains H. It contains also $\tilde{G}^1$, hence it is open in G. We denote by $\mathfrak{a}_{G,\mathbf{F}}$ the image of H_G. Let $X(G)_\sigma$ be the connected component of the group of antiinvariant elements of $X(G)$. Then $X(G)_\sigma$ is the analytic subgroup of $X(G)$ with Lie algebra $(\mathfrak{a}'_G)_{\mathbb{C}} \subset (\tilde{\mathfrak{a}}'_G)_{\mathbb{C}}$. The elements of $X(G)_\sigma$ are precisely the characters of G of the form

$$\chi_\nu(g) = e^{\langle \nu, H_G(g) \rangle}, \nu \in (\mathfrak{a}'_G)_{\mathbb{C}}, g \in G.$$

They are exactly the characters of the lattice $G^1 \backslash G$. The group $X(G)_\sigma$ has a natural structure of complex algebraic group. We denote by $X(G)_{\sigma,u}$ the group of unitary elements of $X(G)_\sigma$.

One has

The group $\Lambda(A_G)$ is identified by H_G with $H_G(A_G)$. (2.10)

Let $\tilde{A}$ be a maximal split torus of G. Let M be the centralizer of $\tilde{A}$ in G. Let us show the following assertion.

$$\tilde{H}_M(\tilde{A}) \text{ contains a multiple by } k \in \mathbb{R}^{+*} \text{ of the coweight lattice of the root system } \Sigma \subset (\tilde{\mathfrak{a}}_M^G)' \text{ of } \tilde{A} \text{ in the Lie algebra of } G. \text{ Here the coweight lattice is the dual lattice in } \tilde{\mathfrak{a}}_M^G \text{ of the root lattice.} \tag{2.11}$$

It is clear that it suffices to prove the assertion for one maximal split torus. Let $\tilde{A}'$ be a maximal split torus of the derived group G_{der} of G. Let $\underline{\tilde{A}} = \underline{\tilde{A}'}\underline{\tilde{A}_G}$. This is a maximal $\mathbf{F}$-split torus of $\underline{G}$ for reasons of dimension. The intersection F of $\underline{\tilde{A}'}$ and $\underline{\tilde{A}_G}$ is finite. Hence one has the exact sequence

$$1 \to F \to \underline{\tilde{A}'} \times \underline{\tilde{A}_G} \to \underline{\tilde{A}} \to 1.$$

Going to $\mathbf{F}$-points, the long exact sequence in cohomology implies that $\tilde{A}\tilde{A}'_G$ is of finite index in $\tilde{A}$. Hence the image of $\tilde{A}'\tilde{A}_G$ by $\tilde{H}_M$ is of finite index in the image of $\tilde{A}$. The image of $\tilde{A}'$ (resp., $\tilde{A}_G$) in $\tilde{\mathfrak{a}}_M$ by $\tilde{H}_M$ is contained in $\tilde{\mathfrak{a}}_M^G$ (resp., $\tilde{\mathfrak{a}}_G$) and is a lattice Λ_1 (resp., Λ_G) generating $\tilde{\mathfrak{a}}_M^G$ because $\Lambda_1 + \Lambda_G$ is of finite index in $\Lambda = \tilde{H}_M(\tilde{A})$ which generates $\tilde{\mathfrak{a}}_M$. Hence the rank of Λ_1 is equal to the dimension of $\tilde{\mathfrak{a}}_M^G$. The values of the normalized absolute value of $\mathbf{F}$ are of the form $q^n, n \in \mathbb{Z}$. From the definition of $\tilde{H}_M$, one sees that Λ_1 is included in $(\log q)\Lambda_2$ where $\Lambda_2 \subset \tilde{\mathfrak{a}}_M^G$ is the coweight lattice of Σ. Both are lattices of the same rank, for reasons of dimension. Our claim follows from the following assertion:

$$\text{Let } \Lambda_1 \subset \Lambda_2 \text{ be two lattices of the same rank. Then there exists } n \in \mathbb{N}^* \text{ such that } n\Lambda_2 \subset \Lambda_1, \tag{2.12}$$

which follows by inverting the matrix, with integral entries, expressing a basis of Λ_1 in a basis of Λ_2.
Let A be a maximal σ-split torus of G and let $\tilde{A}$ be a σ-stable maximal split torus of G which contains A. The roots of A in the Lie algebra of G form a root system (cf. [HW], Proposition 5.9). Let M be the centralizer in G of A, which is σ-invariant. One has $A = A_M$. One deduces like (2.11) that:

$$\Lambda(A) \subset \mathfrak{a} \text{ contains a multiple by } k \in \mathbb{R}^{+*} \text{ of the coweight lattice of the root system of } A \text{ in the Lie algebra of } G. \tag{2.13}$$

A parabolic subgroup P of G is called a σ-parabolic subgroup if P and $\sigma(P)$ are opposite parabolic subgroups. Then $M := P \cap \sigma(P)$ is the σ-stable Levi subgroup of P. If P is such a parabolic subgroup, P^- will denote $\sigma(P)$.
The sentence : "Let $P = MU$ be a parabolic subgroup of G" will mean that U is the unipotent radical of P and M a Levi subgroup of G. If moreover P is a σ-parabolic subgroup of G, M will denote its σ-stable Levi subgroup.
Let $P = MU$ be a σ-parabolic subgroup of G. Recall that A_M is the maximal σ-split torus of the center of M.
Let A_P^-, be the set of P-antidominant elements in A_M. More precisely, if $\Sigma(P)$ is the set of roots of A_M in the Lie algebra of P, and $\Delta(P)$ is the set of simple roots, one has:

$$A_P^- = \{a \in A_M \| \alpha(a)|_{\mathbf{F}} \leq 1, \alpha \in \Delta(P)\ \}.$$

One defines also for $\varepsilon > 0$:

$$A_P^-(\varepsilon) = \{a \in A_M | |\alpha(a)|_{\mathbf{F}} \leq \varepsilon, \ \alpha \in \Delta(P)\}.$$

2.2. Some functions on $H\backslash G$.
Let $\tilde{A}_0$ be a σ-stable maximal split torus of G, which contains a maximal σ-split torus A_0 of G. Let P_0 be a minimal parabolic subgroup of G which contains $\tilde{A}_0$. Let K_0 be the fixator of a special point in the apartment of $\tilde{A}_0$ in the Bruhat-Tits building of G. We fix an algebraic embedding

$$\tau : G \to GL_n(\mathbf{F}). \tag{2.14}$$

We may and we will assume that $\tau(K_0) \subset GL_n(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of $\mathbf{F}$ ([W] I.1)). For $g \in G$, we write:

$$\tau(g) = (a_{i,j})_{i,j=1,..,n}, \ \tau(g^{-1}) = (b_{i,j})_{i,j=1,..,n}.$$

We set

$$\|g\| = sup_{i,j} sup(|a_{i,j}|_{\mathbf{F}}, |b_{i,j}|_{\mathbf{F}}). \tag{2.15}$$

We have (cf. [W] I.1) :

$$\begin{aligned} &\|g\| \geq 1 \text{ for } g \in G, \quad \|g_1 g_2\| \leq \|g_1\|\|g_2\| \text{ for } g_1, \ g_2 \in G \text{ and} \\ &\|k_1 g k_2\| = \|g\| \text{ for } k_1, k_2 \in K_0, \ g \in G. \end{aligned} \tag{2.16}$$

We denote by $(\varepsilon_{M_0}, \mathbb{C})$ the trivial representation of the centralizer M_0 of $\tilde{A}_0$ in G and $(\pi_0, V_0) = (i_{P_0}^G \varepsilon_{M_0}, i_{P_0}^G \mathbb{C})$ the normalized induced representation. Let e_0 be the unique element of V_0 invariant by K_0 and such that $e_0(1) = 1$.
We remark that the contragredient representation $(\check{\pi}_0, \check{V}_0)$ is isomorphic to (π_0, V_0). For $g \in G$, we set :

$$\Xi_G(g) = \langle \pi_0(g) e_0, e_0 \rangle.$$

The function Ξ_G is biinvariant by K_0.

We will say that two functions f_1 and f_2 defined on a set E with values in $\mathbb{R}+$ are equivalent on a subset E' of E (we write $f_1(x) \asymp f_2(x), \ x \in E'$), if there exist $C, C' > 0$ such that: (2.17)

$$C' f_2(x) \leq f_1(x) \leq C f_2(x), \ x \in E'.$$

We recall (cf. [W], Lemma II.1.2):

There exist $d \in \mathbb{N}$ and for all $g_1, \ g_2 \in G$, a constant $c > 0$ such that

$$\Xi_G(g_1 g g_2) \leq c \Xi_G(g)(1 + log\|g\|)^d, g \in G. \tag{2.18}$$

We set :

$$\|Hg\| := \|\sigma(g^{-1})g)\|, \ g \in G. \tag{2.19}$$

For a compact subset Ω' of G, we deduce from (2.16):

$$\|Hg\omega\| \asymp \|Hg\|, \ \omega \in \Omega', \ g \in G. \tag{2.20}$$

Let us define the functions Θ_G and N_d, $d \in \mathbb{Z}$ by

$$\Theta_G(Hg) = (\Xi_G(\sigma(g^{-1})g))^{1/2}, \ g \in G. \tag{2.21}$$

and

$$N_d(Hg) = (1 + log\|Hg\|)^d, \ g \in G. \tag{2.22}$$

(2.20) implies (with $N = N_1$):

$$N(Hg\omega) \asymp N(Hg), g \in G, \omega \in \Omega'. \tag{2.23}$$

The next assertion follows from the definitions and (2.18).

There exists $d \in \mathbb{N}$, and for all $g_1 \in G$ there exists $c > 0$ such that:

$$\Theta_G(Hgg_1) \leq c\Theta_G(g)N_d(Hg), g \in G. \tag{2.24}$$

It follows from the Cartan decomposition for $H\backslash G$ (cf. [BeOh] Theorem 1.1) that there exist a compact subset Ω of G and a finite set $\mathcal{P}$ of minimal σ-parabolic subgroups of G such that:

$$H\backslash G = \cup_{P \in \mathcal{P}} HA_P^- \Omega. \tag{2.25}$$

Let $P = MU$ be a minimal σ-parabolic subgroup of G and let Ω' be a compact subset of G. We choose a norm on $\mathfrak{a}_M$. By ([L], Lemma 7 and Proposition 6), we have:

(i) There exist $c, c', C, C' > 0$ such that:

$$Ce^{c\|H_M(a)\|} \leq \|Ha\omega\| \leq C'e^{c'\|H_M(a)\|}, \omega \in \Omega', \ a \in A_P^-, \tag{2.26}$$

(ii)

$$N(Ha\omega) \asymp (1 + \|H_M(a)\|), a \in A_M, \omega \in \Omega.$$

(iii) The function Θ_G is right invariant by $K_0 \cap \sigma(K_0)$.

(iv) There exist $C, C' > 0$ and $d, d' \in \mathbb{N}$ such that for $g = a\omega$ with $\omega \in \Omega'$, $a \in A_P^-$, one has (2.27)

$$C\delta_P^{1/2}(a)N_{-d}(Ha) \leq \Theta_G(Hg) \leq C'\delta_P^{1/2}(a)N_{d'}(Ha) \, .$$

Lemma 2.1 *Let dx be a non zero G-invariant measure on $H\backslash G$. There exists $d \in \mathbb{N}$ such that:*

$$\int_{H\backslash G} \Theta_G^2(x)N_{-d}(x)dx < \infty.$$

Proof :

Let $P = MU \in \mathcal{P}$ and Ω as in (2.25). From (2.27) one deduces that there exist $C' > 0$ and $d' \in \mathbb{N}$ such that :

$$\Theta_G(Ha\omega) \leq C'\delta_P^{1/2}(a)N_{d'}(Ha), a \in A_P^-, \omega \in \Omega.$$

We can choose Ω large enough in order to have

$$A_P^-\Omega \subset \Lambda_P^-\Omega,$$

where Λ_P^- is the set of P-antidominant elements in $\Lambda(A_M)$. It follows from [KT2], Proposition 2.6, that

There exist constants $C_1, C_2 > 0$ such that:

$$C_1\delta_P^{-1}(\lambda) \leq \mathrm{vol}(H\backslash H\lambda\Omega) \leq C_2\delta_P^{-1}(\lambda), \lambda \in \Lambda_P^-, \tag{2.28}$$

where $\mathrm{vol}(H\backslash H\lambda\Omega)$ is the volume of the subset $H\backslash H\lambda\Omega$ of $H\backslash G$.

From (2.26) (ii) one deduces that for $d'' \in \mathbb{N}$ large enough:

$$\sum_{\lambda\in\Lambda(A_M)} N_{-d''}(H\lambda) < \infty.$$

This implies easily the Lemma. $\square$

3 Tempered functions on $H\backslash G$

3.1. On the Cartan decomposition and lattices.

Let $P = MU$ be a σ-parabolic subgroup of G. Let $\Sigma(P)$ be the set of roots of A_M in the Lie algebra of U and let $\Delta(P)$ be the set of simple roots. It will be viewed as a subset of $\mathfrak{a}'_M$. Let us denote by ${}^+\overline{\mathfrak{a}}_P{}'$ (resp., ${}^+\mathfrak{a}'_P$) the set of $\chi \in \mathfrak{a}'_M$ of the form:

$$\chi = \sum_{\alpha\in\Delta(P)} x_\alpha \alpha,$$

where $x_\alpha \geq 0$ (resp., $x_\alpha > 0$) for all $\alpha \in \Delta(P)$.
Let us assume that P is a minimal σ-parabolic subgroup of G. If $Q = LV$ is a σ-parabolic subgroup of G such that $P \subset Q$, let Δ^L be the set of elements of $\Delta := \Delta(P)$ which are roots of A_M in the Lie algebra of L. We remark that A_Q^- is equal to the intersection $A_L \cap A_P^-$. For $\varepsilon > 0$, we define

$$A_P^-(Q,\varepsilon) := \{a \in A_P^- || \alpha(a)|_{\mathbf{F}} \geq \varepsilon, \alpha \in \Delta^L \textit{ and } |\alpha(a)|_{\mathbf{F}} < \varepsilon, \alpha \in \Delta \setminus \Delta^L\}.$$

Let $\mathcal{P}(P)$ be the set of σ-parabolic subgroups of G which contain P. For $\varepsilon > 0$, one has a partition of A_P^-:

$$A_P^- = \cup_{Q\in\mathcal{P}(P)} A_P^-(Q,\varepsilon). \tag{3.1}$$

Moreover for any $Q \in \mathcal{P}(P)$ there exists a compact subset $\omega_{\varepsilon,Q}$ of A_M such that:

$$A_P^-(Q,\varepsilon) \subset A_Q^-\omega_{\varepsilon,Q}, \tag{3.2}$$

and further, introducing Λ_Q^- the set of the Q-antidominant elements of $\Lambda(A_L)$, there exists a compact set $\omega'_{\varepsilon,Q}$ of A_M such that :

$$A_P^-(Q,\varepsilon) \subset \Lambda_Q^- \omega'_{\varepsilon,Q}. \tag{3.3}$$

One uses (2.13) and one introduces a multiple by $k \in \mathbb{R}^{+*}$ of the coweight lattice.
Let $\delta_\alpha \in \mathfrak{a}_M, \alpha \in \Delta(P)$, the fundamental coweights.
Then $\Lambda(A_L)$ contains a sublattice Λ'_L of finite index in $\Lambda(A_L)$, which is generated by $\delta'_\alpha := -k\delta_\alpha \in \Lambda_Q^-$, $\alpha \in \Delta \setminus \Delta^L$ and by $\Lambda(A_G)$. Let $\omega_1, \dots, \omega_p$ be a basis of $\Lambda(A_G)$ and $\omega'_1, \dots, \omega'_p$ be the dual basis in $\mathfrak{a}'_G$. Let Λ'^-_Q be the semigroup generated by the $\delta'_\alpha, \alpha \in \Delta \setminus \Delta^L$ and $\Lambda(A_G)$, i.e. :

$$\Lambda'^-_Q = \{ \prod_{\alpha \in \Delta \setminus \Delta^L} (\delta'_\alpha)^{n_\alpha} | n_\alpha \in \mathbb{N} \} \Lambda(A_G).$$

Then we will see that there exists a finite set F_Q in Λ'_L such that

$$\Lambda_Q^- \subset \Lambda'^-_Q F_Q. \tag{3.4}$$

In fact if $\lambda \in \Lambda_Q^-$, for each $\alpha \in \Delta \setminus \Delta^L$ (resp., $j = 1, \dots, p$), one defines n_α (resp., n_j) the largest integer such that $\langle \lambda, \alpha \rangle$ (resp., $\langle \lambda, \omega'_j \rangle$) is less than or equal to $-kn_\alpha$ (resp., n_j). Then $\lambda' = \prod_{\alpha \in \Delta \setminus \Delta^L} (\delta'_\alpha)^{n_\alpha} \prod_{j=1,\dots p} \omega_j^{n_j}$ is in Λ'^-_Q. Moreover $\lambda(\lambda')^{-1}$ lies in a bounded subset of $\Lambda(A_Q)$, as λ varies in Λ_Q^-, hence it lies in a finite set F_Q.
Summarizing, one sees that there exists a compact subset $\omega''_{\varepsilon,Q}$ of A_M such that :

$$A_P^-(Q,\varepsilon) \subset \Lambda'^-_Q \omega''_{\varepsilon,Q}. \tag{3.5}$$

3.2. $\mathcal{A}(H\backslash G)$, $\mathcal{A}_{temp}(H\backslash G)$, $\mathcal{A}_2(H\backslash G)$.

The proof of the following Lemma is analogous to the proof of [D1], Lemma 3.

Lemma 3.1 *Let f be a function on $H\backslash G$ which is right invariant by a compact open subgroup. The following conditions are equivalent:*
(i) The G-module V_f, generated by the right translates $\rho(g)f, g \in G$, is admissible.
(ii) There exist an admissible representation (π, V) of G, an element v of V and an H-fixed linear form ξ on V such that $f = c_{\xi,v}$ where:

$$c_{\xi,v}(Hg) = \langle \xi, \pi(g)v \rangle, g \in G.$$

(iii) The function f is $ZB(G)$-finite, where $ZB(G)$ is the Bernstein's center of G.

We denote by $\mathcal{A}(H\backslash G)$ the vector space of such functions. An element of this space is A_G-finite, hence there exists a finite set $\mathrm{Exp}(f)$ of smooth characters of A_G such that

$$f = \sum_{\chi \in \mathrm{Exp}(f)} f_\chi,$$

where the f_χ are non zero and satisfy for some $n \in \mathbb{N}^*$:

$$(\rho(a) - \chi(a))^n f_\chi = 0, a \in A_G.$$

The elements of $\mathrm{Exp}(f)$ are called the exponents of f.
Let (π, V) be a smooth representation of G of finite length. Then it is a finite direct sum of generalized eigenspaces under A_G. If ν is a character of A_G, let us denote by $V(\nu)$ the corresponding generalized eigenspace of V and by $\xi(\nu)$ the restriction to $V(\nu)$ of any element ξ of V', which can be extended to an element of V', denoted also $\xi(\nu)$, which is zero on the other generalized eigenspaces. If $\xi \in V'^H$, $\mathrm{Exp}(\xi)$ will denote the subset of ν such that $\xi(\nu)$ is non zero. The elements of $\mathrm{Exp}(\xi)$ are called the A_G-exponents or exponents of ξ.

For any σ-parabolic subgroup P, the constant term f_P of f along P has been defined in [L], Proposition 2. For an H-invariant linear form ξ on V, $j_P^*(\xi)$ has been defined in [L], Theorem 1. It is an $M \cap H$-invariant linear form on the normalized Jacquet module $j_P(V)$. One denotes by j_P the canonical projection from V to $j_P(V)$. If $f = c_{\xi,v}$, one has the equality:

$$f_P = c_{j_P^*(\xi), j_P(v)}. \tag{3.6}$$

Let us recall a property of the constant term (cf. [D2] Proposition 3.7), in which one has to change right H-invariance to left invariance by changing $g \mapsto f(g)$ into $g \mapsto f(g^{-1})$.

> Let $P = MU$ be a minimal σ-parabolic subgroup of G and let $Q = LV$ be a σ-parabolic subgroup of G which contains P. Let K be an open compact subgroup of G. Then there exists $\varepsilon > 0$ such that, for any right K-invariant element f of $\mathcal{A}(H\backslash G)$, one has (3.7)
>
> $$f(a) = \delta_Q^{1/2}(a) f_Q(a), a \in A_M^-(Q < \varepsilon),$$
>
> where $A_M^-(Q < \varepsilon) := \{a \in A_P^- \| \alpha(a)|_{\mathbf{F}} < \varepsilon, \alpha \in \Delta(P) \setminus \Delta^L(P)\}$.

One defines

$$f_P^{ind}(g) := (\rho(g)f)_P, g \in G.$$

As the Jacquet module of an admissible representation is admissible, one deduces from (3.6) that the constant term f_P is an element of $\mathcal{A}(M \cap H\backslash M)$. The union $\mathrm{Exp}_P(f)$ of the set of exponents of $f_P^{ind}(g), g \in G$ is finite, as the Jacquet module of the G-module generated by f is of finite length. This set is called the set of exponents of f along P. If ξ is an H-fixed linear form on a smooth G-module of finite length, one defines similarly $\mathrm{Exp}_P(\xi) = \mathrm{Exp}(j_P^*(\xi))$.

One says that an element f of $\mathcal{A}(H\backslash G)$ is tempered (resp., square integrable) if for every σ-parabolic subgroup P of G, the real part of the elements of $\mathrm{Exp}_P(f)$ are contained in ${}^+\bar{\mathfrak{a}}_P{}'$ (resp., ${}^+\mathfrak{a}'_P$).

We denote by $\mathcal{A}_{temp}(H\backslash G)$ (resp., $\mathcal{A}_2(H\backslash G)$) the subspace of tempered elements (resp., square integrable) of $\mathcal{A}(H\backslash G)$. Obviously one has:

> The spaces $\mathcal{A}_2(H\backslash G) \subset \mathcal{A}_{temp}(H\backslash G)$ are G-invariant subspaces of $\mathcal{A}(H\backslash G)$. (3.8)

Moreover, from [KT2], Theorem 4.7, one deduces:

> If $A_G = \{1\}$, an element f of $\mathcal{A}(H\backslash G)$ is element of $\mathcal{A}_2(H\backslash G)$ if and only if it is an element of $L^2(H\backslash G)$. (3.9)

Let V be a smooth G-module of finite length. Similarly, one says that an H-fixed linear form ξ on V is tempered (resp., square integrable) if for every σ-parabolic subgroup P of G the real part of the elements of $\mathrm{Exp}_P(\xi)$ are contained in ${}^+\overline{\mathfrak{a}}_P{}'$ (resp., ${}^+\mathfrak{a}'_P$). We denote by V'^H_{temp} (resp., V'^H_2) the set of tempered (resp., square-integrable) H-invariant linear forms on V.

Lemma 3.2 *The following conditions are equivalent:*
(i) The function f is an element of $\mathcal{A}_{temp}(H\backslash G)$ (resp., $\mathcal{A}_2(H\backslash G)$).
(ii) There exist an admissible representation (π, V) of G, an element v of V and an element ξ in V'^H_{temp} (resp., V'^H_2) such that:

$$f(Hg) = \langle \xi, \pi(g)v \rangle, g \in G.$$

Proof :

One uses Lemma 3.1 and (3.6). □

Definition 3.3 *Let $f \in \mathcal{A}_{temp}(H\backslash G)$ and let P be a σ-parabolic subgroup of G. Let $\mathrm{Exp}^{\mathrm{w}}_P(f)$ (resp., $\mathrm{Exp}^{+}_P(f)$) be the set of elements χ of $\mathrm{Exp}_P(f)$ such that $\mathrm{Re}\chi = 0$ (resp., is different from zero). The weak constant term f^{w}_P of f along P is the sum of the $(f_P)_\chi$ where χ varies in $\mathrm{Exp}^{\mathrm{w}}_P(f)$. We set $f^+_P = f_P - f^{\mathrm{w}}_P$ and $f^{\mathrm{w},ind}_P(g) = (\rho(g)f)^{\mathrm{w}}_P$ for $g \in G$.*

Lemma 3.4 *With the notation of the definition, let $P = MU$, $Q = LV$ be two σ-parabolic subgroups of G such that $P \subset Q$. Let $R = P \cap L$. Then one has:*
(i) $f^{\mathrm{w}}_Q \in \mathcal{A}_{temp}(L \cap H\backslash L)$.
(ii)

$$f^{\mathrm{w}}_P = (f^{\mathrm{w}}_Q)^{\mathrm{w}}_R.$$

Proof :

(i) From the definition of f^{w}_Q and the fact that $f_Q \in \mathcal{A}(L \cap H\backslash L)$, one sees that f^{w}_Q is also an element of $\mathcal{A}(L\cap H\backslash L)$. The set of exponents $\mathrm{Exp}_R(f_Q)$ is the disjoint union of $\mathrm{Exp}_R(f^{\mathrm{w}}_Q)$ and $\mathrm{Exp}_R(f^+_Q)$. From the transitivity of the constant term (cf. [L], Corollary 1 of Theorem 3), one has $\mathrm{Exp}_R(f_Q) \subset \mathrm{Exp}_P(f)$. Hence if $\chi \in \mathrm{Exp}_R(f^{\mathrm{w}}_Q)$, one has $\mathrm{Re}(\chi) \in^+ \overline{\mathfrak{a}}_{\mathrm{P}}{}'$ and $\mathrm{Re}(\chi)$ restricted to $\mathfrak{a}_L$ is equal to zero. This implies $\mathrm{Re}(\chi) \in^+ \overline{\mathfrak{a}}_{\mathrm{R}}{}'$. One deduces (i).
Let us prove (ii). We have

$$f_Q = f^{\mathrm{w}}_Q + f^+_Q.$$

Then by the transitivity of the constant term, one has:

$$f_P = (f^{\mathrm{w}}_Q)_R + (f^+_Q)_R = (f^{\mathrm{w}}_Q)^{\mathrm{w}}_R + (f^{\mathrm{w}}_Q)^+_R + (f^+_Q)_R.$$

Looking to exponents, one concludes that

$$f^{\mathrm{w}}_P = (f^{\mathrm{w}}_Q)^{\mathrm{w}}_R, \quad f^+_P = (f^{\mathrm{w}}_Q)^+_R + (f^+_Q)_R.$$

□

3.3. Families of type I of tempered functions.

Definition 3.5 *Let $X(\mathbb{C})$ be a complex algebraic torus. We denote by B the algebra of polynomial functions on $X(\mathbb{C})$. We denote by X the maximal compact subgroup of $X(\mathbb{C})$. A family (F_x), parametrized by X, of elements of $\mathcal{A}_{temp}(H\backslash G)$ is called a family of type I of tempered functions on $H\backslash G$ if:*
a) There exists a compact open subgroup J of G such that for all $x \in X$, F_x is right invariant by J.
b) For all $g \in G$, the map $x \mapsto F_x(Hg)$ is C^∞ on X.
c) For all constant coefficient differential operator D on X and $x \in X$, the map $Hg \mapsto D(F_x(Hg))$ is an element of $\mathcal{A}(H\backslash G)$.
d) For every σ-parabolic subgroup $Q = LV$ of G, there exists a finite family $\Xi_Q = \{\xi_1, \dots \xi_n\}$, with possible repetitions, of characters of A_L with values in the group of invertible elements $B^\times$ of B, such that:
(d-i)

$$(\rho(a) - (\xi_1(a))(x)) \dots (\rho(a) - (\xi_n(a))(x)).F^{ind}_{x,Q}(g) = 0, a \in A_L, g \in G, x \in X, \quad (3.10)$$

(d-ii) for $i = 1, \dots, n$, the real part of $\xi_i(\cdot)(x)$ is independent of $x \in X$ and is an element of ${}^+\bar{\mathfrak{a}}_Q{}'$.
In the following, we will denote $\xi_{i,x}(a)$ instead of $(\xi_i(a))(x)$.
The family $\mathcal{E}$ of Ξ_Q will be called a set of exponents of the family F.

We will see later (cf. Theorem 5.1) examples of such families related to Eisenstein integrals.
The following properties are easy consequences of the definitions.

If F is a family of type I, parametrized by X, of tempered functions on $H\backslash G$, the same is true for the family $\rho(g)F$, for every $g \in G$, with the same set of exponents. (3.11)

Lemma 3.6 *Let F be a family of type I, parametrized by X, of tempered functions on $H\backslash G$ and $Q = LV$ be a σ-parabolic subgroup.*
(i) For all $l \in L$, the map $x \mapsto (F_x)_Q((H \cap L)l)$ is C^∞ on X.
(ii) If D is a differential operator with constant coefficients on X of degree d, one has:

$$[(\rho(a) - (\xi_{1,x}(a)) \dots (\rho(a) - (\xi_{n,x}(a))]^{2^d} D(F_x)_Q = 0, a \in A_L, g \in G, x \in X,$$

(iii) One has the equality:

$$(DF_x)_Q = D(F_x)_Q, x \in X.$$

In particular $D(F_x)_Q \in \mathcal{A}((H \cap L)\backslash L)$
(iv) The family (DF_x) is a family of type I with a set of exponents given by the $2^d\Xi_Q$,where $2^d\Xi_Q$ means Ξ_Q repeated 2^d times.

Proof :

(i) Using translations, it is enough to prove that $(F_x)_Q(a), a \in A_L$ is C^∞ on X. Let $P = MU$ be a σ-parabolic subgroup contained in Q. By (3.7), there exists $\varepsilon > 0$ such that

$$(F_x)_Q((H \cap L)a) = \delta_Q(a)^{-1/2} F_x(Ha), a \in A_M^-(Q < \varepsilon).$$

Hence the assertion of the Lemma is true for $a \in A_M^-(Q < \varepsilon)$. But the relation (3.10) applied to a_0 strictly P-dominant instead of a implies a linear recursion relation for the sequence $((F_x)_Q(aa_0^{-p}))$ which allows to compute $(F_x)_Q$ on $(H \cap L)A_L$ from its values on $A_M^-(Q < \varepsilon)$ (cf. [D2] proof of Proposition 3.11 for details). Then (i) follows.
(ii) By using induction, it suffices to prove the assertion for $d = 1$. In that case, we apply D to (3.10) and then we apply the product of operators $(\rho(a) - (\xi_{1,x}(a)) \dots (\rho(a) - (\xi_{n,x}(a))$ to the equality obtained. This gives the result.
(iii) We fix $x \in X$. Let V be the linear span of the set $\{D(\rho(g)F_x) | g \in G\}$. As D and $\rho(g)$ commute, this space is invariant by right translation by elements of G. The elements of V are of the form DF'_x with a family F' of type I satisfying (3.10).

We first show that the map $DF'_x \mapsto D(F'_x)_Q$ is well defined. For this it is enough to prove that if $DF'_x = 0$ then $D(F'_x)_Q = 0$. From (3.7), with the notation of (ii), one has $(F'_y)_Q((H \cap L)a) = F'_y((H \cap L)a)\delta_Q(a)^{-1/2}$ on $A_M^-(Q < \varepsilon)$ for some $\varepsilon > 0$. Hence by derivation $D(F'_x)_Q = 0$ on $A_M^-(Q < \varepsilon)$. Using recursion relations as in (ii), one gets that $D(F'_x)_Q = 0$ on A_L. Then using translations, one sees that $D(F'_x)_Q = 0$ on L.

Hence the map $DF'_x \mapsto D(F'_x)_Q$ is well defined on V. From the properties of the constant term of the F'_x (cf. [D2], Proposition 3.14), it is easily seen that this map has the characteristic properties of the constant term map on V (cf. l.c.). This proves (iii). Then (iv) follows from (ii) applied to right translates of F by elements of G and from (iii). □

Lemma 3.7 *Let F be a family of type I, parametrized by X, of tempered functions on $H \backslash G$. Let $Q = LV$ a σ-parabolic subgroup. Then, one has*
(i) The family $(F_x)_Q^{\mathrm{w}}, x \in X$ is a family of tempered functions on $(L \cap H) \backslash L$ of type I.
(ii) Let D be a differential operator on X with constant coefficients. Then, one has $D(F_x)_Q^{\mathrm{w}} = (DF_x)_Q^{\mathrm{w}}, x \in X$.

Proof :
(i) Let $a' \in A_L$ be such that $|\alpha(a')|_{\mathbf{F}} < 1$ for all α in $\Delta(Q)$. Let Ξ_Q^{w} be the set of elements ξ of Ξ_Q such that ξ_x is a unitary character of A_L for all $x \in X$. We set $\Xi_Q^+ = \Xi_Q \setminus \Xi_Q^{\mathrm{w}}$. We recall that there might be repetitions in these families. From the theory of the resultant there exist elements R, S of $B[T]$ such that:

$$R(T) \prod_{\xi \in \Xi_Q^{\mathrm{w}}} (T - \xi(a')) + S(T) \prod_{\xi' \in \Xi_Q^+} (T - \xi'(a')) = b,$$

where

$$b = \prod_{\xi \in \Xi_Q^{\mathrm{w}}, \xi' \in \Xi_Q^+} (\xi(a') - \xi'(a')).$$

We define

$$\Gamma_x = S_x(\rho(a')) \prod_{\xi' \in \Xi_Q^+} (\rho(a') - \xi'_x(a')).$$

where ρ denotes the right regular representation on the space of functions on $(L \cap H)\backslash L$. One sees easily that, from the definition of R, S, the definition of the constant term and of Ξ_Q^{w}:

$$\Gamma_x(F_x)_Q = b(x)(F_x)_Q^{\mathrm{w}}, x \in X. \tag{3.12}$$

From the properties of a' and the definition of Ξ_Q^+, one sees that $b(x)$ does not vanish for $x \in X$ and is C^∞ on X. Hence

$$(F_x)_Q^{\mathrm{w}} = b(x)^{-1}\Gamma_x(F_x)_Q, x \in X.$$

By Lemma 3.6 (i), for $l \in L$, the map $x \mapsto (F_x)_Q((H \cap L)l)$ is C^∞ on X.
One has to prove that for a differential operator D with constant coefficients on X and $x \in X$, $D(F_x)_Q^w \in \mathcal{A}((L \cap H)\backslash L)$. First, from Lemma 3.6, second part of (iii), $D(F_x)_Q$ is an element of $\mathcal{A}((L \cap H)\backslash L)$. Then our claim follows by applying D to the preceeding equality.
Separating the exponents of $(F_x)_Q^{\mathrm{w}}$ and $(F_x)_Q^+$, one deduces from (3.10) that:

$$\prod_{\xi \in \Xi_Q^{\mathrm{w}}} (\rho(a) - \xi_x(a))(F_x)_Q^{\mathrm{w}} = 0, a \in A_L, x \in X.$$

Similarly, if R is a σ-parabolic subgroup of L, one gets a relation like (3.10) for $((F_x)_Q^{\mathrm{w}})_R$. Altogether this shows that $(F_x)_Q^{\mathrm{w}}$ is a family of type I of tempered functions on $(L \cap H)\backslash L$. This proves (i).
From Lemma 3.6 (iii), $(DF_x)_Q = D(F_x)_Q$. By (i), $(F_x)_Q^{\mathrm{w}}$ is C^∞ in $x \in X$. As this is also true for $(F_x)_Q$ this implies that $(F_x)_Q^+ = (F_x)_Q - (F_x)_Q^{\mathrm{w}}$ is also C^∞ in $x \in X$. Hence

$$(DF_x)_Q = D(F_x)_Q^{\mathrm{w}} + D(F_x)_Q^+.$$

But the exponents of $D(F_x)_Q^+$ are (up to multiplicities) the exponents of $(F_x)_Q^+$ and similarly for $D(F_x)_Q^{\mathrm{w}}$ (cf. Lemma 3.6 (ii)). From the definition of the weak constant term one deduces (ii). $\square$

Proposition 3.8 *Let F be a family of type I, parametrized by X, of tempered functions on $H\backslash G$. Then, there exist $d \in \mathbb{N}$ and $C > 0$ such that :*

$$|F_x(Hg)| \leq C\Theta_G(Hg)N_d(Hg), g \in G, x \in X.$$

Proof :

By using the Cartan decomposition (cf. (2.25)) and a finite number of right translates of F, one sees, using (3.11), (2.23) and (2.24), that it is enough to prove for each element $P = MU$ of $\mathcal{P}$, and each family of type I, parametrized by X, of tempered functions on

$H\backslash G$, an inequality of this type for $a \in A_P^-$. Now, it follows from (3.7) and Definition 3.5 a) that there exists $\varepsilon > 0$ such that for all $Q \in \mathcal{P}(P)$ and for all $x \in X$:

$$F_{x|A_P^-(Q,\varepsilon)} = (\delta_Q)^{1/2}(F_x)_{Q|A_P^-(Q,\varepsilon)}. \tag{3.13}$$

By (3.1),(3.5), we have $A_P^- \subset \cup_{Q\in\mathcal{P}(P)}\Lambda_Q^-\omega''_{\varepsilon,Q}$. Using a finite number of right translates, (2.23) again and the estimate (2.27) of Θ_G, it is enough to prove that there exist $C > 0$ and $d \in \mathbb{N}$ such that:

$$|(F_x)_Q(\lambda)| \leq CN_d(H\lambda), \lambda \in \Lambda_Q'^-. \tag{3.14}$$

By assumption on the real part of $\xi_{i,x}$, the eigenvalues $\xi_{i,x}(\lambda), \lambda \in \Lambda_Q'^- \subset \Lambda_L'$ have a modulus less or equal to 1 which does not depend on $x \in X$. We will see that (3.14) follows from the following Lemma applied to the lattice Λ_L'.

Lemma 3.9 *Let Λ be a lattice with basis $\lambda_1, \dots, \lambda_q$. If $\lambda = i_1\lambda_1 + \dots + i_q\lambda_q$, we set $|\lambda| = |i_1| + \dots + |i_q|$. Denote by Λ^+ the set of λ such that the i_j are in $\mathbb{N}$. Let $\xi_{1,x}, \dots \xi_{n,x}, x \in X$ be a C^∞ family of characters of Λ such that:*

$$|\xi_{i,x}(\lambda_j)| \leq 1, x \in X.$$

Let $(f_x), x \in X$ be a C^∞ family of functions on Λ such that

$$(\rho(\lambda) - \xi_{1,x}(\lambda)) \dots (\rho(\lambda) - \xi_{n,x}(\lambda))f_x = 0, x \in X, \lambda \in \Lambda.$$

Then there exist $C > 0, d \in \mathbb{N}$ such that:

$$|f_x(\lambda)| \leq C(1 + |\lambda|)^d, x \in X, \lambda \in \Lambda.$$

Proof :
If $i = (i_1, \dots, i_q) \in \mathbb{Z}^q$ we define

$$\lambda^i = i_1\lambda_1 + \dots i_q\lambda_q.$$

Let $E_{n,q}$ be the space of maps from $\{0, \dots, n-1\}^q$ to $\mathbb{C}$. We fix a norm on this vector space. To $x \in X$, we associate the element g_x of $E_{n,q}$ defined by $g_x(i) = f_x(\lambda^i), i \in \{0, \dots, n-1\}^q$. Then (cf. [D1] before Lemma 14) there exists a representation ξ_x of Λ on $E_{n,q}$, depending only on the family characters $\xi_{1,x}, \dots \xi_{n,x}$ of Λ and which depends smoothly on $x \in X$, such that for $\lambda \in \Lambda$, the eigenvalues of $\xi_x(\lambda)$ are $\xi_{1,x}(\lambda), \dots \xi_{n,x}(\lambda)$ and

$$f_x(\lambda) = ((\xi_x(\lambda)g_x)(0, \dots, 0), \lambda \in \Lambda.$$

The eigenvalues of $\xi_x(\lambda_1), \dots, \xi_x(\lambda_l)$ are of modulus less than or equal to 1. Moreover, from the smoothness of ξ_x in $x \in X$, one sees that the norms of the endomorphisms $\xi_x(\lambda_i)$ are bounded by a constant independent from $x \in X$, as well as their inverse.
From [DOp] Lemma 8.1, one sees that, for some $d' \in \mathbb{N}$, the norm of $\xi_x(\lambda^i)$ is bounded by the product of a constant, independent of $x \in X$, with $(1 + |i_1|)^{d'} \dots (1 + |i_q|)^{d'}$ for $i \in \mathbb{Z}^q$. But the latter is bounded by $(1 + |i_1| + \dots + |i_q|)^d$, with $d = d'q \in \mathbb{N}$. $\square$

End of the proof of the Proposition. From (2.26), we have:

$$N(Ha) \asymp (1 + \|H_M(a)\|), a \in A_M. \tag{3.15}$$

From the equivalence of norms for finite dimensional vector spaces, one sees that:

$$1 + |i_1| + \cdots + |i_l| \asymp N(H\lambda^i), i \in \mathbb{Z}^l, \tag{3.16}$$

where l is the rank of Λ'_L. Then the Lemma 3.9 implies easily (3.14). This finishes the proof of the Proposition. □

We have the following Proposition.

Proposition 3.10 *Let $f \in \mathcal{A}(H\backslash G)$. The following conditions are equivalent:*
(i) The function f is an element of $\mathcal{A}_{temp}(H\backslash G)$.
(ii) There exist $C > 0$ and $d \in \mathbb{N}$ such that:

$$|f(x)| \leq C\Theta_G(x)N_d(x), x \in H\backslash G.$$

Proof :
(i) implies (ii) follows from the Proposition applied to X reduced to one point.
One sees that (ii) implies (i) is the analogue of (i) implies (ii) in [W], Proposition III.2.2. Let us give a detailed proof.
Let f as in (ii). Let P be a σ-parabolic subgroup of G. Let us denote by V_f the linear span of the set $\{\rho(g)f | g \in G\}$.
From (2.24) and (2.23), one sees that, for any element f' of V_f, there exist $C' > 0$ and $d' \in \mathbb{N}$ such that:

$$|f'(x)| \leq C'\Theta_G(x)N_{d'}(x), x \in H\backslash G. \tag{3.17}$$

Let $E_f := \{(f'_P)_{|A_M} | f' \in V_f\}$. It is A_M-invariant and each element of E_f is A_M-finite as $V_f \subset \mathcal{A}(H\backslash G)$. One sees easily that the set $\mathrm{Exp}_P(f)$ is exactly the set of characters of A_M which appear as a subrepresentation of E_f. Let $\chi \in \mathrm{Exp}_P(f)$ and let $f' \in V_f$ such that $(f'_P)_{|A_M}$ transforms under A_M by χ. From (3.7), one sees that there exists $\varepsilon > 0$ such that:

$$f'(a) = \delta_P^{1/2}(a)f'_P(a), a \in A_M^-(P < \varepsilon)$$

From this, (3.17) and(2.27), one deduces that there exists $d'' \in \mathbb{N}$, $C'' > 0$ such that:

$$|\chi(a)| \leq C'' N_{d''}(a), a \in A_M^-(P < \varepsilon).$$

Let

$$\mathfrak{a}_M^-(P < \varepsilon) = \{X \in \mathfrak{a}_M | \, |\alpha(X)| < \varepsilon, \alpha \in \Delta(P)\}$$

Writing $|\chi| = \chi_\nu$ for $\nu \in \mathfrak{a}'_M$, one deduces from this and (2.26) (ii), the existence of $C''' > 0$ such that:

$$e^{\nu(X)} \leq C'''(1 + \|X\|)^{d''}, X \in \mathfrak{a}_M^-(P < \varepsilon).$$

This implies that $\nu(X) \leq 0$ for $X \in \mathfrak{a}_M^-(P < \varepsilon)$. Hence by applying homotheties, one sees that ν is an element of ${}^+\overline{\mathfrak{a}}_P{}'$. This proves that (ii) implies (i). □

We want to define some kind of seminorms on the space of families of type I with good properties of comparison when looking to Levi subgroups. For this, we introduce suitable sets of σ-parabolic subgroups.

Let G_1 be the group of $\mathbf{F}$-points of an algebraic group defined over $\mathbf{F}$. Let σ_1 be a rational involution of this group defined over $\mathbf{F}$ and let H_1 be the $\mathbf{F}$-points of the neutral component of the group of fixed points of σ_1. If $G_1 = (G_1)_{der}$, we choose a set $\mathcal{P}_{min}(G_1, \sigma_1)$ of minimal σ_1-parabolic subgroups of G_1 which gives a Cartan decomposition for $H_1 \backslash G_1$ (cf. 2.25). In general, using (6.12), let $\mathcal{P}_{min}(G_1, \sigma_1)$ be the set of minimal σ_1-parabolic subgroups of G_1 whose intersection with $(G_1)_{der}$ is an element of $\mathcal{P}_{min}((G_1)_{der}, \sigma_1)$. Then from (6.13), this set of minimal σ_1-parabolic subgroups of G_1 leads to a Cartan decomposition for $H_1 \backslash G_1$. Let $\mathcal{P}(G_1, \sigma_1)$ be the set of σ_1-parabolic subgroups of G_1 containing an element of $\mathcal{P}_{min}(G_1, \sigma_1)$ and $\mathcal{L}(G_1, \sigma_1)$ be the set of the σ_1-stable Levi subgroups of elements of $\mathcal{P}(G_1, \sigma_1)$. If there is no ambiguity on the involution σ_1, we drop it from the notation.

We return to G and σ, which induces an involution on each σ-stable subgroups of G. If L is the σ-stable Levi subgroup of a σ-parabolic subgroup of G, we set:

$$\mathcal{L}_1(L) = \mathcal{L}(L), \mathcal{L}_{i+1}(L) = \cup_{L_1 \in \mathcal{L}_i(L)} \mathcal{L}(L_1).$$

If $L_1 \in \mathcal{L}(L)$ is different from L then $dim(A_{L_1}) > dim A_L$. Hence, there exists i_0 such that $\mathcal{L}_{i_0}(L) = \mathcal{L}_j(L)$ for $j \geq i_0$. We set $\mathcal{L}_\infty(L) = \mathcal{L}_{i_0}(L)$. If $L \in \mathcal{L}_\infty(G) = \mathcal{L}_{p_0}(G)$ for some p_0, then $\mathcal{L}(L) \subset \mathcal{L}_{p_0+1}(G) = \mathcal{L}_\infty(G)$. Let us assume that $\mathcal{L}_i(L) \subset \mathcal{L}_{p_0}(G)$ for some $i \geq 1$. Then $\mathcal{L}_{i+1}(L) = \cup_{L_1 \in \mathcal{L}_i(L)} \mathcal{L}(L_1) \subset \cup_{L_1 \in \mathcal{L}_{p_0}(G)} \mathcal{L}(L_1) = \mathcal{L}_{p_0+1}(G) = \mathcal{L}_\infty(G)$. This implies:

For $L \in \mathcal{L}_\infty(G)$, one has $\mathcal{L}_\infty(L) \subset \mathcal{L}_\infty(G)$. (3.18)

For $L \in \mathcal{L}_\infty(G)$, we denote by $\mathcal{P}_\infty(L)$ the set of σ-parabolic subgroups of L whose σ-stable Levi component belongs to $\mathcal{L}_\infty(L)$. Similarly, we define $\mathcal{L}_\infty(L_{der})$. Then

The map $M \mapsto M \cap L_{der}$ is a bijection between $\mathcal{L}_\infty(L)$ and $\mathcal{L}_\infty(L_{der})$.
We say that $\mathcal{L}_\infty(L)$ is adapted to L_{der}. (3.19)

We introduce the following "seminorms" on the space of families of type I. Notice that these seminorms might be infinite. Let D be a finite set of differential operator on X with constant coefficients and $n \in \mathbb{N}$. If F is a family of type I parametrized by X, we set

$$\nu^X(G, D, n, F) = \sup_{x \in X} \sup_{d \in D} \sup_{g \in G} N_n(Hg)^{-1} \Theta_G(Hg)^{-1} |(d \cdot F_x)(g)|, \tag{3.20}$$

and

$$\mu^X(G, D, n, F) = \sup_{Q = LV \in \mathcal{P}_\infty(G)} \nu^X(L, D, n, (F)_Q^{\mathrm{w}}). \tag{3.21}$$

Remark 3.11 *Notice that in considering the right hand side of (3.21), we have chosen the function N on L defined by:*

$$N((H \cap L)l) := N(Hl), l \in L.$$

Another choice simply produces functions equivalent to this one, from (2.26).

The following Proposition is the analogue of [W] Lemmas VI.2.1, VI.2.3. The proof is essentially similar but takes into account the dependence on the family F.

Proposition 3.12 *We fix a set of exponents $\mathcal{E}$ and a compact open subgroup J of G. Let $Q = LV \in \mathcal{P}(G)$ and $P = MU \in \mathcal{P}_{min}(G)$ such that $P \subset Q$. Let $\Delta = \Delta(P)$, $\Delta^L = \Delta(P \cap L) \subset \Delta$ and for $\delta > 0$, let*

$$D^L(\delta) = \{a \in A_P^- | \langle \alpha, H_M(a)\rangle \leq -\delta \|H_M(a)\|, \alpha \in \Delta \setminus \Delta^L\}.$$

There exists a compact subset $C^L(\delta)$ of A_P^- and for all $n \in \mathbb{N}$, there exist $\varepsilon > 0, C_n > 0$ such that, for all families F of type I, parametrized by X, of tempered functions on $H\backslash G$, with the given set of exponents, and right invariant by J, one has

$$|(F_x)_Q^+((H \cap L)a)| \leq C_n \mu^X(G, 1, n, F)\Theta_L((H \cap L)a)e^{-\varepsilon\|H_M(a)\|}$$

for $a \in D^L(\delta) \setminus C^L(\delta)$ and $x \in X$.

Proof :
One can assume that Q is proper otherwise $(F_x)_Q^+ = 0$. We fix $n \in \mathbb{N}$.
Let us prove that there exist $t > 0, C_1 > 0$ and $d \in \mathbb{N}$ such that, if $a \in A_P^-$ satisfies $\langle \alpha, H_M(a)\rangle \leq -t$ for $\alpha \in \Delta \setminus \Delta^L$, one has for all families of the Proposition:

$$|(F_x)_Q^+((H \cap L)a)| \leq C_1 \mu^X(G, 1, n, F)\Theta_L((L \cap H)a)N_{n+d}(Ha), x \in X. \tag{3.22}$$

By (3.7), there exists $t > 0$ (independent of F) such that for a satisfying the above hypothesis, one has the equality:

$$(F_x)_Q((H \cap L)a) = \delta_Q(a)^{-1/2}F_x(a).$$

By definition of the seminorms, one has

$$|(F_x)_Q((H \cap L)a)| \leq \mu^X(G, 1, n, F)\delta_Q(a)^{-1/2}\Theta_G(Ha)N_n(Ha).$$

Applying the right inequality of (2.27) to G and the left inequality to L, and the equality $\delta_Q(a)^{-1}\delta_P(a) = \delta_{P\cap L}(a)$, one gets that there exist $C_2 > 0$ and $d \in \mathbb{N}$ such that:

$$\delta_Q(a)^{-1/2}\Theta_G(Ha)N_n(Ha) \leq C_2\Theta_L((L \cap H)a)N_{n+d}(Ha).$$

One deduces from this an inequality like (3.22) for $(F_x)_Q$.
A similar inequality for $(F_x)_Q^{\mathrm{w}}$ follows from the definition of the seminorms. Hence (3.22) follows by difference.

With the notations of the proof of lemma 3.7, let us define:

$$r_x(T) := \prod_{\xi \in \Xi_Q^+} (T - \xi_x(a')), x \in X.$$

By expanding these polynomials, one gets:

$$r_x(T) = \sum_{i=0,\ldots,N} r_{i,x}T^{N-i}.$$

For all $\xi \in \Xi_Q^+$, $|\xi_x(a')|$ is independent of $x \in X$ and belongs to the interval $]0,1[$. Changing a' to a suitable power, one can assume that:

$$|r_{i,x}| \leq 2^{-i}N^{-1}, i = 1, \dots, N-1. \tag{3.23}$$

Let us show the following property.

There exists $C_3 > 0$ such that, for all $k \in \mathbb{N}$ and all $a \in A_P^-$ satisfying $\langle \alpha, H_M(a) \rangle \leq -t$ for $\alpha \in \Delta \setminus \Delta^L$, one has:

$$|(F_x)_Q^+(a(a')^k)| \leq C_3 \mu^X(G, 1, n, F) 2^{-k} \Theta_L((L \cap H)a) N_{n+d}(Ha). \tag{3.24}$$

If $N = 0$, this implies that Ξ_Q^+ is empty, hence $(F_x)_Q^+ = 0$. So one can assume that $N \in \mathbb{N}^*$. Let

$$C_3 = C_1 Sup\{2^k(N_{n+d}(H(a')^k)|k = 0, \dots, N-1\}.$$

If $k < N$, (3.24) follows from (3.22) applied to $a(a')^k$, from the definition of C_3, from the equality

$$\Theta_L((L \cap H)la') = \Theta_L((L \cap H)l), l \in L,$$

as $a' \in A_L$, and from the inequality

$$N(Haa') \leq N(Ha)N(Ha'),$$

which follows easily from the definitions (2.19) and (2.22).
Let $k \geq N$ and let us assume that the inequality (3.24) is true for $k' < k$. It follows from the definitions that $r_x(\rho(a'))(F_x)_Q^+ = 0$ for all $x \in X$, hence one gets:

$$(F_x)_Q^+(a(a')^k) = -\sum_{i=1,\dots,N} r_{i,x}(F_x)_Q^+(a(a')^{k-i}).$$

The inequality (3.24) for the left side of this equality follows from the induction hypothesis and (3.23).
Let $C^L(\delta) := \{a \in D^L(\delta) | \, \|H_M(a)\| \leq t\delta^{-1}\}$. It is compact. Let $D = D^L(\delta) \setminus C^L(\delta)$. Hence one has:

$$D = \{a \in D^L(\delta) | \, \|H_M(a)\| > t\delta^{-1}\}.$$

For $a \in D$, let k be the largest integer which is less or equal to

$$(\delta \|H_M(a)\| - t)(-\langle \alpha, H_M(a') \rangle)^{-1},$$

when α varies in $\Delta \setminus \Delta^L$. From the definition of D and the choice of a', k is an element of $\mathbb{N}$. From the definition of $D^L(\delta)$, $a(a')^{-k}$ is in A_P^- and satisfies:

$$\langle \alpha, H_M(a(a')^{-k}) \rangle \leq -t, \alpha \in \Delta \setminus \Delta^L. \tag{3.25}$$

By applying (3.24) to $a(a')^{-k}$ instead of a and to the integer k, one gets :

$$(F_x)_Q^+((L \cap H)a) \leq C_3 \mu^X(G, 1, n, F) 2^{-k} \Theta_L((L \cap H)a(a')^{-k}) N_{d+n}(Ha(a')^{-k}).$$

As it was already observed $\Theta_L((L \cap H)a(a')^{-k}) = \Theta_L((L \cap H)a)$. From (3.25), one deduces:

$$|\langle \alpha, H_M(a'^k)\rangle| \leq t + |\langle, \alpha, H_M(a)\rangle|.$$

From this and (2.23), one sees that there exists $C_4 > 0$

$$N_{d+n}(Ha(a')^{-k}) \leq C_4(1 + \|H_M(a)\|)^{d+n}.$$

Writing that

$$(\delta\|H_M(a)\| - t)(-\langle \alpha, H_M(a')\rangle)^{-1} \leq k + 1,$$

for some $\alpha \in \Delta \setminus \Delta^L$, one sees that there exist $r > 0$ and $l \in \mathbb{N}$, independent of $a \in D$, such that:

$$r\|H_M(a)\| \leq l + k.$$

From this it follows that for $a \in D$:

$$(F_x)^+_Q(Ha) \leq C_3C_4\mu^X(G, 1, n, F)\Theta_L((L \cap H)a)2^{-r\|H_M(a)\|+l}(1 + \|H_M(a)\|)^{d+n}.$$

In order to finish the proof of (ii), it is enough to remark that there exist $C_5 > 0$ and $\varepsilon > 0$ such that for all $x > 0$,

$$2^{-rx+l}(1 + x)^{d+n} \leq C_5e^{-\varepsilon x}.$$

□

4 Wave packets in the Schwartz space

Definition 4.1 *The Schwartz space $\mathcal{C}(H\backslash G)$ is the space of functions f on $H\backslash G$, which are right invariant by a compact open subgroup of G and such that for any $d \in \mathbb{N}$, there exists a constant $C_d > 0$ such that:*

$$|f(x)| \leq C_d\Theta_G(x)(N_d(x))^{-1}, x \in H\backslash G.$$

The smallest constant C_d is denoted by $p_d(f)$. It defines a seminorm on $\mathcal{C}(H\backslash G)$.

Lemma 4.2 *One has*

$$\mathcal{A}_2(H\backslash G) \subset \mathcal{C}(H\backslash G).$$

Proof :

One proceeds as in the proof of Proposition 3.8 with X reduces to a single point. One has to replace Lemma 3.9 by the following property, which follows from [DOp], Corollary 8.2 (ii).

> Let A be an endomorpism of a finite dimensional normed vector space whose eigenvalues are of modulus strictly less than 1. Then for any d in $\mathbb{N}$, there exists a constant $C > 0$ such that: (4.26)

$$\|A^n\| \leq C(1 + n)^{-d}, n \in \mathbb{N}.$$

This achieves to prove the Lemma. □

Lemma 4.3 *If $f \in \mathcal{A}_{temp}(H\backslash G)$ and $f' \in \mathcal{C}(H\backslash G)$ the integral*

$$\int_{H\backslash G} f(x)f'(x)dx$$

converges absolutely.

Proof :
The lemma follows from Proposition 3.10 and Lemma 2.1. □

Let M (resp., L) be the σ-stable Levi subgroup of a σ-parabolic subgroup P (resp., Q) of G. Let $\tilde{A}$ (resp., $\tilde{A}'$) be a maximal split torus of M (resp., L) such that the set A (resp., A') of its antiinvariant elements is a maximal σ-split torus of M (resp., L) . By ([CD] (4.9)), we can choose a set of representatives $W(Q\backslash G/P)$ of $Q\backslash G/P$ such that its elements satisfy $w.\tilde{A} = \tilde{A}'$.

Let $(Q\backslash G/P)_\sigma$ be the set of (Q, P)-double cosets in G having a representative w such that $w.A = A'$ and $w.\tilde{A} = \tilde{A}'$. We denote by $W(L\backslash G/M)_\sigma$ a set of representatives of $(Q\backslash G/P)_\sigma$ with this property and we assume that $W(L\backslash G/M)_\sigma \subset W(Q\backslash G/P)$.

Let $(Q|G|P)_\sigma$ be the set of elements of $(Q\backslash G/P)_\sigma$ having a representative w such that $w.\tilde{A} = \tilde{A}', w.A = A'$ and $A_L \subset w.A_M$. Let $W(L|G|M)_\sigma$ be a set of representatives of $(Q|G|P)_\sigma$ with these properties and we assume $W(L|G|M)_\sigma \subset W(L\backslash G/M)_\sigma$. We want to identify $W(L|G|M)_\sigma$ with a set independent of choices.

First we prove the following property.

Let $s, s' \in W(L|G|M)_\sigma$ such that

$$(s.\chi)_{|A_L} = (s'.\chi)_{|A_L}, \chi \in X(M)_{\sigma,u}. \tag{4.1}$$

Then $s = s'$.

As conjugacy by s defines an isomorphism from A to A', it determines a linear isomorphism $s : \mathfrak{a} \to \mathfrak{a}'$. One has a similar map for s'. As A_L is contained in $s.A_M$ and in $s'.A_M$, one has $s^{-1}\mathfrak{a}_L \subset \mathfrak{a}_M$ and $s'^{-1}\mathfrak{a}_L \subset \mathfrak{a}_M$. The condition (4.1) implies that for all λ in $\mathfrak{a}'_M$, $(s\lambda)_{|\mathfrak{a}_L} = (s'\lambda)_{|\mathfrak{a}_L}$. Evaluating in $X \in \mathfrak{a}_L$, one deduces

$$\langle \lambda, s^{-1}X \rangle = \langle \lambda, s'^{-1}X \rangle, X \in \mathfrak{a}_L.$$

This implies, by varying λ in $\mathfrak{a}'_M$, the equality

$$s^{-1}X = s'^{-1}X, X \in \mathfrak{a}_L.$$

In other words, one has:

$$s's^{-1}X = X, X \in \mathfrak{a}_L.$$

This gives $s's^{-1} \in L$ and s, s' are representatives of the same (Q, P)-double coset. Hence one has $s = s'$. This achieves to prove (4.1).

Let us remark that one has the following immediate corollary of the proof of (4.1).

Let s, s' be distinct elements of $W(L|G|M)_\sigma$ and μ, μ' two characters of A_L. For χ in an open subset of $X(M)_{\sigma,u}$, one has : (4.2)

$$\mu(s.\chi)_{|A_L} \neq \mu'(s'.\chi)_{|A_L}.$$

Now, we will identify $W(L|G|M)_\sigma$ with a set which does not depends on choices. Let $N(A,A')_\sigma$ be the set of $g \in G$ such that $g.A = A'$, $g.\tilde{A} = \tilde{A}'$. Let M_0 (resp., L_0) be the centralizer of $\tilde{A}$ (resp., $\tilde{A}'$). Let $W(A,A')_\sigma$ be the quotient $N(A,A')_\sigma/M_0$ which is identified with $L_0\backslash N(A,A')_\sigma/M_0$. It appears as a set of isomorphisms between $\tilde{A}$ and $\tilde{A}'$.
Let us prove the following result:

Let $(W^L)_\sigma$ be the subgroup of the Weyl group W^L of L for $\tilde{A}'$ whose elements preserve A'. Let

$$\overline{W}(L|G|M)_\sigma := (W^L)_\sigma \backslash \{s \in W(A,A')_\sigma | A_L \subset s.A_M\}. \tag{4.3}$$

The natural map from $W(L|G|M)_\sigma$ to $\overline{W}(L|G|M)_\sigma$ is bijective.

Let us prove that this map is surjective. Let $s \in \overline{W}(L|G|M)_\sigma$ and let s_1 be a representative of s in $N(A,A')_\sigma$. In the (Q,P)-double coset Qs_1P, there is an element s' in $W(L|G|M)_\sigma$ by definition of the latter set. Then one has two elements $s', s_1 \in N(A,A')_\sigma$ such that $QsP = Qs_1P$. We want to show that $s' = ls_1m$ for some $l \in L$ and $m \in M$. Using conjugacy by an element of $N(A,A')_\sigma$, one can reduce to the case $\tilde{A} = \tilde{A}'$. But, by the Bruhat decomposition, s' and s_1 represent elements of the Weyl group which have the same (W^L, W^M)-double coset. This proves the existence of l and m. As $s_1.M \subset L$ one can omit the m and write $s' = ls_1$ for some $l \in L$. From the properties of s', s_1, one deduces that l normalize $\tilde{A}'$ and A'. Hence the image of s' by our map is s. Hence this map is surjective.
Let us prove the injectivity. If $s, s' \in W(L|G|M)_\sigma$ have the same image by our map, they satisfy the condition (4.1). Hence they are equal. This achieves to prove (4.3).

We recall that $X(G)_{\sigma,u}$ has been identified with a subgroup of $X(M)_{\sigma,u}$. Let $X(M)^G_\sigma$ (resp., $X(M)^G_{\sigma,u}$) be the neutral connected component of the group of elements χ of $X(M)_\sigma$ (resp., $X(M)_{\sigma,u}$) whose restriction to A_G is trivial. The group $X(M)^G_{\sigma,u}$ is the maximal compact subgroup of the algebraic complex torus $X(M)^G_\sigma$ and its Lie algebra is equal to $(i\mathfrak{a}^G_M)'$. Hence one has $X(M)_{\sigma,u} = X(G)_{\sigma,u}X(M)^G_{\sigma,u}$ and $X(M)^G_{\sigma,u} \cap X(G)_{\sigma,u}$ is finite.

Let X be the maximal compact subgroup of a complex algebraic torus $X(\mathbb{C})$. We assume that $X(\mathbb{C})$ is a finite covering of $X(M)_\sigma$ i.e. there exists a surjective morphism of algebraic groups $p : X(\mathbb{C}) \to X(M)_\sigma$ whose kernel is finite. Let X_G be the neutral connected component of $p^{-1}(X(G)_{\sigma,u})$ and X' be the neutral connected component of $p^{-1}(X(M)^G_{\sigma,u})$. Then X_G (resp., X') is the maximal compact subgroup of the complex algebraic torus equal to the connected component of $p^{-1}(X(G)_\sigma)$ (resp., $p^{-1}(X(M)^G_\sigma)$. One has $X = X_GX'$ and $X_G \cap X'$ is finite. For $x \in X$, we set $\chi_x = p(x) \in X(M)_{\sigma,u}$.

Definition 4.4 *Let F be a family of type I of tempered functions on $H\backslash G$ parametrized by X.*
(1) The family F is called an M-family of type I' if
(i) there exists a unitary character μ_G of A_G such that

$$F_x(Hga) = \mu_G(a)\chi_x(a)F_x(Hg), a \in A_G, g \in G, x \in X, \tag{4.4}$$

(ii) Let $Q = LV$ be a σ-parabolic subgroup. There exists a finite set Ξ^Q, independent of $x \in X$, of characters of A_L with $Re(\xi) \in^+ \bar{\mathfrak{a}}_L{}'$ such that all exponents of F_x along Q are of the form

$$\mu(w.\chi_x)_{|A_L}, \mu \in \Xi^Q, w \in W(Q\backslash G/P).$$

(2) An M-family of tempered functions on $H\backslash G$ of type I' is said to be of type II' if for any Q as above

$$(F_x)_Q^{\mathrm{w},ind}(g) = \sum_{s \in \overline{W}(L|G|M)_\sigma} (F_{Q,s}(g))_{s.x}, x \in X, g \in G, \tag{4.5}$$

where for all $s \in \overline{W}(L|G|M)_\sigma$, $F_{Q,s}(g)$ is a $s.M$-family of type I' of tempered functions on $(L \cap H\backslash L)$ parametrized by $s.X := \{(s,x)|x \in X\}$ with the multiplication induced by the multiplication on X and with a canonical projection on $X(s.M)_\sigma$ given by $s.x := (s,x) \to s.\chi_x$.
From the definition it follows that if F is of type II' and $g \in G$, $\rho(g)F$ is also of type II'.

We will give examples of such families, derived from Eisenstein integrals (cf. Theorem 5.1). Condition (4.5) is motivated by this example.
Let us remark that (4.2) implies the unicity of $(F_{Q,s})_{s.x}$ for x in an open dense subset of X and then everywhere by continuity.
Let us prove the following assertion in which one sets $F_{Q,s} := F_{Q,s}(e)$:

> Let $Q = LV$ be a σ-parabolic subgroup of G and $s \in \overline{W}(L|G|M)_\sigma$. Let $R = SN$ be a σ-parabolic subgroup of L and $s' \in \overline{W}(S|L|s.M)_\sigma$. Let $Q_R = RV$. Then $s's \in \overline{W}(S|G|M)_\sigma$ and
>
> $$\big((F_{Q,s})_{s.x}\big)_R^{\mathrm{w}} = \sum_{s' \in \overline{W}(S|L|s.M)_\sigma} [(F_{Q,s})_{R,s'}]_{s's.x} \tag{4.6}$$
>
> with
>
> $$[(F_{Q,s})_{R,s'}]_{s'sx} = (F_{Q_R,s's})_{s's.x}.$$

To prove this one uses (4.5) for Q_R to express directly $F_{Q_R}^{\mathrm{w}}$, involving the second member of the equality to prove. Then, one uses (4.5) for Q and R and the transitivity of the weak constant term (Lemma 3.4 (ii)) to compute in another way $F_{Q_R}^{\mathrm{w}}$. Then (4.6) follows from the identification of the terms with the same action of A_S using (4.2).
This implies easily that:

> If F is a family, parametrized by X, of type II' on $H\backslash G$, then $F_{Q,s}$ is a family, parametrized by $s.X$, of type II' on $H \cap L\backslash L$. (4.7)

Let $\mathcal{D}^X$ be the set of finite families of invariant differential operators with constant coefficients on X. If $D \in \mathcal{D}^X$ and $\psi \in C^\infty(X)$, we define:

$$q(D,\psi) = Sup\{|d\psi(\chi)||d \in D, \chi \in X\}.$$

For $D \in \mathcal{D}^X$ and $n \in \mathbb{N}$, we introduce the following "seminorms" on families of type II'.

$$p^X(G, D, n, F) = \sup_{Q=LV \in \mathcal{P}_\infty(G)} \sup_{s \in \overline{W}(L|G|M)_\sigma} \nu^{s \cdot X}(L, D^s, n, F_{Q,s}), \tag{4.8}$$

where D^s is the family of differential operators on $s \cdot X$ deduced from D by the action of s and ν is defined in (3.20).
If $Q = LV \in \mathcal{P}_\infty(G)$ then

$$\nu^X(L, D, n, (F)^{\mathrm{w}}_Q) \leq |\overline{W}(L|G|M)_\sigma| \sup_{s \in \overline{W}(L|G|M)_\sigma} \nu^{s.X}(L, D^s, n, F_{Q,s}).$$

As $\mathcal{P}_\infty(G)$ is finite, there exists a constant $C_0 > 0$ such that

$$\mu^X(G, D, n, F) \leq C_0 p^X(G, D, n, F). \tag{4.9}$$

We keep the previous notation. Let dx be the Haar measure of X of volume 1. For an M-family F, parametrized by X, of tempered functions on $H \backslash G$ of type II' and a C^∞ function ψ on X, we define

$$\mathcal{W}_{\psi,F}(Hg) = \int_X \psi(x) F_x(Hg) dx, g \in G. \tag{4.10}$$

Theorem 4.5 *We fix a set $\mathcal{E}$ of exponents and a compact open subgroup J of G and let $k \in \mathbb{N}$. There exist $D, D_0 \in \mathcal{D}^X$ and, for all $n \in N$, there exists $C > 0$ such that, for all M-family F of type II' with the given set of exponents, and right invariant by J, one has*

$$\sup_{g \in G} |N_k(Hg) \Theta_G(Hg)^{-1} \mathcal{W}_{\psi,F}(Hg)| \leq C p^X(G, D, n, F) q(D_0, \psi), \psi \in C^\infty(X). \tag{4.11}$$

Proof :

Proceeding as in the proof of Proposition 3.8, using a finite number of right translations, one is reduced to prove a similar statement for $g \in A_1^-$, where $P_1 = M_1 U_1 \in \mathcal{P}_{min}(G)$ and A_1 is the maximal σ-split torus of the center of M_1. By an argument similar to (2.11), there exists a split torus A'_1 of G_{der}, the derived group of G, and a finite set F_1 such that $A_1 = A'_1 A_G F_1$. Using a finite number of translations again, one is reduced to prove the following assertion:

> Let $k \in \mathbb{N}$ and $\mathcal{E}$, J as in the Theorem. Then, there exist $D, D_0 \in \mathcal{D}^X$, and for all $n \in \mathbb{N}$, there exists a constant $C > 0$, such that, for all M-family F of type II' with the given set of exponents, and right invariant by J, one has, for $g \in A_1'^{-} A_G$ and $\psi \in C^\infty(X)$, (4.12)
>
> $$|N_k(Hg) \Theta_G(Hg)^{-1} \mathcal{W}_{\psi,F}(Hg)| \leq C p^X(G, D, n, F) q(D_0, \psi).$$

We first reduce the proof of the Theorem to the case where G is semisimple and then we prove it by induction on the semisimple rank of $H \backslash G$.

Reduction to semi-simple case. Let us assume that the Theorem is true for the derived group G^{der} of G.
As $X = X_G X'$ and $X' \cap X_G$ is a finite abelian group, there exists $C_1 > 0$ such that

$$\mathcal{W}_{\psi,F}(Haa_1) = C_1 \int_{X_G} (\int_{X'} \psi(x_G x') F_{x_G x'}(Haa_1) dx') dx_G, a \in A_G, a_1 \in A_1'^{-}.$$

By (4.4), one has

$$\mathcal{W}_{\psi,F}(Haa_1) = C_1 \mu_G(a) \int_{X_G} x_G(a) (\int_{X'} \psi(x_G x') F_{x_G x'}(Ha_1) dx') dx_G,$$

as $\chi_{x'|A_G} = 1$.
By properties of the classical Fourier transform on X_G, for $k \in \mathbb{N}$, there exists $D_1 \in \mathcal{D}^{X_G}$ such that

$$(1 + \|H_G(a)\|)^k |\mathcal{W}_{\psi,F}(Haa_1)| \leq C_1 \sup_{d \in D_1} \sup_{x_G \in X_G} |\int_{X'} d\big(\psi(x_G x') F_{x_G x'}(Ha_1)\big) dx'|,$$

and by Leibnitz formula, there exist two families $d_1', \dots, d_t'$ and $d_1'', \dots, d_t''$ in $\mathcal{D}^{X_G}$ such that

$$(1 + \|H_G(a)\|)^k |\mathcal{W}_{\psi,F}(Haa_1)| \leq C_1 \sup_{i=1,\dots t} \sup_{x_G \in X_G} |\int_{X'} d_i' \cdot \psi(x_G x') d_i'' \cdot F_{x_G x'}(Ha_1)\big) dx'|. \tag{4.13}$$

We fix $i \in \{1, \dots, t\}$. For $x_G \in X_G$, we set

$$\psi'_{x_G}(x') = d_i' \cdot \psi(x_G x'); x' \in X'$$

and

$$(F'_{x_G})_{x'}(g) = d_i'' \cdot F_{x_G x'}(g), g \in G_{der}.$$

For any subgroup I of G, we set $I' = I \cap G_{der}$. We will use the notation G' instead of G_{der}. As $X(M)^G_\sigma$ is a finite covering of $X(M')^G_\sigma$, X' is the maximal compact subgroup of a finite covering of $X(M')^G_\sigma$. Let us prove the following assertion :

The families $(F'_{x_G})_{x' \in X'}$ are families of type II' on $H' \backslash G'$ with the same set of exponents $\mathcal{E}'$ independent of x_G. Moreover, they are right invariant by J'. (4.14)

Let $Q = LV$ be a σ-parabolic subgroup of G. It follows from Lemma 3.6 and (6.15) that $(F'_{x_G})_{x'}$ is of type I and its exponents along Q' are the restrictions to $A_L \cap G'$ of the exponents of $F_{x_G x'}$ along Q (with different multiplicities). By definition of type I', these exponents are of the form $(\mu w.(\chi_{x_G} \chi_{x'}))_{|A_L \cap G'} = (\mu w.\chi_{x'})_{|A_L \cap G'}$. One deduces that

$(F'_{x_G})_{x' \in X'}$ is of type I' on $H' \backslash G'$ and has a set of exponents along Q' independent of x_G. (4.15)

By (6.14) and Lemma 3.7 (ii), one has

$$((F'_{x_G})_{x'})^{\mathrm{w}}_{Q'} = \big[d_i'' \cdot (F_{x_G x'})^{\mathrm{w}}_Q\big]_{|L \cap G'} = \sum_{s \in \overline{W}(L|G|M)_\sigma} \big[d_i''.(F_{Q,s})_{s.x_G\, s.x'}\big]_{|L \cap G'}.$$

As $(F_{Q,s})_{s.x_G\, s.x'}$ is of type I', the same argument as before proves that $\left[d''_i.(F_{Q,s})_{s.x_G\, s.x'}\right]_{|L\cap G'}$ is of type I'. So we have proved (4.14).

We can apply our assumption. In the definition of the seminorms p^X for G', we choose the function N on G' defined by $N(H'g) = N(Hg)$ for $g \in G'$ (another choice produces an equivalent function).

$$\begin{aligned}&\text{Let } k_1 \in \mathbb{N}. \text{ There exists } D_i, D'_i \in \mathcal{D}^{X'}, \text{ and for all } n_1 \in \mathbb{N}, \text{ there exists a}\\ &\text{constant } C_2 > 0 \text{ such that, for all } x_G \in X_G \text{ and for all } a_1 \in A_1^{'-}, \text{ one has}\\ &|N_{k_1}(Ha_1)\Theta_{G'}(H'a_1)^{-1}\mathcal{W}_{\psi'_{x_G},F'_{x_G}}(a_1)| \leq C_2 p^{X'}(G', D_i, n_1, F'_{x_G})q^{X'}(D'_i, \psi'_{x_G}).\end{aligned} \tag{4.16}$$

Let $D = \cup_{i=1,\dots,t}\{dd''_i; d \in D_i\}$ and $D_0 = \cup_{i=1,\dots,t}\{dd'_i; d \in D'_i\}$. One has

$$\sup_{i=1,\dots t}\sup_{x_G\in X_G} p^{X'}(G', D_i, n_1, F'_{x_G})q^{X'}(D'_i, \psi'_{x_G}) \leq p^X(G', D, n_1, F_{|G'})q^X(D_0, \psi).$$

By (4.13) and (4.16), one deduces for all $a \in A_G$ and $a_1 \in A_1^{'-}$

$$\begin{aligned}&(1 + \|H_G(a)\|)^k N_{k_1}(Ha_1)\Theta_{G'}(H'a_1)^{-1}|\mathcal{W}_{\psi,F}(Haa_1)|\\ &\qquad \leq C_1C_2p^X(G', D, n_1, F_{|G'})q^X(D_0, \psi).\end{aligned} \tag{4.17}$$

By (6.16), there exist $C_L, C'_L > 0$ and r_L, s_L in $\mathbb{N}$ such that, for $l \in (H \cap L')\backslash L'$,

$$\begin{aligned}C_L^{-1}N_{-r_L}((H \cap L)l)\Theta_L((H \cap L)l) &\leq \Theta_{L'}((H \cap L')l)\\ &\leq C'_L N_{s_L}((H \cap L)l)\Theta_L((H \cap L)l).\end{aligned} \tag{4.18}$$

Let $r_0 = \sup_{L\in\mathcal{L}_\infty(G)} r_L$ and $C_0 = \sup_{L\in\mathcal{L}_\infty(G)} C_L$. Taking the inverse of the left inequality together with the fact that $\mathcal{L}_\infty(G)$ is adapted to G' (cf. (3.19)), we obtain

$$p^X(G', D, n_1, F_{|G'}) \leq C_0 p^X(G, D, n_1 - r_0, F).$$

Taking C'_G and s_G for $L = G$ in (4.18), we choose $k_1 = k + s_G$ and $n_1 = r_0 + n$ in (4.16). By (4.17), we obtain

$$\begin{aligned}&(1 + \|H_G(a)\|)^k N_k(Ha_1)\Theta_G(Ha_1)^{-1}|\mathcal{W}_{\psi,F}(Haa_1)|\\ &\qquad \leq C_1C_2C_0C'_G p^X(G, D, n, F)q^X(D_0, \psi).\end{aligned}$$

By (2.16) and (2.26), there exists $C_3 > 0$ such that

$$N_k(Haa_1) \leq C_3(1 + \|H_G(a)\|)^k N_k(Ha_1).$$

Recall that $\Theta_G(Haa_1) = \Theta_G(Ha_1)$ for $a \in A_G$. Then one deduces (4.12) from the previous inequalities.

Semisimple case. We prove the Theorem by induction on $dim A_1$. If $dim A_1 = 0$ then $H\backslash G$ is compact and the result is clear. Let us assume that $dim A_1 > 0$. Let

$\Delta_1 := \Delta(P_1)$. If $a \in A_1$, we define $s(a) = inf\{\langle \alpha, H_{M_1}(a)\rangle | \alpha \in \Delta_1\}$. For $\Phi \subset \Delta_1$, we define $A_1^-(\Phi)$ to be the set of all $a \in A_1^-$ such that $\Phi = \{\alpha \in \Delta_1 | \langle \alpha, H_{M_1}(a)\rangle = s(a)\}$. So, one has $A_1^- = \cup_{\Phi \subset \Delta_1} A_1^-(\Phi)$. Let $Q = LV \in \mathcal{P}(G)$ be such that $P_1 \subset Q$ and $\Phi = \Delta_1 - \Delta_1^L$. If $\Phi = \emptyset$ (which corresponds to $Q = G$) then $A_1^-(\Phi) = \emptyset$, from our hypothesis on the semisimple rank of $H\backslash G$. Hence we can assume $Q \neq G$.

As the set $\{a \in A_1^-(\Phi) | t \leq s(a) \leq 0\}$ is a compact subset of A_1^-, the inequality (4.12) on this set is clear. It is enough to prove the statement (4.12) on $A_1^-(\Phi, < t) := \{a \in A_1^-(\Phi) | s(a) < t\}$ for some $t < 0$. By (3.13), there exists $t < 0$, which depends only on J, such that, for all σ-parabolic subgroup containing P_1 and $a \in A_1^-$ with $s(a) < t$, one has

$$\begin{aligned} F_x(Ha) &= \delta_Q(a)^{1/2}(F_x)_Q((H\cap L)a) \\ &= \delta_Q(a)^{1/2}(F_x)_Q^{\mathrm{w}}((H\cap L)a) + \delta_Q(a)^{1/2}(F_x)_Q^{+}((H\cap L)a). \end{aligned} \tag{4.19}$$

We fix such $t < 0$.
By (4.7), one has $F_Q^{\mathrm{w}} = \sum_{s \in \overline{W}(L|G|M)_\sigma} F_{Q,s}$ where $F_{Q,s}$ is of type II' on $(L \cap H)\backslash L$.
By the induction hypothesis applied to $L^{der} \cap H\backslash L^{der}$ and the reduction to the semisimple case, the Theorem is true for $L \cap H\backslash L$. Let $k_1 \in \mathbb{N}$. For $s \in \overline{W}(L|G|M)_\sigma$, there exist $D_1, D'_0 \in \mathcal{D}^X$ and for $n_1 \in \mathbb{N}$, there exists $C'_0 > 0$ such that, for $a \in A_1^-$,

$$|N_{k_1}(Ha)\Theta_L((L\cap H)a)^{-1}|\mathcal{W}_{\psi, F_{Q,s}}(a)| \leq C'_0 p^{s.X}(L, D_1^s, n_1, F_{Q,s}) q(D'_0, \psi).$$

Recall that

$$p^{s.X}(L, D_1^s, n_1, F_{Q,s}) = \sup_{R=SN \in \mathcal{P}_\infty(L)} \sup_{s' \in \overline{W}(S|L|s.M)_\sigma} \nu^{s's.X}(S, D_1^{s's}, n_1, (F_{Q,s})_{R,s'}).$$

As $\mathcal{L}_\infty(L) \subset \mathcal{L}_\infty(G)$ (cf. (3.18)), by (4.6) and the finiteness of $\overline{W}(L|G|M)_\sigma$, there exist $D, D_0 \in \mathcal{D}^X$, and for $n_1 \in \mathbb{N}$ there exists $C_0 > 0$ such that

$$|N_{k_1}(Ha)\Theta_L((L\cap H)a)^{-1}|\mathcal{W}_{\psi, F_Q^{\mathrm{w}}}(a)| \leq C_0 p^X(G, D, n_1, F) q(D_0, \psi).$$

By (2.26) and (2.27), there exist $C_2 > 0$ and $r \in \mathbb{N}$ such that

$$\delta_Q^{1/2}(a)\Theta_L((L\cap H)a) \leq C_2 N_r(Ha)\Theta_G(Ha), a \in A_1^-. \tag{4.20}$$

Taking $k_1 = k + r$ and $n_1 = n$, this gives an upper bound like (4.12) for $|\delta_Q(a)^{1/2}\int_X \psi(x)(F_x)_Q^{\mathrm{w}}(a)dx|$ on A_1^-.
With the notation of Proposition 3.12, there exists $\delta > 0$ such that $A_1^-(\Phi, < t) \subset D^L(\delta) \subset A_1^-$. By Proposition 3.12 and (4.20), for $n \in \mathbb{N}$, there exist $\varepsilon > 0$ and $C_n > 0$ such that, for $a \in D^L(\delta) \setminus C^L(\delta)$, one has

$$\delta_Q^{1/2}(a)|\int_X \psi(x)(F_x)_Q^{+}(a)dx| \leq$$

$$C_n \sup_{x\in X} |\psi(x)|\mu^X(G,1,n,F)\Theta_G(Ha)e^{-\varepsilon\|H_{M_1}(a)\|}N_r(Ha),$$

for $a \in D^L(\delta) \setminus C^L(\delta)$.
As $\mu^X(G,1,n,F) \leq Cp^X(G,1,n,F)$ for some constant $C > 0$ and for $k \in \mathbb{N}$, there exists C_k such that $N_r(Ha)e^{-\varepsilon\|H_1(a)\|} \leq C_k N^{-k}(Ha)$, one deduces an upper bound like (4.12) for $|\delta_Q(a)^{1/2}\int_X \psi(x)(F_x)_Q^+(a)dx|$ for $a \in A_1^-(\Phi, < t) \setminus C^L(\delta)$. Together with the result above for F_Q^{w} and (4.19), one gets (4.12) for $a \in A_1^-(\Phi, < t) \setminus C^L(\delta)$.
As $C^L(\delta)$ is compact, one gets a similar inequality for $a \in C^L(\delta)$. This achieves the proof. □

Theorem 4.6 *Let F be an M-family, parametrized by X, of tempered functions on $H\backslash G$ of type II'. Let ψ be a C^∞ function on X.*
(i) $\mathcal{W}_{\psi,F}$ is an element of $\mathcal{C}(H\backslash G)$.
(ii) For each $k \in \mathbb{N}$, there exists a continuous semi norm q_k on $C^\infty(X)$ such that (with the notation of Definition 4.1):

$$p_k(\mathcal{W}_{\psi,F}) \leq q_k(\psi), \psi \in C^\infty(X).$$

Proof :
Let $k \in \mathbb{N}$. We fix D and D_0 in $\mathcal{D}^X$ as in the Theorem 4.5. Let $Q = LV \in \mathcal{P}_\infty(G)$, $s \in \overline{W}(L|G|M)_\sigma$ and $d \in D^s$. By assumption and Lemma 3.6 (iv), $d.F_{Q,s}$ is a $s.M$ family of type I on $L \cap H\backslash L$. By Proposition 3.8, there exist $n = n(Q,s,d) \in \mathbb{N}$ and $C_n > 0$ such that

$$\sup_{x\in X}\sup_{l\in L\cap H\backslash L} \Theta_L^{-1}(L\cap Hl)N_{-n}(H\cap Ll)|d.(F_{Q,s})_x(H\cap Ll)| \leq C_n.$$

As $\mathcal{P}_\infty(G)$ is finite as well as D and $\overline{W}(L|G|M)_\sigma$ for $L \in \mathcal{L}_\infty(G)$, there exist $n_1 \in \mathbb{N}$ and $C_{n_1} > 0$ such that, for $L \in \mathcal{L}_\infty(G)$ and $s \in \overline{W}(L|G|M)_\sigma$, one has

$$\nu^{s.X}(L, D^s, n_1, F_{Q,s}) \leq C_{n_1}.$$

One deduces

$$p^X(G, D, n_1, F) \leq C_{n_1}.$$

The Theorem follows from the Theorem 4.5 with $q_k = C_{n_1}q(D_0,.)$. □

5 Some properties of Eisenstein integrals

5.1. Eisenstein integrals.
Let us recall some results of [BD]. Let $P = MU$ be a σ-parabolic subgroup of G, (δ, E) be a smooth representation of finite length of M. Let I_δ be the space of the induced representation $i_{K_0\cap P}^{K_0}\delta_{|K_0\cap P}$. Let $i_P^G E_\chi$ or simply I_{δ_χ} be the space of the normalized induced representation $\pi_\chi := i_P^G(\delta_\chi)$, $\chi \in X(M)_\sigma$, where $\delta_\chi = \delta\otimes\chi$. The restriction of functions from G to K_0 determines an isomorphism of K_0-modules between I_{δ_χ} and

I_δ. One denotes by $\overline{\pi}_\chi$ the representation of G on I_δ deduced from π_χ by " transport de structure" by this isomorphism.

If $\varphi \in I_\delta$ and $\chi \in X(M)_\sigma$, one denotes by φ_χ the element of the space I_{δ_χ} corresponding to φ by this isomorphism.

Let B be the algebra of polynomial functions on $X(M)_\sigma$, which is generated by the functions $b_m, m \in M$ defined by $b_m(\chi) = \chi(m)$. One has:

$$\text{For all } \varphi \in I_\delta \text{ and } g \in G,\ \chi \mapsto \overline{\pi}_\chi(g)\varphi \text{ is an element of } I_\delta \otimes B. \tag{5.1}$$

Let $\mathcal{O}$ be the union of the open (P, H)-double cosets in G. There exists a set of representatives, $\overline{\mathcal{W}}^G_M$, of these open (P, H)-double cosets which depends only on M and not on P. Moreover for all $x \in \overline{\mathcal{W}}^G_M$, $x^{-1}.P$ is a σ-parabolic subgroup of G (cf [BD] Lemma 2.4). Let A be a maximal σ-split torus of M. We may (cf. [BD] , beginning of section 2.4 and Lemma 2.4) and we will assume that for all $x \in \overline{\mathcal{W}}^G_M$, $x^{-1}.A$ is a σ-split torus. One says that x is A-good. Then $x^{-1}.M$ is the σ-stable Levi subgroup of $x^{-1}.P$ (cf [CD], Lemma 2.2).

One sets $J_\chi = \{\varphi \in I_{\delta_\chi} | \mathrm{Supp}(\varphi) \subset \mathcal{O}\}$ and we define:

$$\mathcal{V}_\delta := \oplus_{x \in \overline{\mathcal{W}}^G_M} (E'_\delta)^{M \cap x.H}.$$

Let $\chi \in X(M)_\sigma$. To $\eta \in {E'_\delta}^{M\cap x.H}$, one associates $j(P, \delta_\chi, \eta) \in J'_\chi$ defined by:

$$j(P, \delta_\chi, \eta)(\varphi) = \int_{H \cap (x^{-1}.M)\backslash H} \langle \varphi(xh), \eta \rangle d\dot{h}, \varphi \in J_\chi. \tag{5.2}$$

Then one has (cf. [BD], Theorem 2.8):

> For χ in an open dense subset O of $X(M)_\sigma$, $j(P, \delta_\chi, \eta)$ extends uniquely to an H-invariant linear form $\xi(P, \delta_\chi, \eta)$ on I_{δ_χ}. There exists a non zero polynomial q on $X(M)_\sigma$ such that for all $\varphi \in I_\delta$, the map $\chi \mapsto q(\chi)\langle \xi(P, \delta_\chi, \eta), \varphi_\chi \rangle$, defined on O, extends to a polynomial function on $X(M)_\sigma$. (5.3)

The Eisenstein integrals are defined, as rational functions of $\chi \in X(M)_\sigma$, by

$$E^G_P(\eta \otimes \varphi_\chi)(Hg) = \langle \xi(P, \delta_\chi, \eta), \pi_\chi(g)\varphi_\chi \rangle, g \in G, \varphi \in I_\delta. \tag{5.4}$$

Let $x \in \overline{\mathcal{W}}^G_M$ and $\eta \in E'^{M\cap x.H}_\delta$. Then from our choice, $x^{-1}.P$ is a σ-parabolic subgroup and $x^{-1}.M$ is its σ-stable Levi. One can choose 1 as an element of $\overline{\mathcal{W}}^G_{x^{-1}.M}$ and one has $E'^{M\cap x.H}_\delta = E'^{(x^{-1}.M)\cap H}_{x^{-1}\delta} \subset \mathcal{V}_{x^{-1}.\delta}$. Let $\chi \in X(M)_\sigma$. The map $\varphi \to \lambda(x^{-1})\varphi$ is a bijective intertwining map between $i^G_P(\delta_\chi)$ and $i^G_{x^{-1}.P}((x^{-1}\delta)_{x^{-1}\chi})$. By "transport de structure", one sees

$$E^G_P(\eta \otimes \varphi_\chi) = E^G_{x^{-1}.P}(\eta \otimes (\lambda(x^{-1})\varphi_\chi)), g \in G, \varphi \in I_\delta. \tag{5.5}$$

5.2. Examples of families of type II' of tempered functions.

We keep the notation of the previous subsection (cf. also after (3.8)). Let $E'(\delta, H)_2 =$

$\oplus_{x\in \overline{\mathcal{W}}_M^G} {E'_2}^{M\cap x.H}$. One introduces the C functions as in [CD], Proposition 6.2 (resp., Theorem 8.4). One has

$$\begin{array}{l}\text{If } \phi \in E'(\delta, H)_2 \otimes i_P^G E_\chi \text{ and } \delta \text{ and } \chi \text{ are unitary, } E_P^G(\phi) \text{ is tempered when}\\ \text{it is defined and one has the following equalites of rational functions in } \chi\text{:}\\ \text{Let } Q = LV \text{ be a } \sigma\text{-parabolic subgroup. Then}\\ E_P^G(\phi)_Q^{\mathrm{w}} = 0 \text{ if } W(L|G|M)_\sigma \text{ is empty.}\\ \text{Otherwise } E_P^G(\phi)_Q^{\mathrm{w}} \text{ is equal, with the notation of l.c., to:}\\ \qquad \sum_{w\in W(L|G|M)_\sigma} E^L_{L\cap w.P}(r_L(C(w, Q, P, \delta_\chi)\phi)).\end{array} \tag{5.6}$$

Theorem 5.1 *Let $\varphi \in I_\delta$ and $\eta \in E'(\delta, H)_2$. There exists a non zero polynomial p on $X(M)_\sigma$, such that $\chi \mapsto p(\chi)E_P^G(\eta \otimes \varphi_\chi), \chi \in X(M)_{\sigma,u}$ is a family F_χ of type II' of tempered functions on $H\backslash G$.*

Proof :

From the definition of C-functions and Proposition 6.2 in l.c., one sees that, for all σ-parabolic subgroup Q, there is a non zero polynomial p_Q in $\chi \in X(M)_\sigma$ such that its product with each term of the sum in (5.6) is polynomial. As the set of H-conjugacy classes of maximal σ-split tori is finite, the set of H-conjugacy classes of minimal σ-parabolic subgroups is finite. Let $\mathcal{F}$ be a set of representatives of these H-conjugacy classes.

Let p be the product of the polynomials p_Q when Q varies in the (finite) set $\mathcal{P}$ of σ-parabolic subgroups of G containing an element of $\mathcal{F}$. We will see that this polynomial satisfies the property of the Theorem. One has to check the various properties of the Definition 4.4.

Let μ_G be the restriction to A_G of the central character of δ. The property 1) (i) is true with μ_G as the central character of $i_P^G\delta_\chi$ restricted to A_G is the restriction to A_G of the central character of δ_χ. The exponents of $(F_\chi)_Q$ are restriction to A_Q of exponents of the Jacquet module of $i_P^G\delta_\chi$. These have the required form (see e.g. [D1], Lemma 7 (i) which is valid without assumptions on δ and χ).

It remains to check the property c) of the Definition of families of functions of type I (cf. Definition 3.5).

For this, we need to use the notation of Definition 4 in [D1]. Namely, let $\mathcal{O}$ be a C^∞ manifold, V be a vector space and for all $\nu \in \mathcal{O}$, π_ν be an admissible representation of G on V, such that the action of some maximal compact subgroup does not depend on ν. For all $v \in V$ and $g \in G$, the map $\nu \to \pi_\nu(g)v$ varies in a finite dimensional vector space of vectors fixed by some compact open subgroup. Let us assume that it is C^∞ for all $v \in V$. We say that (π_ν) is a C^∞-family of representations of G in V.

Let D be a C^∞ vector field on $\mathcal{O}$. Let us define a smooth family $(D.\pi_\nu)$ of representations of G in $V \times V$ as in [D1], Lemma 16, by:

$$(D.\pi_\nu)(g)(v_1, v_2) = (\pi_\nu(g)v_1 + D(\pi_\nu(g)v_2), \pi_\nu(g)v_2), g \in G, v_1, v_2 \in V.$$

Let $(\xi_\nu)_{\nu\in\mathcal{O}}$ be a family of linear forms on V, such that for all ν, ξ_ν is H-fixed by the dual representation of π_ν, and for all $v \in V$, $\nu \mapsto \langle v, \xi_\nu\rangle$ is C^∞ on $\mathcal{O}$. Let $v \in V$ and let us denote by F_ν the generalized coefficient coefficient $g \mapsto \langle \pi_\nu(g)v, \xi_\nu \rangle$. Then DF_ν appears as a generalized coefficient of $D.\pi_\nu$. More precisely let $\tilde{\xi_\nu} = (\xi_\nu, D\xi_\nu) \in V' \times V'$. Then

$$DF_\nu = \langle D.\pi_\nu(g)(0, v), \tilde{\xi_\nu} \rangle.$$

A simple computation shows that $\tilde{\xi_\nu}$ is H-fixed by the dual representation of $D.\pi_\nu$. This formula implies that DF_ν is in $\mathcal{A}(H \backslash G)$. Applying an induction process, one sees that it is true for any C^∞ differential operators on $\mathcal{O}$.
This applies immediately to Eisenstein integrals and this proves that our family F satisfies condition c) of Definition 3.5. Hence F is of type I'.
The decomposition of $F_Q^{\mathrm{w}, ind}$ as in (4.5) follows from (5.6). If $Q \in \mathcal{P}$ each term of this sum is of type I', as it follows from the properties of p_Q and the definition of p. If Q is H-conjugate to an element of $\mathcal{P}$, it is also true due to ([D2], Proposition 3.16 (ii)). Hence F is of type II' and the Theorem is proved. $\square$

6 Appendix: some properties of the derived group

Recall that we denote by G_{der} the group of $\mathbf{F}$-points of the derived group $\underline{G}_{der}$ of $\underline{G}$. If J is a subgroup of G we denote, unless otherwise specified, by J' the intersection of J with G_{der}. In particular, one has $G_{der} = G'$. Let $Z(G)$ the group of $\mathbf{F}$-points of the center of $\underline{G}$. We recall the following facts:

The group $G'\tilde{A}_G$ is cocompact in G and the group $(H \cap \tilde{A}_G)A_G$ is of finite index in $\tilde{A}_G$. (6.1)

If $\tilde{A}_0$ is maximal split torus of G, there exists a maximal split torus $\tilde{A}'_0$ of G' such that $\underline{\tilde{A}}_0 = \underline{\tilde{A}}_G \underline{\tilde{A}}'_0$: this has been proved for at least one $\tilde{A}_0$ in the proof of (2.11) and the result follows from the fact that all maximal split tori of G are G-conjugate. It is clear that $\tilde{A}'_0$ is the maximal split torus of $\tilde{A}_0 \cap G'$ and one has

The map $\tilde{A}_0 \to \tilde{A}'_0$ is a bijection between the set of maximal split torus of G and the set of maximal split torus of G'. (6.2)

Hence one has:

All maximal split tori of G are G'-conjugate. (6.3)

If $\lambda \in \Lambda(\tilde{A}_0)$, let P_λ be the parabolic subgroup of G which contains $\tilde{A}_0$, such that the roots of $\tilde{A}_0$ in the Lie algebra of P_λ are the roots α such that $|\alpha(\lambda)|_{\mathbf{F}} \leq 1$.
If P is a parabolic subgroup of G and $\tilde{A}_0 \subset P$, there exists $\lambda \in \Lambda(\tilde{A}_0)$ such that $P = P_\lambda$. One has seen (cf. after (2.11)) that the lattice $\Lambda(\tilde{A}'_0)\Lambda(A_G)$ is of finite index in $\Lambda(\tilde{A}_0)$. Then a power of λ is an element of this lattice, hence of the form $\lambda'\mu$ where $\lambda' \in \Lambda(\tilde{A}'_0)$ and $\mu \in \Lambda(A_G)$. One deduces from the definitions the equality:

$$P_\lambda = P_{\lambda'}. \tag{6.4}$$

Hence one can even choose $\lambda \in \Lambda(\tilde{A}'_0)$. From this and [BD] Equation (2.7), it follows easily that $P \cap G'$ is a parabolic subgroup of G'. Reciprocally if P' is a parabolic subgroup of G' then there exists $\lambda \in \Lambda(\tilde{A}'_0)$ such that $P' = P_\lambda \cap G'$. Looking to Lie algebras, one sees that P_λ is the unique parabolic subgroup of G such that $P' = P_\lambda \cap G'$. Altogether we have shown:

The map $P \mapsto P \cap G'$ is a bijection between the sets of parabolic subgroups of G and G'. (6.5)

If P and Q are opposed parabolic subgroups of G, one can choose a maximal split torus $\tilde{A}_0 \subset P \cap Q$ and $\lambda \in \Lambda(\tilde{A}_0)$ such that $P = P_\lambda$ and $Q = P_{\lambda^{-1}}$. As above we can take $\lambda \in \Lambda(\tilde{A}'_0)$. This implies that $P \cap G'$ and $Q \cap G'$ are opposed parabolic subgroups of G'. One shows similarly that if P', Q' are opposed parabolic subgroups of G' and P (resp., Q) is the unique parabolic subgroup of G which contains P' (resp., Q') then P and Q are opposed.

It follows easily that the map $P \mapsto P' = P \cap G'$ is a bijection between the sets of σ-parabolic subgroups of G and G', and in particular between the sets of minimal σ-parabolic subgroups. Then it follows:

The map $M \mapsto M \cap G'$ is a bijection for the sets Levi subgroups of σ-parabolic subgroups of G and G', (6.6)

which can be specialized to Levi subgroups of minimal σ-parabolic subgroups. The map which associates to such a Levi subgroup its unique maximal σ-split torus is a bijection (cf. [HW] Proposition 4.7 and Lemma 4.5). Hence it follows that the correspondence which associates to a maximal σ-split torus A of G the maximal split torus A' of its intersection with G' is a bijection between the sets of maximal σ-split tori of G and G'. Then one has:

The split torus A is the unique maximal σ-split torus such that $A' \subset A$. (6.7)

This implies, for reason of dimensions, that $\underline{A} = \underline{A'}\underline{A_G}$. From which it follows:

Let A_1 be a maximal σ-split torus. If $A'_1 = g'.A'$, for some $g' \in G'$, one has $A_1 = g'.A$. (6.8)

Hence it follows from (6.3) that:

All the maximal σ-split tori of G are G'-conjugate. (6.9)

Let $P = MU$ be a σ-parabolic subgroup of G. Recall that $M' := M \cap G'$. Let us show that

$$A_{M'} \subset A_M. \tag{6.10}$$

One has only to check that $A_{M'}$ is in the center of M. But the derived group $\underline{M}_{der}$ of $\underline{M}$ is contained in $\underline{G}_{der}$, hence contained in $\underline{M'}$. As $\underline{M}$ is the almost product of $\underline{M}_{der}$ and its center, an element of $\underline{M}$ which commutes with $\underline{M}_{der}$ is an element of the center. Our claim follows easily.

There exists $\lambda \in \Lambda(A_M)$ such that $P = P_\lambda$. As in the proof of (6.5), one shows that:

There exists $\lambda \in A_{M'}$ such that $P = P_\lambda$. (6.11)

Applying (6.5) and (6.6) to a σ-stable Levi subgroup L of a σ-parabolic subgroup of G and to L' one sees:

The map $P \mapsto P \cap L'$ is a bijection between the sets of σ-parabolic subgroups of L and L'. The map $M \mapsto M \cap L'$ is a bijection for the sets of Levi subgroups of σ-parabolic subgroups of L and L', (6.12)

As the derived group of L is contained in L', by (6.1) applied to L instead of G, one deduces that

If a finite set of minimal σ-parabolic subgroups of L' leads to a Cartan decomposition for $(L' \cap H)\backslash L'$, the corresponding family of minimal σ-parabolic subgroup of L leads to a Cartan decomposition for $(L \cap H)\backslash L$. (6.13)

Let us prove:

If $f \in \mathcal{A}((L \cap H)\backslash L)$ then $f_{|L'} \in \mathcal{A}((H \cap L')\backslash L')$ and for any σ-parabolic subgroup $P = MU$ of L, one has $(f_P)_{|M\cap L'} = (f_{|L'})_{P\cap L'}$. Moreover, if $f \in \mathcal{A}_{temp}((H \cap L)\backslash L)$ then $f_{|L'} \in \mathcal{A}_{temp}((H \cap L')\backslash L')$ and $(f_P)^{\mathrm{w}}_{|M\cap L'} = (f_{|L'})^{\mathrm{w}}_{P\cap L'}$. (6.14)

It follows from Lemma 2.1 in [GK], that a finitely generated admissible L-module is also an admissible finitely generated L_{der}-module. Hence the same property is true for L'. This implies easily the former half of our first claim.
The space V of restriction to $(L' \cap H)\backslash L'$ of smooth functions on $(L \cap H)\backslash L$ is L'-invariant. From the properties of the constant term of smooth functions on $(L \cap H)\backslash L$ and the characterization of the constant term of the elements of V ([D2] Proposition 3.14), one deduces the latter half of our first claim.
As the exponents of $(f_P)_{|M\cap L'}$ are the restrictions to $A_M \cap L'$ of exponents of f_P, one deduces the second part of our claim. This achieves to prove (6.14).
Together with (6.12) this implies

If F be a family of type I of tempered functions on $(H \cap L)\backslash L$ then $F_{|L'}$ is a family of type I of tempered functions on $(H \cap L')\backslash L'$. (6.15)

Let us prove that there exists constant $C, C' > 0$ and $d \in \mathbb{N}$ such that

$$\begin{aligned} CN_d((H \cap L)l)^{-1}\Theta_L((H \cap L)l) &\leq \Theta_{L'}((H \cap L')l) \\ &\leq C'N_d((H \cap L)l)\Theta_L((H \cap L)l), l \in L'. \end{aligned} \tag{6.16}$$

From (2.27) applied to L and L' this holds for $l \in A^-_{P'}$ for every minimal σ-parabolic subgroup P' of L'. From this fact, from the Cartan decomposition and the invariance of $\Theta_{L'}$ (resp., Θ_L) by a compact open subgroup of L' (resp., L) the required inequality is a consequence of the following assertion applied to L and L'.

There exists $d \in \mathbb{N}$, and for all $g_1 \in G$ there exists $c, c' > 0$ such that: (6.17)

$$cN_{-d}(Hg)\Theta_G(Hg) \leq \Theta_G(Hgg_1) \leq c'\Theta_G(g)N_d(Hg), g \in G.$$

The right inequality is simply (2.24). The left inequality follows from the right one applied to gg_1 instead of g and g_1^{-1} instead of g_1 as $N(Hgg_1) \leq CN(Hg)N(Hg_1)$ for some $C > 0$.

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